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Conserved quantities in the ocean surface boundary layer: A fresh perspective on a classical problem

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A B S T R A C T

Noether's (1918) (first) Theorem reveals three conservation laws from physical symmetries in the wind-induced ocean's surface boundary layer, as described by the Classical Ekman's (1905) Theory. The Lagrangian function used for the Ekman model is comprised of two terms. A term that accounts for vertical mixing, parametrized by a constant eddy viscosity coefficient, and a second term for the rotation of the current field due to Coriolis force. The derived conservation laws, which involve relationships between the helicity, enstrophy, and kinetic energy within the surface boundary layer, allow recovering and explaining well-known and new features of the Classical Ekman's (1905) Theory. Enstrophy, which is a property of the entire water column, can be readily obtained from the surface deflection angle at the surface alone. Conservation laws provide a theoretical explanation for the Bjerknes experiment, according to which the phase angle grows linearly with depth. Remarkably, a unique symmetry-preserving constant eddy viscosity coefficient can be determined from observations, provided that observations are described by Ekman's Theory. This outcome suggests that the determined value converges more closely to the true physical value compared to crude estimates by statistical fitting.

1. Introduction

The Classical Ekman's (1905) Theory [1] is a building block of physical oceanography that explains wind-driven currents in the ocean's surface boundary layer. Ekman formulated his model as:

$$\begin{cases} -f\,v = K_0 \dfrac{\mathrm{d}^2 u}{\mathrm{d}z^2}, \quad f\,u = K_0 \dfrac{\mathrm{d}^2 v}{\mathrm{d}z^2}, & 0 < z < -\infty \\ \rho K_0 \left.\dfrac{\mathrm{d}u}{\mathrm{d}z}\right|_0 = \tau_w^x, \quad \rho K_0 \left.\dfrac{\mathrm{d}v}{\mathrm{d}z}\right|_0 = \tau_w^y & z = 0 \\ u(-\infty) = 0, \quad v(-\infty) = 0 & z \to -\infty \end{cases} \tag{1}$$

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where $u = u(z)$ and $v = v(z)$ are the non-geostrophic time-averaged velocities (so, the above differential equations represent the Reynolds-averaged Navier−Stokes (RANS) equations); f, the Coriolis parameter; and ρ, the water density; both constants. These steady currents are forced by a uniform wind on the sea surface (τ_w^x and τ_w^y). The vertical coordinate z is assumed here to be positive upwards and its reference $z = 0$ is set at the surface. The turbulent stress is modelled in a similar manner to that of the viscous stress: a velocity gradient multiplied by a vertical eddy viscosity coefficient, namely $K_0 > 0$, which depends on the local flow conditions. Therefore, modelling the eddy viscosity is the core of the turbulent closure and fundamental for matching the velocity field of the model with actual observations [e.g., 2]. This is still a challenge and, in most practical cases, the coefficient is adjusted by empirical statistical methods or averaged values are considered to reproduce the observations [e.g. 3–5]. In fact, Ekman assumed a constant K_0 value, which, despite of the simplification, works reasonably well in many situations [e.g., 6]. The constant eddy viscosity coefficient (K_0) will be referred to as the Ekman eddy viscosity henceforth. With this assumption, the boundary-value problem (BVP) (1) shows a velocity field that rotates as depth increases, thereby displaying the well-known Ekman Spiral.

Noether's (1918) Theorems [7] play a fundamental role in modern-day physics, and represent an elegant connection between the action of a physical system and continuous transformations, in the sense of Lie group, that provide continuous symmetries to differential equations [8–10]. These symmetries usually have physical meaning and help to constrain model parameters. The first theorem states that every such global symmetry has an associated conservation law. The second theorem states that a local symmetry, depending upon an arbitrary function, corresponds to a non-trivial differential relation of the Euler–Lagrange equations. The application of Noether's (1918) Theorems [7] is proven to be of huge importance also in Geophysical Fluid Dynamics [11].

In the present study, we formulate the Classical Ekman's (1905) Theory [1] into a variational framework. Even though the exact solution of the Ekman model is known, the variational framework provides a new perspective on this classical building block of physical oceanography. The aim of the research is to find conserved quantities, through the Noether's (1918) (first) theorem [7], that allow for unraveling novel (and explaining well-known) features of the Ekman surface boundary layer (Section 2). In particular, a research question that this study addresses is: Is it possible to determine a more robust estimate of eddy viscosity (assumed constant) based on newly derived conserved quantities in the Ekman boundary layer? (Section 3). The most convenient approach to present this research is to deal with a relatively simple scenario in which the eddy viscosity does not depend on depth.

2. Conserved quantities in the Classical Ekman's (1905) Theory

Before deriving conservation laws for the Ekman boundary layer and discuss their physical meaning, several definitions are needed. Given a generic variable $a = a(z)$, $\dot{a} \equiv \mathrm{d}a/\mathrm{d}z$. A continuous transformation takes the form $z \to z' = Z(z, a; \varepsilon)$ and $a \to a' = A(z, a; \varepsilon)$ for small real values of a parameter ε. The functions Z and A describe Taylor expansion of the transformation on $\varepsilon = 0$. The vector notation used is $\mathbf{u} = (u, v, 0)^{\mathrm{T}}$, $\boldsymbol{\tau}_w = (\tau_w^x, \tau_w^y, 0)^{\mathrm{T}}$, and $\boldsymbol{\omega} := \nabla \times \mathbf{u} = (-\dot{v}, \dot{u}, 0)^{\mathrm{T}}$. These are the velocity field, the wind stress, and the vorticity, respectively. The current definition of $\boldsymbol{\omega}$ relies on x- and y-independence. Also, we describe $\mathbf{u}^{\perp} = (-v, u, 0)^{\mathrm{T}}$ as the counterclockwise perpendicular vector of the velocity field. Scalar variables of the former are $\mathrm{u}^2 = \mathbf{u} \cdot \mathbf{u}$, $\tau_w^2 = \boldsymbol{\tau}_w \cdot \boldsymbol{\tau}_w$, and $\omega^2 = \boldsymbol{\omega} \cdot \boldsymbol{\omega}$. In addition, we use the complex velocity $\psi = u + \mathrm{i}v$ where i is the imaginary unit. The complex conjugate of ψ is $\psi^* = u - \mathrm{i}v$.

2.1. Noether's conservation laws

The Lagrangian function that corresponds to the Ekman equations (1), properly cast in Euler-Lagrange form, is

$$\mathscr{L}(z, \dot{\mathbf{u}}) = \frac{1}{2} K_0 \dot{\mathrm{u}}^2 + f \dot{\mathbf{u}} \cdot \mathbf{m}^{\perp}(z)\,, \tag{2}$$

where the mass transport at depth z is defined as $\mathbf{m}^{\perp}(z) := \int_z^0 \mathbf{u}^{\perp} \mathrm{d}\xi$. Equation (2) shows that $\mathscr{L}$ consists of two parts: a viscous term that essentially represents the vertical turbulent mixing, and a Coriolis term showing the rotation of the velocity field through the gradient of $\mathbf{u}$ and the mass transport. The dummy variable ξ has units of length. Note that the Lagrangian is specifically defined as a functional of z and $\dot{\mathbf{u}}$, whereas $\mathbf{u}$ is not considered. This definition overcomes the difficulty of changing the sign in the momentum equations (1) due to the Coriolis term. As $\mathscr{L}$ remains invariant by translations $\mathbf{u}' = \mathbf{u} + \mathbf{1}\varepsilon$, $\mathbf{1} = (1, 1, 0)^{\mathrm{T}}$, the canonical momentum ($\mathbf{p} := \partial \mathscr{L} / \partial \dot{\mathbf{u}}$) is conserved, thereby recovering Ekman's equations (1).

In the following, we seek the invariance of $\mathscr{L}$ under symmetries in space, scale, and phase. The conserved (Noether) quantity is a linear combination of the canonical momentum and the Hamiltonian ($\mathscr{H} := \mathbf{p} \cdot \dot{\mathbf{u}} - \mathscr{L}$), that is, $\mathbf{p} \cdot \boldsymbol{\eta} - \mathscr{H} \tau - \Phi = \text{const.}$ with τ and $\boldsymbol{\eta}$ being the generators. The function $\Phi(z)$ is introduced to ensure that $\mathscr{L}$ remains invariant under the transformation, see further details in Logan [12]. Any other symmetry written as a combination of a symmetry in space, scale, or phase would result in a conservation law composed of the later-on presented Eqns. (3a), (4a), and (8a). The derivation of these conservation laws can be found in Appendix A.

2.1.1. Symmetry in space

First, we explore how Ekman dynamics changes for infinitesimal displacements in the vertical axis. To do so, we need to write the generators of the infinitesimal transformation and find a function Φ that makes invariant $\mathscr{L}$ under this transformation. Once Φ is achieved, we invoke the Noether Theorem and obtain the corresponding conservation law.

Therefore, under space translation, i.e. $z' = z + \varepsilon$ and $\mathbf{u}' = \mathbf{u}$, the Lagrangian (2) is divergence invariant, i.e. $\mathscr{L}' - \mathscr{L} = f \boldsymbol{\omega} \cdot \mathbf{u}\, \varepsilon + \mathcal{O}(\varepsilon^2)$, and yields the following Noether conservation law:

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(K_0\frac{\omega^2}{2} - f\int_z^0 \boldsymbol{\omega}\cdot\mathbf{u}\,\mathrm{d}\xi\right) = 0\,, \tag{3a}$$

which express the idea that the Noether current $j_z = j_z(z)$ defined by:

$$j_z := K_0\frac{\omega^2}{2} - f\int_z^0 \boldsymbol{\omega}\cdot\mathbf{u}\,\mathrm{d}\xi\,, \tag{3b}$$

is a conserved quantity. That means j_z is vertically constant. In particular, the following result is achieved: $\omega^2 = \text{const.}$ if and only if $f = 0$ (no rotation).

2.1.2. *Symmetry in scale*

Subsequently, we explore how Ekman dynamics changes for infinitesimal scaling in the current. Under scaling translation, i.e. $z' = z$ and $\mathbf{u}' = \mathbf{u} + \mathbf{u}\varepsilon$, the Lagrangian (2) is divergence invariant, i.e. $\mathscr{L}' - \mathscr{L} = \mathbf{p}\cdot\dot{\mathbf{u}}\,\varepsilon + \mathcal{O}(\varepsilon^2)$, and produces the following Noether conservation law:

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(K_0\left[\mathbf{u}\cdot\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}z} + \int_z^0 \omega^2\,\mathrm{d}\xi\right]\right) = 0\,, \tag{4a}$$

which express the idea that the Noether current $j_{\mathbf{u}} = j_{\mathbf{u}}(z)$ defined by:

$$j_{\mathbf{u}} := \mathbf{u}\cdot\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}z} + \int_z^0 \omega^2\,\mathrm{d}\xi\,, \tag{4b}$$

is a conserved quantity. That means $j_{\mathbf{u}}$ is vertically constant. Notice that, by examining the conservation law (4a), the Ekman eddy viscosity (K_0) acts as a scale factor for $j_{\mathbf{u}}$.

2.1.3. *Symmetry in phase*

Finally, we explore Ekman dynamics changes for infinitesimal angular displacements $\theta = \theta(z)$. In this case, the problem is best analyzed in terms of the complex velocity ψ, with θ being the phase, i.e. $\tan\theta = v/u$. The Lagrangian (2) reformulates for this complex field as follows:

$$\mathscr{L}(z,\dot\psi,\dot\psi^*) = \frac{1}{2}K_0\dot\psi\dot\psi^* + \frac{\mathrm{i}}{2}f\left[\dot\psi^*\chi(z) - \dot\psi\chi^*(z)\right]\,, \tag{5}$$

where $\chi(z) = \int_z^0 \psi\,\mathrm{d}\xi$ and $\chi^*(z) = \int_z^0 \psi^*\,\mathrm{d}\xi$. The complex canonical momentum and the complex conjugate canonical momentum are $p := \partial\mathscr{L}/\partial\dot\psi$ and $p^* := \partial\mathscr{L}/\partial\dot\psi^*$, respectively. The phase transformation reads:

$$\begin{cases} z' = z \\ \psi' = \psi\exp\left(\mathrm{i}\dfrac{f}{|f|}\varepsilon\right) = \psi + \mathrm{i}\dfrac{f}{|f|}\psi\varepsilon + \mathcal{O}(\varepsilon^2) \\ \psi^{*\prime} = \psi^*\exp\left(-\mathrm{i}\dfrac{f}{|f|}\varepsilon\right) = \psi^* - \mathrm{i}\dfrac{f}{|f|}\psi^*\varepsilon + \mathcal{O}(\varepsilon^2). \end{cases} \tag{6}$$

Appendices A and B provide a discussion on this infinitesimal transformation.

Under transformation (6), the Lagrangian (5) is divergence invariant, i.e., $\mathscr{L}' - \mathscr{L} = \mathrm{i}[f/|f|]\,[\dot\psi p - \dot\psi^* p^*]\,\varepsilon + \mathcal{O}(\varepsilon^2)$, and yields the following Noether conservation law:

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{i}}{2}K_0\frac{f}{|f|}\left[\psi\dot\psi^* - \psi^*\dot\psi\right] + f\frac{f}{|f|}\int_z^0 \psi\psi^*\,\mathrm{d}\xi\right) = 0\,. \tag{7}$$

Developing the products between complex velocities, the conservation law (7) is written as follows:

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{f}{|f|}K_0\mathrm{u}^2\frac{\mathrm{d}\theta}{\mathrm{d}z} + |f|\int_z^0 \mathrm{u}^2\,\mathrm{d}\xi\right) = 0\,, \tag{8a}$$

which express the idea that the Noether current $j_\theta = j_\theta(z)$ defined by:

$$j_\theta := \frac{f}{|f|}K_0\mathrm{u}^2\frac{\mathrm{d}\theta}{\mathrm{d}z} + |f|\int_z^0 \mathrm{u}^2\,\mathrm{d}\xi\,, \tag{8b}$$

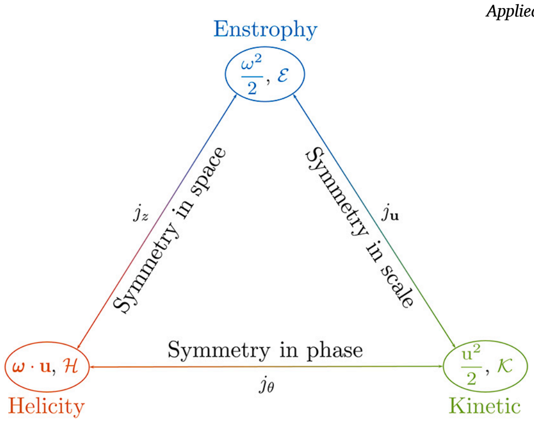

Fig. 1. Relation between enstrophy, helicity, and kinetic energy through symmetries in space, scale, and phase.

is a conserved quantity. That means j_θ is vertically constant. This provides the following result: $\mathrm{u}^2 \mathrm{d}\theta/\mathrm{d}z = \text{const.}$ if and only if $f = 0$ (no rotation).

2.2. Physical interpretation of j_z, $j_{\mathbf{u}}$, and j_θ

The theoretical well-known solution for the BVP (1) reads

$$\frac{u}{U_0} = \exp\left(\frac{z}{d}\right) \cos\left(\frac{f}{|f|}\left[\frac{z}{d} - \varphi\right]\right), \tag{9a}$$

$$\frac{v}{U_0} = \exp\left(\frac{z}{d}\right) \sin\left(\frac{f}{|f|}\left[\frac{z}{d} - \varphi\right]\right), \tag{9b}$$

with $U_0 := \sqrt{2}\tau_w/(\rho|f|d)$ being a reference velocity at the surface ($\mathrm{u}_0 = U_0$), $d := \sqrt{2K_0/|f|}$ the depth of the surface boundary Ekman layer, and $\varphi := \pi/4 - \arctan\left(f/|f|\,\tau_w^y/\tau_w^x\right)$. Solutions (9) have been rewritten from those in Ekman's paper [1] to be presented in a more general and compact version. This set of solutions provides a validation of the derived conserved quantities.

By substituting the set (9) into the conserved quantities, we obtain

$$\boxed{j_z = \frac{\tau_w^2}{2\rho^2 K_0}, \quad j_{\mathbf{u}} = \frac{\tau_w^2}{\rho^2 K_0\sqrt{2K_0|f|}}, \quad j_\theta = \frac{\tau_w^2}{\rho^2\sqrt{2K_0|f|}}.} \tag{10}$$

A simple dimensional analysis reveals that the conserved quantities (10) scale as, respectively, $j_z \sim \nu \mathtt{U}^2/\mathtt{L}^2$, $j_{\mathbf{u}} \sim \mathtt{U}^2/\mathtt{L}$, and $j_\theta \sim \nu\mathtt{U}^2/\mathtt{L}$ where ν is a kinematic viscosity, $\mathtt{U}$ is a reference velocity, and $\mathtt{L}$ is a reference length. The conserved quantities (10) are depth-independent, i.e. they are conserved in the water column, and we can conclude the following: $j_z, j_{\mathbf{u}}, j_\theta > 0 \ \forall f \gtrless 0$ and only j_z is latitude-independent.

The relationship between the theoretical values of j_z, $j_{\mathbf{u}}$, and j_θ reads

$$j_\theta = K_0 j_{\mathbf{u}} = d\, j_z\,. \tag{11}$$

In every conserved quantity (Equations (3b), (4b), and (8b)), there is a balance between two terms. Such balance involves a pair of variables of the following set $\{\omega^2/2, \boldsymbol{\omega}\cdot\mathbf{u}, \mathrm{u}^2/2, \mathrm{u}^2\mathrm{d}\theta/\mathrm{d}z\}$, where $\omega^2/2$ is the enstrophy density that relates to the intensity of turbulence –more precisely with vorticity–, $\boldsymbol{\omega}\cdot\mathbf{u}$ is the helicity density that relates rotation and momentum [13], $\mathrm{u}^2/2$ is the kinetic energy density, and $\mathrm{u}^2\mathrm{d}\theta/\mathrm{d}z$ is a reformulation of $\boldsymbol{\omega}\cdot\mathbf{u}$ for the Ekman layer but in terms of the vertical phase gradient (see Appendix C and Tan and Wu [14]). Therefore, j_z represents the enstrophy-helicity balance; $j_{\mathbf{u}}$ represents the kinetic energy-enstrophy balance; and j_θ that for helicity-kinetic energy; see Fig. 1.

The conservation law (3a), which is comprised by two terms with different signs, may be interpreted as one term compensating for the other. Similar interpretations may be applied to the conservation laws (4a) and (8a) by changing the integration limits:

$$j_{\mathbf{u}}^- := j_{\mathbf{u}} - 2\mathcal{E} = \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{2}\mathrm{u}^2\right) - \int_{-\infty}^{z} \omega^2\,\mathrm{d}\xi\,, \quad \mathcal{E} := \int_{-\infty}^{0} \frac{\omega^2}{2}\,\mathrm{d}z \geq 0 \tag{12a}$$

$$j_\theta^- := j_\theta - 2|f|\mathcal{K} = \frac{f}{|f|}K_0\mathrm{u}^2\frac{\mathrm{d}\theta}{\mathrm{d}z} - |f|\int_{-\infty}^{z} \mathrm{u}^2\,\mathrm{d}\xi\,, \quad \mathcal{K} := \int_{-\infty}^{0} \frac{\mathrm{u}^2}{2}\,\mathrm{d}z \geq 0 \tag{12b}$$

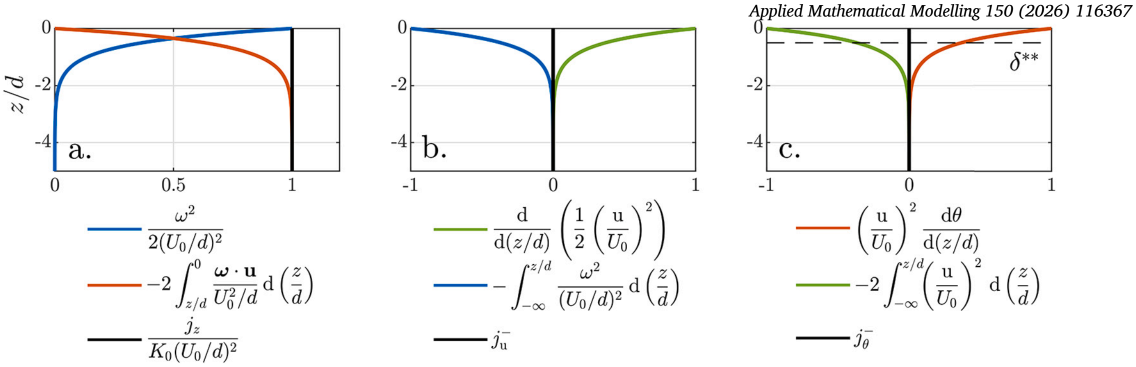

Fig. 2. Representation of each term on the right-hand side of Equations (3b) (panel a), (12a) (panel b) and (12b) (panel c) as a function of z/d. In black solid lines, it is shown that the sum of each term is vertically constant and its value coincides with the theoretical value (10) of the conserved quantities j_z, $j_{\mathbf{u}}$ and j_θ. In panel c, $\delta^{**} = d/2$ is the momentum thickness of the Ekman model.

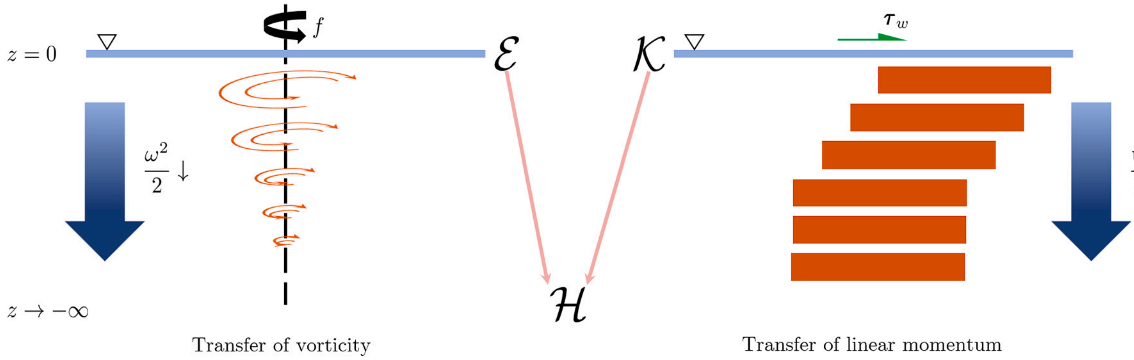

Fig. 3. Schematic drawing of how enstrophy and kinetic energy is transformed into helicity.

where $\mathcal{E}$ and $\mathcal{K}$ are the enstrophy and kinetic energy of the entire water column, respectively. The above conserved quantities $j_{\mathbf{u}}^-$ and j_θ^- are still conserved quantities, i.e., constant values along the water column, whose values are easily evaluated when $z \to -\infty$, viz. $j_{\mathbf{u}}^- = j_\theta^- = 0$. Therefore, Equations (12) quantify the net energy exchanges along the water column. Notice that, the first term of $j_{\mathbf{u}}$ (Equation (4a)) is written here as $\mathrm{d}(\mathrm{u}^2/2)/\mathrm{d}z$ for convenience.

Fig. 2 shows the (dimensionless) conserved quantities j_z (3b), $j_{\mathbf{u}}^-$ (12a), and j_θ^- (12b), and their respective terms as a function of depth for the Ekman solutions (9). Based on these conserved quantities, the following interpretation of the physical behaviour of the Ekman layer arises: On the one hand, the shear stress due to the wind generates a linear momentum transfer along z (panel b, green line, in Fig. 2). That means that there is more kinetic energy near the surface than far from it. On the other hand, the rotation of the Earth (through the Coriolis force in a mathematical model) generates turbulence in the ocean by transferring that vorticity along z (panel a, blue line, of Fig. 2). This means that there is more enstrophy near the surface than far from it. The decay of kinetic energy and enstrophy along z yields helicity density at z (panel c, orange line, of Fig. 2 with identity (22) in Appendix C) generating the well-known Ekman spiral. In a nutshell, what these conservation laws describe is that the kinetic energy and enstrophy of the water column, $\mathcal{K}$ and $\mathcal{E}$ respectively, yields helicity of the water column (panel a, orange line, in Fig. 2),

$$\mathcal{H} := \int_{-\infty}^{0} \boldsymbol{\omega}\cdot\mathbf{u}\,\mathrm{d}z\,. \tag{13}$$

A schematic representation of this dynamic is shown in Fig. 3. These conservation laws therefore complement the Classical Ekman's (1905) Theory [1] where the spiral is revisited through three integrals: $\mathcal{E}$ (12a), $\mathcal{K}$ (12b) and $\mathcal{H}$ (13).

The helicity, to some extent, characterises the Ekman spiral. $\mathcal{H}$ is positive (right-handed) if the direction of the vorticity vector is the same −on average− as the direction of the velocity vector, and negative (left-handed), if opposite. Therefore, $\boldsymbol{\omega}\cdot\mathbf{u}$ (and its analogous $\mathrm{u}^2\mathrm{d}\theta/\mathrm{d}z$ for the symmetry in phase) should have a sign that comes from the sign of the Coriolis parameter. Notice that when $\boldsymbol{\omega}\cdot\mathbf{u}$ or $\mathrm{u}^2\mathrm{d}\theta/\mathrm{d}z$ appears, f or $f/|f|$ also appears multiplying this term in the conservation laws; only the first term of Equation (4a) is not multiplied by f of $f/|f|$. This means that the conserved quantities preserve the magnitude and therefore $\mathbf{u}\cdot\mathrm{d}\mathbf{u}/\mathrm{d}z$ do not depend on the sign of the Coriolis parameter (see Appendix B).

Another result that emerges from conservation law (8a) using solutions (9) is:

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{f}{|f|}\frac{1}{d}\,, \tag{14}$$

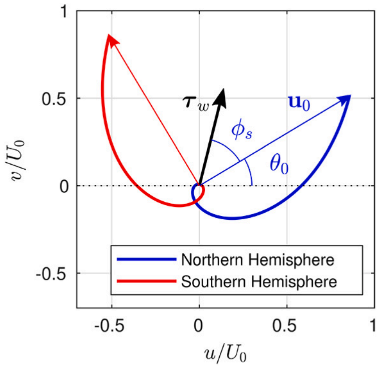

Fig. 4. Hodograph solution (15). The particular conditions for these hodographs are: The wind components are $w_x = 0.25\,\mathrm{m\,s^{-1}}$ and $w_y = 1\,\mathrm{m\,s^{-1}}$; The latitude is $37°\,1'$ N (blue), S (red); The density of water and air are, respectively, $1025\,\mathrm{kg\,m^{-3}}$ and $1.22\,\mathrm{kg\,m^{-3}}$; And the eddy viscosity K_0 is set to $0.01\,\mathrm{m^2\,s^{-1}}$.

which means that the rate of change of the phase angle is constant at any depth, due to the conservation of j_θ. This provides a theoretical basis for the Bjerknes' experiment [1] in which the phase angle grows linearly with depth from an initial phase angle at the surface (θ_0). The integration of the differential equation (14) yields an analytical expression for the hodograph:

$$\theta_0 - \theta(z) = -\frac{f}{|f|}\frac{z}{d} \Longrightarrow \frac{v}{u} = \tan\left(\theta_0 + \frac{f}{|f|}\frac{z}{d}\right), \tag{15}$$

whose typical shape within the Ekman layer is the well-known Ekman Spiral (Fig. 4). Notice that θ_0 is an arbitrary angle based on the coordinate system of the solution set (9), i.e. $\theta_0 = -f/|f|\varphi$. In contrast, the surface deflection angle, ϕ_s, has traditionally been a far more relevant angle as it is relative to the wind direction. The trigonometric relationship between both angles is

$$\theta_0 + \frac{f}{|f|}|\phi_s| = \left|\arctan\left(\frac{\tau_w^y}{\tau_w^x}\right)\right|,$$

which does not depend on the shape of the spiral (see Fig. 4).

2.2.1. Depth-integrated balance equations

Further physical interpretations can be obtained by integrating the conservation laws along the water column ($\int_{-\infty}^{0}(\mathrm{d}j_\bullet/\mathrm{d}z)\,\mathrm{d}z = 0$, so $j_\bullet(0) = j_\bullet(-\infty)$), which requires application of the boundary conditions of the BVP (1). The integration of the conservation law (3a) yields

$$\tau_w^2 = -2\rho^2 K_0 f \mathcal{H}, \tag{16a}$$

where the vorticity is assumed to tend to zero with depth, according to Fig. 2. Equation (16a) accounts for a balance between wind stress and the rotation of the velocity field throughout the whole water column. As $\tau_w^2 > 0$, the right hand-side of the equation must be positive, giving us the following sign criterion: $f > 0 \to \mathcal{H} < 0$ and $f < 0 \to \mathcal{H} > 0$. That is, in the Northern (Southern) Hemisphere the helicity must be negative (positive).

The integration of the conservation law (4a) results in

$$\cos\phi_s = \sqrt{2}\frac{d}{\mathrm{u}_0^2}\mathcal{E}. \tag{16b}$$

This important result indicates that currents at the surface deviate by a certain angle relative to the wind stress depending on the turbulence, quantified by the enstrophy $\mathcal{E}$. Equation (16b) can be used to predict the surface current angle depending of the water column structure, which controls vertical transfer of turbulence. Alternatively, making enstrophy the subject of Equation (16b) allows it to be obtained from the surface deflection angle which can be measured with drifting buoys. This is remarkable, since the enstrophy, which is a physical property of the entire water column, can be determined from surface information alone, provided that the water column behaves according to the Ekman model. In the Classical Ekman's (1905) Theory [1]: $\mathcal{E} = j_{\mathbf{u}}/2$, hence $\cos\phi_s = \sqrt{2}/2$, resulting in the well-known theoretical value of $\phi_s = \pm\pi/4$.

Finally, the integration of the conservation law (8a) results in

$$\left.\frac{\mathrm{d}\theta}{\mathrm{d}z}\right|_0 = \frac{f}{|f|}\left(\frac{2}{\mathrm{u}_0 d}\right)^2\mathcal{K}, \tag{16c}$$

where the kinetic energy controls the rate of change of the phase angle at the surface. Equation (16c) can be reformulated using the definition of momentum thickness (δ^{**}). For the surface boundary layer, the momentum thickness could be defined as the distance that, when multiplied by the square of the magnitude of the surface momentum, equals the integral of the total momentum flux, that is, $\mathrm{u}_0^2\delta^{**} = 2\mathcal{K}$. Thus, the Equation (16c) reformulated with the momentum thickness gives as a result:

$$\left.\frac{\mathrm{d}\theta}{\mathrm{d}z}\right|_0 = \frac{f}{|f|}\frac{2\delta^{**}}{d^2},$$

the greater the veering of the velocity field on the surface, the larger the momentum thickness (i.e. the larger the boundary layer).

As expected, the phase gradient on the surface has a sign criterion: $f > 0 \rightarrow \mathrm{d}\theta/\mathrm{d}z|_0 > 0$ and $f < 0 \rightarrow \mathrm{d}\theta/\mathrm{d}z|_0 < 0$. This rule of the sign is in accordance with the sign of $\mathcal{H}$, the phase gradient and the helicity density have opposite signs by identity (22) in Appendix C. According to the definition of θ in Fig. 4, if $\mathrm{d}\theta/\mathrm{d}z|_0 > 0$ ($\mathrm{d}\theta/\mathrm{d}z|_0 < 0$), the currents spin clockwise in the Northern Hemisphere (counterclockwise in the Southern Hemisphere) with depth.

3. Estimation of the eddy viscosity based on conserved quantities

Quantifying the vertical turbulent mixing is a fundamental challenge for oceanographers, as it influences the dynamics of surface currents. It is therefore customary to estimate the eddy viscosity from observational data. Such an approach can be considered a diagnostic-like method, as the determination of the eddy viscosity requires prior knowledge of the ocean's circulation.

As far as this study is concerned, a practical calculation is to perform a regression for the x- and y-momentum equations (1) to infer the Ekman eddy viscosity (K_0). However, in general, this estimate (hereafter $K_0^\dagger$) may not comply with the conservation of j_z, $j_\mathbf{u}$ and j_θ throughout the water column. Robust estimates of the Ekman eddy viscosity must unambiguously verify these conserved quantities.

According to relationship (11), let us define $K_0^{\theta\mathbf{u}}$ as a constant eddy viscosity coefficient relating j_θ and $j_\mathbf{u}$, i.e. $j_\theta = K_0^{\theta\mathbf{u}} j_\mathbf{u}$, $K_0^{\theta z}$ as a coefficient relating j_θ and j_z, i.e. $j_\theta = (2K_0^{\theta z}/|f|)^{1/2} j_z$, and $K_0^{\mathbf{u}z}$ as that relating $j_\mathbf{u}$ and j_z, i.e. $K_0^{\mathbf{u}z} j_\mathbf{u} = (2K_0^{\mathbf{u}z}/|f|)^{1/2} j_z$. Knowing that $j_z = -f\mathcal{H}$, $j_\mathbf{u} = 2\mathcal{E}$, and $j_\theta = 2|f|\mathcal{K}$ at large depths (i.e., $z \rightarrow -\infty$), these three constants eddy viscosity coefficients can be independently calculated as a function of each conserved quantity:

$$K_0^{\theta\mathbf{u}} = |f|\frac{\mathcal{K}}{\mathcal{E}},\quad K_0^{\theta z} = 2|f|\frac{\mathcal{K}^2}{\mathcal{H}^2},\quad K_0^{\mathbf{u}z} = \frac{1}{2}|f|\frac{\mathcal{H}^2}{\mathcal{E}^2}.$$

For the Ekman model solutions, these three coefficients must be equal: solution (9) provides $\mathcal{K} = \tau_w^2/(2\rho^2(2K_0|f|^3)^{1/2})$, $\mathcal{E} = \tau_w^2/(2\rho^2(2K_0^3|f|)^{1/2})$ and $\mathcal{H} = -\tau_w^2/(2\rho^2 K_0 f)$, so $K_0^{\theta\mathbf{u}} = K_0^{\theta z} = K_0^{\mathbf{u}z} = K_0$. However, these three eddy viscosity coefficients are not necessarily equal when dealing with actual observations, but the differences are bounded. This is because helicity, enstrophy and kinetic energy satisfy the Schwarz inequality, i.e. $4\mathcal{H}^2 \leq \mathcal{K}\mathcal{E}$ see Moffatt [15], and provide the following bound:

$$\frac{1}{2}K_0^{\mathbf{u}z} \leq K_0^{\theta\mathbf{u}} \leq 2K_0^{\theta z}. \tag{17}$$

The central question guiding this study is to determine, from the observations made, which of the three coefficients would give a value for the turbulent viscosity, assuming, for the moment, that the turbulent viscosity remains constant. $K_0^{\theta\mathbf{u}}$ could stand as a good estimation of the Ekman eddy viscosity (K_0) for two primary reasons. The first is that $K_0^{\theta\mathbf{u}}$ is the only coefficient that is bounded, and the second is that it is physically consistent with turbulence modelling formulations. Let us expand on this: In general, ocean general circulation models (OGCMs) widely employ turbulence models that parameterize vertical turbulent mixing, typically using differential equations for turbulent kinetic energy (k) and associated variables such as turbulent frequency (ϖ) or turbulence scale (l), which are linked to the dissipation rate of turbulent kinetic energy (ϵ) [see 2]. Consequently, the vertical eddy viscosity, named ν_T, is defined based on the specific turbulence model applied: $\nu_T \propto k^2/\epsilon$ ($k-\epsilon$ model), $\nu_T \propto k/\varpi$ where $\epsilon \propto k\varpi$ ($k-\varpi$ model) and $\nu_T \propto k^{1/2}l$ where $\epsilon \propto k^{3/2}l$ ($k-kl$ model). On the other hand, as noted by Yeung et al. [16], enstrophy and rate of dissipation of turbulent kinetic energy show a certain degree of correlation. Therefore, estimates of the Ekman eddy viscosity (K_0) from observations seem to be best estimated by $K_0^{\theta\mathbf{u}}$ (its calculation involves kinetic energy through $\mathcal{K}$ and dissipation through $\mathcal{E}$) and, moreover, accounts for the conservation of j_z, $j_\mathbf{u}$, and j_θ, provided that observational data is well-described by the Ekman model.

3.1. Discussion on the validity of the approach

The validity of $K_0^{\theta\mathbf{u}}(= |f|\mathcal{K}/\mathcal{E})$ as a robust estimate of the Ekman eddy viscosity (K_0) is tested both upon theoretical current data based on the Ekman solutions (hereafter Ekman data) and observations by Chereskin [3] (hereafter C1995 data). The computation of derivatives and integrals for the calculations of $\mathcal{K}$ and $\mathcal{E}$ from current data is performed by a fourth-order compact finite difference [17] and a fourth-order compact integration rule [18], respectively, to reduce numerical errors.

Ekman data consist of the computation of the Ekman solutions (9) along the entire water column for an arbitrary experimental condition. The wind components are $w_x = 0.43\,\mathrm{m\,s^{-1}}$ and $w_y = -7.16\,\mathrm{m\,s^{-1}}$, the latitude is $37°1'0''$ N, the density is homogeneous ($\rho = 1025\,\mathrm{kg\,m^{-3}}$) and the target eddy viscosity K_0 is set to $0.01\,\mathrm{m^2\,s^{-1}}$. The vertical domain is equally spaced in depth ($z_i = -[i-1]\Delta z\ \forall i = 1,\dots,N$, $\Delta z = 100/[N-1]$, and $N = 100$).

The computation of $K_0^{\theta\mathbf{u}}$ using the field of currents of the Ekman data yields a value of $0.00990\,\mathrm{m^2\,s^{-1}}$, which is far more accurate than the estimation by statistical fitting ($K_0^\dagger = 0.00680\,\mathrm{m^2\,s^{-1}}$). Fig. 5 (a1) illustrates the l_2-norm of the error ($:= (\sum_i (j_\bullet(z_i) - j_\bullet)^2/N)^{1/2}$) for a wide range of eddy viscosity values, where it is seen that both $K_0^{\theta\mathbf{u}}$ and $K_0^\dagger$ are within the bounded values (17). For $K_0^{\theta\mathbf{u}}$, the conserved quantities are largely conserved since the l_2-norm decays drastically for j_z, $j_\mathbf{u}$ and j_θ (see Fig. 5 (a1)), and the current profiles closely match the actual Ekman data (see Fig. 5 (a2)). For $K_0^\dagger$, no such conservation levels are achieved, and consequently, higher differences exist between the resulting and actual current profiles.

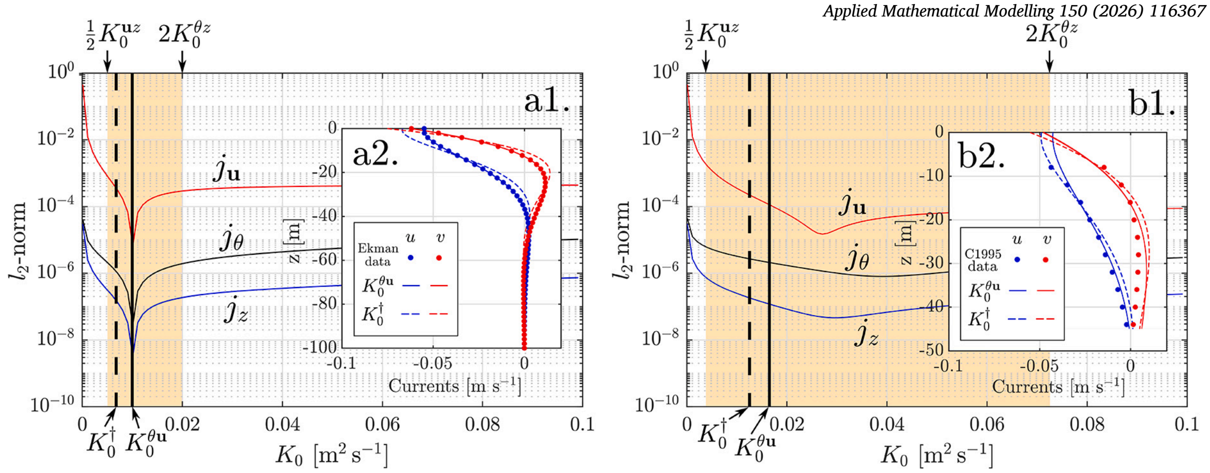

Fig. 5. Error analysis in the conserved quantities for the Ekman data (a1) and C1995 data (b1). The dashed and dotted lines indicate $K_0^\dagger$ and $K_0^{\theta\mathbf{u}}$, respectively, whereas the shaded area is the region between the bounded values $0.5K_0^{\mathbf{u}z}$ and $2K_0^{\theta z}$. The inset a2 illustrates the current profiles resulting of $K_0^\dagger$ and $K_0^{\theta\mathbf{u}}$ compared to the actual Ekman data. The inset b2 illustrates the current profiles resulting of $K_0^\dagger$ and $K_0^{\theta\mathbf{u}}$ compared to the actual [3] data.

C1995 data consist of the observations by Chereskin [3] given in her Figure 9. For these observations, the target value of the eddy viscosity is unknown, although a range of values is provided by Chereskin [3] (between $0.0274\,\mathrm{m^2\,s^{-1}}$ and $0.1011\,\mathrm{m^2\,s^{-1}}$) based on a coarse estimation of the Ekman layer depth. The computation of $K_0^{\theta\mathbf{u}}$ based on C1995 observations results in a value of $0.0165\,\mathrm{m^2\,s^{-1}}$, whereas the estimation by a crude statistical fitting $K_0^\dagger$ results in a value of $0.0125\,\mathrm{m^2\,s^{-1}}$. Fig. 5 (b1) shows that there is no unique value of the eddy viscosity that makes the error associated to j_z, $j_{\mathbf{u}}$ and j_θ to be minimal, i.e., the values of eddy viscosity that makes l_2-norm of j_z, $j_{\mathbf{u}}$ and j_θ to be minimal are different (although not very different to each other). It is evident that the assumption of vertically-uniform eddy viscosity in the model (1) does not match observations. As Chereskin [3] indicated (see Figure 10 therein), a vertically-varying eddy viscosity profile reproduce better the observed C1995 data. Even in this case (vertically-varying eddy viscosity), using $K_0^{\theta\mathbf{u}}$ as a simplification is closer to the minimum errors in the conserved quantities than $K_0^\dagger$ (see Fig. 5 b1). The resulting velocity profiles (9) due to $K_0^{\theta\mathbf{u}}$ seem to fairly agree with observations; and better than those due to $K_0^\dagger$ (see Fig. 5 b2).

The value $K_0^{\theta\mathbf{u}}$ behaves closely as an averaged value of the varying eddy viscosity profile showed by Chereskin [3] because $K_0^{\theta\mathbf{u}}$ averages the kinetic energy density and enstrophy density throughout the water column. Therefore, even if the quantities j_z, $j_{\mathbf{u}}$ and j_θ are not conserved, but $K_0^{\theta\mathbf{u}}$ remains within the range (17), this new eddy viscosity coefficient is a good estimate of the average of the actual eddy viscosity profile. Results based on the author in [6] have evaluated the applicability of the Ekman theory for wind-driven ocean currents, comparing it with the Mellor-Yamada turbulence model. Despite the fact that considering a constant vertical eddy viscosity is considered an unrealistic assumption of the Ekman model, the author has verified that using this model with a depth-averaged (constant) eddy viscosity the results are consistent with observations. Therefore, it seems feasible to use observations to estimate an average eddy viscosity based on $K_0^{\theta\mathbf{u}}$. This estimate could indeed also serve to parameterize more sophisticated oceanic models.

The approach presented in this study can be readily extended, and allows for a vertical depth-dependent eddy viscosity parametrization. It is well-established that the Ekman spiral with constant eddy viscosity is not generally observed in the ocean. Stratification due to vertical temperature-salinity gradients and surface and internal forcings (such as wind, waves and internal waves) may affect vertical mixing rates [e.g., 19]. To incorporate a depth-dependent eddy viscosity in the formulation, the parameter K_0 in the first term of the Lagrangian function (2) should simply be replaced by $K(z)$. Applying the Euler-Lagrange equations, the extended model of surface currents with varying turbulent viscosity would be obtained.

Other ocean processes can be considered explicitly, not only though any parameterization of $K(z)$. For instance, Stokes drift [e.g., 20] or K-Profile parameterizations (KPP) [21,22] could be explicitly incorporated in the formalism by introducing additional terms in the Lagrangian function, i.e., $\mathscr{L} = \mathscr{L}_{\text{Turbulence}} + \mathscr{L}_{\text{Coriolis}} + \sum_i \mathscr{L}_i$ where $\mathscr{L}_{\text{Turbulence}}$ and $\mathscr{L}_{\text{Coriolis}}$ are the first and second term of Equation (2), respectively, and $\mathscr{L}_i$ gather the extra terms. It is important to note that the introduction of $\mathscr{L}$ into the Euler-Lagrange equations produces the particular mathematical ocean model typically formulated in terms of partial differential equations.

Finally, both the approach and the results can also be easily extrapolated to the atmospheric Ekman layer: The velocity will now be the unbalanced part with the geostrophic wind (assumed constant), and the limits of integration will change by modifying the sign of the integrals.

3.2. Uniqueness of the eddy viscosity

Finally, one question that may arise when obtaining $K_0^{\theta\mathbf{u}}$ is whether the Ekman eddy viscosity (K_0) is the only one that conserves quantities j_z, $j_{\mathbf{u}}$, j_θ. We provide the following result:

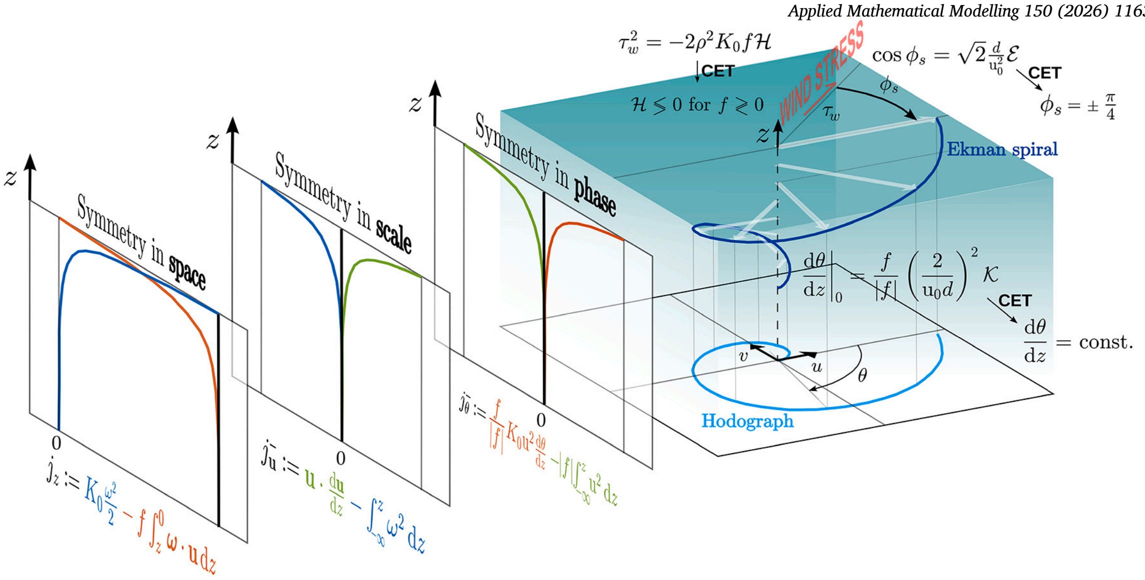

Fig. 6. Summary of the conserved quantities ($j_\bullet$) derived from space, scale and phase symmetries to the Lagrangian function (Eq. (2)), their integration in the water column and the particular features when the Classical Ekman's (1905) Theory [1] (CET) is applied.

Theorem 1. *Given a vertically uniform water column (constant eddy viscosity coefficient), there is only one value of K_0 that makes quantity $j_\bullet$ to be conserved.*

Proof. Assume that $\mathbf{u}$ and $\boldsymbol{\tau}_w$ are known, $f \neq 0$, and K_0 is the correct value associated with the set $\{\mathbf{u}, \boldsymbol{\tau}_w\}$ that $j_\bullet = \text{const.}\ \forall z \leq 0$. Assume now that the constant eddy viscosity coefficient is modified to $\overline{K}_0 = K_0 + \delta K_0$ due to a perturbed value $\delta K_0 \neq 0$, for example. The Lagrangian (2) is modified as $\overline{\mathscr{L}} = \mathscr{L} + 0.5\delta K_0 \dot{\mathrm{u}}^2$ and the Ekman model in Euler-Lagrange form becomes

$$\frac{\partial \mathscr{L}}{\partial \mathbf{u}} - \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\partial \mathscr{L}}{\partial \dot{\mathbf{u}}}\right) = \delta K_0 \ddot{\mathbf{u}}$$

where $\ddot{\mathbf{u}} \equiv \mathrm{d}^2\mathbf{u}/\mathrm{d}z^2$. Under a generic infinitesimal transformation $z' = z + \tau\varepsilon + \mathcal{O}(\varepsilon^2)$ and $\mathbf{u}' = \mathbf{u} + \boldsymbol{\eta}\varepsilon + \mathcal{O}(\varepsilon^2)$, applying the Rund-Trautman identity [12] (identity (19) in Appendix A):

$$\frac{\mathrm{d}j_\bullet}{\mathrm{d}z} = \delta K_0 \ddot{\mathbf{u}} \cdot [\boldsymbol{\eta} - \dot{\mathbf{u}}\tau]\ .$$

A solution of BVP (1) does not have to meet $\boldsymbol{\eta} - \dot{\mathbf{u}}\tau = 0$ for any $\tau = \tau(z, \mathbf{u})$ and $\boldsymbol{\eta} = \boldsymbol{\xi}(z, \mathbf{u})$, and $\ddot{\mathbf{u}} \neq 0$ since otherwise $\mathbf{u} = 0$ due BVP (1). Therefore, $\mathrm{d}j_\bullet/\mathrm{d}z \neq 0$. The existence of a perturbed value generates a source that makes $j_\bullet$ nonconservable, and because the conserved quantities (10) depend on K_0, it is not possible to get two values of $j_\bullet$ for one value of K_0. □

Therefore, a unique constant eddy viscosity coefficient can be determined from observations without the ambiguity of any fitting procedure to the Classical Ekman's (1905) Theory [1].

4. Conclusions

This research brings together the Classical Ekman's (1905) Theory [1] and the Noether's (1918) (first) Theorem [7] to theoretically reformulate into conservation laws from physical symmetries (in space, scale and phase) the wind-induced Ekman ocean boundary layer. The conservation laws, which are summarized in Fig. 6, and Table 1, state a detailed balance between enstrophy density and integral of helicity density from the surface to z (derived from space symmetry), between kinetic energy density gradient and integral of enstrophy density from the surface to z (scale symmetry), and between helicity density and integral of kinetic energy density from the surface to z (phase symmetry). The conserved quantities are not independent from each other. Wind stress at the surface generates enstrophy and kinetic energy. As depth increases, the turbulent dissipation of kinetic energy and enstrophy causes a rotation of the velocity field, i.e., it develops the well-known Ekman spiral, which is characterized by its helicity. The vertical integration of the conservation laws evidences that: helicity $\mathcal{H}$ is driven by the wind stress τ_w (relationship derived from space symmetry), the deflection angle at the surface ϕ_s can be predicted from enstrophy $\mathcal{E}$ (scale symmetry), and the rate of change of the phase angle at the surface $\mathrm{d}\theta/\mathrm{d}z$ is driven from the kinetic energy $\mathcal{K}$ (phase symmetry).

The relationship between enstrophy and surface deflection angle may have practical applications. The surface current angle can be derived from enstrophy determined from observations. However, most importantly, the enstrophy can be obtained from the surface deflection angle which can be measured with Lagrangian or drifting buoys, for example. This is remarkable, since the enstrophy,

Table 1
The application of space, scale and phase symmetries (first column) to the Lagrangian function (Eq. (2)) leads to the conserved quantities ($j_\bullet$) in CET (second column). Their theoretical (constant) values are $j_z = \tau_w^2/(2\rho^2 K_0)$, $j_{\mathbf{u}} = \tau_w^2/(\rho^2 K_0\sqrt{2K_0|f|})$, and $j_\theta = \tau_w^2/(\rho^2\sqrt{2K_0|f|})$. The vertical integration of the conservation law yields the condition $j_\bullet(0) = j_\bullet(-\infty)$, which results in the relationships shown in the third column. These relationships imply the remarkable features shown in the fourth column.

Symmetry in	Conserved quantities ($j_\bullet$)	Depth-integrated balance	Key features
space	$j_z := K_0\dfrac{\omega^2}{2} - f\int_z^0 \boldsymbol{\omega}\cdot\mathbf{u}\,\mathrm{d}z$	$\tau_w^2 = -2\rho^2 K_0 f\mathcal{H}$	$\mathcal{H} \lesseqgtr 0$ for $f \gtreqless 0$
scale	$j_{\mathbf{u}} := \mathbf{u}\cdot\dfrac{\mathrm{d}\mathbf{u}}{\mathrm{d}z} + \int_z^0 \omega^2\,\mathrm{d}z$	$\cos\phi_s = \sqrt{2}\dfrac{d}{\mathrm{u}_0^2}\mathcal{E}$	$\phi_s = \pm\dfrac{\pi}{4}$
phase	$j_\theta := \dfrac{f}{\lvert f\rvert}K_0\mathrm{u}^2\dfrac{\mathrm{d}\theta}{\mathrm{d}z} + \lvert f\rvert\int_z^0 \mathrm{u}^2\,\mathrm{d}z$	$\left.\dfrac{\mathrm{d}\theta}{\mathrm{d}z}\right\rvert_0 = \dfrac{f}{\lvert f\rvert}\left(\dfrac{2}{\mathrm{u}_0 d}\right)^2\mathcal{K}$	$\dfrac{\mathrm{d}\theta}{\mathrm{d}z} = \text{const. }\forall z$

which is a physical property of the entire water column, can be determined from surface information alone, provided that the water column behaves according to the Ekman model. Another interesting finding is that the conservation of j_θ theoretically explains the Bjerknes' experiment [1] (phase angle grows linearly with depth).

The constant eddy viscosity coefficient (Ekman eddy viscosity) that parameterizes vertical turbulent mixing in the Classical Ekman's (1905) Theory [1] is found to be proportional to the ratio between kinetic energy and enstrophy. Indeed, there is a unique value of the Ekman eddy viscosity coefficient that complies with the conserved quantities. A robust and objective estimate of Ekman eddy viscosity derived from the conserved quantities in the Ekman boundary layer is proven to be easily quantified from observations of horizontal current profiles. The common approach in which Ekman eddy viscosity is obtained from a crude statistical fit or optimization based on the momentum equations, may not comply with the conservation laws in the ocean boundary layer. The novel theoretical conserved quantities presented in this study contribute to better constrain eddy viscosity estimates from observations, thereby improving the modelling of wind-induced ocean circulation.

CRediT authorship contribution statement

Víctor J. Llorente: Writing – original draft, Validation, Supervision, Software, Project administration, Methodology, Investigation, Formal analysis, Conceptualization. **Enrique M. Padilla:** Writing – original draft, Visualization, Validation, Supervision, Investigation, Formal analysis. **Manuel Díez-Minguito:** Writing – original draft, Visualization, Validation, Investigation, Funding acquisition, Formal analysis. **Arnoldo Valle-Levinson:** Writing – original draft, Visualization, Supervision, Funding acquisition.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Derivation of Noether's conservation laws for the classical Ekman's (1905) theory

The proposed Lagrangian function $\mathscr{L}$ for the Classical Ekman's (1905) Theory [1] is formulated as follows:

$$\mathscr{L}(z,\dot{u},\dot{v}) = \frac{1}{2}K_0\left[\dot{u}^2+\dot{v}^2\right] + f\left[\dot{v}m_x(z) - \dot{u}m_y(z)\right], \tag{18}$$

which corresponds with Equation (2) written in compact form: $\dot{\mathrm{u}}^2 = \dot{u}^2+\dot{v}^2$ and $\dot{\mathbf{u}}\cdot\mathbf{m}^\perp(z) = \dot{v}m_x(z) - \dot{u}m_y(z)$ where $m_{\{x,y\}}(z) = \int_z^0\{u,v\}\,\mathrm{d}\xi$. It is asserted that if the action $\mathcal{S} = \int\mathscr{L}\,\mathrm{d}z$ of the Ekman model has an extremum, then the components of the vector $\mathbf{u} = (u,v)$ are solutions of the Euler-Lagrange equations. It is straightforward to calculate that by inserting the Lagrangian (18) into the Euler-Lagrangian equations, that is,

$$\frac{\partial\mathscr{L}}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\partial\mathscr{L}}{\partial\dot{u}}\right) = 0,$$
$$\frac{\partial\mathscr{L}}{\partial v} - \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\partial\mathscr{L}}{\partial\dot{v}}\right) = 0,$$

results in the differential equations of the Ekman model (1). Given the Lagrangian (18), the canonical momentum $\mathbf{p} = (p_u, p_v)$ results in

$$p_u(z) = \frac{\partial \mathscr{L}}{\partial \dot{u}} = K_0 \dot{u} - f m_y(z),$$
$$p_v(z) = \frac{\partial \mathscr{L}}{\partial \dot{v}} = K_0 \dot{v} + f m_x(z),$$

and the Hamiltonian as

$$\mathscr{H}(z, p_u, p_v) = p_u \dot{u} + p_v \dot{v} - \mathscr{L} = \frac{(p_u + f m_y(z))^2 + (p_v - f m_x(z))^2}{2K_0},$$

where we can conclude $\mathscr{H} > 0$.

On the other hand, the action $\mathscr{S}$ is invariant under infinitesimal transformation

$$\begin{cases} z' = z + \tau(z, u, v)\varepsilon + \mathcal{O}(\varepsilon^2) \\ u' = u + \eta_u(z, u, v)\varepsilon + \mathcal{O}(\varepsilon^2) \\ v' = v + \eta_v(z, u, v)\varepsilon + \mathcal{O}(\varepsilon^2) \end{cases}$$

for $\varepsilon \in \mathbb{R}$ with infinitesimal generators τ, η_u and η_v if and only if the following identity [12] holds:

$$\frac{\partial \mathcal{L}}{\partial u}\eta_u + \frac{\partial \mathcal{L}}{\partial v}\eta_v + p_u \frac{\mathrm{d}\eta_u}{\mathrm{d}z} + p_v \frac{\mathrm{d}\eta_v}{\mathrm{d}z} + \frac{\partial \mathcal{L}}{\partial z}\tau - \mathcal{H}\frac{\mathrm{d}\tau}{\mathrm{d}z} - \frac{\mathrm{d}\Phi}{\mathrm{d}z} = 0, \tag{19}$$

for some function $\Phi = \Phi(z)$. This function is a non-homogeneous term appearing in the definition of invariance, i.e. $(\mathrm{d}z'/\mathrm{d}z)\mathscr{L}' - \mathscr{L} = (\mathrm{d}\Phi/\mathrm{d}z)\varepsilon + \mathcal{O}(\varepsilon^2)$, with which one can force the action or the Lagrangian to be invariant under an infinitesimal transformation. This kind of relaxation definition is called divergence invariance/symmetry; see, for example, Olver [9]. The previous identity can be rewritten, according to Logan [12], as follows:

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(p_u\eta_u + p_v\eta_v - \mathscr{H}\tau - \Phi\right) = -\left(\frac{\partial \mathscr{L}}{\partial u} - \dot{p}_u\right)(\eta_u - \dot{u}\tau) - \left(\frac{\partial \mathscr{L}}{\partial v} - \dot{p}_v\right)(\eta_v - \dot{v}\tau).$$

Finally, as the action $\mathscr{S}$ has an extremum, the right-hand side of the previous equation becomes zero, yielding the conservation law of Noether's (first) Theorem [7].

In the following, we develop the conservation law for each infinitesimal transformation considered in Section 2.1.

A.1. Symmetry in space

An infinitesimal translation in space means

$$\begin{cases} z' = z + \varepsilon \\ u' = u \\ v' = v \end{cases}$$

for $\varepsilon \in \mathbb{R}$. Their infinitesimal generators of the transformation are $\tau = 1$ and $\eta_u = \eta_v = 0$. The condition (19) for $\mathscr{L}$ to be invariant under the space transformation reads:

$$f\left[\dot{u}v - u\dot{v}\right] - \frac{\mathrm{d}\Phi}{\mathrm{d}z} = 0 \Longrightarrow \Phi(z) = -f\int_z^0 (\dot{u}v - u\dot{v})\,\mathrm{d}\xi.$$

The term in parentheses can be expressed in vector notation as the dot product of the vorticity vector and the velocity vector, denoted by $\boldsymbol{\omega} \cdot \mathbf{u}$. Invoking the Noether Theorem, the conservation law associated with the Euler-Lagrange form is

$$\frac{\mathrm{d}}{\mathrm{d}z}(-\mathcal{H} - \Phi) = \frac{\mathrm{d}}{\mathrm{d}z}\underbrace{\left(\frac{1}{2}K_0\left[\dot{u}^2 + \dot{v}^2\right] - f\int_z^0 (\dot{u}v - u\dot{v})\,\mathrm{d}\xi\right)}_{=j_z} = 0, \tag{20}$$

which corresponds in vector notation to Equation (3a). The term $K_0(\dot{u}^2 + \dot{v}^2)/2$ is expressed in vector notation as $K_0\omega^2/2$. It is important to note that the Ekman equation can be represented in the following manner:

$$-fv = K_0\frac{\mathrm{d}^2u}{\mathrm{d}z^2} = K_0\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{d}u}{\mathrm{d}z}\right), \quad fu = K_0\frac{\mathrm{d}^2v}{\mathrm{d}z^2} = K_0\frac{\mathrm{d}}{\mathrm{d}z}\left(-\frac{\mathrm{d}v}{\mathrm{d}z}\right) \Longrightarrow -f\mathbf{u} = K_0\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}z}$$

Multiplying the given equation by the vorticity vector yields

$$-f\boldsymbol{\omega}\cdot\mathbf{u} = K_0\boldsymbol{\omega}\cdot\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}z} = K_0\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\omega^2}{2}\right),$$

which corresponds to Equation (20). Consequently, the expression $\dot{u}^2 + \dot{v}^2$ denotes the squared magnitude of the vorticity.

A.2. Symmetry in scale

The second continuous transformation involves scaling the velocity field, represented by

$$\begin{cases} z' = z \\ u' = u + u\varepsilon \\ v' = v + v\varepsilon \end{cases}$$

for $\varepsilon \in \mathbb{R}$. Consequently, we have $\tau = 0$, $\eta_u = u$, and $\eta_v = v$. The requirement for $\mathscr{L}$ to be invariant under the scaling transformation, as stated in condition (19), is:

$$p_u \dot{u} + p_v \dot{v} - \frac{\mathrm{d}\Phi}{\mathrm{d}z} = 0 \Longrightarrow \Phi(z) = -K_0 \int_z^0 \left(\dot{u}^2 + \dot{v}^2\right) \mathrm{d}\xi + f \int_z^0 \left(\dot{u} m_y - \dot{v} m_x\right) \mathrm{d}\xi \,.$$

The method of integration by parts is applied to simplify the second integral expression:

$$\int_z^0 \left(\dot{u} m_y - \dot{v} m_x\right) \mathrm{d}\xi = \left(u m_y - v m_x\right)\Big|_z^0 - \int_z^0 \left(u \dot{m}_y - v \dot{m}_x\right) \mathrm{d}\xi = -\left(u m_y - v m_x\right) \,.$$

The conservation law derived from Noether's theorem, corresponding to the Euler-Lagrange equation under a scaling transformation, is expressed as follows,

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(p_u u + p_v v - \Phi\right) = \frac{\mathrm{d}}{\mathrm{d}z}\underbrace{\left(K_0\left[u\dot{u} + v\dot{v} + \int_z^0 \left(\dot{u}^2 + \dot{v}^2\right) \mathrm{d}\xi\right]\right)}_{=K_0 j_{\mathbf{u}}} = 0 \,,$$

which corresponds in vector notation to Equation (4a).

A.3. Symmetry in phase

The next study of the invariance of $\mathscr{L}$ to an infinitesimal transformation would be that of a continuous rotation. However, given that the group of complex numbers $\exp(\mathrm{i}x) = \cos x + \mathrm{i}\sin x$ of absolute value 1 is isomorphic to the special orthogonal group SO(2),

$$\begin{pmatrix} \cos(x) & -\sin(x) \\ \sin(x) & \cos(x) \end{pmatrix} ,$$

and given that the Ekman equation can be expressed using a complex variable, then the transformation is written as

$$\begin{cases} z' = z \\ \psi' = \psi \exp(\mathrm{i}\lambda\varepsilon) \\ \psi^{*\prime} = \psi^* \exp(-\mathrm{i}\lambda\varepsilon) \end{cases}$$

where λ denotes an arbitrarily coupling constant, whose value will be determined later (Appendix B). ψ and ψ^* represent the complex velocity and its complex conjugate, respectively, as previously explained in the article. Therefore, the invariance of $\mathscr{L}$ to infinitesimal angular displacements (or phase shifts) is studied.

The Lagrangian $\mathscr{L}$ must be real-valued for the equations of motion to be well-defined and physically meaningful. Consequently, in Equation (5), the expression $\mathscr{L}$ should include the complex conjugate velocity.

Using Taylor's series expansion, $\exp(x) = 1 + x + \mathcal{O}(x^2)$, the infinitesimal generators of the transformation are $\tau = 0$, $\eta = \mathrm{i}g\psi$, and $\eta^* = -\mathrm{i}g\psi^*$. Condition (19) specifies the criteria for $\mathscr{L}$, as outlined in Equation (5), to remain invariant under phase transformation:

$$\mathrm{i}\lambda\left[\dot{\psi} p - \dot{\psi}^* p^*\right] - \frac{\mathrm{d}\Phi}{\mathrm{d}z} = 0 \Longrightarrow \Phi(z) = -\frac{f\lambda}{2} \int_z^0 \left[\chi^*(\xi)\dot{\psi} + \chi(\xi)\dot{\psi}^*\right] \mathrm{d}\xi \,.$$

Applying the integration by parts method, the result is

$$\Phi(z) = \frac{f\lambda}{2}\left[\chi^*(z)\psi + \chi(z)\psi^*\right] - f\lambda \int_z^0 \psi^*\psi \,\mathrm{d}\xi \,.$$

According to Noether's theorem, the conservation law resulting from the Euler-Lagrange equation subject to phase transformation can be written as follows:

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(p\eta + p^*\eta^* - \Phi\right) = \frac{\mathrm{d}}{\mathrm{d}z}\underbrace{\left(\frac{\mathrm{i}}{2}K_0\lambda\left[\psi\dot{\psi}^* - \psi^*\dot{\psi}\right] + f\lambda\int_z^0 \psi\psi^*\,\mathrm{d}\xi\right)}_{=j_\theta} = 0\,,$$

which corresponds to Equation (7). Although this conserved quantity is similar to the previous ones, it is not unique as it relies on a parameter, λ. What does this dependence on λ entail? Does λ have any physical significance? The conserved quantities must satisfy another invariance which is discussed in Appendix B.

Appendix B. Invariance concerning the sign of Coriolis

The conserved quantities should be invariant to the sign of the Coriolis force. If one takes observational data and calculates these conserved quantities, they should be the same regardless of the hemisphere in which one has made these measurements. Therefore, if in the Northern Hemisphere ($f > 0$) the Ekman equations are

$$-fv = K_0\frac{\mathrm{d}^2u}{\mathrm{d}z^2}\,,$$
$$fu = K_0\frac{\mathrm{d}^2v}{\mathrm{d}z^2}\,,$$

the transformation $u' = u$ and $v' = -v$ when applied in the Southern Hemisphere ($f < 0$) would result in the same equations,

$$-|f|v' = K_0\frac{\mathrm{d}^2u'}{\mathrm{d}z^2}\,,$$
$$|f|u' = K_0\frac{\mathrm{d}^2v'}{\mathrm{d}z^2}\,.$$

Hereinafter, the prime symbol refers to the Southern Hemisphere, while absence of the prime indicates the Northern Hemisphere.

Analysing how each term of each conserved quantity changes under the previous transformation: $\omega^2 = \omega'^2$, $\mathrm{u}^2 = \mathrm{u}'^2$, $\mathbf{u}\cdot\mathrm{d}\mathbf{u}/\mathrm{d}z = \mathbf{u}'\cdot\mathrm{d}\mathbf{u}'/\mathrm{d}z$, $\boldsymbol{\omega}\cdot\mathbf{u} = -\boldsymbol{\omega}'\cdot\mathbf{u}'$ and $\mathrm{u}^2\mathrm{d}\theta/\mathrm{d}z = -\mathrm{u}'^2\mathrm{d}\theta'/\mathrm{d}z$; the result is $j_z = j'_z$, $j_{\mathbf{u}} = j'_{\mathbf{u}}$ and $j_\theta = -j'_\theta$. The only quantity that changes sign is j_θ, that is, when $\mathscr{L}$ is invariant to infinitesimal phase shifts. If j_θ is to be invariant to sign(f), then $\lambda = f/|f|$. In simpler terms, if in the Northern Hemisphere the phase is shifted to the right, in the Southern Hemisphere the phase must be shifted to the left for j_θ to remain invariant.

Appendix C. Helicity in the Ekman layer

For a fluid flow in $\mathbb{R}^3$, the relative motion near a point states that

$$\mathrm{d}\mathbf{u} = \mathbf{E}\mathrm{d}\mathbf{x} + \frac{1}{2}\boldsymbol{\omega}\times\mathrm{d}\mathbf{x}\,,$$

where $\mathbf{E} = 0.5[\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}}]$ is the strain-rate tensor. Operating the cross product on both sides of the equality,

$$\mathbf{u}\times\mathrm{d}\mathbf{u} = \mathbf{u}\times(\mathbf{E}\mathrm{d}\mathbf{x}) + \frac{1}{2}\left[(\mathbf{u}\cdot\mathrm{d}\mathbf{x})\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\mathbf{u})\mathrm{d}\mathbf{x}\right]\,. \tag{21}$$

Particularizing the variables to the Ekman layer, i.e. $\mathbf{u} = (u, v, 0)^{\mathrm{T}}$, $\boldsymbol{\omega} = (-\dot{v}, \dot{u}, 0)^{\mathrm{T}}$, $\mathrm{d}\mathbf{u} = (\mathrm{d}u, \mathrm{d}v, 0)^{\mathrm{T}}$, $\mathrm{d}\mathbf{x} = (0, 0, \mathrm{d}z)^{\mathrm{T}}$ and

$$\mathbf{E} = \frac{1}{2}\begin{pmatrix} 0 & 0 & \dot{u} \\ 0 & 0 & \dot{v} \\ \dot{u} & \dot{v} & 0 \end{pmatrix},$$

the equality (21) writes

$$u\mathrm{d}v - v\mathrm{d}u = -(\boldsymbol{\omega}\cdot\mathbf{u})\mathrm{d}z\,.$$

Finding the differential of $\tan(\theta) = v/u$ and substituting it in the left-hand side of the previous equality yields the following identity:

$$\mathrm{u}^2\mathrm{d}\theta = -\boldsymbol{\omega}\cdot\mathbf{u}\,\mathrm{d}z\,. \tag{22}$$

The existence of $\boldsymbol{\omega}\cdot\mathbf{u}$ spins the velocity field with depth: $\theta(z) - \theta(z + \mathrm{d}z) \neq 0$. In the way the identity (22) is written, θ can be regarded as the veering angle (Equation (8) in Rosas-Villegas et al. [23]).

Data availability

Data will be made available on request.

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