Zero sum sign-central matrices and applications

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Abstract

A matrix with a nonzero nonnegative vector in its null space is called *central*. We study classes of central matrices having zero column sums. The study is motivated by an engineering application concerning induction heating where central matrices provide a way to control the energy flow over time. A (± 1)-matrix A is called a *ZSC-matrix* (zero sum sign-central) if each matrix with the same sign pattern as A and having zero column sums is central. We establish several classes of ZSC-matrices, and give separate sufficient and necessary conditions for a matrix to be ZSC. Moreover, we give algorithms for finding central matrices that are used for power control in induction heating, and illustrate these by some numerical examples.

Key words. Qualitative matrix theory, central matrices, engineering power control.

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1 Introduction

Let $A = [a_{ij}]$ be an $m \times n$ matrix. A is called *central* if there is a nonzero nonnegative vector in its null space. The row space of A is denoted by Row A and its null space is denoted by Nul A. Let csum A denote the column sum vector of A, so its jth component is $\sum_{i=1}^{m} a_{ij}$ ($j \leq n$). Similarly, rsum A is the row sum vector of A. The sign matrix of a matrix A, denoted sign A, is the matrix obtained by replacing each entry in A by its sign ($+$, $-$ or 0). The qualitative class $\mathcal{Q}(A)$ of A consists of all matrices with the same sign matrix as A. A matrix A is called *sign-central* if each matrix in the qualitative class of A is central. Sign-central matrices were introduced and studied in [1], and a characterization was found. This matrix class is also treated in [2] along with qualitative matrix theory in general. The related notion of strict sign-central matrices was studied in [5]. For related combinatorial properties of matrices, we refer to [6, 7].

Throughout the paper, we are concerned with central matrices, and a new notion, closely related to sign-centrality. We call a (± 1)-matrix A a *ZSC-matrix* (zero sum sign-central) if each matrix with the same sign pattern as A and having zero column sums is central. The investigation is motivated by an engineering problem where one wants to perform efficient power control in induction heating. It turns out that central matrices with zero column sums play an important role for such control. We outline this application in Section 4. Since there is a great uncertainty in the quantitative data of this application, qualitative matrix theory is used as a key tool for finding central matrices to perform the control. Qualitative matrix theory is rooted in economics, and it is interesting that it also plays a role in an important industrial engineering application, as discussed in this paper.

The paper is organized as follows. In Section 2 central matrices are characterized and a connection to the reduced echelon form of matrix is discussed. Section 3 contains an analysis of ZSC matrices. It gives separate sufficient and necessary conditions for a matrix to be ZSC. Several classes of ZSC matrices are established. These matrix classes have a combinatorial structure. In Section 4 we describe the engineering application, and algorithms based on results from Section 3 for power control. Some numerical examples are also given.

We treat vectors in $\mathbb{R}^n$ as column vectors and identify these with corresponding n-tuples. $M_{m,n}$ denotes the set of all real $m \times n$ matrices, and when $m = n$ we just write M_n. O denotes the zero matrix or vector. I_n (or just I)

is the identity matrix of order n. We let e_i denote the ith unit (coordinate) vector in $\mathbb{R}^n$, and e denotes the all ones vector. A vector $x = (x_1, x_2, \ldots, x_n)$ is *nonnegative* if each component x_i is nonnegative. Similarly, we define *positive*, *negative*, and *nonpositive* vectors. The kth smallest component of the vector x is denoted by $x_{(k)}$.

2 Central matrices

We first study the class of central matrices. A characterization of this class may be derived using separation of convex sets ([9]), or by Farkas' lemma/duality. Note that if a matrix contains a zero column, then it is trivially central. Thus, in this section we restrict the attention to matrices with no zero column (unless otherwise stated).

Theorem 1 *Let $A \in M_{m,n}$. Then A is a central matrix if and only if the row space* $\mathrm{Row}\, A$ *does not contain any positive vector.*

Proof. Assume that A is central, and let z be a nonzero nonnegative vector in $\mathrm{Nul}\, A$, so $Az = O$. Then, for any $w \in \mathbb{R}^m$, $0 = w^T(Az) = (w^TA)z$. But this implies, as z is nonzero and nonnegative, that the vector w^TA cannot be positive. Thus, the row space of A does not contain any positive vector.

Conversely, assume that $\mathrm{Row}\, A$ does not contain any positive vector. Then $\mathrm{Row}\, A$ and the positive orthant $P_+ = \{x \in \mathbb{R}^n : x = (x_1, x_2, \ldots, x_n),\ x_i > 0\ (i \leq n)\}$ are disjoint convex sets. It follows from a separation theorem in convex analysis ([9]) that there exists a nonzero $y \in \mathbb{R}^n$ such that $y^Tx \leq 0$ for all $x \in \mathrm{Row}\, A$, and $y^Tx > 0$ for all $x \in P_+$. Actually, since $\mathrm{Row}\, A$ is a subspace, it follows that

$$y^Tx = 0 \ \text{ for all } x \in \mathrm{Row}\, A, \text{ and } \ y^Tx > 0 \ \text{ for all } x \in P_+.$$

Thus, $y \in \mathrm{Nul}\, A$. Moreover, y must be nonnegative, otherwise we could find a positive vector x with $y^Tx < 0$ (by letting $x_j = 1$ for some j where $y_j < 0$, and x_k very small otherwise). So y is a nonzero nonnegative vector in $\mathrm{Nul}\, A$, and therefore A is central. □

The next theorem gives a sufficient condition for centrality, and shows how this condition may be checked efficiently by the row reduction algorithm. We let $\mathrm{rref}\, A$ denote the (row) reduced echelon form of a matrix A.

Theorem 2 *Let $A \in M_{m,n}$, and let $R = \text{rref}\, A$. If R contains a nonpositive column, then A is central, and one can find a nonzero nonnegative vector in* $\text{Nul}\, A$ *directly from that column in matrix R.*

Proof. Since A has no zero column the reduced echelon form R has the following structure

$$R = \begin{bmatrix} 1 & \dots & 0 & \dots & & 0 & \dots \\ 0 & \dots & 1 & \dots & & 0 & \dots \\ & & & & \ddots & & \\ 0 & \dots & 0 & \dots & & 1 & \dots \\ 0 & \dots & 0 & \dots & & 0 & \dots \end{bmatrix}$$

with (possible) zero rows as the last rows. Let the pivots (leading ones) be in columns $j_1, j_2, \dots, j_k$, and define $J = \{j_1, j_2, \dots, j_k\}$. So, k is the rank of A. The entries in non-pivot columns, and to the right of some pivot can be any numbers. Let

$$R = \begin{bmatrix} R_1 & R_2 & \cdots & R_n \end{bmatrix}$$

be the column partition of R. By assumption, there exists a nonpivot column R_j of R which is nonpositive. Then $j \notin J$, so there is a unique s such that $j_s < j < j_{s+1}$ (where we define $j_{k+1} = n + 1$). Let r_{ij} be the entry in row i of R_j. Since R_{j_i} equals the ith unit vector for $i \leq k$, we see that R_j can be written as a linear combination of the pivot columns in R

$$R_j = \sum_{i=1}^{s} r_{ij} R_{j_i}$$

or, equivalently,

$$R_j + \sum_{i=1}^{s} (-r_{ij}) R_{j_i} = O.$$

So, $Rx = O$ where x is the vector whose j_ith component is $-r_{ij}$ ($i \leq s$), $x_j = 1$, and $x_p = 0$ otherwise. Then x is nonzero and nonnegative (as column R_j was nonpositive), and $x \in \text{Nul}\, R = \text{Nul}\, A$. Therefore A is a central matrix, as desired. □

The condition in the theorem is sufficient for centrality. It is not necessary, however. For instance, consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \end{bmatrix}.$$

Then $R = \operatorname{rref} A = A$. A is central as $x = (0, 0, 1, 1)$ is a nonzero nonnegative vector in $\operatorname{Nul} A$. But R has no column which is nonpositive.

An efficient computational approach for testing if a given matrix A is central is via linear optimization (linear programming, [8]). Consider the linear program

$$\sup\{\sum_{j=1}^{n} x_j : Ax = O,\ x \geq O,\ x \in \mathbb{R}^n\}.$$

Then A is central if and only if this supremum is positive (and then it is $+\infty$). Adding upper bounds $x_j \leq 1$ ($j \leq n$) works just as well here (or add the equation $\sum_j x_j = 1$ in the constraints, and put 0 in the objective function).

An algorithmic idea based on Theorem 2 is described next. Later we will use this idea in the mentioned power control application. Let A be a matrix which is "of interest" (e.g., a possible power control). Assume we do not know if it is central, so we compute $R = \operatorname{rref} A$. If the criterion of the theorem holds, the matrix is central and we stop. Otherwise, we choose a non-pivot column which is close to being nonpositive and modify it so that it becomes nonpositive (and nonzero). Let R^* be the resulting matrix. Then define

$$A^* = M^{-1} R^*$$

where M is the invertible matrix found during the row reduction algorithm so that $MA = R$. (This matrix is not unique, but it may be found as in the LU-factorization of the matrix). Then $\operatorname{rref} A^* = R^*$ so A^* is central, and it is close to A in some sense.

The following theorem deals with square $n \times n$ matrices of rank $n-1$. This is of interest in connection with certain matrices whose column sums are all zero. (We here allow a possible zero column in the matrix.)

Theorem 3 *Let $A \in M_n$, and assume* $\operatorname{rank} A = n - 1$. *Then A is central if and only if the unique nonpivot column in* $\operatorname{rref} A$ *is nonpositive.*

Proof. Let $R = \operatorname{rref} A$. It follows from the assumption that A has a unique column which is a nonpivot column, say this is column j. We can then explicitly describe the null space of A (and that of R) as follows.

Case 1. $j = 1$. Since the first column is the nonpivot column, that column must be zero. Then $\operatorname{Nul} A$ consists of vectors $x = (x_1, x_2, \ldots, x_n)$

where x_1 is arbitrary, and $x_i = 0$ $(2 \le i \le n)$. Then A is central, and $x = e_1 = (1, 0, \ldots, 0)$ is a nonzero nonnegative vector in the null space.

Case 2. $j > 1$. Suppose r_{ij} is the ith component of column j of R, where $r_{ij} = 0$ for $i \ge j$ (by the echelon structure). Then $\mathrm{Nul}\, A$ consists of the vectors $x = (x_1, x_2, \ldots, x_n)$ with

$$x_i = -r_{ij} x_j \ (i < j), \ \ x_i = 0 \ (i > j), \ \ \text{and } x_j \text{ free.}$$

The only way to obtain a nonzero and nonnegative $x \in \mathrm{Nul}\, A$ is to choose $x_j > 0$ and have $r_{ij} \le 0$ for $i < j$, i.e., the jth column of R is nonpositive. The theorem follows from this discussion. □

Finally, in this section, we make some observations:

- If the row sum vector $\mathrm{rsum}\, A$ of a matrix A is zero, then A is clearly central, as $Ae = O$ where e is the all ones vector. Similarly, if A has a submatrix A' consisting of some of the columns of A, and $\mathrm{rsum}\, A' = O$, then A is central.
- Let P and Q be permutation matrices of order m and n, and let $A \in M_{m,n}$. The A is central if and only if PAQ is central. Thus, permuting rows and/or columns of A does not affect the centrality property.

3 Zero sum sign-central matrices

We now turn to ZSC-matrices. For a matrix A, its sign matrix, denoted $\mathrm{sign}\, A$, is obtained by replacing each entry in A by its sign, $+$, $-$ or 0.

Let $n \ge 1$ and define the sign matrix

$$S^{(n)} = \begin{bmatrix} + & - & - & \cdots & - \\ - & + & - & \cdots & - \\ - & - & + & \cdots & - \\ \vdots & & & \ddots & \vdots \\ - & - & - & \cdots & + \end{bmatrix} \tag{1}$$

so diagonal entries are $+$ and off-diagonal entries are $-$.

Theorem 4 *For each n the matrix $S^{(n)}$ is ZSC. Moreover, if a matrix A satisfies* $\operatorname{sign} A = S^{(n)}$ *and* $\operatorname{csum} A = O$*, then it is central,* $\operatorname{rank} A = n-1$ *and*

$$\operatorname{rref} A = \left[\begin{array}{c|c} I_{n-1} & v \\ \hline O & 0 \end{array}\right] \tag{2}$$

where $v \in \mathbb{R}^{n-1}$ is a negative vector.

Proof. We use row reduction on A. First we add suitable multiples of the first row to each of the other rows, in order to obtain zeros below entry $(1,1)$. Let $\tilde{A} = [\tilde{a}_{ij}]$ denote the resulting matrix. Then we have $\tilde{a}_{1j} = a_{1j}$ for $j \leq n$, $\tilde{a}_{i1} = 0$ for $i \geq 2$, and

$$\tilde{a}_{ij} = a_{ij} - a_{1j}\frac{a_{i1}}{a_{11}} \qquad (2 \leq i \leq n,\ 1 \leq j \leq n).$$

So, for $i, j \geq 1$, this gives $\tilde{a}_{ij} < a_{ij}$ as $a_{1j}, a_{i1} < 0$, and $a_{11} > 0$, and, in particular,

$$\tilde{a}_{ij} < a_{ij} < 0 \qquad (i, j \geq 2,\ i \neq j).$$

Let $\tilde{A}_1$ be the submatrix of $\tilde{A}$ obtained by deleting the first row and column. We compute its column sums. Let $j > 1$. Then, using that the column sums in A are zero, we get

$$\begin{aligned} \textstyle\sum_{i=2}^n \tilde{a}_{ij} &= \textstyle\sum_{i=2}^n (a_{ij} - a_{1j}\frac{a_{i1}}{a_{11}}) \\ &= \textstyle\sum_{i=2}^n a_{ij} - \frac{a_{1j}}{a_{11}} \sum_{i=2}^n a_{i1} \\ &= \textstyle\sum_{i=2}^n a_{ij} - \frac{a_{1j}}{a_{11}}(-a_{11}) \\ &= \textstyle\sum_{i=1}^n a_{ij} \\ &= 0. \end{aligned}$$

This also implies that $\tilde{a}_{ii} > 0$ for $i \geq 2$ (since off-diagonal entries are negative). In summary, these row operations have transformed A into $\tilde{A}$ where the submatrix $\tilde{A}_1$ satisfies

$$(*) \quad \operatorname{sign} \tilde{A}_1 = S^{n-1}, \text{ and } \operatorname{csum} \tilde{A}_1 = O.$$

We go on similarly with row operations to produce zeros below the diagonal in column 2, then column 3 and eventually column $n-1$. It follows by induction that property $(*)$ remains for the suitable submatrix throughout this process. Therefore we end up with a matrix $\hat{A} = [\hat{a}_{ij}]$ satisfying (i) $\hat{A}$

has positive entries in the diagonal positions (j,j) for $j \le n-1$, (ii) the last row is zero (as all column sums in A are 0 so A cannot have rank n) and (iii) $\hat{a}_{in} < 0$ for $i < n$. Then we continue the process by adding suitable positive multiples of row $n-1$ to the previous rows: this introduces zeros in column $n-1$ above the diagonal, and does not change anything in the first $n-2$ columns. Moreover, the entries in the last column stay negative (apart from in position (n,n) where the entry is 0). We do the same for columns $n-2$ etc. and the final result is the reduced echelon form as in (2). In particular rank $A = n-1$ and A is central, due to Theorem 2, as the last column of rref A is nonpositive. □

We note that, in Theorem 4, the condition that csum $A = O$ is crucial. In fact, there are matrices $A \in M_n$ such that sign $A = S^{(n)}$, but where A is not central. One can simply choose such A with positive column sums, and then A is not central. This is a consequence of Theorem 1 (as the sum of the row vectors is positive).

Example 1 Let $a_i > 0$ for $i \le 6$. From Theorem 4 the following matrix is central

$$A = \begin{bmatrix} a_1 + a_2 & -a_3 & -a_5 \\ -a_1 & a_3 + a_4 & -a_6 \\ -a_2 & -a_4 & a_5 + a_6 \end{bmatrix}.$$

□

Define the $n \times n$ sign matrix

$$T^{(n)} = \begin{bmatrix} - & + & + & \cdots & + \\ + & - & + & \cdots & + \\ + & + & - & \cdots & + \\ \vdots & & & \ddots & \vdots \\ + & + & + & \cdots & - \end{bmatrix}. \tag{3}$$

Corollary 5 *For each n the matrix $T^{(n)}$ is ZSC.*

Proof. Note that $T^{(n)} = -S^{(n)}$, and that, for any matrix A, Nul A = Nul $(-A)$ and csum $A = O$ if and only if csum $(-A) = O$. The result then follows from Theorem 4. □

We now present a main result of this paper. It describes a large class of ZSC-matrices where the sign pattern has a certain combinatorial structure.

Let $S^{(k,n)}$ denote the (± 1)-matrix of size $n \times \binom{n}{k}$ whose columns are all distinct permutations of the vector

$$(1, 1, \ldots, 1, -1, -1, \ldots, -1)$$

having k leading 1's followed by $(n-k)$ components being -1. The number of such permutations is $\binom{n}{k}$ as it corresponds to all possible subsets of $\{1, 2, \ldots, n\}$ of cardinality k where the 1's are placed. (Recall that permutation of rows and/or columns does not influence the ZSC-property of a matrix, so we do not have to specify the matrix $S^{(k,n)}$ explicitly.) As before, the all ones vector is denoted by e. We also write $u > O$ for a vector u with only positive components.

Theorem 6 *Let $1 \leq k < n$. Then $S^{(k,n)}$ is a ZSC-matrix.*

Proof. Let $A = [a_{ij}]$ be a matrix with $\operatorname{sign} A = \operatorname{sign} S^{(k,n)}$ and with $\operatorname{csum}(A) = O$ (zero column sums). Assume that A is not central; we shall deduce a contradiction from this.

Since A is not central, by Theorem 1, there is a (strictly) positive vector v in the row space of A, so for some $u \in \mathbb{R}^n$

$$u^T A = v^T > O.$$

Let $1 \leq p \leq n$ be such that $u_p = u_{(k)}$, i.e., the kth smallest component of u. Define $I_1 = \{i \leq n : u_i < u_p\}$ and $I_2 = \{i \leq n : u_i > u_p\}$. Then $|I_1| < k$ and $|I_2| \leq n - k$ (and one or both these sets may be empty). Since $\operatorname{sign} A = \operatorname{sign} S^{(k,n)}$, there exists an index j such that the jth column of A satisfies

$$(*) \quad \operatorname{sign} a_{ij} = + \text{ for all } i \in I_1, \text{ and } \operatorname{sign} a_{ij} = - \text{ for all } i \in I_2.$$

Define the vector

$$\hat{u} = u + \sum_{i \in I_1} (u_p - u_i) e_i - \sum_{i \in I_2} (u_i - u_p) e_i$$

where e_i is the ith unit vector. Then $\hat{u} = u_p\, e$, i.e., each component equals u_p. Therefore

$$\hat{u}^T A = u_p\, e^T A = O.$$

But for the jth column $A^{(j)}$ of A we get from $(*)$

$$\hat{u}^T A^{(j)} = u^T A^{(j)} + \sum_{i \in I_1} (u_p - u_i) a_{ij} - \sum_{i \in I_2} (u_i - u_p) a_{ij} \geq u^T A^{(j)} = v_j > 0$$

But, from above, $\hat{u}^T A^{(j)} = 0$, so we have arrived at a contradiction. This proves the theorem. □

Note that this theorem contains Theorem 4 and Corollary 5 as special cases, corresponding to $k = 1$ and $k = n - 1$, respectively. Two other such examples of ZSC matrices are

$$S^{(2,4)} = \begin{bmatrix} + & + & + & - & - & - \\ + & - & - & + & + & - \\ - & + & - & + & - & + \\ - & - & + & - & + & + \end{bmatrix}$$

and

$$S^{(3,5)} = \begin{bmatrix} + & + & + & + & + & + & - & - & - & - \\ + & + & + & - & - & - & + & + & + & - \\ + & - & - & + & + & - & + & + & - & + \\ - & + & - & + & - & + & + & - & + & + \\ - & - & + & - & + & + & - & + & + & + \end{bmatrix}.$$

Another class of ZSC-matrices is introduced next. Let $1 \leq k < n$ and define a (± 1)-matrix S by

$$S = \begin{bmatrix} e^T & -e^T \\ & \\ PS^{(k,n)} & QS^{(k,n)} \end{bmatrix} \tag{4}$$

where P and Q are permutation matrices of order n. So S has size $(n+1) \times 2\binom{n}{k}$.

Theorem 7 *Let S be as in* (4). *Then S is a ZSC-matrix.*

Proof. This proof is a variation of the proof of Theorem 6. Let $A = [a_{ij}]$ be a matrix with $\operatorname{sign} A = \operatorname{sign} S$ and with $\operatorname{csum}(A) = O$. Assume that A is not central. Then, by Theorem 1, there is a (strictly) positive vector v in the row space of A, so for some $u \in \mathbb{R}^{n+1}$

$$u^T A = v^T > O.$$

Let $1 \le p \le n+1$ be such that $u_p = u_{(k+1)}$, i.e., the $(k+1)$th smallest component of u. Here, if $u_1 = u_{(k+1)}$, we let $p = 1$. Define $I_1 = \{i \le n+1 : u_i < u_p\}$ and $I_2 = \{i \le n+1 : u_i > u_p\}$. Then $|I_1| \le k$ and $|I_2| \le n-k$.

Since $\operatorname{sign} A = \operatorname{sign} S$, there exists an index j such that the jth column of A satisfies

$$(*) \;\; \operatorname{sign} a_{ij} = + \;\text{ for all } i \in I_1, \text{ and } \;\operatorname{sign} a_{ij} = - \;\text{ for all } i \in I_2.$$

In fact, consider first the case when $p = 1$. Then $I_1, I_2 \subseteq \{2, 3, \ldots, n+1\}$, and there are at least two such choices of j, corresponding to either of the two blocks in A (the first or the last $\binom{n}{k}$ columns). Alternatively, when $p > 1$, then, I_1 or I_2 contains 1. Then we choose suitable $j \le \binom{n}{k}$ if $1 \in I_1$, and suitable $j > \binom{n}{k}$ if $1 \in I_2$. Otherwise, the properties of $(*)$ are due to the structure the (permuted) matrix $S^{(k,n)}$.

Define the vector $\hat{u} = u + \sum_{i \in I_1}(u_p - u_i)e_i - \sum_{i \in I_2}(u_i - u_p)e_i$. Then $\hat{u} = u_p\, e$, i.e., each component equals u_p. Therefore

$$\hat{u}^T A = u_p\, e^T A = O.$$

But for the j'th column $A^{(j)}$ of A we get from $(*)$

$$\hat{u}^T A^{(j)} = u^T A^{(j)} + \sum_{i \in I_1}(u_p - u_i)a_{ij} - \sum_{i \in I_2}(u_i - u_p)a_{ij} \ge u^T A^{(j)} > 0$$

which contradicts that $\hat{u}^T A^{(j)} = 0$. The results now follows. □

We remark that the matrix obtained from S in (4) by changing the signs in the first row (multiplying by -1) is also a ZSC-matrix. This follows from the general fact that the negative of a ZSC-matrix is again a ZSC-matrix (as explained before) and using that $-S^{k,n} = S^{n-k,n}$.

Example 2 Let $n = 3$ and $k = 1$. Then

$$A = \begin{bmatrix} + & + & + & - & - & - \\ + & - & - & - & - & + \\ - & + & - & - & + & - \\ - & - & + & + & - & - \end{bmatrix}.$$

has the form (4) and is a ZSC-matrix.

∎

We now extend the previous ZSC-class. Let $n > k \geq l \geq 1$ and define a (± 1)-matrix $S^{(k,l,n)}$ by

$$S^{(k,l,n)} = \begin{bmatrix} e^T & -e^T \\ \\ PS^{(k,n)} & QS^{(l,n)} \end{bmatrix} \tag{5}$$

where P and Q are permutation matrix of order n. $S^{(k,l,n)}$ depends on P and Q, but we suppress that in our notation, and its size is $(n+1) \times (\binom{n}{k} + \binom{n}{l})$.

Theorem 8 *For each $n > k \geq l \geq 1$, and permutation matrices P and Q, $S^{(k,l,n)}$ is a ZSC-matrix.*

Proof. We have already proved this result in the case $k = l$, so assume that $k > l$. Let $A = [a_{ij}]$ be a matrix with $\operatorname{sign} A = \operatorname{sign} S^{(k,l,n)}$ and with $\operatorname{csum}(A) = O$. Assume that A is not central; we shall derived a contradiction. Then, by Theorem 1, there is a (strictly) positive vector v in the row space of A, so for some $u = (u_1, u_2, \ldots, u_{n+1}) \in \mathbb{R}^{n+1}$ $u^T A = v^T > O$. We consider different cases, depending on the value of u_1.

Case 1: u_1 equals either $u_{(k+1)}$ or $u_{(l+1)}$. We only need to consider when $u_1 = u_{(k+1)}$, the other case is similar. Let $\alpha = u_1$. Define $I_1 = \{i \leq n+1 : u_i < \alpha\}$ and $I_2 = \{i \leq n+1 : u_i > \alpha\}$. Then $|I_1| \leq k$ and $|I_2| \leq n-k$, and $I_1, I_2 \subseteq \{2, 3, \ldots, n+1\}$. Thus there exists a $j \leq \binom{n}{k}$ such that

$$(*) \ \operatorname{sign} a_{ij} = + \ \text{ for all } i \in I_1, \text{ and } \ \operatorname{sign} a_{ij} = - \ \text{ for all } i \in I_2.$$

Define the vector $\hat{u} = u + \sum_{i \in I_1} (\alpha - u_i) e_i - \sum_{i \in I_2} (u_i - \alpha) e_i$. Then $\hat{u} = \alpha\, e$ and therefore

$$\hat{u}^T A = \alpha\, e^T A = O.$$

But for the j'th column $A^{(j)}$ of A we get from $(*)$

$$\hat{u}^T A^{(j)} = u^T A^{(j)} + \sum_{i \in I_1} (\alpha - u_i) a_{ij} - \sum_{i \in I_2} (u_i - \alpha) a_{ij} \geq u^T A^{(j)} > 0$$

which contradicts that $\hat{u}^T A^{(j)} = 0$.

Case 2: $u_1 > u_{(k+1)}$. Then, let $1 \leq p \leq n+1$ be such that $u_p = u_{(l+1)}$. Let $\alpha := u_{(l+1)}$. Then $p > 1$, as $u_1 > u_{(k+1)} \geq u_{(l+1)} = \alpha$. Define $I_1 = \{i \leq$

$n+1 : u_i < \alpha\}$ and $I_2 = \{i \leq n+1 : u_i > \alpha\}$. As $u_1 > \alpha$, $1 \in I_2$. Moreover, $|I_1| \leq l$ and $|I_2| \leq n-l$. Since $\operatorname{sign} A = \operatorname{sign} S^{(k,l,n)}$, there exists an index $j > \binom{n}{k}$ such that the jth column of A satisfies

$$(*)\ \ \operatorname{sign} a_{ij} = + \ \text{ for all } i \in I_1, \text{ and } \ \operatorname{sign} a_{ij} = - \ \text{ for all } i \in I_2.$$

Note that this was possible as $1 \in I_2$ and $\operatorname{sign} a_{1j} = -1$. Then, let $\hat{u} = u + \sum_{i \in I_1}(u_p - u_i)e_i - \sum_{i \in I_2}(u_i - u_p)e_i = u_p\, e$ and we proceed exactly as in Case 1 to derive a contradiction.

Case 3: $u_1 < u_{(k+1)}$ and $u_1 \neq u_{(l+1)}$. Then, let $1 \leq p \leq n+1$ be such that $u_p = u_{(k+1)}$. Let $\alpha := u_{(k+1)}$. Then $p > 1$, as $u_1 < u_{(k+1)} = \alpha$. Define $I_1 = \{i \leq n+1 : u_i < \alpha\}$ and $I_2 = \{i \leq n+1 : u_i > \alpha\}$. As $u_1 < \alpha$, $1 \in I_1$. Moreover, $|I_1| \leq k$ and $|I_2| \leq n-k$. Since $\operatorname{sign} A = \operatorname{sign} S^{(k,l,n)}$, there exists an index $j \leq \binom{n}{k}$ such that the jth column of A satisfies

$$(*)\ \ \operatorname{sign} a_{ij} = + \ \text{ for all } i \in I_1, \text{ and } \ \operatorname{sign} a_{ij} = - \ \text{ for all } i \in I_2.$$

This was possible as $1 \in I_1$ and $\operatorname{sign} a_{1j} = 1$. Then, let $\hat{u} = u + \sum_{i \in I_1}(u_p - u_i)e_i - \sum_{i \in I_2}(u_i - u_p)e_i = u_p\, e$ and we proceed exactly as in Case 1 to derive a contradiction.

Thus, in any case, we obtained a contradiction, and this proves the theorem. ∎

Example 3 Let $n = 3$, $k = 2$ and $l = 1$. Then

$$A = \begin{bmatrix} + & + & + & - & - & - \\ - & + & + & - & - & + \\ + & - & + & - & + & - \\ + & + & - & + & - & - \end{bmatrix}.$$

has the form (5) and is a ZSC-matrix.

Example 4 Let $n = 4$, $k = 3$ and $l = 2$. Then

$$A = \begin{bmatrix} + & + & + & + & - & - & - & - & - & - \\ - & + & + & + & - & - & - & + & + & + \\ + & - & + & + & + & + & - & + & - & - \\ + & + & - & + & + & - & + & - & + & - \\ + & + & + & - & - & + & + & - & - & + \end{bmatrix}.$$

has the form (5) and is a ZSC-matrix.

In the remaining part of this section we establish a necessary condition for a matrix to be ZSC.

By a *strict sign matrix* we mean a matrix with entries ± 1. A *strict sign vector* is defined similarly. The jth column of a matrix A is denoted by $A^{(j)}$. Let $v = (v_1, v_2, \ldots, v_m)$ and $w = (w_1, w_2, \ldots, w_m)$ be two strict sign vectors. We say that v and w are *sign-close* if there is *no* pair (i, k) with

$$v_i = 1,\ w_i = -1, \ \text{ and }\ v_k = -1,\ w_k = 1. \tag{6}$$

For instance, $(1, 1, 1, -1)$ and $(1, -1, -1, -1)$ are sign-close, while $(1, 1, 1, -1)$ and $(1, 1, -1, 1)$ are not sign-close because of the pair $(i, k) = (3, 4)$. A vector is *balanced* if it contains both a positive component and a negative component. A vector which is not balanced is called *unisigned*. Clearly, if a strict sign matrix A has a matrix $\tilde{A} \in \mathcal{Q}(A)$ with $\operatorname{csum} \tilde{A} = O$, then every column of A must be balanced.

The next theorem contains a necessary condition for the sign-centrality concept we consider here.

Theorem 9 *Let $m \geq 2$ and let $A \in M_{m,n}$ be a strict sign matrix with balanced columns. Assume that A is ZSC. Then, for each balanced strict sign vector $w \in \mathbb{R}^m$, there exists a $j \leq n$ such that $A^{(j)}$ and w are sign-close.*

Proof. Let first $m = 2$. Then each column must be $(1, -1)$ or $(-1, 1)$. So, each matrix $\tilde{A} \in \mathcal{Q}(A)$ with $\operatorname{csum} \tilde{A} = O$ is central if and only if both these vectors are columns of A (otherwise $\operatorname{Nul} A$ would only contain O), and we see that this is equivalent to (6).

Let $m \geq 3$. Assume there is a balanced strict sign vector $w = (w_1, w_2, \ldots, w_m)$ such that for each $j \leq n$ column $A^{(j)}$ and w are not sign-close. We shall define a certain matrix $\tilde{A} = [\tilde{a}_{rs}] \in \mathcal{Q}(A)$ with $\operatorname{csum} \tilde{A} = O$. Let $j \leq n$. Since $A^{(j)}$ and w are not sign-close, there are $i(j), k(j) \leq m$ (depending on j) such that

$$a_{i(j),j} = 1,\ w_{i(j)} = -1, \ \text{ and }\ a_{k(j),j} = -1,\ w_{k(j)} = 1.$$

Choose $\tilde{a}_{ij}$ for $i \neq i(j), k(j)$ such that $\operatorname{sign} \tilde{a}_{ij} = \operatorname{sign} a_{ij}$, $|\tilde{a}_{ij}| \leq 1$, and $|\alpha_j| = 1$ where $\alpha_j := \sum_{i \neq i(j),k(j)} \tilde{a}_{ij}$. If $\alpha_j = 1$, define $\tilde{a}_{i(j)j} = m - 1$ and $\tilde{a}_{k(j)j} = -m$. If $\alpha_j = -1$, define $\tilde{a}_{i(j)j} = m$ and $\tilde{a}_{k(j)j} = -m + 1$. Then we clearly have $\tilde{A} \in \mathcal{Q}(A)$ and $\sum_{i=1}^m \tilde{a}_{ij} = 0$ for each j, so $\operatorname{csum} \tilde{A} = O$.

Moreover, for each j,

$$\begin{aligned}\textstyle\sum_{i=1}^m w_i\tilde{a}_{ij} &= w_{i(j)}\tilde{a}_{i(j),j} + w_{k(j)}\tilde{a}_{k(j),j} + \textstyle\sum_{i\neq i(j),k(j)} w_i\tilde{a}_{ij}\\ &= -2m+1+\textstyle\sum_{i\neq i(j),k(j)} w_i\tilde{a}_{ij}\\ &\leq -2m+1+(m-2)\\ &\leq -m.\end{aligned}$$

Therefore

$$w^T\tilde{A}^{(j)} \leq -m \quad \text{for each } j \leq n$$

which implies that the convex hull of the set of columns $\{\tilde{A}^{(1)}, \tilde{A}^{(2)}, \ldots, \tilde{A}^{(n)}\}$ is contained in the halfspace $\{x \in \mathbb{R}^m : w^T x \leq -m\}$, and therefore the origin does not lie in this convex hull. This means that $\tilde{A}$ is not central. □

4 Power control in induction heating

Induction heating is an electronic technology that provides a fast and efficient contactless way of heating conductive materials. The application of this technology to home appliances is usually called domestic induction heating. The latest trends in this area have introduced new technological challenges such as achieving flexible cooking surfaces, where the user can place different pots anywhere. This section is focused on the power control of a flexible cooking surface. For more information on this application, see [3].

The power delivered to each pot is controlled by the excitation frequency of the magnetic field, among other parameters. A modulation is defined as the set of control parameters applied together. A set of powers is delivered when a modulation is applied, where each power is associated to a pot. The target powers are those that have to be delivered to each pot. The control algorithm should determine which modulations to use in order to deliver the target powers. For the power control of one pot, the control algorithm determines a single modulation and the target power is delivered continuously. However, when several pots have to be controlled, several couplings and constraints arise so, instead, one delivers the target powers averagely, over time. The time span depends on the system thermal inertia and it is about a few seconds.

Consider a matrix $P = [p_{ij}]$ where p_{ij} represents the power delivered to pot i at slot j $(i \leq m, j \leq n)$. Every p_{ij} is nonnegative. The jth column of P represents the powers delivered to each pot at slot j, where a certain modulation is applied. Each row i of P represents the powers delivered to pot i over time. A vector $b \in \mathbb{R}^m$ is given, whose ith component b_i represents the target power for each pot i. Consider the linear equation

$$Px = b. \tag{7}$$

The vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is the weights set of the power averaging over time, where x_j represents the duration of slot j. To obtain a feasible solution x must be nonnegative.

The Flicker standard [4] limits the grid power time variation in domestic equipment, so that the grid is not destabilized. Therefore, the maximum deviation in $\operatorname{csum} P$ over time is limited. An inconvenient case may occur when $\sum_i x_i < 1$ in (7), what means that there exists a slot of duration $1 - \sum_i x_i$ without grid power consumption. This yields to a considerable grid power time variation. Another inconvenient case may occur when $\sum_i x_i > 1$ in (7), which means that there are at least two slots j, k that occur simultaneously. However, this would lead to apply simultaneously two different control actions, which is not physically feasible. In order to avoid these situations we add the constraint

$$\sum_j^n x_j = 1. \tag{8}$$

Due to (8) we can reformulate equation (7) into

$$Ax = O \tag{9}$$

where $A = [a_{ij}]$ is given by $a_{ij} = p_{ij} - b_i$ $(i \leq m, j \leq n)$. Here, the entries a_{ij} of the matrix A can be positive or negative.

As observed in (9), the original problem can be reformulated as finding a central matrix A. Thus, the control algorithm should determine a modulations set that defines a central matrix A. There is a finite number of modulations that satisfy the application constraints, and we let K denote this (finite) set of modulations. Then, the matrix is constructed gathering columns $A^{(j)}$ of A associated to certain modulations in K. Only A matrices with $\operatorname{csum} A = O$ are considered because these are the more beneficial solutions for the Flicker standard.

We now present an algorithm to find an central matrix A. Only strict sign matrices are considered. The algorithm searches in K for a column vector $A^{(j)}$ with the sign pattern associated to a specified qualitative class. These column vectors $A^{(j)}$ vary a lot depending on each set of pots and operating conditions. However, they show a repetitive sign-pattern. To improve the search for $A^{(j)}$ in K, modulations are classified by their most likely sign-pattern in simulations. Our method is a prototype algorithm that considers every qualitative class described in the paper. We believe that with further application tests and implementation work the algorithm may be used in the power control application. Here we present the basic ideas of the algorithm.

ZSC Algorithm

Step 1: Find A in K with $\operatorname{sign} A = T^{(n)}$. If such A is found, then *stop*.
Otherwise, if Theorem 9 holds, then goto step 5.
Step 2. Find A in K with $\operatorname{sign} A = S^{(n)}$. If such A is found, then *stop*.
Otherwise, if Theorem 9 holds, then goto step 5.
Step 3. **for** k=n-2:-1:2
    Find A in K with $\operatorname{sign} A = S^{(k,n)}$. If such A is found, then *stop*.
    Otherwise, if Theorem 9 holds, then goto step 5.
end for
Step 4. **for** k=n-1:-1:1
    for l=k:-1:1
        Find A in K with $\operatorname{sign} A = S^{(k,l,n)}$. If such A is found, then *stop*.
        Otherwise, if Theorem 9 holds, then goto step 5.
    end for
end for
Step 5. Compute $R = \operatorname{rref} A$. If Theorem 2 holds, then *stop*.
If step 6 has been executed, then return. Otherwise, goto step 6.
Step 6. Modify A as explained in section 2 to satisfy Theorem 2.
Find A in K as close as possible to the modified A^*. Goto step 5.

Steps 1 to 4 try to find the qualitative classes described in the paper. Qualitative classes $\operatorname{sign} A \in M_n$ are prioritized over $\operatorname{sign} A \in M_{m,n}$ because they are smaller and simpler to find. Also, $A^{(j)}$ with more positive elements are preferred because they usually have smaller norm and thus a better power distribution. Every $A^{(j)}$ found with a different sign pattern is stored to avoid searches of that $\operatorname{sign} A^{(j)}$ later. Step 5 is executed when $A \in M_{m,n}$ holds Theorem 9, i.e., when necessary conditions for sign-centrality are satisfied. Then, centrality is checked computing its rref, see Theorem 2.

Central matrices with $\operatorname{csum} A \neq O$ are not the most beneficial due to the Flicker Standard limits [4]. However, one may find a valid solution with csum $A \neq O$ if the components of csum A do not differ more than a specific level established by the standard. For strict sign matrices, the only sign-central matrix is the $m \times 2^m$ matrix E_m with each m-tuple of $+1$'s and -1's as columns ([2]). Thus, this sign matrix may be used for finding central matrices in a qualitative class $M_{m,n}$ with csum $A \neq O$.

In the following some numerical examples are presented that show different ZSC matrices given in the power control application. We provide the vector b, matrix A and the minimum norm solution x.

Example 5 A solution based on $S^{(5)}$:

$$b = \begin{bmatrix} 1408 \\ 968 \\ 968 \\ 1760 \\ 2200 \end{bmatrix}, \quad A = \begin{bmatrix} 297 & -257 & -146 & -421 & -377 \\ -33 & 446 & -109 & -127 & -450 \\ -104 & -40 & 370 & -304 & -199 \\ -63 & -8 & -74 & 855 & -132 \\ -97 & -141 & -41 & -3 & 1158 \end{bmatrix}, \quad x = \begin{bmatrix} 0.4507 \\ 0.1786 \\ 0.2367 \\ 0.0659 \\ 0.0681 \end{bmatrix}.$$

Example 6 A solution based on $T^{(4)}$:

$$b = \begin{bmatrix} 896 \\ 896 \\ 1120 \\ 616 \end{bmatrix}, \quad A = \begin{bmatrix} 122 & 87 & 238 & -424 \\ 44 & 249 & -312 & 27 \\ 50 & -357 & 52 & 351 \\ -216 & 21 & 22 & 46 \end{bmatrix}, \quad x = \begin{bmatrix} 0.1179 \\ 0.3208 \\ 0.2956 \\ 0.2657 \end{bmatrix}.$$

Example 7 A solution based on $S^{(1,1,2)}$:

$$b = \begin{bmatrix} 2200 \\ 792 \\ 1760 \end{bmatrix}, \quad A = \begin{bmatrix} 99 & 177 & -176 & -315 \\ -240 & 60 & -80 & 349 \\ 141 & -237 & 256 & -34 \end{bmatrix}, \quad x = \begin{bmatrix} 0.4741 \\ 0.2413 \\ 0.0 \\ 0.2846 \end{bmatrix}.$$

Example 8 A zero sum central matrix for 5 pots whose rref contains a nonpositive nonpivot column:

$$b = \begin{bmatrix} 792 \\ 648 \\ 1800 \\ 540 \\ 1440 \end{bmatrix}, \quad A = \begin{bmatrix} 370 & -256 & -133 & -92 & 399 \\ 268 & 325 & -310 & -166 & -183 \\ -317 & 10 & 419 & -30 & -336 \\ -131 & -63 & -124 & 295 & -107 \\ -190 & -16 & 148 & -7 & 227 \end{bmatrix}, \quad x = \begin{bmatrix} 0.2291 \\ 0.2070 \\ 0.2361 \\ 0.2672 \\ 0.0606 \end{bmatrix},$$

$$R = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & -3.7790 \\ 0.0 & 1.0 & 0.0 & 0.0 & -3.4130 \\ 0.0 & 0.0 & 1.0 & 0.0 & -3.8950 \\ 0.0 & 0.0 & 0.0 & 1.0 & -4.4070 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}.$$

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