SeMA Journal
https://doi.org/10.1007/s40324-025-00392-w

On the accurate basis conversion of univariate polynomials

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Received: 2 December 2024 / Accepted: 19 April 2025

Abstract
This paper summarizes the bidiagonal factorizations of change of basis matrices between polynomial bases commonly used in interpolation and computer-aided geometric design. These factorizations enable the efficient and highly accurate resolution of linear algebra problems, offering significant advantages in terms of precision. The article also highlights a variety of applications, including the accurate computation of divided differences and the efficient solution of algebraic problems involving collocation matrices or Gram matrices, underscoring the practical relevance of these factorizations in mathematical and computational contexts.

Keywords High relative accuracy · Totally positive matrices · Bidiagonal decompositions · Lagrange bases · Newton bases · Bernstein bases · Said-Ball bases · q-Bernstein bases

Mathematics Subject Classification 65G50 · 41A05 · 65D17 · 65F05 · 65F15

1 Introduction

Many interpolation, numerical quadrature, and approximation problems involve algebraic computations with collocation matrices derived from specific bases. For instance, in Lagrange interpolation, collocation matrices play a crucial role. Similarly, Gramian matrices are essential for transforming nonorthogonal bases into orthogonal ones and are used in least-squares approximations, where their inversion supports curve fitting through basis functions.

However, as the dimension grows, these matrices often become highly ill-conditioned, rendering standard numerical methods incapable of achieving accurate solutions. Addressing

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Published online: 05 May 2025

this, a key challenge in Numerical Linear Algebra is the development of algorithms that ensure high relative accuracy (HRA), maintaining errors on the order of machine precision regardless of matrix size or traditional conditionings.

One of the key milestones in this field is the work of Gasca and Peña on nonsingular totally positive matrices, characterized by nonnegative minors. In [22–24], it is shown that such matrices can be decomposed into products of bidiagonal factors. This factorization leverages the inherent total positivity property to achieve highly accurate computations. Providing to HRA the bidiagonal factorization of a totally positive matrix enables solving related algebraic problems with exceptional precision. Recent research has actively explored bidiagonal decompositions for various totally positive bases, further advancing this field (see [15, 16, 47], and references therein).

Vandermonde matrices have relevant applications in interpolation and numerical quadrature (see [21, 51]). Studies on bidiagonal factorizations of Vandermonde matrices explore efficient ways to handle these matrices in various computational applications. This type of factorization often relies on algorithms such as the Björck–Pereyra algorithm, which provides a structured approach to solve linear systems involving Vandermonde matrices (cf. [48]). Many authors have extended these factorization methods, connecting them with Neville elimination, where intermediate matrix transformations yield bidiagonal forms.

The Vandermonde matrix associated with a sequence of distinct nodes is the collocation matrix of a monomial basis but also defines the linear transformation between a monomial basis and the corresponding Lagrange polynomial basis. Building on the second idea, [42] investigates the change of basis matrices between the monomial and Newton polynomial bases. These matrices exhibit a triangular structure, and their total positivity is fully characterized based on the signs of the associated nodes. A bidiagonal decomposition is derived, demonstrating that linear algebra problems involving these matrices can be solved to HRA as long as all nodes share the same sign, even if the matrices are not totally positive.

Subsequently, [30] explores the bidiagonal factorization of the collocation matrices for Newton bases corresponding to a given sequence of interpolation nodes. Conditions for their total positivity are established, along with a fast algorithm for computing both their bidiagonal factorization and divided differences.

The Bernstein basis is a cornerstone in numerical analysis and computer-aided geometric design (CAGD), providing a robust framework for representing polynomials. Named after Sergei Bernstein, these polynomials are defined over compact intervals and are highly valued for their numerical stability and geometric properties (cf. [9, 19]). As a result, Bernstein polynomials have practical relevance not only in CAGD but also across various mathematical fields (see [6, 7, 18, 20] and references therein). For example, Bernstein bases play a critical role in solving elliptic and hyperbolic partial differential equations using Galerkin or collocation methods (cf. [6, 7]). They also find applications in optimal control theory (cf. [58]) and stochastic dynamics (cf. [35]). Furthermore, Bernstein polynomials are fundamental in approximation theory, serving as the foundation for proving the Weierstrass approximation theorem (cf. [5]). Overall, Bernstein bases are indispensable in polynomial approximation, interpolation, and curve design, offering stability and intuitive control over polynomial representations.

It is well-established that a Bernstein basis on a compact interval is the normalized B-basis of the corresponding polynomial space. This property ensures that any totally positive polynomial basis of the space can be expressed as the product of the Bernstein basis and a totally positive matrix. On the interval [0, 1], any monomial basis is totally positive and so, can be represented in terms of the corresponding Bernstein basis using a triangular matrix whose rows correspond to Pascal's triangle. These matrices are commonly referred to as Pascal matrices.

Pascal matrices and their generalized forms find applications in a wide range of fields, including filter design, probability, combinatorics, signal processing, and electrical engineering (see [39], and references therein). Notably, Lv et al. [39] explores their role in solving linear systems. A remarkable feature of Pascal matrices is their well-known bidiagonal decomposition (41), which consists entirely of ones.

Other polynomial bases that are totally positive on [0, 1] also hold significant interest for design purposes. Among these are the Said-Ball bases and the q-Bernstein bases. The linear transformation matrices connecting these bases to Bernstein bases provide a means to relate their respective control polygons, further highlighting their utility in geometric and computational applications.

This article illustrates and summarizes the bidiagonal factorizations of the change of basis matrices between polynomial bases of interest in interpolation and CAGD. It compiles the derived bidiagonal factorizations, which enable solving linear algebra problems to HRA. Additionally, it highlights numerous applications, including precise calculations of divided differences and efficient solutions to algebraic problems involving collocation or Gram matrices.

To provide a comprehensive overview, Sect. 2 reviews foundational concepts related to total positivity and HRA. Section 3 examines the polynomial bases commonly used in Lagrange interpolation problems, including the monomial, Lagrange, and Newton bases. Intriguing applications of the proposed bidiagonal factorizations are explored. Section 4 focuses on the bidiagonal factorizations of matrices that relate polynomial bases relevant to CAGD, highlighting their applications in the accurate resolution of algebraic problems involving their Gramian matrices.

2 Basic aspects on total positivity and high relative accuracy

When computing in a given floating-point arithmetic, we say that an algorithm is performed to HRA if the relative errors in the computations have the order of the unit round-off or machine precision, without being affected by the dimension or the conventional conditionings of the problem to be solved.

Algorithms achieving HRA are well known to satisfy the Non-Inaccurate Cancellation (NIC) condition. This condition is met when computations avoid subtractive cancellations, relying instead on operations such as products, quotients, and additions of numbers with the same sign [14, page 52]. Additionally, when the floating-point arithmetic is robustly implemented, even the subtraction of initial data can be performed without compromising HRA [14, page 53].

A matrix is called totally positive (TP) if all its minors are nonnegative, and strictly totally positive (STP) if all its minors are positive (see [2]). In the literature, TP and STP matrices are also referred to as totally nonnegative and totally positive matrices, respectively (see [17, 33]). Applications of TP and STP matrices are very interesting and diverse, with notable examples discussed in [2, 17, 56].

A system $(u_0, \dots, u_n)$ of functions defined on $I \subseteq \mathbb{R}$ is said to be TP if, for any sequence $t_1 < \cdots < t_{n+1}$ in the domain, the corresponding collocation matrix

$$M\begin{bmatrix} u_0, \dots, u_n \\ t_1, \dots, t_{n+1} \end{bmatrix} := \big(u_{j-1}(t_i)\big)_{1\le i,j\le n+1}. \tag{1}$$

is TP. On the other hand, if the functions of a TP system sum up to one, that is,

$$\sum_{i=0}^{n} u_i(t) = 1, \quad t \in I,$$

the system $(u_0, \ldots, u_n)$ is said to be normalized totally positive (NTP).

NTP bases play a fundamental role in CAGD due to their shape-preserving properties. These properties guarantee that the parametric curve $\gamma(t) = \sum_{i=0}^{n} P_i u_i(t)$, $t \in I$, closely resembles the shape of its control polygon $P_0 \cdots P_n$. Within the set of NTP bases for a given space of functions, there exists a unique NTP basis that exhibits optimal shape-preserving characteristics (cf. [10]). This basis, known as the normalized B-basis, ensures that any parametric curve within the space provides the best possible approximation to the shape of its control polygon, surpassing the performance of other NTP bases. Notable examples of B-bases include the Bernstein basis, the B-spline basis, and the basis used in the construction of Non-Uniform Rational B-Spline (NURBS) curves.

As a consequence of Corollary 3.10 and Proposition 3.11 of [10], B-bases can be characterized as follows.

Theorem 1 *A TP basis $(u_0, \ldots, u_n)$ is a B-basis of a vector space of functions U if and only if, for any other TP basis $(v_0, \ldots, v_n)$ of U, the change of basis matrix A such that*

$$(v_0, \ldots, v_n) = (u_0, \ldots, u_n)A$$

is TP.

From Theorem 3.1 of [2], the product of TP matrices is a TP matrix. Consequently, an interesting topic in the literature about TP matrices is their factorization as the product of simpler TP matrices. An efficient tool to derive bidiagonal decompositions of TP matrices is the Neville elimination, which has also been used to improve some well-known characterizations of TP and STP matrices.

According to [24, Theorem 4.2 and page 116] any nonsingular TP matrix $A \in \mathbb{R}^{(n+1)\times(n+1)}$ has the decomposition

$$A = F_n F_{n-1} \cdots F_1 D G_1 G_2 \cdots G_n, \tag{2}$$

where $F_i \in \mathbb{R}^{(n+1)\times(n+1)}$ and $G_i \in \mathbb{R}^{(n+1)\times(n+1)}$, $i = 1, \ldots, n$, are TP, lower and upper triangular bidiagonal matrices. Specifically,

$$F_i = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & m_{i+1,1} & 1 & & \\ & & & \ddots & \ddots & \\ & & & & m_{n+1,n+1-i} & 1 \end{pmatrix}, \quad G_i^T = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & \widetilde{m}_{i+1,1} & 1 & & \\ & & & \ddots & \ddots & \\ & & & & \widetilde{m}_{n+1,n+1-i} & 1 \end{pmatrix}, \tag{3}$$

and $D \in \mathbb{R}^{(n+1)\times(n+1)}$ is a diagonal matrix with positive diagonal entries, which are the diagonal pivots $p_{i,i}$ from the Neville elimination of A (see [22–24]). Neville elimination also yields multipliers $m_{i,j}$ and $\widetilde{m}_{i,j}$ in F_i and G_i, satisfying

$$m_{i,j} = \frac{p_{i,j}}{p_{i-1,j}}, \tag{4}$$

with $p_{i,1} := a_{i,1}$, $1 \le i \le n+1$, and

$$p_{i,j} := \frac{\det A[i-j+1,\dots,i \mid 1,\dots,j]}{\det A[i-j+1,\dots,i-1 \mid 1,\dots,j-1]}, \quad 1 < j \le i \le n+1. \tag{5}$$

Here, $A[i_1,\dots,i_r \mid j_1,\dots,j_s]$ denotes the submatrix of A formed by rows $i_1,\dots,i_r$ and columns $j_1,\dots,j_s$. Similarly, the entries $\widetilde{m}_{i,j}$ are given by

$$\widetilde{m}_{i,j} = \frac{\widetilde{p}_{i,j}}{\widetilde{p}_{i-1,j}}, \tag{6}$$

where $\widetilde{p}_{i,j}$ can be obtained as in (5) but with A^T in place of A.

In addition, if $m_{i,j}$, $\widetilde{m}_{i,j}$ satisfy

$$m_{i,j} = 0 \quad \Rightarrow \quad m_{h,j} = 0,\ \forall h > i \quad \text{and} \quad \widetilde{m}_{i,j} = 0 \quad \Rightarrow \quad \widetilde{m}_{h,j} = 0,\ \forall h > i,$$

then the decomposition (2) is unique. In [32], the bidiagonal factorization (2) is encoded by a matrix $BD(A) \in \mathbb{R}^{(n+1)\times(n+1)}$ such that

$$BD(A)_{i,j} := \begin{cases} m_{i,j}, & i > j, \\ p_{i,i}, & i = j, \\ \widetilde{m}_{j,i}, & i < j, \end{cases} \tag{7}$$

For example, the bidiagonal factorization of the matrix

$$A = \begin{pmatrix} 2 & 6 & 24 \\ 10 & 36 & 198 \\ 20 & 114 & 950 \end{pmatrix}$$

is

$$A = \begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&2&1 \end{pmatrix} \begin{pmatrix} 1&0&0 \\ 5&1&0 \\ 0&7&1 \end{pmatrix} \begin{pmatrix} 2&0&0 \\ 0&6&0 \\ 0&0&8 \end{pmatrix} \begin{pmatrix} 1&3&0 \\ 0&1&9 \\ 0&0&1 \end{pmatrix} \begin{pmatrix} 1&0&0 \\ 0&1&4 \\ 0&0&1 \end{pmatrix},$$

and

$$BD(A) = \begin{pmatrix} 2&3&4 \\ 5&6&9 \\ 2&7&8 \end{pmatrix}.$$

The computational cost of generating the matrix representation (7) for the bidiagonal factorization (2) of a nonsingular TP matrix A is determined by the number of arithmetic operations required to compute the pivots and multipliers during Neville elimination. Compact formulas for these elements, derived from the entries of A, can be obtained using (4) and (6). This paper will demonstrate that recurrence relations for the entries of $BD(A)$ can often be established, enabling their efficient computation.

The matrix $BD(A)$ in (7) can be considered as an alternative parameterization for nonsingular and TP matrices exploiting the total positivity property. In fact, it enables algorithms that preserve the total positivity structure, ensuring accurate computations for A, including its inverse A^{-1}, singular values, eigenvalues, and solutions to $Ax = b$ for specific b, as demonstrated in [33].

For a nonsingular totally positive (TP) matrix A with its $BD(A)$ representation provided to HRA, the software library `TNTool` [34] offers MATLAB functions specifically designed to compute with HRA precision:

- `TNInverseExpand`(B): Returns A^{-1} with $O(n^2)$ cost.

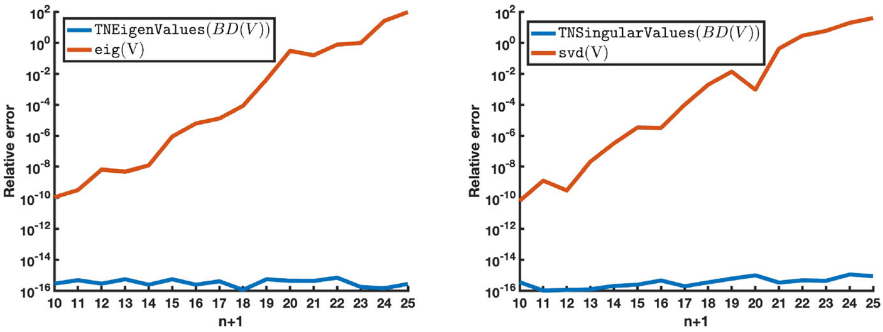

Fig. 1 Relative error in the approximations to the smallest eigenvalue (left) and singular value (right) of $V = (t_i^{j-1})_{1 \le i,j \le n+1}$ with $t_i = i/(n+1)$, $i = 1, \ldots, n+1$

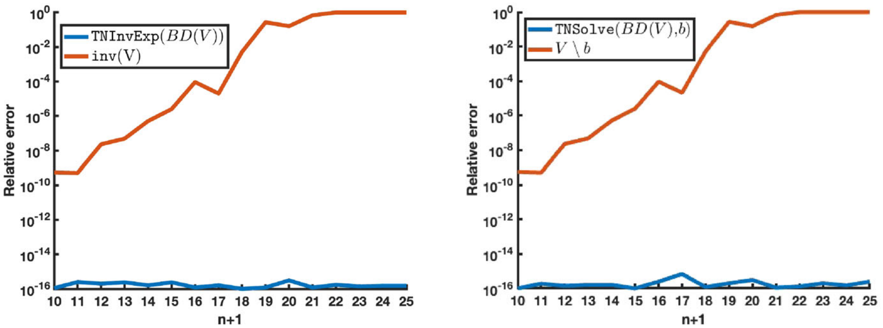

Fig. 2 Relative error in the approximations to V^{-1} and the solution of $Vx = b$ with $V = (t_i^{j-1})_{1 \le i,j \le n+1}$ with $t_i = i/(n+1)$, $i = 1, \ldots, n+1$ and b having alternating signs components

- `TNSingularValues`(B): Computes singular values of A with $O(n^3)$ cost.
- `TNEigenValues`(B): Computes eigenvalues of A with $O(n^3)$ cost.
- `TNSolve`(B, d): Solves $Ax = b$ with $O(n^2)$ cost.

The function `TNSolve` achieves HRA for vectors b with alternating signs, leveraging the TP structure for highly accurate solutions.

The relative errors in the resolution of the above mentioned algebraic problems when considering Vandermonde matrices V for nonnegative nodes in increasing order are depicted in Figs. 1 and 2. Since $BD(V)$ can be computed to HRA, the computation of both the smallest eigenvalue and singular value achieve great accuracy regardless of the 2-norm condition number of the considered matrices. In contrast, the Matlab commands `eig` and `svd` return results that are not accurate at all (see Fig. 1).

As illustrated in Fig. 2, leveraging $BD(V)$ ensures highly accurate results for computing V^{-1} and solving linear systems $Vx = b$, where b components alternate in sign. In contrast, MATLAB's standard methods yield significantly less accurate outcomes, highlighting the advantages of using $BD(V)$ for these computations.

At this point a natural question that arises is if the above nice results can be extended beyond the TP matrices family. In this sense, let us mention as an example a consequence of [42, Theorem 2], stated in the following result.

Lemma 1 *Let $A \in \mathbb{R}^{(n+1)\times(n+1)}$ and $J := diag((-1)^{i-1})_{1\le i\le n+1}$. Then, A is the inverse of a TP matrix if and only if the matrix $A_J := JAJ$ is nonsingular and TP.*

Using Lemma 1, if $BD(A_J)$ can be computed to HRA, we can ensure HRA computations with non TP matrices A whose inverses are TP. Since J is a unitary matrix, the eigenvalues and singular values of A are identical to those of A_J. For accurate computation of A^{-1}, A_J^{-1} can be computed to HRA, and $A^{-1} = JA_J^{-1}J$ is obtained by appropriately adjusting the signs of A_J^{-1}. Then, it is possible to solve $Ax = b$, where all entries of b have the same sign, defining $d := Jb$, computing to HRA the solution $y \in \mathbb{R}^{n+1}$ for $A_J y = d$, and eventually recovering $x = Jy$.

In more general terms, a relevant study of non TP matrices that can be converted into a totally positive version by means of a suitable transformation was performed in [3]. There, the class of matrices that admit a signed bidiagonal decomposition was analyzed, identifying several equivalent properties that characterize this family (see Theorem 3.1 of the mentioned reference). Further discussion on this topic can be found in [4] for sign consistent and sign regular matrices, whereas works as [29] and references therein address consecutive-rank-descending matrices, which also extend the class of TP matrices, analyzing the subject from a different perspective.

3 Accurate matrix conversion for Lagrange interpolation

In the sequel, $\mathbf{P}^n(I)$ denotes the space of polynomials of a degree not greater than n defined on $I \subseteq \mathbb{R}$.

Polynomial interpolation is a classical technique in numerical analysis, providing a means to estimate values of a function based on a set of known data points. For the Lagrange interpolation problem, given a function $f : I \to \mathbb{R}$ and nodes $t_1, \ldots, t_{n+1}$, with $t_i \in I$, $i = 1, \ldots, n+1$, and $t_i \neq t_j$ for $i \neq j$, we seek $p_n \in \mathbf{P}^n(I)$ such that

$$p_n(t_i) = f(t_i), \quad i = 1, \ldots, n+1. \tag{8}$$

When considering a basis $(b_0, \ldots, b_n)$ of $\mathbf{P}^n(I)$, the interpolating polynomial satisfying (8) can be expanded in terms of the considered basis,

$$p_n(t) = \sum_{i=0}^{n} c_{i+1} b_i(t), \quad t \in I,$$

and the coefficients $c_1, \ldots, c_{n+1}$ form the solution of the linear system

$$M\begin{bmatrix} b_0, \ldots, b_n \\ t_1, \ldots, t_{n+1} \end{bmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_{n+1} \end{pmatrix} = \begin{pmatrix} f(t_1) \\ \vdots \\ f(t_{n+1}) \end{pmatrix}.$$

In particular, when considering the monomial basis $(m_0, \ldots, m_n)$ of $\mathbf{P}^n(I)$, with $m_i(t) = t^i$, for $i = 0, \ldots, n$, the interpolant p_n is represented as

$$p_n(t) = \sum_{i=0}^{n} c_{i+1} m_i(t), \tag{9}$$

where $c = (c_1, \ldots, c_{n+1})^T$ solves the linear system

$$Vc = f, \tag{10}$$

with $f := (f(t_1), \dots, f(t_{n+1}))^T$ and V is the collocation matrix (1) of the monomial basis, which coincides with the Vandermonde matrix V at the interpolation nodes,

$$V := M\begin{bmatrix} m_0, \dots, m_n \\ t_1, \dots, t_{n+1} \end{bmatrix} = (t_i^{j-1})_{1\le i,j\le n+1}. \tag{11}$$

The following result can be found in [32, Section 3] or [40, Theorem 3].

Theorem 2 *For positive nodes $t_1 < \cdots < t_{n+1}$, V is STP allowing a bidiagonal factorization of the form* (2),

$$V = F_n F_{n-1} \cdots F_1 D G_1 G_2 \cdots G_n, \tag{12}$$

such that $BD(V) = (BD(V)_{i,j})_{1\le i,j\le n+1}$, with

$$BD(V)_{i,j} := \begin{cases} t_i, & i < j, \\ \prod_{k=1}^{i-1}(t_i - t_k), & i = j, \\ \prod_{k=1}^{j-1} \frac{t_i - t_{i-k}}{t_{i-1}-t_{i-k-1}}, & i > j. \end{cases} \tag{13}$$

Equation (13) yields a matrix that can be computed using only products and subtractions of the initial data, in this case, the nodes $t_1, \dots, t_{n+1}$. This characteristic ensures that no subtractions of intermediate results occur and the NIC condition is satisfied, leading to HRA in the computation of $BD(V)$. As a result, linear algebra problems involving the Vandermonde matrix V can be solved with relative errors close to machine precision. Remarkably, this high accuracy remains unaffected by the size of the Vandermonde matrix, making it a robust approach for computations involving V (see Figs. 1 and 2). The recursive relation

$$BD(V)_{i,j} = \frac{t_i - t_{i-j+1}}{t_{i-1} - t_{i-j}} BD(V)_{i,j-1}, \quad i > j,$$

allows for the computation of the matrix $BD(V)$ in $O(n^2)$ arithmetic operations. Consequently, this representation of V allows for solving the aforementioned algebraic problems with a computational cost comparable to that of the corresponding functions in the `TNTool` library [34].

Expressing the interpolating polynomial in terms of monomials reveals significant insights into its structure and behavior. By studying these monomial-based representation, we gain a better understanding of the stability and efficiency of interpolation methods, especially when applied to large sets of data points.

Alternatively, p_n can also be formulated in the Lagrange form, where the polynomial is represented as a sum of terms that reflect the product of monomials associated with the interpolation nodes. Each term in the polynomial captures the influence of a specific data point, and the use of monomials as the building blocks for each term allows for a straightforward interpretation of the behavior of the polynomial. This form provides both computational and theoretical advantages, as the structure of the polynomial becomes clearer and easier to manipulate in many contexts.

When considering interpolation nodes $t_1, \dots, t_{n+1}$ such that $t_i \ne t_j$ for $i \ne j$, and the corresponding Lagrange polynomial basis $(\ell_0, \dots, \ell_n)$, with

$$\ell_i(t) := \prod_{j\ne i} \frac{t - t_{j+1}}{t_{i+1} - t_{j+1}}, \quad i = 0, \dots, n, \tag{14}$$

the Lagrange formula for the polynomial interpolant p_n is

$$p_n(t) = \sum_{i=0}^{n} f(t_{i+1})\ell_i(t). \tag{15}$$

Let us observe that the collocation matrix (1) of the Lagrange basis $(\ell_0, \dots, \ell_n)$ at the interpolation nodes $t_1, \dots, t_{n+1}$ is the $(n+1) \times (n+1)$ identity matrix.

From (15), (9), and (10), we have

$$p_n(t) = (\ell_0(t), \dots, \ell_n(t))f = (m_0(t), \dots, m_n(t))V^{-1}f,$$

where $f = (f(t_1), \dots, f(t_{n+1}))^T$. Thus, we deduce that

$$(m_0, \dots, m_n) = (\ell_0, \dots, \ell_n)V. \tag{16}$$

Identity (16) shows that the Vandermonde matrix $V \in \mathbb{R}^{(n+1)\times(n+1)}$ in (11) acts as the change of basis matrix between the monomial and the Lagrange bases of the polynomial space $\mathbf{P}^n(I)$ for the given interpolation nodes.

Instead of Lagrange formulas, one can use the Newton form of the interpolant, which is derived by expressing the interpolating polynomial in terms of a specific basis known as the Newton basis, defined by terms that successively incorporate each data point in the interpolation set.

Let us recall that the Newton basis $(w_0, \dots, w_n)$ determined by the interpolation nodes $t_1, \dots, t_{n+1}$ is

$$w_0(t) := 1, \quad w_i(t) := \prod_{k=1}^{i}(t - t_k), \quad i = 1, \dots, n. \tag{17}$$

Then, the interpolating polynomial p_n can be written as follows,

$$p_n(t) = \sum_{i=0}^{n}[t_1, \dots, t_{i+1}]f\, w_i(t), \tag{18}$$

where $[t_1, \dots, t_i]f$ represents the divided difference of f at the nodes $t_1, \dots, t_i$.

Among the various approaches, the Newton form of the interpolating polynomial is notable for its recursive structure, which leverages divided differences to construct the polynomial incrementally. If f is n-times continuously differentiable on $[t_1, t_{n+1}]$, the divided differences $[t_1, \dots, t_i]f$, for $i = 1, \dots, n+1$, can be computed using the following recursion,

$$[t_i, \dots, t_{i+k}]f = \begin{cases} \frac{[t_{i+1},\dots,t_{i+k}]f - [t_i,\dots,t_{i+k-1}]f}{t_{i+k}-t_i}, & t_{i+k} \neq t_i, \\ \frac{f^{(k)}(t_i)}{k!}, & t_{i+k} = t_i. \end{cases} \tag{19}$$

The divided differences depend only on the interpolation nodes and, once computed, the interpolating polynomial (18) can be evaluated in $O(n)$ flops per evaluation. Additionally, for two functions f and g defined on an interval containing the nodes $t_1, \dots, t_{n+1}$, the following Leibniz-type formula for divided differences holds:

$$[t_1, \dots, t_{n+1}](fg) = \sum_{k=1}^{n+1}[t_1, \dots, t_k]f\,[t_k, \dots, t_{n+1}]g.$$

The following result can be found in [30, Theorem 1] and provides conditions so that the divided differences can be computed to HRA using the recursion (19).

Theorem 3 *Let $t_1, \ldots, t_{n+1}$ be strictly ordered nodes and f a function such that the entries of the vector $(f(t_1), \ldots, f(t_{n+1}))$ can be computed to HRA and have alternating signs. Then, the divided differences of order $k = 0, \ldots, n$,*

$$[t_i, \ldots, t_{i+k}]f, \quad i = 1, \ldots, n+1-k,$$

have alternating signs and, using recurrence (19), can be computed to HRA.

Let us note that, since $m_i(t) = t^i$, $i = 0, \ldots, n$, coincides with its interpolating polynomial at the set of nodes $\{t_1, \ldots, t_{j+1}\}$, for $j = i, \ldots, n$, we can deduce using the Newton interpolation formula (18), that the change of basis matrix U, which satisfies

$$(m_0, \ldots, m_n) = (w_0, \ldots, w_n)U, \tag{20}$$

is given by $U = (u_{i,j})_{1 \le i,j \le n+1}$, where

$$u_{i,j} = [t_1, \ldots, t_i]m_{j-1}.$$

Consequently, U can be expressed explicitly as:

$$U = \begin{pmatrix} 1 & [t_1]m_1 & [t_1]m_2 & \cdots & [t_1]m_n \\ 0 & 1 & [t_1, t_2]m_2 & \cdots & [t_1, t_2]m_n \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & [t_1, \ldots, t_n]m_n \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}. \tag{21}$$

On the other hand, by induction, it can be verified that

$$[t_1, \ldots, t_{i+1}]m_j = \sum_{\alpha_1 + \cdots + \alpha_{i+1} = j-i+1} t_1^{\alpha_1} \cdots t_{i+1}^{\alpha_{i+1}}, \quad j > i.$$

Thus, the collocation matrix (1) of the Newton basis $(w_0, \ldots, w_n)$ at nodes $t_1, \ldots, t_{n+1}$ is a lower triangular matrix $L = (l_{i,j})_{1 \le i,j \le n+1}$ whose entries are given by

$$l_{i,j} = w_{j-1}(t_i) = \prod_{k=1}^{j-1}(t_i - t_k), \quad 1 \le i, j \le n+1.$$

Explicitly, L can be expressed as:

$$L = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & t_2 - t_1 & 0 & \cdots & 0 \\ 1 & t_3 - t_1 & (t_3 - t_1)(t_3 - t_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 1 & t_{n+1} - t_1 & (t_{n+1} - t_1)(t_{n+1} - t_2) & \cdots & (t_{n+1} - t_1) \cdots (t_{n+1} - t_n) \end{pmatrix}. \tag{22}$$

Taking into account (16) and (20), we obtain the following Crout factorization of the Vandermonde matrix V at nodes $t_1, \ldots, t_{n+1}$:

$$V = M\begin{bmatrix} \ell_0, \ldots, \ell_n \\ t_1, \ldots, t_{n+1} \end{bmatrix} = M\begin{bmatrix} m_0, \ldots, m_n \\ t_1, \ldots, t_{n+1} \end{bmatrix} = M\begin{bmatrix} w_0, \ldots, w_n \\ t_1, \ldots, t_{n+1} \end{bmatrix} U = LU. \tag{23}$$

This factorization can be utilized to solve the linear Vandermonde system $Vx = f$ by solving the systems $Ld = f$ and $Ux = d$ sequentially.

In the context of Lagrange interpolation problems, the vectors $d = (d_1, \dots, d_{n+1})^T$ and $f = (f_1, \dots, f_{n+1})^T$, where

$$d_i = [t_1, \dots, t_i]f \quad \text{and} \quad f_i = f(t_i), \quad i = 1, \dots, n+1,$$

are related by the system:

$$Ld = f.$$

Thus, the change of basis matrix U in (21) relates the solution vector x with an intermediate vector d of divided differences (see [8]).

In [42, Theorem 3], the expressions for the diagonal pivots and multipliers in the Neville elimination of the matrix U in (21) are derived. Subsequently, the decomposition (2) is obtained.

Theorem 4 *Let U be the $(n+1) \times (n+1)$ change of basis matrix in (21), representing the transformation between the monomial basis and the Newton basis corresponding to the nodes $t_1, \dots, t_{n+1}$. Then,*

$$U = G_1 \cdots G_n, \tag{24}$$

and $BD(U) = \big(BD(U)_{i,j}\big)_{1 \le i,j \le n+1}$ is given by:

$$BD(U)_{i,j} = \begin{cases} t_{i-1}, & 2 \le i < j \le n+1, \\ 1, & 1 \le i = j \le n+1, \\ 0, & \textit{otherwise}. \end{cases} \tag{25}$$

Taking into account the factorization derived in Theorem 4, the total positivity of U and U^{-1} is analyzed.

Corollary 1 *Let U be the $(n+1) \times (n+1)$ change of basis matrix in (21), representing the transformation between the monomial basis and the Newton basis corresponding to the nodes $t_1, \dots, t_{n+1}$. Then,*

a) U (respectively, U^{-1}) *is TP if and only if* $t_i \ge 0$ (respectively, $t_i \le 0$), $i = 1, \dots, n$.
b) JUJ (respectively, $JU^{-1}J$) *is TP if and only if* $t_i \le 0$ (respectively, $t_i \ge 0$), $i = 1, \dots, n$.

Taking into account the identities in (23), along with the bidiagonal factorization of the Vandermonde matrix (11) (see (12) or (13)) and the $(n+1) \times (n+1)$ change of basis matrix U in (21) (see (25)), the bidiagonal factorization of the collocation matrix L in (22) for the Newton basis can be derived (see [30, Theorem 2]).

Theorem 5 *Let $L \in \mathbb{R}^{(n+1)\times(n+1)}$ be the collocation matrix described by (22) of the Newton basis (17) corresponding to interpolation nodes $t_1, \dots, t_{n+1}$, with $t_i \ne t_j$ for $i \ne j$. Then,*

$$L = F_n F_{n-1} \cdots F_1 D, \tag{26}$$

with $BD(L) = (BD(L)_{i,j})_{1 \le i,j \le n+1}$ given by

$$BD(L)_{i,j} := \begin{cases} 0, & i < j, \\ \prod_{k=1}^{i-1} (t_i - t_k), & i = j, \\ \prod_{k=1}^{j-1} \frac{t_i - t_{i-k}}{t_{i-1} - t_{i-k-1}}, & i > j. \end{cases} \tag{27}$$

The analysis of the signs of the entries in (27) enables the characterization of the total positivity of the collocation matrix (1) of the Newton basis in terms of the ordering of the nodes. This result is formalized in [30, Corollary 2].

Corollary 2 *Let $L \in \mathbb{R}^{(n+1)\times(n+1)}$ be the collocation matrix described by (22) of the Newton basis (17) corresponding to interpolation nodes $t_1, \ldots, t_{n+1}$, with $t_i \neq t_j$ for $i \neq j$ and J be the diagonal matrix $J := diag((-1)^{i-1})_{1\leq i\leq n+1}$.*

a) *The matrix L is TP if and only if $t_1 < \cdots < t_{n+1}$. Furthermore, in this case, both L and the matrix $BD(L)$ in (27) can be computed to HRA.*
b) *The matrix $L_J := LJ$ is TP if and only if $t_1 > \cdots > t_{n+1}$. Moreover, in this case, the matrix $BD(L_J)$ can also be computed to HRA.*

The divided differences $d_i := [t_1, \ldots, t_i]f$, $i = 1, \ldots, n+1$, associated with the Newton form of the Lagrange interpolant,

$$p_n(t) = \sum_{i=0}^{n} d_{i+1} w_i(t), \tag{28}$$

are typically computed using the recursion in (19). However, they can alternatively be obtained by solving the linear system:

$$Ld = f,$$

where $d := (d_1, \ldots, d_{n+1})^T$ and $f := (f_1, \ldots, f_{n+1})^T$. This can be efficiently done using the MATLAB function `TNSolve` available on Koev's web page [34], with the bidiagonal decomposition of L, given in (27), provided as an input argument.

Taking into account Corollary 2, if the elements of f are provided to HRA and exhibit alternating signs, the computation of d can also achieve HRA, provided that the nodes are strictly ordered.

The numerical experimentation illustrated in [30, Section 2] compare the vectors of divided differences obtained using the provided bidiagonal factorization, the recurrence (19) and, finally, the Matlab command \ for the resolution of linear systems.

3.1 Applications to Touchard bases

Given $n \in \mathbb{N} \cup \{0\}$, the falling factorial is the polynomial defined by:

$$(t)_0 := 1, \quad (t)_n := t(t-1)\cdots(t-n+1), \quad n \in \mathbb{N}. \tag{29}$$

On the other hand, the rising factorial is the polynomial defined by:

$$t^{\bar{0}} := 1, \quad t^{\bar{n}} := t(t+1)\cdots(t+n-1).$$

Let us observe that the polynomial systems $((t)_0, (t)_1, \ldots, (t)_n)$ and $(t^{\bar{0}}, t^{\bar{1}}, \ldots, t^{\bar{n}})$ are bases of $\mathbf{P}^n(\mathbb{R})$ that can be seen as Newton bases (17) corresponding to particular sequences of interpolation nodes: $t_i = i$, $i = 0, \ldots, n-1$, for the falling factorial polynomial bases, or $t_i = -i$, $i = 0, \ldots, n-1$, for the rising factorial polynomial bases.

First kind Stirling numbers $s(n,k)$ can be computed recursively, with

$$s(n+1,k) = -n \cdot s(n,k) + s(n,k-1), \tag{30}$$

and the conventions $s(0,0) := 1$, $s(0,n) = s(n,0) := 0$, $n \in \mathbb{N}$.

Stirling numbers obtained by (30) arise in combinatorics, when analyzing permutations, and coincide with the coefficients of the expansion of the falling factorial polynomials in

(29) in terms of the monomial basis $(m_0, \ldots, m_n)$ of $\mathbf{P}^n(\mathbb{R})$. In fact,

$$(t)_n = \sum_{k=0}^{n} s(n,k)t^k = \sum_{k=0}^{n} s(n,k)m_k(t). \tag{31}$$

It can be checked that $\operatorname{sign}(s(n,k)) = (-1)^{n-k}$ and so, Stirling numbers of the first kind are also called signed Stirling numbers. The absolute values of the first kind Stirling numbers are known as unsigned Stirling numbers and are usually denoted by $c(n,k)$ or $\left[{n \atop k}\right]$,

$$c(n,k) = \left[{n \atop k}\right] = (-1)^{n-k}s(n,k).$$

Unsigned Stirling numbers can be computed recursively since they satisfy the following identities:

$$c(n+1,k) = -n \cdot c(n,k) + c(n,k-1),$$

with the conventions $c(0,0) := 1$, $c(0,n) = c(n,0) := 0$, $n \in \mathbb{N}$. They are the coefficients in the expansion of the rising factorial polynomial in terms of the monomial basis $(m_0, \ldots, m_n)$ of $\mathbf{P}^n(\mathbb{R})$:

$$t^{\bar{n}} = \sum_{k=0}^{n} c(n,k)t^k = \sum_{k=0}^{n} c(n,k)m_k(t).$$

Conversely, the Stirling numbers of the second kind, usually denoted by $S(n,k)$ or $\left\{{n \atop k}\right\}$, are the coefficients in the expansion of the monomials polynomials in terms of the polynomial basis form by the falling factorial polynomials (29),

$$\sum_{k=0}^{n} S(n,k)(t)_k = t^n = m_n(t). \tag{32}$$

These numbers count the number of partitions of a set of size n into k disjoint non-empty subsets and can be computed using the relation

$$S(n+1,k) = k \cdot S(n,k) + S(n,k-1), \tag{33}$$

with the conventions $S(n,n) := 1$, $n \geq 0$, $S(0,n) = S(n,0) := 0$, $n \in \mathbb{N}$.

Stirling numbers of the second kind can be seen as divided differences of monomials with respect to the set of nodes formed by the first consecutive nonnegative integers. In particular,

$$S(n,k) = [0,1,\ldots,k]m_n,$$

for $m_n(t) = t^n$. Moreover, the linear transformation between the monomial and the falling factorial polynomial bases can be analyzed through the matrix $U = (u_{i,j})_{1\leq i,j\leq n+1}$ of the change of bases, with

$$u_{i,j} = \begin{cases} [0,\ldots,i-1]m_{j-1} = S(j-1,i-1), & i \leq j, \\ 0, & i > j. \end{cases} \tag{34}$$

We shall say that the matrix U, whose entries are given in (34), is the $(n+1)\times(n+1)$ second kind Stirling matrix. As a direct consequence of Theorem 4 and Corollary 1, we can deduce that $BD(U) = \big(BD(U)_{i,j}\big)_{1\leq i,j\leq n+1}$ is

$$BD(U)_{i,j} = \begin{cases} i-1, & i < j, \\ 1, & i = j, \\ 0, & i > j. \end{cases}$$

Clearly, the decomposition (2) of U can be computed to HRA. Consequently, the eigenvalues and singular values of U, as well as the solution of the linear systems $Ux = b$, where the entries of b have alternating signs, can be obtained to HRA.

Furthermore, using (31), we can also deduce that the inverse of the second kind Stirling matrix U described by (34) is the upper triangular $\widetilde{U} := U^{-1} = (\widetilde{u}_{i,j})_{1\le i,j\le n+1}$ such that

$$\widetilde{u}_{i,j} = \begin{cases} s(j-1, i-1), & i \le j, \\ 0, & i > j, \end{cases} \tag{35}$$

where $s(n,k)$ denotes the corresponding (signed) first kind Stirling number provided by (30). We shall say that this matrix is the $(n+1)\times(n+1)$ signed Stirling matrix.

Let us consider $J := \operatorname{diag}((-1)^{i-1})_{1\le i\le n+1}$ and the signed Stirling matrix $\widetilde{U}$ in (35). Using Theorem 4, we derive that $\widehat{U} := J\widetilde{U}J$ is TP and admits a bidiagonal factorization that can be represented by

$$BD(\widehat{U})_{i,j} = \begin{cases} j-i-1, & i < j, \\ 1, & i = j, \\ 0, & i > j. \end{cases}$$

The Touchard basis of $\mathbf{P}^n(\mathbb{R})$ is the polynomial system $(T_0, \dots, T_n)$, with

$$T_n(t) = \sum_{k=0}^{n} S(n,k)t^k, \tag{36}$$

where $S(n,k)$, $k = 0, \dots, n$ are the second type Stirling numbers in (32). The polynomials T_k, $k = 0, \dots, n$, are also called the exponential polynomials or Bell polynomials (cf. [59]).

Touchard polynomials are also called the exponential polynomials and generalize the Bell polynomials for the enumeration of the permutations when the cycles possess certain properties. Algebraic, combinatorial and probabilistic properties of these polynomials are described in [11, 49, 54, 59]. Let us observe that

$$(T_0, \dots, T_n) = (m_0, \dots, m_n)U, \tag{37}$$

where $(m_0, \dots, m_n)$ is the monomial basis of $\mathbf{P}^n$ and $U = (u_{i,j})_{1\le i,j\le n+1}$ is the $(n+1)\times(n+1)$ second kind Stirling matrix, that is,

$$u_{i,j} = \begin{cases} S(j-1, i-1), & i \le j, \\ 0, & i > j, \end{cases} \tag{38}$$

(see (34)).

Given $0 < t_1 < \cdots < t_{n+1}$, by formula (37), the corresponding collocation matrix (1) of the Touchard basis satisfies

$$T = VU,$$

where V is the Vandermonde matrix (11) and U is the $(n+1)\times(n+1)$ second kind Stirling matrix described by (38).

Taking into account the total positivity of the Vandermonde matrices at positive nodes in increasing ordering and the total positivity of the second kind Stirling matrices, we can deduce the the collocation matrices (1) of Touchard bases are STP, since they are products of an STP matrix and a nonsingular TP matrix (see Theorem 3.1 of [2]). Consequently, we can guarantee that the Touchard polynomial basis $(T_0, \dots, T_n)$ is STP on $(0, \infty)$.

Moreover, using Algorithm 5.1 of [33], if the decomposition (2) of two nonsingular TP matrices is provided to HRA, then the decomposition of the product can be obtained to HRA. Consequently, T and its decomposition (2) can be obtained to HRA.

4 Accurate matrix conversions with polynomial bases used in CAGD

For a given degree n, the Bernstein polynomials on the interval [0, 1] are defined as:

$$B_{i,n}(t) := \binom{n}{i} t^i (1-t)^{n-i}, \quad t \in [0,1], \quad i = 0, 1, \ldots, n, \tag{39}$$

where $\binom{n}{i}$ represents the binomial coefficient. It can be easily checked that the following partition of unity property holds:

$$\sum_{i=0}^{n} B_{i,n}(t) = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} = (1-t+t)^n = 1, \quad t \in [0,1].$$

The monomial basis can be represented in terms of the Bernstein basis of $\mathbf{P}^n([0,1])$. Specifically, the monomial $m_k(t) = t^k$ can be expressed as a linear combination of Bernstein basis polynomials of degree $n \geq k$, using the following formula:

$$t^k = \frac{1}{\binom{n}{k}} \sum_{i=k}^{n} \binom{i}{k} B_i^n(t), \quad k \leq n.$$

The identities above are extensively considered in approximation theory and geometric modeling, and they can be succinctly expressed in the following matrix form:

$$(m_0, \ldots, m_n) = (B_0^n, \ldots, B_n^n) DL, \tag{40}$$

where $D = (d_{i,j})_{1 \leq i,j \leq n+1}$ is a diagonal matrix with $d_{i,i} = \binom{n}{i-1}^{-1}$, $i = 1, \ldots, n+1$, and L is a lower triangular matrix with the rows of Pascal's triangle, that is, $L = (l_{i,j})_{1 \leq i,j \leq n+1}$ with

$$l_{i,j} = \begin{cases} 0, & i < j, \\ \binom{i-1}{j-1}, & i \geq j. \end{cases} \tag{41}$$

The well-known bidiagonal decomposition of a Pascal matrix (41) possesses the notable feature of being composed entirely of ones, that is,

$$BD(L)_{i,j} = \begin{cases} 0, & i < j, \\ 1, & i = j, \\ 1, & i > j, \end{cases} \tag{42}$$

(cf. [1, 33]). Furthermore, [12, Section 3] provides the bidiagonal factorization (2) for generalized triangular Pascal matrices and lattice path matrices, which encompass numerous classical generalized Pascal matrices. In many cases, it has been shown that these matrices are TP or STP, allowing various algebraic computations with these matrices to be performed to HRA.

Let us observe that, taking into account (42), it can be easily checked that the matrix DL relating the monomial and the Bernstein basis of $\mathbf{P}^n([0,1])$ (see (40)) is TP and its bidiagonal factorization (2) can be represented by the matrix $BD(DL) = (BD(DL)_{i,j})_{1\le i,j\le n+1}$, such that

$$BD(DL)_{i,j} = \begin{cases} 0, & i < j, \\ \binom{n}{i-1}^{-1}, & i = j, \\ 1, & i > j. \end{cases} \tag{43}$$

Bernstein polynomials are square-integrable functions with respect to the inner product

$$\langle f, g\rangle := \int_0^1 f(t)g(t)\,dt. \tag{44}$$

Although the Bernstein basis $(B_0^n, \dots, B_n^n)$ is not orthogonal, it can be transformed into an orthogonal basis of $\mathbf{P}^n([0,1])$ using its Gramian (mass) matrix $M = (M_{i,j})_{1\le i,j\le n+1}$, defined as

$$M_{i,j} = \int_0^1 B_i^n(t)B_j^n(t)\,dt = \frac{\binom{n}{i-1}\binom{n}{j-1}(i+j-2)!(2n-i-j+2)!}{(2n+1)!}, \quad 1\le i,j\le n+1, \tag{45}$$

(see [31]).

The constrained dual Bernstein basis $(D_0^n, \dots, D_n^n)$, which satisfies $\langle D_i^n, B_j^n\rangle = \delta_{i,j}$, can be expressed in terms of the mass matrix M as

$$(B_0^n, \dots, B_n^n)^T = M(D_0^n, \dots, D_n^n)^T,$$

where M serves as the change of basis matrix between the constrained dual Bernstein basis and the Bernstein basis (cf. [36–38, 60, 61]).

According to [41, Theorem 2], the mass matrix M is STP. Furthermore, M admits a bidiagonal factorization

$$M = F_nF_{n-1}\cdots F_1DG_1\cdots G_{n-1}G_n, \tag{46}$$

with entries given explicitly by

$$BD(M)_{i,j} = \begin{cases} \frac{(n-i+2)(2n-i+3)}{(2n-i-j+3)(2n-i-j+4)}, & i > j, \\ \left(\binom{n}{i-1}\frac{(i-1)!}{(2n-i+2)!}\right)^2 (2n-2i+2)!(2n-2i+3)!, & i = j, \\ \frac{(n-j+2)(2n-j+3)}{(2n-i-j+3)(2n-i-j+4)}, & i < j. \end{cases} \tag{47}$$

The diagonal pivots $p_i := BD(M)_{i,i}$, $i = 1, \dots, n+1$, can be computed recursively:

$$p_1 = \frac{1}{2n+1}, \quad p_{i+1} = \frac{(2n-i+2)^2}{4(2n-2i+1)(2n-2i+3)}p_i, \quad i = 1, \dots, n,$$

allowing the construction of the matrix $BD(M)$ in $O(n^2)$ arithmetic computations. This bidiagonal factorization can be computed to HRA, as demonstrated in numerical experiments in [41, Section 5].

Finally, the following corollary extends the total positivity property to Gram matrices of any TP polynomial basis.

Corollary 3 *Let $(p_0^n, \dots, p_n^n)$ be a TP basis of $\mathbf{P}^n([0,1])$. The Gram matrix M_p of $(p_0^n, \dots, p_n^n)$ with respect to the inner product defined in (44) is TP. If the change of basis matrix A, such that*

$$(p_0^n, \dots, p_n^n)^T = A(B_0^n, \dots, B_n^n)^T,$$

can be computed to HRA, then M_p and its inverse can also be computed to HRA.

4.1 Matrix conversion between Bernstein and Said-Ball bases

The Bernstein basis is well-suited for applications in Bézier curve construction due to its close association with the geometric representation of curves. Bézier curves are defined as linear combinations of the Bernstein polynomials, with the control points serving as coefficients. This geometric perspective establishes Bernstein polynomials as a fundamental tool in CAGD. In fact, Bernstein bases are the polynomial bases most used in CAGD because they have optimal shape preserving and stability properties (see [19, 20]). These nice properties are related to the fact that $(B_0^n, \dots, B_n^n)$ is the normalized B-basis of the polynomial space $\mathbf{P}^n([0,1])$ (cf. [9, 10]) and then, any TP basis $(p_0^n, \dots, p_n^n)$ of $\mathbf{P}^n([0,1])$ can be written in terms of the B-basis as follows,

$$(p_0^n, \dots, p_n^n) = (B_0^n, \dots, B_n^n) A_n, \tag{48}$$

where the change of bases matrix A_n is nonsingular and TP (see Theorem 1). Moreover, if the polynomial basis $(p_0^n, \dots, p_n^n)$ is normalized, A_n is stochastic (see Theorem 4.3 of [10]).

Another important class of polynomial bases with applications in CAGD is formed by the Said-Ball bases, which were defined by Said in [57] for spaces of odd degree polynomials on $[0, 1]$ and later, extended to spaces of polynomials of even degree in [28]. Said-Ball polynomial bases have the same type of shape preserving properties as the Bernstein bases (see [9, 25, 45]) and provide an algorithm with less computational cost than the evaluation algorithm for the Bernstein basis (see [28]). Computations to HRA with the collocation matrices (1) of the Said-Ball bases were achieved in [45] and [46]. In [53] an explicit formula for the dual basis functions for generalized Ball bases is provided.

The Said-Ball basis $(s_0^n, \dots, s_n^n)$ of $\mathbf{P}^n([0,1])$ is defined by:

$$s_i^n(t) := \binom{\lfloor n/2 \rfloor + i}{i} t^i (1-t)^{\lfloor n/2 \rfloor + 1}, \quad i = 0, \dots, \lfloor (n-1)/2 \rfloor,$$

$$s_i^n(t) := \binom{\lfloor n/2 \rfloor + n - i}{n - i} t^{\lfloor n/2 \rfloor + 1} (1-t)^{n-i}, \quad i = \lfloor n/2 \rfloor + 1, \dots, n,$$

and, for an even n,

$$s_{n/2}^n(t) := \binom{n}{n/2} t^{n/2} (1-t)^{n/2},$$

where $\lfloor a \rfloor$ denotes the greatest positive integer less than or equal to a, for a given $a > 0$.

In Theorem 1 of [25] it was proved that the Said-Ball basis $(s_0^n, \dots, s_n^n)$ is a NTP basis of $\mathbf{P}^n([0,1])$ for odd degrees n. Later, in Proposition 3 of [13], this property was also proved for even degrees. Taking into account that the Bernstein basis on $[0, 1]$ (see (39)) is the normalized B-basis of $\mathbf{P}^n([0,1])$, we can immediately deduce from Theorem 1 that the change of basis matrix A such that

$$(s_0^n, \dots, s_n^n)^T = A (B_0^n, \dots, B_n^n)^T$$

is TP. Observe that, since both bases are normalized, we also have that A^T is also stochastic. Using formula (7) of [28], A can be described as $A = (a_{i,j})_{1 \le i, j \le n+1}$, with

$$a_{i,j} = \begin{cases} \binom{k+i-1}{i-1} \binom{k-i+1}{j-i} \Big/ \binom{2k+1}{j-1}, & 1 \le i \le j \le k+1, \\ \binom{3k-i+2}{2k-i+2} \binom{i-k-2}{i-j} \Big/ \binom{2k+1}{j-1}, & k+2 \le j \le i \le 2k+2, \\ 0, & \text{elsewhere}, \end{cases} \tag{49}$$

for odd degree $n = 2k + 1$, $k \in \mathbb{N}$ and

$$a_{i,j} = \begin{cases} \binom{k+i-1}{i-1}\binom{k-i}{j-i}/\binom{2k}{j-1}, & 1 \le i \le j \le k, \\ \binom{3k-i+1}{2k-i+1}\binom{i-k-2}{i-j}/\binom{2k}{j-1}, & k+2 \le j \le i \le 2k+1, \\ 1, & i = j = k+1, \\ 0, & \text{elsewhere}, \end{cases} \tag{50}$$

for even degree $n = 2k$, $k \in \mathbb{N}$.

A corner cutting algorithm for obtaining the Bézier polygon of a polynomial curve from its control polygon with respect to the generalized Ball basis is constructed in Sect. 3 of [25]. This corner cutting provides a factorization of the stochastic and TP matrix A^T in terms of TP, bidiagonal and stochastic matrices. In contrast, the diagonal pivots and the multipliers of the Neville ellimination of A and then its bidiagonal factorization (2) are obtained in [44, Section 4], showing that the computation of this new factorization satisfies the NIC condition.

In [44, Theorem 4], the Neville elimination of A is studied and its bidiagonal factorization is provided.

Theorem 6 *Let A be the change of basis matrix described by (49) for $n = 2k + 1$ and by (50) for $n = 2k$. Then its bidiagonal factorization (2) can be represented by means of $BD(A) = (BD(A)_{i,j})_{1 \le i,j \le n+1}$ such that*

$$BD(A)_{i,j} := \begin{cases} \binom{k+i-1}{i-1}/\binom{2k+1}{i-1}, & 1 \le i = j \le k+1, \\ (k-j+2)/(2k-j+3), & 1 \le i < j \le k+1, \\ \binom{3k-i+2}{2k-i+2}/\binom{2k+1}{i-1}, & k+2 \le i = j \le 2k+2, \\ (2k-i+3)/(3k-i+3), & k+2 \le j < i \le 2k+2, \\ 0, & \textit{elsewhere}, \end{cases}$$

for $n = 2k + 1$ and $BD(A) = (BD(A)_{i,j})_{1 \le i,j \le n+1}$ such that

$$BD(A)_{i,j} := \begin{cases} \binom{k+i-1}{i-1}/\binom{2k}{i-1}, & 1 \le i = j \le k, \\ (k-j+1)/(2k-j+2), & 1 \le i < j \le k, \\ 1, & \textit{if } i = j = k+1, \\ \binom{3k-i+1}{2k-i+1}/\binom{2k}{i-1}, & k+2 \le i = j \le 2k+1, \\ (2k-i+2)/(3k-i+2), & k+2 \le j < i \le 2k+1, \\ 0, & \textit{elsewhere}, \end{cases}$$

for even $n = 2k$.

For the efficient computation of the diagonal pivots $p_{i,i}$, recursive formulas can be derived depending on whether n is odd or even.

For $n = 2k + 1$,

$$p_{1,1} = 1, \quad p_{i+1,i+1} = \frac{k+i}{2k-i+2}\, p_{i,i}, \quad i = 1, \ldots, k,$$
$$p_{2k+2,2k+2} = 1, \quad p_{i,i} = \frac{3k-i+2}{i}\, p_{i+1,i+1}, \quad i = 2k+1, \ldots, k+2.$$

For $n = 2k$,

$$p_{1,1} = 1, \quad p_{i+1,i+1} = \frac{k+i}{2k-i+1}\, p_{i,i}, \quad i = 1, \ldots, k,$$
$$p_{k+1,k+1} = 1, \quad p_{2k+1,2k+1} = 1, \quad p_{i,i} = \frac{3k-i+1}{i}\, p_{i+1,i+1}, \quad i = 2k+1, \ldots, k+2.$$

These recursions enable the computation of the matrix $BD(A)$ with a computational cost of $O(n^2)$ arithmetic computations.

The TP change of basis matrix A, relating the Said-Ball basis $(s_0^n, \ldots, s_n^n)^T$ to the Bernstein basis $(B_0^n, \ldots, B_n^n)^T$, satisfies

$$(s_0^n, \ldots, s_n^n)^T = A(B_0^n, \ldots, B_n^n)^T$$

and can be computed to HRA. The Gram matrix M of the Bernstein basis and the Gram matrix M_{SB} of Said-Ball basis are related by

$$M_{SB} = AMA^T.$$

The previous property guarantees that M_{SB} can also be computed to HRA.

Theorem 7 *The Gram matrix M_{SB} of the Said-Ball basis for $\mathbf{P}^n([0, 1])$ with respect to the inner product (44) is TP. Furthermore, M_{SB} and its inverse can be computed to HRA.*

4.2 Matrix conversion between Bernstein and q-Bernstein bases

In recent years, quantum calculus (q-calculus) has garnered significant attention due to its potential applications in mathematics, mechanics and physics. This branch of mathematics replaces classical concepts with their q-analogues, such as q-integers, q-binomial coefficients, and other q-derived constructs, offering a versatile framework for exploring discrete phenomena (cf. [27]).

Let us recall that the q-binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q$, $k = 0, \ldots, n$, are given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q!\,[n-k]_q!},$$

where, for any non-negative integer n, the q-factorial $[n]_q!$ is defined by

$$[0]_q! := 1, \quad [n]_q! := [n]_q\,[n-1]_q \cdots [1]_q, \quad n \in \mathbb{N},$$

and the q-integer $[n]_q$ is

$$[n]_q := \begin{cases} 1 + q + \cdots + q^{n-1} = \dfrac{1-q^n}{1-q}, & q \neq 1 \\ n, & q = 1. \end{cases}$$

Clearly, $[n]_q$ is a polynomial in q and $[n]_q > 0$, for any $q \in (0, 1]$, $n \in \mathbb{N}$. Moreover, the q-binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q$, $k = 0, \ldots, n$, are also polynomials in q with integer coefficients, which are known as Gaussian polynomials.

It can be checked that the q-binomial coefficients satisfy useful identities, such as,

$$\frac{[\alpha]_q}{[n]_q}\begin{bmatrix} \alpha-1 \\ n-1 \end{bmatrix}_q = \begin{bmatrix} \alpha \\ n \end{bmatrix}_q, \quad \frac{[\alpha-n]_q}{[n]_q}\begin{bmatrix} \alpha-1 \\ n-1 \end{bmatrix}_q$$

$$= \begin{bmatrix} \alpha-1 \\ n \end{bmatrix}_q, \frac{[\alpha]_q}{[\alpha-n+1]_q} \begin{bmatrix} \alpha-1 \\ n-1 \end{bmatrix}_q$$
$$= \begin{bmatrix} \alpha \\ n-1 \end{bmatrix}_q.$$

The q-Bernstein polynomials are a generalization of classical Bernstein polynomials, incorporating the principles of quantum calculus. These polynomials depend on a parameter q, which allows for more flexible and nuanced representations, particularly in discrete or non-standard settings. They are widely applied in approximation theory, where they exhibit shape-preserving properties, and in the modeling of Bézier curves with shifted nodes. Additionally, their iterative properties and convergence rates have been studied extensively, proving useful in both theoretical and applied contexts (cf. [50, 52, 55]).

The q-Bernstein polynomials of degree n on [0, 1] are defined as

$$Q_k^n(t) := \begin{bmatrix} n \\ k \end{bmatrix}_q t^k \prod_{r=0}^{n-k-1} (1-q^r\, t), \quad k=0,\dots,n. \tag{51}$$

For the particular case $q = 1$, the q-Bernstein basis (51) coincides with the Bernstein basis (39).

By Corollary 3.3 of [10], for any $q \in (0, 1]$, the basis $(Q_0^n, \dots, Q_n^n)$ is TP on the interval [0, 1] and STP on (0, 1). Moreover, the partition of unity property satisfied by $(Q_0^n, \dots, Q_n^n)$ can be deduced using Proposition 5.2 of [26]. Then we can guarantee that the change of basis matrix A such that

$$(Q_0^n, \dots, Q_n^n) = (B_0^n, \dots, B_n^n)L, \tag{52}$$

is nonsingular, stochastic and TP. In [43, Section 3], it is shown that $L = (l_{i,j})_{1\le i,j\le n+1}$ can be described by

$$\widetilde{l}_{i,j} = \begin{cases} 0, & i<j, \\ \dfrac{\left[{n \atop i-1}\right]_q}{\binom{n}{i-1}}, & i=j, \\ \dfrac{\left[{n \atop j-1}\right]_q}{\binom{n}{i-1}} (1-q)^{i-j} \displaystyle\sum_{\alpha_1<\cdots<\alpha_{i-j}\subset \mathcal{I}_{n-j}} \prod_{k=1}^{i-j} [\alpha_k]_q, & j<i, \end{cases}$$

with the notation $\mathcal{I}_k := \{1, \dots, k\}$, for $k \in \mathbb{N}$ (see also Sect. 2 of [50]).

Let us recall that the converse of a matrix $A = (a_{i,j})_{1\le i,j\le n}$ is defined as

$$A^\# = (a^\#_{i,j})_{1\le i,j\le n} := (a_{n+1-i,n+1-j})_{1\le i,j\le n}. \tag{53}$$

Clearly, the converse $A^\#$ can be written as $A^\# = RAR$, where R is the $n \times n$ matrix obtained by reversing the order of the rows of the identity $n \times n$ matrix.

In order to achieve a suitable factorization of the matrix L satisfying (52), let us define the matrix $U := L^\#$, that can be considered as the change of basis matrix such that

$$(Q_n^n, \dots, Q_0^n) = (B_n^n, \dots, B_0^n)U. \tag{54}$$

In [43, Corollary 6], the Neville elimination of U is studied and its bidiagonal factorization is provided.

Theorem 8 *For $q \in (0,1]$, the matrix U satisfying* (54) *is TP and its bidiagonal factorization* (2) *can be represented through $BD(U) = \left(BD(U)_{i,j}\right)_{1\le i,j\le n+1}$, with*

$$BD(U)_{i,j} = \begin{cases} \frac{[n-j+2]_q}{[j-1]_q}(1-q^{j-i}), & i<j, \\ \frac{\left[{n \atop n-i+1}\right]_q}{\binom{n}{n-i+1}}, & i=j, \\ 0, & i>j. \end{cases}$$

The computation of the matrix $BD(U)$ can be performed with $O(n^2)$ arithmetic computations. Theorem 8 is applied in [43, Section 5] to prove that q-Bernstein mass matrices are TP and derive accurate computations when solving algebraic problems with these matrices.

Author Contributions The authors contributed equally to this work.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature. This work was partially supported through the Spanish research grants PID2022-138569NB-I00 and RED2022-134176-T (MCI/AEI) and by Gobierno de Aragón (E41_23R).

Data availability Not applicable.

Declarations

Conflict of interest The authors have no conflict of interest to declare that are relevant to the content of this article.

Ethical approval Not applicable.

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