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Spectrum of invertible weighted composition operators on the unit disk ☆

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ABSTRACT

The spectrum of invertible weighted composition operators uC_φ acting on classical Banach spaces of holomorphic functions in the unit disk $\mathbb{D}$ has been studied intensively over the years. Complete descriptions of that spectrum have been given in the elliptic or parabolic cases, that is, for φ either elliptic or parabolic, but only partial results have been obtained in the remaining case, that is, for hyperbolic φ. In this paper, we give the spectrum and the essential spectrum of uC_φ for hyperbolic φ. Our results answer in the positive several conjectures posed by different authors.

In order to deal with the above questions, we present new techniques which involve the embedding of the weight u into a cocycle $(u_t)_{t\in\mathbb{R}}$ associated to an hyperbolic flow $(\varphi_t)_{t\in\mathbb{R}}$. We also provide information about the range spaces and null spaces of $\lambda - uC_\varphi$ for λ lying in the interior of $\sigma(uC_\varphi)$.

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1. Introduction

Let $\mathcal{O}(\mathbb{D})$ denote the Fréchet algebra of holomorphic functions on the complex unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Each self-analytic mapping $\varphi : \mathbb{D} \to \mathbb{D}$ (in short, $\varphi \in Mor(\mathbb{D})$) induces a *composition operator* C_φ on $\mathcal{O}(\mathbb{D})$ given by $C_\varphi f = f \circ \varphi$. These operators have been studied intensively, see the monographs [11,28] for two excellent reviews on composition operators on classical spaces of analytic functions. Generalizations of composition operators are the so-called *weighted composition operators* uC_φ, where $u \in \mathcal{O}(\mathbb{D})$, given by

$$(uC_\varphi f)(z) = u(z)(f \circ \varphi)(z), \qquad z \in \mathbb{D},\ f \in \mathcal{O}(\mathbb{D}).$$

Weighted composition operators arise naturally in many situations in mathematical analysis. Just to mention a few examples, every surjective isometry on Hardy spaces $H^p(\mathbb{D})$, $1 < p < \infty$, $p \neq 2$, and on weighted Bergman spaces $\mathcal{A}^p_\sigma(\mathbb{D})$, $\sigma > -1$, $0 < p < \infty$, $p \neq 2$, is given by a weighted composition operator, see [17,25]. They also appear in the study of adjoints of (unweighted) composition operators and in the study of commutants of multiplication operators. They also play a role in the theory of C_0-semigroups [26,30].

The study of the connection between operator theoretic properties of uC_φ and function theoretic properties of u and φ has been a subject of great interest over the years; see, e.g., [6,8,18,24,29]. Here, we are interested in the spectral properties of *invertible* weighted composition operators uC_φ. The full description of the spectrum of such operators uC_φ acting on a Banach space of holomorphic functions on $\mathbb{D}$ has been an open question since the seminal paper [23] where the author obtains the spectra of arbitrary invertible weighted composition operators acting on the disk algebra $\mathfrak{A}(\mathbb{D})$. It is remarkable that $\sigma(uC_\varphi)$ (as an operator on $\mathfrak{A}(\mathbb{D})$) only depends on the nature of the fixed point(s) of φ and the evaluation of u at the fixed point(s) of φ. Recall that an automorphism $\varphi : \mathbb{D} \to \mathbb{D}$ is said to be: elliptic if φ has a fixed point in $\mathbb{D}$ (which is unique); parabolic if the continuous extension of φ to $\overline{\mathbb{D}}$ has a unique fixed point on the torus $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$; and hyperbolic if the continuous extension of φ to $\overline{\mathbb{D}}$ has two distinct fixed points in $\mathbb{T}$.

Motivated by [23], many researchers studied the spectra of operators uC_φ on a fairly wide list of classical Banach spaces of analytic functions, obtaining the complete description of $\sigma(uC_\varphi)$ if the automorphism φ is either elliptic or parabolic and under the assumption that u can be continuously extended to the fixed point of φ, see [10,15,19,20,22]. However, there are only partial results for the spectrum of uC_φ in the case φ is hyperbolic. For instance, assuming u can be continuously extended to $\mathbb{D} \cup \{a, b\}$, where a, b are the attractive fixed point and the repulsive fixed point of φ respectively, it is known that

$$\sigma(uC_\varphi) \subseteq \left\{ \lambda \in \mathbb{C} \,:\, \min\left\{ \frac{|u(a)|}{\varphi'(a)^\gamma}, \frac{|u(b)|}{\varphi'(b)^\gamma} \right\} \le |\lambda| \le \max\left\{ \frac{|u(a)|}{\varphi'(a)^\gamma}, \frac{|u(b)|}{\varphi'(b)^\gamma} \right\} \right\}, \qquad (1.1)$$

where $\gamma \geq 0$ is a parameter which only depends on the Banach space X of analytic functions on which uC_φ is acting. In general, the spectrum of uC_φ (for φ hyperbolic) has been obtained only in special cases. More precisely, it is known that the inclusion "$\subseteq$" of (1.1) can be replaced with equality "$=$" if

- X is either the classical Hardy space $H^2(\mathbb{D})$ or the Hilbertian weighted Bergman space $\mathcal{A}^2_\sigma(\mathbb{D})$, see [19, Th. 3.11 & Cor. 3.12]; or if
- X is the Hardy space $H^p(\mathbb{D})$, the weighted Bergman space $\mathcal{A}^p_\sigma(\mathbb{D})$ or the Korenblum class $\mathcal{K}^{-\rho}(\mathbb{D})$, and $\frac{|u(b)|}{\varphi'(b)^\gamma} \leq \frac{|u(a)|}{\varphi'(a)^\gamma}$, see [22, Sect. 4].

In this paper, we obtain the spectrum $\sigma(uC_\varphi)$ in all the remaining cases, that is, for φ hyperbolic, and for every Banach space of holomorphic functions X satisfying the axioms listed in Section 2, see Theorem 8.1. In particular, our results are valid for Hardy spaces, weighted Bergman spaces, weighted Dirichlet spaces, (little) Korenblum classes and the (little) Bloch space, and we answer in the positive the conjectures on $\sigma(uC_\varphi)$ posed in [10,15,22]. Moreover, we give a detailed description of the range spaces and null spaces of $\lambda - uC_\varphi$ for λ lying in the interior of $\sigma(uC_\varphi)$, which enables us to prove that the essential spectrum of uC_φ, given by

$$\sigma_{ess}(uC_\varphi) = \{\lambda \in \mathbb{C} \, : \, \lambda - uC_\varphi \text{ has either infinite-dimensional kernel or infinite-codimensional range}\},$$

coincides with the spectrum of uC_φ, i.e.,

$$\begin{aligned}\sigma(uC_\varphi) &= \sigma_{ess}(uC_\varphi) \\ &= \left\{\lambda \in \mathbb{C} \, : \, \min\left\{\frac{|u(a)|}{\varphi'(a)^\gamma}, \frac{|u(b)|}{\varphi'(b)^\gamma}\right\} \leq |\lambda| \leq \max\left\{\frac{|u(a)|}{\varphi'(a)^\gamma}, \frac{|u(b)|}{\varphi'(b)^\gamma}\right\}\right\}.\end{aligned}$$

On the one hand, the techniques and proofs given in the papers [2,10,15,22] served as inspiration to obtain our results for the case $\frac{|u|(b)}{\varphi'(b)^\gamma} \leq \frac{|u|(a)}{\varphi'(a)^\gamma}$. More precisely, we make use of a family of weighted Banach spaces $X_{\mu,\nu}$ (see Section 4) to simplify, refine and unify their proofs, and to improve their results (we should point out that the work [19] relies on reproducing kernel Hilbert space theory, which does not work in our setting). On the other hand, in order to deal with the case $\frac{|u|(a)}{\varphi'(a)^\gamma} < \frac{|u|(b)}{\varphi'(b)^\gamma}$, we show first that uC_φ can be embedded into a one-parameter group $(u_t C_{\varphi_t})_{t\in\mathbb{R}}$, i.e. $uC_\varphi = u_1 C_{\varphi_1}$, where $(\varphi_t)_{t\in\mathbb{R}}$ is an hyperbolic flow and $(u_t)_{t\in\mathbb{R}}$ is a cocycle for $(\varphi_t)_{t\in\mathbb{R}}$, see Subsection 6.1 for more details. After that, we represent $(u_t)_{t\in\mathbb{R}}$ as a coboundary, that is,

$$u_t = \frac{\omega \circ \varphi_t}{\omega}, \qquad t \in \mathbb{R},$$

where ω is a non-vanishing holomorphic function on $\mathbb{D}$, and we use such a function ω to provide the spectrum of uC_φ. This method is heavily inspired by the results given in [1] for C_0-groups of weighted composition operators of hyperbolic symbol.

The paper is organized as follows. We present in Section 2 the axiomatic properties of the Banach spaces of holomorphic functions on $\mathbb{D}$ that are dealt with through the paper, and give several examples of classical Banach spaces satisfying such axioms. We adapt in Section 3 some well-known results regarding the spectral radius of uC_φ to our setting. In Section 4 we address the case when $\frac{|u|(b)}{\varphi'(b)^\gamma} < \frac{|u|(a)}{\varphi'(a)^\gamma}$. Section 5 is devoted to the case $\frac{|u|(b)}{\varphi'(b)^\gamma} = \frac{|u|(a)}{\varphi'(a)^\gamma}$. In Section 6 we cover the remaining case $\frac{|u|(a)}{\varphi'(a)^\gamma} < \frac{|u|(b)}{\varphi'(b)^\gamma}$. We also give in Subsection 6.1 a characterization of the embeddability of a weight u into a (semi)cocycle $(u_t)_{t\in\mathbb{R}}$ for a (semi)flow $(\varphi_t)_{t\in\mathbb{R}}$ which we find interesting on its own. Next, we give in Section 7 sufficient conditions for a family of eigenfunctions of uC_φ to belong to the function Banach space under consideration. In particular, we answer in the positive a conjecture posed in [1, Remark 7.3(1)]. We collect our main results in Section 8. Lastly, we provide in Appendix A some auxiliary results.

Before moving on to the next section, let us recall some basic facts about hyperbolic automorphisms of $\mathbb{D}$ which will be used throughout this paper. Recall again that a biholomorphic mapping $\varphi : \mathbb{D} \to \mathbb{D}$ (hence a Möbius transform) is said to be *hyperbolic* if it has no fixed points on $\mathbb{D}$ and (its continuous extension to $\overline{\mathbb{D}}$) has exactly two distinct fixed points a (attractive) and b (repulsive) in $\mathbb{T}$. Hence, $\lim_{n\to\infty} \varphi_n(z) = a$ and $\lim_{n\to-\infty} \varphi_n(z) = b$ for each $z \in \mathbb{D}$ where $\varphi_n := \underbrace{\varphi \circ \ldots \circ \varphi}_{n \text{ times}}$ and $\varphi_{-n} := (\varphi^{-1})_n$ for $n \in \mathbb{N}$. It is well known that φ is determined by a, b and $\varphi'(a)$. Namely,

$$\varphi(z) = \frac{(b\varphi'(a) - a)z + ab(1 - \varphi'(a))}{(\varphi'(a) - 1)z + b - a\varphi'(a)}, \qquad z \in \mathbb{D}. \tag{1.2}$$

Even more, φ is holomorphic in a disk (centered at the origin) of radius strictly greater than 1 and $\varphi'(a) \in (0, 1), \varphi'(b) \in (1, \infty)$ with $\varphi'(a)\varphi'(b) = 1$.

2. Banach spaces of holomorphic functions

In this section, we put up the setting where to work by introducing a number of conditions on a non-zero function Banach space $X \hookrightarrow \mathcal{O}(\mathbb{D})$. We also show that most classical function spaces satisfy such conditions. Some of these conditions concern *multipliers*. We denote by $Mul(X)$ the algebra of multipliers of X, that is, a function g belongs to $Mul(X)$ if the pointwise product operator $f \mapsto g \cdot f$ defines a bounded operator on X. We also denote by $\mathcal{L}(X)$ the Banach algebra of linear bounded operators on X, and we use $Aut(\mathbb{D})$ to denote the group of (holomorphic) automorphisms of $\mathbb{D}$ onto itself. In addition, for an open set $U \subseteq \mathbb{C}$, we let $\mathcal{O}(U)$ stand for the algebra of holomorphic functions with domain U.

The axioms we consider here on the function Banach space X are the following ones:

(**P1**) For each open set $U \subset \mathbb{C}$ with $\overline{\mathbb{D}} \subset U$, one has $\mathcal{O}(U) \subseteq Mul(X)$.

(**P2**) For each $\delta \in \mathbb{C}$ with $\mathfrak{Re}\,\delta > 0$ and $c \in \mathbb{T}$, the function $z \mapsto (c - z)^\delta$ belongs to $Mul(X)$.

(**P3**) Let φ be a hyperbolic automorphism with fixed points a (attractive), b (repulsive) in $\mathbb{T}$. For each $u \in Mul(X)$ for which $|u|$ has continuous extension to $\mathbb{D} \cup \{a, b\}$, one has $u_n := \prod_{j=0}^{n-1} u \circ \varphi_j \in Mul(X)$ for all $n \in \mathbb{N}$, and

$$\limsup_{n\to\infty} \|u_n\|_{Mul(X)}^{1/n} \leq \max\{|u|(a), |u|(b)\}.$$

(**P4**) There exists $\gamma \geq 0$ such that, for all $\varphi \in Aut(\mathbb{D})$, one has $(\varphi')^\gamma C_\varphi \in \mathcal{L}(X)$. Also, for each $\varepsilon > 0$,

$$\sup_{\varphi\in Aut(\mathbb{D})} (1 - |\varphi(0)|^2)^\varepsilon \, \|(\varphi')^\gamma C_\varphi\|_{\mathcal{L}(X)} < \infty.$$

For the sake of the briefness of our statements, we are using the following terminology to denote such spaces X.

Definition 2.1. Let $\gamma \geq 0$, and let $X \hookrightarrow \mathcal{O}(\mathbb{D})$ be a non-zero Banach space of holomorphic functions. We say that X is a γ-space if the pair (X, γ) satisfies the properties (**P1**)-(**P4**), where γ acts as the parameter associated to (**P4**).

Below we give some direct consequences of the above properties. From now on, we use the symbol "$\ldots \lesssim \ldots$" as an abbreviation for "*there exists a constant* $C > 0$ *such that* $\ldots \leq C \ldots$".

Remark 2.2. Korenblum classes $\mathcal{K}^{-\rho}(\mathbb{D})$ (see their definition in Subsect. 2.1) enjoy the property that, for each $\gamma \geq 0$ and $\varepsilon > 0$, $\mathcal{K}^{-\gamma-\varepsilon}(\mathbb{D})$ contains every Banach space X satisfying (**P4**). In effect, one has

$$\begin{aligned}
\sup_{z\in\mathbb{D}}(1 - |z|^2)^{\gamma+\varepsilon}|f(z)| &= \sup_{\varphi\in Aut(\mathbb{D})} (1 - |\varphi(0)|^2)^{\gamma+\varepsilon}|f(\varphi(0))| \\
&= \sup_{\varphi\in Aut(\mathbb{D})} (1 - |\varphi(0)|^2)^{\varepsilon}|((\varphi')^\gamma C_\varphi f)(0)| \\
&\lesssim \sup_{\varphi\in Aut(\mathbb{D})} (1 - |\varphi(0)|^2)^{\varepsilon}\|(\varphi')^\gamma C_\varphi f\|_X \lesssim \|f\|_X, \qquad f \in X,
\end{aligned}$$

where Schwarz-Pick's Lemma has been used in the second equality. This bound obviously implies $X \hookrightarrow \mathcal{K}^{-\gamma-\varepsilon}(\mathbb{D})$ as claimed.

Remark 2.3. Properties (**P1**) and (**P4**) imply that the composition operator C_φ is an isomorphism on X for each $\varphi \in Aut(\mathbb{D})$. Indeed, since φ' is holomorphic and non-vanishing in a disk of radius strictly greater than 1, one has $(\varphi')^{-\gamma} \in Mul(X)$ and $C_\varphi = (\varphi')^{-\gamma}((\varphi')^\gamma C_\varphi)$, from which the boundedness follows. The invertibility is deduced from $(C_\varphi)^{-1} = C_{\varphi^{-1}}$.

Remark 2.4. Let X be a γ-space for some $\gamma \geq 0$ and take $\varphi \in Aut(\mathbb{D})$. Then $\lim_{n\to\infty} \left\|((\varphi')^\gamma C_\varphi)^n\right\|_{\mathcal{L}(X)}^{1/n} = 1$ and $\sigma((\varphi')^\gamma C_\varphi) \subseteq \mathbb{T}$.

This is trivial if φ is the identity mapping. Otherwise, fix $\varepsilon > 0$. As $((\varphi')^\gamma C_\varphi)^n = (\varphi_n')^\gamma C_{\varphi_n}$, one has for all $n \in \mathbb{N}$,

$$\|((\varphi')^\gamma C_\varphi)^n\| \lesssim (1-|\varphi_n(0)|^2)^{-\varepsilon} = \left(\prod_{j=0}^{n-1} \frac{1-|\varphi_{j+1}(0)|^2}{1-|\varphi_j(0)|^2}\right)^{-\varepsilon}$$

$$= \left(\prod_{j=0}^{n-1} |\varphi'(\varphi_j(0))|\right)^{-\varepsilon} \leq \left(\inf_{z\in\mathbb{D}} |\varphi'(z)|\right)^{n\varepsilon},$$

where we have used the Schwarz-Pick lemma and (**P4**). Since $\inf_{z\in\mathbb{D}} |\varphi'(z)| > 0$ and $\varepsilon > 0$ can be taken arbitrarily small, it follows that $\lim_{n\to\infty} \left\|((\varphi')^\gamma C_\varphi)^n\right\|_{\mathcal{L}(X)}^{1/n} \leq 1$, so by the spectral radius formula $\sigma((\varphi')^\gamma C_\varphi) \subseteq \overline{\mathbb{D}}$. On the other hand, $((\varphi')^\gamma C_\varphi)^{-1} = ((\varphi^{-1})')^\gamma C_{\varphi^{-1}}$. Therefore, the claim is obtained by applying what we have already proven to $((\varphi')^\gamma C_\varphi)^{-1}$.

Remark 2.5. Let X be a γ-space for some $\gamma \geq 0$, and let uC_φ be a bounded and invertible weighted composition operator on X, where φ is an automorphism of $\mathbb{D}$. Then both u and $1/u$ belong to $Mul(X)$. Indeed, $uf = uC_\varphi(C_{\varphi^{-1}} f) \in X$, so u is a multiplier of X since $C_{\varphi^{-1}} \in \mathcal{L}(X)$ by Remark 2.3. Similarly, $u^{-1} f = C_\varphi (uC_\varphi)^{-1} f \in X$, where we have used that $(uC_\varphi)^{-1} = \frac{1}{u\circ\varphi^{-1}} C_{\varphi^{-1}}$.

As an immediate consequence of $u, 1/u \in Mul(X)$, we have

$$\sup_{z\in\mathbb{D}} |u(z)| < \infty, \qquad \inf_{z\in\mathbb{D}} |u(z)| > 0,$$

see for instance [13, Lemma 11].

2.1. Examples

Here we list examples of (classical) Banach spaces of holomorphic functions that are γ-spaces for some $\gamma \geq 0$.

(1) *Hardy spaces.* For $1 \leq p < \infty$, let $H^p(\mathbb{D})$ be the Hardy space on $\mathbb{D}$ formed by all functions $f \in \mathcal{O}(\mathbb{D})$ such that

$$\|f\|_{H^p} := \sup_{0<r<1} \left(\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right)^{1/p} < \infty,$$

endowed with the norm $\|\cdot\|_{H^p}$. Then $H^p(\mathbb{D})$ is a $1/p$-space for each $p \in [1,\infty)$ (we will see that $H^\infty(\mathbb{D})$ is a 0-space). Indeed, (**P1**), (**P2**) and (**P3**) are a straightforward consequence of the fact that the algebra of multipliers of $H^p(\mathbb{D})$ is given by $H^\infty(\mathbb{D}) = \{f \in \mathcal{O}(\mathbb{D}) : \sup_{z\in\mathbb{D}} |f(z)| < \infty\}$ (see Lemma A.1). Regarding (**P4**), it is well known that $(\varphi')^{1/p} C_\varphi$ is an isometric isomorphism on $H^p(\mathbb{D})$ for each $\varphi \in Aut(\mathbb{D})$, see for instance [17].

(2) *Weighted Bergman spaces.* Let $1 \le p < \infty$ and $\sigma > -1$. $\mathcal{A}^p_\sigma(\mathbb{D})$ denotes the weighted Bergman space formed by all holomorphic functions in $\mathbb{D}$ such that

$$\|f\|_{\mathcal{A}^p_\sigma} := \left(\int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\sigma dA(z) \right)^{1/p} < \infty,$$

where A is the Lebesgue measure of $\mathbb{D}$. It is well known that the space $\mathcal{A}^p_\sigma(\mathbb{D})$, with norm $\|\cdot\|_{\mathcal{A}^p_\sigma}$, is a $\frac{\sigma+2}{p}$-space for all $\sigma > -1$ and $p \in [1,\infty)$. Indeed, that $\mathcal{A}^\sigma_p(\mathbb{D})$ satisfies the properties (**P1**)-(**P3**) follows from the fact that $Mul(\mathcal{A}^p_\sigma) = H^\infty(\mathbb{D})$, see Lemma A.1. On the other hand, $(\varphi')^{\frac{\sigma+2}{p}} C_\varphi$ is an isometric isomorphism on $\mathcal{A}^p_\sigma(\mathbb{D})$ for all $\varphi \in Aut(\mathbb{D})$, see [25]. Thus $\mathcal{A}^p_\sigma(\mathbb{D})$ also satisfies property (**P4**).

(3) *Korenblum classes and little Korenblum classes.* For $\rho \ge 0$, the Korenblum growth class $\mathcal{K}^{-\rho}(\mathbb{D})$ is the Banach space of analytic functions on $\mathbb{D}$ given by

$$\mathcal{K}^{-\rho}(\mathbb{D}) := \{f \in \mathcal{O}(\mathbb{D}) : \|f\|_{\mathcal{K}^{-\rho}} := \sup_{z\in\mathbb{D}} (1-|z|^2)^\rho |f(z)| < \infty\},$$

which is a Banach space when endowed with the norm $\|\cdot\|_{\mathcal{K}^{-\rho}}$. Note that $\rho = 0$ corresponds to the Banach algebra of bounded holomorphic functions on $\mathbb{D}$, $H^\infty(\mathbb{D})$. If $\rho > 0$, then the closure of polynomials in $\mathcal{K}^{-\rho}(\mathbb{D})$ is the Little Korenblum growth class $\mathcal{K}_0^{-\rho}(\mathbb{D})$ given by

$$\mathcal{K}_0^{-\rho}(\mathbb{D}) := \{f \in \mathcal{K}^{-\rho}(\mathbb{D}) : \lim_{|z|\to 1} (1-|z|^2)^\rho |f(z)| = 0\},$$

with the norm inherited from $\mathcal{K}^{-\rho}(\mathbb{D})$.
It is readily seen that $\mathcal{K}^{-\rho}(\mathbb{D})$ if a ρ-space for each $\rho \ge 0$. Properties (**P1**)-(**P3**) again follow immediately since $Mul(\mathcal{K}^{-\rho}(\mathbb{D})) = H^\infty(\mathbb{D})$ (see Lemma A.1), and an application of the Schwarz-Pick Lemma shows that $(\varphi')^\rho C_\varphi$ is an isometric isomorphism on $\mathcal{K}^{-\rho}(\mathbb{D})$ for all $\varphi \in Aut(\mathbb{D})$. Similarly, it can be proven that $\mathcal{K}_0^{-\rho}(\mathbb{D})$ is a ρ-space for all $\rho > 0$.

(4) *The disk algebra.* The disk algebra $\mathfrak{A}(\mathbb{D})$ is the space consisting of holomorphic functions on $\mathbb{D}$ which have a continuous extension to $\overline{\mathbb{D}}$. It is well known that $\mathfrak{A}(\mathbb{D})$ is a Banach algebra when endowed with the supremum norm, i.e. $\|f\|_{\mathfrak{A}} := \sup_{z\in\mathbb{D}} |f(z)|$. Also, $\mathfrak{A}(\mathbb{D})$ is the topological closure of the polynomials on $H^\infty(\mathbb{D})$.
It is readily seen that $\mathfrak{A}(\mathbb{D})$ is a 0-space (to check (**P3**), one may use Lemma A.1 and the fact that $Mul(\mathfrak{A}(\mathbb{D}))$ is continuously embedded in $H^\infty(\mathbb{D})$).

(5) *Weighted Dirichlet spaces.* For $p \ge 1$ and $\sigma > -1$, let $\mathcal{D}^p_\sigma(\mathbb{D})$ denote the weighted Dirichlet space of all functions $f \in \mathcal{O}(\mathbb{D})$ such that $f' \in \mathcal{A}^p_\sigma(\mathbb{D})$ and endow it with the norm

$$\|f\|_{\mathcal{D}^p_\sigma} := \left(|f(0)|^p + \|f'\|^p_{\mathcal{A}^p_\sigma}\right)^{1/p} < \infty.$$

Then $\mathcal{D}^p_\sigma(\mathbb{D})$ is a Banach space. When $\sigma > p-1$ one has $\mathcal{D}^p_\sigma(\mathbb{D}) = \mathcal{A}^p_{\sigma-p}(\mathbb{D})$ with equivalent norms, see e.g. [16, Th. 6]. Thus, for $\sigma > p-1$, $\mathcal{D}^p_\sigma(\mathbb{D})$ is a $\left(\frac{\sigma+2}{p}-1\right)$-space.

We claim that $\mathcal{D}^p_\sigma(\mathbb{D})$ is a $\left(\frac{\sigma+2}{p}-1\right)$-space whenever $p-2 \le \sigma \le p-1$ (and $p \ge 1$, $\sigma > -1$). In effect, direct calculations show that $\mathcal{D}^p_\sigma(\mathbb{D})$ satisfies (**P1**). That $\mathcal{D}^p_\sigma(\mathbb{D})$ satisfies (**P2**) is proven in [3,4] for $\sigma > p-2$, and is extended to $\sigma = p-2$ in [1, Lemma 2.5]. That $\mathcal{D}^p_\sigma(\mathbb{D})$ fulfills (**P3**) follows by Lemma A.3. Finally, it is essentially proven in [4, Prop. 3.1] that the spaces $\mathcal{D}^p_\sigma(\mathbb{D})$ satisfy (**P4**) for $p-2 \le \sigma \le p-1$ and $\gamma = \frac{\sigma+2}{p} - 1$, see also [1, Lemma 2.6].

(6) *Bloch space and little Bloch space.* Let $B(\mathbb{D})$ denote the Bloch space, that is, the Banach space of holomorphic functions f on $\mathbb{D}$ with the property

$$\|f\|_B := |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty,$$

which is endowed with the norm $\|\cdot\|_B$. Also, let $B_0(\mathbb{D})$ denote the little Bloch space, consisting of the closure of polynomials in $B(\mathbb{D})$, or equivalently

$$B_0(\mathbb{D}) = \{f \in B(\mathbb{D}) \,:\, \lim_{|z|\to 1}(1-|z|^2)|f'(z)| = 0\},$$

see [32, Prop. 2]. $B(\mathbb{D})$ and $B_0(\mathbb{D})$ are 0-spaces. In effect, the multipliers of these spaces are given by

$$Mul(B(\mathbb{D}) = Mul(B_0(\mathbb{D})) = \{f \in H^\infty(\mathbb{D}) \,:\, (1-|\cdot|^2)\log(1-|\cdot|^2)f' \in H^\infty(\mathbb{D})\},$$

see [32, Th. 27]. Hence, $B(\mathbb{D})$ and $B_0(\mathbb{D})$ fulfill properties (**P1**) and (**P2**). They also satisfy (**P3**) by Lemma A.3. Finally, it is readily seen that (**P4**) also holds, see for instance [1, Subsect. 2.1].

(7) *Functions with bounded primitive.* We consider here a space which will be useful to study the eigenvectors of weighted composition operators in Section 7. Set

$$\mathscr{A}(\mathbb{D}) := \{f' \,:\, f \in H^\infty(\mathbb{D})\}.$$

The mapping $T : H^\infty(\mathbb{D}) \to \mathscr{A}(\mathbb{D})$ given by $T(f) = f'$ is a linear and surjective mapping. Identifying the constant functions with $\mathbb{C}$, one has $\ker T = \mathbb{C}$, so $\mathscr{A}(\mathbb{D}) \simeq H^\infty(\mathbb{D})/\mathbb{C}$ is a Banach space when endowed with the norm

$$\|f\|_{\mathscr{A}} := \left\|\int f\right\|_{H^\infty(\mathbb{D})/\mathbb{C}} = \inf_{K\in\mathbb{C}}\left(\sup_{z\in\mathbb{D}}\left|\int_0^z f(\xi)\,d\xi + K\right|\right), \qquad f \in \mathscr{A}(\mathbb{D}).$$

On the other hand, let $H_0^\infty(\mathbb{D}) := \{f \in H^\infty(\mathbb{D}) \,:\, f(0) = 0\}$, which is a closed subspace (even more, a closed ideal) of $H^\infty(\mathbb{D})$. It is readily seen that the mapping $T|_{H_0^\infty(\mathbb{D})} : H_0^\infty(\mathbb{D}) \to \mathscr{A}(\mathbb{D})$ is a continuous bijection. Thus, by the open mapping theorem, $T|_{H_0^\infty(\mathbb{D})}$ is a Banach space isomorphism and we obtain the following equivalence of norms

$$\|f\|_{\mathscr{A}} \simeq \left\| \int_0^{(\cdot)} f(\xi)\, d\xi \right\|_{H^\infty}, \qquad f \in \mathscr{A}(\mathbb{D}). \tag{2.1}$$

Proposition 2.6. *$\mathscr{A}(\mathbb{D})$ is a 1-space.*

The claim follows by the following lemmas and Lemma A.3.

Lemma 2.7. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a morphism of $\mathbb{D}$. Then $\varphi' C_\varphi$ is a well-defined bounded operator on $\mathscr{A}(\mathbb{D})$ with $\|\varphi' C_\varphi\|_{\mathcal{L}(\mathscr{A}(\mathbb{D}))} \le 1$. Moreover, if φ is an automorphism of $\mathbb{D}$, then $\varphi' C_\varphi$ is an isometric isomorphism of $\mathscr{A}(\mathbb{D})$ onto itself.*

In consequence, $\mathscr{A}(\mathbb{D})$ satisfies (**P4**) for $\gamma = 1$.

Proof. Note that $\int (\varphi' C_\varphi f) = C_\varphi \left(\int f\right)$. Since $C_\varphi(H^\infty(\mathbb{D})) \subseteq H^\infty(\mathbb{D})$, we obtain $\varphi' C_\varphi(\mathscr{A}) \subseteq \mathscr{A}$. Moreover,

$$\begin{aligned} \|\varphi' C_\varphi f\|_{\mathscr{A}} &= \inf_{K \in \mathbb{C}} \left\| \int (\varphi' C_\varphi f) + K \right\|_{H^\infty} = \inf_{K \in \mathbb{C}} \left\| C_\varphi \left(\int f \right) + K \right\|_{H^\infty} \\ &\le \inf_{K \in \mathbb{C}} \left\| \int f + K \right\|_{H^\infty} = \|f\|_{\mathscr{A}}, \qquad f \in \mathscr{A}, \end{aligned}$$

where we have used $\|C_\varphi F + K\|_{H^\infty} \le \|F + K\|_{H^\infty}$ for all $F \in H^\infty(\mathbb{D})$ and $K \in \mathbb{C}$. Now, if φ is an automorphism of $\mathbb{D}$, then $\|C_\varphi F + K\|_{H^\infty} = \|F + K\|_{H^\infty}$ for all $F \in H^\infty(\mathbb{D})$ and $K \in \mathbb{C}$. Thus, reasoning as above one obtains $\|\varphi' C_\varphi f\|_{\mathscr{A}} = \|f\|_{\mathscr{A}}$ for all $f \in \mathscr{A}$ as claimed.

If φ is an automorphism of $\mathbb{D}$, then $(\varphi' C_\varphi)^{-1} = (\varphi^{-1})' C_{\varphi^{-1}}$, and the claim follows by what we have already proven. $\square$

Lemma 2.8. *One has*

$$\begin{aligned} &Mul(\mathscr{A}(\mathbb{D})) \\ &= \left\{ u \in H^\infty(\mathbb{D}) \;:\; \sup_{z \in \mathbb{D}} \left| \int_0^z u'(\xi) f(\xi)\, d\xi \right| \lesssim \|f\|_{H^\infty} \text{ for all } \; f \in H^\infty(\mathbb{D}) \right\}. \end{aligned}$$

Let $\|u'\|_{\mathcal{C}^\infty}$ *denote the infimum of the set of all constants* $K > 0$ *for which* $\sup_{z\in\mathbb{D}} \left|\int_0^z u'(\xi)f(\xi)\,d\xi\right| \le K\|f\|_{H^\infty}$, $f \in H^\infty(\mathbb{D})$. *Then* $\|u\|_{Mul(\mathscr{A})} \lesssim \|u\|_{H^\infty} + \|u'\|_{\mathcal{C}^\infty}$ *for every* $u \in Mul(\mathscr{A}(\mathbb{D}))$.

In particular, $\mathscr{A}(\mathbb{D})$ satisfies (**P1**) and (**P2**).

Proof. It is clear by an integration by parts, i.e., $\int(uf) = u\int f - \int(u'\int f)$, that a holomorphic function $u : \mathbb{D} \to \mathbb{C}$ belongs to $Mul(\mathscr{A}(\mathbb{D}))$ whenever $u \in H^\infty(\mathbb{D})$ and $\|u'\|_{\mathcal{C}^\infty} < \infty$. In this case, the equivalence of norms (2.1) yields the inequality $\|u\|_{Mul(\mathscr{A})} \lesssim \|u\|_{H^\infty} + \|u'\|_{\mathcal{C}^\infty}$, $u \in Mul(\mathscr{A}(\mathbb{D}))$.
Take now $u \in Mul(\mathscr{A}(\mathbb{D}))$ and let us show that then u satisfies the properties of the claim. Note first that necessarily $u \in H^\infty(\mathbb{D})$, see [13, Lemma 11]. Again, integration by parts and the norm equivalence (2.1) implies that, for every $f \in H^\infty$ (so $f' \in \mathscr{A}(\mathbb{D})$)

$$\begin{aligned}\left\|\int_0^{(\cdot)} u'f\right\|_{H^\infty} &\le \left\|\int_0^{(\cdot)} uf'\right\|_{H^\infty} + \|uf\|_{H^\infty} \lesssim \|uf'\|_{\mathscr{A}} + \|uf\|_{H^\infty} \\ &\le \left(\|u\|_{Mul(\mathscr{A})} + \|u\|_{H^\infty}\right)\|f\|_{H^\infty}.\end{aligned}$$

Therefore, $\|u'\|_{\mathcal{C}^\infty} < \infty$ as claimed. □

Next we give two families of functions which are contained in $\mathscr{A}(\mathbb{D})$.

Proposition 2.9. *Let* $f \in \mathfrak{A}(\mathbb{D})$ *(i.e.,* f *is continuous in* $\overline{\mathbb{D}}$*) be such that* $\inf_{z\in\mathbb{D}} |f(z)| > 0$. *Then* $f'/f \in \mathscr{A}(\mathbb{D})$.

Proof. Note that $\operatorname{Log} f$ is a primitive of f'/f, so it is enough to prove that $\operatorname{Log} f \in H^\infty(\mathbb{D})$. Since $|f|$ is bounded and bounded away from 0, it is clear that $\sup_{z\in\mathbb{D}} |\mathfrak{Re}\,((\operatorname{Log} f)(z))| < \infty$. Thus, the proof will be completed if we prove that $\sup_{z\in\mathbb{D}} |\mathfrak{Im}\,((\operatorname{Log} f)(z))| < \infty$.
Since f is uniformly continuous in $\overline{\mathbb{D}}$, we can take $n \in \mathbb{N}$ such that, if $z, w \in \mathbb{D}$ with $|z - w| \le 1/n$, then $|f(z) - f(w)| \le \inf_{\xi\in\mathbb{D}} |f(\xi)|$. Given $z, w \in \mathbb{D}$ with $z \ne w$, the function

$$t \mapsto \mathfrak{Im}\,((\operatorname{Log} f)(t)), \qquad t \in [z, w],$$

where $[z, w]$ denotes the segment joining z and w, is a continuous argument of the curve with range $f([z, w])$. Note that by the choice of n, $f([z, w])$ lies in some rotated half-plane if $|z - w| \le 1/n$ (namely, in $\{\xi \in \mathbb{C} : \mathfrak{Re}\,(z\overline{\xi}) \ge 0\}$). Hence,

$$|\mathfrak{Im}\,((\operatorname{Log} f)(z) - (\operatorname{Log} f)(w))| \le \frac{\pi}{2}, \qquad z, w \in \mathbb{D} \text{ with } |z - w| \le \frac{1}{n},$$

which, applied to partitions of $[0, z]$ for $z \in \mathbb{D}$ with length less than $1/n$, yields

$$\sup_{z\in\mathbb{D}} |\mathfrak{Im}\,((\mathrm{Log}\, f)(z))| \leq n\frac{\pi}{2} + |\mathfrak{Im}\,(\mathrm{Log}\, f)(0)| < \infty,$$

completing the proof. □

Proposition 2.10. *Let $f \in \mathcal{O}(\mathbb{D})$ be such that $\inf_{z\in\mathbb{D}} |f(z)| > 0$. Assume $f' \in \mathcal{K}^{-\rho}(\mathbb{D})$ for some $\rho \in [0, 1)$. Then $f'/f \in \mathscr{A}(\mathbb{D})$.*

Proof. One has

$$\begin{aligned}\sup_{z\in\mathbb{D}} \left| \int_0^z \frac{f'(\xi)}{f(\xi)}\, d\xi \right| &\leq \|1/f\|_{H^\infty} \sup_{z\in\mathbb{D}} \int_0^z |f'(\xi)||d\xi| \\ &\leq \|1/f\|_{H^\infty} \|f'\|_{\mathcal{K}^{-\rho}} \int_0^1 (1-r)^{-\rho}\, dr < \infty,\end{aligned}$$

so $f'/f \in \mathscr{A}$ and the proof is done. □

3. Spectral radius estimate

Following the notation of (**P3**), given a hyperbolic automorphism φ and a holomorphic function $u \in \mathcal{O}(\mathbb{D})$, we set

$$u_n(z) := \prod_{j=0}^{n-1} u \circ \varphi_j, \qquad u_{-n} := \prod_{j=1}^{n} \frac{1}{u \circ \varphi_{-j}}, \qquad n \in \mathbb{N},$$

and $u_0(z) = 1, z \in \mathbb{D}$. Then, one gets $(uC_\varphi)^n = u_n C_{\varphi_n}$ for $n \in \mathbb{Z}$.

The following result is essentially given in [22, Lemma 4.6] and [2, Prop. 3.2] for $H^p(\mathbb{D}), \mathcal{A}^p_\sigma(\mathbb{D})$ and $\mathcal{K}^{-\rho}(\mathbb{D})$; and in [15, Th. 4.5 & Th. 5.2] for $B(\mathbb{D})$ and the classical Dirichlet space $\mathcal{D}^2_0(\mathbb{D})$. Following the same ideas, we extend such a result to our framework.

Proposition 3.1. *Let X be a γ-space for some $\gamma \geq 0$, and let uC_φ be an invertible weighted composition operator on X. Assume that the automorphism φ is hyperbolic, with fixed points a (attractive), b (repulsive) in $\mathbb{T}$. Suppose furthermore that the absolute value $|u|$ can be continuously extended to $\mathbb{D} \cup \{a, b\}$. Then,*

$$\sigma(uC_\varphi) \subseteq \left\{ \lambda \in \mathbb{C} \,:\, \min\left\{ \frac{|u|(a)}{\varphi'(a)^\gamma}, \frac{|u|(b)}{\varphi'(b)^\gamma} \right\} \leq |\lambda| \leq \max\left\{ \frac{|u|(a)}{\varphi'(a)^\gamma}, \frac{|u|(b)}{\varphi'(b)^\gamma} \right\} \right\}.$$

Proof. For all $f \in X$, $n \in \mathbb{N}_0$, one has

$$\|(uC_\varphi)^n f\|_X = \|u_n C_{\varphi_n} f\|_X \leq \left\| \frac{u_n}{(\varphi_n')^\gamma} \right\|_{Mul(X)} \|(\varphi_n')^\gamma C_{\varphi_n} f\|_X$$
$$= \left\| \left(\frac{u}{(\varphi')^\gamma} \right)_n \right\|_{Mul(X)} \|(\varphi_n')^\gamma C_{\varphi_n} f\|_X \,,$$

where we have used $(\varphi_n)' = (\varphi')_n$, so $\frac{u_n}{((\varphi_n)')^\gamma} = \left(\frac{u}{(\varphi')^\gamma} \right)_n$, $n \in \mathbb{N}$. Since φ' is holomorphic and non-vanishing in a disk of radius greater than 1, $\frac{u}{(\varphi')^\gamma} \in Mul(X)$ by (**P1**) and Remark 2.5. Also, $\lim_{n\to\infty} \|(\varphi_n')^\gamma C_{\varphi_n} f\|_X^{1/n} \leq 1$ by Remark 2.4. As $|u/(\varphi')^\gamma|$ has continuous extension to $\mathbb{D} \cup \{a, b\}$, we conclude $\lim_{n\to\infty} \|(uC_\varphi)^n\|_{\mathcal{L}(X)}^{1/n} \leq \max\left\{ \frac{|u|(a)}{\varphi'(a)^\gamma}, \frac{|u|(b)}{\varphi'(b)^\gamma} \right\}$ by (**P3**), and the upper bound for the spectral radius follows by the spectral radius formula.

Regarding the lower bound on $\{|\lambda| \,:\, \lambda \in \sigma(uC_\varphi)\}$, recall that

$$\sigma((uC_\varphi)^{-1}) = (\sigma(uC_\varphi))^{-1} = \{1/\lambda \,:\, \lambda \in \sigma(uC_\varphi)\}.$$

Since $(uC_\varphi)^{-1} = \frac{1}{u\circ\varphi^{-1}} C_{\varphi^{-1}}$, by what we have already proven

$$\lim_{n\to\infty} \|(uC_\varphi)^{-n}\|_{\mathcal{L}(X)}^n \leq \max\left\{ \frac{\varphi'(a)^\gamma}{|u|(a)}, \frac{\varphi'(b)^\gamma}{|u|(b)} \right\},$$

and our claim follows. Note that we have used above that $\lim_{\mathbb{D}\ni z\to a} \left| \frac{1}{u\circ\varphi^{-1}} \right| = (|u|(a))^{-1}$, $(\varphi^{-1})'(a) = (\varphi'(a))^{-1}$, and the analogous identities evaluated at b. $\square$

4. Case 1: $\frac{|u|(b)}{\varphi'(b)^\gamma} < \frac{|u|(a)}{\varphi'(a)^\gamma}$

In this section we focus on the case $\frac{|u|(b)}{\varphi'(b)^\gamma} < \frac{|u|(a)}{\varphi'(a)^\gamma}$. It turns out that we need to solve this case before addressing its counterpart where $\frac{|u|(b)}{\varphi'(b)^\gamma} > \frac{|u|(a)}{\varphi'(a)^\gamma}$, see Corollaries 6.6 and 6.7. Here, we follow the ideas presented in [2] to study the universality of invertible weighted composition operators on the Hilbertian Hardy space $H^2(\mathbb{D})$ and on the Hilbertian weighted Bergman space $\mathcal{A}^2_\sigma(\mathbb{D})$. These ideas can be easily adapted in the case $Mul(X) = H^\infty(\mathbb{D})$, see [2, Sect. 5], but require non-trivial arguments to be extended to our more general setting. Also, our main result of this section (Theorem 4.4) considerably improves the following remark.

Remark 4.1. Let $Y = H^p(\mathbb{D}), \mathcal{A}^p_\sigma(\mathbb{D})$ or $\mathcal{K}^{-\rho}(\mathbb{D})$, and let $\gamma \geq 0$ be such that Y is a γ-space. Assume u is holomorphic in $\mathbb{D}$ and continuous in $\mathbb{T}$ and that u is bounded away from zero. It is proven in [22, Th. 4.9] that, assuming $\frac{|u(b)|}{\varphi'(b)^\gamma} < \frac{|u(a)|}{\varphi'(a)^\gamma}$,

$$\sigma(uC_\varphi) = \left\{ \lambda \in \mathbb{C} \,:\, \frac{|u(b)|}{\varphi'(b)^\gamma} \leq |\lambda| \leq \frac{|u(a)|}{\varphi'(a)^\gamma} \right\},$$

as an operator on Y. Assume furthermore that u' is bounded and $1 < \left|\frac{u(a)}{u(b)}\right| < \varphi'(a)^{-2\gamma}$. Following the ideas of [20, Subsect. 3.5], it is proven in [22, Remark 4.10] that $\sigma(uC_\varphi) = \sigma_{ess}(uC_\varphi)$ and that every interior point of this annulus is an eigenvalue of infinite multiplicity for uC_φ.

Fix a hyperbolic automorphism $\mathbb{D}$ with fixed points a (attractive), b (repulsive) in $\mathbb{T}$. For $\mu, \nu \in \mathbb{R}$, set $\rho_{\mu,\nu}(z) := (a-z)^\mu (b-z)^\nu$, $z \in \mathbb{D}$. It is readily seen from (1.2) that

$$\frac{\rho_{\mu,\nu} \circ \varphi(z)}{\rho_{\mu,\nu}(z)} = \varphi'(a)^\mu \left(\frac{b-a}{(\varphi'(a)-1)z + b - a\varphi'(a)} \right)^{\mu+\nu}, \qquad z \in \mathbb{D}.$$

(Recall that $\varphi'(a) \in (0,1)$.) In particular, $\dfrac{\rho_{\mu,\nu} \circ \varphi}{\rho_{\mu,\nu}}$ is a holomorphic function in a disk of radius strictly greater than 1 and one has

$$\frac{\rho_{\mu,\nu} \circ \varphi}{\rho_{\mu,\nu}}(a) = \varphi'(a)^\mu, \quad \text{and} \quad \frac{\rho_{\mu,\nu} \circ \varphi}{\rho_{\mu,\nu}}(b) = \varphi'(a)^{-\nu} = \varphi'(b)^\nu. \tag{4.1}$$

We set $X_{\mu,\nu} := \{\rho_{\mu,\nu} f \,:\, f \in X\}$. Then $X_{\mu,\nu}$ is a Banach space endowed with the norm

$$\|f\|_{X_{\mu,\nu}} := \left\| \frac{f}{\rho_{\mu,\nu}} \right\|_X, \qquad f \in X_{\mu,\nu},$$

and the multiplication operator $M_{\mu,\nu} : X \to X_{\mu,\nu}$, given by $M_{\mu,\nu} f = \rho_{\mu,\nu} f$, is an isometric isomorphism.

We need the following lemmas for the main result of this section.

Lemma 4.2. *Let X be a γ-space for some $\gamma \geq 0$, and let uC_φ be an invertible weighted composition operator on X. Assume that the automorphism φ is hyperbolic and pick $\mu, \nu \in \mathbb{R}$. Then the weighted composition operator uC_φ is bounded and invertible acting on $X_{\mu,\nu}$, and is isometrically equivalent to the operator $\frac{\rho_{\mu,\nu} \circ \varphi}{\rho_{\mu,\nu}} uC_\varphi$ acting on X.*

Proof. Note that

$$uC_\varphi(M_{\mu,\nu} f) = \rho_{\mu,\nu} \frac{\rho_{\mu,\nu} \circ \varphi}{\rho_{\mu,\nu}} uC_\varphi f = M_{\mu,\nu} \left(\frac{\rho_{\mu,\nu} \circ \varphi}{\rho_{\mu,\nu}} uC_\varphi f \right), \qquad f \in X.$$

That is, $uC_\varphi = M_{\mu,\nu} \frac{\rho_{\mu,\nu} \circ \varphi}{\rho_{\mu,\nu}} uC_\varphi (M_{\mu,\nu})^{-1}$. Note that $\frac{\rho_{\mu,\nu} \circ \varphi}{\rho_{\mu,\nu}} uC_\varphi$ is a well defined and bounded operator on X since $\frac{\rho_{\mu,\nu} \circ \varphi}{\rho_{\mu,\nu}} \in Mul(X)$ by (**P1**). As $M_{\mu,\nu} : X \to X_{\mu,\nu}$ is an isometric isomorphism, the claim follows. □

Lemma 4.3. *Let X be a γ-space for some $\gamma \geq 0$. For $\mu, \nu \geq 0$, $X = X_{\mu,0} + X_{0,\nu}$.*

Proof. Take $m, n \in \mathbb{N}_0$ such that $m \geq \mu$ and $n \geq \nu$. Then, for $f \in X$ and $z \in \mathbb{D}$,

$$f(z) = \frac{(a - z - (b - z))^{m+n}}{(a - b)^{m+n}} f(z)$$
$$= \frac{1}{(a - b)^{m+n}} \sum_{j=0}^{m+n} \binom{m+n}{j} (a - z)^j (-1)^{m+n-j} (b - z)^{m+n-j} f(z).$$

By (**P2**), $(a - (\cdot))^j (b - (\cdot))^{m+n-j} f$ belongs to $X_{0,\nu}$ if $j \leq m$, whereas such a function belongs to $X_{\mu,0}$ if $j \geq m$. Thus, the proof is done. □

Theorem 4.4. *Let X be a γ-space for some $\gamma \geq 0$, and let uC_φ be an invertible weighted composition operator on X. Assume that the automorphism φ is hyperbolic, with fixed points a (attractive), b (repulsive) in $\mathbb{T}$. Suppose furthermore that the absolute value $|u|$ can be continuously extended to $\mathbb{D} \cup \{a, b\}$ with $\frac{|u|(b)}{\varphi'(b)^\gamma} < \frac{|u|(a)}{\varphi'(a)^\gamma}$. Then*

$$\sigma(uC_\varphi) = \sigma_{ess}(uC_\varphi) = \left\{ \lambda \in \mathbb{C} \,:\, \frac{|u|(b)}{\varphi'(b)^\gamma} \leq |\lambda| \leq \frac{|u|(a)}{\varphi'(a)^\gamma} \right\}.$$

Moreover, for each λ lying in the interior of the above annulus, $\lambda - uC_\varphi$ is a surjective and non-injective operator with infinite-dimensional kernel.

Proof. Take $\mu, \nu > 0$ for which

$$|u|(a)\, \varphi'(a)^{\mu-\gamma} = \frac{|u|(b)}{\varphi'(b)^\gamma}, \qquad \frac{|u|(a)}{\varphi'(a)^\gamma} = |u|(b)\, \varphi'(b)^{\nu-\gamma}.$$

Such μ, ν exist since $\varphi'(a) \in (0, 1)$ and $\varphi'(b) \in (1, \infty)$. Note that

$$\lim_{\mathbb{D} \ni z \to a} \left| \frac{\rho_{\mu,0} \circ \varphi(z)}{\rho_{\mu,0}(z)} u(z) \right| = \varphi'(a)^\mu |u|(a), \qquad \lim_{\mathbb{D} \ni z \to b} \left| \frac{\rho_{\mu,0} \circ \varphi(z)}{\rho_{\mu,0}(z)} u(z) \right| = |u|(b),$$

and

$$\lim_{\mathbb{D} \ni z \to a} \left| \frac{\rho_{0,\nu} \circ \varphi(z)}{\rho_{0,\nu}(z)} u(z) \right| = |u|(a), \qquad \lim_{\mathbb{D} \ni z \to b} \left| \frac{\rho_{0,\nu} \circ \varphi(z)}{\rho_{0,\nu}(z)} u(z) \right| = \varphi'(b)^\nu |u|(b),$$

see (4.1). Hence, Proposition 3.1 applied to the operators $\frac{\rho_{\mu,0} \circ \varphi}{\rho_{\mu,0}} uC_\varphi$, $\frac{\rho_{0,\nu} \circ \varphi}{\rho_{0,\nu}} uC_\varphi$ yields

$$\sigma\left(\frac{\rho_{\mu,0} \circ \varphi}{\rho_{\mu,0}} uC_\varphi \right) \subseteq \left\{ \lambda \in \mathbb{C} \,:\, |\lambda| = \frac{|u|(b)}{\varphi'(b)^\gamma} \right\},$$
$$\sigma\left(\frac{\rho_{0,\nu} \circ \varphi}{\rho_{0,\nu}} uC_\varphi \right) \subseteq \left\{ \lambda \in \mathbb{C} \,:\, |\lambda| = \frac{|u|(a)}{\varphi'(a)^\gamma} \right\}, \tag{4.2}$$

regarding both operators as operators in X. (Recall that $\frac{\rho_{\mu,0}\circ\varphi}{\rho_{\mu,0}}, \frac{\rho_{0,\nu}\circ\varphi}{\rho_{0,\nu}} \in Mul(X)$ by (**P1**).)

Now we prove that $\lambda - uC_\varphi$ is surjective on X for each $\lambda \in \mathbb{C}$ with $\frac{|u|(b)}{\varphi'(b)^\gamma} < |\lambda| < \frac{|u|(a)}{\varphi'(a)^\gamma}$. By Lemma 4.2, the operator uC_φ is a bounded invertible operator on $X_{\mu,0}$ ($X_{0,\nu}$), which is isometrically equivalent to the operator $\frac{\rho_{\mu,0}\circ\varphi}{\rho_{\mu,0}} uC_\varphi$ ($\frac{\rho_{0,\nu}\circ\varphi}{\rho_{0,\nu}} uC_\varphi$ respectively) acting on X. Then (4.2) implies that $\lambda - uC_\varphi$ is surjective as an operator both on $X_{\mu,0}$ and on $X_{0,\nu}$. Note that $X = X_{\mu,0} + X_{0,\nu}$ by Lemma 4.3 with $X_{\mu,0}, X_{0,\nu} \subseteq X$ since $\rho_{\mu,0}, \rho_{0,\nu} \in Mul(X)$ by (**P2**). Thus

$$(\lambda - uC_\varphi)(X) = (\lambda - uC_\varphi)(X_{\mu,0}) + (\lambda - uC_\varphi)(X_{0,\nu}) = X_{\mu,0} + X_{0,\nu} = X,$$

so indeed $\lambda - uC_\varphi$ is surjective on X.

Let us now prove the rest of the claim. Take a non-zero $f \in X$, pick $\widetilde{\mu} > \mu$, $\widetilde{\nu} > \nu$, and set

$$f_k(z) = (a - z)^{\widetilde{\mu} - \frac{2\pi i}{\delta} k} (b - z)^{\widetilde{\nu} + \frac{2\pi i}{\delta} k} f(z), \qquad z \in \mathbb{D},\ k \in \mathbb{Z},$$

where $\delta = -\log \varphi'(a) \in (0, \infty)$. By (**P2**), $\{f_k\}_{k\in\mathbb{Z}} \subset X_{\mu,\nu} = X_{\mu,0} \cap X_{0,\nu}$. As the inclusions $X_{\mu,0} \hookrightarrow X$, $X_{0,\nu} \hookrightarrow X$ are continuous, one has from (4.2) and Lemma 4.2 that the X-valued mappings F_k, G_k given by

$$F_k(\lambda) = (\lambda - uC_\varphi|_{X_{\mu,0}})^{-1} f_k, \qquad |\lambda| > \frac{|u|(b)}{\varphi'(b)^\gamma},$$

$$G_k(\lambda) = (\lambda - uC_\varphi|_{X_{0,\nu}})^{-1} f_k, \qquad |\lambda| < \frac{|u|(a)}{\varphi'(a)^\gamma},$$

are well defined and holomorphic for each $k \in \mathbb{Z}$. Note that $(\lambda - uC_\varphi)F_k(\lambda) = f_k$ and $(\lambda - uC_\varphi)G_k = f_k$ for every $k \in \mathbb{Z}$ and λ in the respective domain of F_k, G_k.

Assume now by contradiction that $\lambda_0 \notin \sigma(uC_\varphi)$ for some $\lambda_0 \in \mathbb{C}$ with $\frac{|u|(b)}{\varphi'(b)^\gamma} < |\lambda_0| < \frac{|u|(a)}{\varphi'(a)^\gamma}$. Then, $F_k(\lambda) = G_k(\lambda) = (\lambda - uC_\varphi)^{-1} f_k$ for all λ belonging to an open neighborhood of λ_0 and all $k \in \mathbb{Z}$. By the uniqueness of analytic continuation, $F_k(\lambda) = G_k(\lambda)$ for all λ in the annulus of radii $\frac{|u|(b)}{\varphi'(b)^\gamma}, \frac{|u|(a)}{\varphi'(a)^\gamma}$ (see for instance [5, Cor. A.4]), and we can define the X-valued entire functions H_k, $k \in \mathbb{Z}$, given by

$$H_k(\lambda) := \begin{cases} F_k(\lambda), & |\lambda| > \frac{|u|(b)}{\varphi'(b)^\gamma}, \\ G_k(\lambda), & |\lambda| < \frac{|u|(a)}{\varphi'(a)^\gamma}, \end{cases} \qquad k \in \mathbb{Z}.$$

Moreover, such functions H_k are bounded on $\mathbb{C}$ since $H_k(\lambda) = (\lambda - uC_\varphi)^{-1} f_k$ for all $|\lambda| > \frac{|u|(a)}{\varphi'(a)^\gamma}$ and $\|(\lambda - uC_\varphi)^{-1}\|_{\mathcal{L}(X)} \to 0$ as $|\lambda| \to \infty$ (this is readily seen from the representation $(\lambda - A)^{-1} = \sum_{n=0}^{\infty} \frac{A^n}{\lambda^{n+1}}$ for every bounded operator A and $|\lambda| > r(A)$, where $r(A)$ stands for the spectral radius of A).

By the vector-valued Liouville theorem (see for example [7, Th. 6]), one concludes that the entire functions H_k, $k \in \mathbb{Z}$, are constant. Since $\|F_k(\lambda)\| \to 0$ as $|\lambda| \to \infty$ as mentioned above, H_k is the zero function for each $k \in \mathbb{Z}$, which contradicts the identity $(\lambda - uC_\varphi)H_k(\lambda) = (\lambda - uC_\varphi)F_k(\lambda) = f_k \neq 0$ for $|\lambda| > \frac{|u|(a)}{\varphi'(a)^\gamma}$. Hence, $\lambda_0 \in \sigma(uC_\varphi)$ for every $\frac{|u|(b)}{\varphi'(b)^\gamma} \leq |\lambda_0| \leq \frac{|u|(a)}{\varphi'(a)^\gamma}$ as claimed (recall that $\sigma(uC_\varphi)$ is a closed subset of $\mathbb{C}$).

It only remains to prove the assertion regarding the kernel. Note that, by what we have already proven and the uniqueness of analytic continuation again, the sets $\mathbb{C} \setminus \Omega_k$, where

$$\Omega_k := \left\{ \lambda \in \mathbb{C} \,:\, F_k(\lambda) \neq G_k(\lambda) \quad \text{and} \quad \frac{|u|(b)}{\varphi'(b)^\gamma} < |\lambda| < \frac{|u|(a)}{\varphi'(a)^\gamma} \right\}, \qquad k \in \mathbb{Z},$$

have no accumulation points in the interior of the annulus of radii $\frac{|u|(b)}{\varphi'(b)^\gamma}, \frac{|u|(a)}{\varphi'(a)^\gamma}$. Then, $F_k(\lambda) - G_k(\lambda)$ is a non-zero vector lying in the kernel of $\lambda - uC_\varphi$ for each $\lambda \in \Omega_k$ as $(\lambda - uC_\varphi)(F_k(\lambda) - G_k(\lambda)) = f_k - f_k = 0$. It follows from (1.2) that

$$C_\varphi \left(\frac{b - (\cdot)}{a - (\cdot)} \right)^{\frac{2\pi i}{\delta}\theta} = e^{2\pi i \theta} \left(\frac{b - (\cdot)}{a - (\cdot)} \right)^{\frac{2\pi i}{\delta}\theta}, \qquad \theta \in \mathbb{R}, \tag{4.3}$$

so the functions $\left(\frac{b-(\cdot)}{a-(\cdot)}\right)^{\frac{2\pi i}{\delta}k}$, $k \in \mathbb{Z}$, are fixed points in $\mathcal{O}(\mathbb{D})$ for the composition operator C_φ. Therefore, one has

$$\begin{aligned}
(F_k(\lambda))(z) &= \sum_{n=0}^{\infty} \frac{(uC_\varphi|_{X_{\mu,0}})^n f_k(z)}{\lambda^{n+1}} = \sum_{n=0}^{\infty} \frac{(uC_\varphi|_{X_{\mu,0}})^n f_0(z)}{\lambda^{n+1}} \left(C_\varphi^n \left(\frac{b - (\cdot)}{a - (\cdot)} \right)^{\frac{2\pi i}{\delta}k} \right)(z) \\
&= \left(\frac{b - z}{a - z} \right)^{\frac{2\pi i}{\delta}k} \sum_{n=0}^{\infty} \frac{(uC_\varphi|_{X_{\mu,0}})^n f_0(z)}{\lambda^{n+1}} = \left(\frac{b - z}{a - z} \right)^{\frac{2\pi i}{\delta}k} (F_0(\lambda))(z),
\end{aligned}$$

for all $k \in \mathbb{Z}$, $z \in \mathbb{D}$ and $|\lambda| > \frac{|u|(b)}{\varphi'(b)^\gamma}$. Similar reasoning with the identity $G_k(\lambda) = -\sum_{n=0}^{\infty} \lambda^n (uC_\varphi|_{X_{0,\nu}})^{-n-1} f_k$ yields

$$(G_k(\lambda))(z) = \left(\frac{b - z}{a - z} \right)^{\frac{2\pi i}{\delta}k} (G_0(\lambda))(z), \qquad k \in \mathbb{Z},\ z \in \mathbb{D},\ |\lambda| < \frac{|u|(a)}{\varphi'(a)^\gamma}.$$

In consequence, $F_k(\lambda) - G_k(\lambda) = \left(\frac{b-(\cdot)}{a-(\cdot)}\right)^{\frac{2\pi i}{\delta}k} (F_0(\lambda) - G_0(\lambda))$ for all $k \in \mathbb{Z}$ and $\frac{|u|(b)}{\varphi'(b)^\gamma} < |\lambda| < \frac{|u|(a)}{\varphi'(a)^\gamma}$. This shows that Ω_k is independent of $k \in \mathbb{Z}$ and that the family $\{F_k(\lambda) - G_k(\lambda) \,:\, \lambda \in \Omega_k\}_{k \in \mathbb{Z}}$ is linearly independent (note that the family of functions $\left\{ \left(\frac{b-(\cdot)}{a-(\cdot)}\right)^{\frac{2\pi i}{\delta}k} \,:\, k \in \mathbb{Z} \right\}$ is linearly independent). Thus $\dim \ker(\lambda - uC_\varphi) = \infty$ for $\lambda \in \Omega_k$, and as Ω_k is dense in the annulus of radii $\frac{|u|(b)}{\varphi'(b)^\gamma}, \frac{|u|(a)}{\varphi'(a)^\gamma}$ we conclude that

$$\sigma_{ess}(uC_\varphi) = \left\{ \lambda \in \mathbb{C} \,:\, \frac{|u|(b)}{\varphi'(b)^\gamma} \le |\lambda| \le \frac{|u|(a)}{\varphi'(a)^\gamma} \right\},$$

where we have used that $\sigma_{ess}(uC_\varphi)$ is a closed subset of $\mathbb{D}$. Now take λ in the interior of such an annulus, so $\lambda - uC_\varphi$ is surjective by what we have already proven. Since $\lambda \in \sigma_{ess}(uC_\varphi)$ by the above, one deduces that $\dim\ker(\lambda - uC_\varphi) = \infty$ as claimed. (Note that we had proved this only for $\lambda \in \Omega_k$.) □

5. Case 2: $\frac{|u|(a)}{\varphi'(a)^\gamma} = \frac{|u|(b)}{\varphi'(b)^\gamma}$

Under the assumption $\frac{|u|(a)}{\varphi'(a)^\gamma} = \frac{|u|(b)}{\varphi'(b)^\gamma}$, the spectrum of uC_φ is obtained in [22, Th. 4.8] for $X = H^p(\mathbb{D}), \mathcal{A}^p_\sigma(\mathbb{D}), \mathcal{K}^{-\rho}(\mathbb{D})$ and in [15, Th. 4.6] for $X = B(\mathbb{D})$. Here, we note first that such a result follows easily from the spectral inclusion given in Proposition 3.1 whenever $Mul(X) = H^\infty(\mathbb{D})$. Indeed, if this is the case, multiplication by the function $\left(\frac{b-(\cdot)}{a-(\cdot)}\right)^{\frac{2\pi i}{\delta}\theta}$ defines an isomorphism on X, where $\delta = -\log\varphi'(a)$ and $\theta \in \mathbb{R}$. Thus, it follows by (4.3) that $e^{2\pi i\theta}uC_\varphi$ is a similar operator to uC_φ on X for every $\theta \in \mathbb{R}$. In consequence, the spectral sets of uC_φ are invariant to rotations. Since $\sigma(uC_\varphi)$ and $\sigma_{ess}(uC_\varphi)$ are non-empty subsets of $\mathbb{C}$, it follows by Proposition 3.1 that

$$\sigma(uC_\varphi) = \sigma_{ess}(uC_\varphi) = \left\{ \lambda \in \mathbb{C} \,:\, |\lambda| = \frac{|u|(a)}{\varphi'(a)^\gamma} = \frac{|u|(b)}{\varphi'(b)^\gamma} \right\}.$$

Inspired by this argument, we make use of the functions $\left(\frac{b-(\cdot)}{a-(\cdot)}\right)^{\frac{2\pi i}{\delta}\theta}$ to extend such a result to our framework.

Theorem 5.1. *Let X be a γ-space for some $\gamma \ge 0$, and let uC_φ be an invertible weighted composition operator on X. Assume that the automorphism φ is hyperbolic, with fixed points a (attractive), b (repulsive) in $\mathbb{T}$. Suppose furthermore that the absolute value $|u|$ can be continuously extended to $\mathbb{D} \cup \{a, b\}$ with $\frac{|u|(a)}{\varphi'(a)^\gamma} = \frac{|u|(b)}{\varphi'(b)^\gamma}$. Then*

$$\sigma(uC_\varphi) = \sigma_{ess}(uC_\varphi) = \left\{ \lambda \in \mathbb{C} \,:\, |\lambda| = \frac{|u|(a)}{\varphi'(a)^\gamma} = \frac{|u|(b)}{\varphi'(b)^\gamma} \right\}.$$

Proof. We know by Proposition 3.1 that $\sigma(uC_\varphi) \subseteq \left\{ \lambda \in \mathbb{C} \,:\, |\lambda| = \frac{|u|(a)}{\varphi'(a)^\gamma} = \frac{|u|(b)}{\varphi'(b)^\gamma} \right\}$. Let us prove the reverse inclusion. Fix a non-zero $f \in X$ and $\varepsilon > 0$. For each $\theta \in \mathbb{R}$, set

$$f_\theta(z) := (a - z)^{\varepsilon - \frac{2\pi i}{\delta}\theta} (b - z)^{\varepsilon + \frac{2\pi i}{\delta}\theta} f(z), \qquad z \in \mathbb{D},$$

where $\delta = -\log\varphi'(a) \in (0, \infty)$. By (**P2**), $\{f_\theta\}_{\theta\in\mathbb{R}} \subset X$. Fix $z_0 \in \mathbb{D}$ for the rest of the proof, and set the holomorphic functions H_θ, $\theta \in \mathbb{R}$, given by

$$H_\theta(\lambda) = ((\lambda - uC_\varphi)^{-1} f_\theta)(z_0), \qquad \lambda \in \rho(uC_\varphi) \subset \mathbb{C}.$$

Fixed $\theta \in \mathbb{R}$, clearly $\|H_\theta(\lambda)\|_X \to 0$ as $|\lambda| \to 0$ since $\|(\lambda - uC_\varphi)^{-1}\|_{\mathcal{L}(X)} \to 0$ as $|\lambda| \to 0$. By the Liouville theorem, H_θ does not extend to an entire function for each $\theta \in \mathbb{R}$. Hence the sets

$$\Omega_\theta := \{\lambda \in \mathbb{C} \, : \, H_\theta \text{ cannot be analytically extended in a neighborhood of } \lambda\} \subseteq \sigma(uC_\varphi),$$

are not empty for all $\theta \in \mathbb{R}$. Reasoning as in the proof of Theorem 4.4, we have

$$\begin{aligned} H_\theta(\lambda) &= \sum_{n=0}^{\infty} \frac{(uC_\varphi)^n f_\theta}{\lambda^{n+1}}(z_0) = \sum_{n=0}^{\infty} \frac{(uC_\varphi)^n f_0}{\lambda^{n+1}}(z_0) \left(C_\varphi^n \left(\frac{b-(\cdot)}{a-(\cdot)} \right)^{\frac{2\pi i}{\delta}\theta} \right)(z_0) \\ &= \left(\frac{b-z_0}{a-z_0} \right)^{\frac{2\pi i}{\delta}\theta} \sum_{n=0}^{\infty} e^{2\pi i n\theta} \frac{(uC_\varphi)^n f_0}{\lambda^{n+1}}(z_0) \\ &= e^{-2\pi i\theta} \left(\frac{b-z_0}{a-z_0} \right)^{\frac{2\pi i}{\delta}\theta} \sum_{n=0}^{\infty} \frac{(uC_\varphi)^n f_0}{(e^{-2\pi i\theta}\lambda)^{n+1}}(z_0) \\ &= e^{-2\pi i\theta} \left(\frac{b-z_0}{a-z_0} \right)^{\frac{2\pi i}{\delta}\theta} H_0(e^{-2\pi i\theta}\lambda), \qquad \theta \in \mathbb{R},\, |\lambda| > \frac{|u|(a)}{\varphi'(a)^\gamma}, \end{aligned}$$

where we have used (4.3) at the third "=" sign. Similarly one gets $H_\theta(\lambda) = e^{-2\pi i\theta} \left(\frac{b-z_0}{a-z_0} \right)^{\frac{2\pi i}{\delta}\theta} H_0(e^{-2\pi i\theta}\lambda)$ for all $|\lambda| < \frac{|u|(a)}{\varphi'(a)^\gamma}$ and $\theta \in \mathbb{R}$. In consequence, $\Omega_\theta = e^{2\pi i\theta}\Omega_0$ for all $\theta \in \mathbb{R}$. Since these sets are non-empty, we conclude

$$\sigma(uC_\varphi) \supseteq \bigcup_{\theta\in\mathbb{R}} \Omega_\theta = \bigcup_{\theta\in\mathbb{R}} e^{2\pi i\theta}\Omega_0 = \left\{ \lambda \in \mathbb{C} \, : \, |\lambda| = \frac{|u|(a)}{\varphi'(a)^\gamma} = \frac{|u|(b)}{\varphi'(b)^\gamma} \right\},$$

as we wanted to show.

To end the proof, note that every point in $\sigma(uC_\varphi)$ is an accumulation point of both $\sigma(uC_\varphi)$ and $\mathbb{C} \setminus \sigma(uC_\varphi)$. In consequence, $\sigma(uC_\varphi) = \sigma_{ess}(uC_\varphi)$ as claimed, see for instance [14, Th. I.3.25]. □

6. Case 3: $\frac{|u|(a)}{\varphi'(a)^\gamma} < \frac{|u|(b)}{\varphi'(b)^\gamma}$

Here we deal with the last case, when $\frac{|u|(a)}{\varphi'(a)^\gamma} < \frac{|u|(b)}{\varphi'(b)^\gamma}$. It should be pointed out that the spectrum of uC_φ in this case has remained unknown for all the spaces here considered except for the disk algebra $\mathfrak{A}(\mathbb{D})$, see [23]. To address this last case, we take a short detour to study the embeddability of the weight u into a cocycle $(u_t)_{t\in\mathbb{R}}$ for the (unique) hyperbolic flow $(\varphi_t)_{t\in\mathbb{R}}$ for which $\varphi_1 = \varphi$. This approach is heavily inspired by the study on C_0-groups of weighted composition operators carried out in [1]. We also note that the surjectivity of the operators $\lambda - uC_\varphi$ for $\frac{|u|(b)}{\varphi'(b)^\gamma} < \frac{|u|(a)}{\varphi'(a)^\gamma}$ and λ in the interior of $\sigma(uC_\varphi)$ plays a key role in this section, see Corollary 6.6.

6.1. Embeddability into (semi)cocycles

In this subsection we consider general (semi)flows and general (semi)cocycles (not only hyperbolic flows). We start with their definitions first. We give in Theorem 6.4 a characterization of those weights that can be embedded into a (semi)cocycle for a given (semi)flow. We particularize our results for hyperbolic automorphisms in Corollary 6.6, see also Corollary 6.7.

Definition 6.1. A family $(\varphi_t)_{t\geq 0}$ of morphisms of $\mathbb{D}$ is a (holomorphic) *semiflow* if

(1) $\varphi_0(z) = z$ for all $z \in \mathbb{D}$;
(2) $\varphi_{s+t} = \varphi_s \circ \varphi_t$ for all $s, t \geq 0$;
(3) the mapping $(t, z) \mapsto \varphi_t(z)$ is continuous on $[0, \infty) \times \mathbb{D}$.

When t runs over the whole real line in (φ_t), and (2) and (3) hold for every $s, t \in \mathbb{R}$ the family $(\varphi_t)_{t\in\mathbb{R}}$ is called (holomorphic) *flow*.

The infinitesimal generator of a given (semi)flow $(\varphi_t)_t$ is the function Φ defined by $\Phi(z) := \lim_{t\to 0} t^{-1}(\varphi_t(z) - z)$, $z \in \mathbb{D}$. Actually, the limit exists uniformly on compact subsets of $\mathbb{D}$, the mapping $t \mapsto \varphi_t(z)$ is differentiable on $[0, \infty)$ for every $z \in \mathbb{D}$, and

$$\frac{\partial \varphi_t(z)}{\partial t} = \Phi(\varphi_t(z)) = \Phi(z)\frac{\partial \varphi_t(z)}{\partial z}, \qquad z \in \mathbb{D},\ t \geq 0. \tag{6.1}$$

Definition 6.2. Let $(\varphi_t)_{t\geq 0}$ be a semiflow. A one-parameter family $(u_t)_{t\geq 0}$ of analytic functions $u_t\colon \mathbb{D} \to \mathbb{C}$ is called a (differentiable) *semicocycle* for $(\varphi_t)_{t\geq 0}$ if

(1) $u_0(z) = 1$ for all $z \in \mathbb{D}$;
(2) $u_{s+t} = u_t \cdot (u_s \circ \varphi_t)$ for all $s, t \geq 0$ (the symbol "$\cdot$" denotes the point-wise product);
(3) the mapping $t \mapsto u_t(z)$ is differentiable on $[0, \infty)$ for every $z \in \mathbb{D}$.

Suppose $(\varphi_t)_{t\in\mathbb{R}}$ is a flow. If u_t is given for all $t \in \mathbb{R}$ and the above properties hold for every $t \in \mathbb{R}$ we say that $(u_t)_{t\in\mathbb{R}}$ is a *cocycle* for $(\varphi)_{t\in\mathbb{R}}$.

We use the notation $(\varphi_t)_t$ to denote either $(\varphi_t)_{t\geq 0}$ in the context of semiflows or $(\varphi_t)_{t\in\mathbb{R}}$ in the context of flows. We do analogously for semicocycles and cocycles.

The infinitesimal generator G of a differentiable (semi)cocycle $(u_t)_t$ is defined by $G(z) := \frac{\partial}{\partial t} u_t(z)\big|_{t=0}$, and satisfies the following identity

$$u_t(z) = \exp\left(\int_0^t G(\varphi_s(z))\, ds\right) = \exp\left(\int_z^{\varphi_t(z)} \frac{G(w)}{\Phi(w)}\, dw\right), \qquad z \in \mathbb{D},\ t \geq 0,\ (t \in \mathbb{R}), \tag{6.2}$$

where we have used (6.1) at the second "=" sign above. We refer the reader to [26,30] for these and more facts about (semi)flows and (semi)cocycles.

Definition 6.3. Let $(\varphi_t)_t$ be a (semi)flow. We say that a holomorphic function $u : \mathbb{D} \to \mathbb{C}$ is *embeddable* into a (semi)cocycle $(u_t)_t$ for $(\varphi_t)_t$ if there exists a (semi)cocycle $(u_t)_t$ for $(\varphi_t)_t$ with $u_1 = u$.

The following result characterizes the embeddability of a function into a differentiable (semi)cocycle in terms of the range space of a translation of the composition operator C_{φ_1}.

Theorem 6.4. *Let $(\varphi_t)_t$ be a (semi)flow on $\mathbb{D}$ with generator Φ, and let $u \in \mathcal{O}(\mathbb{D})$. Then u is embeddable into a differentiable (semi)cocycle $(u_t)_t$ for $(\varphi_t)_t$ if and only if the following holds*

i) *$u(z) \neq 0$ for all $z \in \mathbb{D}$,*
ii) *there exists $G \in \mathcal{O}(\mathbb{D})$ such that*

$$\Phi \frac{u'}{u} = (C_{\varphi_1} - I)G.$$

In the positive case, then there exists $K \in \mathbb{C}$ such that $G + K$ is the generator of such a (semi)cocycle $(u_t)_t$, where G is the function of item ii) above.

Proof. First, we note that u is embeddable into a differentiable (semi)cocycle for $(\varphi_t)_t$ if and only if, for each $C \in \mathbb{C} \setminus \{0\}$, Cu is embeddable into a differentiable (semi)cocycle for $(\varphi_t)_t$. In fact, let G be the generator of a differentiable (semi)cocycle $(u_t)_t$ for $(\varphi_t)_t$ with $u_1 = u$. Then $G + \log C$ is the generator of the differentiable semicocycle $(e^{t \log C} u_t)_t$, see (6.2).

By [26, Lemma 2.1(b)], if u is embeddable into a (semi)cocycle then u cannot have zeroes on $\mathbb{D}$. Thus, we can assume that u has no zeroes without loss of generality. By the paragraph above and (6.2), u is embeddable into differentiable a (semi)cocycle $(u_t)_t$ for $(\varphi_t)_t$ if and only if there exists $G \in \mathcal{O}(\mathbb{D})$ such that the holomorphic function given by

$$z \mapsto \frac{\exp\left(\int_0^1 G(\varphi_s(z))\, ds\right)}{u(z)}, \qquad z \in \mathbb{D},$$

is a constant function with value $C \in \mathbb{C} \setminus \{0\}$. Differentiating the above expression, this happens if and only if

$$\frac{u'(z)}{u(z)} = \frac{\partial}{\partial z}\left(\int_0^1 G(\varphi_s(z))\, ds\right) = \int_0^1 G'(\varphi_s(z))\varphi_s'(z)\, ds$$

$$= \frac{1}{\Phi(z)} \int_0^1 G'(\varphi_s(z)) \frac{\partial \varphi_s(z)}{\partial s}\, ds = \frac{1}{\Phi(z)} (G(\varphi_1(z)) - G(z)), \qquad z \in \mathbb{D},$$

and the proof is done. Note that we have used (6.1) at the third "=" sign above. □

Remark 6.5. Let $(\varphi_t)_t$ be a (semi)flow on $\mathbb{D}$, and let $u \in \mathcal{O}(\mathbb{D})$ be embeddable into a differentiable (semi)cocycle $(u_t)_t$ for $(\varphi_t)_t$. We note that $(u_t)_t$ is not the unique (semi)cocycle u is embedded into. For instance, u is embeddable into $(e^{2k\pi i t}u_t)_t$ for each $k \in \mathbb{Z}$.

More generally, assume $h \in \mathcal{O}(\mathbb{D})$ is a non constant function such that $C_{\varphi_1} h = h$. By Theorem 6.4, there exists $K \in \mathbb{C}$ such that the differentiable (semi)cocycle $(v_t)_t$ given by

$$v_t(z) = u_t(z) \exp\left(Kt + \int_0^t h(\varphi_s(z))\, ds \right), \qquad z \in \mathbb{D},\ t \in \mathbb{R},$$

is different from $(u_t)_t$ and satisfies $v_1 = u_1 = u$.

We now focus again on hyperbolic automorphisms. Let φ be a hyperbolic automorphism with fixed points a (attractive), b (repulsive) in $\mathbb{T}$. It is well known that φ can be embedded into a (unique) flow $(\varphi_t)_{t\in\mathbb{R}}$, which is given by

$$\varphi_t(z) = \frac{(b - ae^{\delta t})z + ab(e^{\delta t} - 1)}{(1 - e^{\delta t})z + be^{\delta t} - a}, \qquad z \in \mathbb{D},\ t \in \mathbb{R}, \tag{6.3}$$

where $\delta := -\log \varphi'(a) \in (0, \infty)$, see for example [9, Cor. 8.2.7] and compare with (1.2). The following identity, which will be used multiple times through this paper, is straightforward to check

$$\frac{b - \varphi_t(z)}{a - \varphi_t(z)} = e^{\delta t} \frac{b - z}{a - z}, \qquad z \in \mathbb{D},\ t \in \mathbb{R}. \tag{6.4}$$

Also, the generator Φ of $(\varphi_t)_{t\in\mathbb{R}}$ is given by

$$\Phi(z) := \left.\frac{\partial \varphi_t(z)}{\partial t}\right|_{t=0} = \frac{\delta}{b - a}(z - a)(z - b), \qquad z \in \mathbb{D}. \tag{6.5}$$

Finally, the so called Koenigs model is key for the understanding of $(\varphi_t)_{t\in\mathbb{R}}$, see for instance [9, Prop. 9.3.12]. Namely, let $h : \mathbb{D} \to \left\{ z \in \mathbb{C} \,:\, 0 < \mathfrak{Re}\,(z) < \frac{\pi}{\delta} \right\}$ be the biholomorphism given by

$$h(z) = \frac{i}{\delta} \log\left(\frac{a + z}{a - z} - \frac{a + b}{a - b} \right) + \frac{\pi}{2\delta}, \qquad z \in \mathbb{D}. \tag{6.6}$$

Then, one has $h(\varphi_t(z)) = h(z) + it$ for $z \in \mathbb{D}$, $t \in \mathbb{R}$. We refer the reader to [9] for more details about flows and semiflows of $\mathbb{D}$.

Given a γ-space X and a hyperbolic automorphism φ with fixed points a (attractive), b (repulsive) in $\mathbb{T}$, we define the inductive space $\widetilde{X} \subset \mathcal{O}(\mathbb{D})$ as

$$\widetilde{X} := \bigcup_{\mu,\nu\in\mathbb{R}} X_{\mu,\nu} = \bigcup_{\mu,\nu\in\mathbb{R}} \{\rho_{\mu,\nu} f \,:\, f \in X\},$$

where $\rho_{\mu,\nu}(z) = (a-z)^{\mu}(b-z)^{\nu}$, $z \in \mathbb{D}$. Then, as an immediate consequence of Theorem 6.4, we get the following result.

Corollary 6.6. *Let $\gamma > 0$ and let X be a γ-space containing the constant functions, and pick a hyperbolic automorphism φ with fixed points a (attractive), b (repulsive) in $\mathbb{T}$. Let $(\varphi_t)_{t\in\mathbb{R}}$ the (unique) hyperbolic flow for which $\varphi_1 = \varphi$ and let Φ be its generator. Pick a non-vanishing holomorphic function u on $\mathbb{D}$.*

i) *If $\Phi\frac{u'}{u} \in X$, then u is embeddable into a differentiable cocycle $(u_t)_{t\in\mathbb{R}}$ for $(\varphi_t)_{t\in\mathbb{R}}$ with generator G lying in X.*
ii) *If $u'/u \in \widetilde{X}$, then u is embeddable into a cocycle $(u_t)_{t\in\mathbb{R}}$ for $(\varphi_t)_{t\in\mathbb{R}}$ with generator G lying in $\widetilde{X}$.*

Proof. i) By Theorem 4.4, the operator $C_\varphi - I = C_{\varphi_1} - I$ is surjective on X. Then the claim follows by Theorem 6.4.

ii) By hypothesis, there exist $\mu, \nu \in \mathbb{R}$ such that $u'/u \in X_{\mu,\nu}$. By (**P2**), $\Phi u'/u \in X_{\widetilde{\mu},\widetilde{\nu}}$ for all $\widetilde{\mu} \le \mu + 1$, $\widetilde{\nu} \le \nu + 1$ (recall that Φ is given by (6.5)). By Theorem 4.4 and (4.1), the operator $I - \dfrac{\rho_{\widetilde{\mu},\widetilde{\nu}} \circ \varphi}{\rho_{\widetilde{\mu},\widetilde{\nu}}} C_\varphi$ is surjective on X for each $\widetilde{\mu}, \widetilde{\nu} < 0$ (recall that $\varphi'(a) \in (0,1)$ and $\varphi'(b) \in (1,\infty)$). As the operator $I - C_\varphi$ on $X_{\widetilde{\mu},\widetilde{\nu}}$ is isometrically equivalent to the operator $I - \dfrac{\rho_{\widetilde{\mu},\widetilde{\nu}} \circ \varphi}{\rho_{\widetilde{\mu},\widetilde{\nu}}} C_\varphi$ on X (see Lemma 4.2), we conclude that $I - C_\varphi$ is surjective on $X_{\widetilde{\mu},\widetilde{\nu}}$. Thus there exists $G \in X_{\widetilde{\mu},\widetilde{\nu}}$ such that $\Phi\frac{u'}{u} = (I - C_\varphi)G$, and our claim follows by Theorem 6.4. □

6.2. Spectra of uC_φ via cocycles

Now we use the results given in the preceding subsection to study the spectrum of uC_φ in the remaining case $\frac{|u|(a)}{\varphi'(a)^\gamma} < \frac{|u|(b)}{\varphi'(b)^\gamma}$. To start with, an immediate (but important) consequence of Corollary 6.6 is the following.

Corollary 6.7. *Let X be a γ-space for some $\gamma \ge 0$, and let uC_φ be an invertible weighted composition operator on X. Assume that the automorphism φ is hyperbolic and let $(\varphi_t)_{t\in\mathbb{R}}$ the (unique) hyperbolic flow for which $\varphi_1 = \varphi$. Then u is embeddable into a differentiable cocycle $(u_t)_{t\in\mathbb{R}}$ for $(\varphi_t)_{t\in\mathbb{R}}$ with generator G lying in $\mathcal{K}^{-1}(\mathbb{D})$.*

Proof. Recall that the boundedness and invertibility of uC_φ implies $\sup_{z\in\mathbb{D}}|u(z)| < \infty$ and $\inf_{z\in\mathbb{D}}|u(z)| > 0$, see Remark 2.5. Thus, a simple estimate with Cauchy's integral formula for derivatives shows $u' \in \mathcal{K}^{-1}(\mathbb{D})$. Hence $\Phi\frac{u'}{u} \in \mathcal{K}^{-1}(\mathbb{D})$, where Φ is the generator of $(\varphi_t)_{t\in\mathbb{R}}$, and the claim follows by item i) in Corollary 6.6. □

Remark 6.8. Let $(u_t)_{t\in\mathbb{R}}$ be a differentiable cocycle for a hyperbolic flow $(\varphi_t)_{t\in\mathbb{R}}$. It is well known that $(u_t)_{t\in\mathbb{R}}$ is a coboundary, that is, there exists a non-vanishing function $\omega \in \mathcal{O}(\mathbb{D})$, which we refer to as a *non-vanishing function* associated to $(u_t)_{t\in\mathbb{R}}$, such that

$$u_t(z) = \frac{\omega\circ\varphi_t(z)}{\omega(z)}, \qquad z\in\mathbb{D},\, t\in\mathbb{R},$$

see [26, Lemma 2.2]. Even more, such ω is given by

$$\omega(z) = \exp\left(\int_c^z \frac{G(\xi)}{\Phi(\xi)}\,d\xi\right), \qquad z\in\mathbb{D}, \tag{6.7}$$

where c is an arbitrary number in $\mathbb{D}$, G is the generator of $(u_t)_{t\in\mathbb{R}}$ and Φ is the generator of $(\varphi_t)_{t\in\mathbb{R}}$.

Given $a \neq b \in \mathbb{T}$, we denote by Γ_b^a the hyperbolic geodesic in $\mathbb{D}$ starting in b and ending in a. In other words, Γ_b^a is the arc of the circle containing a, b and intersecting orthogonally $\mathbb{T}$ at a, b (see for instance [9, Section 1.3]). A parametrization of Γ_b^a is given by $t \mapsto \varphi_t(c)$ for $t \in \mathbb{R}$, where $(\varphi_t)_{t\in\mathbb{R}}$ is any hyperbolic flow with fixed points a (attractive), b (repulsive), and where c is an arbitrary point in Γ_b^a. Note also that $\varphi_t(\Gamma_b^a) = \Gamma_b^a$ for all $t\in\mathbb{R}$.

The result below, inspired by [1, Th. 3.11], provides that non-vanishing functions associated to suitable cocycles have polynomial-like behavior near the fixed points of the corresponding hyperbolic flow, see also Remark 6.10.

Proposition 6.9. *Let $(u_t)_{t\in\mathbb{R}}$ be a differentiable cocycle for a hyperbolic flow $(\varphi_t)_{t\in\mathbb{R}}$ with fixed points a (attractive), b (repulsive) in $\mathbb{T}$. Suppose that the absolute value $|u_1|$ can be continuously extended to $\mathbb{D}\cup\{a,b\}$ with $|u|(a), |u|(b) \in (0,\infty)$. Set $\alpha := \log|u|(a)$, $\beta := \log|u|(b)$ and $\delta := -\log\varphi'(a) \in (0,\infty)$. Let ω be a non-vanishing holomorphic function associated to $(u_t)_{t\in\mathbb{R}}$. For every $\varepsilon > 0$, one has*

$$|\omega(z)| \lesssim |a-z|^{-\alpha/\delta-\varepsilon}|b-z|^{\beta/\delta-\varepsilon}, \qquad |\omega(z)| \gtrsim |a-z|^{-\alpha/\delta+\varepsilon}|b-z|^{\beta/\delta+\varepsilon}, \qquad z\in\Gamma_b^a.$$

Proof. Since ω is non-vanishing, it is bounded and bounded away from 0 in compact subsets of $\mathbb{D}$. Thus, we only have to prove the inequalities of the claim for all $z \in \Gamma_a^b \cap (\mathcal{U}_a \cup \mathcal{U}_b)$, where $\mathcal{U}_a, \mathcal{U}_b$ are neighborhoods of a, b respectively. We prove it for all $z \in \Gamma_b^a \cap \mathcal{U}_a$, being the case $z \in \Gamma_b^a \cap \mathcal{U}_b$ analogous. Set $\mathbb{D}_a = \{z \in \mathbb{D} : \mathfrak{Im}\,(h(z)) > 0\}$, where h is given by the Koenigs model of $(\varphi)_{t\in\mathbb{R}}$, see (6.6). Take $\varepsilon > 0$. By Lemma A.2, there exists $N \in \mathbb{N}$ such that

$$|\omega(\varphi_n(z))| = |u_n(z)||\omega(z)| \leq e^{(\alpha+\varepsilon)n}|\omega(z)|, \qquad z \in \mathbb{D}_a,\ n \in \mathbb{N} \text{ with } n \geq N. \tag{6.8}$$

By (6.3), one has

$$\frac{a-\varphi_t(z)}{a-z}e^{\delta t} = \frac{b-a}{b-z+e^{-\delta t}(z-a)}, \qquad z \in \mathbb{D},\ t \in \mathbb{R}.$$

In consequence, $\dfrac{a-\varphi_t(z)}{a-z}e^{\delta t} \to 1$ as $z \to a$ uniformly for all $t \geq 0$. This fact and (6.8) yield, for a suitable neighborhood $\mathcal{V}_a$ in $\mathbb{D}_a$ of a,

$$|\omega(\varphi_n(z))| \lesssim \left|\frac{a-\varphi_n(z)}{a-z}\right|^{-(\alpha+\varepsilon)/\delta} |\omega(z)|, \qquad z \in \mathcal{V}_a,\ n \in \mathbb{N} \text{ with } n \geq N. \tag{6.9}$$

Now take a neighborhood $\mathcal{U}_a \in \mathbb{D}$ of a such that $\mathcal{U}_a \subsetneq \mathcal{V}_a$ and for which, for every $\xi \in \mathcal{U}_a$, there exists $n_\xi \geq N$ so $\varphi_{-n_\xi}(\xi) \in \mathcal{V}_a \setminus \mathcal{U}_a$. Note that such a neighborhood $\mathcal{U}_a$ exists by (6.6). Applying (6.9) to $z = \varphi_{-n_\xi}(\xi)$ and $n = n_\xi$,

$$|\omega(\xi)| \lesssim \left|\frac{a-\xi}{a-\varphi_{-n_\xi}(\xi)}\right|^{-(\alpha+\varepsilon)/\delta} |\omega(\varphi_{-n_\xi}(\xi))|, \qquad \xi \in \mathcal{U}_a. \tag{6.10}$$

Recall that $\varphi(\Gamma_b^a) = \Gamma_b^a$. Thus, for each $\xi \in \Gamma_b^a \cap \mathcal{U}_a$, one has $\varphi_{-n_\xi}(\xi) \in \Gamma_b^a \cap (\mathcal{V}_a \setminus \mathcal{U}_a)$. As the closure of $\Gamma_b^a \cap (\mathcal{V}_a \setminus \mathcal{U}_a)$ is a compact subset of $\mathbb{D}$, one has

$$\sup_{\xi \in \Gamma_b^a \cap \mathcal{U}_a} |a - \varphi_{-n_\xi}(\xi)|^{(\alpha+\varepsilon)/\delta} \leq \sup_{z \in \Gamma_b^a \cap (\mathcal{V}_a \setminus \mathcal{U}_a)} |a-z|^{(\alpha+\varepsilon)/\delta} < \infty,$$

and similarly

$$\sup_{\xi \in \mathcal{U}_a} |\omega(\varphi_{-n_\xi}(\xi))| \leq \sup_{z \in \Gamma_b^a \cap (\mathcal{V}_a \setminus \mathcal{U}_a)} |\omega(z)| < \infty.$$

Hence, by (6.10),

$$|\omega(\xi)| \lesssim |a-\xi|^{-(\alpha+\varepsilon)/\delta}, \qquad \xi \in \Gamma_b^a \cap \mathcal{U}_a.$$

As said at the beginning of the proof, an analogous reasoning proves the existence of neighborhood $\mathcal{U}_b$ in $\mathbb{D}$ of b such that $|\omega(\xi)| \lesssim |b-\xi|^{(\beta-\varepsilon)/\delta}$ for all $\xi \in \Gamma_b^a \cap \mathcal{U}_b$. By the arbitrariness of $\varepsilon > 0$, we obtain the upper bound of the theorem.

Finally, the inequality $\gtrsim$ of the claim follows by an application of what we have already proven to the differentiable cocycle $(1/u_t)_{t\in\mathbb{R}}$, whose associated non-vanishing holomorphic function is given by $1/\omega$. $\square$

Remark 6.10. A Stolz region $S(\sigma, R)$ of vertex $\sigma \in \mathbb{T}$ and amplitude $R > 1$ is given by

$$S(\sigma, R) := \left\{ z \in \mathbb{D} \, : \, \frac{|\sigma - z|}{1 - |z|} < R \right\}.$$

Let ω be as in Proposition 6.9. Adding a few extra steps in the proof of Proposition 6.9, one can prove that the polynomial bounds on ω hold non-tangentially at a, b. That is, for each $\varepsilon > 0$ and $R > 1$, one has

$$|\omega(z)| \lesssim |a-z|^{-\alpha-\varepsilon}|b-z|^{\beta-\varepsilon}, \qquad |\omega(z)| \gtrsim |a-z|^{-\alpha+\varepsilon}|b-z|^{\beta+\varepsilon}, \qquad z \in S(a,R) \cup S(b,R).$$

For suitable $\mu \in \mathbb{C}$, $\omega \in \mathcal{O}(\mathbb{D})$ and $f \in \mathcal{O}(\mathbb{D})$, we define

$$L^\mu_\omega f := \int\limits_{\Gamma^a_b} \frac{(a-z)^{\mu-1}}{(b-z)^{\mu+1}} \omega(z) f(z)\, dz. \tag{6.11}$$

Recall that Γ^a_b is the hyperbolic geodesic in $\mathbb{D}$ starting in b and ending in a, being both points in $\mathbb{T}$. Our notation omits the dependence of L^μ_ω on a, b for the sake of readability.

The following lemma is essentially contained in [1, Sect. 5].

Lemma 6.11. *Let X be a γ-space for some $\gamma \geq 0$ and fix $a \neq b \in \mathbb{T}$. Let $\omega \in \mathcal{O}(\mathbb{D})$ be a non-zero function such that there exist $\alpha, \beta \in \mathbb{R}$ for which, for each $\varepsilon > 0$, one has*

$$|\omega(z)| \lesssim |a-z|^{-\alpha-\varepsilon}|b-z|^{\beta-\varepsilon}, \qquad z \in \Gamma^a_b.$$

For $\mu \in \mathbb{C}$ with $\alpha + \gamma < \mathfrak{Re}\,\mu < \beta - \gamma$, L^μ_ω defines a non-zero continuous functional on X.

Proof. Recall that, for each $\widetilde{\varepsilon} > 0$, $|f(z)| \lesssim (1-|z|)^{-\gamma-\widetilde{\varepsilon}}\|f\|_X$ for all $z \in \mathbb{D}$, $f \in X$, see Remark 2.2. Thus

$$|f(z)| \lesssim |a-z|^{-\gamma-\widetilde{\varepsilon}}|b-z|^{-\gamma-\widetilde{\varepsilon}}\|f\|_X, \qquad z \in \Gamma^a_b,\ f \in X.$$

Hence, for $\varepsilon > 0$ small enough,

$$\begin{aligned} &\int\limits_{\Gamma^a_b} \left| \frac{(a-z)^{\mu-1}}{(b-z)^{\mu+1}} \omega(z) f(z) \right| |dz| \\ &\lesssim \|f\|_X \int\limits_{\Gamma^a_b} |a-z|^{\mathfrak{Re}\,\mu-\alpha-\gamma-1-\varepsilon-\widetilde{\varepsilon}}|b-z|^{\beta-\mathfrak{Re}\,\mu-\gamma-1-\varepsilon-\widetilde{\varepsilon}}\,|dz| \\ &\lesssim \|f\|_X, \qquad f \in X, \end{aligned}$$

thus L_ω^μ is a continuous functional on X. Let us see that $L_\omega^\mu \neq 0$ by contradiction, so assume otherwise and let $f \in X \setminus \{0\}$. By the same reasoning as above, one has that the function $g := \left| \frac{(a-(\cdot))^{\mu-1}}{(b-(\cdot))^{\mu+1}} \omega f \right|$ is a non-zero positive function in $L^1(\Gamma_b^a)$, satisfying

$$\int_{\Gamma_b^a} pg = L_\omega^\mu(pf) = 0,$$

for every polynomial p, where we used above that $p \in Mul(X)$ by (**P1**). Therefore, $g = 0$, reaching the aforementioned contradiction. □

Now we are ready to prove the main result of this section.

Theorem 6.12. *Let X be a γ-space for some $\gamma \geq 0$, and let uC_φ be an invertible weighted composition operator on X. Assume that the automorphism φ is hyperbolic, with fixed points a (attractive), b (repulsive) in $\mathbb{T}$. Suppose furthermore that the absolute value $|u|$ can be continuously extended to $\mathbb{D} \cup \{a, b\}$ with $\frac{|u|(a)}{\varphi'(a)^\gamma} < \frac{|u|(b)}{\varphi'(b)^\gamma}$. Then*

$$\sigma(uC_\varphi) = \sigma_{ess}(uC_\varphi) = \left\{ \lambda \in \mathbb{C} \, : \, \frac{|u|(a)}{\varphi'(a)^\gamma} \leq |\lambda| \leq \frac{|u|(b)}{\varphi'(b)^\gamma} \right\}.$$

Moreover, for each λ lying in the interior of the above annulus, $\lambda - uC_\varphi$ is an injective and non-surjective operator with infinite-codimensional range. Even more, there exists a non-vanishing function $\omega \in \mathcal{O}(\mathbb{D})$ such that for every λ in the interior of $\sigma(uC_\varphi)$, L_ω^μ is a bounded functional on X for every $\mu \in W_\lambda := \{\mu \in \mathbb{C} \, : \, \varphi'(a)^\mu = 1/\lambda\}$ and one has

$$\operatorname{ran}(\lambda - uC_\varphi) \subseteq \cap_{\mu \in W_\lambda} \ker L_\omega^\mu.$$

Proof. By Proposition 3.1, we know $\lambda \notin \sigma(uC_\varphi)$ either if $|\lambda| < \frac{|u|(a)}{\varphi'(a)^\gamma}$ or $|\lambda| > \frac{|u|(b)}{\varphi'(b)^\gamma}$. Let $\alpha := \log |u|(a)$, $\beta := \log |u|(b)$ and $\delta := -\log \varphi'(a) \in (0, \infty)$. Let $(\varphi_t)_{t \in \mathbb{R}}$ be the (unique) hyperbolic flow for which $\varphi_1 = \varphi$, see (6.3). By Corollary 6.7, Remark 6.8 and Proposition 6.9 we have that u is embeddable into a differentiable cocycle $(u_t)_{t \in \mathbb{R}}$ for $(\varphi_t)_{t \in \mathbb{R}}$ with a non-vanishing associated function ω satisfying that, for each $\varepsilon > 0$,

$$|\omega(z)| \lesssim |a - z|^{-\alpha/\delta - \varepsilon} |b - z|^{\beta/\delta - \varepsilon}, \qquad |\omega(z)| \gtrsim |a - z|^{-\alpha/\delta + \varepsilon} |b - z|^{\beta/\delta + \varepsilon}, \qquad z \in \Gamma_b^a.$$

Let $\lambda \in \mathbb{C}$ with $\frac{|u|(a)}{\varphi'(a)^\gamma} < |\lambda| < \frac{|u|(b)}{\varphi'(b)^\gamma}$ and take $\mu \in W_\lambda$, so $e^{\delta\mu} = \lambda$ and $\alpha/\delta + \gamma < \mathfrak{Re}\, \mu < \beta/\delta - \gamma$. By Lemma 6.11, L_ω^μ is a continuous functional on X. Since $u = u_1 = (\omega \circ \varphi)/\omega$, we obtain using the change of variable $\xi = \varphi(z)$,

$$L_\omega^\mu(uC_\varphi f) = \int_{\Gamma_b^a} \frac{(a - z)^{\mu-1}}{(b - z)^{\mu+1}} (\omega \circ \varphi)(z)(f \circ \varphi)(z) \, dz$$

$$= \int_{\varphi(\Gamma_b^a)} \frac{(a-\varphi_{-1}(\xi))^{\mu-1}}{(b-\varphi_{-1}(\xi))^{\mu+1}} \omega(\xi) f(\xi) \frac{d\xi}{\varphi'(\varphi_{-1}(\xi))}$$

$$= \int_{\Gamma_b^a} e^{\delta\mu} \frac{(a-\xi)^{\mu-1}}{(b-\xi)^{\mu+1}} \omega(\xi) f(\xi)\, d\xi = \lambda\, L_\omega^\mu f, \qquad f \in X,$$

where we have used (6.4) with $t = -1$ and the identity $\varphi'(\varphi_{-1}(\xi)) = \frac{\Phi(\xi)}{\Phi(\varphi_{-1}(\xi))}$, see (6.1) and (6.5). Therefore, $\operatorname{ran}(\lambda - uC_\varphi) \subseteq \ker L_\omega^\mu$ for all $\mu \in W_\lambda$, as we wanted to prove. In particular, $\lambda - uC_\varphi$ is not surjective and $\lambda \in \sigma(uC_\varphi)$. Moreover, the functionals $\{L_\omega^\mu : \mu \in W_\lambda\}$ generate an infinite dimensional space in the dual space of X. Indeed, this is readily seen from the fact that

$$\left\{ \frac{(a-(\cdot))^{\mu-1}}{(b-(\cdot))^{\mu+1}} \ : \ \mu \in W_\lambda \right\},$$

is a linearly independent family on $\mathcal{O}(\mathbb{D})$. Then the subspace $\cap_{\mu\in W_\lambda} \ker L_\omega^\mu$ has infinite codimension, see for example [27, Lemma 3.9], and we conclude $\lambda \in \sigma_{ess}(uC_\varphi)$ for all $\lambda \in \mathbb{C}$ with $\frac{|u|(a)}{\varphi'(a)^\gamma} < |\lambda| < \frac{|u|(b)}{\varphi'(b)^\gamma}$. As $\sigma(uC_\varphi), \sigma_{ess}(uC_\varphi)$ are closed subsets of $\mathbb{C}$, we obtain $\sigma(uC_\varphi) = \sigma_{ess}(uC_\varphi) = \left\{\lambda \in \mathbb{C} \ : \ \frac{|u|(a)}{\varphi'(a)^\gamma} \le |\lambda| \le \frac{|u|(b)}{\varphi'(b)^\gamma}\right\}$.

All is left to prove is that $\lambda - uC_\varphi$ is an injective operator for every λ in the interior of $\sigma(uC_\varphi)$. To see this, fix any such λ and take $\mu < 0$ for which $|\lambda| < |u|(a)\varphi'(a)^{\mu-\gamma}$. Note that such a μ exists since $\varphi'(a) \in (0,1)$. By (4.1), one has

$$|\lambda| < \min\left\{ \frac{\left|\frac{\rho_{\mu,0}\circ\varphi}{\rho_{\mu,0}} u\right|(a)}{\varphi'(a)^\gamma}, \frac{\left|\frac{\rho_{\mu,0}\circ\varphi}{\rho_{\mu,0}} u\right|(b)}{\varphi'(b)^\gamma} \right\},$$

so $\lambda - \frac{\rho_{\mu,0}\circ\varphi}{\rho_{\mu,0}} uC_\varphi$ is an invertible operator on X by Proposition 3.1 (with $\rho_{\mu,0}(z) = (a-z)^\mu$, $z \in \mathbb{D}$, see Section 4). It follows by Lemma 4.2 that the operator $\lambda - uC_\varphi$ acting on $X_{\mu,0} = \{\rho_{\mu,0} f \ : \ f \in X\}$ is isometrically equivalent to the operator $\lambda - \frac{\rho_{\mu,0}\circ\varphi}{\rho_{\mu,0}} uC_\varphi$ acting on X. In particular, $\lambda - uC_\varphi$ is injective on $X_{\mu,0}$, and so is in X since $X \subset X_{\mu,0}$ (note that $1/\rho_{\mu,0} \in Mul(X)$ for each $\mu < 0$ by (**P2**)). $\square$

7. Eigenvectors of uC_φ via cocycles

The embedding of u into a cocycle (and more precisely, into a coboundary) $(u_t)_{t\in\mathbb{R}}$ enabled us to obtain precise information regarding the range spaces of $\lambda - uC_\varphi$ in Theorem 6.12 in the case $\frac{|u|(a)}{\varphi'(a)^\gamma} < \frac{|u|(b)}{\varphi'(b)^\gamma}$. In this section, we ponder whether such an embedding also gives important information to obtain the eigenspaces of uC_φ for the case $\frac{|u|(b)}{\varphi'(b)^\gamma} < \frac{|u|(a)}{\varphi'(a)^\gamma}$ studied in Section 4. (Recall that we did not use the cocycle $(u_t)_{t\in\mathbb{R}}$ in that section.)

To start with, the embedding of u into a coboundary $(u_t)_{t\in\mathbb{R}} = \left(\frac{\omega\circ\varphi_t}{\omega}\right)_{t\in\mathbb{R}}$ gives us the following immediate characterization of the eigenspaces of uC_φ in terms of the eigenspaces of the unweighted composition operator C_φ.

Proposition 7.1. *Let X be a γ-space for some $\gamma \geq 0$, and let uC_φ be an invertible weighted composition operator on X. Assume that the automorphism φ is hyperbolic, with fixed points a (attractive), b (repulsive) in $\mathbb{T}$. Let $(\varphi_t)_{t\in\mathbb{R}}$ be the (unique) hyperbolic flow for which $\varphi_1 = \varphi$, and let $\omega \in \mathcal{O}(\mathbb{D})$ be a non-vanishing function associated to a differentiable cocycle $(u_t)_{t\in\mathbb{R}}$ for $(\varphi_t)_{t\in\mathbb{R}}$ such that $u_1 = u$.*

i) *Take $\lambda \in \mathbb{C}$. Then*

$$\ker(\lambda - uC_\varphi) = \left\{\frac{1}{\omega}f \,:\, f\in\mathcal{O}(\mathbb{D}) \textit{ with } \quad C_\varphi f = \lambda f \quad\right\}\cap X.$$

ii) *Suppose furthermore that the absolute value $|u|$ can be continuously extended to $\mathbb{D}\cup\{a,b\}$ with $\frac{|u|(b)}{\varphi'(b)^\gamma} < \frac{|u|(a)}{\varphi'(a)^\gamma}$, and fix $\mu,\nu > 0$ such that $|u|(a)\,\varphi'(a)^{\mu-\gamma} < |u|(b)\,\varphi'(b)^{\nu-\gamma}$. For each $f \in X_{\mu,\nu} = \{\rho_{\mu,\nu} g \,:\, g\in X\}$, the function*

$$J_{f,\lambda} := \frac{1}{\omega}\sum_{n=-\infty}^{\infty}\frac{(\omega f)\circ\varphi_n}{\lambda^{n+1}}$$

is well defined, belongs to X and lies in $\ker(\lambda - uC_\varphi)$ *for every λ in the (open) annulus*

$$\Omega_{\mu,\nu} = \left\{\widetilde{\lambda}\in\mathbb{C} \,:\, \max\left\{\frac{|u|(b)}{\varphi'(b)^\gamma}, |u|(a)\,\varphi'(a)^{\mu-\gamma}\right\} < |\widetilde{\lambda}| < \min\left\{\frac{|u|(a)}{\varphi'(a)^\gamma}, |u|(b)\,\varphi'(b)^{\nu-\gamma}\right\}\right\}.$$

Moreover, the set $\{\lambda\in\Omega_{\mu,\nu} \,:\, J_{f,\lambda} = 0\}$ has no accumulation points in $\Omega_{\mu,\nu}$.

Proof. i) Set $\Lambda_{\omega,\lambda} := \left\{\frac{1}{\omega}f \,:\, f\in\mathcal{O}(\mathbb{D}) \text{ with } \quad C_\varphi f = \lambda f \quad\right\}\cap X$, and let us show that $\Lambda_{\omega,\lambda} = \ker(\lambda - uC_\varphi)$.

As $u = u_1 = \frac{\omega\circ\varphi}{\omega}$, it is readily seen that every function $g = \frac{1}{\omega}f \in \Lambda_{\omega,\lambda} \subset X$ satisfies $uC_\varphi g = \lambda g$, so λ is an eigenvalue for uC_φ and $\Lambda_{\omega,\lambda}$ is contained in the eigenspace of λ. For the reverse inclusion, assume that λ is an eigenvalue of uC_φ and let g be a non-trivial eigenvector associated to λ. Then $g = \frac{1}{\omega}f$, where $f := \omega g$ and

$$C_\varphi f = \omega\frac{\omega\circ\varphi}{\omega}(g\circ\varphi) = \omega\, uC_\varphi g = \lambda\omega g = \lambda f.$$

Thus $g\in\Lambda_{\omega,\lambda}$ as claimed.

ii) This is a straightforward consequence of the following observation: mimicking the steps of the proof of Theorem 4.4, one concludes that, for each $f \in X_{\mu,\nu}$, the function

$$\lambda \mapsto (\lambda - uC_\varphi|_{X_{\mu,0}})^{-1} f - (\lambda - uC_\varphi|_{X_{0,\nu}})^{-1} f = \sum_{n=0}^{\infty} \frac{(uC_\varphi)^n}{\lambda^{n+1}} f + \sum_{n=0}^{\infty} \lambda^n (uC_\varphi)^{-n-1} f$$
$$= \sum_{n=-\infty}^{\infty} \frac{(uC_\varphi)^n}{\lambda^{n+1}} f = \frac{1}{\omega} \sum_{n=-\infty}^{\infty} \frac{(\omega f)\circ \varphi_n}{\lambda^{n+1}} = J_{f,\lambda},$$

is a well defined non-zero X-valued holomorphic function on $\Omega_{\mu,\nu}$ for which $(\lambda - uC_\varphi)J_{f,\lambda} = 0$. □

One of the simplest families of eigenfunctions of C_φ is given by $\left(\frac{b-(\cdot)}{a-(\cdot)}\right)^\eta$, $\eta \in \mathbb{C}$. Thus, in view of the proposition above (and using the same notation), we raise the following question:

Q: Let $(u_t)_{t\in\mathbb{R}}$ be a differentiable cocycle for $(\varphi_t)_{t\in\mathbb{R}}$, for which $u_1 = u$, and with associated non-vanishing function ω. Do the (λ-eigen)functions

$$g_{\omega,\eta} := \frac{1}{\omega}\left(\frac{b-(\cdot)}{a-(\cdot)}\right)^\eta, \qquad \eta \in W_\lambda, \qquad W_\lambda = \{\mu \in \mathbb{C} \,:\, \varphi'(a)^\mu = 1/\lambda\},$$

belong to X for every eigenvalue λ of uC_φ?

Using similar gimmicks as in the proofs of Theorem 4.4 and Proposition 7.1 ii), one obtains the following partial answer to the question above.

Corollary 7.2. *Let X be a γ-space for some $\gamma \geq 0$, and let uC_φ be an invertible weighted composition operator on X. Assume that the automorphism φ is hyperbolic, with fixed points a (attractive), b (repulsive) in $\mathbb{T}$. Let $(\varphi_t)_{t\in\mathbb{R}}$ be the (unique) hyperbolic flow for which $\varphi_1 = \varphi$. Suppose furthermore that the absolute value $|u|$ can be continuously extended to $\mathbb{D}\cup\{a,b\}$ with $\frac{|u|(b)}{\varphi'(b)^\gamma} < \frac{|u|(a)}{\varphi'(a)^\gamma}$.*

Then, there exists a non-vanishing meromorphic function $\varpi : \mathbb{D} \to \mathbb{C}$ such that $u = \frac{\varpi\circ\varphi}{\varpi}$ and for which, for every $\lambda \in \mathbb{C}$ with $\frac{|u|(b)}{\varphi'(b)^\gamma} < |\lambda| < \frac{|u|(a)}{\varphi'(a)^\gamma}$, the functions

$$g_{\varpi,\eta} = \frac{1}{\varpi}\left(\frac{b-(\cdot)}{a-(\cdot)}\right)^\eta, \qquad \eta \in W_\lambda,$$

belong to X and are λ-eigenvectors for uC_φ.

Proof. Using the notation of the proofs of Theorem 4.4 and Proposition 7.1 ii), take $\mu,\nu > 0$ such that $|u|(a)\varphi'(a)^{\mu-\gamma} = \frac{|u|(b)}{\varphi'(b)^\gamma}$ and $|u|(b)\varphi'(b)^{\nu-\gamma} = \frac{|u|(a)}{\varphi'(a)^\gamma}$, fix $f \in X_{\mu,\nu}$ and λ_0 with $\frac{|u|(b)}{\varphi'(b)^\gamma} < |\lambda_0| < \frac{|u|(a)}{\varphi'(a)^\gamma}$ for which $J_{f,\lambda_0} \neq 0$. Now pick $\eta_0 \in W_{\lambda_0}$. Then the function $\varpi := \frac{1}{J_{f,\lambda_0}} \left(\frac{b-(\cdot)}{a-(\cdot)}\right)^{\eta_0}$ is a meromorphic function on $\mathbb{D}$. Since J_{f,λ_0} is a λ_0-eigenvector for uC_φ, it is readily seen that indeed $u = \frac{\varpi\circ\varphi}{\varpi}$, from which immediately follows that the (holomorphic) functions $g_{\varpi,\eta}$, $\eta \in W_\lambda$, are λ-eigenvectors for uC_φ in $\mathcal{O}(\mathbb{D})$. Finally, reasoning as in the proof of Theorem 4.4, one has that, for each $\lambda \in \mathbb{C}$ with $\frac{|u|(b)}{\varphi'(b)^\gamma} < |\lambda| < \frac{|u|(a)}{\varphi'(a)^\gamma}$,

$$g_{\varpi,\eta} = \left(\frac{b-(\cdot)}{a-(\cdot)}\right)^{\eta-\eta_0} J_{f,\lambda_0} = \frac{\lambda}{\lambda_0} J_{\left(\frac{b-(\cdot)}{a-(\cdot)}\right)^{\eta-\eta_0} f,\lambda}, \qquad \eta \in W_\lambda.$$

Since $\left(\frac{b-(\cdot)}{a-(\cdot)}\right)^{\eta-\eta_0} f \in X_{\mu-\mathfrak{Re}\,(\eta-\eta_0)-\varepsilon,\nu+\mathfrak{Re}\,(\eta-\eta_0)-\varepsilon}$ for all $\varepsilon > 0$ by (**P2**), and

$$|u|(a)\,\varphi'(a)^{\mu-\mathfrak{Re}\,(\eta-\eta_0)-\gamma} = \frac{|u|(b)}{\varphi'(b)^\gamma}\frac{|\lambda|}{|\lambda_0|} < |\lambda|,$$
$$|u|(b)\,\varphi'(b)^{\nu+\mathfrak{Re}\,(\eta-\eta_0)-\gamma} = \frac{|u|(a)}{\varphi'(a)^\gamma}\frac{|\lambda|}{|\lambda_0|} > |\lambda|,$$

we conclude (again, mimicking the proofs of Theorem 4.4 and Proposition 7.1 ii)) that indeed the functions $g_{\varpi,\eta}$, $\eta \in W_\lambda$, belong to X, and the proof is finished. □

In the two subsequent subsections, we give sufficient conditions on the differentiable cocycle $(u_t)_{t\in\mathbb{R}}$ and the space X so the answer to (**Q**) is positive, at least for every λ lying in the interior of $\sigma(uC_\varphi)$.

7.1. C_0-groups of weighted composition operators

Recall that a C_0-group of operators $(T(t))_{t\in\mathbb{R}}$ on a Banach space Y is a one-parameter family of bounded operators on Y which satisfies the (algebraic) group law and which is strongly continuous on the parameter $t \in \mathbb{R}$.

Proposition 7.3. *Let X be a γ-space for some $\gamma \geq 0$, and let uC_φ be an invertible weighted composition operator on X. Assume that the automorphism φ is hyperbolic, with fixed points a (attractive), b (repulsive) in $\mathbb{T}$. Let $(\varphi_t)_{t\in\mathbb{R}}$ be the (unique) hyperbolic flow for which $\varphi_1 = \varphi$, and let $\omega \in \mathcal{O}(\mathbb{D})$ be a non-vanishing function associated to a differentiable cocycle $(u_t)_{t\in\mathbb{R}}$ for $(\varphi_t)_{t\in\mathbb{R}}$ such that $u_1 = u$. Suppose furthermore that the absolute value $|u|$ can be continuously extended to $\mathbb{D} \cup \{a, b\}$ with $\frac{|u|(b)}{\varphi'(b)^\gamma} < \frac{|u|(a)}{\varphi'(a)^\gamma}$, and that there exists a closed subspace $Y \subseteq X$ for which the following holds*

- i) *Y satisfies (**P2**);*
- ii) *for any $\mu, \nu \geq 0$, the functions $\left\{\frac{\rho_{\mu,\nu}\circ\varphi_t}{\rho_{\mu,\nu}} : t \in \mathbb{R}\right\}$ belong to $Mul(Y)$ and the mapping from $\mathbb{R}$ to $Mul(Y)$ given by*

$$t \mapsto \frac{\rho_{\mu,\nu} \circ \varphi_t}{\rho_{\mu,\nu}} = e^{\delta\nu t}\left(\frac{b-a}{(1-e^{\delta t})(\cdot)+be^{\delta t}-a}\right)^{\mu+\nu},$$

is Bochner-measurable;

iii) $(u_t C_{\varphi_t})_{t\in\mathbb{R}}$ *defines a* C_0*-group of bounded operators on* Y*.*

Then, for each $\lambda \in \mathbb{C}$ *with* $\frac{|u|(b)}{\varphi'(b)^\gamma} < |\lambda| < \frac{|u|(a)}{\varphi'(a)^\gamma}$*, the functions*

$$g_{\omega,\eta} = \frac{1}{\omega}\left(\frac{b-(\cdot)}{a-(\cdot)}\right)^\eta, \qquad \eta \in W_\lambda,$$

belong to X *and are* λ*-eigenfunctions for* uC_φ*, i.e.,* $uC_\varphi g_{\omega,\eta} = \lambda g_{\omega,\eta}$*.*

Note that assumption *ii)* above is fulfilled whenever the following natural condition (which is satisfied by all the spaces listed in Subsection 2.1) holds for Y:

For each open set $U \subseteq \mathbb{C}$ with $\overline{\mathbb{D}} \subset U$, the inclusion $H^\infty(U) \hookrightarrow Mul(Y)$ holds and is continuous, i.e., there exists $K_U > 0$ such that $\|v\|_{Mul(Y)} \leq K_U \|v\|_{H^\infty(U)}$ for all $v \in H^\infty(U)$.

Proof. Fix $\lambda \in \mathbb{C}$ with $\frac{|u|(b)}{\varphi'(b)^\gamma} < |\lambda| < \frac{|u|(a)}{\varphi'(a)^\gamma}$, $\eta \in W_\lambda$ and take $\mu, \nu > 0$ such that $|u|(a)\,\varphi'(a)^{\mu-\gamma} = \frac{|u|(b)}{\varphi'(b)^\gamma}$ and $|u|(b)\,\varphi'(b)^{\nu-\gamma} = \frac{|u|(a)}{\varphi'(a)^\gamma}$. Recall that such μ, ν exist as $\varphi'(a) \in (0,1)$ and $\varphi'(b) \in (1,\infty)$. By the assumptions on Y, for all $\widetilde{\mu}, \widetilde{\nu} \geq 0$ the one-parameter family $\left(\frac{\rho_{\widetilde{\mu},\widetilde{\nu}} \circ \varphi_t}{\rho_{\widetilde{\mu},\widetilde{\nu}}} uC_{\varphi_t}\right)_{t\in\mathbb{R}}$ is a well defined family of bounded operators on Y which is strongly Bochner-measurable, hence it is a C_0-group on Y (see for example [21, Th. 10.2.3]). Fix a non-zero $f \in Y_{\mu,\nu} = \{\rho_{\mu,\nu} g : g \in Y\} = Y_{\mu,0} \cap Y_{0,\nu}$, which exists since Y satisfies (**P2**). Recall that $\lim_{t\to\infty} \|T(t)\|_{\mathcal{L}(Y)}^{1/t}$ exists for each C_0-group $(T(t))_{t\in\mathbb{R}}$, see for example [21, Th. 7.6.5]. Then

$$\begin{aligned}
\limsup_{t\to\infty} \|u_t C_{\varphi_t} f\|_Y^{1/t} &= \limsup_{t\to\infty} \left\| \rho_{\mu,0} \frac{\rho_{\mu,0} \circ \varphi_t}{\rho_{\mu,0}} u_t C_{\varphi_t}\left(\frac{f}{\rho_{\mu,0}}\right)\right\|_Y^{1/t} \\
&\leq \lim_{t\to\infty} \left\| \frac{\rho_{\mu,0} \circ \varphi_t}{\rho_{\mu,0}} u_t C_{\varphi_t} \right\|_{\mathcal{L}(Y)}^{1/t} = \lim_{\mathbb{N}\ni n\to\infty} \left\| \frac{\rho_{\mu,0} \circ \varphi_n}{\rho_{\mu,0}} u_n C_{\varphi_n} \right\|_{\mathcal{L}(Y)}^{1/n} \\
&\leq \lim_{\mathbb{N}\ni n\to\infty} \left\| \frac{\rho_{\mu,0} \circ \varphi_n}{\rho_{\mu,0}} u_n C_{\varphi_n} \right\|_{\mathcal{L}(X)}^{1/n} = r\left(\frac{\rho_{\mu,0} \circ \varphi}{\rho_{\mu,0}} uC_\varphi\right) = \frac{|u|(b)}{\varphi'(b)^\gamma},
\end{aligned}$$

where we have used that $\rho_{\mu,0} \in Mul(Y)$ by assumption i), and both Proposition 3.1 and (4.1) in the last "=" sign. Reasoning as above with the identity $u_{-t} = \frac{1}{u_t \circ \varphi_{-t}}$, $t \geq 0$, and the C_0-group $\left(\frac{\rho_{0,\nu} \circ \varphi_t}{\rho_{0,\nu}} uC_{\varphi_t}\right)_{t\in\mathbb{R}}$, one obtains $\limsup_{t\to\infty} \|u_{-t} C_{\varphi_{-t}} f\|_Y^{1/t} \leq \frac{\varphi'(a)^\gamma}{|u|(a)}$. From these bounds we conclude $\int_{-\infty}^{\infty} \left\| e^{-\delta\eta t} u_t C_{\varphi_t} f\right\|_Y dt < \infty$, where $\delta :=$

$-\log \varphi'(a) \in (0,\infty)$ (recall that $\eta \in W_\lambda$ so $|e^{-\delta\eta t}| = |\lambda|^{-t}$). Thus, the function $F := \int_{-\infty}^{\infty} e^{-\delta\eta t} u_t C_{\varphi_t} f\, dt$ is well defined and belongs to $Y \subseteq X$ (recall that the mapping $t \mapsto u_t C_{\varphi_t} f$ is continuous from $\mathbb{R}$ to Y). Note that

$$e^{-\delta\eta t} = \left(\frac{b-\varphi_t(z)}{a-\varphi_t(z)}\right)^{-\eta} \Bigg/ \left(\frac{b-z}{a-z}\right)^{-\eta}, \qquad z \in \mathbb{D},\, t \in \mathbb{R},$$

see (6.4). As $u_t = \frac{\omega\circ\varphi_t}{\omega}$, we have

$$F(z) = \frac{1}{\omega(z)}\left(\frac{b-z}{a-z}\right)^{\eta} \int_{-\infty}^{\infty} \omega(\varphi_t(z)) \left(\frac{b-\varphi_t(z)}{a-\varphi_t(z)}\right)^{-\eta} f(\varphi_t(z))\, dt, \qquad z \in \mathbb{D},$$

so $F(z) = g_{\omega,\eta}(z) H_f(z)$ with $H_f(z) := \int\limits_{-\infty}^{\infty} \frac{f(\varphi_t(z))}{g_{\omega,\eta}(\varphi_t(z))}\, dt$ for $z \in \mathbb{D}$.

Now set $\Gamma(z) = \{\varphi_t(z) : t \in \mathbb{R}\}$ for $z \in \mathbb{D}$ and recall that $\frac{\partial\varphi_t(z)}{\partial t} = \Phi(\varphi_t(z))$ for $z \in \mathbb{D}$, $t \in \mathbb{R}$, where Φ is the generator of $(\varphi_t)_{t\in\mathbb{R}}$, see (6.5). Then, one has

$$H_f(z) = \int_{\Gamma(z)} \frac{f(\xi)}{\Phi(\xi) g_{\omega,\eta}(\xi)}\, d\xi, \qquad z \in \mathbb{D}.$$

In particular, H_f is constant on the paths $\Gamma(z)$ for any $z \in \mathbb{D}$, hence H_f is constant on $\mathbb{D}$ by the uniqueness of analytic continuation. Let us show that we can take $f \in Y_{\mu,\nu}$ so $H_f \neq 0$.

Assume by contradiction that $H_f = 0$ for all $f \in Y_{\mu,\nu}$. Note first that, by Remark 6.10 (see also Remark 2.2 and (6.5)), the function $R_g := \frac{\omega}{\Phi}\left(\frac{b-(\cdot)}{a-(\cdot)}\right)^{-\eta} g$ belongs to $L^1(\Gamma(z))$ for each $g \in Y_{\mu,\nu}$ and all $z \in \mathbb{D}$. Note also that $pg \in Y_{\mu,\nu}$ for every $g \in Y_{\mu,\nu}$ and every polynomial p (this follows by decomposing p in integer powers of $b-(\cdot)$ and applying (**P2**)). As

$$H_{pg}(z) = \int_{\Gamma(z)} R_g(\xi) p(\xi)\, d\xi = 0, \qquad \text{for all polynomial } p,$$

we conclude that $R_g = 0$, which is absurd. Hence we can assume $H_f = 1 \neq 0$, so the function $F = g_{\omega,\eta}$ belongs to X and is an eigenfunction for λ by Proposition 7.1 i) and (6.4). $\square$

Remark 7.4. Assume that, following the notation of Proposition 7.3, X satisfies the assumptions i), ii) and iii), so one can take $Y = X$. Then, it is essentially contained in the proof of Proposition 7.3 that, for every, $\eta \in \mathbb{C}$ with $\frac{\log|u|(b)}{\delta} - \gamma < \mathfrak{Re}\,\eta < \frac{\log|u|(a)}{\delta} + \gamma$, the function $g_{\omega,\eta} = \frac{1}{\omega}\left(\frac{b-(\cdot)}{a-(\cdot)}\right)^{\eta}$ is a η-eigenvector for the infinitesimal generator Δ of the C_0-group $(u_t C_{\varphi_t})_{t\in\mathbb{R}}$, i.e., $g_{\omega,\eta}$ belongs to the domain of Δ and $\Delta g_{\omega,\eta} = \eta g_{\omega,\eta}$. In particular, this fact answers in the positive a question posed in [1, Remark 7.3(1)] for weighted Dirichlet spaces $\mathcal{D}^p_\sigma(\mathbb{D})$.

7.2. a, b-uniformly bounded cocycles

Now we consider a weaker condition on the weight u than the one considered in Subsection 7.1, which we define below. In exchange for weakening this condition, we need to assume an additional property on the Banach space X, namely property (**P5**).

Definition 7.5. Let $(u_t)_{t\in\mathbb{R}}$ be a differentiable cocycle for a hyperbolic flow $(\varphi_t)_{t\in\mathbb{R}}$ with fixed points a (attractive), b (repulsive) in $\mathbb{T}$. We say that $(u_t)_{t\in\mathbb{R}}$ is a, b-uniformly bounded if, for each $T > 0$ and each subset $B \subsetneq \mathbb{D}$ with $\{a, b\} \cap \overline{B} = \emptyset$, one has

$$\sup_{z\in B,\,|t|\le T} |u_t(z)| < \infty \quad \text{and} \quad \inf_{z\in B,\,|t|\le T} |u_t(z)| > 0. \tag{7.1}$$

Below we give a result which is essentially contained in [1, Lemma 3.10]. This result gives a key characterization of a, b-uniformly boundedness of a cocycle $(u_t)_{t\in\mathbb{R}}$ in terms of its non-vanishing associated function ω.

Proposition 7.6. *Let $(u_t)_{t\in\mathbb{R}}$ be a differentiable cocycle for a hyperbolic flow $(\varphi_t)_{t\in\mathbb{R}}$ with fixed points a (attractive), b (repulsive) in $\mathbb{T}$, and let $\omega \in \mathcal{O}(\mathbb{D})$ be a non-vanishing function associated to $(u_t)_{t\in\mathbb{R}}$. Assume that $|u_1|$ can be continuously extended to $\mathbb{D}\cup\{a, b\}$ with $|u_1|(a), |u_1|(b) \in (0, \infty)$. Then, $(u_t)_{t\in\mathbb{R}}$ is a, b-uniformly bounded if and only if, for each subset $B \subsetneq \mathbb{D}$ with $\{a, b\} \cap \overline{B} = \emptyset$, one has*

$$\sup_{z\in B} |\omega(z)| < \infty \qquad \text{and} \qquad \inf_{z\in B} |\omega(z)| > 0. \tag{7.2}$$

Proof. Assume first that (7.2) holds. Take $T > 0$ and $B \subsetneq \mathbb{D}$ with $\{a, b\} \cap \overline{B} = \emptyset$. Using the Koenigs model (6.6) of $(\varphi_t)_{t\in\mathbb{R}}$, it is readily seen that the subset $C := \cup_{|t|\le T}\, \varphi_t(B) \subsetneq \mathbb{D}$ also satisfies $\{a, b\} \cap \overline{C} = \emptyset$. Hence, by (7.2),

$$\sup_{z\in B,\,|t|\le T} |u_t(z)| = \sup_{z\in B,\,|t|\le T} \left|\frac{\omega(\varphi_t(z))}{\omega(z)}\right| \le \frac{\sup_{z\in C} |\omega(z)|}{\inf_{z\in B} |\omega(z)|} < \infty.$$

Similarly, one obtains $\inf_{z\in B,\,|t|\le T} |u_t(z)| > 0$, so (7.1) holds and the "if" part of the claim is proven.

Now suppose that $(u_t)_{t\in\mathbb{R}}$ is a, b-uniformly bounded, so (7.1) holds. Set $U = h^{-1}((0, -\pi/\log\varphi_1'(a))$, where h is the function given by the Koenigs model (6.6) of $(\varphi_t)_{t\in\mathbb{R}}$. Then $\overline{U}$ is a continuous path separating $\overline{\mathbb{D}}$ into two connected components such that a, b belong to different components (note that $a, b \notin \overline{U}$). We claim $\sup_{z\in U} |\omega(z)| < \infty$. Assuming this claim is true, take $B \subsetneq \mathbb{D}$ with $\{a, b\} \cap \overline{B} = \emptyset$. By the Koenigs model (6.6), for each $z \in \mathbb{D}$, there exist unique $\xi_z \in U$ and $t_z \in \mathbb{R}$ for which $\varphi_{t_z}(\xi_z) = z$. It is readily seen that the mappings $z \mapsto \xi_z$, $z \mapsto t_z$ are both continuous, and that $T_B := \sup_{z\in B} |t_z| < \infty$ as $\{a, b\} \cap \overline{B} = \emptyset$. Hence

$$\omega(z) = \omega(\varphi_{t_z}(\xi_z)) = u_{t_z}(\xi_z)\omega(\xi_z), \qquad z \in B,$$

so $\sup_{z\in B} |\omega(z)| \leq (\sup_{z\in U} |\omega(z)|) \left(\sup_{z\in U,\, |t|\leq T_B} |u_t(z)|\right) < \infty$.

Now we prove $\sup_{z\in U} |\omega(z)| < \infty$. Let q_1, q_2 be the two points in $\overline{U} \cap \mathbb{T}$. Since $\sup_{z\in K} |\omega(z)| < \infty$ for every compact subset $K \subsetneq \mathbb{D}$, it is enough to prove $\limsup_{U\ni z\to q_j} |\omega(z)| < \infty$ for $j = 1, 2$.

Fix $j = 1, 2$ and assume by contradiction $\lim_{U\ni z\to q_j} |\omega(z)| = \infty$. Let A be the open arc in $\mathbb{T}$ with endpoints a, b and containing q_j, and for each $q \in A$, let $t_q \in \mathbb{R}$ be such that $\varphi_{t_q}(q_j) = q$. Let $\Gamma : [0, 1] \to U \cup \{q_j\}$ be a parameterization of a path starting at an arbitrary point of U and ending at q_j. Then the continuous mapping $J : [0, 1] \times A \to \mathbb{C}$ given by $J(x, q) = \varphi_{t_q}(\Gamma(x))$ is a continuous family of paths in the sense of [12, pp. 83]. Indeed, $J([0, 1) \times A) \subset \mathbb{D}$ and $J(1, q) = q$. As $\{a, b\} \cap \overline{U} = \emptyset$, $\inf_{z\in U} |u_t(z)| > 0$ for all $t \in \mathbb{R}$. Therefore,

$$\lim_{x\to 1} |\omega(J(x, q))| = \lim_{U\ni z\to q_j} |u_{t_q}(z)||\omega(z)| = \infty, \qquad q \in A,$$

which contradicts the uniqueness of limits along the family of continuous paths J, see [12, pp. 83]. Thus, the statement $\lim_{U\ni z\to q_j} |\omega(z)| = \infty$ is false as claimed above.

Before continuing with the proof, we assume furthermore $|u_1|(a) < 1$ and $|u_1|(b) > 1$. In this case, by Lemma A.2 applied to the cocycle $(u_t)_{t\in\mathbb{R}}$ (or by the expression defining $(u_n)_{n\in\mathbb{Z}}$, see Section 3), there exists $N \in \mathbb{N}$ big enough such that

$$|\omega(\varphi_n(z))| = |u_n(z)||\omega(z)| < |\omega(z)|, \qquad z \in U,\ n \in \mathbb{Z} \text{ with } |n| \geq N. \tag{7.3}$$

Also, note that by hypothesis, $C := \sup_{z\in U, |t|\leq N} |u_t(z)| \in [1, \infty)$.

We continue with the proof of the lemma. Assume by contradiction $\limsup_{U\ni z\to q_j} |\omega(z)| = \infty$ for some $j = 1, 2$. Then there exists a sequence $(z_n)_{n\in\mathbb{N}} \subset U$ for which $\lim_{n\to\infty} z_n = q_j$ and $\lim_{n\to\infty} |\omega(z_n)| = \infty$. By what we have already proven, the limit $\lim_{U\ni z\to q_j} |\omega(z)|$ is not equal to ∞. Hence, there exist a constant $K > 0$ and a sequence $(\xi_n)_{n\in\mathbb{N}} \subset U$ such that $\lim_{n\to\infty} \xi_n = q_j$ and $|\omega(\xi_n)| \leq K$ for all $n \in \mathbb{N}$. Take $\widetilde{x} := \xi_{n_1}$, $\widetilde{y} := \xi_{n_2}$, $\widetilde{z} := z_{n_3}$ for $n_1, n_2, n_3 \in \mathbb{N}$ such that $|\omega(\widetilde{z})| > CK$ and $\widetilde{z}$ lies in the closed arc V contained in U with endpoints $\widetilde{x}, \widetilde{y}$ (both included). Let $R \subset \mathbb{D}$ be the compact subset of $\mathbb{D}$ given by

$$R := \{\varphi_s(z) : z \in V,\ s \in [-N, N]\}.$$

R is indeed compact since $h(R) = [\min\{h(\widetilde{x}), h(\widetilde{y})\}, \max\{h(\widetilde{x}), h(\widetilde{y})\}] \times [-N, N]$. We now prove that $|\omega|$ reaches its maximum in R in its interior, which contradicts the maximum modulus principle. Set $L := \max_{z \in V} |\omega(z)|$, which is attained in the interior of the arc V since $L \geq |\omega(\widetilde{z})| > |\omega(\widetilde{x})|, |\omega(\widetilde{y})|$. Note that the boundary ∂R of R is given by

$$\begin{aligned}\partial R = \{\varphi_s(\widetilde{x}) \,:\, s \in [-N, N]\} \cup \{\varphi_s(\widetilde{y}) \,:\, s \in [-N, N]\} \cup \{\varphi_{-N}(z) \,:\, z \in V\} \\ \cup \{\varphi_N(z) \,:\, z \in V\}.\end{aligned}$$

We have

$$\begin{aligned}\max\{|\omega(\varphi_s(\widetilde{x}))|, |\omega(\varphi_s(\widetilde{y}))|\} &\leq C \max\{|\omega(\widetilde{x})|, |\omega(\widetilde{y})|\} \\ &\leq CK < |\omega(\widetilde{z})| \leq L, \qquad s \in [-N, N],\end{aligned}$$

and, by (7.3),

$$\max\{|\omega(\varphi_{-N}(z))|, |\omega(\varphi_N(z))|\} < |\omega(z)| \leq L, \qquad z \in V,$$

reaching the aforementioned contradiction. In consequence, $\limsup_{U \ni z \to q_j} |\omega(z)| < \infty$ for $j = 1, 2$ if $|u_1|(a) < 1$ and $|u_1|(b) > 1$, so $\sup_{z \in U} |\omega(z)| < \infty$ as we wanted to prove.

If either $|u_1|(a) \geq 1$ or $|u_1|(b) \leq 1$, take $\mu, \nu \in \mathbb{R}$ such that $\varphi_1'(a)^\mu |u_1|(a) < 1$ and $\varphi_1'(b)^\nu |u_1|(b) > 1$. Note that such μ, ν exists since $\varphi_1'(a) \in (0, 1)$ and $\varphi_1'(b) \in (1, \infty)$. Set

$$v_t := \left(\frac{a - \varphi_t(\cdot)}{a - (\cdot)}\right)^\mu \left(\frac{b - \varphi_t(\cdot)}{b - (\cdot)}\right)^\nu u_t, \qquad t \in \mathbb{R}.$$

Then, it is readily seen that $(v_t)_{t \in \mathbb{R}}$ is a cocycle for $(\varphi_t)_{t \in \mathbb{R}}$ satisfying the hypothesis of the statement of this lemma, with $|v_1|(a) < 1$ and $|v_1|(b) > 1$. Indeed, $|v_t|(a) = \varphi_t'(a)^\mu |u_t|(a)$ and $|v_t|(b) = \varphi_t'(b)^\nu |u_t|(b)$ for $t \in \mathbb{R}$. Moreover, a non-vanishing holomorphic function ϖ associated to $(v_t)_{t \in \mathbb{R}}$ is given by $\varpi = (a - (\cdot))^\mu (b - (\cdot))^\nu \omega$. Hence, by what we have already proven

$$\sup_{z \in B} |\omega(z)| = \sup_{z \in B} |a - z|^{-\mu} |b - z|^{-\nu} |\varpi(z)| < \infty,$$

and the proof is done for the supremum.

Finally, the inequality for the infimum in (7.2) is obtained by applying what we have already proven to the a, b-uniformly bounded cocycle $(1/u_t)_{t \in \mathbb{R}}$, with associated non-vanishing holomorphic function given by $1/\omega$. More precisely, $\inf_{z \in B} |\omega(z)| = (\sup_{z \in B} |1/\omega(z)|)^{-1} > 0$. $\square$

Recall that $\mathscr{A}(\mathbb{D})$ denotes the Banach space of holomorphic functions on $\mathbb{D}$ with a bounded primitive, see Subsection 2.1. As in the end of Section 6.1, we set $\widetilde{\mathscr{A}} := \bigcup_{\mu, \nu \in \mathbb{R}} \mathscr{A}(\mathbb{D})_{\mu, \nu} = \bigcup_{\mu, \nu \in \mathbb{R}} \{\rho_{\mu, \nu} f \,:\, f \in \mathscr{A}(\mathbb{D})\}$.

Proposition 7.7. *Let $(\varphi_t)_{t\in\mathbb{R}}$ be a hyperbolic flow with fixed points a (attractive), b (repulsive) in $\mathbb{T}$, and let $(u_t)_{t\in\mathbb{R}}$ be a differentiable cocycle for $(\varphi_t)_{t\in\mathbb{R}}$ with generator $G = \frac{\partial u_t}{\partial t}\big|_{t=0}$ lying in $\widetilde{\mathscr{A}}$. Suppose that $|u_1|$ can be continuously extended to $\mathbb{D}\cup\{a,b\}$ with $|u_1|(a), |u_1|(b) \in (0,\infty)$. Then $(u_t)_{t\in\mathbb{R}}$ is a,b-uniformly bounded.*

Proof. Let $B \subsetneq \mathbb{D}$ be such that $\{a,b\}\cap\overline{B} = \emptyset$. We can assume without loss of generality that B is a starlike set centered at 0, so the segment $[0,z]$ lies in B for each $z \in B$. By (6.7),

$$|\omega(z)| \le \exp\left(\left|\int_0^z \frac{G(\xi)}{\Phi(\xi)}\, d\xi\right|\right), \qquad z \in \mathbb{D}.$$

Hence, by Proposition 7.6, it is enough to prove $\sup_{z\in B}\left|\int_0^z G(\xi)/\Phi(\xi)\, d\xi\right| < \infty$.

As $G \in \widetilde{\mathscr{A}}$, there exist $\mu,\nu \in \mathbb{R}$ such that $G/\rho_{\mu,\nu} \in \mathscr{A}(\mathbb{D})$, so its primitive $\widetilde{G}_{\mu,\nu} := \int_0^{(\cdot)} G(\xi)/\rho_{\mu,\nu}(\xi)\, d\xi$ lies in $H^\infty(\mathbb{D})$. Integrating by parts, we get

$$\begin{aligned}\int_0^z \frac{G(\xi)}{\Phi(\xi)}\, d\xi &= \int_0^z \frac{\rho_{\mu,\nu}(\xi)}{\Phi(\xi)} \frac{G(\xi)}{\rho_{\mu,\nu}(\xi)}\, d\xi \\ &= \frac{\rho_{\mu,\nu}(z)}{\Phi(z)}\widetilde{G}_{\mu,\nu}(z) - \int_0^z \left(\frac{\rho_{\mu,\nu}}{\Phi}\right)'(\xi)\, \widetilde{G}_{\mu,\nu}(\xi)\, d\xi, \qquad z\in\mathbb{D}.\end{aligned}$$

It is readily seen that $\sup_{z\in B}|\rho_{\mu,\nu}(z)/\Phi(z)| < \infty$ and $\sup_{z\in B}|(\rho_{\mu,\nu}/\Phi)'(z)| < \infty$, see (6.5). Therefore,

$$\sup_{z\in B}\left|\int_0^z \frac{G(\xi)}{\Phi(\xi)}\, d\xi\right| \le \|\widetilde{G}_{\mu,\nu}\|_{H^\infty}\left(\sup_{z\in B}\left|\frac{\rho_{\mu,\nu}(z)}{\Phi(z)}\right| + \sup_{z\in B}\left|\left(\frac{\rho_{\mu,\nu}}{\Phi}\right)'(z)\right|\right) < \infty,$$

and we conclude that $(u_t)_{t\in\mathbb{R}}$ is indeed a,b-uniformly bounded. □

Proposition 7.8. *Let $(\varphi_t)_{t\in\mathbb{R}}$ be a hyperbolic flow with fixed points a (attractive), b (repulsive) in $\mathbb{T}$, let $(u_t)_{t\in\mathbb{R}}$ be a differentiable a,b-uniformly bounded cocycle for $(\varphi_t)_{t\in\mathbb{R}}$, and let ω be a non-vanishing holomorphic function associated to $(u_t)_{t\in\mathbb{R}}$. Assume furthermore that $|u_1|$ can be continuously extended to $\mathbb{D}\cup\{a,b\}$ with $|u_1|(a), |u_1|(b) \in (0,\infty)$, and set $\alpha := \log|u_1|(a)$, $\beta := \log|u_1|(b)$ and $\delta := -\log\varphi_1'(a) \in (0,\infty)$. Then, for every $\varepsilon > 0$,*

$$|\omega(z)| \lesssim |a-z|^{-\alpha/\delta-\varepsilon}|b-z|^{\beta/\delta-\varepsilon}, \qquad |\omega(z)| \gtrsim |a-z|^{-\alpha/\delta+\varepsilon}|b-z|^{\beta/\delta+\varepsilon}, \qquad z\in\mathbb{D}.$$

Proof. By Proposition 7.6, we only have to prove that the inequalities of the claim hold in some arbitrary neighborhoods $\mathcal{U}_a, \mathcal{U}_b$ of a, b respectively. For $\varepsilon > 0$, repeating verbatim the steps of the proof of Proposition 6.9, take open neighborhoods $\mathcal{U}_a, \mathcal{V}_a$ in $\mathbb{D}$ of a such that $\{a, b\} \cap \overline{\mathcal{V}_a \setminus \mathcal{U}_a} = \emptyset$ and for which, for each $\xi \in \mathcal{U}_a$, there exists $\eta_\xi \in \mathbb{N}$ satisfying $\varphi_{n_\xi}(\xi) \in \mathcal{V}_a \setminus \mathcal{U}_a$ and (6.10), that is,

$$|\omega(\xi)| \lesssim \left| \frac{a - \xi}{a - \varphi_{-n_\xi}(\xi)} \right|^{-(\alpha+\varepsilon)/\delta} |\omega(\varphi_{-n_\xi}(\xi))|, \qquad \xi \in \mathcal{U}_a. \tag{7.4}$$

Thus,

$$\sup_{\xi \in \mathcal{U}_a} |a - \varphi_{-n_\xi}(\xi)|^{(\alpha+\varepsilon)/\delta} \leq \sup_{z \in \mathcal{V}_a \setminus \mathcal{U}_a} |a - z|^{(\alpha+\varepsilon)/\delta} < \infty,$$

and, by (7.2),

$$\sup_{\xi \in \mathcal{U}_a} |\omega(\varphi_{-n_\xi}(\xi))| \leq \sup_{z \in \mathcal{V}_a \setminus \mathcal{U}_a} |\omega(z)| < \infty.$$

Hence, by (7.4),

$$|\omega(\xi)| \lesssim |a - \xi|^{-(\alpha+\varepsilon)/\delta}, \qquad \xi \in \mathcal{U}_a.$$

An analogous reasoning proves the existence of neighborhood $\mathcal{U}_b$ in $\mathbb{D}$ of b such that $|\omega(\xi)| \lesssim |b - \xi|^{(\beta-\varepsilon)/\delta}$ for all $\xi \in \mathcal{U}_b$. By the arbitrariness of $\varepsilon > 0$, we obtain the upper bound of the theorem.

Finally, the inequality $\gtrsim$ of the claim follows by an application of what we have already proven to the differentiable cocycle $(1/u_t)_{t \in \mathbb{R}}$, whose associated non-vanishing holomorphic function is given by $1/\omega$. $\square$

Now we introduce the additional property we require X to satisfy in this subsection.

(**P5**) Every $f \in \mathcal{O}(\mathbb{D})$ for which there exist $c \neq d \in \mathbb{T}$ and $\varepsilon > 0$ such that

$$|f(z)| \lesssim |c - z|^{-\gamma+\varepsilon} |d - z|^{-\gamma+\varepsilon}, \qquad z \in \mathbb{D},$$

belongs to X.

It is readily seen that the following spaces (all of them are already listed in Subsection 2.1) satisfy (**P5**) with the same $\gamma \geq 0$ for which they are γ-spaces: Hardy spaces $H^p(\mathbb{D})$, $1 \leq p \leq \infty$, weighted Bergman spaces $\mathcal{A}^p_\sigma(\mathbb{D})$, $\sigma > -1$, $1 \leq p < \infty$, Korenblum classes $\mathcal{K}^{-\rho}(\mathbb{D})$, $\rho \geq 0$, and little Korenblum classes $\mathcal{K}_0^{-\rho}(\mathbb{D})$, $\rho > 0$.

Theorem 7.9. *Let X be a γ-space for some $\gamma \geq 0$ such that X, γ also satisfy (**P5**). Let uC_φ be an invertible weighted composition operator on X and assume that the automorphism φ is hyperbolic, with fixed points a (attractive), b (repulsive) in $\mathbb{T}$. Let $(\varphi_t)_{t \in \mathbb{R}}$ be the* *(unique) hyperbolic flow for which $\varphi_1 = \varphi$, and suppose furthermore that u is embeddable into a differentiable a, b-uniformly bounded cocycle $(u_t)_{t\in\mathbb{R}}$ for $(\varphi_t)_{t\in\mathbb{R}}$. Assume also that $|u|$ can be continuously extended to $\mathbb{D} \cup \{a, b\}$ with $\frac{|u|(b)}{\varphi'(b)^\gamma} < \frac{|u|(a)}{\varphi'(a)^\gamma}$.*

Let ω be a non-vanishing function associated to $(u_t)_{t\in\mathbb{R}}$. Then, for all $\lambda \in \mathbb{C}$ with $\frac{|u|(b)}{\varphi'(b)^\gamma} < |\lambda| < \frac{|u|(a)}{\varphi'(a)^\gamma}$, the functions

$$g_{\omega,\eta} = \frac{1}{\omega}\left(\frac{b-(\cdot)}{a-(\cdot)}\right)^\eta, \qquad \eta \in W_\lambda = \{\mu \in \mathbb{C} \,:\, \varphi'(a)^\mu = \lambda^{-1}\},$$

lie in X and are λ-eigenfunctions for uC_φ, i.e., $uC_\varphi g_{\omega,\eta} = \lambda g_{\omega,\eta}$.

Note that, by Proposition 2.9, Corollary 6.6 and Proposition 7.7, u is embeddable into a differentiable a, b-uniformly bounded cocycle whenever u lies in the disk algebra $\mathfrak{A}(\mathbb{D})$, or more generally, whenever $u'/u \in \widetilde{\mathscr{A}}$.

Proof. Let $\alpha := \log|u|(a)$, $\beta := \log|u|(b)$ and $\delta := -\log\varphi'(a) \in (0, \infty)$. Take $\lambda \in \mathbb{C}$ lying in the interior of $\sigma(uC_\varphi)$, so $\frac{|u|(b)}{\varphi'(b)^\gamma} < |\lambda| < \frac{|u|(a)}{\varphi'(a)^\gamma}$ by Theorem 4.4. Take $\eta \in W_\lambda$, so $e^{\delta\eta} = \lambda$ and $\beta/\delta - \gamma < \mathfrak{Re}\,\eta < \alpha/\delta + \gamma$. It follows by Proposition 7.8 that there exists $\varepsilon > 0$ for which,

$$|g_{\omega,\eta}(z)| \lesssim \frac{1}{|\omega(z)|}\left|\frac{b-z}{a-z}\right|^{\mathfrak{Re}\,\eta} \lesssim |a-z|^{-\gamma+\varepsilon}|b-z|^{-\gamma+\varepsilon}, \qquad z \in \mathbb{D}.$$

Thus, for each $\eta \in W_\lambda$, we have $g_{\omega,\eta} \in X$ by (**P5**) with $uC_\varphi g_{\omega,\eta} = \lambda g_{\omega,\eta}$ by Proposition 7.1 i) (see also (6.4)), so the proof is finished. □

8. Main result

For clarity, we collect the main contributions of this paper in the theorem below. Recall that the definition of γ-space is given in Section 2, and that a list of examples of such spaces is given in Subsection 2.1. As main examples of γ-spaces, one has: Hardy spaces $H^p(\mathbb{D})$, $1 \le p \le \infty$, with $\gamma = \frac{1}{p}$; weighted Bergman spaces $\mathcal{A}^p_\sigma(\mathbb{D})$, $1 \le p < \infty$, $\sigma > -1$, with $\gamma = \frac{\sigma+2}{p}$; Korenblum classes $\mathcal{K}^{-\rho}(\mathbb{D})$, $\rho \ge 0$, with $\gamma = \rho$; little Korenblum classes $\mathcal{K}_0^{-\rho}(\mathbb{D})$, $\rho > 0$, with $\gamma = \rho$; the disk algebra $\mathfrak{A}(\mathbb{D})$, with $\gamma = 0$; weighted Dirichlet spaces $\mathcal{D}^p_\sigma(\mathbb{D})$, $1 \le p < \infty$, $\sigma > -1$, $\sigma \ge p-2$, with $\gamma = \frac{\sigma+2}{p} - 1$; Bloch space $B(\mathbb{D})$ and little Bloch space $B_0(\mathbb{D})$, with $\gamma = 0$.

Theorem 8.1. *Let X be a γ-space for some $\gamma \ge 0$, and let uC_φ be an invertible weighted composition operator on X. Assume that the automorphism φ is hyperbolic, with fixed points a (attractive), b (repulsive) in $\mathbb{T}$. Suppose furthermore that the absolute value $|u|$ can be continuously extended to $\mathbb{D} \cup \{a, b\}$. Then*

$$\sigma(uC_\varphi) = \sigma_{ess}(uC_\varphi)$$
$$= \left\{ \lambda \in \mathbb{C} \,:\, \min\left\{ \frac{|u|(a)}{\varphi'(a)^\gamma}, \frac{|u|(b)}{\varphi'(b)^\gamma} \right\} \le |\lambda| \le \max\left\{ \frac{|u|(a)}{\varphi'(a)^\gamma}, \frac{|u|(b)}{\varphi'(b)^\gamma} \right\} \right\}.$$

Moreover, the following holds.

(1) *If* $\frac{|u|(a)}{\varphi'(a)^\gamma} < \frac{|u|(b)}{\varphi'(b)^\gamma}$*, and* λ *lies in the interior of* $\sigma(uC_\varphi)$*, then* $\lambda - uC_\varphi$ *is an injective, non-surjective operator with infinite codimensional range. Even more, for every non-vanishing function* $\omega \in \mathcal{O}(\mathbb{D})$ *with* $u = \frac{\omega \circ \varphi}{\omega}$ *(such a function* ω *exists),* L^μ_ω *is a bounded functional on* X *for all* $\mu \in W_\lambda = \{\mu \in \mathbb{C} \,:\, \varphi'(a)^\mu = 1/\lambda\}$ *and*

$$\operatorname{ran}(\lambda - uC_\varphi) \subseteq \bigcap_{\mu \in W_\lambda} \ker L^\mu_\omega.$$

(2) *If* $\frac{|u|(b)}{\varphi'(b)^\gamma} < \frac{|u|(a)}{\varphi'(a)^\gamma}$*, then* $\lambda - uC_\varphi$ *is a surjective, non-injective operator with infinite dimensional kernel for every* λ *lying in the interior of* $\sigma(uC_\varphi)$*. In addition, there exists a non-vanishing meromorphic function* $\varpi : \mathbb{D} \to \mathbb{C}$ *such that* $u = \frac{\varpi \circ \varphi}{\varpi}$ *and for which, for every* λ *in the interior of* $\sigma(uC_\varphi)$*, the functions* $g_{\varpi,\eta}$ *belong to* X *for all* $\eta \in W_\lambda$ *and*

$$\{g_{\varpi,\eta} \,:\, \eta \in W_\lambda\} \subseteq \ker(\lambda - uC_\varphi).$$

Recall that

$$L^\mu_\omega f := \int_{\Gamma^a_b} \frac{(a-z)^{\mu-1}}{(b-z)^{\mu+1}} \omega(z) f(z)\, dz, \qquad f \in X,$$

where Γ^a_b is the hyperbolic geodesic in $\mathbb{D}$ starting in b and ending in a, and that

$$g_{\varpi,\eta}(z) = \frac{1}{\varpi(z)} \left(\frac{b-z}{a-z} \right)^\eta, \qquad z \in \mathbb{D}.$$

Proof. This is just the statements of Theorems 4.4, 5.1, 6.12 and Corollaries 6.7 and 7.2. □

Appendix A. Some auxiliary results

Through the results of this appendix, we fix a hyperbolic automorphism $\varphi : \mathbb{D} \to \mathbb{D}$ with fixed points a (attractive), b (repulsive) in $\mathbb{T}$. We also set $(u_n)_{n\in\mathbb{Z}}$ as in the beginning of Section 3 for $u \in \mathcal{O}(\mathbb{D})$.

The following lemma can be found in [2, Lemma 3.1], see also [22, Lemma 4.4]. We include here its proof for the sake of completeness.

Lemma A.1. *Let $u : \mathbb{D} \to \mathbb{C}$ be with $\sup_{z\in\mathbb{D}} |u|(z) < \infty$. Assume moreover that the absolute value $|u|$ can be continuously extended to $\mathbb{D} \cup \{a, b\}$. Then*

$$\lim_{n\to\infty} \|u_n\|_{H^\infty}^{1/n} = \max\{|u|(a), |u|(b)\},$$

where n runs over the natural numbers in the above limits.

Proof. For each $\varepsilon > 0$, take neighborhoods U, V of a, b (respectively) in $\mathbb{D}$ such that

$$|u(z)| \le (1+\varepsilon)\max\{|u|(a), |u|(b)\}, \qquad z \in U \cup V.$$

Note that there exists $m \in \mathbb{N}_0$ such that, for all $z \in \mathbb{D}$, at most m elements of $\{\varphi_n(z) : n \in \mathbb{N}_0\}$ belong to $\mathbb{D} \setminus \{U \cup V\}$. Thus

$$\|u_n\|_{H^\infty} \le \|u\|_{H^\infty}^m \left[(1+\varepsilon)\max\{|u|(a), |u|(b)\}\right]^{n-m}, \qquad n \ge m.$$

Hence $\limsup_{n\to\infty} \|u_n\|_{H^\infty}^{1/n} \le \max\{|u|(a), |u|(b)\}$. On the other hand, the absolute value $|u_n|$ is also continuous in $\mathbb{D} \cup \{a, b\}$ and $|u_n|(a) = (|u|(a))^n$, $|u_n|(b) = (|u|(b))^n$ for all $n \in \mathbb{N}$. Therefore $\liminf_{n\to\infty} \|u_n\|_{H^\infty}^{1/n} \ge \max\{|u|(a), |u|(b)\}$ and the claim follows. □

Lemma A.2. *Let $u : \mathbb{D} \to \mathbb{C}$ be with $\sup_{z\in\mathbb{D}} |u(z)| < \infty$ and $\inf_{z\in\mathbb{D}} |u(z)| > 0$. Assume moreover that the absolute value $|u|$ can be continuously extended to $\mathbb{D} \cup \{a, b\}$. Set $\mathbb{D}_a := \{z \in \mathbb{D} : \mathfrak{Im}\,(h(z)) > 0\}$ and $\mathbb{D}_b := \{z \in \mathbb{D} : \mathfrak{Im}\,(h(z)) < 0\}$, where h is given by the Koenigs model (6.6) of $(\varphi_t)_{t\in\mathbb{R}}$. One obtains*

$$\lim_{n\to\infty} \left(\sup_{z\in\mathbb{D}_a} |u_n(z)|\right)^{1/n} = |u|(a), \qquad \lim_{n\to\infty} \left(\inf_{z\in\mathbb{D}_a} |u_n(z)|\right)^{1/n} = |u|(a),$$

and

$$\lim_{n\to\infty} \left(\sup_{z\in\mathbb{D}_b} |u_{-n}(z)|\right)^{1/n} = \frac{1}{|u|(b)}, \qquad \lim_{n\to\infty} \left(\inf_{z\in\mathbb{D}_b} |u_{-n}(z)|\right)^{1/n} = \frac{1}{|u|(b)}.$$

Proof. The claim is proven with an analogous reasoning as in the proof of A.1. The only difference is that, given an open neighborhood U of a in $\mathbb{D}$, h there exists $m \in \mathbb{N}_0$ such that, for all $z \in \mathbb{D}_a$, at most m elements of $\{\varphi_n(z) : n \in \mathbb{N}_0\}$ belong to $\mathbb{D}_a \setminus U$ if $z \in \mathbb{D}_a$ (and similarly for b). □

The lemma below is a straightforward extension of results given in [10,15] (see also [1, Prop. 3.13]) for $\mathcal{D}_0^2(\mathbb{D})$, $B(\mathbb{D})$ and a function $u : \mathbb{D} \to \mathbb{C}$ lying in the disk algebra $\mathfrak{A}(\mathbb{D})$.

Lemma A.3. *Let either $X = \mathcal{D}_\sigma^p(\mathbb{D})$ for $\sigma > -1$, $1 \le p < \infty$ and $\sigma \ge p - 2$, $X = B(\mathbb{D})$, $X = B_0(\mathbb{D})$ or $X = \mathscr{A}(\mathbb{D})$. Let $u : \mathbb{D} \to \mathbb{C}$ be a multiplier of X. Assume moreover that the absolute value $|u|$ can be continuously extended to $\mathbb{D} \cup \{a, b\}$. Then*

$$\limsup_{n\to\infty} \|u_n\|_{Mul(X)}^{1/n} \le \max\{|u|(a), |u|(b)\}.$$

Proof. The proof for all the possible spaces X of the claim go along similar steps. Let us show the claim for $X = \mathcal{D}_\sigma^p(\mathbb{D})$ and leave $B(\mathbb{D})$, $B_0(\mathbb{D})$ and $\mathscr{A}(\mathbb{D})$ as an exercise to the reader.

Let $f \in \mathcal{D}_\sigma^p(\mathbb{D})$ with $\|f\|_{\mathcal{D}_\sigma^p} = 1$. It is readily seen that $\|u_n f\|_{\mathcal{D}_\sigma^p} \lesssim \|u_n\|_{H^\infty}\|f\|_{\mathcal{D}_\sigma^p} + \|u_n' f\|_{\mathcal{A}_\sigma^p}$ for $n \in \mathbb{N}$. We have $\lim_{n\to\infty} \|u_n\|_{H^\infty}^{1/n} = \max\{|u|(a), |u|(b)\}$ by Lemma A.1, thus all that is left to prove is that $\limsup_{n\to\infty} \|u_n' f\|_{\mathcal{A}_\sigma^p}^{1/n} \le \max\{|u|(a), |u|(b)\}$. For each $n \in \mathbb{N}$ we have $u_n \in Mul(\mathcal{D}_\sigma^p(\mathbb{D}))$, thus $|u_n'|^p$ is a Carleson measure (see [31, Th. 4.2]), i.e., there exists $K_n > 0$ such that $\|u_n' g\|_{\mathcal{A}_\sigma^p} \le K_n \|g\|_{\mathcal{D}_\sigma^p}$ for all $g \in \mathcal{D}_\sigma^p(\mathbb{D})$. Let $\|u_n'\|_{\mathcal{C}}$ denote the infimum of such constants K_n (for each $n \in \mathbb{N}$). Hence the claim is proven if we show that $\limsup_{n\to\infty} \|u_n'\|_{\mathcal{C}}^{1/n} \le \max\{|u|(a), |u|(b)\}$, which we do below.

The identity $u_n = \prod_{j=0}^{n-1} u_1 \circ \varphi_j$ yields that

$$\|u_n'\|_{\mathcal{C}} = \left\| \sum_{j=0}^{n-1} \left((u\circ\varphi_j)' \cdot \prod_{k=0,k\neq j}^{n-1} u\circ\varphi_k \right) \right\|_{\mathcal{C}} \le \sum_{j=0}^{n-1} \|(u\circ\varphi_j)'\|_{\mathcal{C}} \left\| \prod_{k=0,k\neq j}^{n-1} u\circ\varphi_k \right\|_{H^\infty}. \tag{A.1}$$

Note that we use the symbol "·" to explicitly denote the pointwise product wherever we consider it is convenient for the clarity of the text. Reasoning as in Lemma A.1, one concludes

$$\lim_{n\to\infty} \left\| \prod_{k=0,k\neq j}^{n-1} u\circ\varphi_k \right\|_{H^\infty}^{1/n} \le \max\{|u|(a), |u|(b)\}. \tag{A.2}$$

Set now $\gamma := \frac{\sigma+2}{p} - 1$. The simple observation $(u\circ\varphi_j)' = \varphi_j' \cdot (u'\circ\varphi_j)$ yields that, for every $g \in \mathcal{D}_\sigma^p(\mathbb{D})$,

$$\begin{aligned} \|(u\circ\varphi_j)' g\|_{\mathcal{A}_\sigma^p} &= \|\varphi_j' \cdot (u'\circ\varphi_j) g\|_{\mathcal{A}_\sigma^p} = \left\|(\varphi_j')^{\gamma+1} C_{\varphi_j}(u' \cdot (\varphi_{-j}')^\gamma C_{\varphi_{-j}} g)\right\|_{\mathcal{A}_\sigma^p} \\ &= \|u'(\varphi_j')^\gamma \cdot C_{\varphi_{-j}} g\|_{\mathcal{A}_\sigma^p} \le \|u\|_{\mathcal{C}} \|(\varphi_{-j}')^\gamma C_{\varphi_{-j}} g\|_{\mathcal{D}_\sigma^p}, \end{aligned}$$

where we have used at the second-last step that $(\varphi_j')^{\gamma+1} C_{\varphi_j}$ is an isometric isomorphism on $\mathcal{A}_\sigma^p(\mathbb{D})$. Moreover, it follows by Remark 2.4 that $\lim_{j\to\infty} \|(\varphi_{-j}')^\gamma C_{\varphi_{-j}}\|_{\mathcal{L}(\mathcal{D}_\sigma^p)}^{1/j} = 1$. All these observations yield

$$\begin{aligned} \limsup_{n\to\infty} \left(\sum_{j=0}^{n-1} \|(u\circ\varphi_j)'\|_{\mathcal{C}} \right)^{1/n} &\le \limsup_{n\to\infty} \left(\sum_{j=0}^{n-1} \|(\varphi_{-j}')^\gamma C_{\varphi_{-j}}\|_{\mathcal{L}(\mathcal{D}_\sigma^p)} \right)^{1/n} \\ &\le \limsup_{n\to\infty} \left(n \max_{j=0,\dots,n-1} \|(\varphi_{-j}')^\gamma C_{\varphi_{-j}}\|_{\mathcal{L}(\mathcal{D}_\sigma^p)} \right)^{1/n} \le 1. \end{aligned} \tag{A.3}$$

Thus, we obtain by (A.1), (A.2) and (A.3)

$$\limsup_{n\to\infty} \|u_n'\|_{\mathcal{C}}^{1/n} \leq \max\{|u|(a), |u|(b)\}.$$

Thus, as mentioned above, the claim is proven for weighted Dirichlet spaces $\mathcal{D}^p_\sigma(\mathbb{D})$. □

Data availability

No data was used for the research described in the article.

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