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New vacuum boundary effects of massive field theories

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ABSTRACT

Analytical arguments suggest that the Casimir energy in 2+1 dimensions for gauge theories decays exponentially with the distance between the boundaries. The phenomenon has also been observed by non-perturbative numerical simulations. The dependence of this exponential decay on the different boundary conditions could help into a better understanding of the infrared behavior of these theories and in particular their mass spectrum. A similar phenomenon is expected to hold in 3+1 dimensions. Motivated by this feature we analyze the dependence of the exponential decay of Casimir energy for different boundary conditions of massive scalar fields in 3+1 dimensional spacetimes. We show that boundary conditions can be classified in two different families according to the rate of exponential decay of the Casimir energy. If the boundary conditions on each boundary are independent (e.g. both boundaries satisfy Dirichlet boundary conditions), the Casimir energy is exponentially decaying two times faster than when the boundary conditions interconnect the two boundary plates (e.g. for periodic or antiperiodic boundary conditions). These results will be useful for a comparison with the Casimir energy in the non-perturbative regime of non-Abelian gauge theories.

1. Introduction

Boundary effects of quantum fields play a crucial role in different quantum phenomena. The Casimir effect is one of the first and clearest examples of these boundary effects [1]. In this case, the renormalized energy of the vacuum becomes dependent on the boundary conditions when the quantum field is confined between two parallel plates. The variation of vacuum energy with the distance between the two plates gives rise to a force between them. This force is determined by the specific conditions satisfied by the quantum fields at the boundaries. Despite of its very tiny magnitude the effect has been observed for massless fields in several different experimental setups [2–8].

Significant progress regarding the computation and understanding of the Casimir effect in different models and setups has been made in recent years. Notable findings were presented in Ref. [9], where the vacuum energy for massless scalar fields under very general boundary conditions was obtained in arbitrary dimensions. Later on, the temperature dependence was computed for 3+1 dimensional massless field theories in Ref. [10]. More recently, the massive case was studied in the low temperature limit for general boundary conditions in 2+1 dimensional space–times [11].

However, the Casimir effect in interacting theories is much less known [12]. Recently, there has been some progress in the analytic approach to a non-perturbative calculation of the Casimir energy for Yang–Mills theories in 2+1 dimensions [13–15]. This computation has been later corroborated by some numerical simulations using Dirichlet boundary conditions for $SU(2)$ gauge fields [16].

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This analytic approach is based on the parametrization of $SU(2)$ gauge field by means of a massive scalar field, whose (magnetic) mass $m = g^2/\pi$ is given in terms of the gauge coupling g. For Dirichlet boundary conditions the non-perturbative Casimir energy derived from numerical simulations does agree with that of a massive scalar field with such a magnetic mass.

In order to clarify whether or not there exists a similar relation between the Casimir energies of 3+1 dimensional Yang–Mills theories and 3+1 dimensional massive scalar fields [17], some numerical simulations are in progress. Preliminary results for Dirichlet boundary conditions can been found in Ref. [18]. In order to make a more comprehensive comparison with the results of a massive scalar field it is necessary first to understand how the Casimir energy behaves under different boundary conditions.

Since comparisons with lattice gauge theory results require working at finite temperature, it is crucial to study the effects of thermal fluctuations on Casimir energy at low temperatures in massive scalar field theories, to provide an accurate analytical reference basis.

The paper is organized as follows. In Section 2 we introduce the setup and the renormalization scheme to be used throughout the paper. In Section 3 we develop the formalism required to obtain the free energy in the low temperature regime. By means of these formalism we compute the Casimir energy in Section 4, and also, we explore its decay in the large distance limit. In Section 5 we analyze some particular boundary conditions and compare their behavior. The conclusions of the paper are summarized in Section 6, and finally, in Appendix to verify the results we compute the Casimir energy using the explicit form of the spacial eigenvalues for Dirichlet and periodic boundary conditions.

2. Renormalized effective action

Let us consider a free massive scalar field ϕ confined between two homogeneous plates separated a distance L. In the Euclidean formalism the effects of finite temperature can be described by a compactification of Euclidean time into a circle of radius $\frac{\beta}{2\pi} = \frac{1}{2\pi T}$ with periodic boundary conditions $\phi(t+\beta,\mathbf{x}) = \phi(t,\mathbf{x})$. The corresponding partition function of the quantum field is

$$Z(\beta) = \det\left(-\partial_0^2 - \nabla^2 + m^2\right)^{-1/2}, \tag{1}$$

where ∂_0 denotes the Euclidean time derivative, ∇^2 the spacial Laplacian and m the mass of the fields. Since the infinite boundary plates are homogeneous we can describe the boundary conditions by means of 2×2 unitary matrices U [19]

$$\psi - i\delta\dot{\psi} = U(\psi + i\delta\dot{\psi}); \qquad U \in \mathrm{U}(2), \tag{2}$$

where

$$\psi = \begin{pmatrix} \psi(L/2) \\ \psi(-L/2) \end{pmatrix}, \quad \dot{\psi} = \begin{pmatrix} \dot{\psi}(L/2) \\ \dot{\psi}(-L/2) \end{pmatrix}, \tag{3}$$

$\psi(\pm L/2) = \phi(t,x_1,x_2,\pm L/2)$ being the values of the fields ϕ at the boundary plates, $\dot{\psi}(\pm L/2) = \pm\partial_3(\phi(t,x_1,x_2,\pm L/2))$ the outward normal derivatives on the plates, and ℓ is an arbitrary scale parameter that we shall set $\ell = 1$ for simplicity.

The $U(2)$ matrices can be parametrized in the following way

$$U(\theta,\eta,\mathbf{n}) = \mathrm{e}^{i\theta}\left(\mathbb{I}\cos\eta + i\mathbf{n}\cdot\boldsymbol{\sigma}\,\sin\eta\right); \quad \theta\in[0,2\pi),\ \eta\in[-\pi/2,\pi/2), \tag{4}$$

where $\boldsymbol{\sigma}$ are the Pauli matrices and $\mathbf{n}$ is a three dimensional unit vector $\mathbf{n}\in S^2$. These parameters are restricted to the domain $0\le\theta\pm\eta\le\pi$ in order to keep the operator $-\nabla^2$ selfadjoint and non-negative [9]. Also, we have to impose that $n_2 = 0$ because the scalar field we are working with is real.

The determinant in Eq. (1) is ultraviolet divergent. We will regularize that determinant by using the zeta function regularization method [20,21]. The regularized effective action is

$$S_{\rm eff} = -\log Z = -\frac{1}{2}\frac{d}{ds}\zeta(s)\,|_{s=0}, \tag{5}$$

where

$$\zeta(\beta,m,L;s) = \mathrm{tr}\left(\mu^{2s}\left(m^2 - \nabla^2 - \partial_0^2\right)^{-s}\right). \tag{6}$$

and μ is a scale parameter that makes the zeta function dimensionless and encodes the standard renormalization group parametrization [22,23]. The renormalization prescription consists in fixing this parameter by physical constrains.

The eigenvalues of the operator $-\Box = -\partial_0^2 - \nabla^2$ in the current set up are given by the sum of the two dimensional longitudinal continuous spacial modes q^2 that are parallel to the plates, the discrete spacial modes q_j on the transverse direction to the plates that depend on the boundary conditions, and the square of the temporal modes $(2\pi l/\beta)^2$ which are related to the Matsubara frequencies, i.e.

$$\lambda = \left(\frac{2\pi l}{\beta}\right)^2 + q^2 + q_j^2 + m^2 \qquad\qquad j\in\mathbb{N}, l\in\mathbb{Z}. \tag{7}$$

Hence, we can express the zeta function (6) in the following form

$$\zeta(\beta,m,L;s) = \mu^{2s}\frac{S}{4\pi(s-1)}\sum_{l=-\infty}^{\infty}\sum_j\left(\left(\frac{2\pi l}{\beta}\right)^2 + q_j^2 + m^2\right)^{-s+1}. \tag{8}$$

where S denotes the area of the boundary plates and we have integrated out the continuous spacial modes. A straightforward analysis shows that the zeros of the following spectral function [9]

$$h_U^L(q) = 2i\left(\sin(qL)\left((q^2-1)\cos\eta + (q^2+1)\cos\theta\right) - 2q\sin\theta\cos(qL) - 2qn_1\sin\eta\right). \tag{9}$$

are the discrete spacial eigenvalues of the Laplacian operator $-\Delta$. Thus, by using Cauchy theory we can write the zeta function as a contour integral

$$\zeta(\beta,m,L;s) = \mu^{2s}\frac{S}{8\pi i(s-1)}\sum_{l=-\infty}^{\infty}\oint dq\left(\left(\frac{2\pi l}{\beta}\right)^2 + q^2 + m^2\right)^{-s+1}\frac{d}{dq}\log h_U^L(q) \tag{10}$$

along a thin loop enclosing all the zeros of the spectral function $h_U(k)$ that are in the positive real axis.

This expression has UV divergences that come from the zero temperature contributions and can be identified in the asymptotic expansion at large L [9,24]

$$S_{\text{eff}}^{l=0} = \beta E_0 = C_v(m)S\beta L + C_b(m)S\beta + S\beta\, C_c(m,L) + \cdots. \tag{11}$$

where $C_b(m)$ is the divergent vacuum energy density of the boundary plates, $C_v(m)$ the also divergent bulk vacuum energy density and $C_c(m,L)$ is the finite coefficient corresponding to the Casimir energy.

A way of removing these UV divergences and extracting the value of Casimir energy from this asymptotic expansion is to introduce a renormalization prescription. Our choice is to define the renormalized action as [24,25]

$$S_{\text{eff}}^{\text{ren}} = -\frac{1}{2}\frac{d}{ds}\zeta_{\text{ren}}(\beta,m,L;s)\Big|_{s=0}, \tag{12}$$

with

$$\zeta_{\text{ren}}(\beta,m,L;s) = \lim_{L_0\to\infty}\left(\zeta(\beta,m,L;s) + \zeta(\beta,m,L+2L_0;s) - 2\zeta(\beta,m,L+L_0;s)\right), \tag{13}$$

where we introduce an auxiliary length L_0. This renormalization scheme not only removes the divergences, but also the residual terms that are independent of the distance between the plates or have a linear dependence on L. The remaining finite part consists of the Casimir energy and some extra terms that do not depend linearly in β and also vanish as the distance between the plates goes to infinity. This vanishing condition in the infinite limit $L\to\infty$ is the physical condition we use to fix the renormalization scheme prescription.

3. Zeta function in low temperature regime

We can explicitly compute the sum of the Matsubara modes in the low temperature regime, i.e. $\beta/L > 1$, postponing the analysis of contribution of the spacial modes. The expression (8) can be rewritten as

$$\zeta(\beta,m,L;s) = \left(\frac{\beta\mu}{2\pi}\right)^{2s}\frac{\pi S}{\beta^2(s-1)}\sum_j\sum_{l=-\infty}^{\infty}\left(l^2 + \left(\frac{q_j\beta}{2\pi}\right)^2 + \left(\frac{\beta m}{2\pi}\right)^2\right)^{-s+1}. \tag{14}$$

After applying Mellin transform and Poisson formula for the temperature dependent modes we get

$$\zeta(\beta,m,L;s) = \left(\frac{\beta\mu}{2\pi}\right)^{2s}\frac{\pi^{3/2}S}{\Gamma(s)\beta^2}\sum_j\sum_{l=-\infty}^{\infty}\int_0^\infty dt\, t^{s-5/2}\, e^{-\left(\left(\frac{q_j\beta}{2\pi}\right)^2+\left(\frac{\beta m}{2\pi}\right)^2\right)t-\frac{(\pi l)^2}{t}}. \tag{15}$$

The result after integration is

$$\zeta(\beta,m,L;s) = \left(\frac{\beta\mu}{2\pi}\right)^{2s}\frac{\pi^{3/2}S}{\Gamma(s)\beta^2}\Bigg(\Gamma\left(s-\frac{3}{2}\right)\sum_i\left(\left(\frac{q_j\beta}{2\pi}\right)^2+\left(\frac{\beta m}{2\pi}\right)^2\right)^{3/2-s}$$
$$+4\sum_j\sum_{l=1}^{\infty}(\pi l)^{-3/2+s}\left(\left(\frac{q_j\beta}{2\pi}\right)^2+\left(\frac{\beta m}{2\pi}\right)^2\right)^{3/4-s/2}K_{3/2-s}\left(\beta l\sqrt{q_j^2+m^2}\right)\Bigg),$$

where the first term is linear on β and is the leading one in the zero temperature limit ($l=0$), whereas the rest ($l\neq 0$) have a non-linear dependence on β.

Let us first focus on the leading term. The sum of the boundary modes can be replaced by an integral modulated by the spectral function (9)

$$\zeta^{l=0}(\beta,m,L;s) = \mu^{2s}\frac{S\beta\Gamma(s-3/2)}{16\pi^{5/2}i\Gamma(s)}\oint dq\,\left(q^2+m^2\right)^{3/2-s}\frac{d}{dq}\log h_U^L(q). \tag{16}$$

The renormalized zeta function (13) in our renormalization scheme is given by

$$\zeta_{\text{ren}}^{l=0}(\beta,m,L;s) = \frac{\mu^{2s}S\beta\Gamma(s-3/2)}{16\pi^{5/2}i\Gamma(s)}\lim_{L_0\to\infty}\oint dq\left(q^2+m^2\right)^{3/2-s}\frac{d}{dq}\log\frac{h_U^L(q)h_U^{2L_0+L}(q)}{\left(h_U^{L_0+L}(q)\right)^2}.$$

The only divergent contribution left in the zeta function is the $\Gamma(s)$ factor of the denominator, which disappears when computing the derivative of $\zeta^{l=0}_{\rm ren}(\beta, m, L; s)$ with respect to s and evaluating it at $s = 0$,

$$\left(\zeta^{l=0}_{\rm ren}\right)'(\beta, m, L; 0) = \frac{S\beta}{12\pi^2 i} \lim_{L_0\to\infty} \oint dq \left(q^2+m^2\right)^{3/2} \left(\frac{d}{dq} \log \frac{h^L_U(q) h^{2L_0+L}_U(q)}{\left(h^{L_0+L}_U(q)\right)^2} \right). \tag{17}$$

Now, the loop integral can be simply reduced to an integral over the imaginary axis by using the fact that the integrand is holomorphic

$$\left(\zeta^{l=0}_{\rm ren}\right)'(\beta, m, L; 0) = -\frac{S\beta}{12\pi^2 i} \lim_{L_0\to\infty} \int_{-\infty}^{\infty} dq \left(m^2-q^2\right)^{3/2} \left(\frac{d}{dq} \log \frac{h^L_U(iq) h^{2L_0+L}_U(iq)}{\left(h^{L_0+L}_U(iq)\right)^2} \right). \tag{18}$$

Notice, that since the integrand is parity odd, the integral between $(-m, m)$ vanishes, but the branching point of the square root $\sqrt{m^2-k^2}$ at $-m$ introduces a change of sign and thus the contributions between $(-\infty, -m)$ and (m, ∞) have the same sign. Taking this into account the integral reduces to

$$\left(\zeta^{l=0}_{\rm ren}\right)'(\beta, m, L; 0) = \frac{S\beta}{6\pi^2} \lim_{L_0\to\infty} \int_{m}^{\infty} dq \left(q^2-m^2\right)^{3/2} \left(\frac{d}{dq} \log \frac{h^L_U(iq) h^{2L_0+L}_U(iq)}{\left(h^{L_0+L}_U(iq)\right)^2} \right). \tag{19}$$

Finally, we can take the limit $L_0 \to \infty$ in the spectral function

$$\lim_{L_*\to\infty} h^{L_*}_U(iq) = \lim_{L_*\to\infty} e^{q(L_*)} \left((q^2+1)\cos\eta + (q^2-1)\cos\theta + 2q\sin\theta \right), \tag{20}$$

and the result can be expressed in terms of the asymptotic spectral function

$$h^{\infty}_U(iq) \equiv \left((q^2+1)\cos\eta + (q^2-1)\cos\theta + 2q\sin\theta \right), \tag{21}$$

which leads to the simpler expression

$$\left(\zeta^{l=0}_{\rm ren}\right)'(\beta, m, L; 0) = -\frac{S\beta}{6\pi^2} \int_{m}^{\infty} dq \left(q^2-m^2\right)^{3/2} \left(L - \frac{d}{dq} \log \frac{h^L_U(iq)}{h^{\infty}_U(iq)} \right). \tag{22}$$

3.1. Contribution of the $l \neq 0$ modes

In this case

$$\zeta^{l\neq 0}(\beta, m, L; s) = \left(\frac{\beta\mu}{2\pi}\right)^{2s} \frac{4\pi^{3/2} S}{\Gamma(s)\beta^2} \sum_j \sum_{l=1}^{\infty} (\pi l)^{-3/2+s} \times \left(\left(\frac{q_j\beta}{2\pi}\right)^2 + \left(\frac{\beta m}{2\pi}\right)^2 \right)^{3/4-s/2} K_{3/2-s}\left(\beta l \sqrt{q_j^2+m^2}\right) \tag{23}$$

the sums over the two series are convergent since the Bessel function $K_{3/2}$ exponentially decays as its argument grows. Therefore, only the factor $\Gamma(s)$ is divergent, which after derivation and evaluation at $s = 0$ gives

$$\left(\zeta^{l\neq 0}\right)'(\beta, m, L; 0) = \frac{S}{\pi\beta^2} \sum_j \left(\beta\sqrt{q_j^2+m^2}\, \mathrm{Li}_2\left(e^{-\sqrt{q_j^2+m^2}\beta}\right) + \mathrm{Li}_3\left(e^{-\sqrt{q_j^2+m^2}\beta}\right) \right). \tag{24}$$

Once again, we can use the spectral formula (9) for the sum of the discrete modes q_j

$$\left(\zeta^{l\neq 0}\right)'(\beta, m, L; 0) = \frac{S}{2\pi^2 i\beta^2} \oint dq \left(\beta\sqrt{q^2+m^2}\, \mathrm{Li}_2\left(e^{-\sqrt{q^2+m^2}\beta}\right) + \mathrm{Li}_3\left(e^{-\sqrt{q^2+m^2}\beta}\right) \right) \frac{d}{dq} \log\left(h^L_U(q)\right), \tag{25}$$

which gives rise to the renormalized zeta function (13)

$$\left(\zeta^{l\neq 0}_{\rm ren}\right)'(\beta, m, L; 0) = \lim_{L_0\to\infty} \frac{S}{2\pi^2 i\beta^2} \oint dq \left(\beta\sqrt{q^2+m^2}\, \mathrm{Li}_2\left(e^{-\sqrt{q^2+m^2}\beta}\right) + \mathrm{Li}_3\left(e^{-\sqrt{q^2+m^2}\beta}\right) \right) \frac{d}{dq} \log \frac{h^L_U(q) h^{2L_0+L}_U(q)}{\left(h^{L_0+L}_U(q)\right)^2}. \tag{26}$$

Following similar steps as in the case $l = 0$ we can reduce this integral to the imaginary axis because the integrand is holomorphic in the domain which allows this contour deformation

$$\left(\zeta^{l\neq 0}_{\rm ren}\right)'(\beta, m, L; 0) = -\lim_{L_0\to\infty} \frac{S}{2\pi^2 i\beta^2} \int_{-\infty}^{\infty} dq \left(\beta\sqrt{m^2-q^2}\, \mathrm{Li}_2\left(e^{-\sqrt{m^2-q^2}\beta}\right)\right.$$

$$+\mathrm{Li}_3\left(e^{-\sqrt{m^2-q^2}\beta}\right)\Bigg)\left(\frac{d}{dq}\log\frac{h_U^L(iq)h_U^{2L_0+L}(iq)}{\left(h_U^{L_0+L}(iq)\right)^2}\right). \tag{27}$$

Now, since the integrand is parity odd, the integral between $(-m, m)$ apparently vanishes. But the branching point of the square root $\sqrt{m^2-q^2}$ introduces a minus sign between the integrals on the intervals $(-\infty, -m)$ and (m, ∞). In summary, the result of the integral is

$$\left(\zeta_{\text{ren}}^{l\neq 0}\right)'(\beta, m, L; 0) = -\lim_{L_0\to\infty}\frac{S}{\pi^2\beta^2}\int_m^\infty dq\Big(\beta\sqrt{q^2-m^2}\,\Re\left(\mathrm{Li}_2\left(e^{-i\sqrt{q^2-m^2}\beta}\right)\right)$$
$$+\Im\left(\mathrm{Li}_3\left(e^{-i\sqrt{q^2-m^2}\beta}\right)\right)\Big)\left(\frac{d}{dq}\log\frac{h_U^L(iq)h_U^{2L_0+L}(iq)}{\left(h_U^{L_0+L}(iq)\right)^2}\right), \tag{28}$$

and using Eq. (20) in the limit $L_0 \to \infty$ we get

$$\left(\zeta_{\text{ren}}^{l\neq 0}\right)'(\beta, m, L; 0) = \frac{S}{\pi^2\beta^2}\int_m^\infty dq\Big(\beta\sqrt{q^2-m^2}\,\Re\left(\mathrm{Li}_2\left(e^{-i\sqrt{q^2-m^2}\beta}\right)\right)$$
$$+\Im\left(\mathrm{Li}_3\left(e^{-i\sqrt{q^2-m^2}\beta}\right)\right)\Big)\left(L-\frac{d}{dq}\log\frac{h_U^L(iq)}{h_U^\infty(iq)}\right). \tag{29}$$

4. Free energy

From the renormalized zeta function we can easily derive the Casimir energy. The temperature independent part of the free energy $F = S_{\text{eff}}/\beta$ is the Casimir energy [26]

$$F_U^{l=0}(\beta, m, L) = E_U^c(m, L) = \frac{S}{12\pi^2}\int_m^\infty dq\left(q^2-m^2\right)^{3/2}\left(L-\frac{d}{dq}\log\frac{h_U^L(iq)}{h_U^\infty(iq)}\right), \tag{30}$$

whereas the $l \neq 0$ terms encode the temperature dependent contributions

$$F_U^{l\neq 0}(\beta, m, L) = -\frac{S}{2\pi^2\beta^3}\int_m^\infty dq\Big(\beta\sqrt{q^2-m^2}\,\Re\left(\mathrm{Li}_2\left(e^{-i\sqrt{q^2-m^2}\beta}\right)\right)$$
$$+\Im\left(\mathrm{Li}_3\left(e^{-i\sqrt{q^2-m^2}\beta}\right)\right)\Big)\left(L-\frac{d}{dq}\log\frac{h_U^L(iq)}{h_U^\infty(iq)}\right). \tag{31}$$

Both contributions to the free energy vanish when the distance between the plates L tends to infinity, which is the physical requirement of the renormalization scheme prescription chosen. Moreover, in the zero temperature limit, $F^{l\neq 0}$ vanishes and we recover the free Casimir vacuum energy.

4.1. *Casimir energy in the asymptotic limit*

Let us now explore the behavior of the Casimir energy in the limit when the effective distance mL tends to infinity $mL \to \infty$. First, we can rewrite the spectral function as

$$h_U^L(iq) = e^{qL}\left((q^2+1)\cos\eta + (q^2-1)\cos\theta + 2q\sin\theta\right)\left(1 + n_1\sin(\eta)\mathcal{X}\,e^{-qL} + \mathcal{Y}\,e^{-2qL}\right),$$

where $\mathcal{X}$ and $\mathcal{Y}$ are

$$\mathcal{X}(q,\theta,\eta) = \frac{4q}{(q^2+1)\cos\eta + (q^2-1)\cos\theta + 2q\sin\theta} \tag{32}$$

$$\mathcal{Y}(q,\theta,\eta) = \frac{-(q^2+1)\cos\eta - (q^2-1)\cos\theta + 2q\sin\theta}{(q^2+1)\cos\eta + (q^2-1)\cos\theta + 2q\sin\theta}. \tag{33}$$

Using the asymptotic expansion in powers of e^{-qL} in the logarithmic ratio of spectral functions

$$\log\frac{h_U^L(iq)}{h_U^\infty(iq)} = qL + n_1\sin\eta\,\mathcal{X}\,e^{-qL} + (\mathcal{Y}-\mathcal{X}'/2)e^{-2qL} + O(e^{-3qL}), \tag{34}$$

and introducing this expansion in the integral of the Casimir energy we get

$$E_U^c = -\frac{S}{12\pi^2}\int_m^\infty dq(q^2-m^2)^{3/2}\frac{d}{dq}\left(n_1\sin\eta\,\mathcal{X}\,e^{-qL} + (\mathcal{Y}-\mathcal{X}'/2)e^{-2qL} + O(e^{-3qL})\right) \tag{35}$$
$$= \frac{S}{4\pi}\int_m^\infty dq\,q(q^2-m^2)^{1/2}\left(n_1\sin\eta\mathcal{X}\,e^{-qL} + (\mathcal{Y}-\mathcal{X}'/2)e^{-2qL} + O(e^{-3qL})\right),$$

where $\mathcal{X}' = (n_1\sin(\eta)\mathcal{X})^2$. This integral can be estimated by using the saddle point approximation for each exponential term

$$\int_m^\infty e^{F(x)}dx \simeq e^{F(x_0)}\int_m^\infty e^{\frac{1}{2}F''(x_0)(x-x_0)^2}\,dx \tag{36}$$

where we consider the quadratic approximation of $F(x)$ around its maximum at x_0. In the limit $mL \to \infty$, the maximum is attained in the integration domain at the point $x_0 = m(1 + a/(mL))$, a being a positive numerical parameter. This means we can express the integral of each exponential term as

$$\int_m^\infty dq\; g(\theta,\eta,n_1,q)e^{-jqL} = \frac{e^{-jmL}}{(mL)^{3/2}}\left(c_{j,1}(\theta,\eta,n_1,m) + \frac{c_{j,2}(\theta,\eta,n_1,m)}{mL} + O\left(\frac{1}{(mL)^2}\right)\right).$$

Then the Casimir energy formula can be written as

$$E_U^c = \frac{Sm^3}{(mL)^{3/2}}\left(n_1 \sin\eta\; e^{-mL}\left(c_{1,1}(\theta,\eta,n_1,m) + \frac{c_{1,2}(\theta,\eta,n_1,m)}{mL} + O\left(\frac{1}{(mL)^2}\right)\right)\right. \tag{37}$$

$$\left. + e^{-2mL}\left(c_{2,1}(\theta,\eta,n_1,m) + \frac{c_{2,2}(\theta,\eta,n_1,m)}{mL} + O\left(\frac{1}{(mL)^2}\right)\right) + O(e^{-3mL})\right). \tag{38}$$

The coefficient of the leading exponential term e^{-mL} vanishes when $n_1 \sin\eta = 0$, which increases the Casimir energy exponential decay rate by a factor 2, e^{-2mL}. Thus, according to the asymptotic behavior of the Casimir energy we have two distinct families of boundary conditions

$$(mL)^{3/2}E_U^c \sim \begin{cases} e^{-mL} & \text{if } \operatorname{tr}(U\sigma_1) \neq 0 \\ e^{-2mL} & \text{if } \operatorname{tr}(U\sigma_1) = 0, \end{cases} \tag{39}$$

that are determined by the dependence or not on σ_1 of the unitary matrix U involved in the definition of the boundary conditions.

The classification of the boundary conditions into two different families according to the speed of the exponential decay of Casimir energy for a massive scalar field in 3+1 dimensions, is the main result of this paper. A similar classification also appears in 2+1 dimensions [11].

These two families differ in the vanishing or not of $\operatorname{tr}(U\sigma_1)$. When $\operatorname{tr}(U\sigma_1)$ does not vanish, the boundary conditions involve a relation between the boundary values at the two boundary plates. In contrast, when $\operatorname{tr}(U\sigma_1)$ is zero, the constraints imposed by the boundary conditions at the boundary values on the two boundary plates are independent one from each other.

This result was found previously for Dirichlet and periodic boundary conditions [27]– [28], but we have shown that the result is more general and has a clear physical meaning: if the boundary conditions on each boundary are independent, the Casimir energy is exponentially decaying two times faster than when the boundary conditions interconnect the two boundary values at the two plates, as occurs for periodic or antiperiodic boundary conditions. In fact, the same result can be proved to hold for any space dimension D. Indeed, in general dimensions instead of (35) we have

$$E_U^c(L,m) = \frac{S}{2^{D+1}\pi^{D/2}\Gamma\left(\frac{D}{2}+1\right)}\int_m^\infty dq\,\left(q^2-m^2\right)^{D/2}\left(L - \frac{d}{dq}\log\frac{h_U^L(iq)}{h_U^\infty(iq)}\right), \tag{40}$$

from where we derive the same type of results.

5. Particular cases of boundary conditions

We can analytically compute the Casimir energy for some special boundary conditions that are of interest because they can also be implemented for gauge fields.

(i) *Periodic boundary conditions*: $\psi(L/2) = \psi(-L/2)$ and $\dot\psi(L/2) = -\dot\psi(-L/2)$. The associated unitary operator is $U_P = \sigma_1$. The contribution of spectral functions has the form

$$\frac{d}{dq}\log\left(h_{U_P}^L(iq)/h_{U_P}^\infty(iq)\right) = L\coth(qL/2), \tag{41}$$

which gives to the expression of the Casimir energy

$$E_P^c(L,m) = -\frac{Sm^2}{2\pi^2 L}\sum_{j=1}^\infty \frac{K_2(jmL)}{j^2}, \tag{42}$$

when $m \neq 0$, and

$$E_P^c(L,0) = -\frac{\pi^2 S}{90L^3} \tag{43}$$

in the massless case. In the large mL limit this expression decays exponentially as e^{-mL} which is the result expected since the unitary matrix U_P depends on σ_1, i.e. $\operatorname{Tr}\sigma_1\sigma_1 = 2 \neq 0$. The temperature dependent terms of the free energy are

$$F_P^{l\neq 0}(\beta,m,L) = -\frac{SL}{2\pi^2\beta^3}\int_m^\infty dq\;\left(\beta\sqrt{q^2-m^2}\;\Re\left(\operatorname{Li}_2\left(e^{-i\sqrt{q^2-m^2}\beta}\right)\right)\right.$$

$$\left. + \Im\left(\operatorname{Li}_3\left(e^{-i\sqrt{q^2-m^2}\beta}\right)\right)\right)(1-\coth(qL/2)). \tag{44}$$

Although this integral cannot be analytically computed, the asymptotic expansion of $1-\coth(qL/2)$ shows that it has the same asymptotic behavior as the Casimir energy

(ii) *Dirichlet boundary conditions*: $\psi(L/2) = \psi(-L/2) = 0$. The corresponding unitary matrix is $U_D = -\mathbb{I}$ and the spectral function factor is

$$\frac{d}{dq} \log \left(h^L_{U_D}(iq)/h^\infty_{U_D}(iq) \right) = L \coth(qL). \tag{45}$$

The Casimir energy is given by

$$E^c_D(m, L) = -\frac{Sm^2}{8\pi^2 L} \sum_{j=1}^{\infty} \frac{K_2(2jmL)}{j^2}, \tag{46}$$

when $m \neq 0$, whereas for the massless case we have

$$E^c_D(0, L) = -\frac{\pi^2 S}{1440 L^3}. \tag{47}$$

Since $-\mathrm{Tr}\,\mathbb{I}\,\sigma_1 = 0$, we have the expected asymptotic behavior where the Casimir energy decays as e^{-2mL}. For $l \neq 0$ terms of the free energy we have

$$F^{l\neq 0}_D(\beta, m, L) = -\frac{SL}{2\pi^2\beta^3} \int_m^\infty dq \, \left(\beta\sqrt{q^2 - m^2}\, \Re \left(\mathrm{Li}_2 \left(e^{-i\sqrt{q^2-m^2}\beta} \right) \right) \right.$$
$$\left. + \Im \left(\mathrm{Li}_3 \left(e^{-i\sqrt{q^2-m^2}\beta} \right) \right) \right) (1 - \coth(qL)). \tag{48}$$

This integral cannot be explicitly analytically computed but it can be shown that decays exponentially in the same way as the Casimir energy.

An alternative calculation for the above boundary conditions by using the exact spectrum of the spacial Laplacian instead of the spectral function is postponed to Appendix.

(iii) *Neumann boundary conditions*: $\dot\psi(L/2) = \dot\psi(-L/2) = 0$. Although the corresponding unitary matrix is $U_N = \mathbb{I}$ and the spectral function differs from that of Dirichlet boundary conditions, the derivative of the logarithm of both spectral functions is the same as (45). Thus, the free energy is the same as for Dirichlet boundary conditions.

(iv) *Anti-periodic boundary conditions*: $\psi(L/2) = -\psi(-L/2)$ and $\dot\psi(L/2) = \dot\psi(-L/2)$. The corresponding unitary matrix is $U_A = -\sigma_1$ and in this case the derivative of the logarithm of the spectral function is

$$\frac{d}{dq} \log \left(h^L_{U_A}(iq)/h^\infty_{U_A}(iq) \right) = L \tanh(qL/2). \tag{49}$$

Thus, the Casimir energy is

$$E^c_{U_A}(L, m) = \frac{Sm^2}{2\pi^2 L} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j^2} K_2(jmL) \tag{50}$$

for $m \neq 0$, and

$$E^c_A(L, 0) = \frac{7\pi^2 S}{720 L^3} \tag{51}$$

in the massless case. We remark that one gets the same asymptotic decay as for periodic boundary conditions, since U_A also depends on σ_1. The temperature dependent part of the free energy is

$$F^{l\neq 0}_{U_A}(\beta, m, L) = -\frac{SL}{2\pi^2\beta^3} \int_m^\infty dq \, \left(\beta\sqrt{q^2 - m^2}\, \Re \left(\mathrm{Li}_2 \left(e^{-i\sqrt{q^2-m^2}\beta} \right) \right) \right.$$
$$\left. + \Im \left(\mathrm{Li}_3 \left(e^{-i\sqrt{q^2-m^2}\beta} \right) \right) \right) (1 - \tanh(qL/2)). \tag{52}$$

with the same exponential decay as the Casimir energy.

(v) *Zaremba boundary conditions*. This boundary condition consists of one boundary wall with Neumann boundary conditions whereas the opposite boundary has Dirichlet boundary conditions. They are described by the operator $U_Z = \pm\sigma_3$. The derivative of the logarithm of spectral functions is given by

$$\frac{d}{dq} \log \left(h^L_{U_Z}(iq)/h^\infty_{U_Z}(iq) \right) = L \tanh(qL), \tag{53}$$

and the Casimir energy is

$$E^c_{U_Z}(L, m) = \frac{Sm^2}{8\pi^2 L} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j^2} K_2(2jmL), \tag{54}$$

when the mass is non-null, whereas

$$E^c_Z(L, 0) = \frac{7\pi^2 S}{11520 L^3} \tag{55}$$

for the massless case. The exponential decay is the same as for Dirichlet or Neumann boundary conditions e^{-2mL}. The rest of the terms of the free energy have the form

$$F^{l\neq 0}_Z(\beta, m, L) = -\frac{SL}{2\pi^2\beta^3} \int_m^\infty dq \, \left(\beta\sqrt{q^2 - m^2}\, \Re \left(\mathrm{Li}_2 \left(e^{-i\sqrt{q^2-m^2}\beta} \right) \right) \right.$$

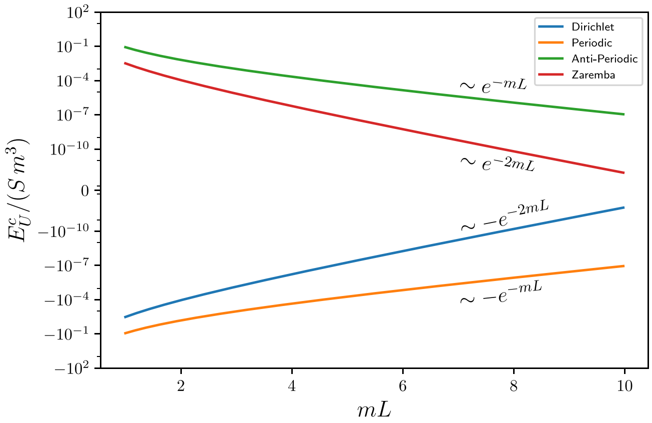

Fig. 1. Behavior of the adimensional Casimir energy (in logarithmic scale) for different boundary conditions as a function of the adimensional distance mL.

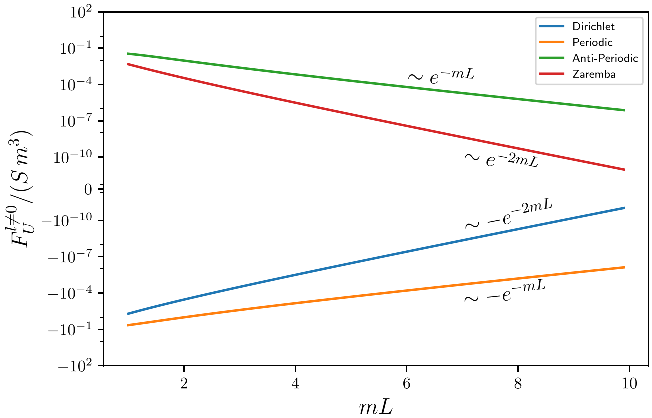

Fig. 2. Behavior of the adimensional free energy in logarithmic scale for Dirichlet, periodic, Neumann and Zaremba boundary conditions, as a function of the effective distance mL with a fixed temperature $m\beta = 1$.

$$+\Im\left(\mathrm{Li}_3\left(e^{-i\sqrt{q^2-m^2}\beta}\right)\right)(1-\tanh(qL))\,. \tag{56}$$

5.1. Asymptotic behavior of Casimir energy

From the previous results we can see how the asymptotic behavior of the Casimir energy for $mL \gg 1$ follows a common rule (39); for Neumann, Dirichlet (46) and Zaremba (54) boundary conditions it is exponentially decaying

$$(mL)^{3/2} E_U^c \sim e^{-2mL} \tag{57}$$

at the same rate, which agrees with the prescription given by the general rule because in all these cases $\mathrm{tr}\left(U\sigma_1\right) = 0$. However, the boundary conditions satisfying that $\mathrm{tr}\left(U\sigma_1\right) \neq 0$ like periodic (42) and anti-periodic (50) the exponential decay is

$$(mL)^{3/2} E_U^c \sim e^{-mL}. \tag{58}$$

The behavior of the Casimir energy for these boundary conditions with different asymptotic behaviors is displayed in Fig. 1.

On the other hand, we can also plot the rest of the terms of the free energy $F_U^{l\neq 0}$, that cannot be analytically calculated, to show how they have the same exponential decay with mL as the Casimir energy (see Fig. 2).

How the boundary conditions are imposed at each wall is what physically differentiates the two families with different asymptotic behavior. In the family with a faster decay (Dirichlet, Neumann and Zaremba) the boundary conditions on each plate are imposed independently, whereas for the second family (periodic, antiperiodic) there is an inter-connection between the boundary values at the two boundary plates, which induces a slower decay.

6. Conclusions

In this paper we have shown that the rate of exponential decay at large distances in the Casimir energy of a massive scalar field in 3+1 dimensions, splits the boundary conditions into two different families. These two types differ by the fact that whether or not there is an inter-connection between the boundary values of the fields or its normal derivatives at the boundary plates.

Although measuring this effect for massive fields seems hopeless because this exponential decay makes it negligible in comparison with the effect for electrodynamics fields (that experience just a potential decay with the distance), from a theoretical viewpoint it can be relevant for discerning whether the infrared behavior in non-abelian gauge theories can be described by massive scalar fields or not.

In fact, in 2+1 dimensions numerical simulations [16] and some analytic arguments [14,15] point out that with Dirichlet boundary conditions the gauge theories have an exponential decay with a mass lower than the lightest glueball. Finding some similar behavior for 3+1 dimensions would provides new hints for a better understanding the infrared regime of non-abelian gauge theories, and in particular can illuminate the solution of the mass-gap problem. In particular, the identification of these two different regimes of exponential decays for the Casimir energy would provide a very strong support of this conjecture and its implications. Some accurate numerical simulations to test the conjecture are in progress.

CRediT authorship contribution statement

Manuel Asorey: Writing – review & editing, Supervision, Project administration, Funding acquisition, Conceptualization. **Fernando Ezquerro:** Writing – original draft, Investigation, Formal analysis, Conceptualization. **Miguel Pardina:** Validation, Supervision, Formal analysis.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix. Explicit calculations of the free energy

In this section we calculate the free energies of the theory for two different boundary conditions (Dirichlet and periodic) directly by using the known spectrum of the spacial Laplacian, and compare with the results obtained by using the general method based on the spectral function and the use of Cauchy theorems.

A.1. Dirichlet boundary conditions

The eigenvalues of the discrete modes of the Laplacian operator $-\Delta$ are of the form $k_n = \pi n/L$ where $n = 1, \dots, \infty$.

The corresponding zeta function (8) is given by

$$\zeta(\beta, m, L; s) = \left(\frac{\beta\mu}{2\pi}\right)^{2s} \frac{\pi S}{\beta^2(s-1)} \sum_{j=1}^{\infty} \sum_{l=-\infty}^{\infty} \left(l^2 + \left(\frac{j\beta}{2L}\right)^2 + \left(\frac{m\beta}{2\pi}\right)^2\right)^{-s+1}, \tag{A.1}$$

and operating in the same way as we did in the general case we get

$$\zeta(\beta, m, L; s) = \left(\frac{\beta\mu}{2\pi}\right)^{2s} \frac{\pi^{3/2} S}{\Gamma(s)\beta^2} \left(\Gamma\left(s-\frac{3}{2}\right) \sum_{j=1}^{\infty} \left(\left(\frac{j\beta}{2L}\right)^2 + \left(\frac{\beta m}{2\pi}\right)^2\right)^{3/2-s} \right.$$
$$\left. +4 \sum_{j,l=1}^{\infty} (\pi l)^{s-3/2} \left(\left(\frac{j\beta}{2L}\right)^2 + \left(\frac{\beta m}{2\pi}\right)^2\right)^{3/4-s/2} K_{3/2-s}\left(\beta l \sqrt{\left(\frac{\pi j}{L}\right)^2 + m^2}\right)\right).$$

Although this expression diverges in the limit $s \to 0$ its first derivative does not. Let us first derive the $\Gamma(s)$ on the $l \neq 0$ terms

$$\left(\zeta^{l\neq 0}\right)'(\beta, m, L; 0) = \frac{S}{\pi\beta^2} \sum_{j=1}^{\infty} \left(\frac{\beta}{L}\sqrt{(j\pi)^2 + (mL)^2}\, \mathrm{Li}_2\left(e^{-\frac{\beta}{L}\sqrt{(j\pi)^2+(mL)^2}}\right)\right.$$
$$\left. +\ \mathrm{Li}_3\left(e^{-\frac{\beta}{L}\sqrt{(j\pi)^2+(mL)^2}}\right)\right), \tag{A.2}$$

whereas from the $l = 0$ terms we get

$$\zeta^{l=0}(\beta, m, L; s) = \left(\frac{\mu L}{\pi}\right)^{2s} \frac{\pi^2 S\beta}{16L^3\Gamma(s)} \left(\Gamma(s-2)\left(\frac{mL}{\pi}\right)^{4-2s} - \frac{\Gamma(s-\frac{3}{2})}{\sqrt{\pi}}\left(\frac{mL}{\pi}\right)^{3-2s} + 4\sum_{j=1}^{\infty}\left(\frac{mL}{j\pi^2}\right)^{2-s} K_{2-s}(2jmL)\right), \tag{A.3}$$

and by using the Gamma function $\Gamma(s)$ properties we get

$$\left(\zeta^{l=0}\right)'(\beta, m, L; 0) = \frac{SL\beta m^4}{32\pi^2}\left(2\log\frac{\mu}{m} + \frac{3}{2}\right) - \frac{Sm^3\beta}{12\pi} + \frac{S\beta m^2}{4\pi^2 L}\sum_{j=1}^{\infty}\frac{K_2(2jmL)}{j^2}. \tag{A.4}$$

The Casimir energy can be computed from this expression by using the renormalization prescription we described in (12)

$$E_D^c(L, m) = -\frac{Sm^2}{8\pi^2 L}\sum_{j=1}^{\infty}\frac{K_2(2jmL)}{j^2}, \tag{A.5}$$

which is the same as that derived in (46) by the spectral function method. In the massless case it reduces to

$$E_D^c(L, 0) = -\frac{\pi^2 S}{1440L^3}. \tag{A.6}$$

The temperature dependent component (A.2)

$$F_D^{l\neq 0}(\beta, m, L) = \frac{-S}{2\pi\beta^3}\sum_{j=1}^{\infty}\left(\frac{\beta}{L}\sqrt{(j\pi)^2+(mL)^2}\,\mathrm{Li}_2\left(e^{-\frac{\beta}{L}\sqrt{(j\pi)^2+(mL)^2}}\right) + \mathrm{Li}_3\left(e^{-\frac{\beta}{L}\sqrt{(j\pi)^2+(mL)^2}}\right)\right) - \frac{1}{2\beta}\lim_{L_0\to\infty}\left(\left(\zeta^{l\neq 0}\right)'(\beta, m, L+2L_0; 0) - 2\left(\zeta^{l\neq 0}\right)'(\beta, m, L+L_0; 0)\right),$$

gives the same result as the one obtained from the expression (48).

A.2. Periodic boundary conditions

In the case of periodic boundary conditions, we have as discrete eigenvalues $k_n = 2\pi n/L$ with $j \in \mathbb{Z}$, and the corresponding zeta function

$$\zeta(\beta, m, L; s) = \left(\frac{\beta\mu}{2\pi}\right)^{2s} \frac{\pi S}{\beta^2(s-1)} \sum_{j,l=-\infty}^{\infty}\left(l^2 + \left(\frac{j\beta}{L}\right)^2 + \left(\frac{m\beta}{2\pi}\right)^2\right)^{-s+1} \tag{A.7}$$

becomes

$$\zeta(\beta, m, L; s) = \left(\frac{\beta\mu}{2\pi}\right)^{2s} \frac{\pi^{3/2} S}{\Gamma(s)\beta^2}\left[\Gamma\left(s-\frac{3}{2}\right)\sum_{j=-\infty}^{\infty}\left(\left(\frac{j\beta}{L}\right)^2 + \left(\frac{\beta m}{2\pi}\right)^2\right)^{3/2-s} + 4\sum_{j=-\infty}^{\infty}\sum_{l=1}^{\infty}(\pi l)^{s-3/2}\left(\left(\frac{j\beta}{L}\right)^2 + \left(\frac{\beta m}{2\pi}\right)^2\right)^{3/4-s/2} \times K_{3/2-s}\left(\beta l\sqrt{\left(\frac{2\pi j}{L}\right)^2 + m^2}\right)\right], \tag{A.8}$$

after some simple operations

For $l \neq 0$ terms we can just take the derivative of the $\Gamma(s)$ and evaluate it in $s = 0$

$$\left(\zeta^{l\neq 0}\right)'(\beta, m, L; 0) = \frac{S}{\pi\beta^2}\sum_{j=-\infty}^{\infty}\left(\frac{\beta}{L}\sqrt{(j\pi)^2+(mL)^2}\,\mathrm{Li}_2\left(e^{-\frac{\beta}{L}\sqrt{(j\pi)^2+(mL)^2}}\right) + \mathrm{Li}_3\left(e^{-\frac{\beta}{L}\sqrt{(j\pi)^2+(mL)^2}}\right)\right) \tag{A.9}$$

whereas the $l = 0$ term can be rewritten as

$$\zeta^{l=0}(\beta, m, L; s) = \left(\frac{\mu L}{2\pi}\right)^{2s} \frac{\pi^{3/2} S\beta\Gamma\left(s-\frac{3}{2}\right)}{8L^3\Gamma(s-1)(s-1)}\sum_{j=-\infty}^{\infty}\left(j^2 + \left(\frac{mL}{2\pi}\right)^2\right)^{3/2-s}, \tag{A.10}$$

and by using the properties of Gamma function $\Gamma(s)$ we get

$$\left(\zeta^{l=0}\right)'(\beta, m, L; 0) = \frac{SL\beta m^4}{32\pi^2}\left(\log\frac{\mu}{m} + \frac{3}{4}\right) + \frac{S\beta m^2}{\pi^2 L}\sum_{j=1}^{\infty}\frac{K_2(jmL)}{j^2}. \tag{A.11}$$

Thus, under the renormalization prescription (12) the Casimir energy is

$$E_P^c(L,m) = -\frac{Sm^2}{2\pi^2 L}\sum_{j=1}^{\infty}\frac{K_2(jmL)}{j^2}, \tag{A.12}$$

which does coincide with (42), and in the massless case we get

$$E_P^c(L,0) = -\frac{\pi^2 S}{90L^3}. \tag{A.13}$$

The temperature dependent component (A.9) is

$$\begin{aligned} F_P^{l\neq 0}(\beta,m,L) = &-\frac{S}{2\pi\beta^3}\sum_{j=1}^{\infty}\left(\frac{\beta}{L}\sqrt{(2j\pi)^2+(mL)^2}\,\mathrm{Li}_2\left(e^{-\frac{\beta}{L}\sqrt{(2j\pi)^2+(mL)^2}}\right)\right.\\ &\left.+\mathrm{Li}_3\left(e^{-\frac{\beta}{L}\sqrt{(2j\pi)^2+(mL)^2}}\right)\right)\\ &-\frac{1}{2\beta}\lim_{L_0\to\infty}\left(\left(\zeta^{l\neq 0}\right)'(\beta,m,L+2L_0;0)-2\left(\zeta^{l\neq 0}\right)'(\beta,m,L+L_0;0)\right), \end{aligned} \tag{A.14}$$

which also does agree with expression (44).

Data availability

No data was used for the research described in the article.

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