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Milnor number of plane curve singularities in arbitrary characteristic

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To the memory of Arkadiusz Płoski

Abstract. Reduced power series in two variables with coefficients in a field of characteristic zero satisfy a well-known formula that relates a codimension related to the normalization of a ring and the Jacobian ideal. In the general case Deligne proved that this formula is only an inequality; García Barroso and Płoski stated a conjecture for irreducible power series. In this work we generalize Kouchnirenko's formula for any reduced power series and also generalize García Barroso and Płoski's conjecture. We prove the conjecture in some cases using in particular Greuel–Nguyen's results.

Let $\mathbb{K}$ be an algebraically closed field of arbitrary characteristic, and let $f \in \mathbb{K}[[x,y]]$ be a reduced power series. Let $\overline{\mathcal{O}}$ be the normalization of the ring $\mathcal{O} := \mathbb{K}[[x,y]]/(f)$, and let $\delta(f) := \dim_{\mathbb{K}} \overline{\mathcal{O}}/\mathcal{O}$. We set

$$\overline{\mu}(f) := 2\delta(f) - r(f) + 1,$$

where $r(f)$ is the number of distinct irreducible factors of f.

The main result of this article is the computation of $\overline{\mu}(f)$ in terms of areas of Newton polygons, in the spirit of Kouchnirenko, without any hypothesis on degeneration. Let

$$\mu(f) := \dim_{\mathbb{K}} \mathbb{K}[[x,y]]/\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right).$$

In characteristic zero, we have $\mu(f) = \overline{\mu}(f)$. Deligne proved that

$$\mu(f) \geq \overline{\mu}(f).$$

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Some authors, including García Barroso and Płoski [GP18], Greuel and Nguyen [GN12], and Hefez, Rodrigues, and Salomão [HRS18], were interested in the question of equality in characteristic $p \neq 0$. We give a conjecture on this question, and using works of these authors, we prove that after adding some hypothesis, it is true. We show some more examples of its validity.

The paper is organized as follows. In §1, we study the Hamburger–Noether algorithm, defined in a form which is very close to the Newton algorithm, but that can be used in any characteristic.

In §2, using the Hamburger–Noether algorithm, we construct trees for any reduced power series in $f(x, y) \in \mathbb{K}[[x, y]]$ in any characteristic. Note that, given f, the tree depends on $\operatorname{Char}\mathbb{K}$. These trees are constructed using the Newton polygon of f and the ones of its transforms at each stage of the Hamburger–Noether algorithm. We define an important invariant: the multiplicity of the tree. It is defined using the decorations of the tree, which are computed from the equations of the faces of the Newton polygons.

In §3, we show that in fact the multiplicity of the tree can also be expressed in terms of the areas below the Newton polygons that appear in the Hamburger–Noether algorithm. This is a generalization of Kouchnirenko's result; see Remark 6.4.

The subsequent three sections aim to prove the main result of the paper, that the multiplicity of the tree is also equal to $r-2\delta(f)$. For this we need to compute the multiplicity of intersection of two power series in terms of the trees, which is done in §4. Then in §5 we study irreducible series and show how to compute their Zariski characteristic series from the tree. The main result is proven in §6, first for irreducible series, and then in general using results from [CD24].

The final §7 is devoted to the study of the following conjecture. Let $f \in \mathbb{K}[[x, y]]$, and let $\mathcal{T}(f)$ be its minimal tree (see §2). Let $\mathcal{V}$ be the set of vertices of $\mathcal{T}(f)$.

Conjecture. *We have*

$$\mu(f) = \overline{\mu}(f)$$

if and only if p does not divide any of the N_v for $v \in \mathcal{V}$.

The previous sections compute the term on the right hand side. Using Greuel and Nguyen's result [GN12], we show that the conjecture is true when f is non-degenerate. Here one has to be careful about the definition of degenerate, since our definition does not coincide with Greuel–Nguyen's.

Using a result of García Barroso and Płoski [GP18], we prove that if $\mathcal{T}$ is a tree and $p > M(\mathcal{T}) + \operatorname{ord}\mathcal{T}$, then for all $f \in \mathbb{K}[[x, y]]$ with characteristic of $\mathbb{K}$ equal to p and $\mathcal{T}(f) = \mathcal{T}$, we have $\mu(f) = \overline{\mu}(f)$. In this case, p divides N_v for no vertex v of $\mathcal{T}$.

We also show that the conjecture agrees with the conjecture of García Barroso and Płoski [GP18] in the case where f is irreducible, and with the result of Hefez, Rodrigues, and Salomão. We give some examples where the conjecture is true.

In a subsequent article in preparation, we will show some other parts of the conjecture.

This article owes a great deal to Arkadiusz Płoski for many reasons. The second author met him about thirty years ago in Bordeaux, when he came for a month as invited Professor at the University. It was the beginning of a strong friendship and collaboration. Over the years, she learnt, discussing with him, not only mathematics, but also history, political sciences, literature, ... She wants also to mention the important part played by his wife, Anna, in this relation, and to thank her.

The book *Plane Algebroid Curves in Arbitrary Characteristic* [PP22], by Gerhard Pfister and Arkadiusz Płoski, has been the main source of inspiration for this article. Thanks to them both. Arkadiusz sent it to the second author in November 2023 and it was the source of many ideas in the paper. The second author wants to thank Michel Raibaut for interesting discussions and the two authors are very grateful to the referees for careful reading of the article.

1. Hamburger–Noether algorithm. The usual Newton algorithm may not work in positive characteristic when the weights and the characteristic are not coprime. The Hamburger–Noether algorithm works in any characteristic, since it uses a sequence of blow-ups that solves the singularity. The Newton algorithm can be interpreted in terms of weighted blow-ups which involve quotient singularities by the action of some abelian group; these singularities are not well-defined when the characteristic of the field is not coprime to the order of the group.

1.1. Preliminaries. Let $p, q \in \mathbb{N}$ be coprime and $p \geq q \geq 1$. Consider the Euclidean algorithm

$$p = k_1 q + r_1,\ q = k_2 r_1 + r_2,\ r_1 = k_3 r_2 + r_3, \ldots,\ r_{\omega-1} = k_{\omega+1} r_\omega + 1,\ r_\omega = k_{\omega+2}.$$

Define $r_0 := q$, $m_1 := k_1$, $n_1 := 1$, $\tilde{m}_1 := k_2$, $\tilde{n}_1 := 1$, and $m_i, n_i, \tilde{m}_i, \tilde{n}_i$, for $i \geq 1$, satisfying

$$p = m_i r_{i-1} + n_i r_i, \qquad q = \tilde{m}_i r_i + \tilde{n}_i r_{i+1}.$$

LEMMA 1.1. *For $i \geq 1$, we have*

$$m_{i+1} = m_i k_{i+1} + n_i, \qquad n_{i+1} = m_i,$$
$$\tilde{m}_{i+1} = \tilde{m}_i k_{i+2} + \tilde{n}_i, \qquad \tilde{n}_{i+1} = \tilde{m}_i.$$

Proof. By induction. ∎

Lemma 1.2. *For* $i \geq 1$, $\Delta_i := n_{i+1}\tilde{m}_i - \tilde{n}_i m_{i+1}$ *equals* $(-1)^i$.

Proof. We can see that $\Delta_1 = -1$ and $\Delta_i + \Delta_{i-1} = 0$ if $i \geq 2$. ∎

1.2. Algorithm. Let

$$\begin{aligned} x &= x_1 y_1^{k_1}, & y &= y_1,\\ x_1 &= x_2, & y_1 &= x_2^{k_2} y_2,\\ x_2 &= x_3 y_3^{k_3}, & y_2 &= y_3. \end{aligned}$$

More generally,

$$\begin{aligned} x_i &= x_{i+1}, & y_i &= x_{i+1}^{k_{i+1}} y_{i+1}, \quad i \text{ odd},\\ x_i &= x_{i+1} y_{i+1}^{k_{i+1}}, & y_i &= y_{i+1}, \quad i \text{ even}. \end{aligned}$$

The following lemma is proved by induction.

Lemma 1.3. *For* $i \geq 0$,

$$\begin{aligned} x &= x_{2i+1}^{n_{2i+1}}, y_{2i+1}^{m_{2i+1}},\\ y &= x_{2i+1}^{\tilde{n}_{2i}} y_{2i+1}^{\tilde{m}_{2i}}; \end{aligned}$$

for $i \geq 1$,

$$\begin{aligned} x &= x_{2i}^{m_{2i}} y_{2i}^{n_{2i}},\\ y &= x_{2i}^{\tilde{m}_{2i-1}} y_{2i}^{\tilde{n}_{2i-1}}. \end{aligned}$$

Since $r_{\omega+1} = 1$, $r_{\omega+2} = 0$, we have $p = m_{\omega+2}$, $q = \tilde{m}_{\omega+1}$.

Lemma 1.4. *If* ω *is odd, then*

$$\begin{aligned} x &= x_{\omega+2}^{n_{\omega+2}} y_{\omega+2}^{p},\\ y &= x_{\omega+2}^{\tilde{n}_{\omega+1}} y_{\omega+2}^{q}; \end{aligned}$$

if ω *is even, then*

$$\begin{aligned} x &= x_{\omega+2}^{p} y_{\omega+2}^{n_{\omega+2}},\\ y &= x_{\omega+2}^{q} y_{\omega+2}^{\tilde{n}_{\omega+1}}. \end{aligned}$$

Proof. If n is odd, then $(x_{\omega+2}, y_{\omega+2}) = (Y_1, X_1)$, and if n is even, then $(x_{\omega+2}, y_{\omega+2}) = (X_1, Y_1)$. Hence $x = X_1^p Y_1^{q'}$, $y = X_1^q Y_1^{p'}$,

$$\det \begin{pmatrix} p & q' \\ q & p' \end{pmatrix} = 1. \ ∎$$

We can check that $p' = n_{\omega+2} \leq p$ and $q' = \tilde{n}_{\omega+1} \leq q$. Let

$$P(x,y) = x^a y^b \prod_{i=1}^{k} (x^q - \mu_i y^p)^{\nu_i}, \qquad N := ap + bq + pq \sum_{i=1}^{k} \nu_i. \tag{1.1}$$

The sequence of Hamburger–Noether maps gives, if ω is odd,

$$P(x,y) := x_{\omega+2}^{an_{\omega+2}+b\tilde{n}_{\omega+1}} y_{\omega+2}^{N} \prod_{i=1}^{k} (x_{\omega+2}^{n_{\omega+2}q} - \mu_i x_{\omega+2}^{\tilde{n}_{\omega+1}p})^{\nu_i}$$
$$= x_{\omega+2}^{\ell} y_{\omega+2}^{N} \prod_{i=1}^{k} (x_{\omega+2} - \mu_i)^{\nu_i},$$

where $\ell = an_{\omega+2} + b\tilde{n}_{\omega+1} + \tilde{n}_{\omega+1} p \sum_{i=1}^{k} \nu_i$. If ω is even, we obtain

$$x_{\omega+2}^{N} y_{\omega+2}^{\ell} \prod_{i=1}^{k} (1 - \mu_i y_{\omega+2})^{\nu_i}.$$

Let us assume that $f \in \mathbb{K}[[x,y]]$ has a (p,q)-edge in its Newton polygon and that its *face polynomial* (see (2.1)) for this edge is P. Hence, if we put

$$x = (Y_1 + \bar{\mu})^{q'} X_1^{p}, \quad y = (Y_1 + \bar{\mu})^{p'} X_1^{q}, \quad \bar{\mu} = \mu_i^{\pm 1}, \tag{1.2}$$

then

$$f(x,y) = X_1^{N} \underbrace{(Y_1^{\nu_i} + \cdots)}_{f_1(X_1,Y_1)}.$$

The germ of plane curve defined by $f_1(X_1, Y_1)$ is called an *HN-transform* of f.

Definition 1.5. A germ of plane curve is *non-degenerate* if no HN-transformation is needed, i.e. the exponents ν_i are all equal to 1.

2. Newton trees. Let $f \in \mathbb{K}[[x,y]]$ be a non-constant reduced series, with $\mathbb{K}$ algebraically closed. Let us consider the Newton polygon of f, with edges $S_1, \ldots, S_k$, ordered from top to bottom. The edge v_i representing S_i is supported by the line $p_i X + q_i Y = N_i$ with $\gcd(p_i, q_i) = 1$.

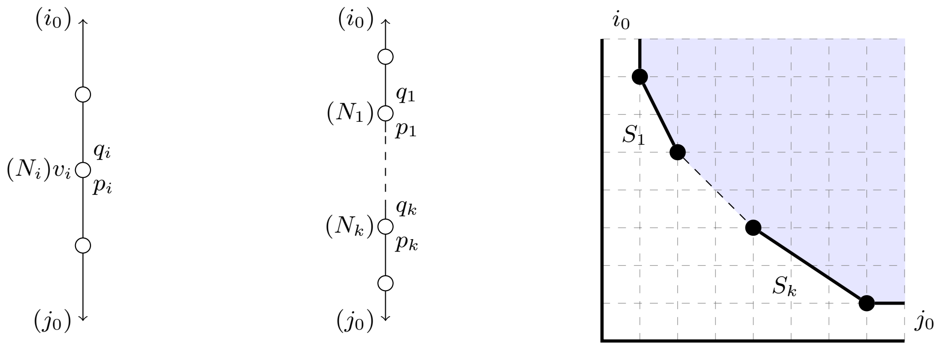


Fig. 1. The left figure corresponds to the part of the Newton tree associated to the edge v_i. The central one is the Newton tree of the right figure.

The Newton polygon is represented by a vertical linear tree. Each face is represented by a vertex of the tree, and two vertices of the tree are connected by an edge if and only if the corresponding faces intersect. Each edge is decorated at its extremities with natural numbers. Near a vertex v_i, representing a face with equation $p_iX+q_iY=N_i$, the edges arising from v_i are decorated with p_i and q_i (see Figure 1). The vertex v_i is decorated with N_i. If x^{i_0} and y^{j_0} are factors of f (with maximal multiplicity, $0\leq i_0,j_0\leq 1$) then the non-compact faces $X=i_0$, $Y=j_0$ are represented by two decorated arrows. The decorated tree contains the same information as the Newton polygon. As in (1.1), each face has an associated homogeneous polynomial (the *face polynomial*)

$$P_i(x,y)=x^{n_i}y^{m_i}\prod_{j=1}^{k_i}(x^{q_i}-\mu_{i,j}y^{p_i})^{\nu_{i,j}},\qquad \gcd(p_i,q_i)=1.$$

Let us fix i,j; if $\nu_{i,j}=1$, we attach to the vertex v_i an edge ending with an arrow (to the right). If $\nu_{i,j}>1$, we apply the Hamburger–Noether algorithm (1.2) for $(p_i,q_i,\mu_{i,j})$ (the choice of the power ±1 depends on the parity of the length of the Euclidean algorithm):

$$f_{i,j}(x_1,y_1)=x_1^{N_i}(y_1^{\nu_{i,j}}+\cdots)\in\mathbb{K}[[x,y]], \tag{2.1}$$

to which a new Newton polytope can be applied (in general with a smaller height) and translated to the right by N_i. We glue the new Newton tree as in Figure 2, including the changes of decorations.

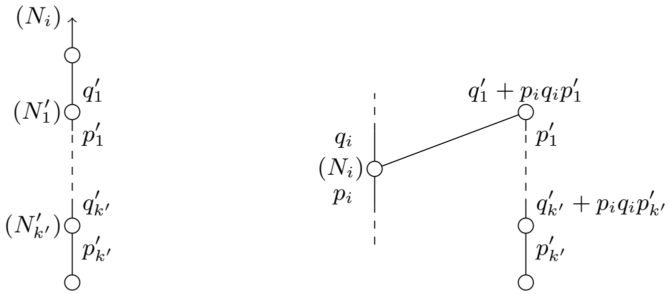


Fig. 2. New Newton tree and gluing

Remark 2.1. The construction of Newton trees in characteristic 0 is similar, but we can use Newton maps instead of Hamburger–Noether maps, with the same result.

2.1. Examples

Example 2.2. Let

$f(x,y):=(x^2-y^3)^4-2(x^2-y^3)^2xy^{11}-y^{19}(1-y^3)(x^2-y^3)+y^{25}.$

The Newton polygon has only one edge, with face polynomial $P(x,y) := (x^2 - y^3)^4$ with $\omega = 0$. There is only one edge and one root, hence only one transformation:

$$x = x_1^3(y_1 + 1), \qquad y = x_1^2(y_1 + 1).$$

Then

$$f(x,y) = x_1^{24}(y_1+1)^8((y_1^2 - x_1^{13})^2 + \cdots).$$

The Newton polytope is associated to $(y_1^2 - x_1^{13})^2$, $\omega = 0$. Hence,

$$x_1 = x_2^2(y_2 + 1), \qquad y_1 = x_2^{13}(y_2 + 1)^6,$$

and

$$f(x,y) = x_2^{100}(y_2+1)^{48}(1 + x_2^{13} + \cdots)^8(y_2^2 + x_2 + \cdots).$$

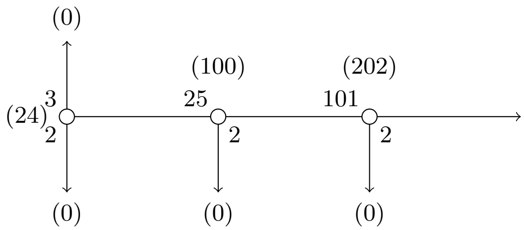


Fig. 3. Newton tree of Example 2.2

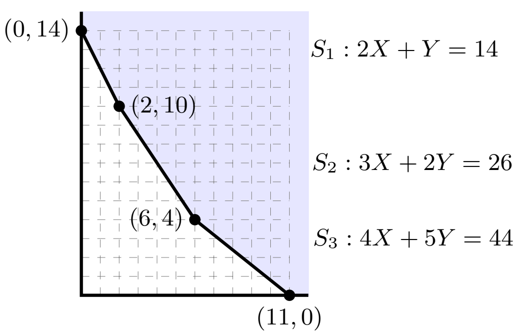


Fig. 4. Newton polygon of Example 2.3

Example 2.3. Let

$$f(x,y) := -x^2y^4(x^2 - y^3)^2 + x^{11} + y^{14} + xy^{13}.$$

The factorized face polynomials are

$$P_1 = \begin{cases} y^{10}(y^2 - x)(y^2 + x) & \text{if } \operatorname{Char} \mathbb{K} \neq 2, \\ y^{10}(y^2 - x)^2 & \text{if } \operatorname{Char} \mathbb{K} = 2, \end{cases}$$

$P_2 = x^2y^4(x^2 - y^3)^2$, and $P_3 = x^6(x^5 - y^4)$. For S_1 and $\operatorname{Char}\mathbb{K} = 2$, we take

$$x = x_1^2(y_1 + 1), \qquad y = x_1,$$

for which we find that, up to a unit in $\mathbb{K}[[x_1, y_1]]$, f is $x_1^{14}(y_1^2 + x_1 + \cdots)$. For S_2,

$$x = x_1^3(y_1 + 1), \qquad y = x_1^2(y_1 + 1).$$

Then

$$f(x, y) = x_1^{26}(y_1 + 1)^{10}(x_1^2 - y_1^2 + \cdots).$$

If $\operatorname{Char}\mathbb{K} = 2$, we perform the transformation

$$x_1 = x_2(y_2 + 1), \qquad y_1 = x_2,$$

and we obtain for f, up to a unit in $\mathbb{K}[[x_2, y_2]]$, $x_2^{28}(y_2^2 + x_2 + \cdots)$. If $\operatorname{Char}\mathbb{K} \neq 2$, we finish with two arrows.

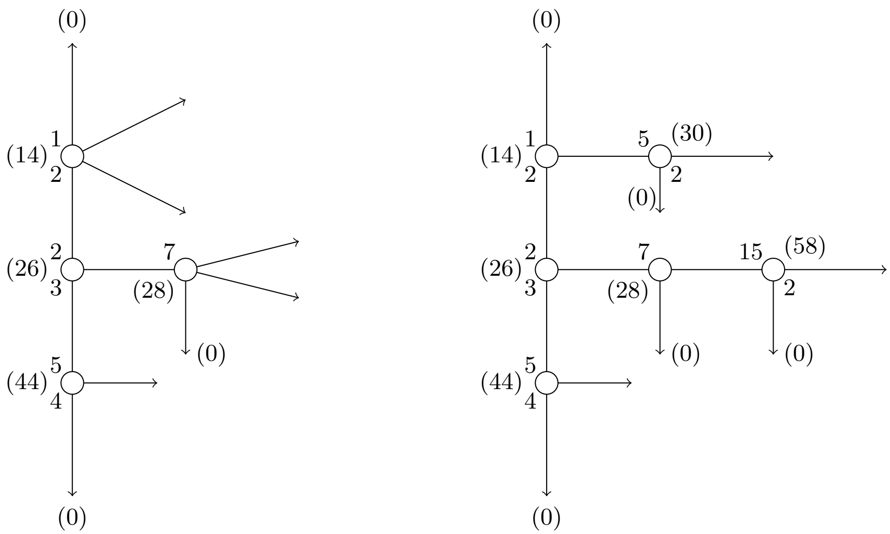


Fig. 5. Newton tree of Example 2.3 if $\operatorname{Char}\mathbb{K} \neq 2$ (left), or $\operatorname{Char}\mathbb{K} = 2$ (right)

2.2. Minimal trees. Let us consider a Newton tree $\mathcal{T}$. Let $\mathcal{V}$ be its set of vertices, $\mathcal{E}$ its set of edges, $\mathscr{A}$ its set of arrows, and $\mathscr{A}_0$ the set of its arrows decorated with (0).

Definition 2.4. A *dead end* is an edge between a vertex and an arrow decorated with (0).

We will proceed with the following conventions:

(a) The dead ends decorated with 1 and the attached arrows will be erased.
(b) Vertices of valency 2 will be erased, while the decorations of the remaining vertices will be kept.

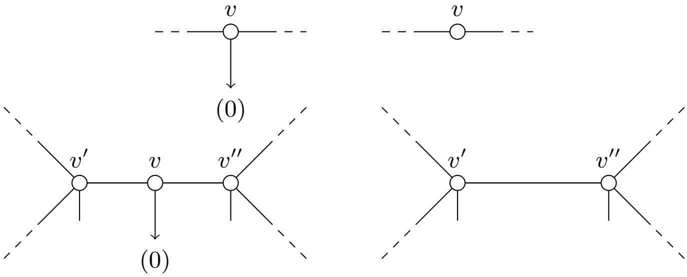


Fig. 6. Erasure operations

REMARK 2.5. These operations are defined in [CD24, Def. 1.1.1], and their properties are studied. A tree is *minimal* if no operation (a)–(b) can be performed.

EXAMPLE 2.6. The tree in Figure 3 is already minimal. In Figure 5 (left) the dead end to the left can be erased and the tree becomes minimal. The minimalization of Figure 5 (right) is in Figure 7.

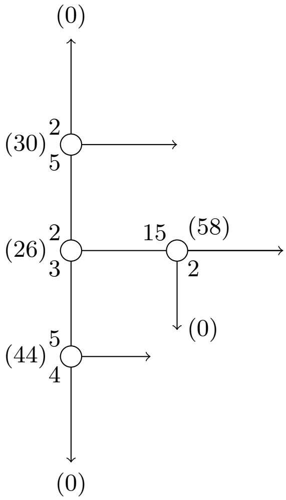


Fig. 7

DEFINITION 2.7. Let $v \in \mathcal{V}$. The *valency* δ_v of v is the number of edges through v (the valency of an arrow is 1). Let $v' \in \mathscr{A}_0$ be attached to $v \in \mathcal{V}$ such that a is the decoration of the edge joining v, v'. Then the *multiplicity* of v' is $N_{v'} := N_v/a$. Finally, the *multiplicity* of $\mathcal{T}$ is

$$M(\mathcal{T}) := -\sum_{v \in \mathcal{V} \cup \mathscr{A}_0} N_v(\delta_v - 2),$$

where δ_v is the valency of v in the tree.

Example 2.8. For Example 2.2, the multiplicity is $M(\mathcal{T}) = -155$; for Example 2.3, it is -103 if $\operatorname{Char}\mathbb{K} = 2$, and -101 otherwise.

3. Multiplicity of a tree and area of Newton polygons

Definition 3.1. A *convenient Newton polygon* is a Newton polygon which hits both axes.Let $\mathcal{A}$ be the area between the Newton polygon and the axes.

A *semiconvenient Newton polygon* is a Newton polygon that hits the lines $y = 0$ and $x = N$. In this case, let $\mathcal{A}$ be the area between the Newton polygon and lines.

Definition 3.2. A *reduced convenient Newton polygon* is a Newton polygon which hits the lines $x = 0$ or $x = 1$ and $y = 0$ or $y = 1$.

Remark 3.3. We can make a Newton polygon convenient in case it does not hit $x = 0$ or $y = 0$. This is done as follows. Assume that the polygon does not hit $x = 0$. Let v be the vertex of the Newton polygon $\mathcal{N}$ with coordinates $(1, \beta)$ and let v_n be a point with coordinates $(0, n)$ with n large enough such that the set of vertices of the convex hull of $\mathcal{V}' := \mathcal{V} \cup \{v_n\}$ is $\mathcal{V}'$. A similar procedure can be applied if the polygon does not hit $y = 0$. If a and b are the lengths on the axes, then $2\mathcal{A} - a - b$ does not depend on how we make the Newton polygon convenient.

Definition 3.4. A *reduced semiconvenient Newton polygon* is a Newton polygon which hits the lines $x = N$ and $y = 0$ or $y = 1$. Again, we can make it semi-convenient and if $a + N$ is the length on the x-axis, then $2\mathcal{A} - a$ does not depend on the chosen semiconvenient Newton polygon.

Definition 3.5. A tree is *non-degenerate* if it is *vertical* except for the non-decorated arrows. For the associated Newton polygon, the face polynomials are as in (1.1) with $\nu_i = 1$. In particular, the Hamburger–Noether algorithm is not needed.

Remark 3.6. In positive characteristic this definition does not imply that the face polynomials do not have singularities in the torus. For example, let

$$f(x, y) := xy(x + y),$$

which is non-degenerate according to this definition but $(1, 1)$ is a common zero of the derivatives in $\mathbb{F}_3^2$.

Lemma 3.7 ([CV14, Lemma 5.20]). *If the tree $\mathcal{T}$ is reduced and non-degenerate, then*

$$-M(\mathcal{T}) = 2\mathcal{A} - a - b.$$

THEOREM 3.8. *Let $f \in \mathbb{K}[[x, y]]$, and let $\mathcal{T}$ be its Newton tree. Then*

$$-M(\mathcal{T}) = 2\mathcal{A}_0 - a - b + \sum_{\ell=1}^{r} (2\mathcal{A}_\ell - a_\ell),$$

where $\mathcal{A}_0$ is the area of the first Newton polygon and a, b are the traces on the axes. The summation is taken over all Hamburger–Noether transforms, $\mathcal{A}_\ell$ is the area of the Newton polygon limited by the line $x = N_\ell$, and $a_\ell + N_\ell$ is the trace on the x-axis.

Proof. Let v be a vertex, and let AB be the face of the Newton polygon which corresponds to v in the HN algorithm. Let δ_v be the valency of v, and let d_v be the number of points with integral coordinates on the face AB.

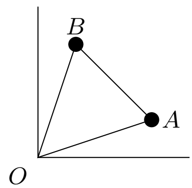


Fig. 8

Let $\mathcal{A}_v$ be the area of the triangle OAB. We have $2\mathcal{A}_v - a - b = N_v(d_v - 2)$ where a is the first coordinate of A if A is on the x-axis and 0 otherwise, and b the second coordinate of B if B is on the y-axis, and 0 otherwise. Since

$$d_v = \sum_{i=1}^{\delta_v} \nu_i \implies d_v - \delta_v = \sum_{i=1}^{\delta_v} (\nu_i - 1),$$

we see that

$$N_v(\delta_v - 1) = N_v(d_v - 1) - N_v(d_v - \delta_v) = N_v(d_v - 1) - N_v \sum_{i=1}^{\delta_v} (\nu_i - 1).$$

If $\nu_i = 1$ for all i, then $2A_v - a - b = N_v(d_v - 2) = N_v(\delta_v - 2)$.

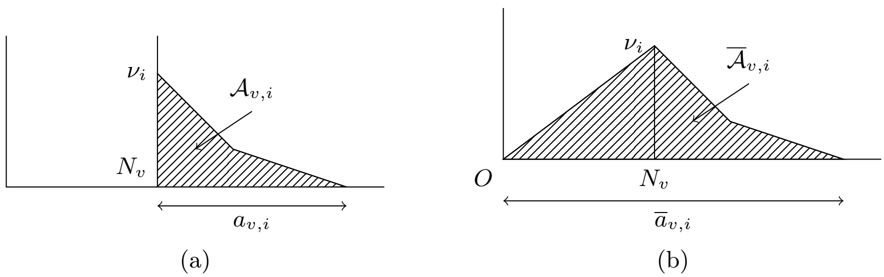


Fig. 9

If there exists i such that $\nu_i > 1$, we consider all HN-transforms, and we compute

$$\mathcal{A}_{v,1} := 2\mathcal{A}_v - a - b + \sum_{i=1}^{\delta_v} (2\mathcal{A}_{v,i} - a_{v,i}).$$

Then $\mathcal{A}_{v,1} = N_v(\delta_v - 2) + \sum_i 2\mathcal{A}_{v,i} + N_v\nu_i - (N_v + a_{v,i})$. Let $\overline{\mathcal{A}}_{v,i} := 2\mathcal{A}_{v,i} + N_v\nu_i$ be the area of the polygon delimited by the Newton polygon $\mathcal{N}_{v,i}$ and O, and let $\overline{a}_{v,i} := N_v + a_{v,i}$ be its trace on the x-axis.

Then we proceed by induction, and we stop when all ν's are 1. ∎

4. Newton trees and intersection multiplicity. In this section, we will show how to compute the intersection multiplicity of two branches using the trees.

NOTATION 4.1. Let v be a vertex of $\mathcal{T}$, and let ε_v be the set of edges incident to v. If $e \in \varepsilon_v$, let $q(v,e)$ be the decoration of e near v. Let v, w be two vertices or arrows of $\mathcal{T}$; we denote the path from v to w in $\mathcal{T}$ as $\varepsilon = \varepsilon_{v,w}$; $\mathcal{V}_\varepsilon$ is the set of vertices and arrows in ε, and $\mathcal{E}_\varepsilon$ is the set of edges in ε. If x is a vertex, then $\mathcal{E}_x$ is the set of edges containing x. We denote

$$\rho(v,w) := \prod_{x \in \mathcal{V}_\varepsilon \setminus \{v,w\}} \prod_{e \in \mathcal{E}_x \setminus \mathcal{E}_\varepsilon} q(x,e),$$

$$\overline{\rho}(v,w) := \prod_{x \in \mathcal{V}_\varepsilon \setminus \{w\}} \prod_{e \in \mathcal{E}_x \setminus \mathcal{E}_\varepsilon} q(x,e).$$

In case v, w are two distinct arrows, we put $i(v,w) := \rho(v,w)$. For $\mathcal{V}_1, \mathcal{V}_2 \subset \mathscr{A}$ we set

$$i(\mathcal{V}_1, \mathcal{V}_2) := \sum_{v_1 \in \mathcal{V}_1} \sum_{v_2 \in \mathcal{V}_2 \setminus \{v_1\}} i(v_1, v_2).$$

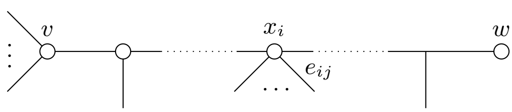


Fig. 10

An irreducible series $f \in \mathbb{K}[[x,y]]$ admits primitive *parametrizations* $t \mapsto (\varphi(t), \psi(t))$. If g is another series, then

$$i(f,g) := \operatorname{ord} g(\varphi(t), \psi(t)).$$

If g is irreducible, we can exchange the roles of f, g, and the value does not change, and the definition can be extended to the case where f, g are reducible assuming that $i(f,f) = \infty$.

THEOREM 4.2. *Let $f, g \in \mathbb{K}[[x, y]]$ be irreducible series, let $\mathcal{T}$ be the Newton tree of fg, and let α_f, α_g be the arrows representing the last HN-transforms of f, g. Then*

$$i(f, g) = i(\alpha_f, \alpha_g).$$

Proof. The proof is in several steps.

(1) Either the Newton polygons of f, g are distinct or their face polynomials do not coincide.

(a) They separate at different vertices as in Figure 11 (left). Under these conditions, we can assume that

$$f(x, y) = (x^{q_i} - \mu y^{p_i})^{\nu_i} + \cdots ,$$

and a parametrization of $g = 0$ is given by

$$\varphi(t) = t^{p_j \nu_j} \tilde{\varphi}(t), \qquad \psi(t) = t^{q_j \nu_j} \tilde{\psi}(t), \qquad \tilde{\varphi}(0), \tilde{\psi}(0) \text{ units}.$$

Then

$$f(\varphi(t), \psi(t)) = (t^{p_j q_i \nu_j} \tilde{\varphi}^{p_i}(t) - \mu t^{q_j p_i \nu_j} \tilde{\psi}_1^{q_i}(t))^{\nu_i} + \cdots .$$

Since $p_i q_j > q_i p_j$, we have $i(f, g) = q_i p_j \nu_i \nu_j$. On the other hand, one can see that the product of the weights on the horizontal part of ε are ν_i, ν_j and the product on the vertical part is $q_i p_j$, i.e., $i(\alpha_f, \alpha_g)$.

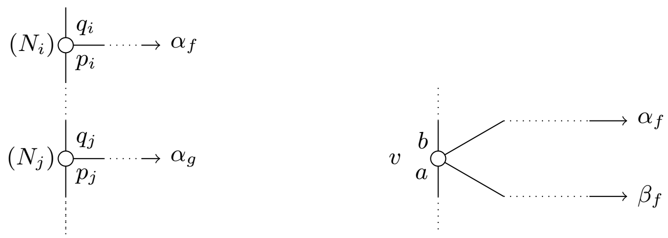


Fig. 11. Separations as in (a) and (b)

(b) They separate on the same face, see Figure 11 (right). We have $i = j$, and we denote $(p_i, q_i) =: (a, b)$. We have

$$P_{f,v}(x, y) = (x^b - \mu y^a)^{\nu_1}, \qquad \varphi(t) = t^{a\nu_2} \tilde{\varphi}(t), \qquad \psi(t) = t^{q_j b \nu_2} \tilde{\psi}(t),$$

with

$$\tilde{\varphi}(0), \tilde{\psi}(0), \tilde{\varphi}(0)^b - \mu \tilde{\psi}(0)^a \neq 0$$

and

$$i(f, g) = \operatorname{ord} t^{ab\nu_1\nu_2}(\tilde{\varphi}(0)^b - \mu \tilde{\psi}(0)^a) = ab\nu_1\nu_2 = i(\alpha_f, \alpha_g).$$

(2) We use induction on the number of steps such that f, g are separated (see Figure 12). The first step has been done. Let us assume the formula holds if the separation is at step $h - 1$, and we check that it holds if the separation is at step h.

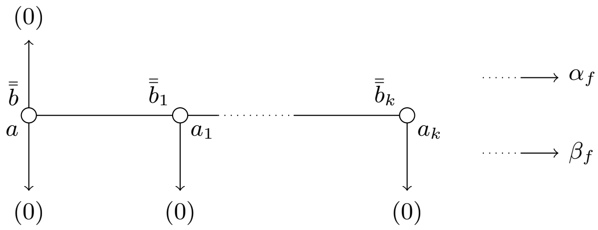


Fig. 12. Case (2)

After the first step, we have

$$f(x,y) = X_1^{N_f} f_1(X_1,Y_1), \qquad g(x,y) = X_1^{N_g} g_1(X_1,Y_1)$$

with

$$x = X_1^a (Y_1 - \mu)^{a'}, \qquad y = X_1^b (Y_1 - \mu)^{b'}.$$

From a primitive parametrization $(\varphi_1(t), \psi_1(t))$ of $f_1(X_1,Y_1)$ we obtain a primitive parametrization $(\varphi(t), \psi(t))$ of $f(X_1,Y_1)$ using the above expression. Then

$$\begin{aligned} i(f,g) &= \operatorname{ord} g(\varphi(t),\psi(t)) = \operatorname{ord} \varphi_1(t)^{N_g} g_1(\varphi_1(t),\psi_1(t)) \\ &= \operatorname{ord} g_1(\varphi_1(t),\psi_1(t)) + N_g \operatorname{ord} \varphi_1(t) = i(f_1,g_1) + N_g i(X_1,f_1); \end{aligned}$$

by symmetry, the second term equals $N_f i(X_1, g_1)$. By induction, we have $i(f_1,g_1) = \bar{b}_k a_k d_1 d_2$.

We claim that

$$\bar{\bar{b}}_i = a \cdot b \cdot a_1^2 \cdot \ldots \cdot a_{i-1}^2 a_i + \bar{b}_i. \tag{4.1}$$

To prove this claim, note that $\bar{\bar{b}}_1 = aba_1 + \bar{b}_1$. We assume (4.1) is true until i and we prove it for $i+1$:

$$\begin{aligned} \bar{\bar{b}}_{i+1} &= \bar{\bar{b}}_i \cdot a_i \cdot a_{i+1} + b_{i+1} \\ &= a \cdot b \cdot a_1^2 \cdot \ldots \cdot a_{i-1} \cdot a_i^2 \cdot a_{i+1} + \underbrace{\bar{b}_i \cdot a_i \cdot a_{i+1} + b_{i+1}}_{\bar{b}_{i+1}}, \end{aligned}$$

and the claim is true.

Let us consider a parametrization for g_1:

$$\varphi_1(t) = t^{a_1} \tilde{\varphi}_1(t), \qquad \psi_1(t) = t^{b_1} \tilde{\psi}_1(t), \qquad \tilde{\varphi}_1(0), \tilde{\psi}_1(0) \text{ units}.$$

The order of $\varphi_1(t)$ equals $a_1 \cdot \ldots \cdot a_k \cdot d_2$, and $N_f = a \cdot b \cdot a_1 \cdot \ldots \cdot a_k \cdot d_1$. Since

$$\varphi(t) = \varphi_1(t)^a \tilde{\varphi}_2(t), \qquad \psi(t) = \psi_1(t)^b \tilde{\psi}_2(t), \qquad \tilde{\varphi}_2(0), \tilde{\psi}_2(0) \text{ units},$$

and $f(\varphi(t),\psi(t)) = \varphi_1(t)^{N_f} f(\varphi_1(t),\psi_1(t))$, the computation of the order gives the statement. ∎

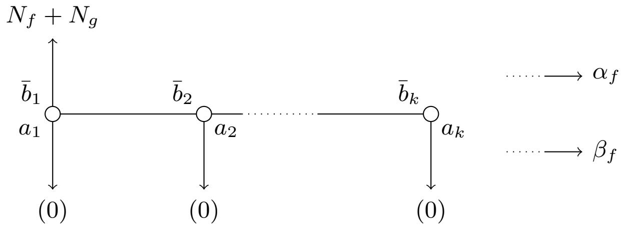


Fig. 13. Induction step for case (2)

5. Irreducible power series

DEFINITION 5.1. A sequence $(v_0, v_1, \dots, v_r)$ of positive integers is said to be a *Zariski characteristic sequence* if it satisfies the following two conditions:

(Z_1) Set $d_k := \gcd(v_0, \dots, v_k)$, $0 \le k \le r$. Then $d_k > d_{k+1}$, $0 \le k < r$, and $d_r = 1$.

(Z_2) Let $n_k := d_{k-1}/d_k$, $1 \le k < r$. Then $n_k v_k < v_{k+1}$, $1 \le k < r$.

Let $\mathcal{T}$ be a tree with one arrow not decorated with 0 as in Figure 14. Assume that $\mathcal{T}$ is a minimal tree. Let $v_0 := N_1/b_1, v_1 := N_1/a_1, \dots, v_r := N_r/a_r$.

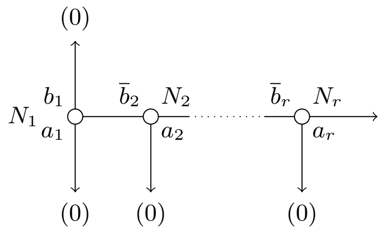


Fig. 14. Tree $\mathcal{T}$

PROPOSITION 5.2. *The sequence $(v_0, v_1, \dots, v_r)$ is a Zariski characteristic sequence.*

Proof. Note that

$$d_0 = v_0 = \frac{N_1}{b_1} = a_1 \cdot a_2 \cdot \ldots \cdot a_r,$$
$$d_1 = \gcd(v_0, v_1) = \gcd(a_1 \cdot a_2 \cdot \ldots \cdot a_r, b_1 \cdot a_2 \cdot \ldots \cdot a_r) = a_2 \cdot \ldots \cdot a_r.$$

Then $d_1 < d_0$. Assume

$$d_i = a_{i+1} \cdot \ldots \cdot a_r.$$

Then

$$d_{i+1} = \gcd(a_1 \cdot a_2 \cdot \ldots \cdot a_r, \dots, \bar{b}_{i+1} \cdot a_{i+2} \cdot \ldots \cdot a_r) = a_{i+2} \cdot \ldots \cdot a_r.$$

Hence, $d_i > d_{i+1}$, and (Z_1) is proved. We have

$$\begin{aligned} n_k v_k < v_{k+1} &\iff N_k < \frac{N_{k+1}}{a_{k+1}}, \\ N_k &= \bar{b}_k \cdot a_k \cdot \ldots \cdot a_r, \\ N_k < \frac{N_{k+1}}{a_{k+1}} &\iff \bar{b}_k \cdot a_k \cdot a_{k+1} \cdot \ldots \cdot a_r < \bar{b}_{k+1} \cdot a_{k+2} \cdot \ldots \cdot a_r, \\ &\iff \bar{b}_k \cdot a_k \cdot a_{k+1} < \bar{b}_{k+1}, \end{aligned}$$

which is true since $\bar{b}_{k+1} = \bar{b}_k \cdot a_k \cdot a_{k+1} + b_{k+1}$ with $b_{k+1} > 0$. ∎

We have the following result.

PROPOSITION 5.3 ([PP22, Ch. 3, Prop. 1.17]). *Let G be the semigroup generated by a Zariski sequence $(v_0, v_1, \ldots, v_r)$. Then the conductor c of G equals*

$$c = \sum_{k=1}^{r} (n_k - 1) v_k - v_0 + 1.$$

Let f be an irreducible power series. Recall that the semigroup $\Gamma(f)$ of f is defined by $\Gamma(f) := \{v_f(g) : g \text{ a power series such that } g \not\equiv 0\ (f)\}$.

PROPOSITION 5.4. *Assume f is an irreducible power series, and $\mathcal{T}$ is its tree. The semigroup generated by $(v_0, \ldots, v_r)$ is the semigroup $\Gamma(f)$.*

Proof. Let g be an irreducible power series separating from f at a vertex of $\mathcal{T}$. Then

$$i(f, g) = N_i d_g = v_i a_i d_g.$$

Hence, $i(f, g) \in \langle v_0, \ldots, v_r \rangle$.

Now assume that g separates from f at a dead end of $\mathcal{T}$. Then

$$i(f, g) = d_g \cdot \bar{b}_i \cdot a_{i+1} \cdot \ldots \cdot a_r = d_g \cdot v_i.$$

Hence, $i(f, g) \in \langle v_0, \ldots, v_r \rangle$. Moreover, we see that if $d_g = 1$, then $v_i \in \Gamma(f)$.

Finally, assume that g separates from f between the vertices v_{i-1} and v_i. We have $i(f, g) > N_{i-1}$.

We want to show that $N_{i-1} > c_{i-1} \cdot a_i \cdot \ldots \cdot a_r$, where c_{i-1} is the conductor of the semigroup generated by

$$\left\langle \frac{v_0}{a_i \cdot \ldots \cdot a_r}, \ldots \frac{v_{i-1}}{a_i \cdot \ldots \cdot a_r} \right\rangle.$$

We have

$$a_i \cdot \ldots \cdot a_r \cdot c_{i-1} = N_{i-1} - \frac{N_{i-1}}{a_{i-1}} + N_{i-2} - \frac{N_{i-2}}{a_{i-2}} + N_{i-3} - \cdots - \frac{N_2}{a_2} + N_1 - v_0 + 1.$$

Then $a_i \cdot \ldots \cdot a_r \cdot c_{i-1} < N_{i-1}$ and $i(f, g) > N_{i-1} > c_{i-1} \cdot a_i \cdot \ldots \cdot a_r$. Hence, $i(f, g) \in \langle v_0, \ldots v_{i-1} \rangle$. ∎

6. Multiplicity of a tree and δ-invariant. In this section we want to compute the δ-invariant of a series f in terms of the multiplicity of its tree. Let $f \in \mathbb{K}[[x,y]]$ be a reduced power series. Let $r(f)$ be the number of arrows of the tree not decorated with (0). In this section, we shall use results from [CD24] and [CV14]. In particular, we need the following:

PROPOSITION 6.1 ([CV14, Prop. 3.3]). *For all $v \in \mathcal{V}$, we have*

$$N_v = \sum_{\alpha \in \mathscr{A} \setminus \mathscr{A}_0} \overline{\rho}(v, \alpha).$$

This proves that the definitions of multiplicity in [CD24] and in the present work are the same, and we can use the results of [CD24].

PROPOSITION 6.2 ([CD24]). *The number $-M(\mathcal{T}_f) + r(f)$ is even.*

Let

$$\mathcal{O}_f := \mathbb{K}[[x,y]]/(f).$$

Let $\overline{\mathcal{O}}_f$ be the integral closure of $\mathcal{O}_f$. Define

$$\delta(f) := \dim_{\mathbb{K}} \overline{\mathcal{O}}_f / \mathcal{O}_f.$$

We can state and prove the main result of the paper.

THEOREM 6.3. $2\delta(f) = -M(f) + r(f)$.

Proof. Assume first that f is an irreducible power series. Then $r(f) = 1$. We have

$$M(\mathcal{T}) = -\sum_{i=1}^{r} \left(N_i - \frac{N_i}{a_i} \right) - \frac{N_1}{a_1}.$$

Then $c(f) = -M(\mathcal{T}) + 1$. It is proven in [PP22, Ch. 4, Thm. 2.1] that $c(f) = 2\delta(f)$. Hence, the result is true when f is irreducible.

Let us recall [PP22, Ch. 4, Thm. 2.1]:

If $f = f_1 \dots f_r$ with irreducible coprime factors f_i, then

$$\delta(f) = \sum_{i=1}^{r} \delta(f_i) + \sum_{1 \le i < j \le r} i(f_i, f_j).$$

If we define

$$\tilde{\delta}(\mathcal{T}) := \frac{-M(\mathcal{T}) + r(f)}{2},$$

then [CD24, Prop. 4.13] says that

$$\tilde{\delta}(\mathcal{T}) = \sum_{\alpha \in \mathscr{A} \setminus \mathscr{A}_0} \tilde{\delta}(\mathcal{T}_\alpha) + \frac{i(\mathscr{A} \setminus \mathscr{A}_0, \mathscr{A} \setminus \mathscr{A}_0)}{2},$$

which proves the general case. ∎

Remark 6.4. The combination of Theorems 3.8 and 6.3 provides a new interpretation of Kouchnirenko's theorem, namely it refers to $\bar{\mu}$ and not μ for some positive characteristics. Moreover, it can be extended without any hypothesis of non-degeneracy.

7. Multiplicity of a tree and Milnor number. Recall that the *Milnor number* of f is

$$\mu(f) := \dim_{\mathbb{K}} \mathbb{K}[[x,y]]/\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right).$$

If $\operatorname{Char} \mathbb{K} = 0$, Milnor's results (in particular [Mil68, Theorem 10.5]) imply that

$$\mu(f) = 1 - M(\mathcal{T}) = 2\delta(f) - r(f) + 1,$$

and Deligne [Del73] showed that in general we only have

$$\mu(f) \geq 1 - M(f) = 2\delta(f) - r(f) + 1. \tag{7.1}$$

Note that in positive characteristic, the Milnor numbers of f and uf (for u a unit) may differ, see e.g. Examples 7.9 and 7.10, but $\bar{\mu}$ does not [PP22, Ch. 4, Prop. 3.1]. For a review of these equalities (and inequalities) we refer to [GP18].

Conjecture. *Let $\mathcal{T}$ be a minimal tree of f. Equality holds in* (7.1) *if and only if* $\operatorname{Char} \mathbb{K}$ *divides N_v for no $v \in \mathcal{V} \cup \mathcal{A}_0$.*

We shall now show that the conjecture was already proven in some particular cases.

Definition 7.1. Let Δ be a face of the Newton polygon of f. We call f *non-degenerate* (*ND*) *along* Δ if the Jacobian ideal of f_Δ has no zero in the torus $(\mathbb{K}^*)^2$. We say that f is *Newton non-degenerate* (*NND*) if f is ND along each face (of any dimension) of the Newton polygon of f.

In 1976, Kouchnirenko [Kou76] proved that if f is NND and convenient, then $\mu(f) = 1 - M(f)$. This result was extended by [BGM12] in 2010 (published in 2012) without the hypothesis of convenience (if $\mu(f) < \infty$).

Proposition 7.2. *Let f be non-degenerate. If* $\operatorname{Char} \mathbb{K}$ *divides N_v for no v, then f is NND.*

Proof. Let Δ be a face of dimension 1 of the Newton polygon of f. Let v be the corresponding vertex on the tree, and N_v the corresponding multiplicity. The equation of the face is $aX + bY = N_v$. We want to show that if $\operatorname{Char}\mathbb{K} = p$ does not divide N_v, then f is ND along Δ. We have

$$f_\Delta(x,y) = x^n y^m \sum_i^k c_i x^{ia} y^{(k-i)b},$$

$$\frac{\partial f_\Delta}{\partial x}(x,y) = \sum_i^k c_i(n+ia)x^{n+ia-1}y^{(k-i)b+m},$$

$$\frac{\partial f_\Delta}{\partial y}(x,y) = \sum_i^k c_i(m+(k-i)b)x^{n+ia}y^{(k-i)b+m-1}.$$

If the Jacobian ideal of f_Δ has a zero (α,β) in $(\mathbb{K}^*)^2$, then

$$\sum_i^k c_i(n+ia)\alpha^{ia}\beta^{(k-i)b} = 0,$$

$$\sum_i^k c_i(m+(k-i)b)\alpha^{ia}\beta^{(k-i)b} = 0,$$

which is impossible: for all i we have

$$(n+ia)(m+(k-i+1)b-(n+(i-1)a)(m+(k-i)b)) = N_v \neq 0,$$

since p does not divide N_v.

Now if f is not ND with respect to a vertex of the Newton polygon, then p divides N_v for a face of the Newton polygon which contains the vertex. ∎

COROLLARY 7.3. *If f is non-degenerate and* $\operatorname{Char}\mathbb{K}$ *divides N_v for no v, then $\mu(f) = 2\delta(f) - r(f) + 1$.*

Proof. Assume f is non-degenerate. Recall that we assume that the tree is minimal. If f is non-degenerate and $\operatorname{Char}\mathbb{K}$ divides N_v for no v, then f is NND (Proposition 7.2) and $\mu(f) = 1 - M(f)$ (see [BGM12]). ∎

COROLLARY 7.4. *The conjecture is true if f is non-degenerate.*

Proof. If p divides some N_v, then f is not ND along Δ, the corresponding face of the Newton polygon. Using [GN12, Prop. 2.12] we deduce that $\mu(f) > 2\delta(f) - r(f) + 1$. ∎

Now we recall two interesting results from [GP18]. Let $p := \operatorname{Char}\mathbb{K}$, $l(x,y) := ax + by$, and $P_l(f) := b\frac{\partial f}{\partial x} - a\frac{\partial f}{\partial y}$.

PROPOSITION 7.5 ([GP18, Prop. 2.1]). *Let l be a regular parameter of $\mathbb{K}[[x,y]]$, and $f = f_1 \cdot \ldots \cdot f_r$ be such that $i(f_i,l) \not\equiv 0 \bmod p$. Then*

$$i(f, P_l(f)) = 2\delta(f) + i(f,l) - r = -M(\mathcal{T}(f)) + i(f,l).$$

Proposition 7.6 ([GP18, Prop. 4.4]). *Assume that there exists a regular parameter l such that $i(f,l) = \operatorname{ord} f$ and $i(f, P_l(f)) < p$. Then $\mu(f) = -M(\mathcal{T}(f)) + 1 = 2\delta - r + 1$.*

Then we deduce the following:

Proposition 7.7. *If $p > -M(\mathcal{T}(f)) + \operatorname{ord} f$, then $\mu(f) = -M(\mathcal{T}) + 1 = 2\delta(f) - r + 1$.*

Now we study the case where f is irreducible. In this case, we have two results. García Barroso and Płoski [GP18, Theorem 5.1] proved the following:

> *Let $n^* := \max(a_1, \ldots, a_r)$ (see Figure 14). If $\operatorname{Char} \mathbb{K} = p > n^*$, then*
>
> $$\mu(f) = 2\delta(f) \iff \forall k,\ 0 \le k \le r,\ \nu_k \not\equiv 0 \bmod p.$$

This proves the conjecture when $p > n^*$. On the other hand, Hefez, Rodrigues, and Salomão [HRS18] proved that if for every $k \in \{0, \ldots, r\}$ we have $\nu_k \not\equiv 0 \bmod p$, then $\mu(f) = 2\delta(f)$.

Example 7.8. We consider

$$f(x,y) := (x - a_1 y)(x - a_2 y)(x - a_3 y)(x - a_4 y) + xy^5 + x^4 y.$$

Let

$$b_i := a_{i+1} - a_1, \quad 1 \le i < 4, \qquad c_i := a_{i+2} - a_2, \quad 1 \le i < 3, \qquad d_1 := a_4 - a_3.$$

We assume $p \neq 2$ and consider several cases.

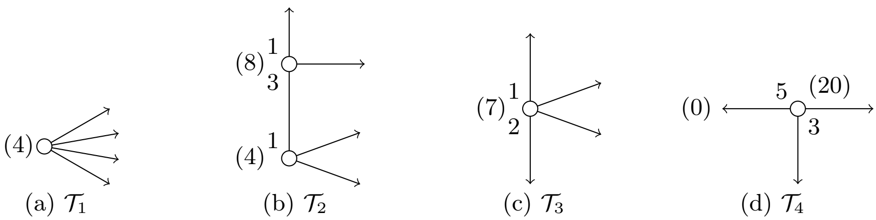


Fig. 15

(1) Assume all the a_i are pairwise distinct mod p. In this case, the tree is $\mathcal{T}_1$ in Figure 15(a), and we have

$$\mu(f) = 1 - M(\mathcal{T}_1) = 2\delta(f) - 3 = 9, \qquad \forall p \neq 2.$$

(2) Assume that $a_1 \equiv a_2 \equiv 0 \bmod p$, $a_3, a_4 \not\equiv 0 \bmod p$, $a_3 \not\equiv a_4 \bmod p$. In this case, the tree is $\mathcal{T}_2$ in Figure 15(b), and we have

$$\mu(f) = 1 - M(\mathcal{T}_2) = 2\delta(f) - 3 = 13, \qquad \forall p \neq 2.$$

(3) Assume only a_4 is non-vanishing mod p. In this case, the tree is $\mathcal{T}_3$ in Figure 15(c), and we have

$$\mu(f) = 1 - M(\mathcal{T}_3) = 2\delta(f) - 3 = 15, \quad \forall p \neq 2, 7; \qquad \mu(f) = 17 \quad \text{if } p = 7.$$

(4) Assume that $a_i \equiv 0 \bmod p$. In this case, the tree is $\mathcal{T}_4$ in Figure 15(d), and we have

$$\mu(f) = 1 - M(\mathcal{T}_4) = 2\delta(f) - 1 = 17, \quad \forall p \neq 2, 5; \quad \mu(f) = 20 \quad \text{if } p = 5.$$

In all cases (1)–(4), we use the fact that f is non-degenerate.

(5) Assume no a_i vanishes mod p, $a_1 \equiv a_2 \bmod p$, and a_2, a_3, a_4 are pairwise distinct mod p. In this case the tree is $\mathcal{T}_{1,2}$ in Figure 16(a), and we have

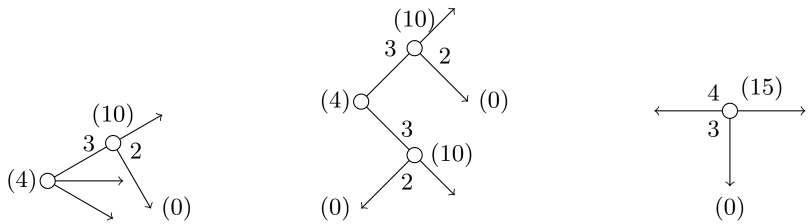
$$\mu(f) = 1 - M(\mathcal{T}_{1,2}) = 2\delta(f) - 2 = 10 \quad \text{if } p = 0; \quad \mu(f) = 11 \quad \text{if } p = 5.$$

From Proposition 7.7, we know that if $p > 13$, then $\mu = 10$. This value is also obtained for the remaining cases $p = 3, 7, 11, 13$.

Assume no a_i vanishes mod p, $a_1 \equiv a_2 \not\equiv a_3 \equiv a_4 \bmod p$. In this case the tree is $\mathcal{T}_{1,3}$ in Figure 16(b), and we have

$$\mu(f) = 1 - M(\mathcal{T}_{1,3}) = 2\delta(f) - 1 = 11 \quad \text{if } p = 0; \quad \mu(f) = 13 \quad \text{if } p = 5.$$

Again, using Proposition 7.7, we know that if $p > 14$, then $\mu = 11$, and we verify for $p = 3, 7, 11, 13$ that $\mu = 11$ (using `Singular` or `Sagemath`).



(a) $\mathcal{T}_{1,2}$ (b) $\mathcal{T}_{1,3}$ (c) $\mathcal{T}_{1,4}$

Fig. 16

(7) Assume no a_i vanishes mod p, $a_1 \equiv a_2 \equiv a_3 \not\equiv a_4 \bmod p$. In this case the tree is $\mathcal{T}_{1,4}$ in Figure 16(c), and we have $\mu(f) = 1 - M(\mathcal{T}_{1,4}) = 2\delta(f) - 1 = 11$ if $p = 0$; $\mu = 12$ if $p = 5$; and $\mu = 13$ if $p = 3$. We know that if $p > 14$, then $\mu = 11$, and we verify that this is also the case for $p = 7, 11, 13$.

(8) Assume no a_i vanishes mod p, and they are equal mod p. In this case the tree is $\mathcal{T}_{1,5}$ in Figure 17(a), and we have

$$\mu(f) = 1 - M(\mathcal{T}_{1,5}) = 2\delta(f) - 1 = 12 \quad \text{if } p = 0; \quad \mu(f) = 13 \quad \text{if } p = 5.$$

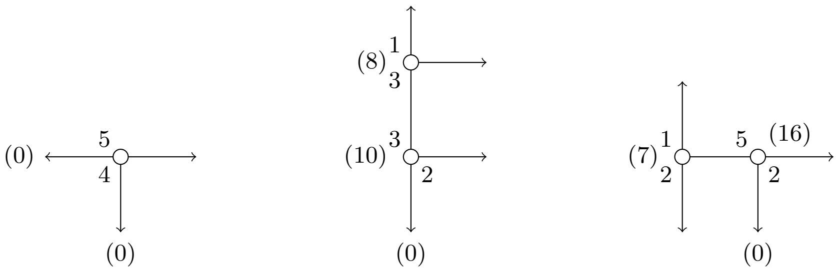
We know that if $p > 15$, then $\mu = 12$, and we verify that this is also the case for $p = 3, 7, 11, 13$.

(9) Assume that $a_1 \equiv a_2 \equiv 0 \bmod p$ and $a_3 \equiv a_4 \not\equiv 0 \bmod p$. In this case the tree is $\mathcal{T}_{2,1}$ in Figure 17(b), and we have

$$\mu(f) = 1 - M(\mathcal{T}_{2,1}) = 2\delta(f) - 2 = 14 \quad \text{if } p = 0; \quad \mu(f) = 15 \quad \text{if } p = 5.$$

We know that for $p > 17$, then $\mu = 14$, and we verify that this is also the case for $p = 3, 7, 11, 13, 17$.

The conjecture is true for this family.



(a) $\mathcal{T}_{1,5}$ (b) $\mathcal{T}_{2,1}$ (c) $\mathcal{T}_{1,6}$

Fig. 17

Now we consider the case where $p = 2$.

(10) Assume that $a_i \neq 0$ for all i. Then the tree is $\mathcal{T}_{1,5}$. The multiplicity of the tree is 11, and we can compute $\mu = 20$.

(11) Assume that exactly one of the a_i vanishes. Then the tree is $\mathcal{T}_{1,4}$. We have $M(\mathcal{T}_{1,4}) = 10$ and $\mu = 11$ since 2 does not divide N_v.

(12) Assume that exactly two of the a_i vanish. The tree is $\mathcal{T}_{2,1}$. Its multiplicity is 13 and $\mu = 20$.

(13) Assume that exactly three of the a_i vanish. The tree is $\mathcal{T}_{1,6}$ in Figure 17(c). Its multiplicity is 15 and we compute $\mu = 19$.

(14) Assume all the a_i vanish. The tree is $\mathcal{T}_4$. The multiplicity is 16 and $\mu = 20$.

Example 7.9. Let us consider Example 2.2. In this example, f is irreducible. The results of [GP18] tell us that if $p > 3$ then $\mu = 156$ if and only if $p \neq 5, 101$, where $\mu = 157$. We can check that $\mu = \infty$ for $p = 2$, and $\mu = 166$ for $p = 3$. Note also that if we multiply f by a *random* unit then $\mu = 168$ for $p = 2$ and $\mu = 157$ for $p = 3$; nothing changes for the other primes. Thus the conjecture is true for this example.

Example 7.10. Let us consider Example 2.3. First we assume $p \neq 2$. If $p > 111$, then $\mu = 102$. The prime numbers which divide N_v for some v are $p = 7$ with $\mu = 105$, $p = 11$ with $\mu = \infty$, and $p = 13$ with $\mu = 104$. For the remaining primes $p < 111$, we also have $\mu = 102$. For $p = 2$, we have $\mu = 133$. Note also that if we multiply f by some unit then $\mu = 118$ for $p = 2$, $\mu = 104$ for $p = 3$, and $\mu = 105$ for $p = 11$; nothing changes for the other primes. Thus the conjecture is true in this example.

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