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Convergence rates in mean ergodic theorems for Cesàro bounded operators

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Abstract. We establish the (polynomially) logarithmic decay of ergodic means of Cesàro bounded operators of any fractional order, under convergence of the one-sided ergodic Hilbert transform. This extends the theorem of Gomilko, Haase and Tomilov for power bounded operators. We also improve the polynomial decay of means involved in the fractional Poisson equation. The theorems are obtained as an application of a general result, also proved here, about rates of decay of means for Cesàro bounded operators.

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1. Introduction

Let X be a Banach space and $\mathcal{B}(X)$ be the usual unital Banach algebra of bounded operators on X, with identity I. For an operator T in $\mathcal{B}(X)$ and $n \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$, let $M_n(T)$ denote the Cesàro mean operator associated with T and given by

$$M_n(T)x := \frac{1}{n+1}\sum_{j=0}^{n} T^j x, \quad x \in X. \tag{1.1}$$

Assume that T is a power-bounded operator, that is, $\sup_n \|T^n\| < \infty$. Then there exists $P_1 x := \lim_n M_n(T)x$ (in the norm of X) if and only if $x \in \mathrm{Ker}(\mathrm{I}-T) \oplus \overline{(\mathrm{I}-T)X}$ (in fact, $\mathrm{Ker}(\mathrm{I}-T) = \{x \in X : P_1 x = x\}$ and $\overline{(\mathrm{I}-T)X} = \{x \in X : P_1 x = 0\}$). So the operator is mean ergodic, that is, $M_n(T)x$ converges for all $x \in X$ iff $X = \mathrm{Ker}(\mathrm{I}-T) \oplus \overline{(\mathrm{I}-T)X}$, and one has a so-called mean ergodic theorem. In this case $P_1 \colon X \to \mathrm{Ker}(\mathrm{I}-T)$ is

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a bounded projection along $\overline{(\mathrm{I}-T)X}$. As an example, every power-bounded operator on a reflexive space is mean ergodic [13, Chapter 2, Theorem 1.2].

When $\lim_n M_n(T)x$ exists the study of its convergence rate is important for applications in probability theory and in itself as part of the ergodic theory and operator theory. It is known that there is no universal or uniform convergence in (1.1), so it is relevant to find classes of vectors $x \in X$ for which the convergence decay in the limit $\lim_n M_n(T)x$ is suitable to handle (polynomial, logarithmic, ...). Such a matter is related with the characterization of domains of operator functions like for instance $(\mathrm{I}-T)^{-s}$, $0 < s \le 1$, arising in the fractional Poisson equation $(\mathrm{I}-T)^s y = x$, or $\log(\mathrm{I}-T)$, linked to the one-sided discrete Hilbert transform $H_T(x) := \sum_{n=1}^{\infty} n^{-1} T^n x$. These characterizations, and so the one of the required vectors, come expressed in terms of series convergence; see the introductory sections of [10,11] for more details about the above facts.

In this order of ideas, it can be shown, for T such that $\sup_n \|T^n\| < \infty$, that

$$\sum_{n=1}^{\infty} \frac{T^n x}{n^{1-s}} \text{ weakly convergent } \Longrightarrow \|M_n(T)x\| = \mathrm{o}(n^{-s}) \text{ as } n \to \infty, \quad (1.2)$$

and that

$$\sum_{n=1}^{\infty} \frac{T^n x}{n} \text{ weakly convergent } \Longrightarrow \|M_n(T)x\| = \mathrm{o}\left(\frac{1}{\log n}\right) \text{ as } n \to \infty, \quad (1.3)$$

see [8, Cor. 2.15] and [10, p. 212] for (1.2), and [10, Th. 4.1] for (1.3).

We pay special attention to the latter result. This was established in [7] for normal contractions T on a Hilbert space and remained an open problem for every power-bounded operator T on a Banach space until it was eventually proven in this general case, in [10, Th. 4.1 a)] (moreover, it is shown in [10, Th. 4.1 b)] that the logarithmic decay of $M_n(T)x$ is the best possible under natural spectral assumptions). Our main goal in the present paper is to largely extend (1.3), in a precise sense that we are going to explain right now. As a consequence of the approach carried out here to produce such an extension we also extend (1.2).

Power-boundedness is a fairly usual, suitable condition or assumption on operators in a number of situations where one naturally deals with iterates of operators. However, there are operators of interest in those settings which do not fulfill such a boundedness property. Thus for instance one has the class of operators T which, not being power-bounded, are (C,α)-bounded for some $\alpha > 0$. Let us see.

For $\alpha \ge 0$, let $(k_\alpha(n))_{n=0}^{\infty}$ denote the sequence of Cesàro numbers of order α, that is, the sequence of coefficients in the power series representation of the function $(1-z)^{-\alpha}$, $z \in \mathbb{D}$. For $T \in \mathcal{B}(X)$ and $n \in \mathbb{N}_0$, let $M_n^\alpha(T)$ be the Cesàro mean of order α defined by

$$M_n^\alpha(T) := \frac{1}{k_{\alpha+1}(n)} \sum_{j=0}^{n} k_\alpha(n-j) T^j.$$

(note that $M_n^1(T) = M_n(T)$). Put $K_\alpha(T) := \sup_{n\in\mathbb{N}_0} \|M_n^\alpha(T)\|$. The operator T is said to be Cesàro bounded of (fractional) order α, or (C,α)-bounded for short, if $K_\alpha(T) < \infty$. Thus T is power-bounded iff it is $(C,0)$-bounded. Similarly, T is called Cesàro ergodic of order α, or (C,α)-ergodic for short, if there exists the limit $\lim_n M_n^\alpha(T)x$ for all $x \in X$, and so mean ergodic and Cesàro ergodic of order 1 mean the same thing.

The study of Cesàro boundedness and Cesàro ergodicity of fractional order dates back to [12] at least, where these properties are considered in connection with the Volterra operator. Actually, growth properties and ergodicity of Cesàro means of fractional order have been extensively studied over the years; see [3], introduction and references therein. Quite recently, these items have been treated regarding operator inequalities as well as functional models of Nagy-Foias type, for the shift operator on weighted Bergman spaces [1]; see also [2,6] for more examples.

It is natural to pose the question about whether or not (1.3) holds, in its own adequate formulation, in the setting of Cesàro means of fractional order. This is interesting in order to analyse in depth and to further reveal what is behind statements like (1.2) and (1.3). Here we present the following result, which corresponds to Theorem 4.2 and Theorem 4.4, see Section 4.

For $T \in \mathcal{B}(X)$, $\alpha \geq 0$ and $n \in \mathbb{N}_0$ set

$$\Delta^{-\alpha}\mathcal{T}(n) := k_{\alpha+1}(n) M_n^\alpha(T).$$

Theorem 1.1. *Let T be a (C,α)-bounded operator on X.*

(i) *Let $x \in X$ be such that the series $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha+1}} \Delta^{-\alpha}\mathcal{T}(n)x$ is weakly convergent in X. Then*

$$\|M_n^\beta(T)x\| = \mathrm{o}\left(\frac{1}{(\log n)^{\beta-\alpha}}\right) \text{ if } \alpha < \beta \leq \alpha+1, \text{ and}$$

$$\|M_n^\beta(T)x\| = \mathrm{o}\left(\frac{1}{\log n}\right) \text{ if } \beta > \alpha+1,$$

as $n \to \infty$.

(ii) *Suppose that, in addition, $0 \leq \alpha < 1$. Let x be such that $\sum_{n=1}^{\infty} \frac{T^n x}{n}$ is weakly convergent in X. Then, as $n \to \infty$,*

$$\|M_n^\beta(T)x\| = \mathrm{o}\left(\frac{1}{(\log n)^{\beta-\alpha}}\right) \text{ if } \alpha < \beta \leq \alpha+1, \text{ and}$$

$$\|M_n^\beta(T)x\| = \mathrm{o}\left(\frac{1}{\log n}\right) \text{ if } \beta > \alpha+1.$$

It is to be emphasized that the assumpion $0 \leq \alpha < 1$ in part (ii) of the theorem allows us to replace the series convergence hypothesis involving the term $\Delta^{-\alpha}\mathcal{T}(n)$ by the convergence of the one-sided discrete Hilbert transform. So we get two consequences that seem to deserve particular attention. One concerns means of order 1 (take $\beta = 1$ in (ii) of the theorem), the other one is about means of *any* order $\beta \in (0,1]$ for power-bounded operators (take $\alpha = 0$ in (ii) above):

Corollary 1.2. *Let $T \in \mathcal{B}(X)$ and $x \in X$.*

(i) *Let $0 \le \alpha < 1$. If T is a (C,α)-bounded operator and the series $\displaystyle\sum_{n=1}^{\infty} \frac{T^n x}{n}$ is weakly convergent in X then $\|M_n(T)x\| = \mathrm{o}\left(\frac{1}{(\log n)^{1-\alpha}}\right)$ as $n \to \infty$.*

(ii) *Let $0 < \beta \le 1$. If $\sup\{\|T^n\| : n \in \mathbb{N}\} < \infty$ and the series $\displaystyle\sum_{n=1}^{\infty} \frac{T^n x}{n}$ is weakly convergent in X then $\|M_n^{\beta}(T)x\| = \mathrm{o}\left(\frac{1}{(\log n)^{\beta}}\right)$ as $n \to \infty$.*

Thus these results are substantial extensions of (1.3). Here we also give extensions of (1.2) which moreover improve previous generalizations of (1.2) given in [3]; see Section 4 below.

The method of proof of the above theorems relies on the strategy carried out in [10]. However, it does not mean that this procedure merely consists of mimicking arguments involving just obvious generalizations of results in [10]. On the contrary, overcoming the difficulties concerning technicalities and theoretical points has demanded certainly nontrivial adaptations to our context. We have organized the paper accordingly.

Thus, in order to explain in proper terms the notions involved in the problems considered in this paper, and to make it as self-contained as possible, we devote Section 2 to introduce Cesàro numbers, fractional differences, admissible analytic functions, as well as functional calculus associated to differences and Cesàro bounded operators. Some primary results on means of fractional order are also included in the section.

Our theorems on logarithmic or polynomial decay of convergence processes are obtained as an application to particular functions of an abstract theorem given for admissible functions subject to certain general conditions. Section 3 contains a proof of such a key theorem. The proof is rather long –so it is divided into two other theorems–, even though we have chosen to avoid long computations in that section, which have been gathered in Section 5. Then Section 4 contains our theorems on logarithmic decay and polynomial decay referred to formerly. As an application, some remarks are made here about the Volterra operator, and the shift operator on weighted Bergman spaces.

2. Setting, tools, and first properties

2.1. Cesàro numbers and fractional differences

2.1.1. Cesàro numbers. For $\alpha \in \mathbb{C}$, let $(k_\alpha(n))_{n=0}^{\infty}$ denote the sequence of Taylor coefficients of the generating function $(1-z)^{-\alpha}$; that is,

$$\sum_{n=0}^{\infty} k_\alpha(n) z^n = \frac{1}{(1-z)^{\alpha}}, \quad |z| < 1. \tag{2.1}$$

The elements of sequences $(k_\alpha(n))_{n=0}^\infty$ are called Cesàro numbers, and are given by $k_\alpha(0)=1$ and

$$k_\alpha(n) := \binom{n+\alpha-1}{\alpha-1} = \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!}, \ n\in\mathbb{N}; \qquad (2.2)$$

see [17, Vol. I, p.77], where $k_\alpha(n)$ is denoted by $A_n^{\alpha-1}$. For $\alpha\in\mathbb{C}\setminus\{0,-1,-2,\dots\}$ one has $k_\alpha(n)=\Gamma(n+\alpha)/(\Gamma(\alpha)\Gamma(n+1))$. For $\alpha\in\mathbb{R}$, $k_\alpha(n)$ as a function of n is increasing for $\alpha>1$, decreasing for $0<\alpha<1$, and $k_1(n)=1$ for all $n\in\mathbb{N}_0$ ([17, Th. III.1.17]). Furthermore, $0\le k_\alpha(n)\le k_\beta(n)$ for $\beta\ge\alpha>0$ and $n\in\mathbb{N}_0$. For $m\in\mathbb{N}_0$,

$$k_{-m}(n) = \begin{cases} (-1)^n\binom{m}{n}, & 0\le n\le m; \\ 0, & n\ge m+1, \end{cases} \qquad (2.3)$$

and, if $m<\alpha<m+1$,

$$\operatorname{sign} k_{-\alpha}(n) = \begin{cases} (-1)^n, & 0\le n\le m; \\ (-1)^{m+1}, & n\ge m+1. \end{cases} \qquad (2.4)$$

As regards the asymptotic behaviour of the sequence $(k_\alpha(n))_{n=0}^\infty$ we have

$$k_\alpha(n) = \frac{n^{\alpha-1}}{\Gamma(\alpha)}(1+\mathrm{O}(1/n)), \quad \text{as } n\to\infty, \quad \text{for every } \alpha\in\mathbb{R}\backslash\{0,-1,-2,\dots\}; \qquad (2.5)$$

see [17, Vol.I, p.77 (1.18)] or [9, Eq.(1)].

Recall that for two sequences $a=(a(n))_{n=0}^\infty$ and $b=(b(n))_{n=0}^\infty$ the convolution is given by

$$(a*b)(n) = \sum_{j=0}^{n} a(n-j)b(j), \ n\ge 0.$$

Then it follows from (2.1) that the sequence $(k_\alpha(n))_{n=0}^\infty$ satisfies the group property, $k_\alpha * k_\beta = k_{\alpha+\beta}$ for every $\alpha,\beta\in\mathbb{C}$. Note that k_0 is the Dirac mass δ_0 on $\mathbb{N}_0$ and so it is the unit for the convolution. In particular we have $k_{\alpha+1}=k_1 * k_\alpha$, which is to say

$$k_{\alpha+1}(n) = \sum_{j=0}^{n} k_\alpha(j), \quad n\ge 0. \qquad (2.6)$$

Formula (2.6) will be used several times through the paper .

2.1.2. Fractional differences. For a sequence $u=(u(n))_{n=0}^\infty$, one defines

$$[W^m u](n) := \sum_{j=0}^{m}(-1)^j\binom{m}{j}u(n+j) = \sum_{j=n}^{\infty} k_{-m}(j-n)u(j), \quad n\in\mathbb{N}_0,\, m\in\mathbb{N}.$$

Differences W^m can be extended to the fractional case as follows; see [3].

Definition 2.1. For given $\alpha > 0$, the *Weyl sum $W^{-\alpha}u$ of order α* of u is defined by

$$[W^{-\alpha}u](n) := \sum_{j=n}^{\infty} k_\alpha(j-n)u(j), \quad n \in \mathbb{N}_0, \tag{2.7}$$

whenever the series converges. The *Weyl difference $W^{\alpha}u$ of order α* of u is defined by

$$[W^{\alpha}u](n) = [W^m[W^{-(m-\alpha)}u]](n), \quad n \in \mathbb{N}_0,$$

for $m = [\alpha] + 1$, with $[\alpha]$ the integer part of α, whenever the corresponding series converges.

Another very useful extension of W^m is the operator D^α given by

$$[D^{\alpha}u](n) := \sum_{j=n}^{\infty} k_{-\alpha}(j-n)u(j), \quad n \in \mathbb{N}_0, \tag{2.8}$$

whenever the series converges. Operators W^α and D^α are different, in general. However, they coincide on ℓ^1-spaces with appropriate weights.

Let $\ell^1(\omega)$ denote the space of absolutely summable sequences on $\mathbb{N}_0$ with respect to a weight $\omega\colon \mathbb{N}_0 \to \mathbb{C}$, endowed with its usual norm $\|\cdot\|_{\ell^1(\omega)}$, that is, $\|u\|_{\ell^1(\omega)} = \sum_{n=0}^{\infty} |u(n)\omega(n)|$. In the folllowing proposition (which is [3, Proposition 2.3]) we consider weights $\omega = k_\mu$, $\mu \in \mathbb{R}$.

Proposition 2.2. *Let $\alpha > 0$ and $m := [\alpha] + 1$. Then:*

(i) *The Weyl sum $W^{-\alpha}$ defines a bounded linear operator $W^{-\alpha}\colon \ell^1(k_\alpha) \to \ell^1(k_{-\alpha})$. Thus W^α is well defined on $\ell^1(k_{m-\alpha})$. Moreover $W^{-\alpha}$ defines a bounded linear operator*

$$W^{-\alpha}\colon \ell^1(k_{\alpha+\beta}) \to \ell^1(k_\beta) \text{ for every } \beta > 0.$$

(ii) *The operator D^α is well defined on the space $\ell^1(k_{-\alpha})$. Moreover, $D^\alpha(\ell^1(k_\alpha)) \subset \ell^1(k_\alpha)$ and $D^\alpha\,|_{\ell^1(k_\alpha)}\colon \ell^1(k_\alpha) \to \ell^1(k_\alpha)$ is bounded.*

(iii) *Operators W^α and D^α coincide on $\ell^1(k_{m-\alpha})$.*

(iv) *For $u \in \ell^1(k_\alpha)$, we have $D^\alpha(W^{-\alpha}u) = u$ and $W^{-\alpha}(D^\alpha u) = u$.*

Example 2.3. For the above and the following examples we refer the reader to [3, Subsection 2.2].

(i) Let $s \in \mathbb{R}$ and $m \in \mathbb{N}_0$. Then

$$[D^m k_s](n) = [W^m k_s](n) = (-1)^m k_{s-m}(n+m), \quad n \in \mathbb{N}_0;$$

see [5, Ex. 3.4]. Also, if $\alpha > 0$ and $s \in (0, 1)$, then by [5, Lemma 1.1] one gets

$$[D^\alpha k_s](n) = \frac{\mathrm{B}(1-s+\alpha, s+n)}{\Gamma(s)\Gamma(1-s)} = \frac{\sin(\pi s)}{\pi}\mathrm{B}(1-s+\alpha, s+n), \quad n \in \mathbb{N}_0, \tag{2.9}$$

where B is the Beta function. Therefore, by [9, Eq.(1)],

$$[D^\alpha k_s](n) = \frac{\Gamma(1-s+\alpha)}{\Gamma(s)\Gamma(1-s)} n^{s-\alpha-1}(1 + \mathrm{O}(1/n)) \quad \text{as } n \to \infty. \tag{2.10}$$

(ii) Let $m \in \mathbb{N}_0$ and let L be the sequence defined by $L(n) = \dfrac{1}{n+1}$ for $n \in \mathbb{N}_0$. Then, for $\alpha > 0$, by [5, Lemma 1.1] we have

$$[D^\alpha L](n) = \frac{\Gamma(\alpha+1)n!}{\Gamma(n+\alpha+2)}, \quad n \in \mathbb{N}_0, \tag{2.11}$$

and by [9, Eq.(1)],

$$[D^\alpha L](n) = \frac{\Gamma(\alpha+1)}{n^{\alpha+1}}(1 + \mathrm{O}(1/n)) \quad \text{as } n \to \infty. \tag{2.12}$$

On the basis of Cesàro numbers and fractional differences we pass to introduce convenient sets of analytic functions in the unit disc.

2.2. Wiener type algebras of fractional order and associated admissible functions.

2.2.1. Algebras of Wiener type of fractional order. For all this section we refer the reader to [3, Section 2, Section 4 and Section 5]. For $\alpha > 0$, put

$$\tau^\alpha(\mathbb{N}_0) := W^{-\alpha}(\ell^1(k_{\alpha+1})).$$

It follows by Proposition 2.2 (v) that $W^{-\alpha}$ is injective on $\ell^1(k_\alpha)$ and so the linear operator

$$W^{-\alpha} \colon \ell^1(k_{\alpha+1}) \to \tau^\alpha(\mathbb{N}_0)$$

is bijective. Also, $\tau^\alpha(\mathbb{N}_0) \subset \ell^1 := \ell^1(k_1)$ by (i), second half, of that proposition. It follows readily by parts (iv) and (v) that $W^\alpha = D^\alpha$ on $\tau^\alpha(\mathbb{N}_0)$ and therefore the mappings

$$W^{-\alpha} \colon \ell^1(k_{\alpha+1}) \to \tau^\alpha(\mathbb{N}_0) \text{ and } W^\alpha \colon \tau^\alpha(\mathbb{N}_0) \to \ell^1(k_{\alpha+1})$$

are inverse one of the other. We endow the space $\tau^\alpha(\mathbb{N}_0)$ with the norm given by the convergent series

$$\|u\|_{1,(\alpha)} := \sum_{n=0}^{\infty} |[W^\alpha u](n)| k_{\alpha+1}(n), \quad u \in \tau^\alpha(\mathbb{N}_0),$$

obtained by transference of the norm $\|\cdot\|_{\ell^1(k_{\alpha+1})}$ in $\ell^1(k_{\alpha+1})$ through $W^{-\alpha}$ (for $\alpha = 0$, the notation $\|u\|_{1,(0)}$ corresponds to the usual norm in ℓ^1). Hence, the space $\tau^\alpha(\mathbb{N}_0)$ can be described as the space of sequences $u = (u(n))$ in ℓ^1 such that $\|u\|_{1,(\alpha)} < \infty$. Since $W^{-\alpha}$ takes the space c_{00} of eventually null sequences onto itself, one has that c_{00} is dense in $\tau^\alpha(\mathbb{N}_0)$. We also have by Proposition 2.2 (i) that

$$\tau^\beta(\mathbb{N}_0) \hookrightarrow \tau^\alpha(\mathbb{N}_0) \hookrightarrow \ell^1, \quad \text{for } \beta > \alpha > 0, \tag{2.13}$$

since $W^{-\beta} = W^{-\alpha}W^{-(\beta-\alpha)}$ on $\ell^1(k_{\beta+1})$. Notice finally that by (2.10) we have $(k_\beta(n)) \in \tau^\alpha(\mathbb{N}_0)$ if $\beta \in \mathbb{C}$ with $\mathfrak{Re}\,\beta < 0$ or $\beta = 0$, for all $\alpha \geq 0$.

The Banach space $\tau^\alpha(\mathbb{N}_0)$ is actually a Banach algebra for the convolution product (here we understand a Banach algebra by a complete normed algebra with submultiplivative norm). This is a consequence of the following formula, which is established, in more generality, in [4, Lemma 2.7]: for φ and ϕ in $\tau^\alpha(\mathbb{N}_0)$ and $v \in \mathbb{N}_0$,

$$[W^\alpha(\varphi * \phi)](v) := \left(\sum_{j=0}^{v} \sum_{l=v-j}^{v} - \sum_{j=v+1}^{\infty} \sum_{l=v+1}^{\infty} \right) k_\alpha(l+j-v)[D^\alpha\varphi](j)[W^\alpha\phi](l). \tag{2.14}$$

For $u = (u(n))_{n=0}^\infty \in \tau^\alpha(\mathbb{N}_0)$, let g denote for a moment the holomorphic function on the unit disc $\mathbb{D}$ (and continuous on $\overline{\mathbb{D}}$) given by $g(z) := \sum_{n=0}^\infty u(n)z^n$. We define the Wiener algebra of order α by

$$A^\alpha(\mathbb{D}) := \{g \,:\, u \in \tau^\alpha(\mathbb{N}_0)\},$$

endowed with pointwise multiplication and the norm $\|g\|_{A^\alpha(\mathbb{D})} := \|u\|_{1,(\alpha)}$. Thus $A^\alpha(\mathbb{D})$ and $\tau^\alpha(\mathbb{N}_0)$ are Banach algebras isometrically isomorphic Henceforth, we denote the elements of $\tau^\alpha(\mathbb{N}_0)$ by $\widehat{g}$, with $g \in A^\alpha(\mathbb{D})$.

For $\alpha \geq 0$ and $\mathcal{Z} = (z^n)_{n\in\mathbb{N}_0}$ set

$$[\Delta^{-\alpha}\mathcal{Z}](n) := \sum_{j=0}^{n} k_\alpha(n-j)z^j, \quad z \in \overline{\mathbb{D}}, n \in \mathbb{N}_0.$$

Note that $[\Delta^0\mathcal{Z}](n) = z^n$ and, for $\alpha \geq 0$, $|[\Delta^{-\alpha}\mathcal{Z}](n)| \leq k_{\alpha+1}(n)$ for all $z \in \overline{\mathbb{D}}$ and $n \in \mathbb{N}_0$.

Take $g(z) = \sum_{n=0}^\infty \widehat{g}(n)z^n$ in $A^\alpha(\mathbb{D})$. Using Fubini's theorem (for series) in the standard way, one obtains

$$g(z) = \sum_{n=0}^{\infty} [W^\alpha\widehat{g}](n)[\Delta^{-\alpha}\mathcal{Z}](n), \quad z \in \overline{\mathbb{D}}, \tag{2.15}$$

where the series converges absolutely in $\overline{\mathbb{D}}$ (see [3, Eq. (4.3)]). Evaluating at $z = 1$ one gets

$$g(1) = \sum_{n=0}^{\infty} [W^\alpha\widehat{g}](n)k_{\alpha+1}(n),$$

which is an expresion to be used several times in the sequel, in particular when $g(1) = 0$.

2.2.2. Admissible analytic functions. For $\alpha > 0$, α-admissible functions were introduced in [3] as a generalization of the notion of admissible analytic function given in [11].

Let $f(z) = \sum_{n=0}^\infty \widehat{f}(n)z^n$ be a holomorphic function on $\mathbb{D}$ with $\widehat{f}(n) \geq 0$, $n \geq 0$. We will write $f(1)$ to denote the limit $f(1) := \lim_{0<z\nearrow 1} f(z)$ in $[0, +\infty) \cup \{+\infty\}$. For $\alpha \geq 0$, f is called α-admissible if $D^\alpha\widehat{f}$ exists and:

- (i) The sequence $(\widehat{f}(n))$ is bounded, $f(n) \geq 0$ and $[D^\alpha\widehat{f}](n) \geq 0$ for every $n \in \mathbb{N}_0$, and $W^{-\alpha}(D^\alpha\widehat{f}) = \widehat{f}$.
- (ii) The function f does not have zeros in $\mathbb{D}$ and, if $g(z) := 1/f(z) = \sum_{n=0}^\infty \widehat{g}(n)z^n$, $z \in \mathbb{D}$, then $W^\alpha\widehat{g}$ exists with $\widehat{g}(n) \leq 0$, $[W^\alpha\widehat{g}](n) \leq 0$ for $n \geq 1$, and $\widehat{g}(0) \geq 0$, $[W^\alpha\widehat{g}](0) \geq 0$.

Under the above conditions one gets $(\widehat{g}(n)) \in \tau^\alpha(\mathbb{N}_0)$, which is to say $g \in A^\alpha(\mathbb{D})$.

We next provide several examples of α-admissible functions which are crucial in the paper. Such examples have been already given in [3] but the explanation there contains several misprints and obscurities which we prefer to clarify now for convenience of readers.

2.2.3. Hausdorff moments. Let ν be a bounded positive Borel measure on $[0,1)$ such that $c_0 := \int_0^1 (1-t)^{-1} d\nu(t) < \infty$. Let $(c_n)_{n\geq 1}$ be the Hausdorff moment sequence associated to ν; that is, $c_n := \int_0^1 t^{n-1} d\nu(t)$, $n \in \mathbb{N}$.

The following proposition is Proposition 5.9 of [3], but, while its stament in [3] provides just partial positiveness or negativeness properties of fractional differences, it now presents explicit identities for such differences for better understanding.

Proposition 2.4. *Let $a, b \geq 0$. Given ν, (c_n) as before, put $\widehat{h}(0) = a + b + c_0$, $\widehat{h}(1) = -b - c_1$ and $\widehat{h}(n) = -c_n$, $n \geq 2$. For $z \in \mathbb{D}$, set*

$$h(z) = \sum_{n=0}^{\infty} \widehat{h}(n) z^n \quad \text{and} \quad f(z) := (1-z)^{-1} h(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n.$$

Then, for $\alpha > 0$,

(1) $$[D^\alpha \widehat{h}](0) = h(0) - \alpha \widehat{h}(1) - \int_0^1 t^{-1} \left[(1-t)^\alpha - 1 + \alpha t\right] d\nu(t),$$

$$[D^\alpha \widehat{h}](1) = -b - \int_0^1 (1-t)^\alpha d\nu(t),$$

$$[D^\alpha \widehat{h}](n) = - \int_0^1 (1-t)^\alpha t^{n-1} d\nu(t), \quad n \geq 2,$$

whence

$$\widehat{h} \in \tau^\alpha(\mathbb{N}_0), \quad [D^\alpha \widehat{h}](0) \geq 0, \quad [D^\alpha \widehat{h}](n) \leq 0, \, n \geq 1.$$

(2) $$\widehat{f}(0) = a + b + c_0, \quad \widehat{f}(n) = a + \int_0^1 t^n (1-t)^{-1} d\nu(t), \quad n \geq 1,$$

$$[D^\alpha \widehat{f}](0) = a + b + \int_0^1 (1-t)^{\alpha-1} d\nu(t),$$

$$[D^\alpha \widehat{f}](n) = \int_0^1 t^n (1-t)^{\alpha-1} d\nu(t), \quad n \geq 1,$$

and therefore the sequence $(\widehat{f}(n))_{n=0}^\infty$ is bounded, with

$$\widehat{f}(n) \geq 0, \; [D^\alpha \widehat{f}](n) \geq 0 \, (n \geq 0), \; \lim_{n\to\infty} n^\alpha [D^\alpha \widehat{f}](n) = 0 \text{ and } W^{-\alpha}(D^\alpha \widehat{f}) = \widehat{f}.$$

The proof of the proposition consists of just writing the coefficient sequences $\widehat{h}$ and $\widehat{f}$ in their integral forms in the formula for differences, and doing some calculations. We leave this task to the reader.

Proposition 2.4 is a source for α-admissible functions constructed from complete Bernstein functions.

2.2.4. Complete Bernstein functions and Hausdorff moments. This part is Remark 5.10 of [3] and it contains the same material. However, there are a couple of misprints and a wrong reference in [3] which have been corrected here.

An analytic function $H\colon \mathbb{C}\setminus(-\infty,0]\to\mathbb{C}$ is a complete Bernstein function if it has the form

$$H(\lambda)=a+b\lambda+\int_0^\infty \lambda(\lambda+s)^{-1}\,d\mu(s),\qquad \text{for } \mathfrak{Im}\,\lambda>0,$$

where $a,b\ge 0$ and μ is a positive Borel measure on $(0,\infty)$ with $\int_0^\infty(1+s)^{-1}d\mu(s)<\infty$.

Under $t=1/(1+s)$ and $d\nu(t)=s(1+s)^{-2}d\mu(s)$ one has, for $\lambda>0$,

$$\int_0^\infty \frac{\lambda}{\lambda+s}d\mu(s)=\int_0^1\frac{\lambda}{1-(1-\lambda)t}\,\frac{d\nu(t)}{1-t}=\int_0^1\frac{d\nu(t)}{1-t}-\int_0^1\frac{(1-\lambda)d\nu(t)}{1-(1-\lambda)t},$$

whence, for $z=1-\lambda$ in $\mathbb{D}$,

$$\begin{aligned}h(z):=H(1-z)&=a+b(1-z)+c_0-\int_0^1\frac{z\,d\nu(t)}{1-zt}\\&=a+b(1-z)+c_0-\sum_{j=1}^\infty\left(\int_0^1 t^{j-1}d\nu(t)\right)z^j,\end{aligned}$$

and so h satisfies the hypotheses of Proposition 2.4. On the other hand, H is complete Bernstein if and only if $G(\lambda):=\lambda H(\lambda)^{-1}$ is a complete Bernstein function, see [16, Proposition 7.1] [1] Applying to G the argument done for H and putting $g(z):=G(1-z)=\sum_{n=0}^\infty\widehat{g}(n)z^n$, $z\in\mathbb{D}$, one obtains

$$\widehat{g}(0)\ge 0,\ [D^\alpha\widehat{g}](0)\ge 0;\quad \widehat{g}(n)\le 0,\ [D^\alpha\widehat{g}](n)\le 0\quad \forall n\ge 1.$$

Assume $h(z)\neq 0$ for $z\in\mathbb{D}$ and let $f(z):=1/g(z)=(1-z)^{-1}h(z)$. Then $\widehat{f}$ is bounded, $[D^j\widehat{f}](n)\ge 0$ for $j\in\{0,\alpha\}$, $n\ge 0$, $W^{-\alpha}(D^\alpha\widehat{f})=\widehat{f}$ and $\lim_{n\to\infty}n^\alpha[D^\alpha\widehat{f}](n)=0$ by Proposition 2.4 again. In conclusion, f is an α-admissible function with the additional property that $\lim_{n\to\infty}n^\alpha[D^\alpha\widehat{f}](n)=0$.

There are many concrete examples of complete Bernstein functions with no zero in the disc $\{|1-\lambda|<1\}$, see [16, Chapter 15] for an extensive list of such examples. We are mainly interested in the two following. These are Example 5.11 and Example 5.12 of [3]. Here we change the presentation slightly, add some information and correct another misprint.

Example 2.5. For $0<r<1$, let $H_r(\lambda):=\lambda^r$, $\lambda\in\mathbb{C}\setminus(-\infty,0]$. Then

$$H_r(\lambda)=\int_0^\infty\frac{\lambda}{\lambda+u}d\mu_r(u)$$

where $d\mu_r(u)=(\sin(\pi r)/\pi)u^{r-1}du$ with its corresponding moment measure given by $d\nu_r(t)=(\sin(\pi r)/\pi)t^{-r}(1-t)^r dt$, is a complete Bernstein function, see [16, Section 15.2, Ex. No 1]. So, the function $\mathfrak{q}_s(z)=(1-z)^{-s}=$

[1] An important misprint of the authors in [3] is to have written $G(\lambda):=\lambda^{-1}H(\lambda)$. Also, the reference cited above substitutes the wrong [16, Th. 6.2].

$(1-z)^{-1}H_{1-s}(1-z)$, $z \in \mathbb{D}$, of Example 2.8 (i) is an α-admissible function, for every $\alpha \geq 0$ and $0 < s < 1$.

Example 2.6. Let G be the complete Bernstein function given by $G(\lambda) := (\lambda-1)(\log(\lambda))^{-1}$, $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ (with measure $d\mu(u) = (u+1)(u(\log^2(u) + \pi^2))^{-1}du$), see [16, Section 15.4, Ex. No. 41], so that

$$H(\lambda) := \lambda/G(\lambda) = \lambda \log(\lambda)(\lambda-1)^{-1} = \int_0^\infty \frac{\lambda}{\lambda+u}\frac{du}{u+1},$$

see [16, Section 15.4, Ex. No. 33]. Then H is a complete Bernstein function and the function Λ defined by

$$\Lambda(z) = \frac{-\log(1-z)}{z} = \sum_{n=0}^{\infty} L(n)z^n, \quad z \in \mathbb{D},$$

where $L(n) = (n+1)^{-1}$ for all $n \geq 0$, is an α-admissible function for every $\alpha \geq 0$.

Remark 2.7. Example 2.6 will be used in Section 4 to show the logarithmic decay of Cesàro means on vectors making the Hilbert transform convergent. For $\alpha = 0$ this is accomplished in [10] by using the admissible function $f_0(z) := 2 - \log(1-z)$, $z \in \mathbb{D}$. This function is not α-admissible for general $\alpha > 0$, although it can be replaced with $f_\alpha(z) := N - \log(1-z)$, $z \in \mathbb{D}$, with $N > \alpha$, which is α-admissible. The (not serious but annoying) trouble here is that the admissibility of f_0 is proved in [10, p. 205] using Kaluza's theorem for log-convex sequences. If we wanted to follow this way here, showing in passing the α-admissibility of f_α, we should introduce a suitable notion of log-convex sequences for any $\alpha > 0$, and the corresponding theorem of Kaluza type. Both things are possible but this would take too much space. Then we keep working on the function Λ.

2.3. Operator means of fractional order and functional calculus

2.3.1. Fractional means. Let $\mathcal{B}(X)$ denote the Banach algebra of bounded operators on X. Given $T \in \mathcal{B}(X)$ and $\alpha \in \mathbb{R}$, put

$$[\Delta^{-\alpha}\mathcal{T}](n) := \sum_{j=0}^{n} k_\alpha(n-j)T^j, \text{ and } M_n^\alpha(T) := \frac{1}{k_{\alpha+1}(n)}[\Delta^{-\alpha}\mathcal{T}](n), \ n \geq 0. \tag{2.16}$$

where in the case $\alpha \geq 0$ they are known as the *Cesàro sum* and *Cesàro mean* of order α of T respectively. A useful formula involving Cesàro sums is

$$(\mathrm{I} - T)[\Delta^{-\alpha}\mathcal{T}](n) = k_\alpha(n+1) - [\Delta^{-(\alpha-1)}\mathcal{T}](n+1), \quad \alpha \in \mathbb{R}, n \geq 0. \tag{2.17}$$

In effect,

$$\begin{aligned}(\mathrm{I} - T)[\Delta^{-\alpha}\mathcal{T}](n) &= k_\alpha(n)\mathrm{I} + \sum_{j=1}^{n}(k_\alpha(n-j) - k_\alpha(n+1-j))T^j - k_\alpha(0)T^{n+1} \\ &= k_\alpha(n)\mathrm{I} - \sum_{j=1}^{n+1} k_{\alpha-1}(n+1-j)T^j\end{aligned}$$

$$= k_\alpha(n)\mathrm{I} + k_{\alpha-1}(n+1)\mathrm{I} - [\Delta^{-(\alpha-1)}\mathcal{T}](n+1)$$
$$= k_\alpha(n+1)\mathrm{I} - [\Delta^{-(\alpha-1)}\mathcal{T}](n+1).$$

For $\alpha \geq 0$, recall that an operator $T \in \mathcal{B}(X)$ is said to be (C,α)-bounded if

$$K_\alpha(T) := \sup\{\|M_n^\alpha(T)\| : n \in \mathbb{N}_0\} < \infty.$$

In this case $\|[\Delta^{-\alpha}\mathcal{T}](n)\| \leq K_\alpha(T)k_{\alpha+1}(n)$ for all $n \geq 0$, and moreover T is (C,δ)-bounded for all $\delta > \alpha$ with the same constant $K_\alpha(T)$. Indeed, given $n \in \mathbb{N}$,

$$\sum_{j=0}^{n} k_\delta(n-j)T^j = \sum_{j=0}^{n}\sum_{q=0}^{n-j} k_\alpha(n-j-q)k_{\delta-\alpha}(q)T^j$$
$$= \sum_{q=0}^{n}\left(\sum_{j=0}^{n-q} k_\alpha(n-q-j)T^j\right) k_{\delta-\alpha}(q)$$

so that

$$\|\sum_{j=0}^{n} k_\delta(n-j)T^j\| \leq K_\alpha(T)\sum_{q=0}^{n} k_{\alpha+1}(n-q)k_{\delta-\alpha}(q) = K_\alpha(T)k_{\delta+1}(n). \quad (2.18)$$

The following estimate for a (C,α)-bounded operator T is to be applied in Section 3, Theorem 3.2. Let β be such that $0 \leq \alpha < \beta \leq \alpha+1$. Then, for some positive constant $K_{\alpha,\beta}$,

$$\|M_n^\beta(T)(\mathrm{I}-T)\| \leq \frac{K_{\alpha,\beta}}{n^{\beta-\alpha}}, \quad n \in \mathbb{N}_0. \quad (2.19)$$

In detail,

$$\|(\mathrm{I}-T)[\Delta^{-\beta}\mathcal{T}](n)\|$$
$$\leq k_\beta(n+1) + \sum_{j=0}^{n+1} |k_{\beta-\alpha-1}(j)|\|\Delta^{-\alpha}\mathcal{T}(n+1-j)\|$$
$$\leq k_\beta(n+1) + K_\alpha(T)\left(2k_{\alpha+1}(n+1) - \sum_{j=0}^{n+1} k_{\beta-\alpha-1}(j)k_{\alpha+1}(n+1-j)\right)$$
$$= 2K_\alpha(T)k_{\alpha+1}(n+1) + (1-K_\alpha(T))k_\beta(n+1).$$

where the first inequality is obtained using (2.17) together with the identity $[\Delta^{-(\beta-1)}\mathcal{T}](n+1) = (k_{\beta-\alpha-1}*\Delta^{-\alpha}\mathcal{T})(n+1)$, $n \geq 0$, and the second inequality follows by (2.4) since $-1 < \beta-\alpha-1 \leq 0$. Then (2.19) follows readily from (2.5).

2.3.2. Functional calculus. We introduce the functional calculus, with initial domain the Wiener algebra $A^\alpha(\mathbb{D})$, such as it is done in [3]. To begin with, one has that $T \in \mathcal{B}(X)$ is a (C,α)-bounded operator if and only if there exists a Banach algebra bounded homomorphism $\Theta_\alpha : \tau^\alpha(\mathbb{N}_0) \to \mathcal{B}(X)$, which furthermore is given by

$$\Theta_\alpha(\widehat{g})x = \sum_{n=0}^{\infty} [W^\alpha \widehat{g}](n)[\Delta^{-\alpha}\mathcal{T}](n)x, \quad x \in X, \widehat{g} \in \tau^\alpha(\mathbb{N}_0),$$

see [4, Corollary 3.7]. Then we define the functional calculus $g \in A^\alpha(\mathbb{D}) \mapsto g(T) \in \mathcal{B}(X)$ by

$$g(T) := \Theta_\alpha(\widehat{g}), \text{ so that } \|g(T)\| \leq K_\alpha(T)\|g\|_{A^\alpha(\mathbb{D})}. \tag{2.20}$$

The above calculus can be extended to functions suitable to be regularized by elements of $A^\alpha(\mathbb{D})$: a function f holomorphic in $\mathbb{D}$ is called α-*regularizable* if there is an element $\mathfrak{e} \in A^\alpha(\mathbb{D})$ such that $\mathfrak{e}f \in A^\alpha(\mathbb{D})$ and $\mathfrak{e}(T)$ is an injective operator. Then we define

$$f(T) := \mathfrak{e}(T)^{-1}(\mathfrak{e}f)(T).$$

The so-defined $f(T)$ does not depend on the α-regularizer $\mathfrak{e}$ and it is a closed operator, generally unbounded.

Example 2.8. Assume T is (C,α)-bounded such that $\mathrm{I} - T$ is injective.

(i) Let $r \in \mathbb{R}$ and set $\mathfrak{q}_r(z) := (1-z)^{-r} = \sum_{n=0}^{\infty} k_r(n) z^n$, $z \in \mathbb{D}$. Let $\alpha > 0$. Clearly, $\mathfrak{q}_r \in A^\alpha(\mathbb{D})$ if $r \leq 0$ and $\mathfrak{q}_r \notin A^\alpha(\mathbb{D})$ for $r > 0$, see (2.5). Then, for $0 < s < 1$, the function $\mathfrak{q}_s$ is α-regularizable by $\mathfrak{q}_{-1}$ since $(1-z)\mathfrak{q}_s = \mathfrak{q}_{s-1} \in A^\alpha(\mathbb{D})$. Thus $(\mathrm{I} - T)^{-s}$ is defined and it is closed on X.

(ii) Similarly to the example in (i), we have that $\log(1-z)$ and $\Lambda(z) := -z^{-1}\log(1-z)$ are α-regularizable by $1-z$. Thus $\log(\mathrm{I} - T)$ and $\Lambda(T)$ are defined and they are closed operators on X.

2.4. Means of fractional order and ergodicity

In [3, Section 3] it is shown that ergodicity is an invariant for the order of convergent means, in the sense that, given a (C,α)-bounded operator $T \in \mathcal{B}(X)$, we have for $x \in X$ and any $\beta > \alpha$,

$$\exists \lim_{n\to\infty} M_n^\beta(T)x \text{ in } X \iff x \in \mathrm{Ker}(\mathrm{I} - T) \oplus \overline{(\mathrm{I} - T)X},$$

and

$$\lim_{n\to\infty} M_n^\beta(T)x = 0 \iff x \in \overline{(\mathrm{I} - T)X}. \tag{2.21}$$

As a consequence, X is ergodic if and only if there is $\beta > 0$ such that $P_\beta x := \lim_n M_n^\beta(T)x$ exists in X for every $x \in X$ (so $P_\beta \colon X \to \mathrm{Ker}(\mathrm{I} - T)$ is a bounded projection along $\overline{(\mathrm{I} - T)X}$). Thus we are interested in the study of decay convergence rates of $\|M_n^\beta(T)x\|$ only for $x \in \overline{(\mathrm{I} - T)X}$ (since $M_n^\beta(T)x = x$ for $x \in \mathrm{Ker}(\mathrm{I} - T)$).

In this respect, one remark is in order. Similarly to which is noticed in [10, Prop. 1.1] for means of order one we have that if there exists a sequence $(r_n)_{n\geq 1} \subset \mathbb{R}$ with $r_n > 0$ for all n and $\lim_n r_n = 0$ such that $\|M_n^\alpha(T)x\| = \mathrm{O}(r_n)$ for every $x \in X$ then $\mathrm{I} - T$ is invertible.

Indeed, the sequence $(r_n^{-1} M_n^\alpha(T))$ is pointwise bounded, so it is norm bounded by the uniform boundedness principle and therefore $\lim_n \|M_n^\alpha(T)\| =$

0. Take now n such that $\|M_n^\alpha(T)\| < 1$ so that $\mathrm{I} - M_n^\alpha(T)$ is invertible. Then, applying (2.17) in the form

$$\frac{1}{k_{\alpha+1}(n)}(\mathrm{I}-T)[\Delta^{-(\alpha+1)}\mathcal{T}](n-1) = \mathrm{I} - [M_n^\alpha(T)]$$

one obtains that $\mathrm{I} - T$ is invertible on X (and $(\mathrm{I}-T)X$ is closed). For such operators T one cannot expect a uniform decay of $\|M_n^\alpha(T)x\|$, that is, uniform rates, in general, see [10, p. 202].

Let f be an admissible function on $\mathbb{D}$ and be $g := 1/f$. We know $\widehat{g} \in \tau^\alpha(\mathbb{N}_0)$, which is to say $g \in A^\alpha(\mathbb{D})$. Let T be a (C,α)-bounded operator on X, $\alpha \geq 0$, and let $g(T)$ denote the closed operator on X obtained from T and g by the action of the functional calculus. Suppose for a moment that $f(1) < \infty$. Then one has $f \in A^\alpha(\mathbb{D})$ and therefore $\mathrm{Ran}(g(T)) = X$, whence in this case, if $x \in \mathrm{Ran}(g(T))$ one cannot expect a general rate of convergence of $\|M_n^\beta(T)x\|$ $(\beta > \alpha)$, after the former discussion.

So we will work on admissible functions f such that $f(1) = \infty$, which is to say $g(1) = 0$. This, by (2.15), implies

$$[W^\alpha\widehat{g}](0) = -\sum_{j=1}^{\infty}[W^\alpha\widehat{g}](j)k_{\alpha+1}(j) \geq 0. \tag{2.22}$$

So, we will deal with Cesàro means on vectors $x \in \mathrm{Ran}(g(T))$. The reason for this is that $\mathrm{Ran}(g(T)) \subset \overline{(\mathrm{I}-T)X}$ when $g(1) = 0$:

$$\begin{aligned} g(T) &= \lim_n \sum_{j=0}^{n}[W^\alpha\widehat{g}](j)[\Delta^{-\alpha}\mathcal{T}](j) \\ &= \lim_n \sum_{j=1}^{n}[W^\alpha\widehat{g}](j)([\Delta^{-\alpha}\mathcal{T}](j) - k_{\alpha+1}(j)) \\ &= \lim_n (T-\mathrm{I})\sum_{j=1}^{n}[W^\alpha\widehat{g}](j)[\Delta^{-(\alpha+1)}\mathcal{T}](j-1). \end{aligned}$$

3. The key theorem

Here we outline the proof of the central theorem about mean convergence. Details of involved calculations are given in Section 5. We also state some consequences and remarks.

In the entire section, α is a nonnegative real number and T is a (C,α)-bounded operator on a Banach space X. Put as before $K_\alpha(T) := \sup_n \|M_n^\alpha(T)\|$. In the following we will assume that $f(z) = \sum_{n=0}^\infty \widehat{f}(n)z^n$ is an α-admissible function such that $f(1) = \infty$ and with inverse $g := 1/f$ in $A^\alpha(\mathbb{D})$. As noticed before, $\mathrm{Ran}(g(T)) \subseteq \overline{(\mathrm{I}-T)X}$. Hence, for $x \in \mathrm{Ran}(g(T))$ one has $\lim_{n\to\infty} M_n^\beta(T)x = 0$ for each $\beta > \alpha$. In the following result a uniform rate of the decay of $M_n^\beta(T)x$ for $x \in \mathrm{Ran}(g(T))$ is given.

Set

$$X_T := \mathrm{Ker}(\mathrm{I}-T) \oplus \overline{(\mathrm{I}-T)X}.$$

Theorem 3.1. *Let $\alpha \geq 0$ and let T be a (C,α)-bounded operator on a Banach space X. Let f be an α-admissible function such that $f(1) = \infty$ with $g := 1/f$.*

(i) *Let $\beta \in \mathbb{R}$ such that $\alpha < \beta \leq \alpha + 1$. Then, for every $x \in \mathrm{Ran}(g(T))$,*

$$\|M_n^\beta(T)x\| = \mathrm{O}\left(\frac{1}{f(1-\frac{1}{n})^{\beta-\alpha}}\right) \quad as \quad n \to \infty.$$

Moreover, assume that $\dfrac{n}{f(1-\frac{1}{n})} \to \infty$ as $n \to \infty$. Then, for $x \in$

$g(T)$
(X_T),

$$\|M_n^\beta(T)x\| = \mathrm{o}\left(\frac{1}{f(1-\frac{1}{n})^{\beta-\alpha}}\right) \quad as \quad n \to \infty.$$

(ii) *Let $\beta \in \mathbb{R}$ such that $\beta \geq \alpha + 1$. Assume now that f is δ-admissible for all $\delta > \alpha$. Then, for every $x \in \mathrm{Ran}(g(T))$,*

$$\|M_n^\beta(T)x\| = \mathrm{O}\left(\frac{1}{f(1-\frac{1}{n})}\right) \quad as \quad n \to \infty.$$

Moreover, assume that $\dfrac{n}{f(1-\frac{1}{n})} \to \infty$ as $n \to \infty$. Then, for all $x \in$

$g(T)$
(X_T),

$$\|M_n^\beta(T)x\| = \mathrm{o}\left(\frac{1}{f(1-\frac{1}{n})}\right) \quad as \quad n \to \infty.$$

The proof of Theorem 3.1 splits into several steps. Some of them are written below in the form of theorems because of their interest in their own. Cases $\alpha < \beta \leq \alpha + 1$ and $\beta > \alpha + 1$ require different treatements. We first deal with the case where $\alpha < \beta \leq \alpha + 1$. The next result is a quantitative version of part (i) in Theorem 3.1.

Theorem 3.2. *Let $\alpha \geq 0$ and let T be a (C,α)-bounded operator on a Banah space X. Let f be an α-admissible function such that $f(1) = \infty$ and put $g := 1/f$. Let $\beta \in \mathbb{R}$ such that $\alpha < \beta \leq \alpha+1$. Then, for every $x \in \mathrm{Ran}(g(T))$,*

$$\|M_n^\beta(T)x\| \leq \frac{D_\alpha}{f(1-1/n)^{\beta-\alpha}}\|y\|,$$

where $g(T)y = x$, with $D_\alpha = 2K_\alpha(T)(\mathrm{e}^\alpha+2)(2\mathrm{e}(\alpha+1)(\alpha+2))^{\beta-\alpha}\|g\|_{A^\alpha(\mathbb{D})}^{\alpha+1-\beta}$.

Moreover, assume that $\dfrac{n}{f(1-\frac{1}{n})} \to \infty$ as $n \to \infty$. Then, for $x \in$ $g(T)(X_T)$,

$$\|M_n^\beta(T)x\| = \mathrm{o}\left(\frac{1}{f(1-\frac{1}{n})^{\beta-\alpha}}\right) \text{ as } n \to \infty.$$

Proof. Let n be a natural number in all of the argument which follows right now. For $g = 1/f$ one has $g \in A^\alpha(\mathbb{D})$ and so $M_n^\beta \cdot g \in A^\alpha(\mathbb{D})$, with $M_n^\beta(z) = \frac{1}{k_{\beta+1}(n)}\sum_{j=0}^n k_\beta(n-j)z^j$ for $|z| \leq 1$. Most of the proof is devoted to estimate the norm $\|M_n^\beta \cdot g\|_{A^\alpha(\mathbb{D})}$.

Let β such that $\alpha < \beta \leq \alpha + 1$. The following claim, like the others coming, will be shown in Section 5. This claim does not seem to be true for $\beta > \alpha + 1$. $\square$

Claim 1.

$$\|M_n^\beta \cdot g\|_{A^\alpha(\mathbb{D})} = \frac{2}{k_{\beta+1}(n)} \sum_{v=0}^{n} [W^\alpha \widehat{h}_n](v) k_{\alpha+1}(v), \quad \alpha < \beta \leq \alpha + 1, \tag{3.1}$$

where $h_n(z) = [\Delta^{-\beta} \mathcal{Z}](n) g(z)$, $z \in \mathbb{D}$.

The above allows us to establish the following central equality.

Claim 2.

$$\|M_n^\beta \cdot g\|_{A^\alpha(\mathbb{D})} = \frac{2}{k_{\beta+1}(n)} \sum_{l=1}^{\infty} [-W^\alpha \widehat{g}](l)\, c_l. \tag{3.2}$$

where, for $1 \leq l \leq n$,

$$c_l = k_{\alpha+1}(l) k_{\beta+1}(n) + \left(\sum_{v=0}^{l-1} \sum_{j=v+1}^{n} - \sum_{v=l}^{n} \sum_{j=v-l}^{v} \right) k_{\alpha+1}(v) k_\alpha(l+j-v) k_{\beta-\alpha}(n-j),$$

and, for $l \geq n+1$,

$$c_l = k_{\alpha+1}(l) k_{\beta+1}(n) + \sum_{v=0}^{n-1} \sum_{j=v+1}^{n} k_{\alpha+1}(v) k_\alpha(l+j-v) k_{\beta-\alpha}(n-j).$$

Now we need to estimate the coefficients c_l, $l \geq 1$, accordingly to our aim.

Claim 3.

$$0 \leq c_l \leq (\mathrm{e}^\alpha + 2)\; k_{\alpha+1}(n) k_{\beta+1}(l), \quad \forall\ 1 \leq l \leq n. \tag{3.3}$$

and

$$0 \leq c_l \leq \mathrm{e}^\alpha k_{\beta+1}(n) k_{\alpha+1}(l), \quad \forall\ l \geq n+1. \tag{3.4}$$

Hence

$$\begin{aligned} \|M_n^\beta \cdot g\|_{A^\alpha(\mathbb{D})} &\leq N_\alpha \left(\sum_{l=1}^{n} [-W^\alpha \widehat{g}](l) \frac{k_{\alpha+1}(n) k_{\beta+1}(l)}{k_{\beta+1}(n)} \right. \\ &\qquad \left. + \sum_{l=n+1}^{\infty} [-W^\alpha \widehat{g}](l) k_{\alpha+1}(l) \right) \\ &= N_\alpha \left(\sum_{l=1}^{n} [-W^\alpha \widehat{g}](l) k_{\alpha+1}(l) \frac{k_{\alpha+1}(n) k_{\beta+1}(l)}{k_{\alpha+1}(l) k_{\beta+1}(n)} \right. \\ &\qquad \left. + \sum_{l=n+1}^{\infty} [-W^\alpha \widehat{g}](l) k_{\alpha+1}(l) \right). \end{aligned}$$

where $N_\alpha = 2(\mathrm{e}^\alpha + 2)$.

Let $R_n(\alpha,\beta;g)$ denote the above big bracket and, for a moment, let us take the specific value $\beta=\alpha+1$. In this case, it is readily seen that

$$\begin{aligned}
R_n(\alpha,\alpha+1;g) &\le \sum_{l=1}^{n}[-W^\alpha\widehat{g}](l)k_{\alpha+1}(l)\frac{(\alpha+2)l}{n} \\
&\quad + \sum_{l=n+1}^{\infty}[-W^\alpha\widehat{g}](l)k_{\alpha+1}(l) \\
&\le (\alpha+2)\left(\sum_{l=1}^{n}[-W^\alpha\widehat{g}](l)k_{\alpha+1}(l)\frac{l}{n}\right. \\
&\quad \left. + \sum_{l=n+1}^{\infty}[-W^\alpha\widehat{g}](l)k_{\alpha+1}(l)\right)
\end{aligned}$$

Now, we need to estimate the preceding big bracket, which we denote by $U_n(\alpha;g)$. To do this, set the symbol $\mathcal{Z}_n=(z_n^j)_{j\in\mathbb{N}_0}$ where $z_n=(n-1)/n$.

Claim 4.

$$\begin{aligned}
\frac{1}{\mathrm{e}(\alpha+1)}k_{\alpha+1}(l)\frac{l}{n} &\le k_{\alpha+1}(l)-[\Delta^{-\alpha}\mathcal{Z}_n](l) \\
&\le \frac{1}{\alpha+1}k_{\alpha+1}(l)\frac{l}{n}, \quad l=1,\dots,n, \qquad (3.5)
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{2\mathrm{e}^{\alpha+1}(\alpha+1)}k_{\alpha+1}(l) &\le k_{\alpha+1}(l)-[\Delta^{-\alpha}\mathcal{Z}_n](l) \\
&\le k_{\alpha+1}(l), \quad l\ge n+1. \qquad (3.6)
\end{aligned}$$

Assume that claim 4 holds true. Since $g(1)=0$ we have by (2.15) and (2.22)

$$g(z)=\sum_{l=1}^{\infty}[-W^\alpha\widehat{g}](l)\left(k_{\alpha+1}(l)-[\Delta^{-\alpha}\mathcal{Z}](l)\right), \quad z\in\mathbb{D},$$

so that, taking $z=z_n$ and using (3.5) and (3.6), one obtains

$$\frac{1}{2\mathrm{e}^{\alpha+1}(\alpha+1)}U_n(\alpha;g)\le g(z_n)\le U_n(\alpha;g).$$

In particular it implies, for $\beta=\alpha+1$,

$$\begin{aligned}
\|M_n^{\alpha+1}\cdot g\| &\le N_\alpha R_n(\alpha,\alpha+1;g)\le(\alpha+2)N_\alpha U_n(\alpha;g) \\
&\le 4(\alpha+1)(\alpha+2)\mathrm{e}^{\alpha+1}(\mathrm{e}^\alpha+2)\ g(z_n). \qquad (3.7)
\end{aligned}$$

The question now is to estimate $\|M_n^\beta\cdot g\|$ for every β such that $\alpha<\beta<\alpha+1$. To do this we employ an argument from complex variable theory.

Let F_n be the function on $\Omega:=\{\alpha\le\Re z\le\alpha+1\}$ given by

$$F_n(z):=\sum_{l=1}^{n}[-W^\alpha\widehat{g}](l)k_{\alpha+1}(l)\frac{k_{\alpha+1}(n)}{k_{\alpha+1}(l)}\frac{k_{z+1}(l)}{k_{z+1}(n)}$$

$$+\sum_{l=n+1}^{\infty}[-W^{\alpha}\widehat{g}](l)k_{\alpha+1}(l), \quad z\in\overline{\Omega}.$$

Clearly, the function F_n is holomorphic on Ω and continuous on $\overline{\Omega}$. Also, for if $1\le l\le n$ and $z\in\overline{\Omega}$, one has

$$\frac{k_{\alpha+1}(n)}{k_{\alpha+1}(l)}\frac{|k_{z+1}(l)|}{|k_{z+1}(n)|}\le\frac{(\alpha+l+1)\dots(\alpha+n)}{(r+l+1)\dots(r+n)}$$

with $r=\Re z$; that is, for all $\alpha\le r\le\alpha+1$. It follows that $|F_n(z)|\le\|g\|_{A^\alpha(\mathbb{D})}$ for all $z\in\overline{\Omega}$.

More precisely, for $r=\alpha+1$ one has

$$\frac{(\alpha+l+1)\dots(\alpha+n)}{(\alpha+l+2)\dots(\alpha+n+1)}=\frac{\alpha+1+l}{\alpha+1+n}\le(\alpha+2)\frac{l}{n}.$$

In other words, $F_n\colon\overline{\Omega}\to\mathbb{C}$ is continuous and holomorphic in Ω, such that $|F_n(z)|\le\|g\|_{A^\alpha(\mathbb{D})}$ for every $z\in\overline{\Omega}\setminus\{\alpha+1+it:t\in\mathbb{R}\}$ and

$$\sup_{t\in\mathbb{R}}|F_n(\alpha{+}1{+}it)|\le(\alpha+2)U_n(\alpha;g)\le 2\mathrm{e}^{\alpha+1}(\alpha+1)(\alpha+2)g(z_n).$$

Then it follows by the Hadamard theorem for vertical strips, see [15, Theorem 12.8], that

$$|F_n(r)|\le\|g\|_{A^\alpha(\mathbb{D})}^{\alpha+1-r}(2\mathrm{e}^{\alpha+1}(\alpha+1)(\alpha+2))^{r-\alpha}g(z_n)^{r-\alpha}, \quad \alpha\le r\le\alpha+1.$$

Applying the above bound to $r=\beta$ we obtain

$$\|M_n^\beta\cdot g\|_{A^\alpha(\mathbb{D})}\le N_\alpha(2\mathrm{e}^{\alpha+1}(\alpha+1)(\alpha+2))^{\beta-\alpha}\|g\|_{A^\alpha(\mathbb{D})}^{\alpha+1-\beta}\,g(z_n)^{\beta-\alpha},$$
$$\text{for all } \alpha\le\beta\le\alpha+1. \tag{3.8}$$

This is the time to pass to dealing with the operator T. Let $y\in X$ such that $x=g(T)y$. Then

$$\|M_n^\beta(T)x\|=\|(M_n^\beta\cdot g)(T)y\|\le K_\alpha(T)\|M_n^\beta\cdot g\|_{A^\alpha(\mathbb{D})}\|y\|.$$

and therefore it follows from (3.8) that

$$\|M_n^\beta(T)x\|\le D_\alpha\|y\|\,g(z_n)^{\beta-\alpha} \quad \text{for all} \quad \alpha\le\beta\le\alpha+1, \tag{3.9}$$

where $D_\alpha=K_\alpha(T)N_\alpha(2\mathrm{e}^{\alpha+1}(\alpha+1)(\alpha+2))^{\beta-\alpha}\|g\|_{A^\alpha(\mathbb{D})}^{\alpha+1-\beta}$.

This proves the first part of the statement.

For the second half, assume further that $ng(z_n)=n/f(1-\frac{1}{n})\to\infty$ as $n\to\infty$ and let $x=g(T)y$ with $y\in X_T$. First, we do notice that one only has to check elements x such that $y\in\overline{(\mathrm{I}-T)X}$. Indeed, $y\in\mathrm{Ker}(\mathrm{I}-T)$ implies $[\Delta^{-\alpha}\mathcal{T}(j)]y=k_{\alpha+1}(j)y$, for $j\ge 0$, whence $g(T)y=\sum_{j=0}^{\infty}[W^\alpha\widehat{g}](j)[\Delta^{-\alpha}\mathcal{T}](j)y=\sum_{j=0}^{\infty}[W^\alpha\widehat{g}](j)k_{\alpha+1}(j)y=g(1)y=0$ since $f(\infty)=0$. Hence $x=0$.

So let $x=g(T)y$ with $y\in\overline{(\mathrm{I}-T)X}$. Note that (3.9) can be read as

$$g(z_n)^{-(\beta-\alpha)}\|M_n^\beta(T)g(T)\|\le D_\alpha. \tag{3.10}$$

Suppose first that y is of the form $y = (\mathrm{I} - T)u$ with $u \in X$. Then

$$\begin{aligned}\frac{1}{g(z_n)^{\beta-\alpha}}\|M_n^\beta(T)x\| &= \frac{1}{g(z_n)^{\beta-\alpha}}\|M_n^\beta(T)(\mathrm{I}-T)g(T)u\| \\ &\le \frac{K_{\alpha,\beta}\|g(T)u\|}{g(z_n)^{\beta-\alpha}}\frac{1}{n^{\beta-\alpha}} \\ &\sim \frac{K_{\alpha,\beta}\|g(T)u\|}{(ng(z_n))^{\beta-\alpha}} \longrightarrow 0 \text{ as } n \to \infty,\end{aligned}$$

where the inequality is a consequence of (2.19).

Then, for generic $y \in \overline{(\mathrm{I}-T)X}$, we apply (3.10) to obtain $g(z_n)^{-(\beta-\alpha)}\| M_n^\beta(T)g(T)y\| \to 0$ as $n \to \infty$, by density. Thus the theorem has been proved for $\alpha < \beta \le \alpha + 1$. □

The following result gives part (ii) of Theorem 3.1. Actually, it turns out to be a corollary of preceding Theorem 3.2. We do notice that there exist analytic functions on $\mathbb{D}$ which are δ-admissible for every $\delta > 0$; see Example 2.5 and Example 2.6, for instance.

Theorem 3.3. *Let $\alpha \ge 0$ and let T be a (C,α)-bounded operator on a Banach space X. Let f be a δ-admissible function for every $\delta \ge \alpha$, such that $f(1) = \infty$ with $g := 1/f$. Let $\beta \in \mathbb{R}$ such that $\beta > \alpha+1$. Then, for every $x \in \mathrm{Ran}(g(T))$,*

$$\|M_n^\beta(T)x\| \le \frac{E_\beta}{f(1-\frac{1}{n})}\|y\|,$$

where $E_\beta = 4\mathrm{e}K_\alpha(T)\beta(\beta+1)(\mathrm{e}^{\beta-1}+2)$.

Moreover, assume that $\dfrac{n}{f(1-\frac{1}{n})} \to \infty$ as $n \to \infty$. Then, for $x \in g(T)(X_T)$,

$$\|M_n^\beta(T)x\| = \mathrm{o}\left(\frac{1}{f(1-\frac{1}{n})}\right) \quad as \quad n \to \infty.$$

Proof. Let $\beta \in \mathbb{R}$ such that $\beta > \alpha + 1$ and f be a γ-admissible function for every $\gamma > \alpha$. Take $\delta := \beta - 1$ so that $\delta > \alpha$ and thus f is in particular δ-admissible, and T is (C,δ)-bounded. Moreover, since $\beta = \delta + 1$ one can apply Theorem 3.2 replacing α with $\beta - 1$ (so $\alpha + 1$ with β) to obtain

$$\|M_n^\beta(T)x\| \le \frac{D_{\beta-1}}{f(1-\frac{1}{n})}\|y\|, \text{ for all } x \in \mathrm{Ran}(g(T)).$$

Finally, $K_{\beta-1}(T) \le K_\alpha(T)$ in $D_{\beta-1}$ by (2.18).

The remainder of the proof proceeds along similar lines as the proof of the second half of Theorem 3.2. □

4. Logarithmic decay and polynomial decay of means

Here we apply the results of the above section to Cesàro means on vectors making convergent the discrete (one-sided) Hilbert transform of (C,α)-bounded operators, as well as on vectors which are solutions of the fractional Poisson equation. We begin with a simple fact whose central argument is contained in [13, pp. 72, 73]

Lemma 4.1. *Let X be a Banach space and $T \in \mathcal{B}(X)$. Put $Y := \overline{(\mathrm{I}-T)X}$. Suppose $x \in X$ is such that the series*

$$\sum_{n=1}^{\infty} n^{-(\delta+\varepsilon)}[\Delta^{-\delta}\mathcal{T}(n)]x \text{ is weakly convergent in } X,$$

for some $\delta \geq 0$ and $0 < \varepsilon \leq 1$. Then $x \in Y$ and the given series converges weakly in Y.

Proof. Let $y \in X$ such that $\sum_{n=1}^{\infty} n^{-(\delta+\varepsilon)}[\Delta^{-\delta}\mathcal{T}(n)]x$ converges to y weakly. Suppose that $Y \neq X$ and $x \notin Y$. By the Hahn-Banach separation theorem there exists ξ in the dual space X^* of X such that $\xi(x) \neq 0$ and $\xi(z) = 0$ for all $z \in Y$. In particular $\xi(u - Tu) = 0$ for all $u \in X$, which is equivalent to $(\xi - T^*\xi)(u) = 0$ for all $u \in X$, where T^* is the adjoint operator of T. So $T^*\xi = \xi$ and therefore $\xi\left([\Delta^{-\delta}\mathcal{T}(n)]x\right) = \left([\Delta^{-\delta}\mathcal{T}(n)^*]\xi\right)(x) = k_{\delta+1}(n)\xi(x)$ for every $n \in \mathbb{N}$. This implies that, for $m \in \mathbb{N}$,

$$\xi\left(\sum_{n=1}^{m} n^{-(\delta+\varepsilon)}[\Delta^{-\delta}\mathcal{T}(n)]x\right) = \sum_{n=1}^{m} n^{-(\delta+\varepsilon)} k_{\delta+1}(n)\xi(x),$$

which is not convergent by (2.5), unless $\xi(x) = 0$ so contradicting the initial choice of ξ. Thus $x \in Y$. That $y \in Y$ too is a (well known) consequence of the Hahn-Banach theorem. Moreover, $\sum_{n=1}^{m} n^{-(\delta+\varepsilon)}[\Delta^{-\delta}\mathcal{T}(n)]x$ converges to y in the own weak topology of Y by the Hahn-Banach theorem on extension of bounded functionals. □

The following result extends [10, Th. 4.1].

Theorem 4.2. *Let $\alpha \geq 0$. Let X be a Banach space and $T \in \mathcal{B}(X)$ be a (C,α)-bounded operator. Let $x \in X$ such that*

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha+1}}[\Delta^{-\alpha}\mathcal{T}(n)]x \text{ converges weakly.} \tag{4.1}$$

(i) *Let $\beta \in \mathbb{R}$ such that $\alpha < \beta \leq \alpha + 1$. Then*

$$\|M_n^{\beta}(T)x\| = \mathrm{o}\left(\frac{1}{(\log n)^{\beta-\alpha}}\right) \text{ as } n \to \infty.$$

(ii) *Let $\beta \in \mathbb{R}$ such that $\beta > \alpha + 1$. Then*

$$\|M_n^{\beta}(T)x\| = \mathrm{o}\left(\frac{1}{\log n}\right) \text{ as } n \to \infty.$$

Proof. Both items (i) and (ii) will be a direct consequence of Theorem 3.1 once we had shown that we have the main ingredients at hands. Let us see.

Assume that $x \in X$ satisfies condition (4.1). By Lemma 4.1 $x \in Y := \overline{(\mathrm{I}-T)X}$ and the weak convergence holds in Y. Thus, passing to the T-invariant subspace Y if necessary, we can assume $X = Y$ and therefore that $\mathrm{I} - T$ is injective.

Let f be the holomorphic function Λ given by $f(z) = \Lambda(z) := -z^{-1}\log(1-z)$, $z \in \mathbb{D}$. This function is α-regularizable by $1-z$, see Example 2.8 (ii), so that one can define the closed operator $f(T)$ by the formula $f(T) :=$

$(\mathrm{I}-T)^{-1}[(1-z)f](T)$. Also, f is α-admissible, see Example 2.6, with $f(1)=\infty$ and $\lim_{n\to\infty} n^\alpha[D^\alpha\widehat{f}](n)=0$ where $[D^\alpha\widehat{f}](n)=\frac{\Gamma(\alpha+1)n!}{\Gamma(n+\alpha+2)}$, $n\ge 0$; see (2.11) and (2.12).

On the other hand the weak convergence of the series $\sum_{n=1}^\infty n^{-(\alpha+1)}[\Delta^{-\alpha}\mathcal{T}(n)]x$ implies the weak convergence of $\sum_{n=1}^\infty [D^\alpha\widehat{f}](n)[\Delta^{-\alpha}\mathcal{T}(n)]x$ by (2.12) which, in turn, implies that $x\in \mathrm{Dom} f(T)$ by [3, Th. 7.3] (in passing, we do notice that there is a misprint in the last part of the proof of [3, Th. 7.3], where the insuficient factor $(1-z)$ must be replaced with the correct factor $(1-z)^2$).

Thus $x=g(T)y$ for $y:=f(T)x$ where $g=1/f$ and we can apply Theorem 3.1. Finally, we have $f(1-\frac{1}{n})=(\frac{n}{n-1})\log n\sim\log n$, with $n/\log n\to\infty$ as $n\to\infty$ obviously. Thus the proof of (i) and (ii) is over. $\square$

Remark 4.3. The argument carried out in the latter part of the above proof shows indeed that the weak convergence in (4.1) is actually norm convergence: if the series in (4.1) converges weakly then the series $\sum_{n=1}^\infty[D^\alpha\widehat{f}](n)$ $[\Delta^{-\alpha}\mathcal{T}(n)]x$ is weakly convergent and so it is norm convergent, also by [3, Th. 7.3]. By (2.12) again, the series in (4.1) is norm convergent.

The hypothesis in Theorem 4.2 assumes convergence of the generalization $\sum_{n=1}^\infty n^{-(\alpha+1)}[\Delta^{-\alpha}\mathcal{T}(n)]$ of the one-sided discrete Hilbert transform. When $0\le\alpha<1$ we can state a similar theorem where the convergence hypothesis is assumed on the Hilbert transform itself.

Theorem 4.4. *Let $\alpha\in\mathbb{R}$ such that $0\le\alpha<1$. Let X be a Banach space and $T\in\mathcal{B}(X)$ be a (C,α)-bounded operator. Let $x\in X$ such that*

$$\sum_{n=1}^\infty \frac{1}{n}T^n x \text{ converges weakly.}$$

Then

$$\|M_n^\beta(T)x\| = \mathrm{o}\left(\frac{1}{(\log n)^{\beta-\alpha}}\right) \text{ as } n\to\infty, \text{ if } \alpha<\beta\le\alpha+1,$$

and

$$\|M_n^\beta(T)x\| = \mathrm{o}\left(\frac{1}{\log n}\right) \text{ as } n\to\infty, \text{ if } \beta>\alpha+1.$$

Proof. The case $\alpha=0$ has been proven in Theorem 4.2 since $[\Delta^{-\alpha}\mathcal{T}](n)=T^n$ for all n when $\alpha=0$. Thus suppose $0<\alpha<1$.

Since $\sum_{n=1}^\infty\frac{T^n x}{n}$ is weakly convergent one can assume that $X=\overline{(\mathrm{I}-T)X}$ as in the proof of Lemma 4.1 by taking $\delta=0$ and $\varepsilon=1$ there. On the other hand, $\|T^n\|=\mathrm{O}(n^\alpha)$ as $n\to\infty$ since T is (C,α)-bounded, see [1, Remark 7.7], and then $\sum_{n=1}^\infty\left(\frac{1}{n}-\frac{1}{n+1}\right)\|T^n x\|=\sum_{n=1}^\infty\frac{\|T^n x\|}{n(n+1)}<\infty$ because $\alpha<1$. So $\sum_{n=1}^\infty\frac{T^n x}{n+1}$ is weakly convergent too. Take now $f(z)=z^{-1}\log(1-z)=\sum_{n=1}^\infty\frac{z^n}{n(n+1)}$, $z\in\mathbb{D}$. Then the proof of [3, Th. 7.3] works for f and $\sum_{n=1}^\infty\frac{T^n x}{n(n+1)}x$ as well, so that $x\in\mathrm{Dom} f(T)=\mathrm{Ran}\, g(T)$ with $g:=1/f$. Thus the result follows from Theorem 3.1. $\square$

As in Theorem 4.2 above, the convergence of the series in Theorem 4.4 is in norm, actually.

Remark 4.5. The arguments considered above can equally be used to obtain estimates of the convergence rate of means associated with solutions of the fractional Poisson equation $(\mathrm{I}-T)^s x = y$ with $0 < s < 1$. These rates improve those found in [3, Cor. 8.2 and Cor. 8.5]:

Theorem 4.6. *Let $\alpha \geq 0$ and $0 < s < 1$. Let X be a Banach space and $T \in \mathcal{B}(X)$ be a (C, α)-bounded operator. Let $x \in X$ such that*

$$\sum_{n=1}^{\infty} \frac{\Delta^{-\alpha}\mathcal{T}(n)}{n^{\alpha+1-s}} x \textit{ is weakly convergent in } X.$$

Then

$$\|M_n^{\beta}(T)\| = \mathrm{o}\left(\frac{1}{n^{(\beta-\alpha)s}}\right) \textit{ as } n \to \infty, \textit{ if } \alpha < \beta \leq \alpha + 1,$$

and

$$\|M_n^{\beta}(T)\| = \mathrm{o}\left(\frac{1}{n^{s}}\right) \textit{ as } n \to \infty, \textit{ if } \beta \geq \alpha + 1.$$

Proof. The structure of the proof is as in preceding theorems. The convergence of the series in the statement entails the one of the series $\sum_{n=0}^{\infty} D^{\alpha} k_s(n)$ $\Delta^{-\alpha}\mathcal{T}(n)x$ since, by (2.10),

$$\begin{aligned}\sum_{n=1}^{\infty} &\left|D^{\alpha}k_s(n) - \frac{\Gamma(1-s+\alpha)}{\Gamma(s)\Gamma(1-s)} n^{s-\alpha-1}\right| \|\Delta^{-\alpha}\mathcal{T}(n)\| \, \|x\| \\ &\leq K_s \sum_{n=1}^{\infty} \frac{k_{\alpha+1}(n)}{n^{\alpha+1-s}} \frac{1}{n} \leq K_s' \sum_{n=1}^{\infty} \frac{n^{\alpha}}{n^{\alpha+1-s}} = K_s' \sum_{n=1}^{\infty} \frac{1}{n^{2-s}} < \infty,\end{aligned}$$

where here K_s, K_s' are positive constants independent of n.

On the other hand, by Lemma 4.1 applied with $\varepsilon = 1 - s$ one can assume $X = \overline{(\mathrm{I}-T)X}$ without loss of generality. Then from the convergence of $\sum_{n=0}^{\infty} D^{\alpha} k_s(n)\Delta^{-\alpha}\mathcal{T}(n)x$ it follows by [3, Th. 7.3] that $x \in \mathrm{Dom}(\mathfrak{q}_s(T))$, since $\mathfrak{q}_s(z) = (1-z)^{-s}$, $z \in \mathbb{D}$, as in Example 2.8 (i), is α-regularizable by $1-z$, α-admissible (see Example 2.5), with $f(1) = \infty$ and $\lim_{n\to\infty} n^{\alpha}[D^{\alpha}](n) = 0$ (see (2.10)). Thus $x = \mathfrak{q}_{-s}(T)y$, with $y \in X$ and we can apply Theorem 3.1 to end the proof. Note that for $g := \mathfrak{q}_{-s}$ one has $ng(1-\frac{1}{n}) = n^{1-s} \to \infty$, as $n \to \infty$, as it was also required. $\square$

As regards convergence involving just powers of operators one has the following result.

Theorem 4.7. *Let $0 \leq \alpha < 1 - s$ and $0 < s < 1$. Let X be a Banach space and $T \in \mathcal{B}(X)$ be a (C, α)-bounded operator. Let $x \in X$ such that*

$$\sum_{n=1}^{\infty} \frac{T^n x}{n^{1-s}} \textit{ is weakly convergent.}$$

Then

$$\|M_n^{\beta}(T)\| = \mathrm{o}\left(\frac{1}{n^{(\beta-\alpha)s}}\right) \ \textit{as } n\to\infty, \ \textit{if } \alpha<\beta\le\alpha+1.$$

and

$$\|M_n^{\beta}(T)\| = \mathrm{o}\left(\frac{1}{n^{s}}\right) \ \textit{as } n\to\infty, \ \textit{if } \beta\ge\alpha+1.$$

Proof. Let s,α,x as in the statement. By Lemma 4.1 one can assume $X = \overline{(\mathrm{I}-T)X}$. Now, for some constant K_s,

$$\sum_{n=1}^{\infty} |k_s(n) - (n^{s-1}/\Gamma(s)| \|T^n\| \|x\| \le \frac{K_s}{\Gamma(s)}\sum_{n=1}^{\infty}\frac{n^{s-1}}{n}n^{\alpha} = \frac{K_s}{\Gamma(s)}\sum_{n=1}^{\infty}\frac{1}{n^{2-\alpha-s}} < \infty,$$

by (2.5) and because $\alpha < 1-s$. In consequence, the series $\sum_{n=1}^{\infty} k_s(n)T^n x$ is convergent and so $x\in \mathrm{Ran}(\mathfrak{q}_{-s})$. Then Theorem 3.1 applies and the proof is over. □

We end this section with two examples to which the preceding theorems apply.

Example 4.8. Let $1\le p<\infty$ and V be the Volterra integral operator on $L^p(0,1)$ given by

$$Vf(t) := \int_0^t f(s)ds, \quad t\in[0,1],\ f\in L^p(0,1).$$

Put $T_V := \mathrm{I}-V$. Then powers T^n and means $M^{\alpha}_{T_V}(n)$, for $\alpha>0$, are given by integral expressions involving Laguerre polynomials and, in this way, it has been shown that T_V is power bounded exclusively for $p=2$, and polynomially bounded on $L^p(0,1)$, see [12, Th. 11] for $p=1$ and [14, Th. 2.2] for $p>1, p\neq 2$. Moreover, T is (C,α)-ergodic on $L^p(0,1)$, $1\le p<\infty$, for $\alpha>1/2$, see [12, Th. 11] and [3, Prop. 9.2 (ii)].

Example 4.9. Let $0<\beta<1$ and let $\ell^2_{\beta}(\mathbb{N}_0)$ denote the Hilbert space of sequences a such that $\|a\|^2_{2,\beta} := \sum_{j=0}^{\infty}|a(j)|^2 k_{\beta}(j) < \infty$. Let S be the backward shift operator on $\ell^2_{\beta}(\mathbb{N}_0)$ given by

$$(Sa)(j) = a(j{+}1), \quad a\in\ell^2_{\beta}(\mathbb{N}_0),\ j\in\mathbb{N}_0,$$

so for $n,j\in\mathbb{N}_0$ we have $(S^n a)(j) = a(j{+}1))$ and $\Delta^{-\alpha}\mathcal{T}(n)a = \sum_{j=0}^{n} k_{\alpha}(n{-}j)f(j{+}\cdot)$.

Then T_S is not power-bounded on $\ell^2_{\beta}(\mathbb{N}_0)$, but T_S is (C,α)-bounded and (C,α)-ergodic for $\alpha>(1-\beta)/2$, see [1] and [3, Example 9.4].

5. Proofs of formulae and estimates

We first prove Claim 1, that is, (3.1).

Lemma 5.1. *For every β such that $\alpha < \beta \le \alpha + 1$,*

$$\|M_n^\beta \cdot g\|_{A^\alpha(\mathbb{D})} = \frac{2}{k_{\beta+1}(n)} \sum_{v=0}^{n} [W^\alpha \widehat{h}_n](v) k_{\alpha+1}(v),$$

where $h_n(z) = [\Delta^{-\beta}\mathcal{Z}](n)g(z)$, $z \in \mathbb{D}$.

Proof. The norm of h_n in $A^\alpha(\mathbb{D})$ is $\|h_n\|_{A^\alpha(\mathbb{D})} = \sum_{v=0}^\infty |[W^\alpha \widehat{h}_n](v)| k_{\alpha+1}(v)$. In order to calculate $[W^\alpha \widehat{h}_n](v)$ for $v \in \mathbb{N}_0$ we make use of (2.14) and (2.22).

Thus, since

$$[W^\alpha (\widehat{\Delta^{-\beta}\mathcal{Z})(n)}](j) := \begin{cases} k_{\beta-\alpha}(n-j), & j \le n, \\ 0, & j > n, \end{cases}$$

see [4, Example 2.5 (iii)], we get that, for $v \ge n+1$,

$$[W^\alpha \widehat{h}_n](v) = \sum_{j=0}^{n} \sum_{l=v-j}^{v} k_\alpha(l+j-v) k_{\beta-\alpha}(n-j)[W^\alpha \widehat{g}](l) \le 0,$$

and, for $1 \le v \le n-1$,

$$\begin{aligned}[W^\alpha \widehat{h}_n](v) &= \left(\sum_{l=0}^{v} \sum_{j=v-l}^{v} - \sum_{j=v+1}^{n} \sum_{l=v+1}^{\infty}\right) k_\alpha(l+j-v) k_{\beta-\alpha}(n-j)[W^\alpha \widehat{g}](l) \\ &= -k_{\beta-\alpha}(n-v) \sum_{l=1}^{\infty} [W^\alpha \widehat{g}](l) k_{\alpha+1}(l) \\ &\quad + \left(\sum_{l=1}^{v} \sum_{j=v-l}^{v} - \sum_{j=v+1}^{n} \sum_{l=v+1}^{\infty}\right) k_\alpha(l+j-v) k_{\beta-\alpha}(n-j)[W^\alpha \widehat{g}](l) \\ &= V_1 + V_2,\end{aligned}$$

where

$$V_1 := \sum_{l=1}^{v} [-W^\alpha \widehat{g}](l) \left(k_{\beta-\alpha}(n-v) k_{\alpha+1}(l) - \sum_{j=v-l}^{v} k_\alpha(l+j-v) k_{\beta-\alpha}(n-j) \right) \tag{5.1}$$

and

$$V_2 := \sum_{l=v+1}^{\infty} [-W^\alpha \widehat{g}](l) \left(k_{\beta-\alpha}(n-v) k_{\alpha+1}(l) + \sum_{j=v+1}^{n} k_\alpha(l+j-v) k_{\beta-\alpha}(n-j) \right). \tag{5.2}$$

It is clear that $V_2 \ge 0$ since $[W^\alpha \widehat{g}](l) \le 0$ for all $l \ge v+1$ being f α-admissible. Regarding V_1, we use that $0 < \beta - \alpha \le 1$ so that $k_{\beta-\alpha}$ is nonincreasing. Hence,

$$V_1 \geq \sum_{l=1}^{v} [-W^\alpha \widehat{g}](l) \left(k_{\beta-\alpha}(n-v) k_{\alpha+1}(l) - \sum_{j=v-l}^{v} k_\alpha(l+j-v) k_{\beta-\alpha}(n-v) \right)$$
$$= \sum_{l=1}^{v} [-W^\alpha \widehat{g}](l) \left(k_{\beta-\alpha}(n-v) k_{\alpha+1}(l) - k_{\beta-\alpha}(n-v) \sum_{q=0}^{l} k_\alpha(q) \right) = 0.$$

In summary, $[W^\alpha \widehat{h}_n](v) \geq 0$ for $1 \leq v \leq n-1$.

Also, using similar arguments as above, one obtains

$$[W^\alpha \widehat{h}_n](0) = -\sum_{l=1}^{\infty} [W^\alpha \widehat{g}](l) \left(k_{\beta-\alpha}(n) k_{\alpha+1}(l) + \sum_{j=1}^{n} k_\alpha(l+j) k_{\beta-\alpha}(n-j) \right) \geq 0, \tag{5.3}$$

and, since $\sum_{j=n-l}^{n} k_\alpha(l+j-n) k_{\beta-\alpha}(n-j) = k_\beta(l) \leq k_{\alpha+1}(l)$, we get

$$[W^\alpha \widehat{h}_n](n) = -\sum_{l=1}^{n} [W^\alpha \widehat{g}](l) \left(k_{\alpha+1}(l) - \sum_{j=n-l}^{n} k_\alpha(l+j-n) k_{\beta-\alpha}(n-j) \right)$$
$$- \sum_{l=n+1}^{\infty} [W^\alpha \widehat{g}](l) k_{\alpha+1}(l) \geq 0. \tag{5.4}$$

All in all, since $\sum_{v=0}^{\infty} [W^\alpha \widehat{h}_n](v) k_{\alpha+1}(v) = h_n(1) = 0$ (see (2.15)), we have

$$k_{\beta+1}(n) \| M_n^\beta \cdot g \| = \| h_n \|_{A^\alpha(\mathbb{D})} = 2 \sum_{v=0}^{n} [W^\alpha \widehat{h}_n](v) k_{\alpha+1}(v)$$
$$\left(= -2 \sum_{v=n+1}^{\infty} [W^\alpha \widehat{h}_n](v) k_{\alpha+1}(v) \right).$$

as we wanted to show. □

Remark 5.2. The argument to prove formula (3.1) in the above proposition heavily depends on the sign of differences $[W^\alpha \widehat{h}_n](v)$ and then on the sign of $V_1 + V_2$. The term V_2 is always nonnegative but V_1 is nonpositive if $\beta > \alpha + 1$. Thus we cannot conclude by this method a closed formula for $\| M_n^\beta \cdot g \|_{A_+^\alpha(\mathbb{D})}$ in this case. For this reason we have had to resort to prove part (ii) of Theorem 3.1 as a corollary of Theorem 3.2 in Section 3.

We now prove the key formula given in Claim 2 or (3.2). After Lemma 5.1, it is enough to handle differences $[W^\alpha \widehat{h}_n](v)$ for $0 \leq v \leq n$. For $P \subset \mathbb{N}_0$, let χ_P denote the characteristic or indicator function of P. Notice that equalities (5.1), (5.2), (5.3), (5.4) can be gathered in the compact form

$$[W^\alpha \widehat{h}_n](v) = \sum_{l=1}^{v} \left(k_{\beta-\alpha}(n-v) k_{\alpha+1}(l) - \sum_{j=v-l}^{v} k_\alpha(l+j-v) k_{\beta-\alpha}(n-j) \right)$$
$$\chi_{\{1,\ldots,n\}}(v) [-W^\alpha \widehat{g}](l)$$

$$+\sum_{l=v+1}^{\infty}\Bigg(k_{\beta-\alpha}(n-v)k_{\alpha+1}(l)+\sum_{j=v+1}^{n}k_\alpha(l+j-v)k_{\beta-\alpha}(n-j)$$
$$\chi_{\{0,\ldots,n-1\}}(v)\Big)\,[-W^\alpha\widehat{g}](l),$$

with $0\le v\le n$.

Proposition 5.3. *For $\alpha<\beta\le\alpha+1$,*

$$\frac{1}{2}\;k_{\beta+1}(n)\|M_n^\beta\cdot g\|_{A^\alpha(\mathbb{D})}=\sum_{l=1}^{\infty}[-W^\alpha\widehat{g}](l)\,c_l,$$

where, for $1\le l\le n$,

$$c_l=k_{\alpha+1}(l)k_{\beta+1}(n)$$
$$+\left(\sum_{v=0}^{l-1}\sum_{j=v+1}^{n}-\sum_{v=l}^{n}\sum_{j=v-l}^{v}\right)k_{\alpha+1}(v)k_\alpha(l+j-v)k_{\beta-\alpha}(n-j),$$

and, for $l\ge n+1$,

$$c_l=k_{\alpha+1}(l)k_{\beta+1}(n)+\sum_{v=0}^{n-1}\sum_{j=v+1}^{n}k_{\alpha+1}(v)k_\alpha(l+j-v)k_{\beta-\alpha}(n-j).$$

Proof. Take $0\le v\le n$. Applying the equality prior to this proposition in the formula of Lemma 5.1, and exchanging summation order we have

$$\frac{k_{\beta+1}(n)}{2}\|M_n^\beta\cdot g\|_{A^\alpha(\mathbb{D})}$$
$$=\sum_{v=0}^{n}[W^\alpha\widehat{h}_n](v)k_{\alpha+1}(v)$$
$$=\sum_{l=1}^{n}\sum_{v=l}^{n}k_{\beta-\alpha}(n-v)k_{\alpha+1}(v)k_{\alpha+1}(l)[-W^\alpha\widehat{g}](l)$$
$$-\sum_{l=1}^{n}\sum_{v=l}^{n}\sum_{j=v-l}^{v}k_\alpha(l+j-v)k_{\beta-\alpha}(n-j)k_{\alpha+1}(v)[-W^\alpha\widehat{g}](l)$$
$$+\sum_{l=1}^{n+1}\sum_{v=0}^{l-1}k_{\beta-\alpha}(n-v)k_{\alpha+1}(v)k_{\alpha+1}(l)[-W^\alpha\widehat{g}](l)$$
$$+\sum_{l=n+2}^{\infty}\sum_{v=0}^{n}k_{\beta-\alpha}(n-v)k_{\alpha+1}(v)k_{\alpha+1}(l)[-W^\alpha\widehat{g}](l)$$
$$+\sum_{l=1}^{n+1}\sum_{v=0}^{l-1}\sum_{j=v+1}^{n}k_\alpha(l+j-v)k_{\beta-\alpha}(n-j)k_{\alpha+1}(v)$$
$$\chi_{\{0,\ldots,n-1\}}(v)[-W^\alpha\widehat{g}](l)$$
$$+\sum_{l=n+2}^{\infty}\sum_{v=0}^{n}\sum_{j=v+1}^{n}k_\alpha(l+j-v)k_{\beta-\alpha}(n-j)k_{\alpha+1}(v)$$
$$\chi_{\{0,\ldots,n-1\}}(v)[-W^\alpha\widehat{g}](l).$$

Then, summing the first and third rows from $l=1$ to $l=n$ and passing $l=n+1$ to the fourth row, and passing $l=n+1$ from the fifth row to the sixth row, one obtains

$$\begin{aligned}
&\frac{k_{\beta+1}(n)}{2}\|M_n^\beta\cdot g\|_{A^\alpha(\mathbb{D})}\\
&\quad=\sum_{l=1}^{n}\left(\sum_{v=0}^{n}k_{\beta-\alpha}(n-v)k_{\alpha+1}(v)\right)k_{\alpha+1}(l)[-W^\alpha\widehat{g}](l)\\
&\qquad+\sum_{l=n+1}^{\infty}\left(\sum_{v=0}^{n}k_{\beta-\alpha}(n-v)k_{\alpha+1}(v)\right)k_{\alpha+1}(l)[-W^\alpha\widehat{g}](l)\\
&\qquad-\sum_{l=1}^{n}\sum_{v=l}^{n}\sum_{j=v-l}^{v}k_\alpha(l+j-v)k_{\beta-\alpha}(n-j)k_{\alpha+1}(v)[-W^\alpha\widehat{g}](l)\\
&\qquad+\sum_{l=1}^{n}\sum_{v=0}^{l-1}\sum_{j=v+1}^{n}k_\alpha(l+j-v)k_{\beta-\alpha}(n-j)k_{\alpha+1}(v)\chi_{\{0,\dots,n-1\}}(v)[-W^\alpha\widehat{g}](l)\\
&\qquad+\sum_{l=n+1}^{\infty}\sum_{v=0}^{n-1}\sum_{j=v+1}^{n}k_\alpha(l+j-v)k_{\beta-\alpha}(n-j)k_{\alpha+1}(v)[-W^\alpha\widehat{g}](l),
\end{aligned}$$

which is to say

$$\begin{aligned}
&\frac{k_{\beta+1}(n)}{2}\|M_n^\beta\cdot g\|_{A^\alpha(\mathbb{D})}\\
&\quad=\sum_{l=1}^{n}k_{\beta+1}(n)k_{\alpha+1}(l)[-W^\alpha\widehat{g}](l)\\
&\qquad+\sum_{l=1}^{n}\left(\sum_{v=0}^{l-1}\sum_{j=v+1}^{n}-\sum_{v=l}^{n}\sum_{j=v-l}^{v}\right)\\
&\qquad k_\alpha(l+j-v)k_{\beta-\alpha}(n-j)k_{\alpha+1}(v)[-W^\alpha\widehat{g}](l)\\
&\qquad+\sum_{l=n+1}^{\infty}k_{\beta+1}(n)k_{\alpha+1}(l)[-W^\alpha\widehat{g}](l)\\
&\qquad+\sum_{l=n+1}^{\infty}\sum_{v=0}^{n-1}\sum_{j=v+1}^{n}k_\alpha(l+j-v)k_{\beta-\alpha}(n-j)k_{\alpha+1}(v)[-W^\alpha\widehat{g}](l),
\end{aligned}$$

and the proposition follows. □

Our task at this point is to find appropriate bounds of coefficients c_l. This is Claim 3, that is, (3.3) and (3.4).

Lemma 5.4. *For* $(c_l)_{l=1}^\infty$ *as above,*

$$0\le c_l\le(\mathrm{e}^\alpha+2)\ k_{\beta+1}(l)k_{\alpha+1}(n),\quad 1\le l\le n,$$

and

$$0\le c_l\le\mathrm{e}^\alpha\ k_{\alpha+1}(l)k_{\beta+1}(n),\quad l\ge n+1.$$

Proof. For $1 \le l \le n$, write $c_l = a_l + b_l$, with

$$a_l := k_{\alpha+1}(l)k_{\beta+1}(n) - \sum_{v=l}^{n}\sum_{j=v-l}^{v} k_{\alpha+1}(v)k_\alpha(l+j-v)k_{\beta-\alpha}(n-j).$$

and

$$b_l := \sum_{v=0}^{l-1}\sum_{j=v+1}^{n} k_{\alpha+1}(v)k_\alpha(l+j-v)k_{\beta-\alpha}(n-j).$$

Then

$$\begin{aligned} a_l &= k_{\alpha+1}(l)k_{\beta+1}(n) - \sum_{v=l}^{n} k_{\alpha+1}(v)\sum_{q=0}^{l} k_\alpha(q)k_{\beta-\alpha}(n+l-v-q) \\ &= k_{\alpha+1}(l)k_{\beta+1}(n) - \sum_{q=0}^{l} k_\alpha(q)\sum_{v=l}^{n} k_{\alpha+1}(v)k_{\beta-\alpha}(n+l-q-v) \\ &= \sum_{q=0}^{l} k_\alpha(q)\sum_{v=0}^{n} k_{\alpha+1}(v)k_{\beta-\alpha}(n-v) \\ &\quad - \sum_{q=0}^{l} k_\alpha(q)\sum_{v=l}^{n} k_{\alpha+1}(v)k_{\beta-\alpha}(n+l-q-v) \\ &= \sum_{q=0}^{l} k_\alpha(q)\left(\sum_{v=l}^{n} k_{\alpha+1}(v)(k_{\beta-\alpha}(n-v) - k_{\beta-\alpha}(n+l-q-v)) \right. \\ &\quad \left. + \sum_{v=0}^{l-1} k_{\alpha+1}(v)k_{\beta-\alpha}(n-v)\right) \\ &= \sum_{q=0}^{l} k_\alpha(q)A(n,l,q) + \sum_{q=0}^{l} k_\alpha(q)B(n,l), \end{aligned}$$

with

$$A(n,l,q) := \sum_{v=l}^{n} k_{\alpha+1}(v)(k_{\beta-\alpha}(n-v) - k_{\beta-\alpha}(n+l-q-v)), \quad 0 \le q \le l,$$

(notice, $A(n,l,l) = 0$) and

$$B(n,l) := \sum_{v=0}^{l-1} k_{\alpha+1}(v)k_{\beta-\alpha}(n-v).$$

Since $(k_{\beta-\alpha}(n))_{n\in\mathbb{N}_0}$ is a decreasing sequence one has $A(n,l,q) \ge 0$, and $B(n,l) \ge 0$ obviously. So $c_l \ge 0$ for $1 \le l \le n$.

Regarding upper bounds, the term $B(n,l)$ is easy to estimate. Actually, since $(k_{\beta-\alpha}(j))_{j\in\mathbb{N}_0}$ is a nonnegative, decreasing sequence, one has

$$B(n,l) \le \sum_{v=0}^{l} k_{\alpha+1}(v)k_{\beta-\alpha}(l-v) = k_{\beta+1}(l), \quad 1 \le l \le n.$$

The term $A(n,l,q)$ requires some more effort. Notice that, for $l \le v \le n$, $0 \le q \le l-1$,

$$\begin{aligned} k_{\beta-\alpha}(n-v) - k_{\beta-\alpha}(n+l-q-v) &= \sum_{p=0}^{n-v} k_{\beta-\alpha-1}(p) - \sum_{p=0}^{n-v+l-q} k_{\beta-\alpha-1}(p) \\ &= -\sum_{p=n-v+1}^{n-v+l-q} k_{\beta-\alpha-1}(p) = \sum_{j=1}^{l-q}(-k_{\beta-\alpha-1}(n-v+j)). \end{aligned}$$

Hence,

$$\begin{aligned} A(n,l,q) &= \sum_{j=1}^{l-q}\sum_{v=l}^{n} k_{\alpha+1}(v)(-k_{\beta-\alpha-1}(n-v+j)) \\ &= \sum_{j=1}^{l-q}\sum_{p=j}^{j+n-l} k_{\alpha+1}(n+j-p)(-k_{\beta-\alpha-1}(p)) \\ &\le \sum_{j=1}^{l-q}\sum_{p=j}^{j+n-l} k_{\alpha+1}(n)(-k_{\beta-\alpha-1}(p)) \\ &= k_{\alpha+1}(n)\sum_{j=1}^{l-q}(k_{\beta-\alpha}(j-1) - k_{\beta-\alpha}(n+j-l)) \\ &\le k_{\alpha+1}(n)\sum_{j=1}^{l-q} k_{\beta-\alpha}(j-1) = k_{\alpha+1}(n)k_{\beta-\alpha+1}(l-q-1). \end{aligned}$$

In conclusion, for every $1 \le l \le n$,

$$\begin{aligned} a_l &= \sum_{q=0}^{l-1} k_\alpha(q)A(n,l,q) + \sum_{q=0}^{l} k_\alpha(q)B(n,l) \\ &\le k_{\alpha+1}(n)\sum_{q=0}^{l-1} k_\alpha(q)k_{\beta-\alpha+1}(l-q-1) + k_{\alpha+1}(l)k_{\beta+1}(l) \\ &\le k_{\alpha+1}(n)k_{\beta+1}(l-1) + k_{\alpha+1}(l)k_{\beta+1}(l) \\ &\le 2k_{\alpha+1}(n)k_{\beta+1}(l). \end{aligned}$$

Regarding, b_l, note that, for $2 \le l \le n$, one has

$$b_l = \sum_{v=0}^{l-1} k_{\alpha+1}(v) \sum_{q=0}^{n-1-v} k_\alpha(l+q+1)k_{\beta-\alpha}(n-q-1-v),$$

whence, by changing summation order, one gets

$$b_l = \sum_{q=0}^{n-l} k_\alpha(l+q+1)\sum_{v=0}^{l-1} k_{\alpha+1}(v)k_{\beta-\alpha}(n-q-1-v)$$

$$
\begin{aligned}
&+\sum_{q=n-l+1}^{n-1} k_\alpha(l+q+1) \sum_{v=0}^{n-q-1} k_{\alpha+1}(v)k_{\beta-\alpha}(n-q-1-v)\\
&=\sum_{q=0}^{n-l} k_\alpha(l+q+1)\sum_{v=0}^{l-1} k_{\alpha+1}(v)k_{\beta-\alpha}(n-q-1-v)\\
&+\sum_{q=n-l+1}^{n-1} k_\alpha(l+q+1)k_{\beta+1}(n-q-1).
\end{aligned}
$$

We now use that $(k_{\beta-\alpha}(j))_{j\in\mathbb{N}_0}$ is decreasing and $(k_{\beta+1}(j))_{j\in\mathbb{N}_0}$ is increasing to obtain

$$
\begin{aligned}
b_l &\leq \sum_{q=0}^{n-l} k_\alpha(l+q+1)\sum_{v=0}^{l-1} k_{\alpha+1}(v)k_{\beta-\alpha}(l-1-v)\\
&+\sum_{q=n-l+1}^{n-1} k_\alpha(l+q+1)k_{\beta+1}(l-1)\\
&=\sum_{q=0}^{n-1} k_\alpha(l+q+1)k_{\beta+1}(l-1) = k_{\beta+1}(l-1)\sum_{p=l+1}^{n+l} k_\alpha(p)\\
&= k_{\beta+1}(l-1)(k_{\alpha+1}(n+l)-k_{\alpha+1}(l)) \leq k_{\beta+1}(l)k_{\alpha+1}(2n).
\end{aligned}
$$

Since $k_{\alpha+1}(2n) \leq \mathrm{e}^\alpha k_{\alpha+1}(n)$ we have $b_l \leq \mathrm{e}^\alpha k_{\beta+1}(l)k_{\alpha+1}(n)$ for $2 \leq l \leq n$.

Also, for $l=1$,

$$
\begin{aligned}
b_1 &= \sum_{p=2}^{n+1} k_\alpha(p)k_{\beta-\alpha}(n+1-p)\\
&= k_\beta(n+1)-k_\beta(1) \leq k_\beta(n+1) \leq k_{\alpha+1}(n+1)\\
&= k_{\alpha+1}(n)\,\frac{\alpha+n+1}{n+1}\,\frac{k_{\beta+1}(1)}{\beta+1} \leq k_{\alpha+1}(n)k_{\beta+1}(1)
\end{aligned}
$$

Summing a_l and b_l one obtains $c_l \leq (\mathrm{e}^\alpha+2)k_{\alpha+1}(n)k_{\beta+1}(l)$ for every $1 \leq l \leq n$.

Assume now that $l \geq n+1$. Then

$$
\begin{aligned}
&\sum_{v=0}^{n-1}\sum_{j=v+1}^{n} k_{\alpha+1}(v)k_\alpha(l+j-v)k_{\beta-\alpha}(n-j)\\
&=\sum_{v=0}^{n-1} k_{\alpha+1}(v)\sum_{q=0}^{n-1-v} k_\alpha(l+q+1)k_{\beta-\alpha}(n-1-v-q)\\
&=\sum_{q=0}^{n-1} k_\alpha(l+q+1)\sum_{v=0}^{n-1-q} k_{\alpha+1}(v)k_{\beta-\alpha}(n-1-q-v)\\
&=\sum_{q=0}^{n-1} k_\alpha(l+q+1)k_{\beta+1}(n-1-q)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{p=l+1}^{l+n} k_\alpha(p) k_{\beta+1}(n-(p-l)) \le k_{\beta+1}(n) \sum_{p=l+1}^{l+n} k_\alpha(p) \\
&= k_{\beta+1}(n)(k_{\alpha+1}(l+n) - k_{\alpha+1}(l)) \\
&= k_{\beta+1}(n) k_{\alpha+1}(l) \left[\left(1+\frac{\alpha}{l+1}\right) \dots \left(1+\frac{\alpha}{l+n}\right) - 1 \right] \\
&\le k_{\beta+1}(n) k_{\alpha+1}(l) \left[\left(1+\frac{\alpha}{n}\right)^n - 1 \right] \le k_{\beta+1}(n) k_{\alpha+1}(l)(\mathrm{e}^\alpha - 1).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
c_l &= k_{\alpha+1}(l) k_{\beta+1}(n) \\
&\quad + \sum_{v=0}^{n-1} \sum_{j=v+1}^{n} k_{\alpha+1}(v) k_\alpha(l+j-v) k_{\beta-\alpha}(n-j) \le \mathrm{e}^\alpha\ k_{\beta+1}(n) k_{\alpha+1}(l),
\end{aligned}
$$

as we wanted to show. That $c_l \ge 0$ is clear. □

Claim 4, which is to say (3.5), provides the way to establish the (almost) equivalence between $g(z_n)$ and $\|M_n^\beta \cdot g\|_{A^\alpha(\mathbb{D})}$ in the proof of Theorem 3.2. This comes next.

Lemma 5.5. *Let* $z_n := 1 - \frac{1}{n}$, $n \ge 1$, *and put* $[\Delta^{-\alpha} \mathcal{Z}_n](l) := \sum_{j=0}^{l} k_\alpha(l-j) z_n^j$, $l \ge 1$.

Then

$$
\frac{\mathrm{e}^{-1}}{\alpha+1}\ k_{\alpha+1}(l) \frac{l}{n} \le k_{\alpha+1}(l) - [\Delta^{-\alpha} \mathcal{Z}_n](l) \le \frac{1}{\alpha+1}\ k_{\alpha+1}(l) \frac{l}{n}, \quad 1 \le l \le n,
$$

and

$$
\frac{\mathrm{e}^{-(\alpha+1)}}{2(\alpha+1)} k_{\alpha+1}(l) \le k_{\alpha+1}(l) - [\Delta^{-\alpha} \mathcal{Z}_n](l) \le k_{\alpha+1}(l), \quad l \ge n+1.
$$

Proof. Assume that $1 \le l \le n$. Then

$$
\begin{aligned}
k_{\alpha+1}(l) - [\Delta^{-\alpha} \mathcal{Z}_n](l) &= \sum_{j=0}^{l} k_\alpha(l-j)(1-z_n^j) = (1-z_n) \sum_{j=1}^{l} k_\alpha(l-j) \sum_{q=0}^{j-1} z_n^q \\
&= \frac{1}{n} \sum_{q=0}^{l-1} z_n^q \sum_{p=0}^{l-q-1} k_\alpha(p) = \frac{1}{n} \sum_{q=0}^{l-1} k_{\alpha+1}(l-1-q) z_n^q.
\end{aligned}
$$

Therefore, since $z_n^j \le 1$ for all $j \in \mathbb{N}$ we have

$$
k_{\alpha+1}(l) - [\Delta^{-\alpha} \mathcal{Z}_n](l) \le \frac{k_{\alpha+2}(l-1)}{n} = \frac{1}{\alpha+1}\ k_{\alpha+1}(l) \frac{l}{n}
$$

and on the other hand, since $z_n^q \ge z_n^{n-1}$ for $q \le n-1$ and $z_n^{n-1} \ge \mathrm{e}^{-1}$, we have

$$
k_{\alpha+1}(l) - [\Delta^{-\alpha} \mathcal{Z}_n](l) \ge \frac{\mathrm{e}^{-1} k_{\alpha+2}(l-1)}{n} = \frac{\mathrm{e}^{-1}}{\alpha+1}\ k_{\alpha+1}(l) \frac{l}{n}.
$$

Assume now that $l \ge n+1$. Clearly,

$$
k_{\alpha+1}(l) - [\Delta^{-\alpha} \mathcal{Z}_n](l) \le k_{\alpha+1}(l).
$$

Next, looking for a bound from below, we proceed as follows.

To begin with, write

$$k_{\alpha+1}(l)-[\Delta^{-\alpha}\mathcal{Z}_n](l)=\sum_{j=1}^{n}k_\alpha(l-j)(1-z_n^j)+\sum_{j=n+1}^{l}k_\alpha(l-j)(1-z_n^j).$$

By [10, Lemma 5.1], $1-z_n^j\geq \dfrac{\mathrm{e}^{-1}j}{n}$ for $1\leq j\leq n$ and so

$$\sum_{j=1}^{n}k_\alpha(l-j)(1-z_n^j)\geq \frac{\mathrm{e}^{-1}}{n}\sum_{j=1}^{n}k_\alpha(l-j)j.$$

Regarding the sum $\sum_{j=n+1}^{l}$ we deal with two cases. Thus let $l\geq n+1$ and assume first that $n+1\leq l\leq 2n$. For j such that $n+1\leq j\leq l$ one has $1\leq (j/n)\leq 2$, whence $1-z_n^j\geq 1-z_n^n\geq \mathrm{e}^{-1}$ by [10, Lemma 5.1], and also

$$\sum_{j=n+1}^{l}k_\alpha(l-j)(1-z_n^j)\geq\sum_{j=n+1}^{l}k_\alpha(l-j)\mathrm{e}^{-1}\geq\frac{\mathrm{e}^{-1}}{2}\sum_{j=n+1}^{l}k_\alpha(l-j)\frac{j}{n}.$$

Therefore

$$\begin{aligned}k_{\alpha+1}(l)-[\Delta^{-\alpha}\mathcal{Z}_n](l)&\geq\frac{\mathrm{e}^{-1}}{2n}\sum_{j=0}^{l}k_\alpha(l-j)((j+1)-1)\\&=\frac{\mathrm{e}^{-1}}{2n}\left(\sum_{j=0}^{l}k_\alpha(l-j)k_2(j)-k_{\alpha+1}(l)\right)\\&=\frac{\mathrm{e}^{-1}}{2n}\Big(k_{\alpha+2}(l)-k_{\alpha+1}(l)\Big)\\&=\frac{\mathrm{e}^{-1}}{2(\alpha+1)}k_{\alpha+1}(l)\frac{l}{n}\geq\frac{\mathrm{e}^{-1}}{2(\alpha+1)}k_{\alpha+1}(l).\end{aligned}$$

Finally, let consider the case $l>2n$. Then

$$\begin{aligned}k_{\alpha+1}(l)-[\Delta^{-\alpha}\mathcal{Z}_n](l)&\geq\sum_{j=n+1}^{l}k_\alpha(l-j)(1-z_n^j)\\&\geq\mathrm{e}^{-1}\sum_{j=n+1}^{l}k_\alpha(l-j)=\mathrm{e}^{-1}k_{\alpha+1}(l-n-1).\end{aligned}$$

Now, recall that $k_{\alpha+1}(2m)\leq\mathrm{e}^{\alpha}k_{\alpha+1}(m)$ for all $m\in\mathbb{N}$. Then, if l is even with $l\geq 2n+1$, one has

$$k_{\alpha+1}(l-n-1)=k_{\alpha+1}(l/2+l/2-n-1)\geq k_{\alpha+1}(l/2)\geq\mathrm{e}^{-\alpha}k_{\alpha+1}(l).$$

Analogously, if l is odd with $l\geq 2n+1$ then

$$\begin{aligned}k_{\alpha+1}(l-n-1)&=k_{\alpha+1}((l-1)/2+(l-1)/2-n)\geq k_{\alpha+1}((l-1)/2)\\&\geq\mathrm{e}^{-\alpha}k_{\alpha+1}(l-1)=\mathrm{e}^{-\alpha}\frac{l}{\alpha+l}k_{\alpha+1}(l)\geq\frac{\mathrm{e}^{-\alpha}}{\alpha+1}k_{\alpha+1}(l).\end{aligned}$$

Therefore,

$$k_{\alpha+1}(l) - [\Delta^{-\alpha}\mathcal{Z}_n](l) \geq \frac{\mathrm{e}^{-(\alpha+1)}k_{\alpha+1}(l)}{\alpha+1}, \quad l \geq 2n+1,$$

and the proof is over. □

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