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Singularly perturbed convection-diffusion elliptic problems with a non-smooth forcing term

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ABSTRACT

Singularly perturbed elliptic problems, of convection-diffusion type, with a non-smooth forcing term are examined. The lack of smoothness arises from the forcing term either containing an interior layer or being discontinuous across an interface. In addition to the presence of several different kinds of boundary and corner layers, this forcing term introduces an interior layer in the solution. For both problem classes, a decomposition of the continuous solution is constructed, whose components identify the various types of layer functions that can exist in the solution. Parameter-explicit pointwise bounds on the partial derivatives of these components are then established. An appropriate Shishkin mesh is identified and this is combined with upwinding to form a numerical method for each problem class. Parameter-uniform error bounds in the maximum norm are deduced. Numerical results are presented to illustrate the performance of both numerical methods.

1. Introduction

In the case of singularly perturbed problems, even when the solution u_ε is smooth (e.g. $u_\varepsilon \in C^{4,\lambda}(\bar{\Omega})$[1]) over a domain $\bar{\Omega}$, steep gradients can appear in the solution due to the presence of a small parameter $\varepsilon > 0$ multiplying all or some of the highest derivatives appearing in the differential equation. Hence although $u_\varepsilon \in C^{4,\lambda}(\bar{\Omega})$, the solution u_0 of the reduced problem (formally set $\varepsilon = 0$ in the differential equation) may be less regular (e.g. $u_0 \notin C^{1,\lambda}(\bar{\Omega})$). The regions where the solution u_ε is changing rapidly are often referred to as layer regions, which are near some boundary or internal interface of the domain where the reduced solution u_0 is not as smooth as u_ε. If the solution u_ε itself is less regular (e.g. $u_\varepsilon \in C^{1,\lambda}(\bar{\Omega}) \setminus C^{2,\lambda}(\bar{\Omega})$), then additional technical issues arise in the numerical analysis.

In the case of singularly perturbed ordinary differential equations, the lack of regularity in the solution can be due to discontinuous coefficients in the differential equation or a discontinuous forcing term (see, e.g. [4–6,14] and the references therein). This lack of regularity in the continuous solution can also appear in singularly perturbed parabolic problems, but, in addition, we can also have a

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[1] The space $C^\gamma(D)$ is the set of all functions that are Hölder continuous of degree γ with respect to the Euclidean norm $\|\cdot\|_e$. A function $f \in C^\gamma(D)$ if

$$\lceil f \rceil_{0,\gamma,D} = \sup_{\mathbf{u}\neq\mathbf{v},\ \mathbf{u},\mathbf{v}\in \mathbf{D}} \frac{|f(\mathbf{u}) - f(\mathbf{v})|}{\|\mathbf{u}-\mathbf{v}\|_{\mathbf{e}}^\gamma},$$

is finite. The space $C^{k,\gamma}(D)$ is the set of all functions in $C^k(D)$ whose derivatives of order k are Hölder continuous of degree γ.

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solution with low regularity due to discontinuities in the boundary/initial conditions or due to incompatibilities between the initial and boundary conditions (e.g. [8,9,17] and the references therein). Interior layers can appear in the solution of problems where the data has low regularity. In this case, if the reduced solution u_0 is continuous but not $C^1(\bar{\Omega})$, then the interior layer can be weakly singular with respect to ε (i.e. all of the first derivatives of the solution u_ε are bounded independently of ε). On the other hand, if the reduced solution u_0 is discontinuous then the interior layer will be strongly singular.

There has been little investigation in the literature of numerical methods for singularly perturbed elliptic problems with low regularity in the solution (see [1–3,11], [19, Remark 14.6.2] and references therein). In the case of two point boundary value problems with discontinuous data, the solution can be determined in closed form. Moreover, the regularity of the solution is explicitly identified and the nature of the derivatives (especially in the vicinity of any discontinuity in the problem data) can be explicitly determined. However, in the case of elliptic partial differential equations with discontinuous data, the regularity of the solution is not as easily identified.

In this paper the focus will be on the numerical approximation of the following singularly perturbed elliptic problem: Find u such that, over the unit square $\Omega := (0,1)\times(0,1)$,

$$Lu := -\varepsilon\Delta u + a(x,y)u_x + b(x,y)u = f(x,y), \quad (x,y)\in\Omega, \tag{1a}$$

$$u(x,y) = 0,\ (x,y)\in\partial\Omega, \quad a,b\in C^{3,\lambda}(\bar{\Omega}); \tag{1b}$$

$$a(x,y) > \alpha > 0,\ b(x,y)\ge 0,\ (x,y)\in\bar{\Omega}; \quad 0<\varepsilon\le 1, \tag{1c}$$

where a,b are smooth functions and the forcing term f can be discontinuous along the line $x=d,\ 0<d<1$. These problems can occur when dealing with a system of two or more coupled singularly perturbed partial differential equations. For example, these systems arise in mathematical models of semiconductor devices [15] and plasma sheaths [13].

The asymptotic behavior of the solution of (1) is discussed in §2. To this end, we first refer to [10,18], which consider (1) with the forcing term f a smooth function without layers. In [10,18] the solution is decomposed into a regular component v and a layer component w. The later is decomposed into several functions: a regular boundary layer (w_E associated with $x=1$), two characteristic boundary layers (w_S and w_N associated with $y=0$ and $y=1$) and two corner layers (w_{ES} and w_{EN} associated with (1,0) and (1,1)). When the forcing term $f=F_0$ is a discontinuous function at $x=d$, by assuming a certain regularity of the solution (denoted below by $\tilde{u}$), estimates of the derivatives of (1), can be also deduced using an appropriate decomposition of $\tilde{u}$. Compared to [10,18], a new component $\tilde{z}$ appears in the decomposition of $\tilde{u}$, describing the behavior of $\tilde{u}$ in the vicinity of $x=d$ (see Lemma 3).

We can sidestep the regularity difficulties of the solution $\tilde{u}$ by introducing a regularized version of the above elliptic problem. Both problem classes are analyzed in parallel in this paper. In the regularized problem, the forcing term is of the form

$$f \sim e^{-|x-d|/\varepsilon}, \quad \lim_{\varepsilon\to 0^+} f = F_0, \tag{2}$$

and the estimates on the derivatives also reveal the presence of a interior layer z at $x=d$ (see Lemma 2.) The numerical method used in [18] will be modified to approximate the regularized problem due to the presence of these new layers and the resulting scheme is given in §3. Returning to the original problem with discontinuous data F_0, we can apply the same numerical method (designed for the regularized problem) away from the location of the discontinuity and a discrete transmission condition, along the line of the discontinuity, is used for approximation purposes. Parameter-uniform asymptotic error bounds are established for both problem classes, showing that the numerical approximations converge globally and uniformly with almost first order in the maximum norm. Numerical results are presented in §4 and some final conclusions of the paper are given in §5.

Notation. Throughout the paper, C denotes a generic constant that is independent of the singular perturbation parameter ε and the discretization parameters of the numerical scheme.

2. Continuous problem

In this section we present the two problem classes (3) and (4) whose solutions will be approximated with a finite difference scheme. The asymptotic behavior of the solution is proved showing the presence of boundary, corner and interior layers.

Define the following subdomains

$$\Omega_x^- := (0,d)\times(0,1);\ \Omega_x^+ := (d,1)\times(0,1); \quad \Omega_x := \Omega_x^-\cup\Omega_x^+,$$

with $0<d<1$. Consider the following singularly perturbed elliptic problem: Find $u\in C^{3,\lambda}(\bar{\Omega})$ such that,

$$Lu = f(x,y;\varepsilon),\ (x,y)\in\Omega, \quad u=0,\ (x,y)\in\partial\Omega; \quad a,b\in C^{3,\lambda}(\bar{\Omega}); \tag{3a}$$

$$f(x,y;\varepsilon)\in C^{5,\lambda}(\bar{\Omega}), \quad f(1,\ell)=0,\ \ell=0,1; \tag{3b}$$

$$\lim_{\varepsilon\to 0^+} f(x,y) = F_0(x,y) := \begin{cases} f^-(x,y), & (x,y)\in\Omega_x^-, \\ f^+(x,y), & (x,y)\in\Omega_x^+, \end{cases} \quad f^\pm\in C^{5,\lambda}(\bar{\Omega}_x^\pm); \tag{3c}$$

$$\text{and for all} \quad 0\le i+j\le 5, \quad \frac{\partial^{i+j} f}{\partial x^i \partial y^j}(0,\ell) = 0,\ \ell=0,1; \tag{3d}$$

$$\left|\frac{\partial^{i+j}(f-F_0)(x,y)}{\partial x^i\partial y^j}\right| \le C\varepsilon^{-i}(1+\varepsilon^{2-j})e^{-\beta\frac{|d-x|}{\varepsilon}},\ (x,y)\in\Omega_x^{\pm},\quad \alpha\ge\beta. \tag{3e}$$

Hence, although $f\in C^{5,\lambda}(\bar\Omega)$ is smooth for all $\varepsilon>0$ the limiting function $F_0(x,y)$ can be discontinuous along the interface $x=d$. Note also that for $\varepsilon>0$, the steep gradients in f only occur in the vicinity of $x=d$. We make the additional regularity assumption: there exists a $\kappa>0$ such that

$$F_0\in C^{5,\lambda}(\bar\Omega_d),\quad \Omega_d:=(d-\kappa,d+\kappa)\times\big((0,\kappa)\cup(1-\kappa,1)\big). \tag{3f}$$

This final constraint (3f) on f ensures that the limiting function F_0 is smooth along the boundary $\partial\Omega$. In particular, it is smooth in the vicinity of the two points $(d,0)$ and $(d,1)$.

Problem (3) can be viewed as a regularization of a problem with a discontinuous inhomogeneous term. That is, consider the problem

$$L\tilde u=F_0(x,y),\ x\ne d,\quad \tilde u=0,\ (x,y)\in\partial\Omega;\quad [\tilde u_x](d,y)=0; \tag{4a}$$

$$\frac{\partial^{i+j}F_0}{\partial x^i\partial y^j}(0,\ell)=0,\ 0\le i+j\le 5,\ \ell=0,1;\quad F_0(1,\ell)=0,\ \ell=0,1 \tag{4b}$$

$$F_0\in C^{5,\lambda}(\bar\Omega_d),\quad \Omega_d:=(d-\kappa,d+\kappa)\times\big((0,\kappa)\cup(1-\kappa,1)\big),\ \kappa>0. \tag{4c}$$

Motivated by the discussion in [12, Theorem 16.2, pg. 222] we are led to the following:

Assumption. We assume the following regularity of the solution of problem (4)

$$\tilde u\in C^{1,\lambda}(\bar\Omega)\cap\Big(C^{3,\lambda}(\bar\Omega_x^-)\cup C^{3,\lambda}(\bar\Omega_x^+)\Big). \tag{5}$$

The asymptotic behavior of the solution in the case that the forcing term is a smooth function without layers is discussed in Appendix A (cf. [10,18]). We focus the analysis of this section on the behavior of the solution of the elliptic problems (3) and (4). Similarly to [10,18], we consider a decomposition of their solutions into regular v, boundary and corner layers w and interior layer z ($\tilde z$) components

$$u=v+w+z,\quad \tilde u=v+w+\tilde z.$$

However unlike [10,18], the regular component v is discontinuous and the interior layer components $z,\tilde z$ are present. These new components are defined below.

Notation. Throughout this section, we denote extensions of various domains D by D^*, where $\bar D\subset D^*$. Also the data a,b,f will be extended to D^* and denoted by a^*,b^*,f^* in such a way that $a^*(x,y)\equiv a(x,y), b^*(x,y)\equiv b(x,y), f^*(x,y)\equiv f(x,y)$ if $(x,y)\in D$. This extended domain approach is discussed in detail in [1, Section 2] and the references therein.

The construction of the regular component v of the solution u (and $\tilde u$) of problem (3) (and problem (4)) involves two stages. In the first stage, using the extended subdomain

$$(\Omega_x^-)^*:=(0,d+\delta)\times(-\delta,1+\delta),$$

where $\delta>0$ is an arbitrary constant independent of ε, the regular component is constructed in such a way that $v\in C^{3,\lambda}(\bar\Omega_x^-)$. That is:

$$v^*=v_0^*+\varepsilon v_1^*+\varepsilon^2v_2^*,\quad\text{where}$$

$$a^*\frac{\partial v_0^*}{\partial x}+b^*v_0^*=(f^-)^*;\ v_0^*(0,y)=u^*(0,y); \tag{6a}$$

$$a^*\frac{\partial v_1^*}{\partial x}+b^*v_1^*=(\Delta v_0)^*;\ v_1^*(0,y)=0; \tag{6b}$$

$$L^*v_2^*=(\Delta v_1)^*;\ v_2^*(x,y)=0,\ (x,y)\in\partial(\Omega_x^-)^*. \tag{6c}$$

By this construction, the regular component v on the subdomain Ω_x^- satisfies the following problem (for a detailed discussion, see [18])

$$Lv=f^-,\ (x,y)\in\Omega_x^-,\quad v(0,y)=u(0,y),\quad 0\le y\le 1,$$
$$v(x,0)=v^*(x,0),\quad v(x,1)=v^*(x,1),\quad 0\le x\le d,$$
$$v(d,y)=v^*(d,y),\quad 0\le y\le 1.$$

In the second stage, the construction of the regular component over the subdomain Ω_x^+ is as follows: use the extended subdomain

$$(\Omega_x^+)^*:=(d,1+\delta)\times(-\delta,1+\delta),$$

and $v^* = v_0^* + \varepsilon v_1^* + \varepsilon^2 v_2^*$ where

$$a^* \frac{\partial v_0^*}{\partial x} + b^* v_0^* = (f^+)^*; \quad v_0^*(d,y) = v(d,y); \tag{7a}$$

$$a^* \frac{\partial v_1^*}{\partial x} + b^* v_1^* = (\Delta v_0)^*; \quad v_1^*(d,y) = 0; \tag{7b}$$

$$L^* v_2^* = (\Delta v_1)^*; \quad v_2^*(x,y) = 0, \ (x,y) \in \partial(\Omega_x^+)^*. \tag{7c}$$

As v^* was constructed using two extended domains, no additional compatibility conditions were required and from this construction $v \in C^0(\bar{\Omega}) \cap \left(C^{3,\lambda}(\bar{\Omega}_x^-) \cup C^{3,\lambda}(\bar{\Omega}_x^+)\right)$, where $v := v^*|_{\bar{\Omega}}$.

With these two stages completed, we now have that the regular component satisfies the following problem

$$Lv = F_0, \ (x,y) \in \Omega_x, \quad v(0,y) = u(0,y), \ 0 \le y \le 1;$$

$$v(x,0), \ v(x,1), \quad 0 \le x \le d, \ v(d,y), 0 \le \ y \le 1 \quad \text{is determined from (6);}$$

$$v(x,0), \ v(x,1), \quad d < x \le 1, \ v(1,y), \ 0 \le y \le 1 \quad \text{is determined from (7).}$$

As in [18], the bounds in (A.2a) on the regular component remain valid over the regions $\bar{\Omega}_x^-$ and $\bar{\Omega}_x^+$ and we have

$$\left| \frac{\partial^{i+j} v}{\partial x^i \partial y^j}(x,y) \right| \le C \left(1 + \varepsilon^{2-(i+j)}\right), \quad 0 \le i+j \le 4; \qquad x \ne d. \tag{8}$$

Note that, in general, $v \in C^0(\bar{\Omega}) \setminus C^1(\bar{\Omega})$ as

$$[v_x](d,y) \ne 0.$$

However, we have assumed (3f) in neighborhoods of the points $(d,0),(d,1)$. The two extensions f^*, either side of $x=d$, are constructed to preserve this regularity. In this way, we will have that the function v at the boundaries $x=0$ and $x=1$ satisfies $v(x,0), v(x,1) \in C^3(0,1)$.

Given this lack of regularity (along $x=d$) in the regular component, the decomposition (A.1) of the solution of (3) (and of (4)) requires an additional subcomponent z (and $\tilde{z}$), which is associated with the lack of smoothness along the interface $x=d$.

All of the other layer components are constructed as in [18] (see also the Appendix A of this paper). Let us define

$$w := w_E + w_S + w_{ES} + w_N + w_{EN}.$$

Then $Lw = 0, (x,y) \in \Omega$ for each of the five layer functions. That is, the differential equation is also satisfied at the points where $x=d$. Note that the functions $v(x,\ell), \ell = 0,1$ are involved in the definition of w_S, w_N and that is why we assume (3f). In this construction, $w_E, w_S, w_{ES}, w_N, w_{EN} \in C^{3,\lambda}(\bar{\Omega})$. As a consequence, $[w_x](d,y) = 0$. Moreover, the bounds on the derivatives of these layer components (A.2) still apply.

Finally, we define the functions s and z to be

$$s := u - w =: v + z,$$

as $u, w \in C^{3,\lambda}(\bar{\Omega})$ then $s \in C^{3,\lambda}(\bar{\Omega})$. Also s satisfies

$$Ls = f, \ (x,y) \in \Omega; \qquad s = v, \ (x,y) \in \partial\Omega. \tag{9}$$

Hence, in the case of problem (3), $z \in C^0(\bar{\Omega})$ is the solution of

$$Lz(x,y) = f - F_0, \quad (x,y) \in \Omega_x^{\pm}, \quad x \ne d; \tag{10a}$$

$$z(x,0) = z(x,1) = 0, \ 0 \le x \le 1, \ z(0,y) = z(1,y) = 0, \ y \in (0,1); \tag{10b}$$

$$[z_x](d,y) = -[v_x](d,y). \tag{10c}$$

In the case of problem (4), $\tilde{u} - w = v + \tilde{z}$, where $\tilde{z} \in C^0(\bar{\Omega})$ is the solution of

$$L\tilde{z}(x,y) = 0, \quad (x,y) \in \Omega_x^{\pm}, \quad x \ne d; \tag{11a}$$

$$\tilde{z}(x,0) = \tilde{z}(x,1) = 0, \ 0 \le x \le 1, \ \tilde{z}(0,y) = \tilde{z}(1,y) = 0, \ y \in (0,1); \tag{11b}$$

$$[\tilde{z}_x](d,y) = -[v_x](d,y). \tag{11c}$$

To obtain bounds on the functions z and $\tilde{z}$ we will utilize the following maximum principle.

Lemma 1. *If $w \in C^0(\bar{\Omega}) \cap C^2(\Omega_x^- \cup \Omega_x^+)$, $w(x,y) \ge 0$, $(x,y) \in \partial\Omega$, $[w_x](d,y) \le 0, y \in (0,1)$ and $Lw(x,y) \ge 0$, $(x,y) \in \Omega_x$, then $w(x,y) \ge 0$, $(x,y) \in \bar{\Omega}$.*

Proof. The proof is by contradiction. Define the function $\hat{w}$ by

$$w(x,y)=e^{\theta(x)(x-d)}\hat{w}(x,y), \quad \theta(x)=\begin{cases}\theta_1, & x\le d,\\ \theta_2, & x>d,\end{cases}$$

where θ_i, $i=1,2$ are positive parameters such that $\theta_2<\theta_1<\frac{\alpha}{2\varepsilon}$. Assume $\hat{w}(p,q):=\min_{\bar{\Omega}}\hat{w}<0$. Note that

$$Lw=-\varepsilon\Delta\hat{w}+(a-2\varepsilon\theta)\hat{w}_x+(b+a\theta-\varepsilon\theta^2)\hat{w}.$$

As $Lw\ge 0$, the minimum point (p,q) cannot be in the interior. It also cannot be on the boundary as this would contradict $w\ge 0$ on the boundary. Finally, the point (p,q) cannot occur along the interface $x=d$ as

$$[w_x](d,q)=(\theta_2-\theta_1)\hat{w}(p,q)+[\hat{w}_x](p,q)\ge(\theta_2-\theta_1)\hat{w}(p,q)>0,$$

which is a contradiction to $[w_x](d,y)\le 0,\ y\in(0,1)$. This completes the proof. □

Using Lemma 1 and (3e), we establish that, with the strict inequality $a>\alpha\ge\beta>\beta_1>0$,

$$|z(x,y)|\le C\varepsilon\begin{cases}e^{-\beta\frac{d-x}{\varepsilon}}, & x\le d,\\ 2-e^{-\beta_1\frac{x-d}{\varepsilon}}+e^{-\alpha\frac{1-x}{\varepsilon}}-e^{-\alpha\frac{1-d}{\varepsilon}}, & x>d,\end{cases} \tag{12}$$

since

$$Le^{-\beta\frac{d-x}{\varepsilon}}=\frac{\beta(a-\beta)}{\varepsilon}e^{-\beta\frac{d-x}{\varepsilon}}, \quad x<d,$$
$$L\left(e^{-\alpha\frac{1-x}{\varepsilon}}-e^{-\beta_1\frac{x-d}{\varepsilon}}\right)=\frac{\alpha(a-\alpha)}{\varepsilon}e^{-\alpha\frac{1-x}{\varepsilon}}+\frac{\beta_1(a+\beta_1)}{\varepsilon}e^{-\beta_1\frac{x-d}{\varepsilon}}, \quad x>d,$$

and

$$\left(e^{-\beta\frac{d-x}{\varepsilon}}\right)'(d^-)=\frac{\beta}{\varepsilon} \quad \text{and} \quad \left(e^{-\alpha\frac{1-x}{\varepsilon}}-e^{-\beta_1\frac{x-d}{\varepsilon}}\right)'(d^+)=\frac{\alpha}{\varepsilon}e^{-\alpha\frac{1-d}{\varepsilon}}+\frac{\beta_1}{\varepsilon}.$$

In addition to (12), bounds on the derivatives of z will also be required in the error analysis.

Lemma 2. *For all $(x,y)\in\Omega_x$ and $1\le i+j\le 3$, if $x<d$ we have the bounds*

$$\left|\frac{\partial^{i+j}z(x,y)}{\partial x^i\partial y^j}\right|\le C\varepsilon^{2-(i+j)}+C\varepsilon^{1-i}(1+\varepsilon^{2-j})e^{-\beta\frac{d-x}{\varepsilon}}, \tag{13a}$$

and if $x>d$

$$\left|\frac{\partial^{i+j}z(x,y)}{\partial x^i\partial y^j}\right|\le C\varepsilon^{2-(i+j)}+C\varepsilon^{1-i}(1+\varepsilon^{2-j})(e^{-\alpha\frac{(1-x)}{\varepsilon}}+e^{-\beta\frac{x-d}{\varepsilon}}). \tag{13b}$$

Proof. Using the constraint (3f), it follows that $z(d,0)=z(d,1)=0$. Consider the following decomposition

$$z(x,y)=\phi_d(x;y)+\psi_d(x;y)+R(x,y), \tag{14}$$

where, for each $y\in(0,1)$, the function $\phi_d(x;y)\in C^0(\Omega)$ is the solution of the boundary value problem

$$-\varepsilon\frac{\partial^2\phi_d}{\partial x^2}+a(x,y)\frac{\partial\phi_d}{\partial x}=(f-F_0)(x,y), \quad x\in(0,d)\cup(d,1); \tag{15a}$$

$$\phi_d(0,y)=0, \quad \left[\frac{\partial\phi_d}{\partial x}\right](d,y)=-[v_x](d,y), \quad \phi_d(1,y)=0; \tag{15b}$$

and the functions ψ_d and R are defined below. Note that

$$\phi_d(x;y)=\frac{\partial\phi_d}{\partial x}(d^-;y)I_3(x,y)+I_1(x,y), \quad x<d; \tag{16a}$$

$$\phi_d(x;y)=-\frac{\partial\phi_d}{\partial x}(d^+;y)e^{\int_{s=d}^1\frac{a(s,y)}{\varepsilon}\,ds}I_4(x,y)+I_2(x,y), \quad x>d; \tag{16b}$$

where

$$I_1(x,y):=\int_{t=0}^{x}\int_{s=t}^{d}\frac{(f-F_0)(s,y)}{\varepsilon}e^{-\int_{r=t}^{s}\frac{a(r,y)}{\varepsilon}\,dr}\,ds\,dt; \quad |I_1(d,y)|\le C\varepsilon;$$

$$I_2(x,y) := \int_{t=x}^{1}\int_{s=t}^{1} \frac{(f-F_0)(s,y)}{\varepsilon} e^{-\int_{r=t}^{s} \frac{a(r,y)}{\varepsilon}\, dr}\, ds\, dt; \quad |I_2(d,y)| \le C\varepsilon;$$

$$I_3(x,y) := \int_{t=0}^{x} e^{-\int_{s=t}^{d} \frac{a(s,y)}{\varepsilon}\, ds} dt; \quad 0 < I_3(d,y) \le C\varepsilon;$$

$$I_4(x,y) := \int_{t=x}^{1} e^{-\int_{s=t}^{1} \frac{a(s,y)}{\varepsilon}\, ds} dt; \quad 0 < I_4(d,y) \le C\varepsilon.$$

From (16), one has

$$\frac{\partial \phi_d}{\partial x}(d^-;y) I_3(d,y) + \frac{\partial \phi_d}{\partial x}(d^+;y) e^{\int_{s=d}^{1} \frac{a(s,y)}{\varepsilon}\, ds} I_4(d,y) = I_2(d,y) - I_1(d,y),$$

and using (15b)

$$\frac{\partial \phi_d}{\partial x}(d^-;y) - \frac{\partial \phi_d}{\partial x}(d^+;y) = [v_x](d,y). \tag{17}$$

Hence, by collecting terms, we have that

$$\frac{\partial \phi_d}{\partial x}(d^+;y) = \frac{(I_2 - I_1 - I_3\,[v_x])(d,y)}{I_3(d,y) + I_4(d,y) e^{\int_{s=d}^{1} \frac{a(s,y)}{\varepsilon}\, ds}}, \tag{18}$$

and so

$$\left|\frac{\partial \phi_d}{\partial x}(d^+;y)\right| \le C e^{-\int_{s=d}^{1} \frac{a(s,y)}{\varepsilon}\, ds} \le C e^{-\alpha \frac{1-d}{\varepsilon}}.$$

This implies that

$$\left|\frac{\partial \phi_d}{\partial x}(d^-;y)\right| \le C \quad \text{and} \quad |\phi_d(d;y)| \le C\varepsilon.$$

One can deduce, using (16) and the strict inequality $a > \alpha \ge \beta$, that for $x < d$

$$\left|\frac{\partial^i \phi_d}{\partial x^i}\right| \le C\varepsilon^{1-i} e^{-\beta \frac{d-x}{\varepsilon}}.$$

Note that for $t < p$,

$$\left|\frac{\partial}{\partial y} e^{-\int_{s=t}^{p} \frac{a(s,y)}{\varepsilon}\, ds}\right| \le C\|a_y\| \frac{p-t}{\varepsilon} e^{-(\min\{a\}-\beta)\frac{p-t}{\varepsilon}} e^{-\beta \frac{p-t}{\varepsilon}} \le C e^{-\beta \frac{p-t}{\varepsilon}},$$

which can be used to establish

$$\left|\frac{\partial^j}{\partial y^j} I_n(x;y)\right| \le C\varepsilon(1+\varepsilon^{2-j});\ n=1,2; \ \left|\frac{\partial^j}{\partial y^j} I_n(x;y)\right| \le C\varepsilon;\ n=3,4. \tag{19}$$

By differentiating (16), with respect to the variable y and using (17), (18) and (19), we can deduce the bounds

$$\left|\frac{\partial^j \phi_d}{\partial y^j}\right| \le C\varepsilon(1+\varepsilon^{2-j}) e^{-\beta \frac{d-x}{\varepsilon}}.$$

From (16), we can also obtain for $i > 0$ that

$$\left|\frac{\partial^{i+j} I_n}{\partial x^i \partial y^j}\right| \le C\varepsilon^{1-i}(1+\varepsilon^{2-j}) e^{-\beta \frac{d-x}{\varepsilon}}, \quad n = 1,2,$$

$$\left|\frac{\partial^{i+j} I_n}{\partial x^i \partial y^j}\right| \le C\varepsilon^{1-i} e^{-\beta \frac{d-x}{\varepsilon}}, \quad n = 3,4,$$

and, using these estimates, it follows for $i > 0$ that

$$\left|\frac{\partial^{i+j} \phi_d}{\partial x^i \partial y^j}\right| \le C\varepsilon^{1-i}(1+\varepsilon^{2-j}) e^{-\beta \frac{d-x}{\varepsilon}}.$$

Likewise, for $x > d$, using again (16), we have

$$\left|\frac{\partial^i \phi_d}{\partial x^i}\right| \le C\varepsilon^{1-i}(e^{-\alpha \frac{(1-x)}{\varepsilon}} + e^{-\beta \frac{x-d}{\varepsilon}}), \quad \left|\frac{\partial^j \phi_d}{\partial y^j}\right| \le C\varepsilon(1+\varepsilon^{2-j})(e^{-\alpha \frac{(1-x)}{\varepsilon}} + e^{-\beta \frac{x-d}{\varepsilon}}),$$

and for $i > 0$

$$\left|\frac{\partial^{i+j}\phi_d}{\partial x^i \partial y^j}\right| \le C\varepsilon^{1-i}(1+\varepsilon^{2-j})(e^{-\alpha\frac{(1-x)}{\varepsilon}} + e^{-\beta\frac{x-d}{\varepsilon}}).$$

We next examine the component ψ_d in (14). For each $y \in (0,1)$, the function $\psi_d(x;y)$ is the solution of the boundary value problem

$$-\varepsilon\frac{\partial^2\psi_d}{\partial x^2} + a(x,y)\frac{\partial\psi_d}{\partial x} = \varepsilon\frac{\partial^2\phi_d}{\partial y^2} - b\phi_d, \ x \in (0,d)\cup(d,1); \tag{20a}$$

$$\psi_d(0;y) = 0, \ \left[\frac{\partial\psi_d}{\partial x}\right](d,y) = 0, \ \psi_d(1;y) = 0. \tag{20b}$$

Observe that $\left(\varepsilon\frac{\partial^2\phi_d}{\partial y^2} - b\phi_d\right) \in C^0(\Omega)$ and $\psi_d(x;y) \in C^2(\Omega)$. We note that it satisfies

$$\psi_d(x;y) = \frac{\partial\psi_d}{\partial x}(d;y)I_3(x,y) + I_5(x,y), \quad x < d; \tag{21a}$$

$$\psi_d(x;y) = -\frac{\partial\psi_d}{\partial x}(d;y)e^{\int_{s=d}^{1}\frac{a(s,y)}{\varepsilon}\,ds}I_4(x,y) + I_6(x,y), \quad x > d; \tag{21b}$$

where

$$I_5 := \int_{t=0}^{x}\int_{s=t}^{d}\frac{(\varepsilon\frac{\partial^2\phi_d}{\partial y^2} - b\phi_d)(s,y)}{\varepsilon}e^{-\int_{r=t}^{s}\frac{a(r,y)}{\varepsilon}\,dr}\,ds\,dt; \quad |I_5(d,y)| \le C\varepsilon^2;$$

$$I_6 := \int_{t=x}^{1}\int_{s=t}^{1}\frac{(\varepsilon\frac{\partial^2\phi_d}{\partial y^2} - b\phi_d)(s,y)}{\varepsilon}e^{-\int_{r=t}^{s}\frac{a(r,y)}{\varepsilon}\,dr}\,ds\,dt; \quad |I_6(d,y)| \le C\varepsilon^2.$$

As above, by collecting terms, we see that

$$\frac{\partial\psi_d}{\partial x}(d;y) = \frac{(I_6 - I_5)(d,y)}{I_3(d,y) + I_4(d,y)e^{\int_{s=d}^{1}\frac{a(s,y)}{\varepsilon}\,ds}},$$

and we can deduce that

$$\left|\frac{\partial\psi_d}{\partial x}(d;y)\right| \le C\varepsilon e^{-\alpha\frac{1-x}{\varepsilon}} \text{ and } |\psi_d(d;y)| \le C\varepsilon^2.$$

Observe that

$$\left|\frac{\partial^{i+j}}{\partial x^i\partial y^j}\left(\varepsilon\frac{\partial^2\phi_d}{\partial y^2} - b\phi_d\right)\right| \le C\varepsilon^{1-i}(1+\varepsilon^{2-j})\begin{cases} e^{-\beta\frac{d-x}{\varepsilon}}, & x < d,\\ e^{-\alpha\frac{(1-x)}{\varepsilon}} + e^{-\beta\frac{x-d}{\varepsilon}}, & x > d.\end{cases}$$

Hence, for $x < d$

$$\left|\frac{\partial^i\psi_d}{\partial x^i}\right| \le C\varepsilon^{2-i}e^{-\beta\frac{d-x}{\varepsilon}}, \quad \left|\frac{\partial^j\psi_d}{\partial y^j}\right| \le C\varepsilon^2(1+\varepsilon^{2-j})e^{-\beta\frac{d-x}{\varepsilon}};$$

and for $x > d$,

$$\left|\frac{\partial^i\psi_d}{\partial x^i}\right| \le C\varepsilon^{2-i}e^{-\alpha\frac{(1-x)}{\varepsilon}} \quad \text{and} \quad \left|\frac{\partial^j\psi_d}{\partial y^j}\right| \le C\varepsilon^2(1+\varepsilon^{2-j})\left(e^{-\alpha\frac{(1-x)}{\varepsilon}} + e^{-\beta\frac{x-d}{\varepsilon}}\right).$$

The remainder $R \in C^{3,\lambda}(\bar{\Omega})$ satisfies the elliptic problem

$$LR = \varepsilon\frac{\partial^2\psi_d}{\partial y^2} - b\psi_d \in C^2(\Omega), \quad (x,y) \in \Omega,$$

$$R(x,y) = 0, \ (x,y) \in \partial\Omega.$$

Using the stretched variables $\zeta = (d-x)/\varepsilon$, $\eta = y/\varepsilon$ and appropriate bounds from [12, pg.113 (1.13) and pg. 138], we deduce

$$\left|\frac{\partial^{i+j}R(x,y)}{\partial x^i\partial y^j}\right| \le C\varepsilon^{2-(i+j)}, \ i+j \le 3,$$

on the derivatives (up to third order) of R. This completes the proof. □

Remark 1. If problem (4) is regularized only to the left of $x = d$ (i.e. $f^+ \equiv f, f^- \neq f$) then we can remove the term $e^{-\beta\frac{x-d}{\varepsilon}}$ from the bound (13b) noting that $I_2(x, y) \equiv 0$. However, if it is only regularized to the right of $x = d$ (i.e. $f^- \equiv f, f^+ \neq f$), we would still retain the term $e^{-\beta\frac{d-x}{\varepsilon}}$ within the bound (13a).

Based on these bounds on z and v and noting that the function s defined in (9) is in $C^{3,\lambda}(\bar{\Omega})$, we can take one sided limits $x \to d^{\pm}$ to deduce

$$\left|\frac{\partial^3 s(x,y)}{\partial y^3}\right| \le C\varepsilon^{-1}, \quad (x,y) \in \Omega. \tag{22}$$

Finally, we give some estimates for the interior layer component $\tilde{z}$ of the solution $\tilde{u}$ of (4).

Lemma 3. *For problem (4) the interior layer component is bounded as follows:*

$$|\tilde{z}(x,y)| \le C\varepsilon \begin{cases} e^{-\alpha\frac{d-x}{\varepsilon}}, & x \le d, \\ 1 + e^{-\alpha\frac{1-x}{\varepsilon}} - e^{-\alpha\frac{1-d}{\varepsilon}}, & x > d. \end{cases} \tag{23}$$

For all $(x,y) \in \Omega_x$ and $1 \le i+j \le 3$, if $x < d$ we have the bounds

$$\left|\frac{\partial^{i+j}\tilde{z}(x,y)}{\partial x^i \partial y^j}\right| \le C\varepsilon^{2-(i+j)} + C\varepsilon^{1-i}(1+\varepsilon^{2-j})e^{-\alpha\frac{d-x}{\varepsilon}}, \tag{24a}$$

and if $x > d$

$$\left|\frac{\partial^{i+j}\tilde{z}(x,y)}{\partial x^i \partial y^j}\right| \le C\varepsilon^{2-(i+j)} + C\varepsilon^{1-i}(1+\varepsilon^{2-j})e^{-\alpha\frac{(1-x)}{\varepsilon}}. \tag{24b}$$

Proof. Follow the argument that led to (12) and repeat the proof from the previous Lemma 2 that established the bounds (13). □

To obtain a global error bound, we establish (using the decomposition of the corner layer function in [18, §6]) the additional bounds

$$\left|\frac{\partial w_{ES}(x,y)}{\partial y}\right| \le C(1+\varepsilon^{-1/2}e^{-\frac{y}{\sqrt{\varepsilon}}})e^{-\frac{\alpha(1-x)}{\varepsilon}} + C, \tag{25a}$$

$$\left|\frac{\partial w_{EN}(x,y)}{\partial y}\right| \le C(1+\varepsilon^{-1/2}e^{-\frac{1-y}{\sqrt{\varepsilon}}})e^{-\frac{\alpha(1-x)}{\varepsilon}} + C. \tag{25b}$$

3. Numerical method and error analysis

We discretize the domain using a tensor product of two piecewise uniform Shishkin meshes [16]. The motivation for the choice of transition points comes from the bounds on the components in the decomposition of the solution. In the horizontal direction, we split the unit interval into four subintervals $[0, d-\sigma_x] \cup [d-\sigma_x, d+\sigma_x] \cup [d+\sigma_x, 1-\sigma_x] \cup [1-\sigma_x, 1]$ and in the vertical direction we use three subintervals $[0, \sigma_y] \cup [\sigma_y, 1-\sigma_y] \cup 1-\sigma_y, 1]$ where

$$\sigma_x := \min\left\{\frac{d}{4}, \frac{1-d}{4}, \frac{\varepsilon}{\beta}\ln N\right\}, \quad \sigma_y := \min\left\{\frac{1}{4}, \sqrt{\varepsilon}\ln M\right\}. \tag{26}$$

The mesh points are distributed uniformly in the ratio $N/4 : N/4 : N/4 : N/4$ and $M/4 : M/2 : M/4$, for the horizontal and vertical directions, across these subintervals. Note that the discretization parameters N and M are multiples of 4. This construction constitutes the Shishkin mesh $\bar{\Omega}^{N,M} = \{(x_i, y_j)\}_{i,j=0}^{N,M}$. As usual, we denote $\Omega^{N,M} = \bar{\Omega}^{N,M} \cap \Omega$ and $\partial\Omega^{N,M} = \bar{\Omega}^{N,M} \cap \partial\Omega$. We assume that $M = \mathcal{O}(N)$ to simplify the presentation. We discretize the differential operator using a simple upwind finite difference operator. Hence the discrete version of problem (3) is: Find $U(x_i, y_j)$ such that

$$L^{N,M}U = f(x_i, y_j), \quad (x_i, y_j) \in \Omega^{N,M}, \tag{27a}$$

$$U(x_i, y_j) = u(x_i, y_j), \ (x_i, y_j) \in \partial\Omega^{N,M}, \tag{27b}$$

where

$$L^{N,M}U := -\varepsilon(\delta_x^2 + \delta_y^2)U + aD_x^- U + bU,$$

and

$$D_x^- U(x_i, y_j) = \frac{U(x_i, y_j) - U(x_{i-1}, y_j)}{h_i}, \quad D_x^+ U(x_i, y_j) = D_x^- U(x_{i+1}, y_j),$$
$$\delta_x^2 U(x_i, y_j) = \frac{2}{h_i + h_{i+1}} \left(D_x^+ U(x_i, y_j) - D_x^- U(x_i, y_j) \right),$$

with $h_i = x_i - x_{i-1}$, $i = 1, 2, \ldots, N$ and the discrete operator δ_y^2 is defined similarly. Note that (27) satisfies a standard discrete maximum principle.

In the numerical analysis in Theorems 1 and 2, we confine our attention to the case where

$$\sigma_x = \frac{\varepsilon}{\beta} \ln N \quad \text{and} \quad \sigma_y = \sqrt{\varepsilon} \ln M.$$

In the other cases (of $\sigma_x = \frac{d}{4}, \sigma_x = \frac{1-d}{4}$ or $\sigma_y = \frac{1}{4}$), we use a classical argument (as in [18, p.1770]) and we also use $\varepsilon^{-1} \le C \ln N$ when dealing with the error associated with the terms s and $\tilde{z}$.

Theorem 1. *(Regularized problem) Assume $M = \mathcal{O}(N)$, we have the global error bound*

$$\|\bar{U} - u\| \le C N^{-1} (\ln N)^2,$$

where $\bar{U}$ is the bilinear interpolant of the computed solution U of the discrete problem (27) and u is the solution of the continuous problem (3).

Proof. The discrete solution is decomposed in an analogous manner to the continuous solution

$$U = S + W; \quad \text{where}$$
$$L^{N,M} S = f(x_i, y_j), \quad (x_i, y_j) \in \Omega^{N,M}, \ S(x_i, y_j) = v(x_i, y_j), \ (x_i, y_j) \in \partial\Omega^{N,M};$$
$$L^{N,M} W = 0, \quad (x_i, y_j) \in \Omega^{N,M}, \ W(x_i, y_j) = w(x_i, y_j), \ (x_i, y_j) \in \partial\Omega^{N,M}.$$

For each of the five layer functions, the following bounds are satisfied (see [18])

$$\|W - w\| \le C N^{-1} (\ln N)^2. \tag{28}$$

So it remains to bound the error in approximating the function s.

Consider the truncation error $L^{N,M}(S - s)$ over the mesh $\Omega^{N,M}$. Using the bounds (22), we easily deduce, at all mesh points, the error bound

$$\varepsilon \|(\delta_y^2 s - s_{yy})(x_i, y_j)\| \le C M^{-1}.$$

Next, we deduce bounds for the terms $\varepsilon \|(\delta_x^2 s - s_{xx})(x_i, y_j)\|$ and $\|(s_x - D_x^- s)(x_i, y_j)\|$, and we will use the estimates (8) and Lemma 2 for the derivatives of v and z, respectively, in our analysis.

Outside the layer regions where $x_i \le d - \sigma_x$ or $d + \sigma_x \le x_i \le 1 - \sigma_x$ we split the argument into two cases. In the first case where $\varepsilon \le C N^{-1}$, we have

$$\begin{aligned}
|(L^{N,M} - L) s(x_i, y_j)| &= |(L^{N,M} - L)(v + z)(x_i, y_j)| \\
&\le C(\varepsilon \|z_{xx}\|_{(x_{i-1}, x_{i+1})} + \|z_x\|_{(x_{i-1}, x_i)}) + C N^{-1} + C M^{-1} \\
&\le C\varepsilon + C \max \left\{ e^{-\alpha \frac{(1-x_i)}{\varepsilon}}, e^{-\beta \frac{|x_i - d|}{\varepsilon}} \right\} + C N^{-1} + C M^{-1} \\
&\le C N^{-1} + C M^{-1}.
\end{aligned}$$

In the other case where $\varepsilon > C N^{-1}$, we have

$$\begin{aligned}
|(L^{N,M} - L) s(x_i, y_j)| &= |(L^{N,M} - L)(v + z)(x_i, y_j)| \\
&\le C N^{-1} (\varepsilon \|z_{xxx}\| + \|z_{xx}\|)_{(x_{i-1}, x_{i+1})} + C N^{-1} + C M^{-1} \\
&\le C \frac{N^{-2}}{\varepsilon} + C N^{-1} + C M^{-1} \\
&\le C N^{-1} + C M^{-1}.
\end{aligned}$$

We next examine the truncation error in the vicinity of $x = d$. We denote by h the mesh step in the fine mesh area $(d - \sigma_x, d + \sigma_x)$. For the first derivative term we see that for $d - \sigma_x < x_i \le d$

$$|(s_x - D_x^- s)(x_i, y_j)| \le \frac{1}{h} \int_{t=x_{i-1}}^{x_i} \int_{s=t}^{x_i} |(v_{xx} + z_{xx})(s, y_j)| \, ds \, dt$$

$$\le C\frac{e^{-\beta\frac{d-x_i}{\varepsilon}}}{h}\int_{t=x_{i-1}}^{x_i}(1-e^{-\beta\frac{x_i-t}{\varepsilon}})\,dt + Ch$$
$$\le C(1-e^{-\beta\frac{h}{\varepsilon}}) + Ch \le C\frac{h}{\varepsilon} \le CN^{-1}\ln N;$$

and for $d < x_i < d+\sigma_x$

$$|(s_x - D_x^- s)(x_i,y_j)| \le \frac{1}{h}\int_{t=x_{i-1}}^{x_i}\int_{s=t}^{x_i}|(v_{xx}+z_{xx})(s,y_j)|\,ds\,dt$$
$$\le C\frac{e^{-\beta\frac{x_i-d}{\varepsilon}}}{h}\int_{t=x_{i-1}}^{x_i}(1-e^{-\beta\frac{x_i-t}{\varepsilon}})\,dt + Ch$$
$$\le C\frac{h}{\varepsilon} \le CN^{-1}\ln N.$$

For the second derivative term, along the line segment $x_i = d$, we have

$$\varepsilon|(\delta_x^2 s - s_{xx})(d,y_j)| \le \frac{1}{h^2}\int_{t=d}^{d+h}\int_{s=d}^{t}\int_{r=d}^{s}\varepsilon|(v+z)_{xxx}(r,y_j)|\,dr\,ds\,dt$$
$$+\frac{1}{h^2}\int_{t=d-h}^{d}\int_{s=d}^{t}\int_{r=d}^{s}\varepsilon|(v+z)_{xxx}(r,y_j)|\,dr\,ds\,dt$$
$$\le C\frac{h}{\varepsilon} \le CN^{-1}\ln N.$$

Similar bounds are obtained for $x_i \in (d-\sigma_x, d+\sigma_x)\backslash\{d\} \cup (1-\sigma_x, 1)$. Hence, using these two inequalities in the fine mesh areas, we have that

$$|(L^{N,M}-L)s(x_i,y_j)| \le CN^{-1}\ln N, \quad d-\sigma_x < x_i < d+\sigma_x,$$
$$|(L^{N,M}-L)s(x_i,y_j)| \le CN^{-1}\ln N, \quad 1-\sigma_x < x_i < 1.$$

Finally, using a discrete maximum principle over each of the two subintervals, we deduce the nodal error bound

$$|(S-s)(x_i,y_j)| \le CN^{-1}\ln N. \tag{29}$$

With this bound and the bounds (28), we have the nodal error bound

$$|(U-u)(x_i,y_j)| \le CN^{-1}(\ln N)^2, \quad (x_i,y_j) \in \Omega^{N,M}.$$

To extend this nodal error bound to a global error bound, combine the arguments in [7, Theorem 3.12] with the bounds in [20, Lemma 4.1] and the bounds (A.2), (13) and (25), on all the components in the decomposition of the solution. □

Remark 2. If we regularize only to the left of $x = d$ (i.e. $f^+ \equiv f, f^- \neq f$) then we can use a different mesh $\Omega_L^{N,M}$, where the horizontal interval is subdivided as follows: $[0, d-\sigma_x] \cup [d-\sigma_x, d] \cup [d, 1-\sigma_x] \cup [1-\sigma_x, 1]$ and the mesh points are equally distributed across these four subintervals. The above error bound, given in Theorem 1, will still apply on this adjusted mesh when $f^+ \equiv f$.

The mesh $\Omega_L^{N,M}$ is also used to approximate problem (4). Its discrete version is: Find $\tilde{U}(x_i,y_j)$ such that

$$L^{N,M}\tilde{U} = f(x_i,y_j), \quad (x_i,y_j) \in \Omega_L^{N,M}, \ x_i \neq d, \tag{30a}$$
$$D_x^-\tilde{U}(d,y_j) = D_x^+\tilde{U}(d,y_j), \ 0 < y_j < 1; \tag{30b}$$
$$\tilde{U}(x_i,y_j) = \tilde{u}(x_i,y_j), \ (x_i,y_j) \in \partial\Omega_L^{N,M}. \tag{30c}$$

Observe that a different discretization is used for the mesh points along $x = d$ and near $x = d$ we only have a fine mesh on the left of $x = d$.

For problem (30), using the usual proof-by-contradiction argument, we have the following discrete maximum principle:

Lemma 4. *For any mesh function $\tilde{\Phi}$, if $\tilde{\Phi}(x_i,y_j) \ge 0, (x_i,y_j) \in \partial\Omega_L^{N,M}$, $L^{N,M}\tilde{\Phi}(x_i,y_j) \ge 0, (x_i,y_j) \in \partial\Omega_L^{N,M}, x_i \neq d$ and $D_x^+\tilde{\Phi}(d,y_j) \le D_x^-\tilde{\Phi}(d,y_j)$ then $\tilde{\Phi}(x_i,y_j) \ge 0, (x_i,y_j) \in \bar{\Omega}_L^{N,M}$.*

Theorem 2. *Assume (5) and $M = \mathcal{O}(N)$. We have the global error bound:*

$$\|\bar{\tilde{U}} - \tilde{u}\| \le CN^{-1}(\ln N)^2,$$

where $\bar{\tilde{U}}$ is the bilinear interpolant of the computed solution $\tilde{U}$ of the discrete problem (30) and $\tilde{u}$ is the solution of (4).

Proof. The proof has been deferred to the Appendix B. □

Remark 3. We can also use the numerical method (30) to approximate elliptic problems with a non-smooth boundary condition at the characteristic boundary $y = 0$: Find $v \in C^2(\Omega)$ such that

$$Lv(x, y) = f(x, y) \in C^{1,\lambda}(\Omega), \quad (x, y) \in \Omega, \tag{31a}$$

$$v(x, y) = 0, \quad (x, y) \in \partial\Omega \backslash \{y = 0\}, \tag{31b}$$

$$v(x, 0) = \varphi(x), \quad 0 < x < 1. \tag{31c}$$

The function f is now sufficiently smooth in $\bar{\Omega}$, but $v(x, 0) = \varphi(x)$ may not be a smooth function at $(d, 0)$ with $[\varphi](d) = 0$ and $[\varphi'](d) \neq 0$. In order to deal with this problem, we define

$$u(x, y) = v(x, y) - \varphi(x)(1 - y),$$

and u is the solution of the elliptic problem

$$Lu(x, y) = f_1(x, y), \ (x, y) \in \Omega \backslash \{x = d\},$$

$$f_1(x, y) := f(x, y) - (-\varepsilon\varphi''(x) + a(x, y)\varphi'(x) + b(x, y)\varphi(x))(1 - y),$$

$$u(x, y) = 0, \quad (x, y) \in \partial\Omega, \quad [u_x](d, y) = -[\varphi'](d)(1 - y), \quad 0 < y < 1.$$

Numerical results for the problem class (31) are presented in Example 3. However, a rigorous proof of the convergence of the method applied to problem (31) remains an open question.

Remark 4. The same methodology (used in this paper) can be applied to the problem

$$-\varepsilon\Delta\tilde{u} + a_1\tilde{u}_x + a_2\tilde{u}_y + b\tilde{u} = f, \ a_1a_2 \neq 0, \quad (x, y) \in \Omega_x := (0, 1)^2 \setminus \{x = d\},$$

$$[\tilde{u}_x](d, y) = 0, \quad 0 < y < 1, \quad \tilde{u}(x, y) = 0, \quad (x, y) \in \partial\Omega,$$

with f discontinuous along $x = d$ and a_1, a_2 and b are smooth functions satisfying

$$(a_1a_2)(x, y) \neq 0, \quad b(x, y) \ge 0, \quad (x, y) \in \Omega_x.$$

One can employ standard upwinding and a transmission condition $(D_x^+ - D_x^-)(d, y_j) = 0$ for the finite difference operator. This can be combined with a tensor product Shishkin mesh, of order $O(\varepsilon \ln N)$ along the outflow boundaries and along the "outflow" side of $x = d$, to form a suitable numerical method. Under assumption (5) on the regularity of the continuous solution and sufficient compatibility at the corners and at $(d, 0), (d, 1)$, the error bound in Theorem 2 will again apply.

4. Numerical experiments

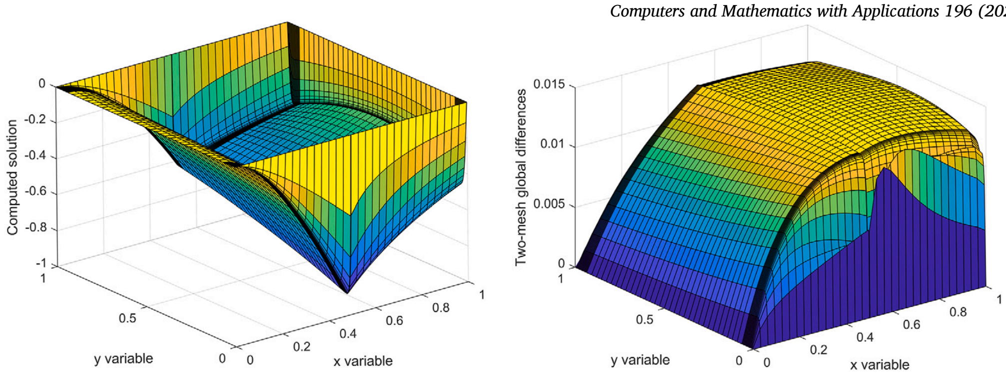
The exact solutions of the test examples below are unknown and the orders of convergence are estimated using the double-mesh principle [7, Chapter 8]: Let $U^{N,M}$ and $U^{2N,2M}$ be the solutions computed on the meshes $\bar{\Omega}^{N,M}$ and $\bar{\Omega}^{2N,2M}$, respectively. The maximum and uniform two-mesh global differences are defined by

$$D_\varepsilon^{N,M} := \|\bar{U}^{N,M} - \bar{U}^{2N,2M}\|_{\bar{\Omega}^{N,M} \cup \bar{\Omega}^{2N,2M}}, \quad D^{N,M} := \max_{\varepsilon \in S_\varepsilon} D_\varepsilon^{N,M}.$$

In all of the examples the singular perturbation parameter ε varies within a wide range of values $S_\varepsilon = \{2^0, 2^{-1}, \ldots, 2^{-30}\}$. Then, the computed orders of global convergence for each value of ε and the computed uniform orders of global convergence are estimated using

$$Q_\varepsilon^{N,M} := \log_2\left(\frac{D_\varepsilon^{N,M}}{D_\varepsilon^{2N,2M}}\right), \quad Q^{N,M} := \log_2\left(\frac{D^{N,M}}{D^{2N,2M}}\right).$$

In the tables, we display the numerical results for a selection of values of $\varepsilon \in S_\varepsilon$ to avoid the length of the paper becoming excessive. The uniform two-mesh global differences $D^{N,M}$ and the uniform order of convergence $Q^{N,M}$ are displayed in the bottom row of each table. The discretization parameters are taken to be $N = M = 32, 64, \ldots, 1024$.



(a) Computed solution (b) Two-mesh global differences

Fig. 1. Example 1: Computed solution and two-mesh global differences for $\varepsilon = 2^{-12}$ and $N = M = 64$.

Table 1
Example 1: Maximum two-mesh global differences and orders of convergence, using the numerical method (27).

	N=M=32	N=M=64	N=M=128	N=M=256	N=M=512	N=M=1024
$\varepsilon = 2^{0}$	4.412E-04 1.260	1.843E-04 1.150	8.302E-05 1.081	3.924E-05 1.043	1.905E-05 1.022	9.379E-06
$\varepsilon = 2^{-2}$	6.852E-03 1.014	3.392E-03 1.004	1.692E-03 1.000	8.457E-04 1.000	4.228E-04 1.000	2.114E-04
$\varepsilon = 2^{-4}$	1.673E-02 1.163	7.470E-03 1.000	3.735E-03 0.997	1.872E-03 0.998	9.368E-04 0.999	4.686E-04
$\varepsilon = 2^{-6}$	3.279E-02 1.069	1.563E-02 0.786	9.063E-03 0.706	5.556E-03 0.759	3.283E-03 0.788	1.902E-03
$\varepsilon = 2^{-8}$	7.961E-02 1.201	3.462E-02 1.751	1.029E-02 0.744	6.140E-03 0.772	3.596E-03 0.797	2.070E-03
$\varepsilon = 2^{-10}$	7.835E-02 1.130	3.580E-02 1.359	1.395E-02 1.059	6.698E-03 0.774	3.918E-03 0.805	2.243E-03
$\varepsilon = 2^{-12}$	7.860E-02 1.128	3.596E-02 1.358	1.403E-02 0.997	7.026E-03 0.770	4.119E-03 0.806	2.356E-03
$\varepsilon = 2^{-14}$	7.865E-02 1.127	3.600E-02 1.358	1.405E-02 0.960	7.219E-03 0.769	4.237E-03 0.806	2.423E-03
$\varepsilon = 2^{-16}$	7.866E-02 1.127	3.601E-02 1.358	1.405E-02 0.940	7.326E-03 0.768	4.302E-03 0.806	2.460E-03
	⋮	⋮	⋮	⋮	⋮	⋮
$\varepsilon = 2^{-28}$	7.866E-02 1.127	3.601E-02 1.358	1.405E-02 0.918	7.437E-03 0.767	4.371E-03 0.805	2.502E-03
$\varepsilon = 2^{-30}$	7.866E-02 1.127	3.601E-02 1.358	1.405E-02 0.917	7.440E-03 0.770	4.364E-03 0.800	2.506E-03
$D^{N,M}$ $Q^{N,M}$	7.961E-02 1.145	3.601E-02 1.358	1.405E-02 0.917	7.441E-03 0.767	4.371E-03 0.801	2.510E-03

Example 1. Consider the following example

$$-\varepsilon\Delta u + (1+x+y)u_x + (2-xy)u = f, \quad (x,y) \in \Omega := (0,1)^2,$$
$$u(x,y) = 0, \quad (x,y) \in \partial\Omega := \bar{\Omega} \setminus \Omega,$$

where

$$f(x,y) = \begin{cases} 12x\tanh((x-0.5)/\varepsilon), & \text{if } 0 < x < 0.5, \\ [(x-1)^2 + 4y(1-y)]\tanh((x-0.5)/\varepsilon), & \text{if } 0.5 < x < 1. \end{cases}$$

In this example, the forcing term f has been designed so that the minimal level of compatibility holds at the four corners. That is, $f(\ell, m) = 0$ for $\ell, m = 0, 1$. However, this example violates most of the constraints on f assumed in the formulation of problem (3). In particular, the constraint (3f) is not satisfied at the boundary points $(0.5, \ell), \ell = 0, 1$ as

Table 2
Example 2: Maximum two-mesh global differences and orders of convergence, using the numerical method (30).

	N=M=32	N=M=64	N=M=128	N=M=256	N=M=512	N=M=1024
$\varepsilon=2^{0}$	2.006E-03 1.140	9.100E-04 1.088	4.281E-04 0.999	2.142E-04 0.999	1.072E-04 1.000	5.360E-05
$\varepsilon=2^{-2}$	1.207E-02 0.941	6.284E-03 0.969	3.211E-03 0.984	1.623E-03 0.992	8.163E-04 0.996	4.094E-04
$\varepsilon=2^{-4}$	1.721E-02 0.956	8.871E-03 0.973	4.518E-03 0.986	2.281E-03 0.993	1.146E-03 0.996	5.746E-04
$\varepsilon=2^{-6}$	3.360E-02 1.040	1.633E-02 0.792	9.437E-03 0.702	5.801E-03 0.756	3.435E-03 0.784	1.994E-03
$\varepsilon=2^{-8}$	8.013E-02 1.197	3.495E-02 1.752	1.038E-02 0.744	6.196E-03 0.771	3.630E-03 0.797	2.090E-03
$\varepsilon=2^{-10}$	7.834E-02 1.127	3.587E-02 1.359	1.399E-02 1.059	6.713E-03 0.774	3.927E-03 0.805	2.248E-03
$\varepsilon=2^{-12}$	7.846E-02 1.125	3.596E-02 1.358	1.403E-02 0.997	7.029E-03 0.770	4.122E-03 0.806	2.357E-03
$\varepsilon=2^{-14}$	7.848E-02 1.125	3.598E-02 1.357	1.405E-02 0.960	7.219E-03 0.769	4.237E-03 0.806	2.423E-03
$\varepsilon=2^{-16}$	7.848E-02 1.125	3.599E-02 1.357	1.405E-02 0.939	7.326E-03 0.768	4.302E-03 0.806	2.460E-03
	⋮	⋮	⋮	⋮	⋮	⋮
$\varepsilon=2^{-28}$	7.848E-02 1.125	3.599E-02 1.357	1.405E-02 0.918	7.438E-03 0.767	4.371E-03 0.806	2.501E-03
$\varepsilon=2^{-30}$	7.848E-02 1.125	3.599E-02 1.357	1.405E-02 0.916	7.444E-03 0.768	4.370E-03 0.803	2.506E-03
$D^{N,M}$ $Q^{N,M}$	8.013E-02 1.155	3.599E-02 1.357	1.405E-02 0.916	7.444E-03 0.768	4.372E-03 0.803	2.506E-03

$$F_0(0.5^-,y)=6\neq F_0(0.5^+,y)=0.25+4y(1-y).$$

The computed solution with the numerical scheme (27) on the Shishkin mesh $\bar{\Omega}^{N,M}$ and the two-mesh global differences for $\varepsilon=2^{-12}$ and $N=M=64$ are shown in Fig. 1. The maximum two-mesh global differences and the orders of convergence for $N=M=32,64,\dots,1024$ and $\varepsilon\in S_\varepsilon$ are given in Table 1. Observe that the numerical scheme (27) converges with almost first order.

Example 2. Consider the following example

$$-\varepsilon\Delta\tilde{u}+(1+x+y)\tilde{u}_x+(2-xy)\tilde{u}=f,\quad (x,y)\in\Omega_x:=(0,1)^2\setminus\{x=0.5\},$$
$$[\tilde{u}_x](0.5,y)=0,\quad 0<y<1,\quad \tilde{u}(x,y)=0,\quad (x,y)\in\partial\Omega,$$

where

$$f(x,y)=\begin{cases}-12x, & \text{if } 0<x<0.5,\\ (x-1)^2+4y(1-y), & \text{if } 0.5<x<1.\end{cases}$$

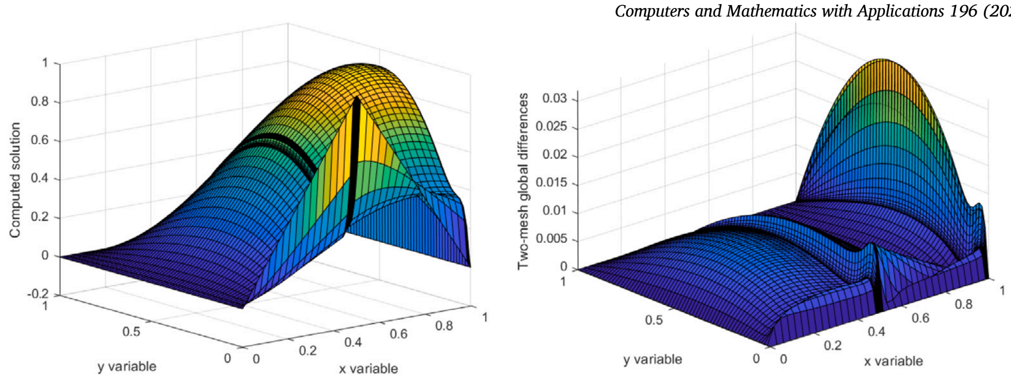
Note that f is discontinuous along $x=0.5$, but the basic compatibility constraints at the corners are satisfied as $f(\ell,m)=0$, for $\ell,m=0,1$. Using the numerical method (30), the maximum two-mesh global differences and the corresponding orders of convergence are given in Table 2 and they suggest that the numerical approximations generated from (30), converge with almost first order.

Example 1 can be considered as a regularization of Example 2. Observe that $\|u-\tilde{u}\|=\|z-\tilde{z}\|\le\|z\|+\|\tilde{z}\|\le C\varepsilon$. Hence, the numerical approximation to Example 1 can be used as an approximation to the solution of Example 2, especially when $\varepsilon\le CN^{-1}$.

In both examples, according to [7, Table 8.4, p.169], the convergence rates appear to be $N^{-1}\ln N$, which are slightly better than those in Theorems 1 and 2.

Example 3. Consider the following problem

$$-\varepsilon\Delta v+(1+x+y)v_x+(2-xy)v=64x(1-x)y(1-y),\quad (x,y)\in\Omega=(0,1)^2,$$
$$v(x,y)=0,\quad (x,y)\in\partial\Omega\setminus\{y=0\},$$



(a) Computed solution (b) Two-mesh global differences

Fig. 2. Example 3: Computed approximation and two-mesh global differences for $\varepsilon = 2^{-8}$ and $N = M = 64$.

Table 3
Example 3: Maximum two-mesh global differences and orders of convergence.

	N = M = 32	N = M = 64	N = M = 128	N = M = 256	N = M = 512	N = M = 1024
$\varepsilon = 2^{0}$	1.999E-02 1.016	9.882E-03 1.014	4.895E-03 1.009	2.432E-03 1.005	1.212E-03 1.003	6.046E-04
$\varepsilon = 2^{-2}$	3.888E-02 0.972	1.982E-02 0.986	1.000E-02 0.993	5.026E-03 0.997	2.519E-03 0.998	1.261E-03
$\varepsilon = 2^{-4}$	8.357E-02 0.975	4.253E-02 0.988	2.145E-02 0.994	1.077E-02 0.997	5.396E-03 0.998	2.701E-03
$\varepsilon = 2^{-6}$	7.563E-02 0.885	4.097E-02 0.769	2.404E-02 0.738	1.441E-02 0.834	8.081E-03 0.828	4.551E-03
$\varepsilon = 2^{-8}$	7.951E-02 0.816	4.516E-02 0.734	2.715E-02 0.731	1.636E-02 0.827	9.222E-03 0.828	5.193E-03
$\varepsilon = 2^{-10}$	8.088E-02 0.789	4.681E-02 0.728	2.826E-02 0.741	1.691E-02 0.823	9.560E-03 0.827	5.389E-03
$\varepsilon = 2^{-12}$	8.133E-02 0.782	4.729E-02 0.727	2.857E-02 0.743	1.707E-02 0.823	9.649E-03 0.827	5.441E-03
$\varepsilon = 2^{-14}$	8.150E-02 0.781	4.745E-02 0.727	2.866E-02 0.744	1.711E-02 0.823	9.673E-03 0.827	5.454E-03
$\varepsilon = 2^{-16}$	8.157E-02 0.780	4.750E-02 0.727	2.869E-02 0.745	1.712E-02 0.823	9.680E-03 0.827	5.458E-03
	⋮	⋮	⋮	⋮	⋮	⋮
$\varepsilon = 2^{-28}$	8.163E-02 0.780	4.754E-02 0.728	2.871E-02 0.745	1.712E-02 0.822	9.684E-03 0.827	5.460E-03
$\varepsilon = 2^{-30}$	8.163E-02 0.780	4.754E-02 0.728	2.870E-02 0.745	1.713E-02 0.824	9.674E-03 0.824	5.465E-03
$D^{N,M}$ $Q^{N,M}$	8.357E-02 0.727	5.048E-02 0.815	2.871E-02 0.745	1.713E-02 0.823	9.684E-03 0.824	5.471E-03

$$v(x,0) = \varphi(x), \quad 0 < x < 1,$$

where

$$\varphi(x) = \begin{cases} 2x, & \text{if } 0 < x < 0.5, \\ 2(1-x), & \text{if } 0.5 < x < 1. \end{cases}$$

Note that in this example f is smooth, but $\varphi'(x)$ is discontinuous at $x = 1/2$. The computed solution $U + \varphi(x)(1-y)$ from the numerical method (30) and the two-mesh global differences for $\varepsilon = 2^{-12}$ and $N = M = 64$ are shown in Fig. 2. The maximum two-mesh global differences and the orders of convergence for $N = M = 32, 64, \ldots, 1024$ and $\varepsilon \in S$ are given in Table 3. Observe that the numerical approximations converge with almost first order.

5. Conclusions

A parameter-uniform numerical method is constructed for a singularly perturbed convection-diffusion problem with a discontinuous inhomogeneous term. By assuming a certain level of regularity of the continuous solution, error bounds are established for this numerical method. By considering a regularization of the problem, one can avoid any regularity assumption and the same error bounds are established for a slightly modified numerical method.

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Appendix A. Bounds on the derivatives of the solution of (1), when f is smooth

We recall in this appendix the behavior of the solution of problem (1) when f is a smooth function without layers. If f is sufficiently smooth ($f \in C^{1,\lambda}(\bar{\Omega})$) and satisfies sufficient compatibility conditions at the four corners (see [18]) then $u \in C^{3,\lambda}(\bar{\Omega})$. Assuming the additional regularity of $f \in C^{5,\lambda}(\bar{\Omega})$ and additional compatibility conditions on the four corners [18], the solution u can be decomposed into a sum of a regular component $v \in C^{3,\lambda}(\bar{\Omega})$, and several layer components (all in the space $C^{3,\lambda}(\bar{\Omega})$)

$$u(x,y) = (v + w_E + w_S + w_{ES} + w_N + w_{EN})(x,y); \tag{A.1}$$

such that $Lv = f, Lw = 0$. The regular boundary layer w_E is significant along the east boundary $\partial\Omega_E := \{(1,y) | 0 \le y \le 1\}$. The characteristic boundary layers w_N and w_S occur, respectively along the north boundary $\partial\Omega_N := \{(x,1) | 0 \le x \le 1\}$ and south boundary $\partial\Omega_S := \{(x,0) | 0 \le x \le 1\}$. The corner layer functions w_{ES} and w_{EN} appear near the outflow corners $(1,0)$ and $(1,1)$.

Moreover, for $f \in C^{5,\lambda}(\bar{\Omega})$, the following bounds on the derivatives of these components have been established [18,10]

$$\left\| \frac{\partial^{i+j} v}{\partial x^i \partial y^j} \right\| \le C\left(1 + \varepsilon^{2-(i+j)}\right), \quad 0 \le i + j \le 3, \tag{A.2a}$$

$$|w_E(x,y)| \le Ce^{-\alpha\frac{1-x}{\varepsilon}}, \quad \left| \frac{\partial^i w_E(x,y)}{\partial x^i} \right| \le C\varepsilon^{-i} e^{-\alpha\frac{1-x}{\varepsilon}}, \ 1 \le i \le 3; \tag{A.2b}$$

$$\left| \frac{\partial^j w_E(x,y)}{\partial y^j} \right| \le C\varepsilon^{1-j} e^{-\alpha\frac{1-x}{\varepsilon}}, \ j = 2,3, \tag{A.2c}$$

$$|w_S(x,y)| \le Ce^{-\frac{y}{\sqrt{\varepsilon}}}, \ |w_N(x,y)| \le Ce^{-\frac{1-y}{\sqrt{\varepsilon}}}, \tag{A.2d}$$

$$\left| \frac{\partial^j w_S(x,y)}{\partial y^j} \right| \le C\varepsilon^{-j/2} e^{-\frac{y}{\sqrt{\varepsilon}}}, \ \left| \frac{\partial^j w_N(x,y)}{\partial y^j} \right| \le C\varepsilon^{-j/2} e^{-\frac{1-y}{\sqrt{\varepsilon}}}, \ j = 2,3, \tag{A.2e}$$

$$\left| \frac{\partial^i w_S(x,y)}{\partial x^i} \right| \le C\varepsilon^{2-i} e^{-\frac{y}{\sqrt{\varepsilon}}}, \ \left| \frac{\partial^i w_N(x,y)}{\partial x^i} \right| \le C\varepsilon^{2-i} e^{-\frac{1-y}{\sqrt{\varepsilon}}}, \ i = 2,3, \tag{A.2f}$$

and, for the corner layer functions,

$$|w_{ES}(x,y)| \le Ce^{-\alpha\frac{1-x}{\varepsilon}} e^{-\frac{y}{\sqrt{\varepsilon}}}, \ |w_{EN}(x,y)| \le Ce^{-\alpha\frac{1-x}{\varepsilon}} e^{-\frac{1-y}{\sqrt{\varepsilon}}}, \tag{A.2g}$$

$$\left\| \frac{\partial^3 w_{ES}}{\partial y^3} \right\|, \left\| \frac{\partial^3 w_{EN}}{\partial y^3} \right\| \le C\varepsilon^{-2}; \tag{A.2h}$$

$$\left\| \frac{\partial^i w_{ES}}{\partial x^i} \right\|, \left\| \frac{\partial^i w_{EN}}{\partial x^i} \right\| \le C\varepsilon^{-i}, \ i = 2,3. \tag{A.2i}$$

Appendix B. Proof of Theorem 2

As for the continuous solution $\tilde{u}$ of (4), the discrete solution $\tilde{U}$ of (30) is decomposed as follows

$$\tilde{U} = \tilde{V} + \tilde{W}_E + \tilde{W}_N + \tilde{W}_S + \tilde{W}_{NE} + \tilde{W}_{SE} + \tilde{Z}.$$

Except for the error $\|\tilde{z} - \tilde{Z}\|$ in approximating the interior layer component, the bounding of the error in each of the other subcomponents follows the argument in [18], with minor modifications in the truncation errors along $x_i = d$ and in the corresponding discrete barrier functions.

The discrete regular component satisfies

$$L^{N,M}\tilde{V} = f,\ (x_i, y_j) \in \Omega_L^{N,M},\ x_i \neq d,$$
$$(D_x^+ - D_x^-)\tilde{V}(d, y_j) = [v_x](d, y_j), \quad 0 < y_j < 1,$$
$$\tilde{V}(x_i, y_j) = v(x_i, y_j), \quad (x_i, y_j) \in \partial\Omega_L^{N,M}.$$

From (8), the truncation error away from $x_i = d$ satisfies

$$|(L^{N,M} - L)v(x_i, y_j)| \le CN^{-1}, \quad x_i \neq d,$$

and at $x_i = d$ we have

$$|(D_x^{\pm} v - v_x)(d^{\pm}, y_j)| \le CN^{-1}\|v_{xx}(\cdot, y_j)\|_{(d-h,d)\cup(d,d+h)} \le CN^{-1}.$$

Hence, the truncation error along $x_i = d$ is bounded by

$$|(D_x^+ - D_x^-)(\tilde{V} - v)(d, y_j)| \le |[v_x](d, y_j) - (D_x^+ - D_x^-)v(d, y_j)| \le CN^{-1}.$$

Consider the barrier function

$$B_1(x) := \begin{cases} \frac{x}{\alpha}, & 0 \le x \le d, \\ \frac{x-d}{2\alpha} + \frac{d}{\alpha}, & 0 \le x \le d, \end{cases}$$

which satisfies

$$(D_x^+ - D_x^-)B_1(d) = -\frac{1}{2\alpha}.$$

Therefore, using Lemma 4, we have the first error bound

$$|(\tilde{V} - v)(x_i, y_j)| \le CN^{-1}B_1(x_i) \le CN^{-1}. \tag{B.1}$$

Next consider the discrete approximation to the regular boundary layer,

$$L^{N,M}\tilde{W}_E = 0, \qquad (x_i, y_j) \in \Omega_L^{N,M},\ x_i \neq d,$$
$$(D_x^+ - D_x^-)\tilde{W}_E(d, y_j) = 0, \quad 0 < y_j < 1,$$
$$\tilde{W}_E(x_i, y_j) = w_E(x_i, y_j), \quad (x_i, y_j) \in \partial\Omega_L^{N,M}.$$

Define the following discrete barrier function

$$-\varepsilon\delta_x^2 B_2 + \alpha D_x^- B_2 = 0, \quad x_i \in (0, d) \cup (d, 1),$$
$$B_2(0) = 0, \quad B_2(d) = N^{-1}, \quad B_2(1) = 1.$$

Note that $\varepsilon D_x^- B_2(d) \ge \frac{\alpha}{2} B_2(d) > 0$, $\varepsilon D_x^+ B_2(d) \le 0$ and, hence,

$$(D_x^+ B_2 - D_x^- B_2)(d) \le 0.$$

Hence,

$$|\tilde{W}_E(x_i, y_j)| \le CB_2(x_i) \quad \text{and} \quad |B_2(1 - \sigma_x)| \le CN^{-1}.$$

Using the same argument as in [18, p.1770], we obtain the second error bound

$$|(\tilde{W}_E - w_E)(x_i, y_j)| \le CN^{-1}\ln N. \tag{B.2}$$

Consider now the discrete approximation to the characteristic boundary layer along $y = 1$, which satisfies:

$$L^{N,M}\tilde{W}_N = 0, \qquad (x_i, y_j) \in \Omega_L^{N,M},\ x_i \neq d,$$
$$(D_x^+ - D_x^-)\tilde{W}_N(d, y_j) = 0, \quad 0 < y_j < 1,$$
$$\tilde{W}_N(x_i, y_j) = w_N(x_i, y_j), \quad (x_i, y_j) \in \partial\Omega_L^{N,M}.$$

Similarly to [18, p.1770], we define the discrete barrier function $\Upsilon(x_i)\Phi(y_j)$, where

$$-\varepsilon\delta_y^2\Phi(y_j) + \Phi(y_j) = 0, \quad 0 < y_j < 1, \quad \Phi(0) = 0,\ \Phi(1) = 1;$$

and

$$\Upsilon(x_i) := \begin{cases} \prod_{j=1}^{i}\left(1+\frac{4h_j}{\alpha}\right), & 1 \le i \le N/2, \\ \prod_{j=1}^{N/2}\left(1+\frac{4h_j}{\alpha}\right)\prod_{j=N/2}^{i}\left(1+\frac{2h_j}{\alpha}\right), & N/2 < i \le N. \end{cases}$$

Noting that $L^{N,M}\Upsilon(x_i)\Phi(y_j) \ge 0$ and $(D_x^+ - D_x^-)(\Upsilon(x_i)\Phi(y_j)) \le 0$, the maximum principle proves

$$|\tilde{W}_N(x_i, y_j)| \le C\Upsilon(x_i)\Phi(y_j) + CN^{-1}, \quad (x_i, y_j) \in \Omega_L^{N,M},$$

and then $|\tilde{W}_N(x_i, y_j)| \le CN^{-1}$ if $y_j \le 1 - \sigma_y$. In the subdomain $(0,1)\times(1-\sigma_y, 1)$

$$|(L^{N,M} - L)w_N(x_i, y_j)| \le CN^{-1}\ln N, \quad x_i \ne d,$$
$$|(D_x^+ - D_x^-)(\tilde{W}_N - w_N)(d, y_j)| \le CN^{-1}\ln N.$$

Finish as for the regular component, to deduce the third error bound

$$|(\tilde{W}_N - w_N)(x_i, y_j)| \le CN^{-1}\ln N. \tag{B.3}$$

Also use the above function $\Upsilon(x_i)$ in bounding the discrete corner layer $\tilde{W}_{NE}$ and follow the argument in [18] to deduce

$$|(\tilde{W}_{NE} - w_{NE})(x_i, y_j)| \le CN^{-1}\ln N. \tag{B.4}$$

In analogous fashion we can obtain the same bound on $|(\tilde{W}_S - w_S)(x_i, y_j)|$ and $|(\tilde{W}_{SE} - w_{SE})(x_i, y_j)|$.
It remains to bound $\|\tilde{Z} - \tilde{z}\|$, where $\tilde{Z}$ satisfies

$$L^{N,M}\tilde{Z} = 0, \ (x_i, y_j) \in \Omega_L^{N,M}, \ x_i \ne d,$$
$$(D_x^+ - D_x^-)\tilde{Z}(d, y_j) = -[v_x](d, y_j), \quad 0 < y_j < 1,$$
$$\tilde{Z}(x_i, y_j) = 0, \quad (x_i, y_j) \in \partial\Omega_L^{N,M}.$$

Consider the barrier function

$$-\varepsilon\delta_x^2 B_3(x_i) + \alpha D_x^- B_3(x_i) = 0, \ x_i \ne d, \quad B_3(0) = 0, \ B_3(d) = B_3(1) = 1.$$

Note that $\varepsilon D_x^- B_3(d) = \mathcal{O}(1) > 0$ and then

$$|\tilde{Z}(x_i, y_j)| \le C\varepsilon B_3(x_i).$$

Hence, using the bound in (23)

$$|(\tilde{Z} - \tilde{z})(x_i, y_j)| \le C\varepsilon N^{-1}, \quad x_i \le d - \sigma_x.$$

We now obtain bounds on $|(\tilde{Z} - \tilde{z})(x_i, y_j)|$, when $x_i > d - \sigma_x$. The truncation error in this subdomain satisfies

$$|(L^{N,M} - L)\tilde{z}(x_i, y_j)| \le CN^{-1}\ln N, \quad d - \sigma_x < x_i < d,$$
$$\varepsilon|(D_x^+ - D_x^-)(\tilde{Z} - \tilde{z})(d, y_j)| \le CN^{-1}\ln N,$$
$$|(L^{N,M} - L)\tilde{z}(x_i, y_j)| \le CN^{-1}, \quad d < x_i \le 1 - \sigma_x,$$
$$|(L^{N,M} - L)\tilde{z}(x_i, y_j)| \le C\frac{N^{-1}\ln N}{\varepsilon}, \quad 1 - \sigma_x < x_i < 1.$$

Consider the barrier function

$$B_4(x_i) := N^{-1}\ln N \begin{cases} \frac{x_i - (d - \sigma_x)}{\varepsilon}, & d - \sigma_x < x_i \le d, \\ \frac{\sigma_x}{\varepsilon} + \frac{x_i - d}{\varepsilon}, & d < x_i < 1. \end{cases}$$

Then,

$$|(\tilde{Z} - \tilde{z})(x_i, y_j)| \le CB_4(x_i).$$

Hence,

$$|(\tilde{Z} - \tilde{z})(x_i, y_j)| \le CN^{-1}\ln^2 N, \quad d - \sigma_x < x_i \le d.$$

It now remains to bound $|(\tilde{Z} - \tilde{z})(x_i, y_j)|$ in Ω_x^+. Consider the additional decomposition

$$\tilde{z}(x, y) = \tilde{z}(d, y) + \tilde{z}_1(x, y), \quad x > d,$$

where

$$L\tilde{z}_1(x,y) = \varepsilon \frac{\partial^2 \tilde{z}}{\partial y^2}(d,y), \quad (x,y) \in \Omega_x^+,$$
$$\tilde{z}_1(1,y) = -\tilde{z}(d,y),\ 0<y<1, \quad \tilde{z}_1(x,y)=0,\ (x,y)\in \partial\Omega_x^+ \setminus \{x=1\}.$$

From a maximum principle and the bound (23), this function satisfies

$$|\tilde{z}_1(x,y)| \le C\varepsilon(x-d).$$

Define the discrete function

$$\tilde{Z}_1(x_i,y_j) = \tilde{Z}(x_i,y_j) - \tilde{z}(d,y_j),$$

which satisfies $L^{N,M}\tilde{Z}_1 = \varepsilon\delta_y^2 \tilde{z}(d,y_j)$ and the bound

$$|\tilde{Z}_1(x_i,y_j)| \le C\prod_{l=i}^{N}\left(1+\alpha\frac{h_l}{\varepsilon}\right)^{l-N} + CN^{-1}\ln N + C\varepsilon(x_i-d), \quad x_i \ge d.$$

Using this estimate, one has

$$|(\tilde{Z}-\tilde{z})(x_i,y_j)| \le CN^{-1}\ln N + C\varepsilon, \quad x_i \le 1-\sigma_x.$$

Using the truncation error bound and the barrier function $\varepsilon^{-1}(x_i-(1-\sigma_x))$ we can establish that

$$|(\tilde{Z}-\tilde{z})(x_i,y_j)| \le CN^{-1}\ln N + C\varepsilon, \quad d \le x_i \le 1.$$

We will use this error bound in the case where $\varepsilon \le CN^{-1}$.

Let us now examine the other case of $N^{-1} \le C\varepsilon$. Let us return to the truncation error bounds on $|(L^{N,M}-L)\tilde{z}(x_i,y_j)|$. In the subregion where $d < x_i \le 1-\sigma_x$, using the bounds (24), we have

$$|(L^{N,M}-L)\tilde{z}(x_i,y_j)| \le \frac{C}{\varepsilon}e^{-\alpha\frac{(1-x_{i+1})}{\varepsilon}} + CN^{-1} \le \frac{CN^{-1}}{\varepsilon}e^{-\alpha\frac{(1-\sigma_x-x_{i-1})}{\varepsilon}} + CN^{-1},$$

and, in the boundary layer region $1-\sigma_x < x_i < 1$, we have

$$|(L^{N,M}-L)\tilde{z}(x_i,y_j)| \le C\frac{N^{-1}\ln N}{\varepsilon}e^{-\alpha\frac{1-x_i}{\varepsilon}} + CN^{-1}\ln N.$$

Our final barrier function is

$$B_5(x_i) := \begin{cases} \left(1+\frac{\alpha H}{\varepsilon}\right)^{i-3N/4}, & d < x_i \le 1-\sigma_x, \\ 1+\left(1+\frac{\alpha h}{\varepsilon}\right)^{i-N} - \left(1+\frac{\alpha h}{\varepsilon}\right)^{-\frac{N}{4}}, & 1-\sigma_x < x_i \le 1, \end{cases}$$

where H and h denote the mesh sizes of the coarse and fine meshes, respectively. Note that

$$L^{N,M}B_5(x_i) \ge \frac{C}{\varepsilon}\left(1+\frac{\alpha H}{\varepsilon}\right)^{i-1-3N/4} \ge \frac{C}{\varepsilon}e^{-\alpha\frac{(1-\sigma_x-x_{i-1})}{\varepsilon}}, \quad d<x_i<1-\sigma_x,$$
$$\varepsilon(D_x^+-D_x^-)B_5(1-\sigma_x) \le C\alpha N^{-1} - \alpha\frac{1}{1+\alpha H/\varepsilon} < 0, \quad \text{if} \quad \alpha H < \varepsilon,$$
$$L^{N,M}B_5(x_i) \ge \frac{C}{\varepsilon}\left(1+\frac{\alpha h}{\varepsilon}\right)^{i-1-N} \ge C\varepsilon^{-1}e^{-\alpha\frac{(1-x_{i-1})}{\varepsilon}}, \quad x_i > 1-\sigma_x.$$

Hence in the case of $N^{-1} \le C\varepsilon$, we have

$$|(\tilde{z}-\tilde{Z})(x_i,y_j)| \le CN^{-1}\ln N\, B_5(x_i) + Cx_iN^{-1}\ln N \le CN^{-1}\ln N, \quad d \le x_i \le 1.$$

This completes the proof of the nodal error bound

$$|(\tilde{z}-\tilde{Z})(x_i,y_j)| \le CN^{-1}\ln N, \quad (x_i,y_j)\in\Omega_L^{N,M}. \tag{B.5}$$

Collecting together the error bounds (B.1), (B.2), (B.3), (B.4) and (B.5), we have

$$|(\tilde{u}-\tilde{U})(x_i,y_j)| \le CN^{-1}(\ln N)^2, \quad (x_i,y_j)\in\Omega_L^{N,M}.$$

This nodal error bound extends to a global error bound as in Theorem 1.

Data availability

No data was used for the research described in the article.

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