Borell's inequality and mean width of random polytopes via discrete inequalities ☆

David Alonso-Gutiérrez *, Luis C. García-Lirola

Área de análisis matemático, Departamento de matemáticas, Facultad de Ciencias, Universidad de Zaragoza, IUMA, Pedro Cerbuna 12, 50009 Zaragoza, Spain

ARTICLE INFO

ABSTRACT

Borell's inequality states the existence of a positive absolute constant $C > 0$ such that for every $1 \le p \le q$

$$\left(\mathbb{E}|\langle X, e_n \rangle|^p\right)^{\frac{1}{p}} \le \left(\mathbb{E}|\langle X, e_n \rangle|^q\right)^{\frac{1}{q}} \le C\frac{q}{p}\left(\mathbb{E}|\langle X, e_n \rangle|^p\right)^{\frac{1}{p}},$$

whenever X is a random vector uniformly distributed on any convex body $K \subseteq \mathbb{R}^n$ and $(e_i)_{i=1}^n$ is the standard canonical basis in $\mathbb{R}^n$. In this paper, we will prove a discrete version of this inequality, which will hold whenever X is a random vector uniformly distributed on $K \cap \mathbb{Z}^n$ for any convex body $K \subseteq \mathbb{R}^n$ containing the origin in its interior. We will also make use of such discrete version to obtain discrete inequalities from which we can recover the estimate $\mathbb{E}w(K_N) \sim w(Z_{\log N}(K))$ for any convex body K containing the origin in its interior, where K_N is the centrally symmetric random polytope $K_N = \text{conv}\{\pm X_1, \ldots, \pm X_N\}$ generated by independent random vectors uniformly distributed on K, $Z_p(K)$ is the L_p-centroid body of K for any $p \ge 1$, and $w(\cdot)$ denotes the mean width.

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1. Introduction and notation

Given a convex body $K \subseteq \mathbb{R}^n$ and $p \geq 1$, its L_p-centroid body, $Z_p(K)$, is defined as the convex body with support function

$$h_{Z_p(K)}(y) = \left(\mathbb{E}|\langle X, y\rangle|^p\right)^{\frac{1}{p}} = \left(\frac{1}{|K|}\int_K |\langle x, y\rangle|^p dx\right)^{\frac{1}{p}} \qquad \forall y \in \mathbb{R}^n,$$

where X is a random vector uniformly distributed on K, $\mathbb{E}Y$ denotes the expectation of a random variable Y, and $|K|$ denotes the volume of K (i.e., its n-dimensional Lebesgue measure). Let us recall that, given any convex body $L \subseteq \mathbb{R}^n$, its support function h_L is defined as $h_L(y) := \sup_{x \in L}\langle x, y\rangle$ for every $y \in \mathbb{R}^n$ and that every sublinear function $f : \mathbb{R}^n \to \mathbb{R}$ is the support function of a unique convex body (see [32, Thm 1.7.1]). These bodies were introduced, under a different normalization, in [24]. Since then, they have shown to play a crucial role on the distribution of the mass on convex bodies (see e.g. [12,15,27]).

By Hölder's inequality, for any convex body $K \subseteq \mathbb{R}^n$ and any $1 \leq p \leq q$, if X is a random vector uniformly distributed on K we have that for any $\theta \in S^{n-1}$, the Euclidean sphere in $\mathbb{R}^n$,

$$h_{Z_p(K)}(\theta) = \left(\mathbb{E}|\langle X, \theta\rangle|^p\right)^{\frac{1}{p}} \leq \left(\mathbb{E}|\langle X, \theta\rangle|^q\right)^{\frac{1}{q}} = h_{Z_q(K)}(\theta). \tag{1.1}$$

Equivalently,

$$Z_p(K) \subseteq Z_q(K). \tag{1.2}$$

As a consequence of Borell's inequality (see [26, Appendix III] and [8, Thm. 2.4.6 and Rmk. 2.4.8]), we have that there exists an absolute constant $C > 0$ such that for any convex body $K \subseteq \mathbb{R}^n$ and any $\theta \in S^{n-1}$, if X is a random vector uniformly distributed on K, then for any $1 \leq p \leq q$,

$$\left(\mathbb{E}|\langle X, \theta\rangle|^q\right)^{\frac{1}{q}} \leq C\frac{q}{p}\left(\mathbb{E}|\langle X, \theta\rangle|^p\right)^{\frac{1}{p}}. \tag{1.3}$$

Equivalently, there exists an absolute constant $C > 0$ such that for any convex body $K \subseteq \mathbb{R}^n$, and for any $1 \leq p \leq q$,

$$Z_q(K) \subseteq C\frac{q}{p}Z_p(K). \tag{1.4}$$

Writing equations (1.2) and (1.4) together, we have that there exists an absolute constant $C > 0$ such that for any convex body $K \subseteq \mathbb{R}^n$ and any $1 \leq p \leq q$,

$$Z_p(K) \subseteq Z_q(K) \subseteq C\frac{q}{p}Z_p(K). \tag{1.5}$$

Let us stress out that the constant C in the equations (1.3) to (1.5) is a positive absolute constant, independent of the dimension n, of the convex body $K \subseteq \mathbb{R}^n$, and of the parameters $1 \leq p \leq q$. Let us also point out that the fact that (1.1) and (1.3) hold for every convex body K and every $\theta \in S^{n-1}$ is equivalent to the fact that they hold for every convex body $K \subseteq \mathbb{R}^n$ and $\theta = e_n$, where $(e_i)_{i=1}^n$ is the canonical basis in $\mathbb{R}^n$. That is, for every convex body $K \subseteq \mathbb{R}^n$ and $1 \leq p \leq q$, if X is a random vector uniformly distributed on K then

$$\left(\mathbb{E}|\langle X, e_n\rangle|^p\right)^{\frac{1}{p}} \leq \left(\mathbb{E}|\langle X, e_n\rangle|^q\right)^{\frac{1}{q}} \leq C\frac{q}{p}\left(\mathbb{E}|\langle X, e_n\rangle|^p\right)^{\frac{1}{p}}. \tag{1.6}$$

Indeed, given $K \subseteq \mathbb{R}^n$ a convex body and $\theta \in S^{n-1}$, there exists an orthogonal map $U \in O(n)$ such that $U^t e_n = \theta$, where U^t denotes the transpose matrix of U. Thus, if X is a random vector uniformly distributed on K, then for any $p \geq 1$

$$\left(\mathbb{E}|\langle X, \theta\rangle|^p\right)^{\frac{1}{p}} = \left(\mathbb{E}|\langle X, U^t e_n\rangle|^p\right)^{\frac{1}{p}} = \left(\mathbb{E}|\langle UX, e_n\rangle|^p\right)^{\frac{1}{p}} = \left(\mathbb{E}|\langle Y, e_n\rangle|^p\right)^{\frac{1}{p}},$$

where Y is a random vector uniformly distributed on UK. Therefore, applying (1.6) to the convex body UK, we obtain (1.1) and (1.3).

The main purpose of this paper is to obtain a discrete version of (1.6), in which the random vector X is uniformly distributed on the intersection of a convex body $K \subseteq \mathbb{R}^n$ with the integer lattice $\mathbb{Z}^n$, following an active line of research whose purpose is to obtain discrete analogues of classical results in convex geometry (see, for instance, the discrete versions of the Brunn-Minkowski inequality proved by Gardner and Zhang in [14], Koldobsky's slicing inequality [3], or Meyer's inequality [13]).

In [21], the authors obtained a discrete Brunn-Minkowski inequality for the lattice point enumerator $G_n(\cdot)$, which is defined as the cardinality of the intersection of a set in $\mathbb{R}^n$ with $\mathbb{Z}^n$ (see Theorem 2.1 below). Such version of the Brunn-Minkowski inequality allows to recover the classical one (see [32, Theorem 7.1.1]), and has been very useful to obtain some other discrete versions of inequalities in convex geometry (see, for instance, [20] for a discrete version of an isoperimetric inequality or [4] for discrete versions of Rogers-Shephard type inequalities and related inequalities). Due to the fact that adding an open unit cube in the large term in this discrete version of the Brunn-Minkowski inequality is necessary, typically an extra open unit cube appears in some terms in these new discrete inequalities. Nevertheless, one can still deduce their continuous versions from them.

In this paper, we will mainly consider convex bodies $K \subseteq \mathbb{R}^n$ with $0 \in \operatorname{int} K$, the interior of K, and $\max_{x \in K \cap \mathbb{Z}^n} |\langle x, e_n\rangle| \geq 1$. Such condition will ensure that $K \cap \mathbb{Z}^n \neq \emptyset$ (i.e., $G_n(K) \neq 0$) and that if X is a random vector uniformly distributed on $K \cap \mathbb{Z}^n$, then $\mathbb{E}|\langle X, e_n\rangle|^p \neq 0$ for every $p \geq 1$. Besides, for any convex body $K \subseteq \mathbb{R}^n$ with $0 \in \operatorname{int} K$, we have that λUK will satisfy such condition for any orthogonal transformation U, if $\lambda > 0$ is large enough, which will allow to recover continuous inequalities from discrete ones for convex bodies containing the origin in its interior.

Unfortunately, given $1 \leq p \leq q$ and $n \in \mathbb{N}$, it is not possible to find $C(p, q, n) > 0$ such that $\left(\mathbb{E}|\langle X, e_n\rangle|^q\right)^{\frac{1}{q}} \leq C(p, q, n) \left(\mathbb{E}|\langle X, e_n\rangle|^p\right)^{\frac{1}{p}}$ for every convex body $K \subset \mathbb{R}^n$ with $0 \in \operatorname{int} K$ and $\max_{x \in K \cap \mathbb{Z}^n} |\langle x, e_n\rangle| \geq 1$. This can be easily checked by considering $K = \operatorname{conv}(\{(x, -1/2) \in \mathbb{R}^{n-1} \times \mathbb{R} : \|x\|_\infty \leq \lambda\} \cup \{e_n\})$ and letting $\lambda \to \infty$, where $\operatorname{conv}(A)$ denotes the convex hull of the set $A \subseteq \mathbb{R}^n$. However, as a direct consequence of the fact that, if $1 \leq p \leq q$, then the ℓ^q-norm is bounded by the ℓ^p-norm in $\mathbb{R}^N$ for any $N \in \mathbb{N}$, we have that for every convex body $K \subseteq \mathbb{R}^n$ with $G_n(K) \neq 0$, then

$$\left(\mathbb{E}|\langle X, e_n\rangle|^q\right)^{\frac{1}{q}} \leq G_n(K)^{\frac{1}{p}-\frac{1}{q}} \left(\mathbb{E}|\langle X, e_n\rangle|^p\right)^{\frac{1}{p}}.$$

Therefore, if we fix $1 \leq p \leq q$ and consider convex bodies such that $G_n(K)$ is bounded above by $C\frac{q}{p}$ with C an absolute constant, we have that $G_n(K)^{\frac{1}{p}-\frac{1}{q}} \leq G_n(K) \leq C\frac{q}{p}$, and for such family of convex bodies we trivially obtain the following discrete version of Borell's inequality:

$$\left(\mathbb{E}|\langle X, e_n\rangle|^q\right)^{\frac{1}{q}} \leq C\frac{q}{p}\left(\mathbb{E}|\langle X, e_n\rangle|^p\right)^{\frac{1}{p}}. \tag{1.7}$$

The following theorem, and its equivalent form given by Theorem 1.2 below, gives a discrete version of Borell's inequality which also holds whenever $G_n(K)$ is not bounded. As we will see, when applied to λK for a fixed convex body $K \subseteq \mathbb{R}^n$ with the origin in its interior and taking $\lambda \to \infty$, we will recover the continuous version of Borell's inequality. Notice that in such case $G_n(\lambda K)$ will tend to infinity and the estimate given by (1.7) cannot be applied.

Theorem 1.1. *There exists an absolute constant $C > 0$ such that for any convex body $K \subseteq \mathbb{R}^n$ with $G_n(K) \neq 0$ and any $1 \leq p \leq q$*

$$\left(\mathbb{E}|\langle X, e_n\rangle|^q\right)^{\frac{1}{q}} \leq C\frac{q}{p}\left(1 + \left(\mathbb{E}|\langle Y, e_n\rangle|^p\right)^{\frac{1}{p}} \left(\frac{G_n(K + (-1, 1)^n)}{G_n(K)}\right)^{\frac{1}{p}}\right), \tag{1.8}$$

where X is a random vector uniformly distributed on $K \cap \mathbb{Z}^n$, and Y is a random vector uniformly distributed on $(K + (-1, 1)^n) \cap \mathbb{Z}^n$.

Let us introduce the following notation, which will allow us to express the right-hand side of Theorem 1.1 in terms of the random vector X uniformly distributed on $K \cap \mathbb{Z}^n$ and, studying the asymptotic behavior of the quantity introduced here, prove that it implies (1.3) whenever $0 \in \text{int } K$: For any convex body $K \subseteq \mathbb{R}^n$ such that $\max_{x \in K \cap \mathbb{Z}^n} |\langle x, e_n \rangle| \geq 1$ and any $p \geq 1$, we define $D_0(K, p)$ as the number given by

$$D_0(K, p) := \frac{\left(1 + (\mathbb{E}|\langle Y, e_n \rangle|^p)^{\frac{1}{p}} \left(\frac{G_n(K+C_n)}{G_n(K)} \right)^{\frac{1}{p}} \right)}{(\mathbb{E}|\langle X, e_n \rangle|^p)^{\frac{1}{p}}},$$

where $C_n = (-1, 1)^n$, $G_n(K)$ is the cardinality of $K \cap \mathbb{Z}^n$, X is a random vector uniformly distributed on $K \cap \mathbb{Z}^n$, and Y is a random vector uniformly distributed on $(K + C_n) \cap \mathbb{Z}^n$. With this notation, and taking into account Hölder's inequality, we can rewrite Theorem 1.1 as follows in the case that $\max_{x \in K \cap \mathbb{Z}^n} |\langle x, e_n \rangle| \geq 1$ (otherwise, Theorem 1.1 is trivial):

Theorem 1.2. *There exists an absolute constant $C > 0$ such that for any convex body $K \subseteq \mathbb{R}^n$ with $\max_{x \in K \cap \mathbb{Z}^n} |\langle x, e_n \rangle| \geq 1$, and any $1 \leq p \leq q$*

$$\left(\mathbb{E}|\langle X, e_n \rangle|^p \right)^{\frac{1}{p}} \leq \left(\mathbb{E}|\langle X, e_n \rangle|^q \right)^{\frac{1}{q}} \leq C \frac{q}{p} D_0(K, p) \left(\mathbb{E}|\langle X, e_n \rangle|^p \right)^{\frac{1}{p}}, \qquad (1.9)$$

where X is a random vector uniformly distributed on $K \cap \mathbb{Z}^n$.

In order to study the asymptotic behavior of the number $D_0(\lambda K, p)$ as $\lambda \to \infty$ and recover continuous inequalities from their discrete versions, we also define, for any convex body $K \subseteq \mathbb{R}^n$ such that the Euclidean closed unit ball, B_2^n, satisfies $B_2^n \subseteq K$, and any $q \geq 1$, $D_q(K)$ as the following number

$$D_q(K) := \sup_{\substack{U \in O(n) \\ 1 \leq p \leq q}} D_0(UK, p).$$

Let us point out that the condition $B_2^n \subseteq K$ in this definition is imposed so that, for every $U \in O(n)$, $e_n \in UK \cap \mathbb{Z}^n$ and then for any $p \geq 1$ and any $U \in O(n)$, $D_0(UK, p)$ is well defined.

The following lemma, involving the role of the number $D_q(\cdot)$, will allow us to recover (1.6) from Theorem 1.2 (with the same value of the constant C), as well as some other continuous inequalities from discrete ones:

Lemma 1.1. *For any convex body $K \subseteq \mathbb{R}^n$ such that $0 \in \text{int } K$ and any $q \geq 1$ we have*

$$\lim_{\lambda \to \infty} D_q(\lambda K) = 1.$$

Remark 1. Let us point out that for every convex body $K \subseteq \mathbb{R}^n$ such that $\max_{x \in K \cap \mathbb{Z}^n} |\langle x, e_n \rangle| \geq 1$ and any $p \geq 1$ we have that $D_0(K, p) \geq 1$. Indeed, if X and Y are random vectors uniformly distributed on $K \cap \mathbb{Z}^n$ and $(K + C_n) \cap \mathbb{Z}^n$ respectively,

$$D_0(K, p) \geq \frac{\left((\mathbb{E}|\langle Y, e_n \rangle|^p)^{\frac{1}{p}} \left(\frac{G_n(K+C_n)}{G_n(K)} \right)^{\frac{1}{p}} \right)}{(\mathbb{E}|\langle X, e_n \rangle|^p)^{\frac{1}{p}}}$$

$$= \left(\frac{\sum_{x \in (K+C_n) \cap \mathbb{Z}^n} |\langle x, e_n \rangle|^p}{\sum_{x \in K \cap \mathbb{Z}^n} |\langle x, e_n \rangle|^p} \right)^{\frac{1}{p}} \geq \left(\frac{\sum_{x \in K \cap \mathbb{Z}^n} |\langle x, e_n \rangle|^p}{\sum_{x \in K \cap \mathbb{Z}^n} |\langle x, e_n \rangle|^p} \right)^{\frac{1}{p}} = 1.$$

For every convex body $K \subseteq \mathbb{R}^n$ such that $0 \in \operatorname{int} K$, there exists $\lambda_0 > 0$ such that $B_2^n \subseteq \lambda_0 K$. Then, for every $\lambda \geq \lambda_0$, $U \in O(n)$, and $p \geq 1$, choosing $q \geq p$, we have that

$$1 \leq D_0(\lambda U K, p) \leq \sup_{\substack{U \in O(n) \\ 1 \leq p \leq q}} D_0(\lambda U K, p) = D_q(\lambda K).$$

Therefore, Lemma 1.1 implies that for every $U \in O(n)$ and $p \geq 1$ we have $\lim_{\lambda \to \infty} D_0(\lambda U K, p) = 1$.

Random polytopes are important objects of study in several areas in mathematics since Sylvester initiated their study with a problem posed in *The Educational Times* in 1864 [33]. This problem was originally ill-posed and the question was modified within a year, and became known as the "four-point problem". It were Rényi and Sulanke who, in their seminal papers [29], [30], [31], focused on the expected volume of random polytopes as the number of points generating it tends to infinity, as well as on the vertex number in the planar case. Random polytopes appear in approximation theory [6], random matrix theory [22], or in other disciplines such as statistics, information theory, signal processing, medical imaging or digital communications (see [11] and the references therein). Besides, since the work of Gluskin [16], they have been known to provide counterexamples to some conjectures, or examples to the sharpness of some estimates on convex bodies (see the survey [25] and its references, or [17] and [23]), as the behavior of some parameters of a linear and geometric nature different from the known deterministic constructions.

Different quantities such as expectations, variances, distribution, concentration inequalities, central limit theorems or large deviation principles of several geometric functionals, such as volume, number of faces, or quermaßintegrals, associated to random polytopes are studied, which obviously depend on the random model from which the random polytope is generated (see, for instance, [1], [2], [7], [10], [18], or [19]). Here we will focus on the mean width and the following model: Given a convex body $K \subseteq \mathbb{R}^n$ with $0 \in \operatorname{int} K$, $N \geq n$, and $X_1, \ldots, X_N$ independent random vectors uniformly distributed on K, let $K_N = \operatorname{conv}\{\pm X_1, \ldots, \pm X_N\}$ be the centrally symmetric random polytope generated by $X_1, \ldots, X_N$. It is well known that there exist absolute constants c_1, c_1', c_2, c_2' such that for any convex body $K \subseteq \mathbb{R}^n$ with $0 \in \operatorname{int} K$, and any $n \leq N \leq e^n$ we have

$$c_1 w(Z_{c_1' \log N}(K)) \leq \mathbb{E} w(K_N) \leq c_2 w(Z_{c_2' \log N}(K)), \tag{1.10}$$

where for any convex body $L \subseteq \mathbb{R}^n$ containing the origin, $w(L)$ denotes its mean width

$$w(L) = \int_{S^{n-1}} h_L(\theta) d\sigma(\theta),$$

being $d\sigma$ the uniform probability measure on S^{n-1}. We refer the reader to [8, Proposition 11.3.10] for a proof of the upper bound and to [8, Theorem 11.3.2] for a proof of a stronger result that implies the lower bound when $N \geq n^2$, which relies on the continuous Borell's inequality given by (1.3). The fact that the lower bound also holds when $n \leq N \leq n^2$ is probably folklore. Nevertheless, since we could not find it in the literature, we will include a proof, which relies on Borell's inequality as well, in Section 2.2. We also refer the reader to [5] for the study of the expected value of the mean width of a large family of random convex sets, which include random polytopes generated by isotropic log-concave random vectors.

In view of the connection of Borell's inequality to the estimate of the mean width of random polytopes given by (1.10), and the relation between Theorem 1.2 and the continuous version of Borell's inequality, it is our purpose to explore the relation between Theorem 1.2, and discrete versions of (1.10). However, since the inequality given by Theorem 1.2 is only provided in the direction given by $\theta = e_n$, in order to explore such relation, we must rewrite (1.10) in a different way.

On the one hand, notice that, by uniqueness of the Haar measure, we have that if $d\nu$ denotes the Haar probability measure on the set of orthogonal matrices $O(n)$, then

$$w(K_N) = \int\limits_{S^{n-1}} h_{K_N}(\theta) d\sigma(\theta) = \int\limits_{S^{n-1}} \max_{1 \le i \le N} |\langle X_i, \theta \rangle| d\sigma(\theta)$$

$$= \int\limits_{O(n)} \max_{1 \le i \le N} |\langle X_i, U^t e_n \rangle| d\nu(U) = \int\limits_{O(n)} \max_{1 \le i \le N} |\langle U X_i, e_n \rangle| d\nu(U).$$

Therefore,

$$\mathbb{E} w(K_N) = \mathbb{E} \int\limits_{O(n)} \max_{1 \le i \le N} |\langle U X_i, e_n \rangle| d\nu(U) = \int\limits_{O(n)} \mathbb{E} \max_{1 \le i \le N} |\langle U X_i, e_n \rangle| d\nu(U)$$

$$= \int\limits_{O(n)} \frac{1}{|K|^N} \int\limits_{K} \cdots \int\limits_{K} \max_{1 \le i \le N} |\langle U x_i, e_n \rangle| dx_N \ldots dx_1 d\nu(U)$$

$$= \int\limits_{O(n)} \frac{1}{|UK|^N} \int\limits_{UK} \cdots \int\limits_{UK} \max_{1 \le i \le N} |\langle x_i, e_n \rangle| dx_N \ldots dx_1 d\nu(U).$$

On the other hand, for any $p \ge 1$ we have that

$$w(Z_p(K)) = \int\limits_{S^{n-1}} h_{Z_p(K)}(\theta) d\sigma(\theta) = \int\limits_{S^{n-1}} \left(\frac{1}{|K|} \int\limits_{K} |\langle x, \theta \rangle|^p dx \right)^{\frac{1}{p}} d\sigma(\theta)$$

$$= \int\limits_{O(n)} \left(\frac{1}{|K|} \int\limits_{K} |\langle x, U^t e_n \rangle|^p dx \right)^{\frac{1}{p}} d\nu(U)$$

$$= \int\limits_{O(n)} \left(\frac{1}{|K|} \int\limits_{K} |\langle U x, e_n \rangle|^p dx \right)^{\frac{1}{p}} d\nu(U)$$

$$= \int\limits_{O(n)} \left(\frac{1}{|UK|} \int\limits_{UK} |\langle x, e_n \rangle|^p dx \right)^{\frac{1}{p}} d\nu(U).$$

Therefore, Equation (1.10) is equivalent to

$$c_1 \int\limits_{O(n)} \left(\frac{1}{|UK|} \int\limits_{UK} |\langle x, e_n \rangle|^{c_1' \log N} dx \right)^{\frac{1}{c_1' \log N}} d\nu(U)$$

$$\le \int\limits_{O(n)} \frac{1}{|UK|^N} \int\limits_{UK} \cdots \int\limits_{UK} \max_{1 \le i \le N} |\langle x_i, e_n \rangle| dx_N \ldots dx_1 d\nu(U)$$

$$\le c_2 \int\limits_{O(n)} \left(\frac{1}{|UK|} \int\limits_{UK} |\langle x, e_n \rangle|^{c_2' \log N} dx \right)^{\frac{1}{c_2' \log N}} d\nu(U). \tag{1.11}$$

In this paper we will prove the following two theorems, which will provide discrete versions of the inequalities in (1.11) where the role of the volume is played by the lattice point enumerator measure given by $G_n(L)$, the cardinality of the set $L \cap \mathbb{Z}^n$ for any Borel set $L \subseteq \mathbb{R}^n$. The two inequalities in (1.11) and, equivalently, equation (1.10) will be recovered from the discrete versions as corollaries.

Let us also point out that, as a consequence of (1.5), inequality (1.10) can be equivalently written with constants $c_1' = c_2' = 1$. Nevertheless, for our purposes of finding analogue discrete inequalities we decided not to do so.

Theorem 1.3. *There exists an absolute constant $C > 0$ such that for every convex body $K \subseteq \mathbb{R}^n$ with $0 \in K$ and every $N \in \mathbb{N}$ with $N \geq 3$,*

$$\int_{O(n)} \frac{1}{(G_n(UK))^N} \sum_{x_1,\ldots,x_N \in UK \cap \mathbb{Z}^n} \max_{1 \leq i \leq N} |\langle x_i, e_n \rangle| d\nu(U)$$

$$\leq C \int_{O(n)} \left(\frac{1}{G_n(UK)} \sum_{x \in UK \cap \mathbb{Z}^n} |\langle x, e_n \rangle|^{\log N} \right)^{\frac{1}{\log N}} d\nu(U).$$

As a consequence, the right-hand side inequality in (1.10) will be recovered.

Theorem 1.4. *There exist absolute constants $c, C > 0$ such that if $K \subseteq \mathbb{R}^n$ is a convex body with $B_2^n \subseteq K$, $q \geq 1$ and $N \in \mathbb{N}$ with $(CD_q(K))^2 \leq N \leq e^q$, then*

$$\frac{1}{(G_n(K))^N} \sum_{x_1,\ldots,x_N \in K \cap \mathbb{Z}^n} \max_{1 \leq i \leq N} |\langle x_i, e_n \rangle|$$

$$\geq c \left(\frac{1}{G_n(K)} \sum_{x \in K \cap \mathbb{Z}^n} |\langle x, e_n \rangle|^{\frac{\log N}{2\log(CD_q(K))}} \right)^{\frac{2\log(CD_q(K))}{\log N}}.$$

Given a convex body $K \subseteq \mathbb{R}^n$ with $B_2^n \subseteq K$, and $U \in O(n)$ the convex body UK satisfies that $B_2^n \subseteq UK$ and $D_q(UK) = D_q(K)$. Applying Theorem 1.4 to UK and integrating in $U \in O(n)$, we obtain the following corollary, which provides a discrete version of the right hand-side inequality in (1.11):

Corollary 1.1. *There exist absolute constants $c, C > 0$ such that if $K \subseteq \mathbb{R}^n$ is a convex body with $B_2^n \subseteq K$, $q \geq 1$ and $N \in \mathbb{N}$ with $(CD_q(K))^2 \leq N \leq e^q$, then*

$$\int_{O(n)} \frac{1}{(G_n(UK))^N} \sum_{x_1,\ldots,x_N \in UK \cap \mathbb{Z}^n} \max_{1 \leq i \leq N} |\langle x_i, e_n \rangle| d\nu(U)$$

$$\geq c \int_{O(n)} \left(\frac{1}{G_n(UK)} \sum_{x \in UK \cap \mathbb{Z}^n} |\langle x, e_n \rangle|^{\frac{\log N}{2\log(CD_q(K))}} \right)^{\frac{2\log(CD_q(K))}{\log N}} d\nu(U).$$

As a consequence, the left-hand side inequality in (1.10) will be recovered.

Remark 2. Let us point out that, given a fixed $N \in \mathbb{N}$, it is necessary that $q \geq \log N$ in order to fulfill the conditions in Theorem 1.4 and Corollary 1.1. However, it might not be possible to find a number $q \geq 1$ such that $(CD_q(K))^2 \leq N$, as the following example in $\mathbb{R}^2$ shows: let, for any $M \in \mathbb{N}$, $K_M = \text{conv}\left(B_2^2 \cup \{(M, 0), (-M, 0)\}\right) \subseteq \mathbb{R}^2$. This convex body satisfies that $B_2^2 \subseteq K_M$ and for any $p \geq 1$

$$D_0(K_M, p) = \frac{1 + (\mathbb{E}|\langle Y, e_2 \rangle|^p)^{\frac{1}{p}} \left(\frac{G_2(K_M + C_2)}{G_2(K_M)} \right)^{\frac{1}{p}}}{(\mathbb{E}|\langle X, e_2 \rangle|^p)^{\frac{1}{p}}}$$

$$
= \frac{G_2(K_M)^{\frac{1}{p}} + \left(\sum_{x \in (K_M + C_2) \cap \mathbb{Z}^2} |\langle x, e_2 \rangle|^p\right)^{\frac{1}{p}}}{\left(\sum_{x \in K_M \cap \mathbb{Z}^2} |\langle x, e_2 \rangle|^p\right)^{\frac{1}{p}}}
$$

$$
= \frac{(2M+3)^{\frac{1}{p}} + (2(2M+1))^{\frac{1}{p}}}{2^{\frac{1}{p}}} = \left(M + \frac{3}{2}\right)^{\frac{1}{p}} + (2M+1)^{\frac{1}{p}}.
$$

Thus, for any $q \geq 1$,

$$
D_q(K_M) \geq \sup_{1 \leq p \leq q} \left(\left(M + \frac{3}{2}\right)^{\frac{1}{p}} + (2M+1)^{\frac{1}{p}} \right) = M + \frac{3}{2} + 2M + 1 = 3M + \frac{5}{2}
$$

and, given a fixed $N \in \mathbb{N}$, taking the convex body K_M with M large enough there exists no $q \geq 1$ for which the condition $(CD_q(K_M))^2 \leq N$ is fulfilled.

However, given a fixed convex body $K \subseteq \mathbb{R}^n$ with $B_2^n \subseteq K$, Hölder's inequality implies that

$$
\begin{aligned}
D_q(K) &= \sup_{\substack{U \in O(n) \\ 1 \leq p \leq q}} \frac{1 + (\mathbb{E}|\langle Y_U, e_n \rangle|^p)^{\frac{1}{p}} \left(\frac{G_n(UK + C_n)}{G_n(UK)}\right)^{\frac{1}{p}}}{(\mathbb{E}|\langle X_U, e_n \rangle|^p)^{\frac{1}{p}}} \\
&\leq \sup_{U \in O(n)} \frac{1 + h_{UK}(e_n) \frac{G_n(UK + C_n)}{G_n(UK)}}{\mathbb{E}|\langle X_U, e_n \rangle|} \\
&= \sup_{U \in O(n)} \frac{2 + h_{UK}(e_n) \frac{G_n(UK + C_n)}{G_n(UK)}}{\mathbb{E}|\langle X_U, e_n \rangle|}
\end{aligned}
$$

where, for every $U \in O(n)$, X_U denotes a random vector uniformly distributed on $UK \cap \mathbb{Z}^n$ and Y_U denotes a random vector uniformly distributed on $(UK + C_n) \cap \mathbb{Z}^n$. Therefore, given a fixed convex body $K \subseteq \mathbb{R}^n$ with $B_2^n \subseteq K$ and calling

$$
D(K) := \sup_{U \in O(n)} \frac{2 + h_{UK}(e_n) \frac{G_n(UK + C_n)}{G_n(UK)}}{\mathbb{E}|\langle X_U, e_n \rangle|},
$$

we have that for every $N \geq (CD(K))^2$, taking $q = \log N$, the conditions in Theorem 1.4 are fulfilled.

Moreover, given a fixed convex body $K \subseteq \mathbb{R}^n$ with $B_2^n \subseteq K$ and a fixed $N \in \mathbb{N}$, with $N > C^2$, taking $q = \log N$, Lemma 1.1 implies that if $\lambda > 0$ is large enough, then the condition $(CD_q(\lambda K))^2 \leq N \leq e^q$ will be fulfilled.

2. Preliminaries

In this section we will introduce some well-known results that will be used in our proofs.

2.1. The lattice point enumerator

Let us recall that the lattice point enumerator measure, dG_n, is the measure on $\mathbb{R}^n$ given, for any Borel set $L \subseteq \mathbb{R}^n$, by

$$
G_n(L) = \sharp(L \cap \mathbb{Z}^n),
$$

where $\sharp(\cdot)$ denotes cardinality of a set.

The measure given by the lattice point enumerator satisfies the following discrete version of the Brunn-Minkowski inequality, which was proved in [21, Theorem 2.1], and from which one can recover the classical one [32, Theorem 7.1.1]. It reads as follows:

Theorem 2.1. *Let $\lambda \in (0,1)$ and let $K, L \subset \mathbb{R}^n$ be non-empty bounded Borel sets. Then*

$$G_n\left((1-\lambda)K + \lambda L + (-1,1)^n\right)^{\frac{1}{n}} \geq (1-\lambda)G_n(K)^{\frac{1}{n}} + \lambda G_n(L)^{\frac{1}{n}}.$$

The measure dG_n also satisfies (see [34, Lemma 3.22] and [4, Section 3.1]) that for any convex body $K \subseteq \mathbb{R}^n$ and any bounded set M containing the origin

$$\lim_{\lambda \to \infty} \frac{G_n(\lambda K + M)}{\lambda^n} = |K|. \tag{2.1}$$

Moreover, for any $f : K \to \mathbb{R}$ which is Riemann-integrable on K, we have that

$$\lim_{\lambda \to \infty} \frac{1}{\lambda^n} \int_{\lambda K} f\left(\frac{x}{\lambda}\right) dG_n(x) = \lim_{\lambda \to \infty} \frac{1}{\lambda^n} \sum_{x \in K \cap \left(\frac{1}{\lambda}\mathbb{Z}^n\right)} f(x) = \int_K f(x)dx, \tag{2.2}$$

where the first identity follows from the definition of the measure dG_n. The second identity can be obtained by extending the function f to a rectangle containing K as $f(x) = 0$ for every $x \notin K$, which is Riemann-integrable on the rectangle, and applying [9, Proposition 6.3], which is valid for Riemann-integrable functions on the rectangle. Notice that $\displaystyle\sum_{x \in K \cap \left(\frac{1}{\lambda}\mathbb{Z}^n\right)} f(x)$ is a Riemann sum of the extension of f to the rectangle, corresponding to the partition given by the rectangle intersected with cubes with vertices on $\frac{1}{\lambda}\mathbb{Z}^n$.

As a consequence of (2.1) and (2.2), many continuous inequalities can be recovered from discrete inequalities. The following consequence of these equations will be useful in the sequel.

Lemma 2.1. *Let $K \subset \mathbb{R}^n$ be a convex body with $0 \in K$, $m \in \mathbb{N}$ and $q \geq 1$. Then*

$$\lim_{\lambda \to \infty} \int_{O(n)} \left(\frac{1}{(G_n(\lambda UK))^m} \sum_{x_1,\ldots,x_m \in \lambda UK \cap \mathbb{Z}^n} \max_{1 \leq i \leq m} \left|\left\langle \frac{x_i}{\lambda}, e_n \right\rangle\right|^q\right)^{1/q} d\nu(U)$$

$$= \int_{O(n)} \left(\frac{1}{|UK|^m} \int_{UK} \cdots \int_{UK} \max_{1 \leq i \leq m} |\langle x_i, e_n \rangle|^q dx_m \ldots dx_1\right)^{1/q} d\nu(U).$$

Proof. For every $U \in O(n)$,

$$\lim_{\lambda \to \infty} \frac{1}{(G_n(\lambda UK))^m} \sum_{x_1,\ldots,x_m \in \lambda UK \cap \mathbb{Z}^n} \max_{1 \leq i \leq m} \left|\left\langle \frac{x_i}{\lambda}, e_n \right\rangle\right|^q$$

$$= \lim_{\lambda \to \infty} \left(\frac{\lambda^n}{G_n(\lambda UK)}\right)^m \frac{1}{\lambda^{nm}} \sum_{x_1,\ldots,x_N \in UK \cap \left(\frac{1}{\lambda}\mathbb{Z}^n\right)} \max_{1 \leq i \leq m} |\langle x_i, e_n \rangle|^q$$

$$= \frac{1}{|UK|^m} \int_{UK} \cdots \int_{UK} \max_{1 \leq i \leq m} |\langle x_i, e_n \rangle|^q dx_m \ldots dx_1.$$

Besides, for every $\lambda > 0$ and any $U \in O(n)$,

$$\left(\frac{1}{(G_n(\lambda UK))^m} \sum_{x_1,\ldots,x_N \in \lambda UK \cap (\mathbb{Z}^n)} \max_{1 \leq i \leq m} \left|\left\langle \frac{x_i}{\lambda}, e_n \right\rangle\right|^q\right)^{\frac{1}{q}}$$

$$= \left(\frac{1}{(G_n(\lambda UK))^m} \sum_{x_1, \ldots, x_N \in UK \cap \left(\frac{1}{\lambda} \mathbb{Z}^n \right)} \max_{1 \le i \le m} |\langle x_i, e_n \rangle|^q \right)^{\frac{1}{q}}$$

$$\le \left(\frac{1}{(G_n(\lambda UK))^m} \sum_{x_1, \ldots, x_m \in UK \cap \left(\frac{1}{\lambda} \mathbb{Z}^n \right)} \max_{x \in UK} |\langle x, e_n \rangle|^q \right)^{\frac{1}{q}}$$

$$= \max_{x \in UK} |\langle x, e_n \rangle| = \max\{h_{UK}(e_n), h_{UK}(-e_n)\} \le R(UK) = R(K),$$

where $R(L) := \inf\{R > 0 : L \subseteq R B_2^n\}$ denotes the circumradius of L, which is integrable in $O(n)$. By the dominated convergence theorem, the result follows. $\square$

Notice also that given a Borel set $L \subseteq \mathbb{R}^n$, for every $x \in L \cap \mathbb{Z}^n$ we have that $x + \frac{1}{2} B_\infty^n$ is a cube with volume $|x + \frac{1}{2} B_\infty^n| = 1$ and if $x_1, x_2 \in L \cap \mathbb{Z}^n$ with $x_1 \neq x_2$, then $(x_1 + \frac{1}{2} B_\infty^n) \cap (x_2 + \frac{1}{2} B_\infty^n)$ has volume 0. Taking also into account that $L \cap \mathbb{Z}^n \subseteq L$ and then $(L \cap \mathbb{Z}^n) + \frac{1}{2} B_\infty^n \subseteq L + \frac{1}{2} B_\infty^n$ we have that

$$G_n(L) = \left| (L \cap \mathbb{Z}^n) + \frac{1}{2} B_\infty^n \right| \le \left| L + \frac{1}{2} B_\infty^n \right|. \tag{2.3}$$

2.2. Expected mean width of random polytopes

In this section we give a proof of the lower bound in (1.10), which holds for every $N \ge n$, if $N \ge 3$, since we could not find this proof in the literature. We state it here as a theorem.

Theorem 2.2. *Let $K \subseteq \mathbb{R}^n$ be a convex body with $0 \in \operatorname{int} K$, $N \ge n$ with $N \ge 3$, let $X_1, \ldots, X_N$ be independent random vectors uniformly distributed on K, and let K_N be the centrally symmetric random polytope generated by $X_1, \ldots X_N$, that is, $K_N = \operatorname{conv}\{\pm X_1, \ldots, \pm X_N\}$. There exists an absolute constant $c_1 > 0$ such that*

$$\mathbb{E} w(K_N) \ge c_1 w(Z_{\log N}(K)).$$

Proof. For any $\theta \in S^{n-1}$ and any $\alpha > 0$ we have

$$\mathbb{P} \left(\max_{1 \le i \le N} |\langle X_i, \theta \rangle| \ge \alpha \right) = 1 - \mathbb{P} \left(\max_{1 \le i \le N} |\langle X_i, \theta \rangle| < \alpha \right)$$

$$= 1 - \mathbb{P} (|\langle X_1, \theta \rangle| < \alpha)^N$$

$$= 1 - (1 - \mathbb{P}(|\langle X_1, \theta \rangle| \ge \alpha))^N.$$

Therefore, taking $\alpha = h_{K_{\frac{1}{N}}}(\theta)$, where for any $0 < \delta < 1$, K_δ is the floating body defined by

$$h_{K_\delta}(\theta) = \sup\{t > 0 : \mathbb{P}(|\langle X_1, \theta \rangle| \ge t) \ge \delta\}$$

we have that

$$\mathbb{P} \left(\max_{1 \le i \le N} |\langle X_i, \theta \rangle| \ge h_{K_{\frac{1}{N}}}(\theta) \right) \ge 1 - \left(1 - \frac{1}{N} \right)^N \ge 1 - \frac{1}{e}.$$

Thus, by Markov's inequality, for any $\theta \in S^{n-1}$, calling $c = 1 - e^{-1}$,

$$\mathbb{E} h_{K_N}(\theta) \ge h_{K_{\frac{1}{N}}}(\theta) \cdot \mathbb{P} \left(\max_{1 \le i \le N} |\langle X_i, \theta \rangle| \ge h_{K_{\frac{1}{N}}}(\theta) \right) \ge c h_{K_{\frac{1}{N}}}(\theta).$$

It was proved in [28, Thm 2.2] that there exist absolute constants $c_1, c_2 > 0$ such that

$$c_1 h_{Z_{\log \frac{e}{2\delta}}(K)}(\theta) \leq h_{K_\delta}(\theta) \leq c_2 h_{Z_{\log \frac{e}{2\delta}}(K)}(\theta).$$

for any $0 < \delta < \frac{1}{2}$. Since there exists an absolute constant $C > 0$ such that $1 \leq \log \frac{1}{\delta} \leq \log \frac{e}{2\delta} \leq C \log \frac{1}{\delta}$ for every $0 < \delta \leq \frac{1}{e} < \frac{1}{2}$, the inclusion relation (1.5) implies that there exist absolute constants $c' = c_1$ and $c'' = c_2 C > 0$ such that

$$c' h_{Z_{\log \frac{1}{\delta}}(K)}(\theta) \leq h_{K_\delta}(\theta) \leq c'' h_{Z_{\log \frac{1}{\delta}}(K)}(\theta).$$

for any $0 < \delta \leq \frac{1}{e}$. In particular, for any $\theta \in S^{n-1}$,

$$\mathbb{E} h_{K_N}(\theta) \geq c h_{K_{\frac{1}{N}}}(\theta) \geq cc' h_{Z_{\log N}(K)}(\theta).$$

if $N \geq 3$. Integrating in $\theta \in S^{n-1}$ and using Fubini's theorem we obtain the result with $c_1 = cc'$. $\quad\square$

3. Discrete Borell's inequalities

In this section we are going to prove Theorem 1.2. As a consequence of Theorem 1.2 and Lemma 1.1, we can obtain inequality (1.3) for convex bodies with $0 \in \text{int} K$.

Proof of Theorem 1.2. The left-hand side inequality in (1.9) is a direct consequence of Hölder's inequality. Let us prove the right-hand side inequality. Fix $1 \leq p \leq q$ and let X be a random vector uniformly distributed on $K \cap \mathbb{Z}^n$ and let Y be a random vector uniformly distributed on $(K + C_n) \cap \mathbb{Z}^n$. Let also, for $s > 1$ to be chosen later,

$$A(s) = \left\{ x \in \mathbb{R}^n : |\langle x, e_n \rangle| \leq s \left(\mathbb{E} |\langle X, e_n \rangle|^p \right)^{\frac{1}{p}} \right\}$$
$$= \left\{ x \in \mathbb{R}^n : |\langle x, e_n \rangle|^p \leq s^p \mathbb{E} |\langle X, e_n \rangle|^p \right\}.$$

Notice that, for any $t \geq 1$ we have

$$(tA(s))^c = A(ts)^c = \left\{ x \in \mathbb{R}^n : |\langle x, e_n \rangle| > ts \left(\mathbb{E} |\langle X, e_n \rangle|^p \right)^{\frac{1}{p}} \right\}$$

and

$$A(s)^c + C_n = \left\{ x \in \mathbb{R}^n : |\langle x, e_n \rangle| > s \left(\mathbb{E} |\langle X, e_n \rangle|^p \right)^{\frac{1}{p}} - 1 \right\}.$$

Since $A(s)$ is convex and centrally symmetric, we have that for every $t > 1$

$$A(s)^c \supseteq \frac{2}{t+1} A(ts)^c + \frac{t-1}{t+1} A(s)$$

and then, since K is convex,

$$A(s)^c \cap K \supseteq \left(\frac{2}{t+1} A(ts)^c + \frac{t-1}{t+1} A(s) \right) \cap K$$
$$\supseteq \frac{2}{t+1} (A(ts)^c \cap K) + \frac{t-1}{t+1} (A(s) \cap K).$$

Therefore,

$$(A(s)^c + C_n) \cap (K + C_n) \supseteq (A(s)^c \cap K + C_n)$$
$$\supseteq \frac{2}{t+1}(A(ts)^c \cap K) + \frac{t-1}{t+1}(A(s) \cap K) + C_n.$$

By the discrete Brunn-Minkowski inequality (see Theorem 2.1), if $A(ts)^c \cap K \neq \emptyset$,

$$G_n((A(s)^c + C_n) \cap (K + C_n))^{\frac{1}{n}} \geq \frac{2}{t+1}G_n((tA(s))^c \cap K)^{\frac{1}{n}}$$
$$+ \frac{t-1}{t+1}G_n(A(s) \cap K)^{\frac{1}{n}}.$$

Thus,

$$G_n((A(s)^c + C_n) \cap (K + C_n)) \geq G_n(A(ts)^c \cap K)^{\frac{2}{t+1}} G_n(A(s) \cap K)^{\frac{t-1}{t+1}},$$

being this inequality also true if $A(ts)^c \cap K = \emptyset$. Therefore, for any $t > 1$

$$\frac{G_n((A(s)^c + C_n) \cap (K + C_n))}{G_n(K)} \geq \left(\frac{G_n(A(ts)^c \cap K)}{G_n(K)}\right)^{\frac{2}{t+1}} \left(\frac{G_n(A(s) \cap K)}{G_n(K)}\right)^{\frac{t-1}{t+1}}.$$

Notice that, by Markov's inequality,

$$\frac{G_n(A(s) \cap K)}{G_n(K)} = 1 - \frac{G_n(A(s)^c \cap K)}{G_n(K)} \geq 1 - s^{-p}.$$

Besides, also by Markov's inequality, if $s\left(\mathbb{E}|\langle X, e_n\rangle|^p\right)^{\frac{1}{p}} > 1$, then

$$\frac{G_n((A(s)^c + C_n) \cap (K + C_n))}{G_n(K + C_n)} \leq \frac{\mathbb{E}|\langle Y, e_n\rangle|^p}{\left(s\left(\mathbb{E}|\langle X, e_n\rangle|^p\right)^{\frac{1}{p}} - 1\right)^p}.$$

Therefore, for every $t > 1$, if $s\left(\mathbb{E}|\langle X, e_n\rangle|^p\right)^{\frac{1}{p}} > 1$ we have

$$\frac{G_n(K + C_n)}{G_n(K)} \frac{\mathbb{E}|\langle Y, e_n\rangle|^p}{\left(s\left(\mathbb{E}|\langle X, e_n\rangle|^p\right)^{\frac{1}{p}} - 1\right)^p} \geq \left(\frac{G_n(A(ts)^c \cap K)}{G_n(K)}\right)^{\frac{2}{t+1}} \left(1 - s^{-p}\right)^{\frac{t-1}{t+1}}$$

and then, if also $s \geq 2$,

$$\frac{G_n(A(ts)^c \cap K)}{G_n(K)} \leq \left(1 - s^{-p}\right) \left(\frac{G_n(K + C_n)}{G_n(K)} \frac{\mathbb{E}|\langle Y, e_n\rangle|^p}{\left(1 - s^{-p}\right)\left(s\left(\mathbb{E}|\langle X, e_n\rangle|^p\right)^{\frac{1}{p}} - 1\right)^p}\right)^{\frac{t+1}{2}}$$

$$\leq \left(\left(\frac{G_n(K + C_n)}{G_n(K)}\right)^{\frac{1}{p}} \frac{(\mathbb{E}|\langle Y, e_n\rangle|^p)^{\frac{1}{p}}}{\left(1 - s^{-p}\right)^{\frac{1}{p}}\left(s\left(\mathbb{E}|\langle X, e_n\rangle|^p\right)^{\frac{1}{p}} - 1\right)}\right)^{\frac{p(t+1)}{2}}$$

$$\leq \left(\left(\frac{G_n(K + C_n)}{G_n(K)}\right)^{\frac{1}{p}} \frac{2(\mathbb{E}|\langle Y, e_n\rangle|^p)^{\frac{1}{p}}}{s\left(\mathbb{E}|\langle X, e_n\rangle|^p\right)^{\frac{1}{p}} - 1}\right)^{\frac{p(t+1)}{2}}.$$

Now, choose

$$s = \frac{4 + 4(\mathbb{E}|\langle Y, e_n\rangle|^p)^{\frac{1}{p}}\left(\frac{G_n(K + C_n)}{G_n(K)}\right)^{\frac{1}{p}}}{(\mathbb{E}|\langle X, e_n\rangle|^p)^{\frac{1}{p}}} = 4D_0(K, p).$$

Recall that, as seen in Remark 1, $D_0(K, p) \geq 1$ and so $s \geq 4$. Also,

$$s \left(\mathbb{E}|\langle X, e_n \rangle|^p \right)^{\frac{1}{p}} - 1 \geq 4 (\mathbb{E}|\langle Y, e_n \rangle|^p)^{\frac{1}{p}} \left(\frac{G_n(K + C_n)}{G_n(K)} \right)^{\frac{1}{p}} > 0.$$

Thus, we have that

$$\mathbb{P} \left(|\langle X, e_n \rangle| > ts \left(\mathbb{E}|\langle X, e_n \rangle|^p \right)^{\frac{1}{p}} \right) = \frac{G_n(A(ts)^c \cap K)}{G_n(K)} \leq \left(\frac{1}{2} \right)^{\frac{p(t+1)}{2}} \leq \left(\frac{1}{2} \right)^{\frac{pt}{2}}$$

$$= e^{-\frac{pt \log 2}{2}}.$$

Therefore,

$$\frac{\mathbb{E}|\langle X, e_n \rangle|^q}{(\mathbb{E}|\langle X, e_n \rangle|^p)^{\frac{q}{p}}} = \frac{q \int_0^\infty t^{q-1} \mathbb{P}\left(|\langle X, e_n \rangle| > t \right) dt}{(\mathbb{E}|\langle X, e_n \rangle|^p)^{\frac{q}{p}}}$$

$$= q \int_0^\infty u^{q-1} \mathbb{P} \left(|\langle X, e_n \rangle| > u \left(\mathbb{E}|\langle X, e_n \rangle|^p \right)^{\frac{1}{p}} \right) du$$

$$= q \int_0^s u^{q-1} \mathbb{P} \left(|\langle X, e_n \rangle| > u \left(\mathbb{E}|\langle X, e_n \rangle|^p \right)^{\frac{1}{p}} \right) du$$

$$+ q \int_s^\infty u^{q-1} \mathbb{P} \left(|\langle X, e_n \rangle| > u \left(\mathbb{E}|\langle X, e_n \rangle|^p \right)^{\frac{1}{p}} \right) du$$

$$\leq s^q + s^q q \int_1^\infty v^{q-1} \mathbb{P} \left(|\langle X, e_n \rangle| > vs \left(\mathbb{E}|\langle X, e_n \rangle|^p \right)^{\frac{1}{p}} \right) dv$$

$$\leq s^q + s^q q \int_1^\infty v^{q-1} e^{-\frac{pv \log 2}{2}} dv$$

$$\leq s^q + s^q q \int_0^\infty v^{q-1} e^{-\frac{pv \log 2}{2}} dv$$

$$= s^q + s^q \left(\frac{2}{p \log 2} \right)^q q \int_0^\infty r^{q-1} e^{-r} dr$$

$$= s^q + s^q \left(\frac{2}{p \log 2} \right)^q \Gamma(1 + q)$$

$$= s^q \left(1 + \left(\frac{2}{p \log 2} \right)^q \Gamma(1 + q) \right).$$

Taking into account that, by Stirling's formula, there exists an absolute constant $K_0 > 0$ such that $\frac{2}{\log 2} \Gamma(1 + q)^{\frac{1}{q}} \leq K_0 q$ for any $q \geq 1$, we obtain

$$\left(\mathbb{E}|\langle X, e_n \rangle|^q \right)^{\frac{1}{q}} \leq s \left(\mathbb{E}|\langle X, e_n \rangle|^p \right)^{\frac{1}{p}} \left(1 + \left(\frac{2}{p \log 2} \right)^q \Gamma(1 + q) \right)^{\frac{1}{q}}$$

$$\leq s \left(\mathbb{E}|\langle X, e_n\rangle|^p\right)^{\frac{1}{p}} \left(1 + \left(\frac{2}{p\log 2}\right)\Gamma(1+q)^{\frac{1}{q}}\right)$$

$$\leq s \left(\mathbb{E}|\langle X, e_n\rangle|^p\right)^{\frac{1}{p}} \left(1 + K_0\frac{q}{p}\right)$$

$$\leq s \left(\mathbb{E}|\langle X, e_n\rangle|^p\right)^{\frac{1}{p}} (K_0+1)\frac{q}{p}$$

$$= 4(K_0+1)\frac{q}{p}D_0(K, p),$$

which proves inequality (1.9) with $C = 4(K_0+1)$. □

Given a convex body K with $0 \in \operatorname{int} K$, we denote $r(K)$ and $R(K)$ the inradius and circumradius of K, that is, the optimal constants satisfying that $r(K)B_2^n \subseteq K \subseteq R(K)B_2^n$.

Lemma 3.1. *Let $K \subset \mathbb{R}^n$ be a convex body with $0 \in \operatorname{int} K$, $\max_{x \in K \cap \mathbb{Z}^n} |\langle x, e_n\rangle| \geq 1$ and $\sqrt{n}R(K) \leq 2r(K)^2$. Then,*

a) $G_n((1+t)K \setminus K) \leq \left(\left(1 + t + \frac{\sqrt{n}}{2r(K)}\right)^n - \left(1 - \frac{\sqrt{n}R(K)}{2r^2(K)}\right)^n\right)|K|$ *for all $t > 0$.*

b) *For any $p \geq 1$ we have*

$$D_0(K, p) \leq 1 + \frac{|K + \frac{\sqrt{n}}{2}B_2^n|}{\sum_{x \in r(K)B_2^n \cap \mathbb{Z}^n}|\langle x, e_n\rangle|}$$

$$+ \frac{\left(1 + \frac{\sqrt{n}}{r(K)}\right)R(K)\left(\left(1 + \frac{3\sqrt{n}}{2r(K)}\right)^n - \left(1 - \frac{\sqrt{n}R(K)}{2r^2(K)}\right)^n\right)^{\frac{1}{p}}|K|^{\frac{1}{p}}}{\left(\frac{1}{G_n(r(K)B_2^n)}\sum_{x \in r(K)B_2^n \cap \mathbb{Z}^n}|\langle x, e_n\rangle|\right)G_n(r(K)B_2^n)^{\frac{1}{p}}}.$$

Proof. Let us first prove a). By (2.3),

$$|G_n((1+t)K \setminus K)| \leq \left|(1+t)K \setminus K + \frac{1}{2}B_\infty^n\right| \leq \left|(1+t)K \setminus K + \frac{\sqrt{n}}{2}B_2^n\right|$$

$$\leq \left|(1+t)K \setminus K + \frac{\sqrt{n}}{2r(K)}K\right|.$$

Now, since

$$-K \subseteq R(K)B_2^n \subseteq \frac{R(K)}{r(K)}K,$$

we have

$$-\frac{\sqrt{n}}{2r(K)}K \subseteq \frac{\sqrt{n}R(K)}{2r^2(K)}K.$$

Recall that $\frac{\sqrt{n}R(K)}{2r^2(K)} \leq 1$. It follows that

$$(1+t)K \setminus K + \frac{\sqrt{n}}{2r(K)}K \subseteq \left(1 + t + \frac{\sqrt{n}}{2r(K)}\right)K \setminus \left(1 - \frac{\sqrt{n}R(K)}{2r^2(K)}\right)K$$

and then

$$\left| (1+t)K \setminus K + \frac{\sqrt{n}}{2r(K)}K \right| \leq \left| \left(1+t+\frac{\sqrt{n}}{2r(K)} \right)K \setminus \left(1 - \frac{\sqrt{n}R(K)}{2r^2(K)} \right)K \right|$$

$$= \left(\left(1+t+\frac{\sqrt{n}}{2r(K)} \right)^n - \left(1 - \frac{\sqrt{n}R(K)}{2r^2(K)} \right)^n \right)|K|.$$

Let us now prove b). We have that if X and Y are random vectors uniformly distributed on $K \cap \mathbb{Z}^n$ and $(K + C_n) \cap \mathbb{Z}^n$ respectively, for any $p \geq 1$,

$$D_0(K, p) = \frac{1 + \left(\mathbb{E}|\langle Y, e_n \rangle|^p \right)^{\frac{1}{p}} \left(\frac{G_n(K+C_n)}{G_n(K)} \right)^{\frac{1}{p}}}{\left(\mathbb{E}|\langle X, e_n \rangle|^p \right)^{\frac{1}{p}}}$$

$$= \frac{1}{\left(\mathbb{E}|\langle X, e_n \rangle|^p \right)^{\frac{1}{p}}} + \left(\frac{\sum_{x \in (K+C_n) \cap \mathbb{Z}^n} |\langle x, e_n \rangle|^p}{\sum_{x \in K \cap \mathbb{Z}^n} |\langle x, e_n \rangle|^p} \right)^{\frac{1}{p}}$$

On the one hand, by Hölder's inequality and (2.3),

$$\left(\mathbb{E}|\langle X, e_n \rangle|^p \right)^{\frac{1}{p}} \geq \mathbb{E}|\langle X, e_n \rangle| = \frac{\sum_{x \in K \cap \mathbb{Z}^n} |\langle x, e_n \rangle|}{G_n(K)} \geq \frac{\sum_{x \in r(K)B_2^n \cap \mathbb{Z}^n} |\langle x, e_n \rangle|}{|K + \frac{1}{2}B_\infty^n|}$$

$$\geq \frac{\sum_{x \in r(K)B_2^n \cap \mathbb{Z}^n} |\langle x, e_n \rangle|}{|K + \frac{\sqrt{n}}{2}B_2^n|}.$$

On the other hand,

$$\left(\frac{\sum_{x \in (K+C_n) \cap \mathbb{Z}^n} |\langle x, e_n \rangle|^p}{\sum_{x \in K \cap \mathbb{Z}^n} |\langle x, e_n \rangle|^p} \right)^{\frac{1}{p}} = \left(1 + \frac{\sum_{x \in ((K+C_n) \setminus K) \cap \mathbb{Z}^n} |\langle x, e_n \rangle|^p}{\sum_{x \in K \cap \mathbb{Z}^n} |\langle x, e_n \rangle|^p} \right)^{\frac{1}{p}}$$

$$\leq 1 + \left(\frac{\sum_{x \in ((K+C_n) \setminus K) \cap \mathbb{Z}^n} |\langle x, e_n \rangle|^p}{\sum_{x \in K \cap \mathbb{Z}^n} |\langle x, e_n \rangle|^p} \right)^{\frac{1}{p}}$$

$$\leq 1 + \left(\frac{\sum_{x \in ((K+\sqrt{n}B_2^n) \setminus K) \cap \mathbb{Z}^n} |\langle x, e_n \rangle|^p}{\sum_{x \in r(K)B_2^n \cap \mathbb{Z}^n} |\langle x, e_n \rangle|^p} \right)^{\frac{1}{p}}$$

$$\leq 1 + \left(\frac{\sum_{x \in ((K+\frac{\sqrt{n}}{r(K)}K) \setminus K) \cap \mathbb{Z}^n} |\langle x, e_n \rangle|^p}{\sum_{x \in r(K)B_2^n \cap \mathbb{Z}^n} |\langle x, e_n \rangle|^p} \right)^{\frac{1}{p}}$$

$$\leq 1 + \frac{h_{(1+\frac{\sqrt{n}}{r(K)})K}(e_n) G_n \left(\left(1 + \frac{\sqrt{n}}{r(K)} \right)K \setminus K \right)^{\frac{1}{p}}}{\left(\sum_{x \in r(K)B_2^n \cap \mathbb{Z}^n} |\langle x, e_n \rangle|^p \right)^{\frac{1}{p}}}$$

$$\leq 1 + \frac{\left(1 + \frac{\sqrt{n}}{r(K)} \right) R(K) G_n \left(\left(1 + \frac{\sqrt{n}}{r(K)} \right)K \setminus K \right)^{\frac{1}{p}}}{\left(\frac{1}{G_n(r(K)B_2^n)} \sum_{x \in r(K)B_2^n \cap \mathbb{Z}^n} |\langle x, e_n \rangle|^p \right)^{\frac{1}{p}} G_n(r(K)B_2^n)^{\frac{1}{p}}}$$

$$\leq 1 + \frac{\left(1 + \frac{\sqrt{n}}{r(K)} \right) R(K) G_n \left(\left(1 + \frac{\sqrt{n}}{r(K)} \right)K \setminus K \right)^{\frac{1}{p}}}{\left(\frac{1}{G_n(r(K)B_2^n)} \sum_{x \in r(K)B_2^n \cap \mathbb{Z}^n} |\langle x, e_n \rangle| \right) G_n(r(K)B_2^n)^{\frac{1}{p}}}.$$

By a), we get

$$\left| G_n \left(\left(1 + \frac{\sqrt{n}}{r(K)} \right) K \setminus K \right) \right| \leq \left(\left(1 + \frac{3\sqrt{n}}{2r(K)} \right)^n - \left(1 - \frac{\sqrt{n}R(K)}{2r^2(K)} \right)^n \right) |K|$$

and the statement follows. $\quad \square$

Let us now prove Lemma 1.1, i.e., that, for every convex body $K \subseteq \mathbb{R}^n$ with $0 \in \text{int}(K)$ and any $q \geq 1$, we have

$$\lim_{\lambda \to \infty} D_q(\lambda K) = 1.$$

Proof of Lemma 1.1. Let $K \subseteq \mathbb{R}^n$ be a convex body with $0 \in \text{int}(K)$, $p \geq 1$ and $U \in O(n)$. Notice that if $\lambda \geq r(K)^{-1}$ we have $B_2^n \subset \lambda U K$ and then $\max_{x \in \lambda U K \cap \mathbb{Z}^n} |\langle x, e_n \rangle| \geq 1$. Therefore, if $\lambda \geq r(K)^{-1}$ the number $D_0(\lambda U K, p)$ is well defined. Since $r(\lambda U K) = \lambda r(K)$, $R(\lambda U K) = \lambda R(K)$ and $|\lambda U K| = \lambda^n |K|$, by Lemma 3.1 we have that, whenever $\lambda \geq \max \left\{ r(K)^{-1}, \frac{\sqrt{n}R(K)}{2r^2(K)} \right\}$,

$$1 \leq D_0(\lambda U K, p) \leq 1 + \frac{|\lambda U K + \frac{\sqrt{n}}{2} B_2^n|}{\sum_{x \in \lambda r(K) B_2^n \cap \mathbb{Z}^n} |\langle x, e_n \rangle|}$$

$$+ \frac{\left(\lambda + \frac{\sqrt{n}}{r(K)} \right) R(K) \left(\left(\lambda + \frac{3\sqrt{n}}{2r(K)} \right)^n - \left(\lambda - \frac{\sqrt{n}R(K)}{2r^2(K)} \right)^n \right)^{\frac{1}{p}} |K|^{\frac{1}{p}}}{\left(\frac{1}{G_n(\lambda r(K) B_2^n)} \sum_{x \in \lambda r(K) B_2^n \cap \mathbb{Z}^n} |\langle x, e_n \rangle| \right) G_n(\lambda r(K) B_2^n)^{\frac{1}{p}}}.$$

Thus, for every $\lambda \geq \max \left\{ r(K)^{-1}, \frac{\sqrt{n}R(K)}{2r^2(K)} \right\}$,

$$1 \leq D_q(\lambda K) = \sup_{\substack{U \in O(n) \\ 1 \leq p \leq q}} D_0(\lambda U K, p)$$

$$\leq 1 + \sup_{U \in O(n)} \frac{|\lambda U K + \frac{\sqrt{n}}{2} B_2^n|}{\sum_{x \in \lambda r(K) B_2^n \cap \mathbb{Z}^n} |\langle x, e_n \rangle|}$$

$$+ \sup_{\substack{U \in O(n) \\ 1 \leq p \leq q}} \frac{\left(\lambda + \frac{\sqrt{n}}{r(K)} \right) R(K) \left(\left(\lambda + \frac{3\sqrt{n}}{2r(K)} \right)^n - \left(\lambda - \frac{\sqrt{n}R(K)}{2r^2(K)} \right)^n \right)^{\frac{1}{p}} |K|^{\frac{1}{p}}}{\left(\frac{1}{G_n(\lambda r(K) B_2^n)} \sum_{x \in \lambda r(K) B_2^n \cap \mathbb{Z}^n} |\langle x, e_n \rangle| \right) G_n(\lambda r(K) B_2^n)^{\frac{1}{p}}}.$$

Let us see that both supremums tend to 0 as $\lambda \to \infty$. First,

$$\sup_{U \in O(n)} \frac{|\lambda U K + \frac{\sqrt{n}}{2} B_2^n|}{\sum_{x \in \lambda r(K) B_2^n \cap \mathbb{Z}^n} |\langle x, e_n \rangle|} = \sup_{U \in O(n)} \frac{|\lambda U (K + \frac{\sqrt{n}}{2} B_2^n)|}{\sum_{x \in \lambda r(K) B_2^n \cap \mathbb{Z}^n} |\langle x, e_n \rangle|}$$

$$= \frac{\lambda^n \left| K + \frac{\sqrt{n}}{2\lambda} B_2^n \right|}{\sum_{x \in \lambda r(K) B_2^n \cap \mathbb{Z}^n} |\langle x, e_n \rangle|}$$

$$= \frac{\frac{1}{\lambda} \left| K + \frac{\sqrt{n}}{2\lambda} B_2^n \right|}{\frac{1}{\lambda^n} \sum_{x \in \lambda r(K) B_2^n \cap \mathbb{Z}^n} |\langle \frac{x}{\lambda}, e_n \rangle|}.$$

Since $\lim\limits_{\lambda\to\infty}\left|K+\frac{\sqrt{n}}{2\lambda}B_2^n\right|=|K|$ and, by (2.2),

$$\lim_{\lambda\to\infty}\frac{1}{\lambda^n}\sum_{x\in\lambda r(K)B_2^n\cap\mathbb{Z}^n}\left|\left\langle\frac{x}{\lambda},e_n\right\rangle\right|=\lim_{\lambda\to\infty}\frac{1}{\lambda^n}\sum_{x\in r(K)B_2^n\cap\frac{1}{\lambda}\mathbb{Z}^n}|\langle x,e_n\rangle|$$
$$=\int_{r(K)B_2^n}|\langle x,e_n\rangle|dx,$$

we get that

$$\lim_{\lambda\to\infty}\frac{\frac{1}{\lambda}\left|K+\frac{\sqrt{n}}{2\lambda}B_2^n\right|}{\frac{1}{\lambda^n}\sum_{x\in r(K)B_2^n\cap\frac{1}{\lambda}\mathbb{Z}^n}|\langle x,e_n\rangle|}=0$$

and then

$$\lim_{\lambda\to\infty}\sup_{U\in O(n)}\frac{|\lambda UK+\frac{\sqrt{n}}{2}B_2^n|}{\sum_{x\in\lambda r(K)B_2^n\cap\mathbb{Z}^n}|\langle x,e_n\rangle|}=0.$$

Let us now see that

$$\lim_{\lambda\to\infty}\sup_{\substack{U\in O(n)\\1\le p\le q}}\frac{\left(\lambda+\frac{\sqrt{n}}{r(K)}\right)R(K)\left(\left(\lambda+\frac{3\sqrt{n}}{2r(K)}\right)^n-\left(\lambda-\frac{\sqrt{n}R(K)}{2r^2(K)}\right)^n\right)^{\frac{1}{p}}|K|^{\frac{1}{p}}}{\left(\frac{1}{G_n(\lambda r(K)B_2^n)}\sum_{x\in\lambda r(K)B_2^n\cap\mathbb{Z}^n}|\langle x,e_n\rangle|\right)G_n(\lambda r(K)B_2^n)^{\frac{1}{p}}}=0.$$

Since

$$\lim_{\lambda\to\infty}\frac{\left(\left(\lambda+\frac{3\sqrt{n}}{2r(K)}\right)^n-\left(\lambda-\frac{\sqrt{n}R(K)}{2r^2(K)}\right)^n\right)|K|}{G_n\left(\lambda r(K)B_2^n\right)}$$
$$=\lim_{\lambda\to\infty}\frac{\frac{1}{\lambda^n}\left(\left(\lambda+\frac{3\sqrt{n}}{2r(K)}\right)^n-\left(\lambda-\frac{\sqrt{n}R(K)}{2r^2(K)}\right)^n\right)|K|}{\frac{1}{\lambda^n}G_n\left(\lambda r(K)B_2^n\right)}$$
$$=\frac{0}{|r(K)B_2^n|}=0,$$

there exists $\lambda_1>0$ such that if $\lambda\ge\lambda_1$ then $\frac{\left(\left(\lambda+\frac{3\sqrt{n}}{2r(K)}\right)^n-\left(\lambda-\frac{\sqrt{n}R(K)}{2r^2(K)}\right)^n\right)|K|}{G_n\left(\lambda r(K)B_2^n\right)}\le 1$ and, for every $1\le p\le q$

$$\left(\frac{\left(\lambda+\frac{3\sqrt{n}}{2r(K)}\right)^n-\left(\lambda-\frac{\sqrt{n}R(K)}{2r^2(K)}\right)^n}{G_n\left(\lambda r(K)B_2^n\right)}|K|\right)^{\frac{1}{p}}\le\left(\frac{\left(\lambda+\frac{3\sqrt{n}}{3r(K)}\right)^n-\left(\lambda-\frac{\sqrt{n}}{r(K)}\right)^n}{G_n\left(\lambda r(K)B_2^n\right)}|K|\right)^{\frac{1}{q}}.$$

Therefore, for every $\lambda\ge\max\left\{r(K)^{-1},\frac{\sqrt{n}R(K)}{2r^2(K)},\lambda_1\right\}$,

$$\sup_{\substack{U\in O(n)\\1\le p\le q}}\frac{\left(\lambda+\frac{\sqrt{n}}{r(K)}\right)R(K)\left(\left(\lambda+\frac{3\sqrt{n}}{2r(K)}\right)^n-\left(\lambda-\frac{\sqrt{n}R(K)}{2r^2(K)}\right)^n\right)^{\frac{1}{p}}|K|^{\frac{1}{p}}}{\left(\frac{1}{G_n(\lambda r(K)B_2^n)}\sum_{x\in\lambda r(K)B_2^n\cap\mathbb{Z}^n}|\langle x,e_n\rangle|\right)G_n(\lambda r(K)B_2^n)^{\frac{1}{p}}}$$

$$\le\frac{\left(1+\frac{\sqrt{n}}{\lambda r(K)}\right)R(K)}{\frac{1}{G_n(\lambda r(K)B_2^n)}\sum_{x\in\lambda r(K)B_2^n\cap\mathbb{Z}^n}|\langle\frac{x}{\lambda},e_n\rangle|}\left(\frac{\left(\lambda+\frac{3\sqrt{n}}{2r(K)}\right)^n-\left(\lambda-\frac{\sqrt{n}R(K)}{2r^2(K)}\right)^n}{G_n\left(\lambda r(K)B_2^n\right)}|K|\right)^{\frac{1}{q}}$$

$$= \frac{\left(1 + \frac{\sqrt{n}}{\lambda r(K)}\right) R(K)}{\frac{1}{\lambda^n} G_n(\lambda r(K) B_2^n) \frac{1}{\lambda^n} \sum_{x \in \lambda r(K) B_2^n \cap \mathbb{Z}^n} \left|\left\langle \frac{x}{\lambda}, e_n \right\rangle\right|} \left(\frac{\left(\lambda + \frac{3\sqrt{n}}{2r(K)}\right)^n - \left(\lambda - \frac{\sqrt{n}R(K)}{2r^2(K)}\right)^n}{G_n(\lambda r(K) B_2^n)} |K|\right)^{\frac{1}{q}}.$$

Since

- $\displaystyle \lim_{\lambda \to \infty} \left(1 + \frac{\sqrt{n}}{\lambda r(K)}\right) = 1,$

- $\displaystyle \lim_{\lambda \to \infty} \frac{\left(\lambda + \frac{3\sqrt{n}}{2r(K)}\right)^n - \left(\lambda - \frac{\sqrt{n}R(K)}{2r^2(K)}\right)^n}{G_n(\lambda r(K) B_2^n)} |K| = 0,$

- $\displaystyle \lim_{\lambda \to \infty} \frac{1}{\lambda^n} \sum_{x \in \lambda r(K) B_2^n \cap \mathbb{Z}^n} \left|\left\langle \frac{x}{\lambda}, e_n \right\rangle\right| = \int_{r(K) B_2^n} |\langle x, e_n \rangle| dx,$ and

- $\displaystyle \lim_{\lambda \to \infty} \frac{1}{\lambda^n} G_n(\lambda r(K) B_2^n) = |r(K) B_2^n|,$

we have that

$$\lim_{\lambda \to \infty} \frac{\left(1 + \frac{\sqrt{n}}{\lambda r(K)}\right) R(K)}{\frac{1}{\lambda^n} G_n(\lambda r(K) B_2^n) \frac{1}{\lambda^n} \sum_{x \in \lambda r(K) B_2^n \cap \mathbb{Z}^n} \left|\left\langle \frac{x}{\lambda}, e_n \right\rangle\right|} \left(\frac{\left(\lambda + \frac{3\sqrt{n}}{3r(K)}\right)^n - \left(\lambda - \frac{\sqrt{n}}{r(K)}\right)^n}{G_n(\lambda r(K) B_2^n)} |K|\right)^{\frac{1}{q}} = 0$$

and then

$$\lim_{\lambda \to \infty} \sup_{\substack{U \in O(n) \\ 1 \le p \le q}} \frac{\left(\lambda + \frac{\sqrt{n}}{r(K)}\right) R(K) \left(\left(\lambda + \frac{3\sqrt{n}}{2r(K)}\right)^n - \left(\lambda - \frac{\sqrt{n}R(K)}{2r^2(K)}\right)^n\right)^{\frac{1}{p}} |K|^{\frac{1}{p}}}{\left(\frac{1}{G_n(\lambda r(K) B_2^n)} \sum_{x \in \lambda r(K) B_2^n \cap \mathbb{Z}^n} |\langle x, e_n \rangle|\right) G_n(\lambda r(K) B_2^n)^{\frac{1}{p}}} = 0.$$

Therefore,

$$\lim_{\lambda \to \infty} D_q(\lambda K) = 1. \quad \square$$

As a consequence of Theorem 1.2 and Lemma 1.1, we can obtain inequality (1.6) for convex bodies containing the origin in its interior.

Corollary 3.1. *There exists an absolute constant $C > 0$ such that for every convex body $K \subseteq \mathbb{R}^n$ with $0 \in \text{int } K$, if X is a random vector uniformly distributed on K, then for any $1 \le p \le q$*

$$\left(\mathbb{E}|\langle X, e_n \rangle|^p\right)^{\frac{1}{p}} \le \left(\mathbb{E}|\langle X, e_n \rangle|^q\right)^{\frac{1}{q}} \le C\frac{q}{p} \left(\mathbb{E}|\langle X, e_n \rangle|^p\right)^{\frac{1}{p}}.$$

Proof. Let $K \subseteq \mathbb{R}^n$ be a convex body with $0 \in \text{int } K$ and $1 \le p \le q$. Let $\lambda \ge r(K)^{-1} > 0$. Then, we have that $\max_{x \in \lambda K \cap \mathbb{Z}^n} |\langle x, e_n \rangle| \ge 1$. By Theorem 1.2, if X_λ is a random vector uniformly distributed on $\lambda K \cap \mathbb{Z}^n$,

$$\left(\mathbb{E}|\langle X_\lambda, e_n \rangle|^p\right)^{\frac{1}{p}} \le \left(\mathbb{E}|\langle X_\lambda, e_n \rangle|^q\right)^{\frac{1}{q}} \le C\frac{q}{p} D_0(\lambda K, p) \left(\mathbb{E}|\langle X_\lambda, e_n \rangle|^p\right)^{\frac{1}{p}},$$

where $C > 0$ is an absolute constant. Equivalently,

$$\left(\mathbb{E}\left|\left\langle \frac{X_\lambda}{\lambda}, e_n \right\rangle\right|^p\right)^{\frac{1}{p}} \le \left(\mathbb{E}\left|\left\langle \frac{X_\lambda}{\lambda}, e_n \right\rangle\right|^q\right)^{\frac{1}{q}} \le C\frac{q}{p} D_0(\lambda K, p) \left(\mathbb{E}\left|\left\langle \frac{X_\lambda}{\lambda}, e_n \right\rangle\right|^p\right)^{\frac{1}{p}}.$$

On the one hand, for any $p \geq 1$ we have

$$
\mathbb{E}\left|\left\langle \frac{X_\lambda}{\lambda}, e_n \right\rangle\right|^p = \frac{1}{G_n(\lambda K)} \sum_{x \in \lambda K \cap \mathbb{Z}^n} \left|\left\langle \frac{x}{\lambda}, e_n \right\rangle\right|^p
$$

$$
= \frac{1}{\frac{1}{\lambda^n} G_n(\lambda K)} \frac{1}{\lambda^n} \sum_{x \in \lambda K \cap \mathbb{Z}^n} \left|\left\langle \frac{x}{\lambda}, e_n \right\rangle\right|^p
$$

$$
= \frac{1}{\frac{1}{\lambda^n} G_n(\lambda K)} \frac{1}{\lambda^n} \sum_{x \in K \cap \frac{1}{\lambda}\mathbb{Z}^n} |\langle x, e_n \rangle|^p \, ,
$$

and, by (2.1) and (2.2), taking the limit as $\lambda \to \infty$ we obtain that

$$
\lim_{\lambda \to \infty} \mathbb{E}\left|\left\langle \frac{X_\lambda}{\lambda}, e_n \right\rangle\right|^p = \frac{1}{|K|} \int_K |\langle x, e_n \rangle|^p dx = \mathbb{E}|\langle X, e_n \rangle|^p,
$$

where X is a random vector uniformly distributed on K. On the other hand, as seen in Remark 1,

$$
\lim_{\lambda \to \infty} D_0(\lambda K, p) = 1
$$

and then, if X is a random vector uniformly distributed on K,

$$
\left(\mathbb{E}|\langle X, e_n \rangle|^p\right)^{\frac{1}{p}} \leq \left(\mathbb{E}|\langle X, e_n \rangle|^q\right)^{\frac{1}{q}} \leq C \frac{q}{p} \left(\mathbb{E}|\langle X, e_n \rangle|^p\right)^{\frac{1}{p}}. \quad \square
$$

4. Expected mean width of random polytopes via discrete inequalities

In this section we will prove Theorems 1.3 and 1.4 and show how they imply the inequalities in (1.11) and, equivalently, (1.10). In order to prove Theorem 1.3 we prove the following lemma.

Lemma 4.1. *Let $L \subseteq \mathbb{R}^n$ be a convex body with $0 \in L$, $N \in \mathbb{N}$ and $a > 0$. Let $X_1, \ldots, X_N$ be independent random vectors uniformly distributed on $L \cap \mathbb{Z}^n$. Then, for any $q > 0$,*

$$
\mathbb{P}\left(\max_{1 \leq i \leq N} |\langle X_i, e_n \rangle| \geq a \left(\mathbb{E}|\langle X_1, e_n \rangle|^q\right)^{\frac{1}{q}}\right) \leq N a^{-q}.
$$

Proof. By the union bound, we have that

$$
\mathbb{P}\left(\max_{1 \leq i \leq N} |\langle X_i, e_n \rangle| \geq a \left(\mathbb{E}|\langle X_1, e_n \rangle|^q\right)^{\frac{1}{q}}\right) \leq N \mathbb{P}\left(|\langle X_1, e_n \rangle| \geq a \left(\mathbb{E}|\langle X_1, e_n \rangle|^q\right)^{\frac{1}{q}}\right)
$$

$$
= N \mathbb{P}\left(|\langle X_1, e_n \rangle|^q \geq a^q \mathbb{E}|\langle X_1, e_n \rangle|^q\right).
$$

By Markov's inequality

$$
\mathbb{P}\left(|\langle X_1, e_n \rangle|^q \geq a^q \mathbb{E}|\langle X_1, e_n \rangle|^q\right) \leq a^{-q}.
$$

Therefore,

$$
\mathbb{P}\left(\max_{1 \leq i \leq N} |\langle X_i, e_n \rangle| \geq a \left(\mathbb{E}|\langle X_1, e_n \rangle|^q\right)^{\frac{1}{q}}\right) \leq N a^{-q}. \quad \square
$$

Proof of Theorem 1.3. For any $a > 0$ and any $q > 1$, let us call

$$
h_q(U) := \left(\frac{1}{G_n(UK)} \sum_{x \in UK \cap \mathbb{Z}^n} |\langle x, e_n \rangle|^q\right)^{\frac{1}{q}} = \left(\mathbb{E}|\langle X_U, e_n \rangle|^q\right)^{\frac{1}{q}},
$$

where X_U is a random vector uniformly distributed on $UK \cap \mathbb{Z}^n$. Let also A and B be the following subsets of $(\mathbb{Z}^n)^N \times O(n)$:

$$A = \left\{ (x_1, \ldots, x_N, U) : x_i \in UK \cap \mathbb{Z}^n, \ \forall 1 \le i \le N, \ \max_{1 \le i \le N} |\langle x_i, e_n \rangle| \ge ah_q(U) \right\},$$

$$B = \left\{ (x_1, \ldots, x_N, U) : x_i \in UK \cap \mathbb{Z}^n, \ \forall 1 \le i \le N, \ \max_{1 \le i \le N} |\langle x_i, e_n \rangle| < ah_q(U) \right\}.$$

Notice that $A \cap B = \emptyset$ and

$$A \cup B = \{ (x_1, \ldots, x_N, U) : x_i \in UK \cap \mathbb{Z}^n, \ \forall 1 \le i \le N \}.$$

Therefore,

$$\int\limits_{O(n)} \frac{1}{(G_n(UK))^N} \sum_{x_1, \ldots, x_N \in UK \cap \mathbb{Z}^n} \max_{1 \le i \le N} |\langle x_i, e_n \rangle| d\nu(U)$$

$$= \int\limits_{O(n)} \frac{1}{(G_n(UK))^N} \sum_{x_1, \ldots, x_N \in UK \cap \mathbb{Z}^n} \max_{1 \le i \le N} |\langle x_i, e_n \rangle| \chi_A(x_1, \ldots, x_N, U) d\nu(U)$$

$$+ \int\limits_{O(n)} \frac{1}{(G_n(UK))^N} \sum_{x_1, \ldots, x_N \in UK \cap \mathbb{Z}^n} \max_{1 \le i \le N} |\langle x_i, e_n \rangle| \chi_B(x_1, \ldots, x_N, U) d\nu(U).$$

On the one hand,

$$\int\limits_{O(n)} \frac{1}{(G_n(UK))^N} \sum_{x_1, \ldots, x_N \in UK \cap \mathbb{Z}^n} \max_{1 \le i \le N} |\langle x_i, e_n \rangle| \chi_B(x_1, \ldots, x_N, U) d\nu(U)$$

$$\le \int\limits_{O(n)} \frac{1}{(G_n(UK))^N} \sum_{x_1, \ldots, x_N \in UK \cap \mathbb{Z}^n} ah_q(U) \chi_B(x_1, \ldots, x_N, U) d\nu(U)$$

$$\le \int\limits_{O(n)} \frac{1}{(G_n(UK))^N} \sum_{x_1, \ldots, x_N \in UK \cap \mathbb{Z}^n} ah_q(U) d\nu(U)$$

$$= a \int\limits_{O(n)} h_q(U) d\nu(U).$$

On the other hand, let us call, for every $k \in \mathbb{N} \cup \{0\}$, A_k the following subset of $(\mathbb{Z}^n)^N \times O(n)$:

$$A_k = \left\{ (x_1, \ldots, x_N, U) \in A : 2^k ah_q(U) \le \max_{1 \le i \le N} |\langle x_i, e_n \rangle| < 2^{k+1} ah_q(U) \right\}$$

Notice that if $k_1 \ne k_2$ then $A_{k_1} \cap A_{k_2} = \emptyset$ and $A = \bigcup_{k=0}^{\infty} A_k$. Then,

$$\int\limits_{O(n)} \frac{1}{(G_n(UK))^N} \sum_{x_1, \ldots, x_N \in UK \cap \mathbb{Z}^n} \max_{1 \le i \le N} |\langle x_i, e_n \rangle| \chi_A(x_1, \ldots, x_N, U) d\nu(U)$$

$$= \int\limits_{O(n)} \frac{1}{(G_n(UK))^N} \sum_{x_1, \ldots, x_N \in UK \cap \mathbb{Z}^n} \max_{1 \le i \le N} |\langle x_i, e_n \rangle| \sum_{k=0}^{\infty} \chi_{A_k}(x_1, \ldots, x_N, U) d\nu(U)$$

$$\le \int\limits_{O(n)} \frac{1}{(G_n(UK))^N} \sum_{x_1, \ldots, x_N \in UK \cap \mathbb{Z}^n} \sum_{k=0}^{\infty} 2^{k+1} ah_q(U) \chi_{A_k}(x_1, \ldots, x_N, U) d\nu(U)$$

$$= 2a \int\limits_{O(n)} h_q(U) \sum_{k=0}^{\infty} 2^k \frac{1}{(G_n(UK))^N} \sum_{x_1,\ldots,x_N \in UK \cap \mathbb{Z}^n} \chi_{A_k}(x_1,\ldots,x_N,U) d\nu(U).$$

Applying, for any $U \in O(n)$, Lemma 4.1 with $L = UK$ we have that if $(X_U)_1, \ldots (X_U)_N$ are independent random vectors uniformly distributed on $UK \cap \mathbb{Z}^n$, then

$$2a \int\limits_{O(n)} h_q(U) \sum_{k=0}^{\infty} 2^k \frac{1}{(G_n(UK))^N} \sum_{x_1,\ldots,x_N \in UK \cap \mathbb{Z}^n} \chi_{A_k}(x_1,\ldots,x_N,U) d\nu(U)$$

$$= 2a \int\limits_{O(n)} h_q(U) \sum_{k=0}^{\infty} 2^k \mathbb{P}\left(2^{k+1}ah_q(U) > \max_{1 \le i \le N} |\langle (X_U)_i, e_n \rangle| \ge 2^k ah_q(U)\right) d\nu(U)$$

$$\le 2a \int\limits_{O(n)} h_q(U) \sum_{k=0}^{\infty} 2^k \mathbb{P}\left(\max_{1 \le i \le N} |\langle (X_U)_i, e_n \rangle| \ge 2^k ah_q(U)\right) d\nu(U)$$

$$\le 2a \int\limits_{O(n)} h_q(U) \sum_{k=0}^{\infty} 2^k N(2^k a)^{-q} d\nu(U)$$

$$= 2N a^{1-q} \int\limits_{O(n)} h_q(U) \sum_{k=0}^{\infty} \frac{1}{2^{(q-1)k}} d\nu(U)$$

$$= \frac{2N a^{1-q}}{1 - \frac{1}{2^{q-1}}} \int\limits_{O(n)} h_q(U) d\nu(U).$$

Taking $a = e$ and $q = \log N \ge \log 3 > 1$ we have that $\dfrac{2}{1 - \frac{1}{2^{q-1}}} \le \dfrac{2}{1 - \frac{1}{2^{\log 3 - 1}}} = C_1$ and

$$\int\limits_{O(n)} \frac{1}{(G_n(UK))^N} \sum_{x_1,\ldots,x_N \in UK \cap \mathbb{Z}^n} \max_{1 \le i \le N} |\langle x_i, e_n \rangle| d\nu(U)$$

$$\le (e + eC_1) \int\limits_{O(n)} h_{\log N}(U) d\nu(U)$$

$$= (e + eC_1) \int\limits_{O(n)} \left(\frac{1}{G_n(UK)} \sum_{x \in UK \cap \mathbb{Z}^n} |\langle x, e_n \rangle|^{\log N}\right)^{\frac{1}{\log N}} d\nu(U),$$

which proves Theorem 1.3 with $C = e + eC_1$. □

As a consequence, we obtain the right-hand side inequality in (1.11) and, equivalently, the right-hand side inequality in (1.10).

Corollary 4.1. *There exists an absolute constant $C > 0$ such that for every convex body $K \subseteq \mathbb{R}^n$ with $0 \in K$ and every $N \in \mathbb{N}$ with $N \ge 3$,*

$$\int\limits_{O(n)} \frac{1}{|UK|^N} \int\limits_{UK} \cdots \int\limits_{UK} \max_{1 \le i \le N} |\langle x_i, e_n \rangle| dx_N \ldots dx_1 d\nu(U)$$

$$\leq C \int\limits_{O(n)} \left(\frac{1}{|UK|} \int\limits_{UK} |\langle x, e_n\rangle|^{\log N} dx \right)^{\frac{1}{\log N}} d\nu(U).$$

Proof. By Theorem 1.3 we have that there exists $C > 0$ such that for any $\lambda > 0$, any convex body $K \subseteq \mathbb{R}^n$ with $0 \in K$ and every $N \in \mathbb{N}$ with $N \geq 3$ we have that

$$\int\limits_{O(n)} \frac{1}{(G_n(U(\lambda K)))^N} \sum_{x_1, \ldots, x_N \in U(\lambda K) \cap \mathbb{Z}^n} \max_{1 \leq i \leq N} |\langle x_i, e_n\rangle| d\nu(U)$$

$$\leq C \int\limits_{O(n)} \left(\frac{1}{G_n(U(\lambda K))} \sum_{x \in U(\lambda K) \cap \mathbb{Z}^n} |\langle x, e_n\rangle|^{\log N} \right)^{\frac{1}{\log N}} d\nu(U).$$

Equivalently,

$$\int\limits_{O(n)} \frac{1}{(G_n(\lambda UK))^N} \sum_{x_1, \ldots, x_N \in \lambda UK \cap \mathbb{Z}^n} \max_{1 \leq i \leq N} \left|\left\langle \frac{x_i}{\lambda}, e_n\right\rangle\right| d\nu(U)$$

$$\leq C \int\limits_{O(n)} \left(\frac{1}{G_n(\lambda UK)} \sum_{x \in \lambda UK \cap \mathbb{Z}^n} \left|\left\langle \frac{x}{\lambda}, e_n\right\rangle\right|^{\log N} \right)^{\frac{1}{\log N}} d\nu(U).$$

Lemma 2.1 with $m = N$ and $q = 1$ yields

$$\lim_{\lambda \to \infty} \int\limits_{O(n)} \frac{1}{(G_n(\lambda UK))^N} \sum_{x_1, \ldots, x_N \in \lambda UK \cap (\mathbb{Z}^n)} \max_{1 \leq i \leq N} \left|\left\langle \frac{x_i}{\lambda}, e_n\right\rangle\right| d\nu(U)$$

$$= \int\limits_{O(n)} \frac{1}{|UK|^N} \int\limits_{UK} \cdots \int\limits_{UK} \max_{1 \leq i \leq N} |\langle x_i, e_n\rangle| dx_N \ldots dx_1 d\nu(U).$$

Using now Lemma 2.1 with $m = 1$ and $q = \log N$ we get

$$\lim_{\lambda \to \infty} \int\limits_{O(n)} \left(\frac{1}{G_n(\lambda UK)} \sum_{x \in \lambda UK \cap \mathbb{Z}^n} \left|\left\langle \frac{x}{\lambda}, e_n\right\rangle\right|^{\log N} \right)^{\frac{1}{\log N}} d\nu(U)$$

$$= \int\limits_{O(n)} \left(\frac{1}{|UK|} \int\limits_{UK} |\langle x, e_n\rangle|^{\log N} dx \right)^{\frac{1}{\log N}} d\nu(U).$$

and the result follows. $\square$

Let us now prove Theorem 1.4.

Proof of Theorem 1.4. Let $C > 0$ be twice the absolute constant given by Theorem 1.2 and let X be a random vector uniformly distributed on $K \cap \mathbb{Z}^n$. Then

$$\frac{\left(\mathbb{E}|\langle X, e_n\rangle|^{2p}\right)^{\frac{1}{2p}}}{\left(\mathbb{E}|\langle X, e_n\rangle|^p\right)^{\frac{1}{p}}} \leq C D_0(K, p).$$

We can assume, without loss of generality, that $C \geq e$. By the Paley-Zygmund inequality, for any $1 \leq p \leq q$,

$$\mathbb{P}\left(|\langle X, e_n\rangle| \geq \frac{1}{2}\left(\mathbb{E}|\langle X, e_n\rangle|^p\right)^{\frac{1}{p}}\right) = \mathbb{P}\left(|\langle X, e_n\rangle|^p \geq \frac{1}{2^p}\mathbb{E}|\langle X, e_n\rangle|^p\right)$$

$$\geq \left(1 - \frac{1}{2^p}\right)^2 \frac{(\mathbb{E}|\langle X, e_n\rangle|^p)^2}{\mathbb{E}|\langle X, e_n\rangle|^{2p}}$$

$$\geq \frac{1}{4(CD_0(K,p))^{2p}} \geq \frac{1}{4(CD_q(K))^{2p}}.$$

Choosing $1 \leq p_1 = \frac{\log N}{2\log(CD_q(K))} \leq \frac{\log N}{2\log C} \leq \log N \leq q$ we obtain

$$\mathbb{P}\left(|\langle X, e_n\rangle| \geq \frac{1}{2}\left(\mathbb{E}|\langle X, e_n\rangle|^{p_1}\right)^{\frac{1}{p_1}}\right) \geq \frac{1}{4N}.$$

Therefore, if $X_1, \ldots, X_N$ are independent random vectors uniformly distributed on $K \cap \mathbb{Z}^n$, we have that

$$\mathbb{P}\left(\max_{1\leq i\leq N}|\langle X_i, e_n\rangle| \geq \frac{1}{2}\left(\mathbb{E}|\langle X, e_n\rangle|^{p_1}\right)^{\frac{1}{p_1}}\right)$$

$$= 1 - \mathbb{P}\left(\max_{1\leq i\leq N}|\langle X_i, e_n\rangle| < \frac{1}{2}\left(\mathbb{E}|\langle X, e_n\rangle|^q\right)^{\frac{1}{p_1}}\right)$$

$$= 1 - \mathbb{P}\left(|\langle X_i, e_n\rangle| < \frac{1}{2}\left(\mathbb{E}|\langle X, e_n\rangle|^{p_1}\right)^{\frac{1}{p_1}}\right)^N$$

$$= 1 - \left(1 - \mathbb{P}\left(|\langle X_i, e_n\rangle| \geq \frac{1}{2}\left(\mathbb{E}|\langle X, e_n\rangle|^{p_1}\right)^{\frac{1}{p_1}}\right)\right)^N$$

$$\geq 1 - \left(1 - \frac{1}{4N}\right)^N = 1 - e^{N\log\left(1-\frac{1}{4N}\right)} \geq 1 - e^{-\frac{1}{4}}.$$

Therefore, by Markov's inequality,

$$\mathbb{E}\max_{1\leq i\leq N}|\langle X_i, e_n\rangle| \geq \frac{1}{2}\left(\mathbb{E}|\langle X, e_n\rangle|^{p_1}\right)^{\frac{1}{p_1}} \cdot \mathbb{P}\left(\max_{1\leq i\leq N}|\langle X_i, e_n\rangle| \geq \frac{1}{2}\left(\mathbb{E}|\langle X, e_n\rangle|^{p_1}\right)^{\frac{1}{p_1}}\right)$$

$$\geq \frac{1 - e^{-\frac{1}{4}}}{2}\left(\mathbb{E}|\langle X, e_n\rangle|^{p_1}\right)^{\frac{1}{p_1}}.$$

This proves Theorem 1.4 with $c = \frac{1-e^{-\frac{1}{4}}}{2}$. $\quad\square$

Let us now see that Corollary 1.1 implies the left-hand side inequality in (1.11).

Corollary 4.2. *There exist absolute constants $c_1, c_1', c > 0$ such that if $K \subseteq \mathbb{R}^n$ is a convex body with $0 \in \text{int } K$ and $N \in \mathbb{N}$ satisfies that $N \geq c$ then*

$$\int_{O(n)} \frac{1}{|UK|^N} \int_{UK} \cdots \int_{UK} \max_{1\leq i\leq N}|\langle x_i, e_n\rangle| dx_N \ldots dx_1 d\nu(U)$$

$$\geq c_1 \int_{O(n)} \left(\frac{1}{|UK|}\int_{UK}|\langle x, e_n\rangle|^{c_1'\log N}dx\right)^{\frac{1}{c_1'\log N}} d\nu(U).$$

Proof. Let $c = 4C^2$, where C is the constant given by Theorem 1.4 and Corollary 1.1. Let $K \subseteq \mathbb{R}^n$ be a convex body with $0 \in \text{int } K$ and let $N \geq c = 4C^2$. Let us take $q = \log N$. Since $\lim_{\lambda\to\infty} D_q(\lambda K) = 1$, there exists $\lambda_0 > 0$ such that for every $\lambda \geq \lambda_0$ we have $B_2^n \subseteq \lambda K$ and $D_q(\lambda K) \leq 2$. Thus

$$(CD_q(\lambda K))^2 \leq 4C^2 \leq N = e^q.$$

Therefore, by Corollary 1.1, there exists an absolute constant $c_1 > 0$ such that we have that for every $\lambda \geq \lambda_0$,

$$\int_{O(n)} \frac{1}{(G_n(\lambda UK))^N} \sum_{x_1,\ldots,x_N \in \lambda UK \cap \mathbb{Z}^n} \max_{1 \leq i \leq N} |\langle x_i, e_n \rangle| d\nu(U)$$

$$\geq c_1 \int_{O(n)} \left(\frac{1}{G_n(\lambda UK)} \sum_{x \in \lambda UK} |\langle x, e_n \rangle|^{\frac{\log N}{2\log(CD_q(K))}} dx \right)^{\frac{2\log(CD_q(K))}{\log N}} d\nu(U)$$

$$\geq c_1 \int_{O(n)} \left(\frac{1}{G_n(\lambda UK)} \sum_{x \in \lambda UK} |\langle x, e_n \rangle|^{\frac{\log N}{2\log(2C)}} \right)^{\frac{2\log(2C)}{\log N}} d\nu(U)$$

$$= c_1 \int_{O(n)} \left(\frac{1}{G_n(\lambda UK)} \sum_{x \in \lambda UK} |\langle x, e_n \rangle|^{c_1' \log N} \right)^{\frac{1}{c_1' \log N}} d\nu(U),$$

with $c_1' = \frac{1}{2\log(2C)}$. Equivalently,

$$\int_{O(n)} \frac{1}{(G_n(\lambda UK))^N} \sum_{x_1,\ldots,x_N \in \lambda UK \cap \mathbb{Z}^n} \max_{1 \leq i \leq N} \left| \left\langle \frac{x_i}{\lambda}, e_n \right\rangle \right| d\nu(U)$$

$$= c_1 \int_{O(n)} \left(\frac{1}{G_n(\lambda UK)} \sum_{x \in \lambda UK \cap \mathbb{Z}^n} \left| \left\langle \frac{x}{\lambda}, e_n \right\rangle \right|^{c_1' \log N} \right)^{\frac{1}{c_1' \log N}} d\nu(U).$$

Taking limit as $\lambda \to \infty$ and using Lemma 2.1 with $m = N$ and $q = 1$ and with $m = 1$ and $q = c' \log N$, we obtain

$$\int_{O(n)} \frac{1}{|UK|^N} \int_{UK} \cdots \int_{UK} \max_{1 \leq i \leq N} |\langle x_i, e_n \rangle| dx_N \ldots dx_1 d\nu(U)$$

$$\geq c_1 \int_{O(n)} \left(\frac{1}{|UK|} \int_{UK} |\langle x, e_n \rangle|^{c_1' \log N} dx \right)^{\frac{1}{c_1' \log N}} d\nu(U). \quad \square$$

Acknowledgements

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