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Regular Articles

Regular fractional weighted Wiener algebras and invariant subspaces

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A B S T R A C T

Since the fifties, the interplay between spectral theory, harmonic analysis and a wide variety of techniques based on the functional calculus of operators, has provided useful criteria to find non-trivial closed invariant subspaces for operators acting on complex Banach spaces. In this article, some standard summability methods (mainly the Cesàro summation) are applied to generalize classical results due to Wermer [51] and Atzmon [8] regarding the existence of invariant subspaces under growth conditions on the resolvent of an operator. To do so, an extension of Beurling's regularity criterion [13] is proved for fractional weighted Wiener algebras $\mathcal{A}_\rho^\alpha$ related with the Cesàro summation of order $\alpha \geq 0$. At the end of the article, other summability methods are considered for the purpose of finding new sufficient criteria which ensure the existence of invariant subspaces, resulting in several open questions on the regularity of fractional weighted Wiener algebras $\mathcal{A}_\rho^\mu$ associated to matrix summation methods defined from non-vanishing complex sequences.

1. Introduction

Consider an infinite-dimensional complex Banach space X. Let $\mathcal{L}(X)$ denote the class of linear operators on X (non-necessarily bounded) and $\mathcal{B}(X)$ the algebra of bounded linear operators on X endowed with the supremum norm. For each densely-defined operator $T \in \mathcal{L}(X)$, its adjoint $T^* \in \mathcal{L}(X^*)$ is a densely-defined operator acting on the dual space X^*.

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A closed subspace (or a linear manifold) $\mathcal{M} \subseteq X$ is said to be *invariant* for a linear operator $T \in \mathcal{L}(X)$, if $Tx \in \mathcal{M}$ for each $x \in \mathcal{M}$. Furthermore, a subspace $\mathcal{M}$ is called *hyperinvariant* for T, if $\mathcal{M}$ is invariant for each operator $S \in \mathcal{L}(X)$ commuting with T. We say that $\mathcal{M}$ is non-trivial whenever $\mathcal{M} \neq \{0\}$ and $\mathcal{M} \neq X$. In this regard, one of the most relevant problems in Operator Theory is the so-called *Invariant Subspace Problem*, which, in its most general form, asked:

Invariant Subspace Problem. Does every linear bounded operator $T : X \to X$ acting on a separable infinite-dimensional complex Banach space X have a non-trivial closed invariant subspace?

As of the mid-seventies, negative solutions to the Invariant Subspace Problem arose. In 1975, Per Enflo announced the existence of a linear bounded operator on a complex Banach space without non-trivial closed invariant subspaces (see [19]). Later, further constructions followed the lines initiated by Enflo, namely: Charles J. Read [42] in 1984, whose approach was repeatedly strengthened over the years (see [43,44] and [24]) so as to find counterexamples to the Invariant Subspace Problem in ℓ^1 (as well as in c_0) with additional features such as *hypercyclicity*; and Beauzamy [10] in 1985, who obtained a counterexample satisfying the *supercyclicity* property. Nevertheless, hitherto, all the counterexamples to the Invariant Subspace Problem have been found in non-reflexive Banach spaces. In the Hilbert space case, an attempt of positive solution to the Invariant Subspace Problem has been recently uploaded by Enflo [20]. For a complete account on the Invariant Subspace Problem, we refer to the monographs [11,14].

A wide variety of techniques have been developed to find invariant subspaces for a given operator $T \in \mathcal{B}(X)$, often relying on deep connections with allied areas. Among these approaches, in this article, we will be interested in the fruitful strategy of *functional calculus*. In a few words, the underlying idea is that suitable bounds for the resolvent of an operator $T : X \to X$ allow one to transfer certain functional properties from the elements of an associated Banach algebra. Due to our interests, we will focus on *regular Banach algebras* (i.e., in a sense, those with partitions of unity) which help to produce hyperinvariant subspaces from the fact that two non-zero functions may have a null product. A cornerstone at this regard is a theorem of Wermer [51] in 1952, who employed *Beurling algebras* to determine that any invertible bounded linear operator $T : X \to X$ whose iterates verify the asymptotic condition

$$\sum_{n\in\mathbb{Z}} \frac{\log \|T^n\|}{1+n^2} < +\infty$$

and whose spectrum $\sigma(T) \subseteq \partial\mathbb{D}$ is not a singleton, has a non-trivial closed hyperinvariant subspace. The proof of Wermer's result is based on a classical regularity criterion due to Beurling [13] which dates back to 1938: given a submultiplicative sequence $\rho : \mathbb{Z} \to [1, +\infty)$ (normalized with $\rho(0) = 1$) the Beurling algebra $\mathcal{A}_\rho := \{f \in C(\partial\mathbb{D}) : \sum_{n\in\mathbb{Z}} |\widehat{f}(n)|\rho(n) < +\infty\}$ has partitions of unity whenever

$$\sum_{n\in\mathbb{Z}} \frac{\log(\rho(n))}{1+n^2} < +\infty. \tag{1.1}$$

Those weights $\rho = (\rho(n))_{n\in\mathbb{Z}}$ satisfying the *Beurling condition* (1.1) are usually known as *Beurling sequences*. Local versions of Wermer's theorem due to Foiaş and Sz.-Nagy [22, p. 74], Colojoară and Foiaş [16], Gellar and Herrero [28], Beauzamy [9], and, specially, Atzmon [7,8] have exploited this idea admitting operators $T : X \to X$ whose spectrum $\sigma(T)$ is not necessarily contained in a curve and discovering deep connections with local spectral theory and decomposable operators. More recently, other existence results have been given by Solomjak [47] and Kellay and Zarrabi [32].

A concrete application based on these methods was first published by Davie [17] in 1973, proving the existence of hyperinvariant subspaces for all *Bishop operators* $T_\alpha : L_p[0, 1) \to L_p[0, 1)$ whose irrational symbol $\alpha \in (0,1)$ was a Liouville number. Davie's approach was subsequently extended to broader classes of *weigthed translation operators* such as *Bishop-type operators* (see the articles by MacDonald [38,39]), as well as to a larger set of Bishop operators T_α, admitting some non-Liouville symbols $\alpha \in (0,1)$ (see [21] and [15]). However, Chamizo et al. [15, Theorem 4.1] have recently established an upper threshold to the applicability of this technique for Bishop operators: in the sense that, there exist so extreme irrationals $\alpha \in (0,1)$ from a diophantine point of view, for which the corresponding Bishop operator T_α cannot be handled using regular Beurling algebras (see also [12] for recent results on the linear dynamics of Bishop operators). This fact, along with other instances, encourages to develop new functional calculi which enable to deal with more general classes of operators.

In this article, we develop a generalization of these methods. Instead of bounds of the form $\|T^n x\|_X \lesssim \rho(n)$ and $\|{T^*}^n y\|_{X^*} \lesssim \rho(n)$ for a Beurling sequence $\rho = (\rho(n))_{n\in\mathbb{Z}}$, our approach will ensure the existence of hyperinvariant subspaces from bounds on the norms of weighted combinations of the powers of the operator:

$$\Big\| \sum_{j=0}^{n} \mu_{j,n} T^j x \Big\|_X \lesssim \rho(n) \quad \text{and} \quad \Big\| \sum_{j=0}^{n} \nu_{j,n} {T^*}^j y \Big\|_{X^*} \lesssim \rho(n), \qquad \text{as } n \to \pm\infty,$$

for suitable classes of weights $(\mu_{j,n})_{n\in\mathbb{Z},\, j\in\{0,\dots,n\}}$ and $(\nu_{j,n})_{n\in\mathbb{Z},\, j\in\{0,\dots,n\}}$. Observe that profitable cancellations within the sums could contribute to a controlled growth of those norms. In particular, inspired by classical tools in summability theory, we will focus on the *Cesàro summation* methods. Other summation methods will be discussed at the end of the article, highlighting that some of the techniques developed in this article could be adapted to operate in more general situations.

Given an operator $T \in \mathcal{B}(X)$, let $\mathcal{T}_T := (\mathcal{T}_T(n))_{n\geq 0}$ denote the discrete semigroup of positive powers $\mathcal{T}_T(n) := T^n$ for each $n \in \mathbb{N}_0$. Recall that the *Cesàro sums of order* $\alpha \geq 0$ of the operator $T : X \to X$ are the elements of the sequence $((\Delta^{-\alpha}\mathcal{T}_T)(n))_{n\geq 0}$ in $\mathcal{B}(X)$ given by

$$(\Delta^{-\alpha}\mathcal{T}_T)(n) := \sum_{j=0}^{n} k_\alpha(n-j)\mathcal{T}_T(j), \quad n \geq 0,$$

where $k_\alpha := (k_\alpha(n))_{n\geq 0}$ denote the *Cesàro kernel of order* α given by $k_\alpha(0) = 1$ and

$$k_\alpha(n) = \binom{n+\alpha-1}{\alpha-1} = \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!}, \quad n \in \mathbb{N}.$$

Cesàro sums are a basic tool to define the class of (C,α)*-operators*, which are a natural extension of *power-bounded operators*. Fixed $\alpha \geq 0$, an operator $T \in \mathcal{B}(X)$ is said to be a (C,α)*-operator* (or a *Cesàro bounded operator of order* α) whenever its *Cesàro means of order* $\alpha \geq 0$

$$\mathcal{M}_T^\alpha(n) := \frac{1}{k_{\alpha+1}(n)} (\Delta^{-\alpha}\mathcal{T}_T)(n), \quad n \geq 0,$$

are uniformly bounded for $n \in \mathbb{N}_0$ in the operator norm. (C,α)-operators have been widely studied in the literature: for example, their ergodic properties were initially discovered by Hille [31] and later exploited by many other authors (see, for instance, [18,48,50,52]), while growth conditions of (C,α)-operators were analyzed, for instance, by Sato et al. [34,46]. We refer to [48] for examples and properties of (C,α)-operators.

Recently, in [4] the authors harness the algebraic structure of the Cesàro sums of order $\alpha \geq 0$ to develop a functional calculus for (C,α)-operators from certain weighted convolution algebras $\tau_+^\alpha(k_{\alpha+1})$ contained in $\ell^1(\mathbb{N}_0)$. Indeed, it turns out that the (C,α)-boundedness is equivalent to the existence of an algebra homomorphism from $\tau_+^\alpha(k_{\alpha+1})$ to $\mathcal{B}(X)$, see [4, Corollary 3.7]. Additional implementations of the functional calculus for (C,α)-operators can be found in [1] but in a bilateral context, where a Katznelson-Tzafriri type theorem for (C, α)-operators is obtained using the isometric isomorphic identification between *fractional Wiener algebras* $\mathcal{A}^\alpha(\partial\mathbb{D})$ and $\tau^\alpha(|n|^\alpha)$ provided by the Fourier transform.

Inspired by such approaches, in this article, we consider bilateral weighted convolution algebras $\tau^\alpha(\rho)$ of order $\alpha \geq 0$ for symmetric weights $\rho : \mathbb{Z} \to [1, +\infty)$ belonging to the class ω_α (see Section 2.3). These weighted convolution algebras $\tau^\alpha(\rho)$ can be naturally identified with *fractional weighted Wiener algebras* $\mathcal{A}^\alpha_\rho$ of functions on the torus (see Definition 2.3), which subsume the notion of Beurling algebras $\mathcal{A}_\rho$. In Theorem 3.1, we obtain a generalization of Beurling's regularity criterion to fractional weighted Wiener algebras $\mathcal{A}^\alpha_\rho$ with $\alpha \in \mathbb{N}_0$. At the beginning of Section 4, we deal with *bilateral Cesàro sums and Cesàro means of order* $\alpha \geq 0$ for an invertible operator $T \in \mathcal{B}(X)$. Subsequently, in Theorem 4.1, we construct an algebra homomorphism $\vartheta^\alpha : \tau^\alpha(\rho) \to \mathcal{B}(X)$ valid for every invertible operator whose Cesàro sums grow as $\|(\Delta^{-\alpha}\mathcal{T}_T)(n)\| \lesssim \rho(n)$ when $n \to \pm\infty$. In Section 5, combining the regularity criterion for fractional weighted Wiener algebras $\mathcal{A}^\alpha_\rho$ and the functional calculus $\vartheta^\alpha : \tau^\alpha(\rho) \to \mathcal{B}(X)$, we obtain a generalization of the aforementioned criteria due to Wermer [51] and Atzmon [8,7] (see, for instance, Theorem 5.3). Lastly, in Section 6, we explore the effective improvement of the application of Cesàro summation methods to produce invariant subspaces and discuss further techniques based on general summation methods.

2. Preliminaries

2.1. Cesàro-Hardy operators and range spaces

Let $\mathbb{U} := \{z \in \mathbb{C} : \Im(z) > 0\}$ denote the upper half of the complex plane and $H_2(\mathbb{U})$ denote the Hardy space of all holomorphic functions $f : \mathbb{U} \to \mathbb{C}$ such that

$$\|f\|_{H_2(\mathbb{U})} := \sup_{y>0} \left(\int_{-\infty}^{\infty} |f(x+iy)|^2 \,\mathrm{d}x \right)^{1/2} < \infty.$$

Fatou's theorem ensures that, given any $f \in H_2(\mathbb{U})$, the limit $\lim_{y\to 0^+} f(x+iy)$ exists for almost every $x \in \mathbb{R}$, so we may define the boundary function on $\mathbb{R}$ as

$$f^*(x) := \lim_{y\to 0^+} f(x+iy).$$

This boundary function f^* always belongs to $L_2(\mathbb{R})$. Accordingly, $H_2(\mathbb{U})$ can be regarded as a closed subspace of $L_2(\mathbb{R})$. Indeed, the classical Paley-Wiener theorem determines that $H_2(\mathbb{U})$ is isometrically isomorphic to $L_2(\mathbb{R}_+)$ under the Fourier transform. In [26, Theorem 6.2], Galé et al. established a Paley-Wiener theorem for Cesàro-Hardy range spaces. Such spaces are defined as range spaces of Cesàro-Hardy operators. For more details of the following construction, we refer to [26].

Let us consider the generalized Cesàro operator and its adjoint in each of their $L_p(\mathbb{R}_+)$, $L_p(\mathbb{R})$, with $1 \leq p < \infty$, and $H_2(\mathbb{U})$ versions. For each $\alpha > 0$, the generalized Cesàro operator $\mathcal{C}_\alpha$ and its adjoint $\mathcal{C}^*_\alpha$ are linear bounded operators on both $L_p(\mathbb{R}_+)$ and $L_p(\mathbb{R})$ (specifically, $\mathcal{C}_\alpha$ for $1 < p \leq \infty$ while $\mathcal{C}^*_\alpha$ for $1 \leq p < \infty$, although the case $p = \infty$ will not be treated in this paper). Namely, on $L_p(\mathbb{R}_+)$ the operators $\mathcal{C}_\alpha$ and $\mathcal{C}^*_\alpha$ are defined respectively as

$$\mathcal{C}_\alpha f(t) := \frac{\alpha}{t^\alpha} \int_0^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s \quad \text{and} \quad \mathcal{C}^*_\alpha f(t) := \alpha \int_t^\infty \frac{(s-t)^{\alpha-1}}{s^\alpha} f(s)\,\mathrm{d}s,$$

for all $t > 0$ and $f \in L_p(\mathbb{R}_+)$. Whilst on the space $L_p(\mathbb{R})$, the generalized Cesàro operator $\mathcal{C}_\alpha$ and its adjoint $\mathcal{C}^*_\alpha$ are defined by the expressions

$$\mathcal{C}_\alpha f(t) := \begin{cases} \dfrac{\alpha}{t^\alpha} \displaystyle\int_0^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s, & \text{if } t > 0, \\ \dfrac{\alpha}{|t|^\alpha} \displaystyle\int_t^0 (s-t)^{\alpha-1} f(s)\,\mathrm{d}s, & \text{if } t < 0, \end{cases}$$

and

$$\mathcal{C}^*_\alpha f(t) := \begin{cases} \alpha \displaystyle\int_t^\infty \frac{(s-t)^{\alpha-1}}{s^\alpha} f(s)\,\mathrm{d}s, & \text{if } t > 0, \\ \alpha \displaystyle\int_{-\infty}^t \frac{(t-s)^{\alpha-1}}{|s|^\alpha} f(s)\,\mathrm{d}s, & \text{if } t < 0, \end{cases}$$

for each $f \in L_p(\mathbb{R})$. In both cases, one can rewrite the operators $\mathcal{C}_\alpha$ and $\mathcal{C}^*_\alpha$ respectively as

$$\mathcal{C}_\alpha f = \int_0^\infty \varphi_{\alpha,1/(1-1/p)}(t) T_p(t) f\,\mathrm{d}t, \quad \text{for } 1 < p < \infty,$$

$$\mathcal{C}^*_\alpha f = \int_0^\infty \varphi_{\alpha,p}(t) T_p(-t) f\,\mathrm{d}t, \quad \text{for } 1 \le p < \infty,$$

where $\varphi_{\alpha,p}(t) := \alpha(1-e^{-t})^{\alpha-1} e^{-t/p}$ belongs to $L_1(\mathbb{R}_+)$ and $T_p(t)f(s) := e^{-t/p} f(e^{-t}s)$. Furthermore, the one-parameter operator family $(T_p(t))_{t\in\mathbb{R}}$ is in fact a C_0-group of isometries on $L_p(\mathbb{R}_+)$ and $L_p(\mathbb{R})$ respectively for $1 \le p < \infty$, whose infinitesimal generator is $\Lambda_p f(s) := -sf'(s) - \frac{1}{p} f(s)$.

For each $1 \le p < \infty$, observe that the Cesàro-Hardy operator $\mathcal{C}^*_\alpha : L_p(\mathbb{R}_+) \to L_p(\mathbb{R}_+)$ is injective for all $\alpha > 0$. Accordingly, we denote by $\mathcal{T}^{(\alpha)}_p(t^\alpha) := \mathcal{C}^*_\alpha(L_p(\mathbb{R}_+))$ its range space, which is a Banach space endowed with the transferred norm:

$$\|f\|_{p,(\alpha)} := \Gamma(\alpha+1) \left\| \mathcal{C}^{*\,-1}_\alpha f \right\|_{L_p(\mathbb{R}_+)}, \quad f \in \mathcal{T}^{(\alpha)}_p(t^\alpha).$$

The range spaces $\mathcal{T}^{(\alpha)}_p(t^\alpha)$ and the Cesàro-Hardy operator $\mathcal{C}^*_\alpha$ are related to fractional calculus. In fact, if one considers the operator $W^\alpha : \mathcal{T}^{(\alpha)}_p(t^\alpha) \to L_p(\mathbb{R}_+, t^{p\alpha})$ (where the space $L_p(\mathbb{R}_+, t^{p\alpha})$ consists of all Lebesgue measurable functions f such that $t \mapsto t^\alpha f(t)$ belongs to $L_p(\mathbb{R}_+)$), given by $W^\alpha f(t) := \Gamma(\alpha+1) t^{-\alpha} (\mathcal{C}^{*\,-1}_\alpha f)(t)$, it is clear that its inverse is

$$W^{-\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_t^\infty (s-t)^{\alpha-1} g(s)\,\mathrm{d}s, \quad g \in L_p(\mathbb{R}_+, t^{p\alpha}).$$

Therefore, one can define $\mathcal{T}^{(\alpha)}_p(t^\alpha)$ as the space of all functions f belonging to $L_p(\mathbb{R}_+)$ for which there exists a unique element in $L_p(\mathbb{R}_+, t^{p\alpha})$ (denoted as $W^\alpha f$) such that $f = W^{-\alpha} W^\alpha f$ with norm

$$\|f\|_{p,(\alpha)} = \left(\int_0^\infty |W^\alpha f(t) t^\alpha|^p \,\mathrm{d}t \right)^{1/p}.$$

When $p = 2$, $\mathcal{T}^{(\alpha)}_2(t^\alpha)$ is a Hilbert space endowed with the inner product

$$\langle f, g\rangle_{2,(\alpha)} := \int_0^\infty W^\alpha f(t)\overline{W^\alpha g(t)}\, t^{2\alpha}\,\mathrm{d}t, \quad \text{for every } f, g \in \mathcal{T}_2^{(\alpha)}(t^\alpha).$$

Indeed, W^α is the Weyl fractional derivative $W^\alpha f = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}t^n} W^{-(n-\alpha)} f$, where $n = \lfloor \alpha \rfloor + 1$ (and W^0 is the identity operator when α is a natural number). Initially, these spaces were studied for $\alpha \in \mathbb{N}$ by Arendt and Kellerman [5], and later in the fractional context by several authors, see for example [26,37,40].

Because $\mathcal{C}_\alpha^*$ is injective on $L_p(\mathbb{R}_+)$, the adjoint of the Cesàro-Hardy operator $\mathcal{C}_\alpha^*$ is also injective on $L_p(\mathbb{R})$ for each $\alpha > 0$ and $1 \le p < \infty$. Accordingly, one can define again the range space $\mathcal{T}_p^{(\alpha)}(|t|^\alpha) := \mathcal{C}_\alpha^*(L_p(\mathbb{R}))$. Note that $\mathcal{T}_p^{(\alpha)}(|t|^\alpha)$ can be regarded as two copies of the space $\mathcal{T}_p^{(\alpha)}(t^\alpha)$. Once again, the space $\mathcal{T}_p^{(\alpha)}(|t|^\alpha)$ is a Banach space with the transferred norm:

$$\|f\|_{p,(\alpha)} := \Gamma(\alpha+1) \left\| \mathcal{C}_\alpha^{*\,-1} f \right\|_{L_p(\mathbb{R})}, \quad f \in \mathcal{T}_p^{(\alpha)}(|t|^\alpha).$$

These spaces were introduced in [37] and are closely related to fractional calculus. In fact, if we consider the operator $W^\alpha : \mathcal{T}_p^{(\alpha)}(|t|^\alpha) \to L_p(\mathbb{R}, |t|^{p\alpha})$ (where $L_p(\mathbb{R}, |t|^{p\alpha})$ denotes those Lebesgue measurable functions $f : \mathbb{R} \to \mathbb{C}$ such that the mapping $t \mapsto |t|^\alpha f(t)$ belongs to $L_p(\mathbb{R})$), given by $W^\alpha f(t) := \Gamma(\alpha+1)|t|^{-\alpha}(\mathcal{C}_\alpha^{*\,-1} f)(t)$, it is clear that it has inverse

$$W^{-\alpha} g(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_t^\infty (s-t)^{\alpha-1} g(s)\,\mathrm{d}s, & \text{if } t > 0, \\ \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{-\infty}^t (t-s)^{\alpha-1} g(s)\,\mathrm{d}s, & \text{if } t < 0, \end{cases}$$

for $g \in L_p(\mathbb{R}, |t|^{p\alpha})$. Hence, one can define the space $\mathcal{T}_p^{(\alpha)}(|t|^\alpha)$ as the space of all functions f belonging to $L_p(\mathbb{R})$ for which there exists a unique element in $L_p(\mathbb{R}, |t|^{p\alpha})$ (denoted by $W^\alpha f$) such that $f = W^{-\alpha} W^\alpha f$ with norm

$$\|f\|_{p,(\alpha)} = \left(\int_{-\infty}^\infty |W^\alpha f(t)|^p |t|^{p\alpha}\,\mathrm{d}t \right)^{1/p}.$$

Also, when $p = 2$, $\mathcal{T}_2^{(\alpha)}(|t|^\alpha)$ is a Hilbert space endowed with the inner product

$$\langle f, g\rangle_{2,(\alpha)} := \int_{-\infty}^\infty W^\alpha f(t)\overline{W^\alpha g(t)}\, |t|^{2\alpha}\,\mathrm{d}t.$$

It can be shown that W^α is the Weyl fractional derivative, that is,

$$W^\alpha f(t) = \begin{cases} (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}t^n} W^{-(n-\alpha)} f(t), & \text{if } t > 0, \\ \frac{\mathrm{d}^n}{\mathrm{d}t^n} W^{-(n-\alpha)} f(t), & \text{if } t < 0 \end{cases}$$

with $n = \lfloor \alpha \rfloor + 1$ (W^0 is the identity operator when α is a natural number).

Furthermore, observe that the domains $D(\Lambda_p)$ of the infinitesimal generator of the C_0-group $(T_p(t))_{t\in\mathbb{R}}$ on each of the spaces $L_p(\mathbb{R}_+)$ and $L_p(\mathbb{R})$ are the dense subspaces $\mathcal{T}_p^{(1)}(t)$ and $\mathcal{T}_p^{(1)}(|t|)$, respectively.

For convenience, on the Hardy space $H_2(\mathbb{U})$, we define the Cesàro-Hardy operator by the subordination formula of the group. The expression $T_2(t)f(z) = e^{-t/2} f(e^{-t}z)$ is well-defined for each holomorphic function on $\mathbb{U}$ and one can easily check that $(T_2(t))_{t\in\mathbb{R}}$ is a C_0-group of isometries on $H_2(\mathbb{U})$. Accordingly, we define

$$\mathfrak{C}_\alpha f = \int_0^\infty \varphi_{\alpha,2}(t) T_2(t) f \,\mathrm{d}t \quad \text{and} \quad \mathfrak{C}_\alpha^* f = \int_0^\infty \varphi_{\alpha,2}(t) T_2(-t) f \,\mathrm{d}t, \qquad f \in H_2(\mathbb{U}),$$

which are clearly linear bounded operators on $H_2(\mathbb{U})$. Given $f \in H_2(\mathbb{U})$ and $z \in \mathbb{U}$, let $z = |z|e^{i\theta}$ with $\theta \in (0, \pi)$, then $f_\theta(t) := f(te^{i\theta})$ belongs to $L_2(\mathbb{R}_+)$. Furthermore, $\mathfrak{C}_\alpha^* f(z) = \mathcal{C}_\alpha^* f_\theta(|z|)$ (see [26, p. 119]). This ensures that $\mathfrak{C}_\alpha^*$ is injective on $H_2(\mathbb{U})$ and hence one can define the range space $H_2^{(\alpha)}(\mathbb{U}) := \mathfrak{C}_\alpha^*(H_2(\mathbb{U}))$, which is a reproducing kernel Hilbert space (RKHS) with norm

$$\|f\|_{2,(\alpha)} := \Gamma(\alpha+1)\, \|\mathfrak{C}_\alpha^{*\,-1} f\|_{H_2(\mathbb{U})}, \quad f \in H_2^{(\alpha)}(\mathbb{U}),$$

see [26] for further details. Arvanitidis and Siskakis [6] studied the operators $\mathfrak{C}_1$ and $\mathfrak{C}_1^*$ (case $\alpha = 1$) showing that

$$\mathfrak{C}_1 f(z) = \frac{1}{z} \int_0^z f(\omega)\,\mathrm{d}\omega \quad \text{and} \quad \mathfrak{C}_1^* f(z) = \int_z^\infty \frac{f(\omega)}{\omega}\,\mathrm{d}\omega, \quad z \in \mathbb{U},$$

for each $f \in H_2(\mathbb{U})$. In the article [26], it is proved a Paley-Wiener theorem for these range spaces, showing that the Laplace transform $\mathcal{L} : \mathcal{T}_2^{(\alpha)}(t^\alpha) \to H_2^{(\alpha)}(\mathbb{U})$ given by

$$(\mathcal{L}f)(z) := \int_0^\infty e^{itz} f(t)\,\mathrm{d}t, \quad z \in \mathbb{U} \quad \text{and} \quad f \in \mathcal{T}_2^{(\alpha)}(t^\alpha)$$

is an isometric isomorphism.

2.2. Cesàro numbers

Fixed $\alpha \in \mathbb{C}$, we denote by $k_\alpha := (k_\alpha(n))_{n\geq 0}$ the sequence of Taylor coefficients of the function $(1-z)^{-\alpha}$ at the origin; that is,

$$\sum_{n=0}^\infty k_\alpha(n) z^n = \frac{1}{(1-z)^\alpha}, \quad |z| < 1. \tag{2.1}$$

The sequences $(k_\alpha(n))_{n\geq 0}$ are usually known as *Cesàro numbers of order* α, and are given by $k_\alpha(0) = 1$ and

$$k_\alpha(n) = \binom{n+\alpha-1}{\alpha-1} = \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!}, \quad n \in \mathbb{N},$$

see, for instance, Zygmund's classical monograph [53, Vol. I, p.77] where $k_\alpha(n)$ is denoted by $A_n^{\alpha-1}$. For $\alpha \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$ one has

$$k_\alpha(n) = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}, \quad n \in \mathbb{N}.$$

The sequences of Cesàro numbers $(k_\alpha(n))_{n\geq 0}$ play an important rôle in summability theory (see [53]), for instance, in the Cesàro summation methods. Recently, non-trivial applications of the Cesàro numbers have been found in the theory of fractional difference equations, see [29,35,36].

Hereunder, we detail some basic properties of the Cesàro numbers that will be used throughout the article. From (2.1), one immediately gets the identity

$$\sum_{n=0}^{\infty} k_\alpha(n) = 0, \quad \text{for } \alpha < 0.$$

As a function, $n \mapsto k_\alpha(n)$ is strictly increasing for $\alpha > 1$, strictly decreasing for $0 < \alpha < 1$, while $k_1(n) = 1$ for all $n \in \mathbb{N}_0$ (see [53, Th. III.1.17]). Furthermore, $0 \le k_\alpha(n) \le k_\beta(n)$ for each $0 < \alpha \le \beta$ and $n \in \mathbb{N}_0$. Fixed $m \in \mathbb{N}_0$, one has

$$k_{-m}(n) = \begin{cases} (-1)^n \binom{m}{n}, & 0 \le n \le m; \\ 0, & n \ge m+1, \end{cases}$$

and, if $m < \alpha < m+1$,

$$\operatorname{sign}\big(k_{-\alpha}(n)\big) = \begin{cases} (-1)^n, & 0 \le n \le m; \\ (-1)^{m+1}, & n \ge m+1. \end{cases}$$

In addition, the asymptotic behavior of the sequence $(k_\alpha(n))_{n\ge 0}$ is

$$k_\alpha(n) = \frac{n^{\alpha-1}}{\Gamma(\alpha)}\left(1 + O\left(\frac{1}{n}\right)\right), \quad \text{as } n \to +\infty, \text{ for every } \alpha \in \mathbb{R} \setminus \{0, -1, -2, \dots\}, \tag{2.2}$$

see [53, Vol.I, p.77 (1.18)].

Using (2.1), it can easily be deduced that the sequences of Cesàro numbers $(k_\alpha(n))_{n\ge 0}$ satisfy the group property $k_\alpha * k_\beta = k_{\alpha+\beta}$ for every $\alpha, \beta \in \mathbb{C}$, where the notation "$*$" stands for the convolution of sequences on $\mathbb{N}_0$. Recall that given two complex sequences $f = (f(n))_{n\ge 0}$ and $g = (g(n))_{n\ge 0}$ the convolution in $\mathbb{N}_0$ is defined by

$$(f * g)(n) = \sum_{j=0}^{n} f(n-j)g(j), \quad n \ge 0.$$

Observe that k_0 is the Dirac mass δ_0 on $\mathbb{N}_0$ and thus it is the unity of the convolution.

2.3. Fractional differences involving Cesàro numbers

For a complex sequence $f = (f(n))_{n\ge 0}$, consider the difference operator

$$[Df](n) := f(n) - f(n+1), \quad n \in \mathbb{N}_0,$$

and its iterates $D^{m+1} := D^m D$ for $m \in \mathbb{N}$ (with $D^0 = I$ the identity operator). Observe that

$$[D^m f](n) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} f(n+j), \quad m, n \in \mathbb{N}_0.$$

Differences D^α can be extended to the fractional case in different ways. Herein, we consider the following definition (see, for instance, [2]): given $f = (f(n))_{n\ge 0}$ a complex sequence and $\alpha > 0$ we define the operator D^α by

$$[D^\alpha f](n) := \sum_{j=n}^{\infty} k_{-\alpha}(j-n) f(j), \quad n \in \mathbb{N}_0,$$

whenever the series converges. The inverse operator of D is given by $[D^{-1}f](n) = \sum_{j=n}^{\infty} f(j)$ and, iterating, it takes the form

$$[D^{-m}f](n) := \sum_{j=n}^{\infty} k_m(j-n)f(j), \quad m \in \mathbb{N}.$$

Therefore, we may consider its fractional expression for $\alpha > 0$

$$[D^{-\alpha}f](n) := \sum_{j=n}^{\infty} k_\alpha(j-n)f(j)$$

whenever the series converges.

In [4], the authors introduced a class of weighted convolution Banach algebras that will play a key rôle in this article. To do so, they needed to coin a class of positive weights ω_α well-behaved with respect to fractional differences. Throughout this article, at certain points, we shall only require some considerations about such weights ω_α, however for convenience we will consider altogether since the beginning.

Definition 2.1. Fixed $\alpha \geq 0$, a sequence $\rho : \mathbb{N}_0 \to [1, +\infty)$ belongs to the class of weights ω_α if:

(i) $\rho = (\rho(n))_{n\geq 0}$ is non-decreasing with $\rho(0) = 1$, $\lim_{n\to+\infty} \rho(n)^{1/n} = 1$ and

$$\inf_{n\geq 0} \left(\frac{\rho(n)}{k_{\alpha+1}(n)} \right) > 0.$$

(ii) There is a constant $c_\rho > 0$ such that

$$\left(\sum_{j=0}^{n} + \sum_{j=m+1}^{n+m} \right) k_\alpha(j)\rho(n+m-j) \leq c_\rho \rho(n)\rho(m), \quad 1 \leq n \leq m.$$

For the sake of brevity, we will use the terminology *weight* to refer to any positive sequence that is submultiplicative up to a constant, i.e. $\rho(m+n) \leq C_\rho \rho(m)\rho(n)$ for some $C_\rho > 0$, and usually normalized by $\rho(0) = 1$. In the bilateral context, we will focus on *symmetric weights*, i.e. with $\rho(n) = \rho(-n)$ for all $n \in \mathbb{N}$.

Below, we highlight some features regarding the class ω_α. Additional properties and examples of this class ω_α appear in the article [4].

Remark 2.2.

(i) Condition (ii) in Definition 2.1 (taking $j = 0$ in the sum) ensures that each sequence $\rho = (\rho(n))_{n\geq 0}$ in the class ω_α is indeed a weight.
(ii) $\omega_\beta \subseteq \omega_\alpha$ for all $0 \leq \alpha \leq \beta$.
(iii) Each sequence of the form $\rho_\beta(n) := k_\beta(n)\rho(n)$ belongs to the class ω_α for $\beta \geq \alpha + 1$ and every weight $\rho = (\rho(n))_{n\geq 0}$ (see [4, p. 482]).

For any weight $\phi = (\phi(n))_{n\geq 0}$, let $\ell^1(\phi)$ denote the space of absolutely summable sequences on $\mathbb{N}_0$ with respect to the weight ϕ and endowed with its usual norm $\|\cdot\|_\phi$. Given any $\alpha > 0$, Abadias et al. [3] established that $D^{-\alpha} : \ell^1(k_\alpha) \to \ell^1(k_{-\alpha})$ is a bounded linear operator, D^α is well-defined on $\ell^1(k_{-\alpha})$ and the restriction $D^\alpha|_{\ell^1(k_\alpha)} : \ell^1(k_\alpha) \to \ell^1(k_\alpha)$ is a bounded linear operator. Furthermore, for each $f \in \ell^1(k_\alpha)$, they also proved that $D^\alpha D^{-\alpha} f = D^{-\alpha} D^\alpha f = f$. In conclusion, $D^{-\alpha}$ is injective in $\ell^1(k_\alpha)$.

According to [4, Prop. 2.10], whenever $\rho \in \omega_\alpha$, then $\ell^1(\rho) \hookrightarrow \ell^1(k_\alpha)$. Therefore, if we define the range space

$$\tau_+^\alpha(\rho) := D^{-\alpha}(\ell^1(\rho)),$$

then the bounded linear operator

$$D^{-\alpha}|_{\ell^1(\rho)} \colon \ell^1(\rho) \to \tau_+^\alpha(\rho)$$

is bijective with bounded inverse $D^\alpha|_{\tau_+^\alpha(\rho)} \colon \tau_+^\alpha(\rho) \to \ell^1(\rho)$. Moreover, since $k_{\alpha+1} \in \omega_\alpha$ and $\inf_{n\geq 0} \big(\rho(n)/k_{\alpha+1}(n)\big) > 0$, it is easy to check that $\tau_+^\alpha(\rho) \hookrightarrow \tau_+^\alpha(k_{\alpha+1}) \hookrightarrow \ell^1(\mathbb{N}_0)$.

We endow the space $\tau_+^\alpha(\rho)$ with the norm given by

$$\|f\|_{\rho,(\alpha)} := \sum_{n=0}^{\infty} \big|[D^\alpha f](n)\big| \rho(n),$$

obtained by transferring the norm $\|\cdot\|_\rho$ in $\ell^1(\rho)$ through $D^{-\alpha}$ (for $\alpha = 0$, the notation $\|f\|_{\rho,(0)}$ corresponds to the usual norm $\|f\|_\rho$ in $\ell^1(\rho)$), so the Banach space $\tau_+^\alpha(\rho)$ can be described as the space of those sequences $f = (f(n))_{n\geq 0}$ in ℓ^1 that have finite $\|f\|_{\rho,(\alpha)}$ norm. Furthermore, note that $D^{-\alpha}$ takes the space $c_{0,0}(\mathbb{N}_0)$ of eventually null sequences onto itself, whence one deduces that $c_{0,0}(\mathbb{N}_0)$ is dense in $\tau_+^\alpha(\rho)$. Additionally, observe that by [4, Thm. 2.11 (iii)], one can easily infer that

$$\tau_+^\beta(\rho_{\beta+1}) \hookrightarrow \tau_+^\alpha(\rho_{\alpha+1}), \quad \text{for every } 0 \leq \alpha \leq \beta \tag{2.3}$$

and each weight $\rho : \mathbb{N}_0 \to [1, +\infty)$ (recall the notation $\rho_\gamma(n) := k_\gamma(n)\rho(n)$ for every $\gamma \geq 0$ introduced in Remark 2.2).

The spaces $\tau_+^\alpha(\rho)$ acquire structure of semi-simple commutative Banach algebras when endowed with both operations of entrywise addition and convolution product, i.e. there exists a constant $M_{\rho,\alpha} > 0$ such that $\|f * g\|_{\rho,(\alpha)} \leq M_{\rho,\alpha} \|f\|_{\rho,(\alpha)} \|g\|_{\rho,(\alpha)}$ for every $f, g \in \tau_+^\alpha(\rho)$ (see [4, Th. 2.11]). These Banach algebras $\tau_+^\alpha(k_{\alpha+1})$ for $\alpha \in \mathbb{N}$ were introduced by Galé and Wawrzyńczyk in [27]. Their extensions to $\alpha > 0$ have been defined in [4,3] and [1, Section 2], though with a slightly different presentation.

Now, we adapt these previous ideas to the bilateral context. Let $f = (f(n))_{n\in\mathbb{Z}}$ be a complex sequence indexed on $\mathbb{Z}$. Following the ideas in [1], we will consider the sequences $f_+ := (f_+(n))_{n\in\mathbb{Z}}$ and $f_- := (f_-(n))_{n\in\mathbb{Z}}$ given by:

$$f_+(n) := \begin{cases} f(n), & \text{for } n \geq 0, \\ 0, & \text{for } n < 0, \end{cases} \quad \text{and} \quad f_-(n) := \begin{cases} 0, & \text{for } n \leq 0, \\ f(-n), & \text{for } n > 0. \end{cases}$$

Accordingly, for any complex bilateral sequence $f = (f(n))_{n\in\mathbb{Z}}$, we define the *fractional difference operator* D^α for any $\alpha \in \mathbb{R}$ as

$$[D^\alpha f](n) := \begin{cases} [D^\alpha f_+](n), & \text{for } n \geq 0, \\ [D^\alpha f_-](-n), & \text{for } n < 0. \end{cases}$$

Hence, if $\rho = (\rho(n))_{n\in\mathbb{Z}}$ is a symmetric weight such that $\rho_+ \in \omega_\alpha$, one can define $\tau^\alpha(\rho) := D^{-\alpha}(\ell^1(\rho))$, which is a semi-simple commutative Banach algebra equipped with the two-sided convolution product

$$(a * b)(n) = \sum_{j=-\infty}^{\infty} a(n-j)b(j) \quad \text{for } n \in \mathbb{Z},$$

and the norm

$$\|f\|_{\rho,(\alpha)} := \sum_{n=-\infty}^{\infty} \big|[D^\alpha f](n)\big|\rho(n).$$

It is straightforward to check that $\tau^\alpha(\rho)$ is contained in $\ell^1(\mathbb{Z})$.

To conclude this part, in order to work more easily with the notion of regularity, we will transfer our construction of the Banach algebras $\tau^\alpha(\rho)$ to a function algebra on unit circle.

Definition 2.3. Let $\alpha \geq 0$ and $\rho = (\rho(n))_{n\in\mathbb{Z}}$ be a symmetric weight with $\rho_+ \in \omega_\alpha$. The *fractional weighted Wiener algebra* $\mathcal{A}^\alpha_\rho$ consists of the space of functions

$$\mathcal{A}^\alpha_\rho = \left\{ f \in C(\partial\mathbb{D}) \,:\, \|f\|_{\rho,(\alpha)} := \sum_{n=-\infty}^{\infty} \big|[D^\alpha \widehat{f}](n)\big|\rho(n) < +\infty \right\}$$

endowed with the usual operations of pointwise addition and multiplication.

Observe that $\mathcal{A}^\alpha_\rho$ is a Banach algebra isometrically isomorphic to $\tau^\alpha(\rho)$ and the correspondence $\mathcal{A}^\alpha_\rho \ni f \mapsto (\widehat{f}(n))_{n\in\mathbb{Z}} \in \tau^\alpha(\rho)$ given by the Fourier coefficients

$$\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})e^{-in\theta}\,\mathrm{d}\theta, \quad n \in \mathbb{Z}$$

is actually the Gelfand transform of $\mathcal{A}^\alpha_\rho$. Of course, for $\alpha = 0$ the fractional weighted Wiener algebra $\mathcal{A}^\alpha_\rho$ corresponds to the usual *Beurling algebra*

$$\mathcal{A}_\rho = \left\{ f \in C(\partial\mathbb{D}) : \sum_{n\in\mathbb{Z}} |\widehat{f}(n)|\rho(n) < +\infty \right\}.$$

Moreover, since $\tau^\alpha(\rho) \hookrightarrow \ell^1(\mathbb{Z})$ for each $\alpha \geq 0$ and every symmetric weight $\rho : \mathbb{Z} \to \mathbb{C}$ such that $\rho_+ \in \omega_\alpha$, it follows that $\mathcal{A}^\alpha_\rho$ is always contained in the *Wiener algebra* $\mathcal{A}(\partial\mathbb{D})$ of absolutely convergent Fourier series.

Finally, note that $\mathcal{A}^\alpha_{\rho_\gamma}$ is a fractional weighted Wiener algebra for each $\gamma \geq \alpha+1$ (in the bilateral context, we also mean $\rho_\gamma(n) := k_\gamma(|n|)\rho(n)$ for each $n \in \mathbb{Z}$) and every symmetric weight $\rho : \mathbb{Z} \to \mathbb{C}$. Besides, the inclusion

$$\mathcal{A}^\beta_{\rho_{\beta+1}} \hookrightarrow \mathcal{A}^\alpha_{\rho_{\alpha+1}} \tag{2.4}$$

still holds whenever $0 \leq \alpha \leq \beta$, as occurred in (2.3).

3. Regularity of fractional weighted Wiener algebras $\mathcal{A}^\alpha_\rho$

Recall that a Banach algebra $\mathcal{A}$ of functions $f : K \to \mathbb{C}$ defined over a compact space K is said to be *regular* if for every $z \in K$ and each compact subset $Q \subsetneq K$ with $z \notin Q$, there exists $f_{z,Q} \in \mathcal{A}$ such that $f_{z,Q}(z) \neq 0$ and $f_{z,Q} \equiv 0$ on Q. As mentioned in the Introduction, a celebrated criterion due to Beurling [13] ensures that a Beurling algebra $\mathcal{A}_\rho$ is regular whenever its weight $\rho = (\rho(n))_{n\in\mathbb{Z}}$ fulfills the asymptotic condition

$$\sum_{n\in\mathbb{Z}} \frac{\log\rho(n)}{1+n^2} < +\infty. \tag{3.1}$$

As mentioned in the Introduction, those weights $\rho = (\rho(n))_{n\in\mathbb{Z}}$ which satisfy the *Beurling condition* (3.1) are known as *Beurling sequences* (or *Beurling weights* following the terminology of this article). Due to the divergence of the harmonic series, observe that Beurling sequences $\rho : \mathbb{Z} \to [1, +\infty)$ must grow subexponentially as $n \to \pm\infty$. For the sake of completeness, below we just mention some examples of Beurling sequences (see, for instance, [15]):

$$\rho(n) = (1 + |n|)^s \quad \text{for } s \geq 1, \quad \rho(n) = \exp\left(\frac{C|n|}{\log^\gamma(2 + |n|)}\right) \quad \text{for } C > 0 \text{ and } \gamma > 1,$$

$$\rho(n) = \exp\left(|n|^\beta\right) \quad \text{for } 0 \leq \beta < 1, \quad \rho(n) = \exp\left(\frac{C|n|}{\log(2 + |n|)\big(\log\log(5 + |n|)\big)^2}\right) \quad \text{for } C > 0.$$

One of the key ideas in the proof of the regularity criterion for Beurling algebras $\mathcal{A}_\rho$ is a construction depending on the classical Paley-Wiener theorem between $H_2(\mathbb{U})$ and $L_2(\mathbb{R}_+)$ via the Fourier transform.

Our main result in Section 3 is a generalization of Beurling's criterion to fractional weighted Wiener algebras $\mathcal{A}^\alpha_\rho$ with $\alpha \in \mathbb{N}$ (see Theorem 3.1 below). Inspired by Beurling's work [13], we find partitions of unity in $\mathcal{A}^\alpha_\rho$ upon applying a Paley-Wiener theorem between the range spaces $\mathcal{T}_2^{(\alpha)}$ and $H_2^{(\alpha)}(\mathbb{U})$ established in [26, Thm. 6.2]. This new regularity criterion for $\mathcal{A}^\alpha_\rho$ is one of the cornerstones upon which our upcoming applications in Operator Theory will be based (see Section 5).

For the sake of simplicity, in the proof of Theorem 3.1, we exclusively deal with the regularity of $\mathcal{A}^\alpha_\rho$ when $\alpha \in \mathbb{N}$. Further down, we will treat the regularity of fractional weighted Wiener algebras of the form $\mathcal{A}^\alpha_{\rho_{\alpha+1}}$ defined by non-integer values of $\alpha \geq 0$ using inclusions of the form (2.4) (see Corollary 3.2).

Theorem 3.1. *Let $\alpha \in \mathbb{N}_0$ and $\rho = (\rho(n))_{n\in\mathbb{Z}}$ be a symmetric weight such that $\rho_+ \in \omega_\alpha$. If*

$$\sum_{n\in\mathbb{Z}} \frac{\log \rho(n)}{1 + n^2} < +\infty,$$

then, the fractional weighted Wiener algebra $\mathcal{A}^\alpha_\rho$ is regular.

Proof. Since the case $\alpha = 0$ is covered by Beurling's theorem [13], without loss of generality, assume that $\alpha \geq 1$ is an integer. We will prove that for fixed $e^{i\theta} \in \partial\mathbb{D}$ and each $\varepsilon \in (0, \pi)$, there is a function $g \in \mathcal{A}^\alpha_\rho$ such that $g(e^{i\theta}) \neq 0$ and $g(e^{it}) = 0$ for all $|\theta - t| \geq \varepsilon$. For convenience, we assume that $\theta \in [-\pi, \pi)$.

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be the even function given by

$$\varphi(t) := \frac{1}{\rho(n)(1 + t^2)}, \quad t \in (n, n + 1) \text{ for each } n \in \mathbb{N}_0.$$

Observe that $\varphi \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ and, according to the hypothesis in the statement,

$$\int_{-\infty}^{\infty} \frac{|\log \varphi(t)|}{1 + t^2}\, \mathrm{d}t < \infty.$$

Taking the Poisson semigroup acting on $\log \varphi$, one can define a harmonic function on the upper half-plane $\mathbb{U}$ as follows. Fixed $z \in \mathbb{U}$, let us denote $z = x + iy$ with $x \in \mathbb{R}$ and $y > 0$. Then

$$u(z) := \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x - t)^2 + y^2} \log \varphi(t)\, \mathrm{d}t.$$

Using the half-plane version of Fatou's Theorem (see, for example, [45, Theorem 5.5, pp. 86–87]), we get that $u : \mathbb{U} \to \mathbb{R}$ is harmonic and that the non-tangential limit

$$\lim_{y\to 0} u(x+iy) = \log \varphi(x)$$

exists for almost every $x \in \mathbb{R}$. Taking its harmonic conjugate $v : \mathbb{U} \to \mathbb{R}$, the function $f(z) = e^{u(z)+iv(z)}$ is holomorphic on the upper half-plane $\mathbb{U}$ and the limit

$$\lim_{y\to 0} |f(x+iy)| = |f^*(x)| = \varphi(x)$$

also holds non-tangentially for almost all $x \in \mathbb{R}$. Taking into account that

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2}\,\mathrm{d}t = 1, \quad \text{for all } x \in \mathbb{R} \text{ and } y > 0,$$

and using the convexity of the exponential function, an application of Jensen's inequality leads to

$$|f(x+iy)| = e^{u(z)} \le \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2}\varphi(t)\,\mathrm{d}t$$

for each $x \in \mathbb{R}$ and $y > 0$. Now, from Hölder's inequality, one gets that

$$|f(x+iy)| \le \frac{1}{\sqrt{\pi}}\left(\int_{-\infty}^{\infty} \frac{|\varphi(t)|^2 y}{(x-t)^2+y^2}\,\mathrm{d}t\right)^{1/2},$$

which, as a consequence of Fubini's theorem, implies that

$$\sup_{y>0}\int_{-\infty}^{\infty} |f(x+iy)|^2\,\mathrm{d}x \le \int_{-\infty}^{\infty} |\varphi(t)|^2\,\mathrm{d}t < \infty.$$

In conclusion, we have proved that $f \in H_2(\mathbb{U})$.

Now, by the Paley-Wiener theorem for the Laplace transform $\mathcal{L} : \mathcal{T}_2^{(\alpha)}(t^\alpha) \to H_2^{(\alpha)}(\mathbb{U})$ (see [26]), there exists a function $F_\alpha \in \mathcal{T}_2^{(\alpha)}(t^\alpha)$ (indeed, we may see F_α as a function in $\mathcal{T}_2^{(\alpha)}(|t|^\alpha)$ vanishing on the interval $(-\infty, 0)$) such that

$$(\mathfrak{C}_\alpha^* f)(z) := (\mathcal{L}F_\alpha)(z) = \int_0^{\infty} F_\alpha(t)e^{itz}\,\mathrm{d}t, \quad z \in \mathbb{U}.$$

Moreover, the dominated convergence theorem entails that

$$\lim_{y\to 0^+}\int_{-\infty}^{+\infty} \left|\mathcal{L}F_\alpha(x+iy) - \mathcal{L}F_\alpha(x)\right|^2\,\mathrm{d}x = 0,$$

and, consequently, the limit $\lim_{y\to 0}(\mathcal{L}F_\alpha)(x+iy) = (\mathcal{F}F_\alpha)(-x)$ holds non-tangentially almost everywhere on $\mathbb{R}$.

Also, reminding that $f \in H_2(\mathbb{U})$, by the classical Paley-Wiener theorem, there exists a function $F \in L_2(\mathbb{R}_+)$ such that $f = \mathcal{L}F$. Now, by [26, Corollary 2.6], the Laplace transform intertwines Cesàro–Hardy operators, then $\mathfrak{C}^*_\alpha f = (\mathfrak{C}^*_\alpha \circ \mathcal{L})F = (\mathcal{L} \circ \mathcal{C}_\alpha)F$. Using again that the limit of the Laplace transform on $L_2(\mathbb{R}_+)$ is the Fourier transform, we get that

$$\lim_{y\to 0} \mathfrak{C}^*_\alpha f(z) = \mathcal{F}(\mathcal{C}_\alpha F)(-x).$$

Once again, by [37, Theorem 6.4], the Fourier transform intertwines Cesàro-Hardy operators, so we have

$$\lim_{y\to 0} \mathfrak{C}^*_\alpha f(z) = (\mathcal{C}^*_\alpha f^*)(x),$$

where we have also used that $f^*(x) = \lim_{y\to 0}(\mathcal{L}F)(z) = (\mathcal{F}F)(-x)$. In conclusion, we get that

$$(\mathcal{C}^*_\alpha f^*)(x) = (\mathcal{F}F_\alpha)(-x) \quad \text{a.e. } x \in \mathbb{R}.$$

Consider $\psi_\alpha(x) := F_\alpha(-x)$ for each $x \in \mathbb{R}$. Observe that ψ_α is a function in $\mathcal{T}_2^{(\alpha)}(|t|^\alpha)$ supported in the interval $(-\infty, 0]$ with Fourier transform satisfying

$$\frac{|x|^\alpha}{\Gamma(\alpha+1)}\left|W^\alpha(\mathcal{F}\psi_\alpha)(x)\right| = \varphi(x) \quad \text{a.e. on } \mathbb{R}. \tag{3.2}$$

Taking into account that $f^* \in L_1(\mathbb{R})$ (since $|f^*(x)| = \varphi(x)$ and $\varphi \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$) and $(\mathcal{F}\psi_\alpha)(x) = (\mathcal{C}^*_\alpha f^*)(x)$, then $\mathcal{F}\psi_\alpha \in \mathcal{T}_1^{(\alpha)}(|t|^\alpha) \subseteq L_1(\mathbb{R})$. Therefore, ψ_α must be a non-vanishing continuous function since it is the Fourier transform of a non-zero $L_1(\mathbb{R})$ function.

Observe that given any $a, b \in \mathbb{R}$ with $a < b$, the function $t \mapsto \psi_\alpha(a-t)\psi_\alpha(t-b)$ is continuous on $\mathbb{R}$ with support on $[a, b]$. Recalling that $\theta \in [-\pi, \pi)$ and $\varepsilon \in (0, \pi)$, we can choose suitable $a, b \in \mathbb{R}$ such that $\psi_\alpha(a-\theta)\psi_\alpha(\theta-b) \neq 0$ but $\psi_\alpha(a-t)\psi_\alpha(t-b) = 0$ for all $|\theta - t| \geq \varepsilon$. Now, we transfer this construction to the unit circle $\partial\mathbb{D}$ defining

$$g(e^{it}) := \begin{cases} 2\pi\left(\dfrac{t}{1-e^{-it}}\right)^\alpha \psi_\alpha(a-t)\psi_\alpha(t-b), & \text{if } t \in (\theta-\pi, 0), \\ \dfrac{2\pi\psi_\alpha(a)\psi_\alpha(-b)}{i^\alpha}, & \text{if } t = 0, \\ 2\pi\left(\dfrac{t}{1-e^{-it}}\right)^\alpha \psi_\alpha(a-t)\psi_\alpha(t-b), & \text{if } t \in (0, \theta+\pi]. \end{cases}$$

Note that g is a continuous function on the unit circle $\partial\mathbb{D}$ that satisfies $g(e^{i\theta}) \neq 0$ and $g(e^{it}) = 0$ for all $|\theta - t| \geq \varepsilon$. Therefore, it only remains to be proved that $g \in \mathcal{A}^\alpha_\rho$.

To do so, we need to estimate the fractional differences of its Fourier coefficients. First, observe that for each $n \geq 0$,

$$D^\alpha \widehat{g}(n) = \int_{-\infty}^{\infty} t^\alpha \psi_\alpha(a-t)\psi_\alpha(t-b)e^{-int}\,\mathrm{d}t = i^\alpha \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}(\mathcal{F}G)(n),$$

while for $n < 0$,

$$D^\alpha \widehat{g}(n) = \int_{-\infty}^{\infty} (-t)^\alpha \psi_\alpha(a-t)\psi_\alpha(t-b)e^{-i(n-\alpha)t}\,\mathrm{d}t = (-i)^\alpha \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}(\mathcal{F}G)(n-\alpha),$$

with $G(t) := \psi_\alpha(a-t)\psi_\alpha(t-b)$, where we have used that the functions $t \mapsto \psi_\alpha(a-t)\psi_\alpha(t-b)$ and $t \mapsto t^\alpha \psi_\alpha(a-t)\psi_\alpha(t-b)$ are continuous on $\mathbb{R}$ with compact support, and thus they belong to $L^1(\mathbb{R})$. Now, using the convolution theorem for the Fourier transform, we have that

$$(\mathcal{F}G)(n) = \int_{-\infty}^{\infty} e^{ia(\eta-n)}(\mathcal{F}\psi_\alpha)(\eta-n)e^{-ib\eta}(\mathcal{F}\psi_\alpha)(\eta)\,\mathrm{d}\eta, \quad n \in \mathbb{Z}.$$

Since $\mathcal{F}\psi_\alpha \in \mathcal{T}_1^{(\alpha)}(|t|^\alpha)$, then $\eta \mapsto e^{i\beta\eta}\mathcal{F}\psi_\alpha(\eta)$ also belongs to $\mathcal{T}_1^{(\alpha)}(|t|^\alpha)$ for any $\beta \in \mathbb{R}$. Hence, $\mathcal{F}G$ is a convolution between functions of $\mathcal{T}_1^{(\alpha)}(|t|^\alpha)$, that we denote by $H^1(\eta) := e^{-ia\eta}(\mathcal{F}\psi_\alpha)(-\eta)$ and $H^2(\eta) := e^{-ib\eta}(\mathcal{F}\psi_\alpha)(\eta)$. Applying [25, Lemma 1.6], the ideas of [25, Theorem 1.8] and the density of $C_c^\infty([0,\infty))$ in $\mathcal{T}_1^{(\alpha)}(t^\alpha)$, one gets that for $n \geq 0$,

$$\begin{aligned} |D^\alpha \widehat{g}(n)| &= |W^\alpha(\mathcal{F}G)(n)| \\ &\leq |W^\alpha(H^1_+ * H^2_+)(n)| + |(W^\alpha H^1_+ * H^2_-)(n)| + |(H^1_- * W^\alpha H^2_+)(n)|, \end{aligned} \tag{3.3}$$

while for $n < 0$,

$$\begin{aligned} |D^\alpha \widehat{g}(n)| &= |W^\alpha(\mathcal{F}G)(n-\alpha)| \\ &\leq |W^\alpha(H^1_- * H^2_-)(n-\alpha)| + |(W^\alpha H^1_- * H^2_+)(n-\alpha)| + |(H^1_+ * W^\alpha H^2_-)(n-\alpha)|. \end{aligned} \tag{3.4}$$

Deriving α times, it is clear that there exists an absolute constant $C_\alpha > 0$ such that

$$|W^\alpha H^1(\eta)| \leq C_\alpha \sum_{j=0}^{\alpha} |W^j(\mathcal{F}\psi_\alpha)(-\eta)|, \qquad \text{a.e. } \eta \in \mathbb{R}. \tag{3.5}$$

Now, for all $j < \alpha$ and $\eta > 0$,

$$\begin{aligned} \left|W^j(\mathcal{F}\psi_\alpha)(-\eta)\right| &= \left|W^{-(\alpha-j)}W^\alpha(\mathcal{F}\psi_\alpha)(-\eta)\right| \\ &\leq \frac{1}{\Gamma(\alpha-j)} \int_{-\infty}^{-\eta} (-\eta-s)^{\alpha-j-1}|W^\alpha(\mathcal{F}\psi_\alpha)(s)|\,\mathrm{d}s \\ &= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-j)} \int_{-\infty}^{-\eta} (-\eta-s)^{\alpha-j-1}\frac{\varphi(s)}{|s|^\alpha}\,\mathrm{d}s, \end{aligned}$$

and, similarly,

$$\left|W^j(\mathcal{F}\psi_\alpha)(-\eta)\right| \leq \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-j)} \int_{-\eta}^{\infty} (s+\eta)^{\alpha-j-1}\frac{\varphi(s)}{|s|^\alpha}\,\mathrm{d}s$$

for each $\eta < 0$. In sum, for each $j < \alpha$ and $\eta \neq 0$, one has

$$\begin{aligned}\left|W^j(\mathcal{F}\psi_\alpha)(\eta)\right| &\le \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-j)}\int\limits_{|\eta|}^{\infty}\frac{(s-|\eta|)^{\alpha-j-1}}{s^\alpha}\varphi(s)\,\mathrm{d}s\\ &\le \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-j)}\int\limits_{|\eta|}^{\infty}\frac{(s-|\eta|)^{\alpha-j-1}}{s^\alpha(1+s^2)\rho(\lfloor s\rfloor)}\,\mathrm{d}s\\ &\le \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-j)\rho(\lfloor\eta\rfloor)}\int\limits_{|\eta|}^{\infty}\frac{1}{s^{j+1}(1+s^2)}\,\mathrm{d}s.\end{aligned}$$

Whenever $0<|\eta|<1$, one has

$$\int\limits_{|\eta|}^{\infty}\frac{1}{s^{j+1}(1+s^2)}\,\mathrm{d}s \le \int\limits_{|\eta|}^{1}\frac{1}{s^{j+1}}\,\mathrm{d}s+\int\limits_{1}^{\infty}\frac{1}{s^{j+3}}\,\mathrm{d}s \lesssim \frac{1}{|\eta|^{j+1}} \lesssim \frac{1}{|\eta|^{\alpha}};$$

while for $|\eta|>1$,

$$\int\limits_{|\eta|}^{\infty}\frac{1}{s^{j+1}(1+s^2)}\,\mathrm{d}s \le \int\limits_{|\eta|}^{\infty}\frac{1}{s^{j+3}}\,\mathrm{d}s \lesssim \frac{1}{1+\eta^2}.$$

Accordingly, for all $\eta\neq 0$ and $0\le j<\alpha$,

$$\left|W^j(\mathcal{F}\psi_\alpha)(\eta)\right| \le \frac{C_\alpha}{\rho(\lfloor\eta\rfloor)}\left(\frac{\chi_{(0,1)}(|\eta|)}{|\eta|^\alpha}+\frac{\chi_{[1,\infty)}(|\eta|)}{1+\eta^2}\right).$$

Furthermore, as a by-product of (3.2), one also gets

$$\left|W^\alpha(\mathcal{F}\psi_\alpha)(\eta)\right| \le \frac{C_\alpha}{\rho(\lfloor\eta\rfloor)}\left(\frac{\chi_{(0,1)}(|\eta|)}{|\eta|^\alpha}+\frac{\chi_{[1,\infty)}(|\eta|)}{1+\eta^2}\right), \qquad \text{for every } \eta\neq 0.$$

Our latter estimations and the sum (3.5) yield

$$|W^\alpha H^1(\eta)| \le \frac{C_\alpha}{\rho(\lfloor|\eta|\rfloor)}\left(\frac{\chi_{(0,1)}(|\eta|)}{|\eta|^\alpha}+\frac{\chi_{[1,\infty)}(|\eta|)}{1+\eta^2}\right), \qquad \text{for all } \eta\neq 0, \tag{3.6}$$

for some absolute constant $C_\alpha>0$ (which only depends on α) and, in a similar way,

$$|W^\alpha H^2(\eta)| \le \frac{C_\alpha}{\rho(\lfloor|\eta|\rfloor)}\left(\frac{\chi_{(0,1)}(|\eta|)}{|\eta|^\alpha}+\frac{\chi_{[1,\infty)}(|\eta|)}{1+\eta^2}\right), \qquad \text{for all } \eta\neq 0. \tag{3.7}$$

Our objective is to prove that $\sum_{n\in\mathbb{Z}}|D^\alpha\widehat{g}(n)|\rho(n)$ is finite. Firstly, we deal with the terms indexed by $n\ge 1$ and bound each of the three summands appearing at (3.3). Taking the first summand in (3.3) and applying [25, Proposition 1.2], one gets

$$\begin{aligned}|W^\alpha(H^1_+ * H^2_+)(n)| \le &\int\limits_0^n\int\limits_{n-r}^{n}(t+r-n)^{\alpha-1}|W^\alpha H^1(r)||W^\alpha H^2(t)|\,\mathrm{d}t\,\mathrm{d}r\\ &+\int\limits_n^\infty\int\limits_n^\infty(t+r-n)^{\alpha-1}|W^\alpha H^1(r)||W^\alpha H^2(t)|\,\mathrm{d}t\,\mathrm{d}r\end{aligned}$$

for every $n \geq 1$. Therefore,

$$\begin{aligned}\sum_{n=1}^{\infty}|W^{\alpha}(H^1_+ * H^2_+)(n)|\rho(n) &\leq \int_0^{\infty}\sum_{n=\lfloor r\rfloor+1}^{\infty}\int_{n-r}^{n}(t+r-n)^{\alpha-1}|W^{\alpha}H^1(r)||W^{\alpha}H^2(t)|\rho(n)\,\mathrm{d}t\,\mathrm{d}r \\ &\quad+\int_1^{\infty}\sum_{n=1}^{\lfloor r\rfloor}\int_n^{\infty}(t+r-n)^{\alpha-1}|W^{\alpha}H^1(r)||W^{\alpha}H^2(t)|\rho(n)\,\mathrm{d}t\,\mathrm{d}r \\ &=\int_0^{1}\sum_{n=1}^{\infty}\int_{n-r}^{n}(t+r-n)^{\alpha-1}|W^{\alpha}H^1(r)||W^{\alpha}H^2(t)|\rho(n)\,\mathrm{d}t\,\mathrm{d}r \\ &\quad+\int_1^{\infty}\sum_{n=\lfloor r\rfloor+1}^{\infty}\int_{n-r}^{n}(t+r-n)^{\alpha-1}|W^{\alpha}H^1(r)||W^{\alpha}H^2(t)|\rho(n)\,\mathrm{d}t\,\mathrm{d}r \\ &\quad+\int_1^{\infty}\sum_{n=1}^{\lfloor r\rfloor}\int_n^{\infty}(t+r-n)^{\alpha-1}|W^{\alpha}H^1(r)||W^{\alpha}H^2(t)|\rho(n)\,\mathrm{d}t\,\mathrm{d}r \\ &=: (I.1)+(I.2)+(I.3).\end{aligned}$$

Now, we bound each of these terms separately. Using (3.6) and (3.7) and Fubini's theorem, we obtain the next inequalities for $(I.1)$:

$$\begin{aligned}(I.1) &\lesssim \int_0^1\frac{1}{r^{\alpha}}\int_{1-r}^1\frac{(t+r-1)^{\alpha-1}}{t^{\alpha}}\,\mathrm{d}t\,\mathrm{d}r+\sum_{n=2}^{\infty}\frac{\rho(n)}{\rho(n-1)}\int_0^1\frac{1}{r^{\alpha}}\int_{n-r}^{n}\frac{(t+r-n)^{\alpha-1}}{t^2}\,\mathrm{d}t\,\mathrm{d}r \\ &\lesssim \int_0^1\frac{1}{r^{\alpha}}\int_0^r\frac{v^{\alpha-1}}{1-v}\,\mathrm{d}v\mathrm{d}r+\sum_{n=2}^{\infty}\int_0^1\frac{1}{r}\int_{n-r}^{n}\frac{1}{t^2}\left(1-\frac{n-t}{r}\right)^{\alpha-1}\mathrm{d}t\,\mathrm{d}r \\ &\leq \int_0^1\frac{1}{r}\int_0^r\frac{1}{1-v}\,\mathrm{d}v\mathrm{d}r+\sum_{n=2}^{\infty}\int_0^1\frac{1}{r}\int_{n-r}^{n}\frac{1}{t^2}\,\mathrm{d}t\,\mathrm{d}r \\ &=\int_0^1\frac{-\log(1-r)}{r}\,\mathrm{d}r+\sum_{n=2}^{\infty}\frac{1}{n}\int_0^1\frac{1}{n-r}\,\mathrm{d}r<+\infty,\end{aligned}$$

where, in order to bound the term corresponding to $n=1$, we use the change of variables $v=1-(1-r)/t$; while for $n \geq 2$, we apply the submultiplicity $\rho(n) \leq \rho(1)\rho(n-1)$. Secondly, to bound the integral $(I.2)$ we use again the inequalities (3.6) and (3.7) and Fubini's theorem:

$$\begin{aligned}(I.2) &= \int_1^{\infty}\int_{\lfloor r\rfloor+1-r}^{\infty}\sum_{n=\max\{\lfloor r\rfloor+1,\lfloor t\rfloor+1\}}^{\lfloor r+t\rfloor}(t+r-n)^{\alpha-1}|W^{\alpha}H^1(r)||W^{\alpha}H^2(t)|\rho(n)\,\mathrm{d}t\,\mathrm{d}r \\ &\lesssim \int_1^{\infty}\frac{\rho(\lfloor r\rfloor+1)}{(1+r^2)\rho(\lfloor r\rfloor)}\int_{\lfloor r\rfloor+1-r}^{1}\frac{(t+r-\lfloor r\rfloor-1)^{\alpha-1}}{t^{\alpha}}\,\mathrm{d}t\,\mathrm{d}r \\ &\quad+\int_1^{\infty}\frac{1}{(1+r^2)\rho(\lfloor r\rfloor)}\int_1^{\infty}\frac{1}{(1+t^2)\rho(\lfloor t\rfloor)}\sum_{n=\max\{\lfloor r\rfloor+1,\lfloor t\rfloor+1\}}^{\lfloor r+t\rfloor}(t+r-n)^{\alpha-1}\rho(n)\,\mathrm{d}t\,\mathrm{d}r\end{aligned}$$

$$=: (I.2.1) + (I.2.2).$$

To estimate the integral $(I.2.1)$, we carry out a similar strategy than the followed for $(I.1)$. Applying the change of variables $v = 1 - (1 - (r - \lfloor r \rfloor))/t$, we obtain

$$(I.2.1) \lesssim \int_1^\infty \frac{1}{1+r^2} \int_0^{r-\lfloor r \rfloor} \frac{v^{\alpha-1}}{1-v}\,\mathrm{d}v\,\mathrm{d}r \le \int_1^\infty \frac{-\log(1-(r-\lfloor r \rfloor))}{1+r^2}\,\mathrm{d}r$$

$$\le \sum_{n=1}^\infty \frac{1}{1+n^2} \int_n^{n+1} -\log(n+1-r)\,\mathrm{d}r < \infty.$$

Now, to deal with $(I.2.2)$, we need to compare it with $(I.3)$. Observe that,

$$(I.3) \lesssim \int_1^\infty \frac{1}{(1+r^2)\rho(\lfloor r \rfloor)} \int_1^\infty \frac{1}{(1+t^2)\rho(\lfloor t \rfloor)} \sum_{n=1}^{\min\{\lfloor r \rfloor, \lfloor t \rfloor\}} (t+r-n)^{\alpha-1}\rho(n)\,\mathrm{d}t\,\mathrm{d}r.$$

Therefore, the sum $(I.2.2) + (I.3)$ can be bounded by

$$\int_1^\infty \frac{1}{(1+r^2)\rho(\lfloor r \rfloor)} \int_1^\infty \frac{1}{(1+t^2)\rho(\lfloor t \rfloor)} \left(\sum_{n=\max\{\lfloor r \rfloor+1, \lfloor t \rfloor+1\}}^{\lfloor r+t \rfloor} + \sum_{n=1}^{\min\{\lfloor r \rfloor, \lfloor t \rfloor\}} \right) (t+r-n)^{\alpha-1}\rho(n)\,\mathrm{d}t\,\mathrm{d}r.$$

Taking into account that $\rho_+ \in \omega_\alpha$ and $k_\alpha(m) \sim m^{\alpha-1}/\Gamma(\alpha)$ as $m \to \infty$, we have

$$\left(\sum_{n=\max\{\lfloor r \rfloor+1, \lfloor t \rfloor+1\}}^{\lfloor r+t \rfloor} + \sum_{n=1}^{\min\{\lfloor r \rfloor, \lfloor t \rfloor\}} \right) (t+r-n)^{\alpha-1}\rho(n) \lesssim \rho(\lfloor r \rfloor + 1)\rho(\lfloor t \rfloor + 1).$$

Hence,

$$(I.2.2) + (I.3) \lesssim \int_1^\infty \frac{\rho(\lfloor r \rfloor + 1)}{(1+r^2)\rho(\lfloor r \rfloor)} \int_1^\infty \frac{\rho(\lfloor t \rfloor + 1)}{(1+t^2)\rho(\lfloor t \rfloor)}\,\mathrm{d}t\,\mathrm{d}r < \infty,$$

and consequently, we have finally proved that the series corresponding to the first summand in (3.3) is finite. Now, we estimate the second summand in (3.3). Note that

$$\sum_{n=1}^\infty |(W^\alpha H_+^1 * H_-^2)(n)|\rho(n) \le \sum_{n=1}^\infty \rho(n) \int_{-\infty}^0 |W^\alpha H^1(n-t)||H^2(t)|\,\mathrm{d}t$$

$$\le C_\alpha \int_{-\infty}^0 |(\mathcal{F}\psi_\alpha)(t)| \sum_{n=1}^\infty \frac{\rho(n)}{(1+(n-t)^2)\rho(\lfloor n-t \rfloor)}\,\mathrm{d}t$$

$$\le \sum_{n=1}^\infty \frac{C_\alpha}{(1+n^2)} \int_{-\infty}^0 |(\mathcal{F}\psi_\alpha)(t)|\,\mathrm{d}t < \infty,$$

since $\mathcal{F}\psi_\alpha \in L_1(\mathbb{R})$. To conclude, observe that taking the third summand in (3.3) and doing the same steps as above, we deduce that $\sum_{n=1}^\infty |(W^\alpha H_+^2 * H_-^1)(n)|\rho(n)$ is also finite. Consequently, as desired, we have proved that the series $\sum_{n=1}^\infty |D^\alpha \widehat{g}(n)|\rho(n)$ is convergent.

On the other hand, in order to bound the negative tail of the series $\sum_{n\in\mathbb{Z}} |D^\alpha \widehat{g}(n)|\rho(n)$, we apply the inequality (3.4):

$$\begin{aligned}\sum_{n=-\infty}^{-1} |D^\alpha \widehat{g}(n)|\rho(n) &= \sum_{n=1}^{\infty} |D^\alpha \widehat{g}(-n)|\rho(n) \le \sum_{n=1}^{\infty} |D^\alpha \widehat{g}(-n)|\rho(n+\alpha) \\ &= \sum_{n=1}^{\infty} |W^\alpha(\mathcal{F}G)(-(n+\alpha))|\rho(n+\alpha) \le \sum_{n=1}^{\infty} |W^\alpha(\mathcal{F}G)(-n)|\rho(n).\end{aligned}$$

Now, following the same steps as above for the positive tail, we conclude that this series is convergent. In addition, note that $|D^\alpha \widehat{g}(0)|$ is also bounded, because it is a sum having finitely many terms.

In conclusion, the series $\sum_{n\in\mathbb{Z}} |D^\alpha \widehat{g}(n)|\rho(n)$ is convergent and thus the function g belongs to $\mathcal{A}^\alpha_\rho$. Moreover, since $g : \partial\mathbb{D} \to \mathbb{C}$ has been built satisfying $g(e^{i\theta}) \neq 0$ and $g(e^{it}) = 0$ for all $|\theta - t| \geq \varepsilon$, we conclude that the algebra $\mathcal{A}^\alpha_\rho$ is regular. □

Now, we apply Theorem 3.1 to ω_α weights of the form $\rho_{\alpha+1}(n) = k_{\alpha+1}(|n|)\rho(n)$ for $n \in \mathbb{Z}$, where $\rho = (\rho(n))_{n\in\mathbb{Z}}$ is a symmetric Beurling sequence.

Corollary 3.2. *Let $\rho = (\rho(n))_{n\in\mathbb{Z}}$ be a symmetric Beurling sequence, i.e. a symmetric weight such that*

$$\sum_{n\in\mathbb{Z}} \frac{\log \rho(n)}{1+n^2} < +\infty.$$

Then, the fractional weighted Wiener algebra $\mathcal{A}^\alpha_{\rho_{\alpha+1}}$ is regular for all $\alpha \geq 0$.

Proof. All cases $\alpha \in \mathbb{N}_0$ are covered by Beurling's theorem [13] and our previous Theorem 3.1. Without loss of generality, assume that $\alpha \notin \mathbb{N}_0$. As noted in Remark 2.2, the positive weight $\rho_{\lceil\alpha\rceil+1}(n) = k_{\lceil\alpha\rceil+1}(|n|)\rho(n)$ belongs to the class $\omega_{\lceil\alpha\rceil}$. Using the asymptotics (2.2)

$$k_{\lceil\alpha\rceil+1}(n) \sim \frac{n^{\lceil\alpha\rceil}}{\lceil\alpha\rceil!}, \quad \text{as } n \to +\infty,$$

one has

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \frac{\log\left(\rho_{\lceil\alpha\rceil+1}(n)\right)}{1+n^2} &= \sum_{n\in\mathbb{Z}} \frac{\log\left(k_{\lceil\alpha\rceil+1}(|n|)\right)}{1+n^2} + \sum_{n\in\mathbb{Z}} \frac{\log \rho(n)}{1+n^2} \\ &\lesssim \sum_{n\in\mathbb{Z}} \frac{\log(|n|)}{1+n^2} + \sum_{n\in\mathbb{Z}} \frac{\log \rho(n)}{1+n^2} < +\infty.\end{aligned}$$

Thus, by Theorem 3.1 we deduce that the fractional weighted Wiener algebra $\mathcal{A}^{\lceil\alpha\rceil}_{\rho_{\lceil\alpha\rceil+1}}$ is regular and so, fixed any $e^{i\theta} \in \partial\mathbb{D}$ and $\varepsilon \in (0, \pi)$, there is a function $f \in \mathcal{A}^{\lceil\alpha\rceil}_{\rho_{\lceil\alpha\rceil+1}}$ such that $f(e^{i\theta}) \neq 0$ and $f(e^{it}) = 0$ for all $|\theta - t| \geq \varepsilon$. Finally, since $\mathcal{A}^{\lceil\alpha\rceil}_{\rho_{\lceil\alpha\rceil+1}} \hookrightarrow \mathcal{A}^\alpha_{\rho_{\alpha+1}}$ by equation (2.4), one concludes that $\mathcal{A}^\alpha_{\rho_{\alpha+1}}$ possesses partitions of unity, and so it is regular. □

4. Functional calculus induced by fractional weighted Wiener algebras $\mathcal{A}^\alpha_\rho$

Following the terminology set in the Introduction, let X denote an arbitrary complex Banach space. Again, let $\mathcal{B}(X)$ be the Banach algebra of linear bounded operators $T : X \to X$ and $\mathcal{L}(X)$ the class of linear operators $T : D(T) \to X$ (defined on an appropriate domain $D(T) \subseteq X$ for each $T \in \mathcal{L}(X)$). Recall that for each linear bounded operator $T : X \to X$, $\mathcal{T}_T := (\mathcal{T}_T(n))_{n\geq 0}$ denote the discrete forward semigroup given by $\mathcal{T}_T(n) := T^n$ for each $n \in \mathbb{N}_0$. The *Cesàro sums of order* $\alpha \geq 0$ of the operator T are given by

$$(\Delta^{-\alpha}\mathcal{T}_T)(n) := \sum_{j=0}^{n} k_\alpha(n-j)\mathcal{T}_T(j), \quad n \geq 0,$$

while its *Cesàro means of order* $\alpha \geq 0$ are

$$\mathcal{M}_T^\alpha(n) := \frac{1}{k_{\alpha+1}(n)}\,(\Delta^{-\alpha}\mathcal{T}_T)(n), \quad n \geq 0.$$

Cesàro summation methods are widely employed in a number of realms. For instance, in Operator Theory, they are used to define (C,α)-*operator*s, which are those $T \in \mathcal{B}(X)$ whose Cesàro means of order $\alpha > 0$ are uniformly bounded:

$$\sup_{n\geq 0}\left\|\frac{1}{k_{\alpha+1}(n)}\,(\Delta^{-\alpha}\mathcal{T}_T)(n)\right\| < +\infty.$$

In [4], the authors showed that for each $\alpha \geq 0$, the Cesàro kernel $k_{\alpha+1} = (k_{\alpha+1}(n))_{n\geq 0}$ belongs to the class ω_α and so one can define a continuous algebra homomorphism for (C,α)-operators $\vartheta_+^\alpha : \tau_+^\alpha(k_{\alpha+1}) \to \mathcal{B}(X)$ given by

$$\vartheta_+^\alpha(f) := \sum_{n=0}^{\infty}[D^\alpha f](n)(\Delta^{-\alpha}\mathcal{T}_T)(n), \quad \text{for each } \; f \in \tau_+^\alpha(k_{\alpha+1}).$$

A similar construction applies for unilateral fractional weighted Wiener algebras $\tau_+^\alpha(\rho)$ whose weight $\rho = (\rho(n))_{n\geq 0}$ belongs to the class ω_α, whenever the Cesàro sums $((\Delta^{-\alpha}\mathcal{T}_T)(n))_{n\geq 0}$ are dominated in norm by such weight, i.e.,

$$\left\|(\Delta^{-\alpha}\mathcal{T}_T)(n)\right\| \lesssim \rho(n), \quad \text{as } n \to \infty.$$

Our main objective in this section is to extend this aforementioned functional calculus to the scope of two-sided fractional weighted Wiener algebras $\tau^\alpha(\rho)$. To do so, consider an invertible operator $T \in \mathcal{B}(X)$ and let $\mathcal{T}_T := (\mathcal{T}_T(n))_{n\in\mathbb{Z}}$ be the discrete group $\mathcal{T}(n) := T^n$ for each $n \in \mathbb{Z}$. Again, one may regard the discrete group $\mathcal{T}_T$ as two semigroups indexed on $\mathbb{N}_0$ and $\mathbb{N}$ respectively: namely, $\mathcal{T}_{T+}(n) := \mathcal{T}_T(n)$ for each $n \geq 0$ while $\mathcal{T}_{T+}(n) := 0$ for all $n < 0$; and $\mathcal{T}_{T-}(n) := \mathcal{T}_T(-n)$ for $n > 0$ while $\mathcal{T}_{T-}(n) := 0$ for every $n \leq 0$. In this context, we define the Cesàro sums of order $\alpha > 0$ of the invertible operator $T : X \to X$ as

$$(\Delta^{-\alpha}\mathcal{T}_T)(n) := \begin{cases} (\Delta^{-\alpha}\mathcal{T}_{T+})(n), & \text{if } n \geq 0, \\ (\Delta^{-\alpha}\mathcal{T}_{T-})(-n), & \text{if } n < 0, \end{cases}$$

while its *Cesàro means of order* $\alpha \geq 0$ are

$$\mathcal{M}_T^\alpha(n) := \frac{1}{k_{\alpha+1}(|n|)}\,(\Delta^{-\alpha}\mathcal{T}_T)(n), \quad n \in \mathbb{Z}.$$

Notice that in the bilateral case, a small asymmetry emerges between the Cesàro sums $(\Delta^{-\alpha}\mathcal{T}_T)(n)$ of positive and negative index due to the rôle of $n = 0$. With these notions at hand, one can adapt the functional calculus ϑ_+^α to the bilateral case.

Theorem 4.1. *Let $T \in \mathcal{B}(X)$ be an invertible operator on a complex Banach space X. Suppose that for some $\alpha \geq 0$, there exists a symmetric weight $\rho := (\rho(n))_{n\in\mathbb{Z}}$ such that $\rho_+ \in \omega_\alpha$ and satisfying*

$$\left\|(\Delta^{-\alpha}\mathcal{T}_T)(n)\right\| \lesssim \rho(n), \quad \text{as } n \to \pm\infty.$$

Then, the mapping $\vartheta^\alpha : \tau^\alpha(\rho) \to \mathcal{B}(X)$ defined by

$$\vartheta^\alpha(f) := \sum_{n=-\infty}^{\infty} [D^\alpha f](n)(\Delta^{-\alpha}\mathcal{T}_T)(n), \quad \text{for each } f \in \tau^\alpha(\rho),$$

is a bounded unital algebra homomorphism.

Proof. To prove the boundedness of the mapping $\vartheta^\alpha : \tau^\alpha(\rho) \to \mathcal{B}(X)$, we need to verify that there exists an absolute constant $C_{\rho,\alpha} > 0$ (only depending on ρ and α) such that

$$\|\vartheta^\alpha(f)\| \leq C_{\rho,\alpha}\|f\|_{\rho,(\alpha)}, \quad \text{for every } f \in \tau^\alpha(\rho).$$

The asymptotic condition $\left\|(\Delta^{-\alpha}\mathcal{T}_T)(n)\right\| \lesssim \rho(n)$ as $n \to \pm\infty$, ensures the existence of a constant $C_{\rho,\alpha} > 0$ such that $\left\|(\Delta^{-\alpha}\mathcal{T}_T)(n)\right\| \leq C_{\rho,\alpha}\rho(n)$ for every $n \in \mathbb{Z}$. Consequently, an easy estimate shows that the mapping ϑ^α is continuous:

$$\|\vartheta^\alpha(f)\| \leq \sum_{n=-\infty}^{\infty} |[D^\alpha f](n)| \left\|(\Delta^{-\alpha}\mathcal{T}_T)(n)\right\| \leq C_{\rho,\alpha} \sum_{n=-\infty}^{\infty} |[D^\alpha f](n)|\rho(n) = C_{\rho,\alpha}\|f\|_{\rho,(\alpha)}.$$

Now, we would like to prove that the mapping ϑ^α fulfills all the required algebraic properties. Firstly, taking the unity of the algebra $\tau^\alpha(\rho)$ (which is given by the Dirac mass δ_0 on $\mathbb{Z}$) it is straightforward to check that

$$[D^\alpha\delta_0](n) = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{if } n \in \mathbb{Z} \setminus \{0\}. \end{cases}$$

Therefore,

$$\vartheta^\alpha(\delta_0) = \sum_{n=-\infty}^{\infty} [D^\alpha\delta_0](n)(\Delta^{-\alpha}\mathcal{T}_T)(n) = (\Delta^{-\alpha}\mathcal{T}_T)(0) = I,$$

where I denotes the identity operator acting on X. Now, since $[D^\alpha(\mu f + \nu g)](n) = \mu[D^\alpha f](n) + \nu[D^\alpha g](n)$ for every $f, g \in \tau^\alpha(\rho)$, $\mu, \nu \in \mathbb{C}$ and $n \in \mathbb{Z}$, it is easy to check that

$$\vartheta^\alpha(\mu f + \nu g) = \mu\vartheta^\alpha(f) + \nu\vartheta^\alpha(g).$$

Finally, we must prove that $\vartheta^\alpha(f * g) = \vartheta^\alpha(f)\vartheta^\alpha(g)$ for every pair $f, g \in \tau^\alpha(\rho)$. In general, for each $m \in \mathbb{Z}$ one has that $\vartheta^\alpha(\delta_m) = T^m$. For instance, given $m \geq 0$, since

$$[D^\alpha\delta_m](n) = \begin{cases} k_{-\alpha}(m-n), & \text{if } 0 \leq n \leq m, \\ 0, & \text{otherwise}, \end{cases}$$

we have

$$\begin{aligned}\vartheta^\alpha(\delta_m) &= \sum_{n=-\infty}^{\infty} [D^\alpha \delta_m](n)(\Delta^{-\alpha}\mathcal{T}_T)(n) = \sum_{n=0}^{m} k_{-\alpha}(m-n)(\Delta^{-\alpha}\mathcal{T}_T)(n)\\ &= \sum_{n=0}^{m}\sum_{j=0}^{n} k_{-\alpha}(m-n)k_\alpha(n-j)T^j = \sum_{j=0}^{m}\sum_{n=j}^{m} k_{-\alpha}(m-n)k_\alpha(n-j)T^j\\ &= \sum_{j=0}^{m} T^j \sum_{n=j}^{m} k_{-\alpha}(m-n)k_\alpha(n-j) = \sum_{j=0}^{m} k_0(m-j)T^j = T^m.\end{aligned}$$

Similarly, for $m < 0$, one has $[D^\alpha \delta_m](n) = k_{-\alpha}(n-m)$ for $m \le n < 0$ while $[D^\alpha \delta_m](n) = 0$ otherwise. Hence,

$$\begin{aligned}\vartheta^\alpha(\delta_m) &= \sum_{n=-\infty}^{\infty} [D^\alpha \delta_m](n)(\Delta^{-\alpha}\mathcal{T}_T)(n) = \sum_{n=m}^{-1} k_{-\alpha}(n-m)(\Delta^{-\alpha}\mathcal{T}_T)(n)\\ &= \sum_{n=m}^{-1} k_{-\alpha}(n-m)\sum_{j=0}^{-n} k_\alpha(-n-j)\mathcal{T}_{T^-}(j) = \sum_{n=m}^{-1} k_{-\alpha}(n-m)\sum_{j=1}^{-n} k_\alpha(-n-j)T^{-j}\\ &= \sum_{j=1}^{-m} T^{-j} \sum_{n=m}^{-j} k_{-\alpha}(n-m)k_\alpha(-n-j) = \sum_{j=1}^{-m} T^{-j} k_0(-m-j) = T^m.\end{aligned}$$

Thus, taking into account that $\delta_m * \delta_n = \delta_{m+n}$ for every $m, n \in \mathbb{Z}$, one immediately concludes that $\vartheta^\alpha(\delta_m * \delta_n) = \vartheta^\alpha(\delta_m)\vartheta^\alpha(\delta_n)$.

Finally, by induction, we show that $\vartheta^\alpha(f * g) = \vartheta^\alpha(f)\vartheta^\alpha(g)$ for each pair of sequences $f, g \in c_{0,0}(\mathbb{Z})$ with the aim of ultimately applying a density argument. Consider two arbitrary finite linear complex combinations of size $M-1 \ge 1$ consisting of Dirac masses supported on $\mathbb{Z}$, that is, $f = \sum_{\ell=1}^{M-1} \mu_\ell \delta_{m_\ell}$ and $g = \sum_{\ell=1}^{M-1} \nu_\ell \delta_{n_\ell}$ with $\mu_\ell, \nu_\ell \in \mathbb{C}$, and assume that $\vartheta^\alpha(f*g) = \vartheta^\alpha(f)\vartheta^\alpha(g)$. Of course, some of the coefficients $\mu_\ell, \nu_\ell \in \mathbb{C}$ could be zero, so that the sizes of the linear combinations that define f and g do not coincide. Then, choosing $m_M \ne m_\ell$ and $n_M \ne n_\ell$ for every $\ell = 1, \dots, M-1$, we have

$$\begin{aligned}&\vartheta^\alpha\big((f + \mu_M \delta_{m_M}) * (g + \nu_M \delta_{n_M})\big)\\ &\quad= \vartheta^\alpha\big(f * g + \nu_M f * \delta_{n_M} + \mu_M \delta_M * g + \mu_M \nu_M \delta_{m_M} * \delta_{n_M}\big)\\ &\quad= \vartheta^\alpha(f * g) + \nu_M \vartheta^\alpha(f * \delta_{n_M}) + \mu_M \vartheta^\alpha(\delta_{m_M} * g) + \mu_M \nu_M \vartheta^\alpha(\delta_{m_M} * \delta_{n_M})\\ &\quad= \vartheta^\alpha(f)\vartheta^\alpha(g) + \nu_M \vartheta^\alpha(f)\vartheta^\alpha(\delta_{n_M}) + \mu_M \vartheta^\alpha(\delta_{m_M})\vartheta^\alpha(g) + \mu_M \nu_M \vartheta^\alpha(\delta_{m_M})\vartheta^\alpha(\delta_{n_M})\\ &\quad= \big(\vartheta^\alpha(f) + \mu_M \vartheta^\alpha(\delta_{m_M})\big)\big(\vartheta^\alpha(g) + \nu_M \vartheta^\alpha(\delta_{n_M})\big)\\ &\quad= \vartheta^\alpha\big(f + \mu_M \delta_{m_M}\big)\,\vartheta^\alpha\big(g + \nu_M \delta_{n_M}\big).\end{aligned}$$

Since each function in $c_{0,0}(\mathbb{Z})$ is a finite linear combination of Dirac masses, this proves that $\vartheta^\alpha(f * g) = \vartheta^\alpha(f)\vartheta^\alpha(g)$ for every $f, g \in c_{0,0}(\mathbb{Z})$ as desired. To conclude the proof, it suffices to consider the assignment

$$\begin{aligned}\eta : \tau^\alpha(\rho) \times \tau^\alpha(\rho) &\longrightarrow \mathcal{B}(X)\\ (f, g) &\longmapsto \vartheta^\alpha(f * g) - \vartheta^\alpha(f)\vartheta^\alpha(g),\end{aligned}$$

which is continuous (since both ϑ^α and convolution are continuous) and identically zero on the dense subset $c_{0,0}(\mathbb{Z}) \times c_{0,0}(\mathbb{Z})$ within $\tau^\alpha(\rho) \times \tau^\alpha(\rho)$. In conclusion, $\vartheta^\alpha(f * g) = \vartheta^\alpha(f)\vartheta^\alpha(g)$ for every $f, g \in \tau^\alpha(\rho)$ and the proof is complete. □

Of course, our last result can be rewritten in terms of fractional weighted Wiener algebras $\mathcal{A}_\rho^\alpha$ in order to work with Banach algebras of functions, where the notion of regularity could be more intuitive. To do so, just consider the Fourier coefficients of $f \in \mathcal{A}_\rho^\alpha$ and the pointwise product of functions in $\mathcal{A}_\rho^\alpha$ (instead of the convolution product of sequences in $\tau^\alpha(\rho)$).

Corollary 4.2. *Let $T \in \mathcal{B}(X)$ be an invertible operator on a complex Banach space X. Suppose that for some $\alpha \geq 0$, there exists a symmetric weight $\rho := (\rho(n))_{n\in\mathbb{Z}}$ such that $\rho_+ \in \omega_\alpha$ and satisfying*

$$\left\|(\Delta^{-\alpha}\mathcal{T}_T)(n)\right\| \lesssim \rho(n), \quad \text{as } n \to \pm\infty.$$

Then, the mapping $\vartheta^\alpha : \mathcal{A}_\rho^\alpha \to \mathcal{B}(X)$ defined by

$$\vartheta^\alpha(f) := \sum_{n=-\infty}^{\infty} [D^\alpha \widehat{f}](n)(\Delta^{-\alpha}\mathcal{T}_T)(n), \quad \text{for each } f \in \mathcal{A}_\rho^\alpha,$$

is a bounded unital algebra homomorphism.

5. An invariant subspace criterion based on fractional weighted Wiener algebras

A great advantage of regular function algebras is that they admit non-zero functions whose product is identically null. This idea, combined with a functional calculus argument, provides a fruitful strategy to produce invariant subspaces. This method was originally developed by Wermer [51] to find invariant subspaces for invertible operators whose forward and backward powers were dominated in norm by a Beurling sequence.

In our case, we shall be interested in working in the generality of [23]. To do so, we will previously introduce some basic concepts of local spectral theory and Gelfand theory of commutative Banach algebras.

5.1. Basics of local spectral theory and Gelfand theory

Given a linear bounded operator $T : X \to X$, let $\sigma(T)$ and $\rho(T) := \mathbb{C} \setminus \sigma(T)$ denote respectively its *spectrum* and its *resolvent set*. The *point spectrum* $\sigma_{\mathrm{p}}(T)$ consists of eigenvalues of T and $\sigma_{\mathrm{com}}(T) = \{\lambda \in \mathbb{C} : (T-\lambda)X$ is not dense in $X\}$ stands for the *compression spectrum*. Recall that the spectral identities $\sigma_{\mathrm{p}}(T) \subseteq \sigma_{\mathrm{com}}(T^*)$ and $\sigma_{\mathrm{com}}(T) = \sigma_{\mathrm{p}}(T^*)$ hold for the *adjoint* $T^* : X^* \to X^*$. Additionally, for our purposes we shall need some basic notions on *local spectral theory* (we refer to [33] for a detailed insight into the topic). Let $\sigma_T(x)$ be the *local spectrum of T at a vector $x \in X$*, i.e. the complement of the set of all $\lambda \in \mathbb{C}$ for which there exists an open neighborhood $U_\lambda \ni \lambda$ and a X-valued holomorphic function $f : U_\lambda \to X$ verifying

$$(T-z)f(z) = x, \quad \text{for every } z \in U_\lambda.$$

Fixed any subset $F \subseteq \mathbb{C}$, the *local spectral manifold* $X_T(F)$ is defined as

$$X_T(F) := \left\{x \in X \, : \, \sigma_T(x) \subseteq F\right\}.$$

Local spectral manifolds $X_T(F)$ are always hyperinvariant linear manifolds, independently on the choice of F. Nevertheless, in general, local spectral manifolds $X_T(F)$ are not closed in the norm topology of X: actually, those operators T having closed local spectral manifolds are said to satisfy *Dunford's property* (C). In any case, a remarkable fact satisfied by local spectral manifolds is (see [33, Prop. 2.5.1]) that whenever $\sigma_{\mathrm{p}}(T) = \emptyset$, then

$$X_T(F) \subseteq {}^{\perp}X^*_{T^*}(G) \quad \text{and} \quad X^*_{T^*}(G) \subseteq X_T(F)^{\perp}$$

for every pair of disjoint closed sets $F, G \subseteq \mathbb{C}$, where $M^{\perp}$ stands for the *annihilator* of a set $M \subseteq X$, while ${}^{\perp}N$ denotes the *preannihilator* of a set $N \subseteq X^*$.

Local spectral theory arose as a fruitful endeavor to transfer some of the most important features of normal operators to more general realms. In that sense, recall that an operator $T \in \mathcal{B}(X)$ is called *decomposable* if for every finite open cover $\{U_1, U_2, \ldots, U_n\}$ of $\sigma(T)$, there exist closed hyperinvariant subspaces $X_1, X_2, \ldots, X_n \subseteq X$ such that

$$\sigma(T|_{X_j}) \subseteq U_j \quad \text{for each } j = 1, \ldots, n,$$

and satisfying $X = X_1 + \ldots + X_n$.

In [23] (see also the work by Neumann [41]), the authors combined well-known aspects of *Gelfand theory* with functional calculus tools to obtain a general strategy to produce invariant subspaces. Recall that for a commutative complex Banach algebra $\mathcal{A}$, its *maximal ideal space* $\Delta(\mathcal{A})$ (also known as *Gelfand spectrum*) is the set of all non-zero multiplicative linear functionals acting boundedly on $\mathcal{A}$. For each $a \in \mathcal{A}$, the *Gelfand transform* $\widehat{a} : \Delta(\mathcal{A}) \to \mathbb{C}$ is given by $\widehat{a}(\varphi) := \varphi(a)$ for all $\varphi \in \Delta(\mathcal{A})$. The *Gelfand topology* in $\Delta(\mathcal{A})$ is defined as the coarsest topology with respect to which all Gelfand transforms are continuous. Whenever $\mathcal{A}$ is unital, $\Delta(\mathcal{A})$ is compact in the Gelfand topology. There exists a strong connection between the topological features of $\Delta(\mathcal{A})$ and the existence of partitions of unity in $\Delta(\mathcal{A})$. In this line, the concept of *regularity* of Banach algebras is particularly remarkable:

Definition 5.1. Let $\mathcal{A}$ be a complex commutative Banach algebra. Then $\mathcal{A}$ is called *regular* if for each Gelfand closed set $S \subseteq \Delta(\mathcal{A})$ and all $\varphi \in \Delta(\mathcal{A}) \setminus S$, there exists an element $a \in \mathcal{A}$ whose Gelfand transform verifies $\widehat{a} \equiv 0$ on S and $\widehat{a}(\varphi) \neq 0$.

Evidently, the notion of regularity in the general context of Banach algebras subsumes the previous one seen for function algebras. According to [23], we shall need to introduce the notions of *algebra action* and *continuity core*. Given a commutative Banach algebra, a map $\vartheta : \mathcal{A} \to \mathcal{L}(X)$ is an algebra action over a linear submanifold $M \subseteq \bigcap_{a \in \mathcal{A}} D(\vartheta(a))$ if for every $x \in M$ one has:

(a) For every $a, b \in \mathcal{A}$ and $\mu, \nu \in \mathbb{C}$, one has $\vartheta(\mu a + \nu b)x = \mu\vartheta(a)x + \nu\vartheta(b)x$.
(b) For all $b \in \mathcal{A}$, one has $\vartheta(b)x \in M$ and $\vartheta(ab)x = \vartheta(a)\vartheta(b)x$ for every $a \in \mathcal{A}$.

Additionally, we will say that a linear manifold $\mathcal{D}_\vartheta \subseteq X$ is a continuity core for $\vartheta : \mathcal{A} \to \mathcal{L}(X)$ if ϑ is an algebra action over $\mathcal{D}_\vartheta$ and the assignment $\mathcal{A} \to X$, $a \mapsto \vartheta(a)x$ is bounded for each $x \in \mathcal{D}_\vartheta$.

5.2. An extension of Wermer's/Atzmon's criterion for invariant subspaces

In the following, we prove our main results concerning the existence of invariant subspaces. As mentioned above, our statements generalize both Wermer theorem [51] and its local counterpart due to Atzmon [8], replacing the growth conditions required in those theorems by their Cesàro summation versions.

The techniques employed in this part are mainly based in spectral decompositions of $\sigma(T)$, developed among several authors by Neumann and Gallardo-Gutiérrez et al. [23,41]. In the invertible case (see Theorems 5.2 and 5.4 below), our results will ensure the decomposability of the operator $T : X \to X$, while in the non-invertible case (see Theorems 5.3 and 5.5 below), it happens that the operator T decomposes over a non-trivial linear manifold $M \subseteq X$ which could be even norm-dense (see [23, Th. 3.4] for an example involving the Bishop operators).

Finally, just mention that in the non-invertible case, we shall repeatedly impose the spectral condition $\sigma_{\mathrm{com}}(T) = \sigma_{\mathrm{com}}(T^*) = \emptyset$. Observe that such a condition does not detract from the generality of our results, since any operator $T \in \mathcal{B}(X)$ having $\sigma_{\mathrm{com}}(T) \neq \emptyset$ or $\sigma_{\mathrm{com}}(T^*) \neq \emptyset$ immediately has non-trivial closed hyperinvariant subspaces. However, one might replace such a condition and just assume the weaker hypothesis that both T and its adjoint T^* are injective with dense range.

We begin with the invertible case, i.e., with the Cesàro summation generalization of Wermer theorem [51].

Theorem 5.2. *Let $T \in \mathcal{B}(X)$ be an invertible operator on a complex Banach space X having $\sigma(T) \subseteq \partial\mathbb{D}$. Suppose that there exist $\alpha \in \mathbb{N}_0$ and a symmetric weight $\rho = (\rho(n))_{n\in\mathbb{Z}}$ such that $\rho_+ \in \omega_\alpha$ satisfying*

$$\text{(i).}\ \ \left\|(\Delta^{-\alpha}\mathcal{T}_T)(n)\right\| \lesssim \rho(n) \ \text{ as } n \to \pm\infty, \qquad\qquad \text{(ii).}\ \ \sum_{n\in\mathbb{Z}} \frac{\log \rho(n)}{1+n^2} < \infty.$$

Then, T is a decomposable operator. In particular, if $\sigma(T)$ is not a singleton, the operator T has a non-trivial hyperinvariant subspace.

Proof. This result is a direct consequence of the regularity criterion proved in Theorem 3.1 for the fractional weighted Wiener algebra $\mathcal{A}^\alpha_\rho$, the algebra homomorphism $\vartheta^\alpha : \mathcal{A}^\alpha_\rho \to \mathcal{B}(X)$ constructed in Corollary 4.2, and Theorem 4.4.1 in the book [33] (see also [41, Theorem 1]). □

The proof in the non-invertible case is not as direct since we are forced to establish a suitable algebra action and continuity core for this Cesàro summation functional calculus.

Theorem 5.3. *Let $T \in \mathcal{B}(X)$ be an operator on a complex Banach space X with $\sigma_{\mathrm{com}}(T) = \sigma_{\mathrm{com}}(T^*) = \emptyset$. Assume that there exist $\alpha \in \mathbb{N}_0$ and two non-zero vectors $x \in X$ and $y \in X^*$ satisfying*

$$\left\|(\Delta^{-\alpha}\mathcal{T}_T)(n)x\right\|_X \lesssim \rho(n) \quad \text{and} \quad \left\|(\Delta^{-\alpha}\mathcal{T}_{T^*})(n)y\right\|_{X^*} \lesssim \rho(n) \qquad \text{as } n \to \pm\infty,$$

for some symmetric weight $\rho = (\rho(n))_{n\in\mathbb{Z}}$ such that $\rho_+ \in \omega_\alpha$ and

$$\sum_{n\in\mathbb{Z}} \frac{\log \rho(n)}{1+n^2} < \infty.$$

Then, if $\sigma_T(x) \cup \sigma_{T^*}(y)$ is not a singleton, the operator T has a non-trivial hyperinvariant subspace.

Proof. Our aim is to apply [23, Th. 2.7] in the context of fractional weighted Wiener algebras $\mathcal{A}^\alpha_\rho$. To do so, we need to construct two algebra actions having non-trivial continuity cores. As mentioned above, observe that since $\sigma_{\mathrm{com}}(T) = \sigma_{\mathrm{com}}(T^*) = \emptyset$, both $T \in \mathcal{B}(X)$ and $T^* \in \mathcal{B}(X)$ are injective operators with dense range in X and X^* respectively. Hence, [11, Cor. 1.B.3] ensures that for each $T : X \to X$ and $T^* : X^* \to X^*$, there exists a dense set of vectors in X and X^*, respectively, having an infinite chain of backward iterates. Indeed, due to the injectivity of T and T^*, these infinite chains of backward iterates are univocally determined. At this regard, let $T^\infty(X)$ and $T^{*\,\infty}(X^*)$ be the hyperranges of the operators T and T^*, i.e.

$$T^\infty(X) := \bigcap_{n\geq 0} T^n(X) \quad \text{ and } \quad T^{*\,\infty}(X^*) := \bigcap_{n\geq 0} T^{*\,n}(X^*).$$

By our latter remark, the hyperranges $T^\infty(X)$ and $T^{*\,\infty}(X^*)$ are both dense linear manifolds consisting of vectors having well-defined infinite backward iterates. So, for the Cesàro sums of the iterates of T and T^* we may choose the domains

$$D\big((\Delta^{-\alpha}\mathcal{T}_T)(n)\big) := \begin{cases} X, & \text{if } n \geq 0, \\ T^{\infty}(X) & \text{if } n < 0, \end{cases} \quad \text{and} \quad D\big((\Delta^{-\alpha}\mathcal{T}_{T^*})(n)\big) := \begin{cases} X^*, & \text{if } n \geq 0, \\ T^{*\,\infty}(X^*) & \text{if } n < 0. \end{cases}$$

With this choice, for each $n < 0$, the Cesàro sums $(\Delta^{-\alpha}\mathcal{T}_T)(n) \in \mathcal{L}(X)$ and $(\Delta^{-\alpha}\mathcal{T}_{T^*})(n) \in \mathcal{L}(X^*)$ are densely-defined linear operators for all $n < 0$.

Now, consider the following mappings taking values in the classes of non-bounded linear operators $\mathcal{L}(X)$ and $\mathcal{L}(X^*)$:

$$\begin{aligned} \vartheta^{\alpha} : \mathcal{A}^{\alpha}_{\rho} &\to \mathcal{L}(X) & \qquad \vartheta^{\alpha}_{*} : \mathcal{A}^{\alpha}_{\rho} &\to \mathcal{L}(X^*) \\ f &\mapsto \sum_{n=-\infty}^{\infty} [D^{\alpha}\widehat{f}](n)(\Delta^{-\alpha}\mathcal{T}_T)(n) & g &\mapsto \sum_{n=-\infty}^{\infty} [D^{\alpha}\widehat{g}](n)(\Delta^{-\alpha}\mathcal{T}_{T^*})(n). \end{aligned}$$

However, to provide a valid definition for each operator in the ranges of ϑ^{α} and ϑ^{α}_{*}, we must specify their domains. For instance, we may consider $\vartheta^{\alpha}(f) \in \mathcal{L}(X)$ defined on the domain

$$D(\vartheta^{\alpha}(f)) := \left\{ x \in T^{\infty}(X) \, : \, \sum_{n=-\infty}^{\infty} \big|[D^{\alpha}\widehat{f}](n)\big| \cdot \big\|(\Delta^{-\alpha}\mathcal{T}_T)(n)x\big\|_X < \infty \right\}.$$

Actually, observe that $D(\vartheta^{\alpha}(f))$ are linear manifolds in X and, due to the absolute convergence of the series, the operator $\vartheta^{\alpha}(f) \in \mathcal{L}(X)$ is well-defined on that domain. An analogous definition applies to specify the domain $D(\vartheta^{\alpha}_{*}(g))$ in the adjoint case.

Now, we must determine continuity cores for each ϑ^{α} and ϑ^{α}_{*}. According to the asymptotic condition stated in the statement, we propose

$$\mathcal{D}_{\vartheta^{\alpha}} := \left\{ x \in T^{\infty}(X) \, : \, \big\|(\Delta^{-\alpha}\mathcal{T}_T)(n)x\big\|_X \lesssim_x \rho(n) \ \text{ as } n \to \pm\infty \right\},$$

where $\lesssim_x$ indicates that the constant $C_x > 0$ such that $\big\|(\Delta^{-\alpha}\mathcal{T})(n)x\big\|_X \leq C_x\rho(n)$ for every $n \in \mathbb{Z}$ may depend on $x \in \mathcal{D}_{\vartheta^{\alpha}}$. Observe that $\mathcal{D}_{\vartheta^{\alpha}}$ is a linear manifold included in $\bigcap_{f \in \mathcal{A}^{\alpha}_{\rho}} D(\vartheta^{\alpha}(f)) \subseteq X$. Furthermore, as a by-product of the algebra homomorphism built in Corollary 4.2, it is plain that ϑ^{α} meets all the algebraic properties required to be an algebra action over $\mathcal{D}_{\vartheta^{\alpha}}$. Furthermore, we must still check that $\vartheta^{\alpha}(f)x$ belongs to $\mathcal{D}_{\vartheta^{\alpha}}$ for every $f \in \mathcal{A}^{\alpha}_{\rho}$ and $x \in \mathcal{D}_{\vartheta^{\alpha}}$. For that purpose, consider the sequence of functions $(s_n)_{n\in\mathbb{Z}}$ in the Banach algebra $\mathcal{A}^{\alpha}_{\rho}$ defined by

$$s_n(e^{it}) := \begin{cases} \displaystyle\sum_{j=0}^{n} k_{\alpha}(n-j)e^{ijt}, & \text{if } n \geq 0, \\ \displaystyle\sum_{j=n}^{-1} k_{\alpha}(j-n)e^{ijt}, & \text{if } n < 0. \end{cases}$$

It is straightforward to prove that

$$[D^{\alpha}\widehat{s}_n](j) = \begin{cases} 1, & \text{if } j = n, \\ 0, & \text{if } j \neq n, \end{cases} \qquad \text{for each } j \in \mathbb{Z}.$$

Accordingly, $\|s_n\|_{\alpha,(\rho)} = \sum_{j\in\mathbb{Z}} |[D^{\alpha}\widehat{s}_n](j)|\,\rho(j) = \rho(n)$ and $\vartheta^{\alpha}(s_n) = (\Delta^{-\alpha}\mathcal{T}_T)(n)$ for all $n \in \mathbb{Z}$. Therefore,

$$\begin{aligned}\big\|(\Delta^{-\alpha}\mathcal{T}_T)(n)\vartheta^\alpha(f)x\big\|_X &= \big\|\vartheta^\alpha(s_n)\vartheta^\alpha(f)x\big\|_X = \big\|\vartheta^\alpha(s_nf)x\big\|_X \\ &= \sum_{m=-\infty}^{\infty}\big|[D^\alpha\widehat{s_nf}\,](m)\big|\cdot\big\|(\Delta^{-\alpha}\mathcal{T}_T)(m)x\big\|_X \\ &\lesssim_x \sum_{m=-\infty}^{\infty}\big|[D^\alpha\widehat{s_nf}\,](m)\big|\,\rho(m) = \|s_nf\|_{\alpha,(\rho)} \\ &\lesssim_\alpha \|f\|_{\alpha,(\rho)}\,\|s_n\|_{\alpha,(\rho)} \lesssim_f \rho(n).\end{aligned}$$

In conclusion, $\vartheta^\alpha(f)x \in \mathcal{D}_{\vartheta^\alpha}$ whenever $f \in \mathcal{A}^\alpha_\rho$ and $x \in \mathcal{D}_{\vartheta^\alpha}$, and so we have already proved that ϑ^α is an algebra action over $\mathcal{D}_{\vartheta^\alpha}$. A similar argument shows the boundedness of the mapping $\mathcal{A}^\alpha_\rho \to X$, $f \mapsto \vartheta^\alpha(f)x$ for each $x \in \mathcal{D}_{\vartheta^\alpha}$:

$$\begin{aligned}\|\vartheta^\alpha(f)x\|_X &= \Big\|\sum_{n=-\infty}^{\infty}[D^\alpha\widehat{f}\,](n)(\Delta^{-\alpha}\mathcal{T}_T)(n)x\Big\|_X \le \sum_{n=-\infty}^{\infty}\big|[D^\alpha\widehat{f}\,](n)\big|\cdot\big\|(\Delta^{-\alpha}\mathcal{T}_T)(n)x\big\|_X \\ &\lesssim_x \sum_{n=-\infty}^{\infty}\big|[D^\alpha\widehat{f}\,](n)\big|\rho(n) = \|f\|_{\alpha,(\rho)}.\end{aligned}$$

In sum, we conclude that $\mathcal{D}_{\vartheta^\alpha}$ is a continuity core for the algebra action ϑ^α. Again, an analogous argument leads to a continuity core $\mathcal{D}_{\vartheta^\alpha_*}$ for the algebra action $\vartheta^\alpha_* : \mathcal{A}^\alpha_\rho \to \mathcal{L}(X^*)$. Furthermore, observe that the condition in the statement which ensures the existence of two non-zero vectors $x \in X$ and $y \in X^*$ such that

$$\big\|(\Delta^{-\alpha}\mathcal{T}_T)(n)x\big\|_X \lesssim \rho(n) \quad \text{and} \quad \big\|(\Delta^{-\alpha}\mathcal{T}_{T^*})(n)y\big\|_{X^*} \lesssim \rho(n) \qquad \text{as } n \to \pm\infty,$$

guarantees that both continuity cores $\mathcal{D}_{\vartheta^\alpha}$ and $\mathcal{D}_{\vartheta^\alpha_*}$ are non-zero. Now, since the weight $\rho = (\rho(n))_{n\in\mathbb{Z}}$ satisfies the Beurling condition,

$$\sum_{n=-\infty}^{\infty}\frac{\log\rho(n)}{1+n^2} < \infty,$$

the fractional weighted Wiener algebra $\mathcal{A}^\alpha_\rho$ is regular. Note that $T = \vartheta^\alpha(Z)$ and $T^* = \vartheta^\alpha_*(Z)$ where $Z \in \mathcal{A}^\alpha_\rho$ is the identity function $Z(e^{it}) := e^{it}$. Recalling that the maximal ideal space of $\mathcal{A}^\alpha_\rho$ can be identified with the torus, i.e. any multiplicative functional in $\Delta(\mathcal{A}^\alpha_\rho)$ is of the form $\varphi_\theta : \mathcal{A}^\alpha_\rho \to \mathbb{C}$ with $\varphi_\theta(f) := f(e^{i\theta})$ for some $\theta \in [0, 2\pi)$, an application of [23, Th. 2.6] implies that for each pair of closed sets $F, G \subseteq \partial\mathbb{D}$, we have

$$X_T(F) \supseteq \big\{\vartheta^\alpha(f)(\mathcal{D}_{\vartheta^\alpha}) \,:\, \text{for every } f \in \mathcal{A}^\alpha_\rho \text{ with } \operatorname{supp}(f) \subseteq F\big\}$$

and

$$X^*_{T^*}(G) \supseteq \big\{\vartheta^\alpha_*(g)(\mathcal{D}_{\vartheta^\alpha_*}) \,:\, \text{for every } g \in \mathcal{A}^\alpha_\rho \text{ with } \operatorname{supp}(g) \subseteq G\big\}.$$

Finally, an argument like the one followed in the proof of [23, Th. 2.7] allows to choose two disjoint closed sets in the torus $F, G \subseteq \mathbb{C}$ with $X_T(F) \neq \{0\}$ and $X^*_{T^*}(G) \neq \{0\}$. This implies that the local spectral manifold $X_T(F)$ satisfies the inequalities

$$\{0\} \neq \overline{X_T(F)} \neq X,$$

and, therefore, as we desired to prove, the operator $T : X \to X$ possesses a non-trivial closed hyperinvariant subspace. $\square$

As done at the end of Section 3, we can restate these results in terms of fractional weighted Wiener algebras of the form $\mathcal{A}^{\alpha}_{\rho_{\alpha+1}}$ to obtain the corresponding Cesàro mean version.

Theorem 5.4. *Let $T \in \mathcal{B}(X)$ be an invertible operator on a complex Banach space X having $\sigma(T) \subseteq \partial\mathbb{D}$. Suppose that there exist $\alpha \geq 0$ and a symmetric Beurling sequence $\rho = (\rho(n))_{n\in\mathbb{Z}}$ such that*

$$\left\|\mathcal{M}^{\alpha}_{T}(n)\right\| \lesssim \rho(n), \quad \text{as } n \to \pm\infty.$$

Then, T is a decomposable operator. In particular, if $\sigma(T)$ is not a singleton, the operator T has a non-trivial closed hyperinvariant subspace.

Theorem 5.5. *Let $T \in \mathcal{B}(X)$ be an operator on a complex Banach space X with $\sigma_{\mathrm{com}}(T) = \sigma_{\mathrm{com}}(T^*) = \emptyset$. Assume that there exist $\alpha \geq 0$ and two non-zero vectors $x \in X$ and $y \in X^*$ satisfying*

$$\left\|\mathcal{M}^{\alpha}_{T}(n)x\right\|_X \lesssim \rho(n) \quad \text{and} \quad \left\|\mathcal{M}^{\alpha}_{T^*}(n)y\right\|_{X^*} \lesssim \rho(n), \quad \text{as } n \to \pm\infty,$$

for some Beurling sequence $\rho = (\rho(n))_{n\in\mathbb{Z}}$. Then, if $\sigma_T(x) \cup \sigma_{T^*}(y)$ is not a singleton, the operator T has a non-trivial closed hyperinvariant subspace.

6. Summability methods and invariant subspaces

A natural question in relation with the Cesàro summability version of Wermer's/Atzmon's theorem is how large is the actual encompassing of these methods. To quantify it, for a given operator $T \in \mathcal{B}(X)$ with $\sigma_{\mathrm{com}}(T) = \sigma_{\mathrm{com}}(T^*) = \emptyset$ (or, similarly, for $T \in \mathcal{B}(X)$ invertible in the non-local case), we will consider the next linear manifolds

$$\mathfrak{X}^{\alpha}_{T} := \left\{x \in X : \left\|(\Delta^{-\alpha}\mathcal{T}_T)(n)x\right\|_X \lesssim_x \rho(n) \text{ for some symmetric Beurling sequence with } \rho_+ \in \omega_\alpha\right\}$$

for each $\alpha \geq 0$. Our next result ensures that all the Cesàro summability versions of Wermer's/Atzmon's theorem are, in a sense, equivalent for each $\alpha \geq 0$. More precisely, it turns out that, unfortunately, all these methods can be applied to the same set of operators. As can be derived from our proof, this happens mainly due to the fact that the Cesàro kernels $k_\alpha = (k_\alpha(n))_{n\geq 0}$ grow polynomially as $n \to +\infty$, since the Beurling sequence is able to absorb the possible cancellations within the norm.

Theorem 6.1. *Let $T \in \mathcal{B}(X)$ be an operator on a complex Banach space X with $\sigma_{\mathrm{com}}(T) = \sigma_{\mathrm{com}}(T^*) = \emptyset$. Then, $\mathfrak{X}^{\alpha}_{T} = \mathfrak{X}^{0}_{T}$ for all $\alpha \geq 0$.*

Proof. First, observe that $\{0\} \subseteq \mathfrak{X}^{\alpha}_{T} \subseteq \mathfrak{X}^{\beta}_{T}$ for all $0 \leq \alpha \leq \beta$. Indeed, taking a vector $x \in \mathfrak{X}^{\alpha}_{T}$, there exists a symmetric Beurling sequence $\rho = (\rho(n))_{n\in\mathbb{Z}}$ with $\rho_+ \in \omega_\alpha$ such that

$$\left\|(\Delta^{-\alpha}\mathcal{T}_T)(n)x\right\|_X \lesssim \rho(n).$$

Then,

$$\begin{aligned}
\left\|\mathcal{M}^{-\beta}_{T}(n)x\right\|_X &= \frac{1}{k_{\beta+1}(|n|)}\left\|(\Delta^{-\beta}\mathcal{T}_T)(n)x\right\|_X = \frac{1}{k_{\beta+1}(|n|)}\left\|(\Delta^{-(\beta-\alpha)}\Delta^{-\alpha}\mathcal{T}_T)(n)x\right\|_X \\
&\leq \frac{1}{k_{\beta+1}(|n|)}\sum_{j=0}^{n} k_{\beta-\alpha}(|n-j|)\left\|(\Delta^{-\alpha}\mathcal{T}_T(j))x\right\|_X \lesssim \frac{1}{k_{\beta+1}(|n|)}\sum_{j=0}^{n} k_{\beta-\alpha}(|n-j|)\rho(j) \\
&\leq \frac{\rho(n)}{k_{\beta+1}(|n|)}\sum_{j=0}^{n} k_{\beta-\alpha}(|n-j|) = \frac{k_{\beta-\alpha+1}(|n|)}{k_{\beta+1}(|n|)}\rho(n) \leq \rho(n),
\end{aligned}$$

and, therefore, $\left\|(\Delta^{-\beta}\mathcal{T}_T)(n)x\right\|_X \lesssim k_{\beta+1}(|n|)\rho(n)$ for all $n\in\mathbb{Z}$ showing that $x\in\mathfrak{X}_T^{\beta}$.

On the other hand, observe that for all $n\geq 1$

$$(\Delta^{-\alpha}\mathcal{T}_T)(n)-(\Delta^{-\alpha}\mathcal{T}_T)(n-1)=(\Delta^{-(\alpha-1)}\mathcal{T}_T)(n),$$

while for all $n<-1$

$$(\Delta^{-\alpha}\mathcal{T}_T)(n)-(\Delta^{-\alpha}\mathcal{T}_T)(n+1)=(\Delta^{-(\alpha-1)}\mathcal{T}_T)(n).$$

Accordingly, if $x\in\mathfrak{X}_T^{\alpha}$, for $n\geq 1$

$$\begin{aligned}\left\|(\Delta^{-(\alpha-1)}\mathcal{T}_T)(n)x\right\|_X &= \left\|(\Delta^{-\alpha}\mathcal{T}_T)(n)x-(\Delta^{-\alpha}\mathcal{T}_T)(n-1)x\right\|_X\\ &\leq \left\|(\Delta^{-\alpha}\mathcal{T}_T)(n)x\right\|_X+\left\|(\Delta^{-\alpha}\mathcal{T}_T)(n-1)x\right\|_X\\ &\lesssim \rho(n)+\rho(n-1)\lesssim\rho(n)\end{aligned}$$

and similarly for $n<-1$. In conclusion, $x\in\mathfrak{X}_T^{\alpha-1}$. Hence, using the chain of inclusions

$$\mathfrak{X}_T^{\alpha}\subseteq\mathfrak{X}_T^{\lceil\alpha\rceil}\subseteq\mathfrak{X}_T^{\lceil\alpha\rceil-1}\subseteq\ldots\subseteq\mathfrak{X}_T^{0}\subseteq\mathfrak{X}_T^{\alpha},$$

one finally obtains the desired identity $\mathfrak{X}_T^{\alpha}=\mathfrak{X}_T^{0}$ for each $\alpha\geq 0$. □

The consequences of our latter result are twofold. At first glance, it enables to generalize Theorems 5.2 and 5.3 to the non-natural case, admitting Cesàro sums of any order $\alpha\geq 0$. We state them as corollaries without proof for the sake of completeness.

Corollary 6.2. *Let $T\in\mathcal{B}(X)$ be an invertible operator on a complex Banach space X having $\sigma(T)\subseteq\partial\mathbb{D}$. Suppose that there exist some $\alpha\geq 0$ and a symmetric weight $\rho=(\rho(n))_{n\in\mathbb{Z}}$ such that $\rho_+\in\omega_\alpha$ satisfying*

$$\text{(i).}\quad \left\|(\Delta^{-\alpha}\mathcal{T}_T)(n)\right\|\lesssim\rho(n)\ \text{ as } n\to\pm\infty,\qquad\qquad \text{(ii).}\quad \sum_{n\in\mathbb{Z}}\frac{\log\rho(n)}{1+n^2}<\infty.$$

Then, T is a decomposable operator. In particular, if $\sigma(T)$ is not a singleton, the operator T has a non-trivial hyperinvariant subspace.

Corollary 6.3. *Let $T\in\mathcal{B}(X)$ be an operator on a complex Banach space X with $\sigma_{\mathrm{com}}(T)=\sigma_{\mathrm{com}}(T^*)=\emptyset$. Assume that there exist some $\alpha\geq 0$ and two non-zero vectors $x\in X$ and $y\in X^*$ satisfying*

$$\left\|(\Delta^{-\alpha}\mathcal{T}_T)(n)x\right\|_X\lesssim\rho(n)\quad\text{ and }\quad\left\|(\Delta^{-\alpha}\mathcal{T}_{T^*})(n)y\right\|_{X^*}\lesssim\rho(n)\qquad\text{ as } n\to\pm\infty,$$

for some symmetric weight $\rho=(\rho(n))_{n\in\mathbb{Z}}$ such that $\rho_+\in\omega_\alpha$ and

$$\sum_{n\in\mathbb{Z}}\frac{\log\rho(n)}{1+n^2}<\infty.$$

Then, if $\sigma_T(x)\cup\sigma_{T^*}(y)$ is not a singleton, the operator T has a non-trivial hyperinvariant subspace.

On the other hand, Theorem 6.1 invites to seek more general summability methods for the purpose of achieving an effective improvement of Wermer's/Atzmon's theorems. In view of the limitations of Cesàro summability methods, it could be of interest to focus on summability methods whose general term grows at least subexponentially. Consequently, the aim of our last subsection is to establish the formal rudiments in order to pursue such a general task in this context.

6.1. General summability methods and invariant subspaces

Given a complex sequence $\mu := (\mu_{nj})_{n\geq 0,\, j=0,\ldots,n}$ with non-zero terms, we can consider the summation method applied to the powers of an operator $T \in \mathcal{B}(X)$:

$$(\Delta^\mu \mathcal{T}_T)(n) := \sum_{j=0}^{n} \mu_{nj} T^j.$$

Despite it is not needed, the reader might impose extra conditions on the summation method in order to be a regular summation method. Recall that *regular summation methods* are those which sum every convergent series (or sequence) to the same value as that to which it converges (see, for instance, the classical monograph [30]). For example, one may consider the standard criteria of regularity for *triangular matrix summation methods* (due to Toeplitz [49]):

(i) $\sup_{n\geq 0} \sum_{j=0}^{n} |\mu_{nj}| < +\infty$.

(ii) $\lim_{n\to\infty} \mu_{nj} = 0$ for all $j \geq 0$.

(iii) $\lim_{n\to\infty} \sum_{j=0}^{n} \mu_{nj} = 1$.

Recall that the Cesàro summation methods given by $\mu_{nj} = \frac{k_\alpha(n-j)}{k_{\alpha+1}(n)}$ for each $\alpha \geq 0$ are basic examples of regular triangular matrix summation methods.

We would like to define *finite differences* of the form $D^\mu : c_{0,0}(\mathbb{N}_0) \to c_{0,0}(\mathbb{N}_0)$ such that the mapping $\vartheta_+^\mu : c_{0,0}(\mathbb{N}_0) \to \mathcal{B}(X)$ given by

$$\vartheta_+^\mu(f) := \sum_{n=0}^{\infty} [D^\mu f](n)(\Delta^\mu \mathcal{T}_T)(n)$$

fulfills all the algebraic properties required for an algebra homomorphism. To do so, related to our summation method, we can consider the lower-triangular infinite matrix

$$M_\mu = \begin{pmatrix} \mu_{00} & 0 & 0 & 0 & \cdots \\ \mu_{10} & \mu_{11} & 0 & 0 & \cdots \\ \mu_{20} & \mu_{21} & \mu_{22} & 0 & \cdots \\ \mu_{30} & \mu_{31} & \mu_{32} & \mu_{33} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{6.1}$$

so that the matrix identity $\Delta^\mu \mathcal{T}_T = M_\mu \cdot \mathcal{T}_T$ holds. To determine the suitable finite differences $D^\mu : c_{0,0}(\mathbb{N}_0) \to c_{0,0}(\mathbb{N}_0)$, we must consider its right inverse M_μ^{-1} which is also a lower-triangular infinite matrix

$$M_\mu^{-1} = \begin{pmatrix} \nu_{00} & 0 & 0 & 0 & \cdots \\ \nu_{10} & \nu_{11} & 0 & 0 & \cdots \\ \nu_{20} & \nu_{21} & \nu_{22} & 0 & \cdots \\ \nu_{30} & \nu_{31} & \nu_{32} & \nu_{33} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{6.2}$$

The finite differences associated to $\mu = (\mu_{nj})_{n\geq 0,\, j=0,\ldots,n}$ are given by the linear operator $D^\mu : c_{0,0}(\mathbb{N}_0) \to c_{0,0}(\mathbb{N}_0)$:

$$[D^\mu f](n) := \sum_{j=0}^{\infty} \nu_{jn} f(j),$$

so that the matrix identity $D^\mu f^\top = f^\top \cdot M_\mu^{-1}$ holds (as usual, here $^\top$ indicates the transpose). Observe that $D^\mu : c_{0,0}(\mathbb{N}_0) \to c_{0,0}(\mathbb{N}_0)$ is injective. In our next result, we check that this definition of finite differences works for our purposes.

Proposition 6.4. *Let $\mu = (\mu_{nj})_{n\geq 0,\, j=0,\ldots,n}$ be a non-vanishing complex sequence. Consider the matrices $M_\mu := (\mu_{nj})_{n,j=0}^{\infty}$ and $M_\mu^{-1} := (\nu_{jn})_{n,j=0}^{\infty}$ given in (6.1) and (6.2), and the finite differences $D^\mu : c_{0,0}(\mathbb{N}_0) \to c_{0,0}(\mathbb{N}_0)$ defined as*

$$[D^\mu f](n) := \sum_{j=0}^{\infty} \nu_{jn} f(j).$$

Then, the mapping $\vartheta_+^\mu : c_{0,0}(\mathbb{N}_0) \to \mathcal{B}(X)$ given by

$$\vartheta_+^\mu(f) := \sum_{n=0}^{\infty} [D^\mu f](n)(\Delta^\mu \mathcal{T}_T)(n)$$

satisfies the algebraic properties $\vartheta_+^\mu(\beta f + \gamma g) = \beta \vartheta_+^\mu(f) + \gamma \vartheta_+^\mu(g)$ and $\vartheta_+^\mu(f * g) = \vartheta_+^\mu(f)\vartheta_+^\mu(g)$ for all $\beta, \gamma \in \mathbb{C}$ and $f, g \in c_{0,0}(\mathbb{N}_0)$.

Proof. Observe that

$$[D^\mu \delta_m](n) = \sum_{j=0}^{\infty} \nu_{jn} \delta_m(j) = \nu_{mn}.$$

We would like to have $\vartheta_+^\mu(\delta_m) = T^m$, since, in that case, $\delta_m * \delta_n = \delta_{m+n}$ would entail the multiplication property for the mapping ϑ_+^μ as in the proof of Theorem 4.1 (note that the addition for ϑ_+^μ holds automatically because of the linearity of D^μ). Accordingly,

$$\vartheta_+^\mu(\delta_m) = \sum_{n=0}^{\infty} [D^\mu \delta_m](n)(\Delta^\mu \mathcal{T}_T)(n) = \sum_{n=0}^{\infty}\sum_{j=0}^{n} \mu_{nj}\nu_{mn} T^j = \sum_{j=0}^{\infty}\sum_{n=j}^{\infty} \mu_{nj}\nu_{mn} T^j = \sum_{j=0}^{\infty} T^j \left(\sum_{n=0}^{\infty} \mu_{nj}\nu_{mn} \right)$$

and so we need that

$$\sum_{n=0}^{\infty} \mu_{nj}\nu_{mn} = \begin{cases} 1, & \text{if } j = m, \\ 0, & \text{if } j \neq m, \end{cases}$$

which clearly holds for the inverse $M_\mu^{-1} = (\nu_{jn})_{j,n=0}^{\infty}$. □

Inspired by [4], we are interested in constructing weighted convolution Banach algebras $\tau_+^\mu(\rho)$ related to our summation method provided by the sequence $\mu = (\mu_{nj})_{n\geq 0,\, j=0,\ldots,n}$. To do so, once again we need to coin a class of weights ω_μ which behaves adequately with respect to our summation method, what at the final stage precisely means that the quantity

$$\|f\|_{\rho,(\mu)} := \sum_{n=0}^{\infty} \big|[D^\mu f](n)\big| \rho(n), \quad f \in c_{0,0}(\mathbb{N}_0)$$

defines a normed algebra over $c_{0,0}(\mathbb{N}_0)$ with respect to the convolution product and the entrywise addition, and whose completion leads to a weighted convolution Banach algebra $\tau_+^\mu(\rho)$ which can be embedded into $\ell^1(\mathbb{N}_0)$.

Definition 6.5. Let $\mu = (\mu_{nj})_{n\geq 0,\, j=0,\dots,n}$ be a complex sequence with non-zero terms and $D^\mu : c_{0,0}(\mathbb{N}_0) \to c_{0,0}(\mathbb{N}_0)$ its finite differences. Then, a weight $\rho = (\rho(n))_{n\geq 0}$ belongs to the class ω_μ whenever

$$\tau_+^\mu(\rho) := \left\{ f \in \ell^1(\mathbb{N}_0) : \|f\|_{\rho,(\mu)} := \sum_{n=0}^{\infty} \big|[D^\mu f](n)\big| \rho(n) < +\infty \right\},$$

is a unital Banach algebra.

With this definition at hand, in the seeking of a regular weighted convolution Banach algebra $\tau^\mu(\rho)$ associated to our summation method, we can transfer this construction to the bilateral context in the same vein as done for the Cesàro summation method. Thus, the fractional differences $D^\mu : c_{0,0}(\mathbb{Z}) \to c_{0,0}(\mathbb{Z})$ are defined in the bilateral context as

$$[D^\mu f](n) := \begin{cases} [D^\mu f_+](n), & \text{for } n \geq 0, \\ [D^\mu f_-](-n), & \text{for } n < 0, \end{cases}$$

while $\Delta^\mu \mathcal{T}_T$ for the powers of the operator T^n with $n \in \mathbb{Z}$ is

$$(\Delta^\mu \mathcal{T}_T)(n) := \begin{cases} (\Delta^\mu \mathcal{T}_{T+})(n), & \text{if } n \geq 0, \\ (\Delta^\mu \mathcal{T}_{T-})(-n), & \text{if } n < 0. \end{cases}$$

Definition 6.6. Let $\mu = (\mu_{nj})_{n\geq 0,\, j=0,\dots,n}$ be a non-vanishing complex sequence and let $D^\mu : c_{0,0}(\mathbb{Z}) \to c_{0,0}(\mathbb{Z})$ be its finite differences. Then, for each symmetric sequence $\rho = (\rho(n))_{n\in\mathbb{Z}}$ with $\rho_+ \in \omega_\mu$, we define

$$\tau^\mu(\rho) := \left\{ f \in \ell^1(\mathbb{Z}) : \|f\|_{\rho,(\mu)} := \sum_{n=-\infty}^{\infty} \big|[D^\mu f](n)\big| \rho(n) < +\infty \right\}.$$

Correspondingly, the *fractional weighted Wiener algebra* $\mathcal{A}_\rho^\mu$ consists of the space of functions

$$\mathcal{A}_\rho^\mu := \left\{ f \in \mathcal{A}(\partial\mathbb{D}) \ : \ \|f\|_{\rho,(\mu)} := \sum_{n=-\infty}^{\infty} \big|[D^\mu \widehat{f}\,](n)\big| \rho(n) < +\infty \right\}$$

endowed with the usual operations of pointwise addition and multiplication.

In light of Proposition 6.4 and following the same steps as in the proof of Theorem 4.1, one can establish an algebra homomorphism from the fractional weighted Wiener algebra $\mathcal{A}_\rho^\mu$ (or, equivalently, from $\tau^\mu(\rho)$) for those operators whose $(\Delta^\mu \mathcal{T}_T(n))_{n\in\mathbb{Z}}$ are dominated in norm by a sequence lying in the class ω_μ. We omit further details for the sake of brevity.

Theorem 6.7. *Let $T \in \mathcal{B}(X)$ be an invertible operator on a complex Banach space X. Given a non-vanishing complex sequence $\mu = (\mu_{nj})_{n\geq 0,\, j=0,\dots,n}$, consider $D^\mu : c_{0,0}(\mathbb{Z}) \to c_{0,0}(\mathbb{Z})$ its finite differences. Suppose that there exists a symmetric weight $\rho = (\rho(n))_{n\in\mathbb{Z}}$ such that $\rho_+ \in \omega_\mu$ and satisfying*

$$\big\|(\Delta^\mu \mathcal{T}_T)(n)\big\| \lesssim \rho(n), \quad \text{as } n \to \pm\infty.$$

Then, the mapping $\vartheta^\mu : \mathcal{A}_\rho^\mu \to \mathcal{B}(X)$ *defined by*

$$\vartheta^\mu(f) := \sum_{n=-\infty}^{\infty} [D^\mu \widehat{f}](n)(\Delta^\mu \mathcal{T}_T)(n), \quad \text{for each } f \in \mathcal{A}_\rho^\mu,$$

is a bounded unital algebra homomorphism.

As in the Cesàro summation case, this construction, along with the spectral techniques developed by Neumann and Gallardo-Gutiérrez et al. [23,41], enables to obtain results regarding the existence of hyperinvariant subspaces using the functional calculus developed in Theorem 6.7 whenever the fractional weighted Wiener algebra $\mathcal{A}_\rho^\mu$ is regular.

Corollary 6.8. *Let* $T \in \mathcal{B}(X)$ *be an invertible operator on a complex Banach space* X *and* $\mu = (\mu_{nj})_{n\geq 0,\, j=0,\dots,n}$ *a non-vanishing complex sequence. Assume that there exists a symmetric weight* $\rho = (\rho(n))_{n\in\mathbb{Z}}$ *such that* $\rho_+ \in \omega_\mu$ *and*

- (i) *The fractional weighted Wiener algebra* $\mathcal{A}_\rho^\mu$ *is regular.*
- (ii) $\left\|(\Delta^\mu \mathcal{T}_T)(n)\right\| \lesssim \rho(n)$ *as* $n \to \pm\infty$.

Then, T *is a decomposable operator. In particular, if* $\sigma(T)$ *is not a singleton, the operator* T *has a non-trivial hyperinvariant subspace.*

Corollary 6.9. *Let* $T \in \mathcal{B}(X)$ *be an operator on a complex Banach space* X *with* $\sigma_{\text{com}}(T) = \sigma_{\text{com}}(T^*) = \emptyset$ *and* $\mu = (\mu_{nj})_{n\geq 0,\, j=0,\dots,n}$ *a non-vanishing complex sequence. Assume that there exists a symmetric weight* $\rho = (\rho(n))_{n\in\mathbb{Z}}$ *such that* $\rho_+ \in \omega_\mu$ *and*

- (i) *The fractional weighted Wiener algebra* $\mathcal{A}_\rho^\mu$ *is regular.*
- (ii) *There exist two non-zero vectors* $x \in X$ *and* $y \in X^*$ *satisfying*

$$\left\|(\Delta^\mu \mathcal{T}_T)(n)x\right\|_X \lesssim \rho(n) \quad \text{and} \quad \left\|(\Delta^\mu \mathcal{T}_{T^*})(n)y\right\|_{X^*} \lesssim \rho(n) \qquad \text{as } n \to \pm\infty.$$

Then, if $\sigma_T(x) \cup \sigma_{T^*}(y)$ *is not a singleton, the operator* T *has a non-trivial hyperinvariant subspace.*

Of course, the underlying big question which remains open is if it could be possible to state an extension of Beurling's regularity criterion to the general context of fractional weighted Wiener algebras of the form $\mathcal{A}_\rho^\mu$. Besides, it will certainly be of interest to know which of those regular fractional weighted Wiener algebras $\mathcal{A}_\rho^\mu$ supplies an effective extension of Wermer's/Atzmon's theorems. We state both questions below to highlight them.

Question 1. Let $\mu = (\mu_{nj})_{n\geq 0,\, j=0,\dots,n}$ be a non-vanishing complex sequence and let $\rho = (\rho(n))_{n\in\mathbb{Z}}$ be a symmetric weight with $\rho_+ \in \omega_\mu$. Then:

- Which fractional weighted Wiener algebras $\mathcal{A}_\rho^\mu$ are regular?
- Does there exist a general Beurling's regularity criterion for weighted Wiener algebras of the form $\mathcal{A}_\rho^\mu$?

Question 2. For which regular fractional weighted Wiener algebras $\mathcal{A}_\rho^\mu$, the corresponding Corollary 6.9 (or Corollary 6.8 in the invertible case) ensures the existence of invariant subspaces for a wider (or alternative) class of operators than the one covered by Atzmon's theorem?

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