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ORIGINAL PAPER

Inventory models with nonlinear costs and random lead times: a general framework and some generalizations using reliability techniques

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Abstract

In this paper, we generalize some of the inventory models with nonlinear costs presented in Rosling (Oper Res 50:1007–1017, 2002). In particular, we consider random lead times and study models with continuous and discrete demand. In the unified model, we study conditions on the lead time and the demand distribution that ensure the quasiconvexity of the cost function, thereby guaranteeing the existence of optimal (s, S) policies. These conditions are applied to specific well-known models. Particularly, we study the case where backlog costs per time equal to zero, thus providing more general conditions than those available in the literature for specific models. Moreover, we introduce a random lead time for the compound renewal model with periodic review. Our results are mainly based on the reliability properties of the random variables under consideration.

Mathematics Subject Classification 62E10 · 60E15

Keywords Reliability · Log-concavity · Compound distribution · Stochastic order · Inventory model

1 Introduction

In the basic stochastic single-item inventory model, the following assumptions are made: the arrival of demands is described by a compound renewal process, and orders can be placed at any renewal epoch. Orders arrive after a given lead time L. All

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stockouts are backlogged. A special case is the periodic review setting, where periods of equal length are considered, and orders can be placed at the beginning of each period. Since the pioneering papers of Arrow et al. (1958) and Scarf (1960), a substantial body of literature has emerged treating this model and its generalizations. Let $G(x)$ be the cost function associated with an inventory position of x units. It is well known that an (s, S) policy is optimal as long as the function G is quasiconvex (a real function G is said to be quasiconvex if $-G$ is unimodal).

Most standard versions of the model assume that the holding costs are proportional to the inventory level and that the backlogging costs are proportional to the size of the backlog.

The seminal paper by Rosling (2002) introduced two additional types of backorder costs that are common in the inventory literature: one is a fixed cost $\bar{\pi}$ for each unit backordered, regardless of how long the unit remains backlogged; the second is a fixed cost b for each time period that ends with a backlog (see Rosling 2002 for more details).

In particular, Rosling (2002) identified sufficient conditions for the demand distribution under which G remains quasiconvex. Later, Badía and Sangüesa (2015) and Badía et al. (2021) considered sufficient conditions for quasiconvexity under stochastic lead times for the periodic review-periodic demands and continuous review-compound renewal demands models. These correspond to Model 1 and Model 2 in Rosling (2002), respectively. Their results were obtained as a consequence of the study of the reliability properties of compound renewal processes stopped at a random time L.

In this paper, we weaken the conditions for the quasiconvexity of the cost function in the above-mentioned models, and also include discrete demands. As noted in Ninh et al. (2021), little attention has been given to optimization problems with discrete random variables, although they appear naturally. We also study the quasiconvexity of the cost function for the periodic review-compound renewal demands model (Model 3 in Rosling 2002). The conditions required are similar to those for the continuous review-compound renewal demands model. As far as we know, this is the first time that a random lead time is considered in Model 3.

For the sake of clarity, the main contributions of this paper to each model are summarized in Table 1. In this table, the initial conditions imposed by Rosling (2002), by Badía and Sangüesa (2015), and the generalization of our proposal are presented. Four different models are considered: (1) periodic review-periodic demands with backlog costs per time equal to zero (i.e., $b = 0$), (2) continuous review-continuous demands and $b = 0$, (3) periodic review-compound renewal demands with $b = 0$ and $b \neq 0$. For the definitions of Decreasing Reversed Hazard Rate (DRHR), Increasing Failure Rate (IFR), Monotone Convolution Ratio (MCR) and discrete log-concave, see Definitions 2.4 and 2.5. Additionally, the abbreviations cdf, pdf, and pmf are used to refer to cumulative distribution function, probability density function, and probability mass function, respectively.

For the periodic review-periodic demands and the continuous review-compound renewal demands models, the conditions provided in Badía et al. (2021) for $b \neq 0$ are simply extended here to include discrete demands.

As shown in Propositions 3.1, 3.3, and 3.4, a unified treatment of the models above, as well as discrete and continuous demand distributions, can be obtained without

Table 1 Conditions for quasiconvexity of the cost function

Components	Rosling (2002)	Badía and Sangüesa (2015)	This research
Periodic review-periodic demands. Case $b = 0$			
Leadtime	Constant	Random (Discrete log-concave)	Random (discrete DRHR)
Demands	Continuous (MCR)	Continuous (MCR)	General (MCR)
Continuous review-compound renewal demands. Case $b = 0$			
Leadtime	Constant	Random (log-concave pdf)	Random (DRHR)
Demands	Continuous (MCR)	Continuous (MCR)	General (MCR)
Interarrivals	log-concave pdf	log-concave pdf	IFR
Periodic review-compound renewal demands. Case $b = 0$			
Leadtime	Constant		Random (DRHR)
Demands	Continuous (MCR)		General (MCR)
Interarrivals	log-concave pdf		IFR
Periodic review-compound renewal demands. Case $b \neq 0$			
Leadtime	Constant		Random (log-concave pdf)
Demands	Unknown		Decreasing pdf/pmf
Interarrivals	Unknown		log-concave pdf

much additional effort. This unified treatment may open the door to investigate other inventory policies. The relaxation of the previously known conditions for the quasi-convexity of the cost function in the case where backlog costs per time equal to zero in the compound renewal models is the question that requires more computational effort.

The remainder of the paper is structured as follows. Section 2 introduces the aging concepts and stochastic orders that will be used throughout the paper. Both topics will play a crucial role in our proofs. Section 3 introduces the general model and the main results, which will be applied to the specific models mentioned above. Section 4 provides possible ways to extend this work in future research.

2 Preliminaries

In this section, several definitions to be used throughout the rest of the paper are introduced. Henceforth, let F and f denote the cdf and pdf, when it exists, of a non-negative random variable X. Write $\bar{F}{:=}1-F$ for the reliability function, and $X =_{st} Y$ to indicate that the random variables X and Y have the same distribution. Let F_M denote the cdf of the sum of M independent and identically distributed (i.i.d.) random variables with cdf F, where M, possibly random, is independent of the summands. As usual, the words 'increasing' ('decreasing') will stand for 'non-decreasing' ('non-increasing').

First, the definitions of the main stochastic orders necessary for the results of this paper are introduced (cf. Müller and Stoyan 2002 or Shaked and Shanthikumar 2007).

Definition 2.1 Let X and Y be random variables, with cdfs F and G, and reliability functions $\bar{F}$ and $\bar{G}$, respectively. Then, X is said to be smaller than Y in the

(a) usual stochastic order ($X \leq_{st} Y$) if $\bar{G}(x) \geq \bar{F}(x)$ for all $x \in \mathbb{R}$;
(b) hazard rate order ($X \leq_{hr} Y$) if $\frac{\bar{G}(x)}{\bar{F}(x)}$ is increasing in x (i.e., $\bar{G}(y)\bar{F}(x) - \bar{G}(x)\bar{F}(y) \geq 0, \ x \leq y$);
(c) reversed hazard rate order ($X \leq_{rh} Y$) if $\frac{G(x)}{F(x)}$ is increasing in x (i.e., $G(y)F(x) - G(x)F(y) \geq 0, \ x \leq y$).

Remark 2.2 Both the hazard rate order and the reversed hazard rate order are stronger than the usual stochastic order.

The main properties considered in this paper are summarized below. Many of these properties are related to the concepts of log-concavity and log-convexity.

Definition 2.3 A non-negative function f is said to be log-concave on an interval I iff

$$f(\alpha x + (1-\alpha)y) \geq f(x)^{\alpha} f(y)^{1-\alpha}, \quad \forall x, y \in I, \quad 0 \leq \alpha \leq 1. \tag{1}$$

The inequality is reversed for log-convexity. That is, f is said to be log-concave (log-convex) if $\log(f)$ is concave (convex) on I.

Definition 2.4 Let X be a non-negative random variable with cdf F and survival function $\bar{F}$. It is said that X has a:

(a) Monotone convolution ratio or MCR if $F_{n+1}(x)/F_n(x)$ is increasing in x, for each $n = 1, 2, \ldots$
(b) Decreasing failure rate or DFR (increasing failure rate or IFR) if $\bar{F}$ is log-convex on $[0, \infty)$ (is log-concave).
(c) Decreasing reversed hazard rate or DRHR if F is log-concave on $[0, \infty)$.

The main properties used in this paper are the following:

1. Log-concave pdf $\Rightarrow$ F log-concave (DRHR) and $\bar{F}$ log-concave (IFR).
2. F log-concave (DRHR) $\Rightarrow$ monotone convolution ratio (MCR).
3. Log-convex pdf $\Rightarrow$ $\bar{F}$ log-convex (DFR).
4. $\bar{F}$ log-convex (DFR) $\Rightarrow$ F concave $\Rightarrow$ F log-concave (DRHR).

Comprehensive and unified proofs of Properties 1, 3, and 4 are available in Marshall and Olkin (2007, p. 694), while the proof of Property 2 can be found in Rosling (2002). See also An (1998) and Sengupta and Nanda (1999). Properties 1–4 above are valid for discrete random variables, whenever the density is replaced by the pmf and the aging properties are replaced by their discrete counterparts.

Many well-known distributions have log-concave or log-convex densities (see Bagnoli and Bergstrom 2005 for a detailed list). Thus, the class of DRHR and hence MCR distributions is large enough for our purpose.

The aging properties for discrete random variables are now being considered. A sequence of non-negative real numbers $(a_k)_{k\geq 1}$ is said to be log-concave (log-convex) if it has no internal zeros and

$$a_{k+1} \geq (\leq) a_k^{\frac{1}{2}} a_{k+2}^{\frac{1}{2}}, \quad k = 0, 1, 2, \ldots \tag{2}$$

Definition 2.5 Let L be a non-negative integer-valued random variable.

(a) L is said to have a discrete DRHR if the sequence $\{P(L \leq k)\}_{k\geq 0}$ is log-concave.
(b) L is said to have a discrete DFR (discrete IFR) if the sequence $\{P(L > k)\}_{k\geq 0}$ is log-convex (log-concave).
(c) L is said to be discrete log-concave (log-convex) if the sequence $\{P(L = k)\}_{k\geq 0}$ is log-concave (log-convex).

Finally, a sequence of non-negative real numbers $(a_k)_{k\geq 1}$ is said quasiconvex iff $a_k \leq \max(a_{k_1}, a_{k_2}), \quad 0 \leq k_1 < k < k_2$. See, for instance, Murota and Shioura (2003).

As mentioned above, we use renewal processes for the description of the arrival of demands in an inventory model. That is, we consider a stochastic process $\{N(t) : t \geq 0\}$, where each $N(t)$ counts the number of arrivals of a demand up to time t (Asmussen 2000, Chap. V, for instance). Then, $(X_n)_{n\geq 1}$ is the sequence of interarrival times, while $T_n = \sum_{i=1}^{n} X_i, \quad n = 1, 2, \ldots$ $(T_0 = 0)$ is the sequence of arrival times. The counting process is defined through the arrival times by the following expression: $N(t) = \max\{n : T_n \leq t\}, \quad t \geq 0$ (i.e., the number of arrival times up to time t). We assume that the interarrival times are independent. When the interarrival times X_n are identically distributed, the counting process is a *renewal process*. Results in

Sects. 3 and 4 consider a *delayed renewal process*, which is a slight modification of a renewal process, assuming an identical distribution for $X_n,\ n \geq 2$. In particular, the so-called *equilibrium renewal process* is a delayed renewal process, in which the first interarrival time X_1^e has the equilibrium distribution of X_2, that is

$$P(X_1^e > t) = \frac{1}{E(X_2)} \int_t^{\infty} P(X_2 > u)\mathrm{d}u, \quad t \geq 0. \tag{3}$$

Note that X_1^e has a decreasing density, and thus a concave cdf. The equilibrium renewal process arises in practice when a renewal process begins to be observed at an arbitrary but sufficiently large instant of time (cf. Asmussen 2000, p. 172).

3 Main results

3.1 The general model

This subsection introduces the general framework for our inventory model. This model, in particular, unifies Models 1, 2, and 3 in Rosling (2002). The main results used in the remaining subsections are also presented.

At some points in time, customers demand a random quantity of a particular item. We denote the demand of customer n by Z_n. We assume that the demands are i.i.d. with cdf F. Demand is satisfied, if possible, otherwise, any shortage is backlogged. At certain points in time, inventory position is reviewed and a replenishment order is placed (if necessary). Orders arrive after a random lead time. We assume that the lead times are independent and sequential (i.e., do not overlap). Backlogs are satisfied, on a first-come, first-served basis.

In inventory systems with backlogs, costs are computed a lead time ahead. To define the cost function, let x be the inventory position at time t, where x represents the sum of stock on hand and on order minus backlogs. Let L be the lead time. We use the notation $S_n := \sum_{i=1}^n Z_i,\ n = 1, 2, \ldots$, and $S_0 = 0$. The 'target period' is specified in each model. The cost function is calculated using the following four terms.

- The expected physical stock at hand in the target period, given by

$$I(x) = E\left[(x - S_{M_1})_+\right], \quad x \in \mathbb{R}, \tag{4}$$

  where $x_+ = \max(0, x), \quad x \in \mathbb{R}$, and M_1 is the number of demands in the target period. A cost $h \geq 0$ is associated with each unit in stock.
- The expected backlog in the target period

$$B_p(x) = E\left[(S_{M_1} - x)_+\right], \quad x \in \mathbb{R}. \tag{5}$$

  A cost $p \geq 0$ is associated with each backlogged unit.
- The expected rate at which new shortages are incurred, given by

$$B_{\bar{\pi}}(x) = cE[(S_{M_3} - x)_+ - (S_{M_2} - x)_+], \quad x \in \mathbb{R}, \tag{6}$$

where $c \geq 0$ and M_2 and M_3 are random variables depending on the model considered. A cost $\bar{\pi} \geq 0$ is associated with each new backlogged unit.

- The probability of backlog in the target period:

$$B_b(x) = E(1_{\{x<S_{M_1}\}}), \quad x \in \mathbb{R}, \tag{7}$$

where 1_A denotes the indicator function on the set A. A cost $b \geq 0$ is associated with each period with a backlog.

The cost function is

$$G(x) = hI(x) + pB_p(x) + \bar{\pi} B_{\bar{\pi}}(x) + bB_b(x), \quad x \in \mathbb{R}. \tag{8}$$

For example, in the periodic review-periodic demands model (Model 1 in Rosling 2002), the time horizon is divided into periods of equal length 1; orders (if any) and demands are produced at the beginning of each period, and a previous order (if any) arrives. As we can see in Rosling (2002), the ‘target period’ is the period between $t + L$ and $t + L + 1$ and the cost function is as in (4)–(8), where

$$M_1 := L + 1, \quad M_2 := L, \quad M_3 := L + 1, \quad \text{and} \quad c = 1. \tag{9}$$

In this case, M_2 and M_3 represent the number of demands at the beginning and at the end of the target period, respectively.

On the other hand, in Models 2 and 3 in Rosling (2002), demand is assumed to follow a renewal process $(N(t),\ t \geq 0)$, in which the mean interarrival time is $1/\lambda$. If we start observing the process at an arbitrary but large instant in time, we will observe the equilibrium renewal process $(\widetilde{N}(t),\ t \geq 0)$. We also assume that the underlying process describing the lead times has a stationary distribution described by the random variable L (see Zipkin 1986 for more details).

In Model 2, orders can be placed in any demand epoch but not between demand epochs. The target period is $[t, t + L]$ and the cost function is as in (4)–(8), where

$$M_1 := \widetilde{N}(L), \quad M_2 := N(L), \quad M_3 := N(L) + 1, \quad \text{and} \quad c = \lambda. \tag{10}$$

Note that M_2 and M_3 describe the number of demands a lead time ahead just before and just after the arrival of the next customer.

In Model 3, the review is periodic. The ‘target period’ is the period between $t + L$ and $t + L + 1$ and the cost function, after some simplifications, is as in (4)–(8), with

$$M_1 := \widetilde{N}(L + U), \quad M_2 := \widetilde{N}(L), \quad M_3 := \widetilde{N}(L + 1), \quad \text{and} \quad c = 1, \tag{11}$$

where U is a uniform random variable on $(0, 1)$ independent of the other random variables in the model. The reason for introducing U is that in Model 3, the physical stock is averaged over $L \leq t \leq L + 1$, since new demands can arrive between $t + L$ and $t + L + 1$. In fact, note that expression (13) in Rosling (2002) corresponds to the distribution function of S_{M_1}, where M_1 is defined in (11). Furthermore, M_2 and M_3

represent the number of new demands in the time intervals $[t, t+L]$ and $[t, t+L+1]$, respectively.

A unified expression for the cost function (8) is provided in the following result.

Proposition 3.1 *Consider a demand system in which the random demands* $(Z_n)_{n\geq 1}$ *are i.i.d. with a general distribution function* F. *Then, the cost function* (8) *has the general expression*

$$G(x) = \begin{cases} h(x - E(M_1)D) + (p+h)\int_x^\infty (1 - F_{M_1}(y))\mathrm{d}y \\ \quad +\bar{\pi} c \int_x^\infty (F_{M_2}(y) - F_{M_3}(y))\mathrm{d}y + b(1 - F_{M_1}(x)), & x \geq 0, \\ \\ p(E(M_1)D - x) + \bar{\pi} c D E(M_3 - M_2) + b, & x < 0, \end{cases} \tag{12}$$

where $D = E(Z_n)$.

Proof The first term in (8) involves the computation of $I(x)$ using (4). Let $\bar{H}$ be the survival function of $(x - S_{M_1})_+$. Standard computations show that

$$\bar{H}(z) = P(S_{M_1} < x - z)1_{\{z \leq x\}}, \quad z \geq 0, \quad x \geq 0.$$

Therefore, taking into account (4) and using the well known formula to compute expected values of non-negative random variables by integrating the survival function (see, for instance, Billingsley 1995, p. 275), we obtain

$$\begin{aligned} I(x) &= \int_0^\infty \overline{H}(z)\mathrm{d}z = \int_0^x P(S_{M_1} < x - z)\mathrm{d}z = \int_0^x (1 - \overline{F}_{M_1}(x - z))\mathrm{d}z \\ &= x - \int_0^x \overline{F}_{M_1}(y)\mathrm{d}y = x - E(S_{M_1}) + \int_x^\infty \overline{F}_{M_1}(y)\mathrm{d}y, \quad x \geq 0. \end{aligned}$$

Therefore, we have

$$I(x) = x - E(M_1)D + \int_x^\infty \overline{F}_{M_1}(y)\mathrm{d}y, \quad x \geq 0. \tag{13}$$

Note that $I(x) = 0$ for $x < 0$, as follows from (4). To calculate $B_p(x)$ in (8), we take into account that

$$x - S_{M_1} = (x - S_{M_1})_+ - (S_{M_1} - x)_+.$$

Thus, taking expectations in the previous expression, we obtain

$$x - E(M_1)D = I(x) - B_p(x),$$

and therefore, taking into account (13), we conclude that

$$B_p(x) = I(x) + E(M_1)D - x = \int_x^\infty \overline{F}_{M_1}(y)\mathrm{d}y, \quad x \geq 0. \tag{14}$$

Note that $B_p(x) = E(M_1)D - x$, when $x < 0$, as follows from (5). For the third term in (8), observe that, taking into account (14), we can write (6) as

$$B_{\bar{\pi}}(x) = c\int_x^{\infty} (\overline{F}_{M_3}(y) - \overline{F}_{M_2}(y))\mathrm{d}y = c\int_x^{\infty} (F_{M_2}(y) - F_{M_3}(y))\mathrm{d}y, \quad x \geq 0. \tag{15}$$

Note that $B_{\bar{\pi}}(x) = c(E(S_{M_3}) - E(S_{M_2})) = cDE(M_3 - M_2)$ for $x < 0$, as follows from (6). Finally, for the fourth term in (8), note that we take expectations involving a Bernoulli random variable with a success probability that corresponds to the probability of backlogging and therefore

$$B_b(x) = \overline{F}_{M_1}(x), \quad x \in \mathbb{R}. \tag{16}$$

Multiplying (13)–(16) by their respective cost terms and adding up, we obtain (12) for $x \geq 0$. The case $x < 0$ is obtained in a similar way. □

Example 3.2 To illustrate (12) in Proposition 3.1, let us consider a compound renewal model with continuous review, where the arrival of demands follows a Poisson process of rate λ, the lead time has an exponential distribution with mean $1/\mu$, and the demands have an exponential distribution with mean D. It is well known that in this case, the equilibrium renewal process coincides with the original Poisson process, and that $N(L)$ has a geometric distribution with success probability $\mu/(\lambda + \mu)$, i.e.,

$$P(N(L) = k) = \left(\frac{\lambda}{\lambda + \mu}\right)^k \frac{\mu}{\lambda + \mu}, \quad k = 0, 1, 2 \ldots$$

It is also well known that $S_{M_3} = S_{N(L)+1}$ has an exponential distribution with mean $\frac{D(\lambda+\mu)}{\mu}$. Therefore, $F_{M_3}(x) = 1 - e^{-\mu x/(D(\lambda+\mu))}$, $x \geq 0$, as well as $S_{M_1} = S_{M_2} = S_{N(L)}$ and

$$F_{M_1}(x) = F_{M_2}(x) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda + \mu}\left(1 - e^{-\mu x/(D(\lambda+\mu))}\right) = 1 - \frac{\lambda}{\lambda + \mu} e^{-\mu x/(D(\lambda+\mu))}.$$

Furthermore, $E(M_1) = E[N(L)] = \lambda/\mu$. Therefore, using Proposition 3.1 and (10), the cost function is

$$G(x) = h\left(x - \frac{\lambda D}{\mu}\right) + \left((p + h)\frac{\lambda D}{\mu} + \lambda\bar{\pi} D + \frac{b\lambda}{\lambda + \mu}\right) e^{-\mu x/(D(\lambda+\mu))}, \quad x \geq 0. \tag{17}$$

Figure 1 shows the cost function for $h = 0.25$, $p = 0.3$, $\bar{\pi} = b = 0.05$, $\lambda = D = 1$, and $\mu = 0.2$.

Note that, in this case, we can find an explicit formula for the cost function, but in general, the distribution function of a compound distribution such as F_{M_1} does not have an explicit expression, and therefore, numerical approximations are needed.

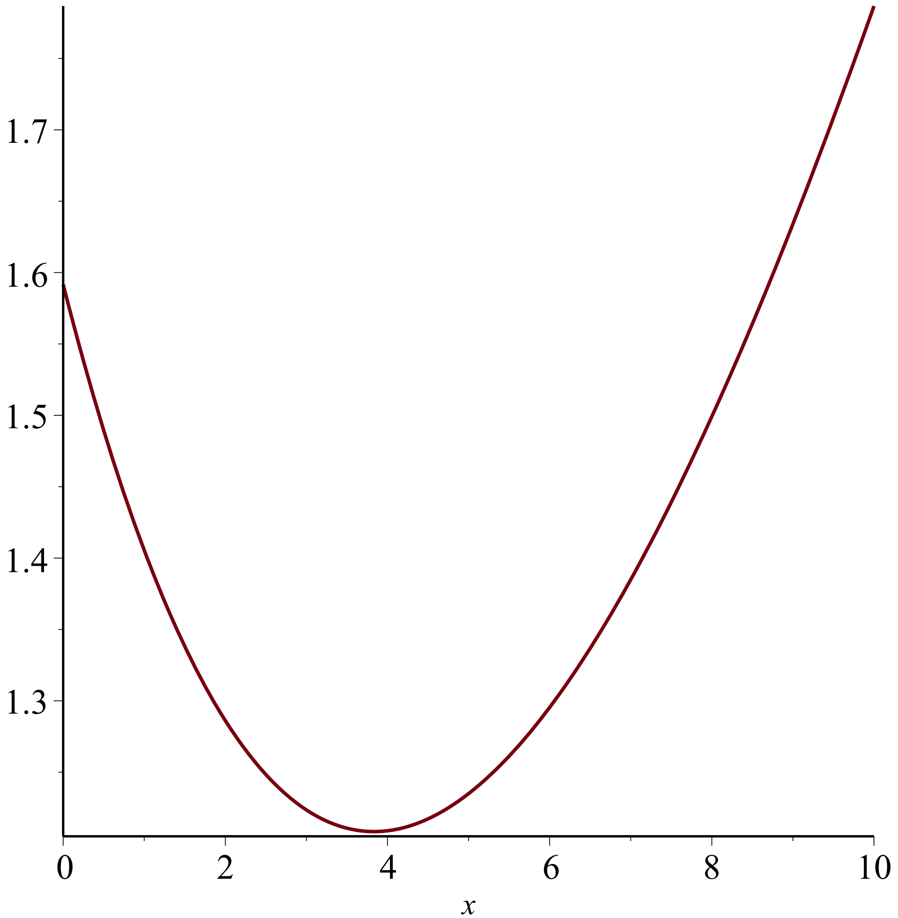

Fig. 1 The cost function in (17) for $h = 0.25,\ p = 0.3,\ \bar{\pi} = b = 0.05,\ \lambda = D = 1,\ \mu = 0.2$

The following result provides conditions for the quasiconvexity of (12), using similar arguments as in Rosling (2002), Prop. 2.1. for the periodic review-periodic demand model and a deterministic lead time L.

Proposition 3.3 *Consider a demand system with backlog as in* (4)–(7)*, where the lead time L is random. Assume that the sequence of i.i.d. random demands* $(Z_n)_{n\geq 1}$ *has a general cdf* F*. Let G be the cost function as in* (12)*. Then*

(a) *G is quasiconvex for all non-negative values of* $h,\ p,\ \bar{\pi}$ *and b if either*

1. Z_n *is continuous, and* $f_{M_1}(x)/F_{M_1}(x)$*, as well as* $(F_{M_2}(x) - F_{M_3}(x))/F_{M_1}(x)$ *are decreasing in x or*
2. $Z_n = \delta K_n$*, where* $\delta > 0$*,* K_n *takes the values* $1, 2, \ldots,$ *and both* $P(S_{M_1} = (k+1)\delta)/F_{M_1}(k\delta)$ *and* $(F_{M_2}(\delta k) - F_{M_3}(\delta k))/F_{M_1}(\delta k)$ *decrease in* $k \in \mathbb{N}$.

(b) *G is quasiconvex for $b = 0$ and all non-negative values of h, p and $\bar{\pi}$ if and only if $(F_{M_2}(x) - F_{M_3}(x))/F_{M_1}(x)$ is decreasing in x.*

Proof For part (a) in the case of continuous demands, as well as for part (b), the proof is similar to that given in Rosling (2002), Prop. 2.1 (i). For part (a) in the case of discrete demands, we need to take into account that the cost function is no longer differentiable if $b > 0$, so that different arguments are needed. First, as mentioned before, we will restrict our attention to the study of $(G(k\delta),\ k \in \mathbb{Z})$, as the inventory position will only take values on $k\delta,\ k \in \mathbb{Z})$. Since the cost function (12) linearly decreases for negative values, with a possible jump downwards at $x = 0$, we only need to prove the quasiconvexity of the sequence $(G(k\delta),\ k \in \mathbb{N})$. To prove this, it is sufficient to show that $G((k+1)\delta) - G(k\delta)$ is increasing. In this case, $G(k\delta)$ can have only one change in monotonicity, from decreasing to increasing, and therefore, the quasiconvexity of $(G(k\delta),\ k \in \mathbb{Z})$ is guaranteed. Notice that a simple computation applied to (12) gives us that

$$G(k\delta) = h(k\delta - E(M_1)D) + (p+h)\delta \sum_{i=k}^{\infty}(1 - F_{M_1}(i\delta))$$
$$+ c\bar{\pi}\delta \sum_{i=k}^{\infty}(F_{M_2}(i\delta) - F_{M_3}(i\delta)) + b(1 - F_{M_1}(k\delta)), \quad k \in \mathbb{N},$$

thus yielding

$$\begin{aligned} &G((k+1)\delta) - G(k\delta) \\ &\quad = h\delta - (p+h)\delta(1 - F_{M_1}(k\delta)) - c\bar{\pi}\delta(F_{M_2}(k\delta) - F_{M_3}(k\delta)) \\ &\qquad - bP(S_{M_1} = (k+1)\delta) \\ &\quad = F_{M_1}(k\delta)\left(\frac{-p\delta}{F_{M_1}(k\delta)} + (p+h)\delta - c\bar{\pi}\delta\frac{F_{M_2}(k\delta) - F_{M_3}(k\delta)}{F_{M_1}(k\delta)}\right. \\ &\qquad \left. -b\frac{P(S_{M_1} = (k+1)\delta)}{F_{M_1}(k\delta)}\right). \end{aligned} \tag{18}$$

By assumption, the second factor in the last term in (18) is increasing in k. This shows the result. □

The next result provides sufficient conditions for Proposition 3.3 to hold.

Proposition 3.4 *Consider a demand system with backlog as in (4)–(7), where the lead time L is random. Assume that the sequence of i.i.d. random demands $(Z_n)_{n\geq 1}$ has a general cdf F. Let G be the cost function as in (12). Then*

(a) *G is quasiconvex for all non-negative values of h, p, $\bar{\pi}$ and b if the following conditions are satisfied:*

1. *M_1 is discrete log-concave and $M_2 \leq_{rh} M_1 \leq_{rh} M_3$.*

2. *Z_n has a decreasing pdf or $Z_n = \delta K_n$, where $\delta > 0$ and K_n takes the values $1, 2, \ldots$ with a decreasing pmf.*

(b) *G is quasiconvex for $b = 0$ and all non-negative values of h, p and $\bar{\pi}$ if Z_n has a MCR and $M_2 \leq_{rh} M_1 \leq_{rh} M_3$.*

Proof Let us prove first the following fact:

$$\text{If } Z_n \text{ has a MCR and } M_2 \leq_{rh} M_1 \leq_{rh} M_3, \text{ then } \frac{F_{M_2}(x) - F_{M_3}(x)}{F_{M_1}(x)} \text{ decreases} \tag{19}$$

In fact the MCR property for Z_n implies that $Z_1 + \cdots + Z_n \leq_{rh} Z_1 + \cdots + Z_{n+1}$, $n = 1, 2, \ldots$, as follows from Definitions 2.4(a) and 2.1(c). This, together with $M_2 \leq_{rh} M_1$, implies that $S_{M_2} \leq_{rh} S_{M_1}$, as follows from Shaked and Shanthikumar (2007), Thm.1.B.52. Thus, $F_{M_2}(x)/F_{M_1}(x)$ decreases. Similarly, if Z_n has an MCR and $M_1 \leq_{rh} M_3$, then $F_{M_3}(x)/F_{M_1}(x)$ increases. These two facts prove (19).

Part (a) when Z_n has a decreasing pdf follows, because if M_1 is discrete log-concave, then S_{M_1} has a DRHR (Badía et al. 2021, Cor. 1). Then, $f_{M_1}(x)/F_{M_1}(x)$ decreases. Moreover, if Z_n has a decreasing pdf, then it has a concave cdf and also an MCR. Thus, (19) follows and Proposition 3.3(a) 1. holds.

Part (a) for discrete demands and $\delta = 1$ follows, because if M_1 is discrete log-concave and Z_n has a decreasing pmf, then S_{M_1} has a discrete DRHR (see Badía et al. 2021, Thm. 1). Therefore, $\frac{P(S_{M_1}=k+1)}{F_{M_1}(k)}$ is decreasing (see Badía et al. 2021, Lem. 2). The case for general δ follows from the fact that $S_{M_1} = \delta \sum_{i=0}^{M_1} K_i$, thus implying that $P(S_{M_1} = (k+1)\delta)/F_{M_1}(k\delta)$ does not depend upon δ. The rest of the proof is similar to the continuous case.

Part (b) follows straightforward from (19) and Proposition 3.3(b). □

It should be noted that the sufficient conditions on the demand distribution in Proposition 3.4(a) are less general when stochastic lead times are considered as opposed to deterministic ones. For the periodic demand model with a deterministic lead time, quasiconvexity of the cost function is ensured by the DRHR property of the demand distribution (see Prop. 2-1 in Rosling 2002). For stochastic lead times, Proposition 3.4(a) requires the less general condition of a decreasing pdf or pmf for the demand distribution. However, this condition is necessary to guarantee the DRHR condition of S_{M_1} for any discrete log-concave M_1. Example 3.5 illustrates the case of discrete demands. See also Rosling (2002) and Cai and Willmot (2005) for the case of continuous demands. In any case, Proposition 3.4(a) still holds for a large class of individual demands, as the equilibrium distribution (3) associated with each random variable has a decreasing pdf.

Part (b) considers a larger class of continuous demand distributions, as previously mentioned in Sect. 2. Furthermore, the MCR property is implied by the discrete DRHR property. The most common discrete distributions have this property, as can be seen in Nair et al. (2018). Examples of these distributions include the binomial, Poisson, geometric, hypergeometric, negative binomial, logarithmic series, hyper-Poisson, Zeta, and Yule distributions.

The next example illustrates how the failure of the conditions in Proposition 3.4 can result in the cost function being non-quasiconvex.

Example 3.5 We start with an inventory model in which the demand condition of Proposition 3.4(a) 2. fails. As in Example 3.2, we consider a continuous review model, in which demand arrivals follow a standard Poisson process. We assume that Z_n is deterministically equal to 2 and that L is deterministically equal to 1, so discrete log-concave. Failure of the demand condition causes Proposition 3.3(a) to fail. In fact, as follows from (10), $S_{M_1} = 2M_1 = 2N(1)$, and, as $P(S_{M_1} = 1) = 0$, we have

$$0 = \frac{P(S_{M_1} = 1)}{F_{M_1}(0)} < \frac{P(S_{M_1} = 2)}{F_{M_1}(1)} = \frac{e^{-1}}{e^{-1}} = 1.$$

We will see that $G(k)$ is not quasiconvex by using similar arguments as those in Rosling (2002) (see the Appendix). Set $h = 1$ and $\bar{\pi} = 0$. Then, we have

$$G(1)-G(0) = 1-(p+1)P(2N(1) > 0)-b[P(2N(1) = 1)] = 1-(p+1)(1-e^{-1}),$$

and

$$G(2)-G(1) = 1-(p+1)P(2N(1) > 1)-b[P(2N(1) = 2)] = G(1)-G(0)-be^{-1}.$$

Therefore, $G(1) > G(0)$, for $p < (1 - e^{-1})^{-1} - 1 = 0.582$. Choosing $p = 0.5$, we see that $G(2) < G(1)$, for $b > e(1 - 1.5(1 - e^{-1})) = 0.1409$, which implies that $G(1) > \max(G(0), G(2))$, and therefore, $G(k)$ is not quasiconvex. Moreover, $\lim_{k\to\infty}(G(k+1) - G(k)) = 1$, as follows from (18). Thus, at least one more change in monotonicity (from decreasing to increasing) will occur.

The next example shows that if the condition $M_2 \leq_{rh} M_1 \leq_{rh} M_3$ fails in Proposition 3.4(b), quasiconvexity may also fail. Consider the periodic review model, where Z_n is deterministically equal to 1. Thus, $S_{M_1} = M_1 = L + 1$, $S_{M_2} = M_2 = L$ and $M_3 = M_1$, as can be seen from (9). We also assume that the lead time L satisfies $P(L = 0) = P(L = 2) = P(L > 2) = 1/3$. Note that $M_2 \not\leq_{rh} M_1$, as

$$1 = \frac{P(L + 1 \leq 1)}{P(L \leq 1)} > \frac{P(L + 1 \leq 2)}{P(L \leq 2)} = \frac{1}{2}.$$

This results in the failure of Proposition 3.3(b), as

$$0 = \frac{F_L(1) - F_{L+1}(1)}{F_{L+1}(1)} < \frac{F_L(2) - F_{L+1}(2)}{F_{L+1}(2)} = 1.$$

Similar to the previous example, we now consider $h = 1$ and $b = 0$ and show that $G(n)$ is not quasiconvex. We have

$$G(2)-G(1)=1-(p+1)P(L+1 > 1)-\bar{\pi}(P(L \leq 1)-P(L+1 \leq 1))=1-\frac{2(p+1)}{3}.$$

Thus, choosing $p = 1/3$, we have $G(2) - G(1) = 1/9 > 0$, as well as

$$G(3) - G(2) = 1 - (p+1)P(L+1 > 2) - \bar{\pi}[P(L \le 2) - P(L+1 \le 2)]$$
$$= 1 - \frac{p+1}{3} - \frac{\bar{\pi}}{3} = \frac{5}{9} - \frac{\bar{\pi}}{3}.$$

Thus, if we set $\bar{\pi} > 5/3$, we have $G(3) - G(2) < 0$. We then conclude that $G(k)$ is not quasiconvex. As in the previous example, a limiting argument shows that another change of monotonicity will take place.

3.2 Periodic review-periodic demands model

In this model, the cost function is as in (8) and (9). The following result holds as a consequence of Proposition 3.4.

Proposition 3.6 *Consider a periodic review-periodic demands model with backlog in which the lead time L is random. Let $(Z_n)_{n\ge 1}$ be the i.i.d. random sequence of demands. Let G be the cost function as defined in* (8). *Then*

(a) *G is quasiconvex for all non-negative values of h, p, $\bar{\pi}$, and b, if*

1. *L is discrete log-concave.*
2. *Z_n has a decreasing pdf or $Z_n = \delta K_n$, where $\delta > 0$ and K_n takes the values $1, 2, \ldots$ with a decreasing pmf.*

(b) *G is quasiconvex for $b = 0$ and all non-negative values of h, p, and $\bar{\pi}$, if Z_n has an MCR and L has a discrete DRHR.*

Proof We will first prove that

$$\text{If } L \text{ has a discrete DRHR, then } M_2 \le_{rh} M_1 \le_{rh} M_3. \quad (20)$$

By (9), $M_1 = M_3 = L+1$ and $M_2 = L$. Thus, we only need to show that $M_2 \le_{rh} M_1$, that is, $L \le_{rh} L+1$. Note that

$$\frac{P(L+1 \le k)}{P(L \le k)} = \frac{P(L \le k-1)}{P(L \le k)},$$

and the previous expression increases, because L has a discrete DRHR, and therefore, $a_k := P(L \le k)$, is a log-concave sequence. This shows (20).

Part (a) follows from Proposition 3.4(a), taking into account that, if L is discrete log-concave, then it has a discrete DRHR and (20) follows. Moreover, if L is discrete log-concave, then it is immediate that $M_1 = L+1$ is also discrete log-concave.

Part (b) immediately follows using Proposition 3.4(b) and (20). □

Some comments are in order regarding the previous result. Part (a) for continuous demands has already been proven in Badía et al. (2021). We therefore extend this result to discrete demands. Part (b) generalizes the result given in Badía and Sangüesa (2015), Thm. 1.2, as it includes a general distribution for the individual demands and relaxes the condition of L being discrete log-concave to L having a discrete DRHR. This allows us to include negative binomial distributions with an arbitrary shape parameter for random lead times, rather than just those with a shape parameter greater than or equal to 1, the restriction required for L to be discrete log-concave within this family.

3.3 Compound renewal demand models

In this subsection, the cost function is as in (8) and (10), or (11), depending on the form in which review is performed. The conditions for the quasiconvexity of the cost function will depend on the shape properties of the interarrival times, as well as on the shape properties of the lead time.

First, we prove a result that gives us the conditions we need in the renewal process to apply Proposition 3.4(b). Part (b) of the next result is similar to Theorem 4.3. in Badía and Sangüesa (2015), but in this case, we are dealing with the reversed hazard rate order instead of the likelihood ratio order. Part (a) is needed for the periodic review policy.

For the sake of simplicity, if we consider a counting process $\{\bar{N}(t) : t \geq 0\}$, and L is a non-negative random variable independent of the process with cdf $\bar{H}$, we will make the technical assumption that $\bar{H}$ has no common discontinuity points with the corresponding cdfs of $\bar{T}_n$, the arrival times in the counting process. In this way, we have (see Esary et al. 1973, p. 642)

$$P(\bar{N}(L) \leq n) = E[\bar{H}(\bar{T}_{n+1})]. \tag{21}$$

Proposition 3.7 *Let $\{N(t) : t \geq 0\}$ be a renewal process with interarrival time X and let $\{N^*(t) : t \geq 0\}$ be its associated delayed renewal process with first interarrival W. If X and W have an IFR and $W \leq_{hr} X$, then*

(a) *$N^*(L_1) \leq_{rh} N^*(L_2)$ for random times $L_i,\ i = 1, 2$, independent of $\{N^*(t) : t \geq 0\}$, such that $L_1 \leq_{rh} L_2$.*
(b) *$N(L) \leq_{rh} N^*(L) \leq_{rh} N(L) + 1$, if L is a random time having a DRHR independent of the processes.*

Proof For part (a), let $(X_n^*)_{n \geq 1}$ and $(\hat{X}_n)_{n \geq 1}$ be two independent copies of the interarrival times in the delayed renewal process, and let $T_n^* = \sum_{i=1}^n X_i^*$ and $\hat{T}_n = \sum_{i=1}^n \hat{X}_i$. Let H_{fi} and H_{se} be the cdfs of L_1 and L_2, respectively. Recalling Definition 2.1(b), we must show that

$$P(N^*(L_2) \leq n)P(N^*(L_1) \leq n+1) \leq P(N^*(L_2) \leq n+1)P(N^*(L_1) \leq n).$$

Using (21), this is equivalent to showing that

$$E[H_{se}(\hat{T}_{n+1})]E[H_{fi}(T^*_{n+2})] \leq E[H_{se}(T^*_{n+2})]E[H_{fi}(\hat{T}_{n+1})], \quad n = 0, 1, \ldots \tag{22}$$

To prove (22), we first check that $\hat{T}_{n+1} \leq_{hr} T^*_{n+2}$. This follows from Shaked and Shanthikumar (2007), Thm. 1.B.4 (a), which states that the *hr* order is preserved under convolutions of IFR random variables. In fact, we use that $\hat{T}_{n+1} = \hat{X}_1 + \cdots + \hat{X}_{n+1} + 0$, that 0, the random variable that accumulates all its probability at 0, is in the IFR class, since its reliability function is log-concave. Moreover, 0 is less than or equal to any non-negative random variable in the hazard rate order, as follows from Definition 2.1(b). Therefore, $\hat{X}_1 \leq_{hr} X_1^*, \ldots, \hat{X}_{n+1} \leq_{hr} X_{n+1}^*$, and $0 \leq_{hr} X_{n+2}^*$ and apply Shaked and Shanthikumar (2007), Thm. 1.B.4 (a) to these random variables, all of them in the IFR class.

Once we have shown that $\hat{T}_{n+1} \leq_{hr} T_{n+2}^*$, we apply the bivariate characterization of the hr order [see Müller and Stoyan 2002, Thm. 1.9.3. (a)] on $\hat{T}_{n+1}$ and T_{n+2}^*. Let us consider the bivariate auxiliary function $b(x, y) = H_{se}(x)H_{fi}(y)$, $x, y \in \mathbb{R}$. First, we show that

$$\text{If } L_1 \leq_{rh} L_2, \text{ then } b(x, y) - b(y, x), \text{ is increasing in } x, \text{ for all } x \geq y. \tag{23}$$

In fact, when $H_{fi}(y) = 0$, the assertion is true, because $H_{se}(y) \leq H_{fi}(y) = 0$ (as the reversed hazard rate order implies the usual stochastic order). Therefore, $b(x, y) - b(y, x) = 0$. When $H_{fi}(y) > 0$, we also have that $H_{fi}(x) > 0$, as $x \geq y$. Then, we can write

$$b(x, y) - b(y, x) = H_{fi}(x)H_{fi}(y)\left(\frac{H_{se}(x)}{H_{fi}(x)} - \frac{H_{se}(y)}{H_{fi}(y)}\right),$$

which obviously increases in x for $x \geq y$, since $L_1 \leq_{rh} L_2$. Thus, as $\hat{T}_{n+1} \leq_{hr} T_{n+2}^*$, we have, by Müller and Stoyan (2002), Thm. 1.9.3. (a) applied to $\hat{T}_{n+1}$, T_{n+2}^* and $b(x, y) = H_{se}(x)H_{fi}(y)$, that (22) holds.

For part (b), let $(X_n)_{n\geq 1}$ and $(X_n^*)_{n\geq 1}$ be two independent sequences of random variables denoting the interarrival times of $\{N(t) : t \geq 0\}$ and $\{N^*(t) : t \geq 0\}$, respectively. Let H be the cdf of L. By (21), the statement $N(L) \leq_{rh} N^*(L)$ is equivalent to prove that

$$E[H(T_{n+1}^*)]E[H(T_{n+2})] \leq E[H(T_{n+2}^*)]E[H(T_{n+1})], \quad n = 0, 1, \ldots \tag{24}$$

From (21), we see that $P(N(L) + 1 \leq n) = E[H(T_n)]$, $n = 2, 3, \ldots$. From this expression, and in a similar way as in (24), we deduce that $N^*(L) \leq_{rh} N(L) + 1$ is equivalent to

$$E[H(T_n)]E[H(T_{n+2}^*)] \leq E[H(T_{n+1})]E[H(T_{n+1}^*)], \quad n = 1, 2, \ldots \tag{25}$$

To prove (24) and (25), we first check that $T_{n+1}^* \leq_{hr} T_{n+1}$ and $T_n \leq_{hr} T_{n+1}^*$. As in part (a), we use Shaked and Shanthikumar (2007), Thm. 1.B.4 (a) again. In particular, to prove $T_n \leq_{hr} T_{n+1}^*$, we use that $T_n = 0 + X_1 + \cdots + X_n$, then $0 \leq_{hr} X_1^*$, $X_1 \leq_{hr} X_2^*$ and so on.

Once we have shown that $T_{n+1}^* \leq_{hr} T_{n+1}$ and $T_n \leq_{hr} T_{n+1}^*$, we apply the bivariate characterization of the hr order (see Thm. 1.9.3. (a) in Müller and Stoyan 2002) to the previous random variables. To do this, consider the bivariate auxiliary function

$b_d(x, y) = H(x)H(y+d)$, $x, y \in \mathbb{R}$, for each $d \geq 0$. Note that $H(y+d) = H_{fi}(y)$, where H_{fi} is the cdf of $L_1 = L - d$. Note also that, if L has a DRHR, then $L_1 \leq_{rh} L_2 = L$ (see Thm. 1.b.62 in Shaked and Shanthikumar 2007). Therefore, using (23), $b_d(x, y) - b_d(y, x) = H(x)H_{fi}(y) - H(y)H_{fi}(x)$ is increasing in x, for all $x \geq y$. Thus, as $T^*_{n+1} \leq_{hr} T_{n+1}$, we can apply (Müller and Stoyan 2002), Thm. 1.9.3. (a) to T^*_{n+1}, T_{n+1} and the function $g(x, y) = b_d(x, y)$ to obtain

$$E[H(T^*_{n+1})]E[H(T_{n+1} + d)] \leq E[H(T_{n+1})]E[H(T^*_{n+1} + d)] \tag{26}$$

and as $T_n \leq_{hr} T^*_{n+1}$

$$E[H(T_n)]E[H(T^*_{n+1} + d)] \leq E[H(T^*_{n+1})]E[H(T_n + d)]. \tag{27}$$

Finally, (24) and (25) follow replacing d by X_{n+2} in (26) and (27) and then taking expectations, and observing that $T^*_{n+1} + X_{n+2}$ has the same distribution as T^*_{n+2}. This concludes the proof of part (b). □

We are now in a position to prove our main result for compound renewal demand models, which includes the two policies indicated in Sect. 3.1.

Theorem 3.8 *Suppose the demand process is compound renewal, with the lead times independent of the demand process. Assume that the random sequence of i.i.d. demands $(Z_n)_{n\geq 1}$ has cdf F. Then, for both the continuous review and the periodic review models, we have the following*:

(a) *G is quasiconvex for all non-negative values of h, p, $\bar{\pi}$, and b if*
   1. *The interdemand times, as well as the stationary lead time L have a log-concave pdf and*
   2. *Z_n has a decreasing pdf or $Z_n = \delta K_n$, where $\delta > 0$ and K_n takes the values $1, 2, \ldots$ with a decreasing pmf.*

(b) *G is quasiconvex for $b = 0$ and all non-negative values of h, p, and $\bar{\pi}$, if Z_n has an MCR, the stationary lead time L has a DRHR, and the interdemand times have an IFR.*

Proof First, we consider the continuous review model. Recalling (10), it follows that $M_1 = \widetilde{N}(L)$, $M_2 = N(L)$ and $M_3 = N(L) + 1$. Let $(X_n)_{n\geq 1}$ be the interarrival times of the renewal process $\{N(t) : t \geq 0\}$. Next, we prove that

$$\text{If } X_1 \text{ has an IFR and } L \text{ has a DRHR, then } N(L) \leq_{rh} \widetilde{N}(L) \leq_{rh} N(L) + 1. \tag{28}$$

Note that if X_1 has an IFR, then the first renewal time in the stationary renewal process (say, X^*_1) has also an IFR, and we have that $X^*_1 \leq_{hr} X_1$ (see Marshall and Olkin 2007, Prop. C.6. p. 111). Thus, (28) follows from Proposition 3.7(b).

Part (a) for discrete demands follows from Proposition 3.4(a). In fact, if the interdemand times have a log-concave pdf, then $M_1 = \widetilde{N}(L)$ is discrete log-concave, as follows from Badía and Sangüesa (2015), Thm.4.5. Moreover, (28) is true, because

X_1 has a log-concave pdf, which implies that X_1 has an IFR and L has a log-concave pdf, and therefore has a DRHR.

Part (a) for continuous demands is shown in Badía et al. (2021), Thm. 2. Alternatively, as in the discrete case, it can be derived directly from Proposition 3.4(a).

Part (b) is immediately derived from (28) and the Proposition 3.4(b).

Now, attention is focused on the periodic review model. In this case, $M_1 = \widetilde{N}(L+U)$, $M_2 = \widetilde{N}(L)$, and $M_3 = \widetilde{N}(L+1)$. On the other hand, we have

$$\text{If } X_1 \text{ has an IFR and } L \text{ has a DRHR, then } \widetilde{N}(L) \leq_{rh} \widetilde{N}(L+U) \leq_{rh} \widetilde{N}(L+1). \tag{29}$$

To show (29), we check easily, applying the definition of the reversed hazard rate order, that $0 \leq_{rh} U \leq_{rh} 1$. Moreover, 0 and 1 are in the DRHR class, because they have a log-concave cdf. Finally, U is in the DRHR class as well, because it has a log-concave pdf. Thus, we apply (Shaked and Shanthikumar 2007), Lem. 1.B.44 which states that the rh order is closed under the convolution of DRHR random variables to show that $L \leq_{rh} L+U \leq_{rh} L+1$. Finally we apply Proposition 3.7(a) to obtain (29).

Part (a) follows by applying Proposition 3.4(a). First, if L has a log-concave pdf, then $L+U$ has a log-concave pdf, as well, since the uniform random variable has a log-concave pdf and this property is preserved by convolutions. Thus, again by Badía and Sangüesa (2015), Thm.4.5, $M_1 = \widetilde{N}(L+U)$ is discrete log-concave. Finally, if the interdemand times have log-concave pdf, they have also an IFR and (29) follows.

Part (b) follows directly from (29) and Proposition 3.4(b). □

Theorem 3.8(a) for the continuous review model and continuous demands was shown in Badía et al. (2021). Here, we extend this to include discrete demands. On the other hand, Theorem 3.8(b) generalizes the result given in Badía and Sangüesa (2015), Thm. 1.4 to lead times with a DRHR and interarrival times with an IFR, instead of both having a log-concave pdf. In particular, gamma random variables with any shape parameter can be used for the lead time, whereas the log-concave pdf condition only allows a shape parameter greater than or equal to one. With respect to the periodic review model, Theorem 3.8 generalizes the result given in Rosling (2002) for this model from a deterministic lead time to a stochastic lead time. In addition, sufficient conditions are included in Theorem 3.8(a), which were not previously known even for deterministic L.

The exponential pdf is both log-concave and decreasing. Therefore, if we have exponential interarrival times and exponential demands, as well as an exponential lead time L, the quasiconvexity of the cost function is guaranteed for both reviewing policies. The cost function for the continuous review model has been explicitly computed in Example 3.2. Further computations would be required to evaluate the cost function for the periodic review model. This is because, as far as we know, the cdf of a sum of exponential distributions up to $M_2 = N(L+U)$ has no simple expression.

4 Concluding remarks

This paper presents a general framework to study the quasiconvexity of the cost function in inventory models with nonlinear costs and random lead times. When particularized to specific inventory models considered in the literature, improvements in the conditions for the model components are obtained. Moreover, this approach includes different review policies, as well as all types of cdf's to model the demands, and not only demands with continuous cdf, as in the previous papers.

An interesting extension of our results would be achieved by considering more general cost functions. For instance, introducing an additional cost depending on the number of periods with backlog, and thus generalizing the fourth term in (8), as in Huh et al. (2011). Moreover, as shown in the case of compound renewal demands, the framework is flexible enough to cover different order policies, such as the periodic review or an ordering policy that only takes place at renewal points. In fact, it is well known that the ordering policies considered in Sect. 3.3 are generally suboptimal, so for specific problems, other policies closer to optimality could be studied.

Another interesting question is to consider models with random lead times, when the demand process, rather than compound renewal, follows a general stochastic process (Rosling 2002, Models 4 and 5).

Finally, as noted after Proposition 3.3, the DRHR condition for S_{M_1} plays a fundamental role in the quasiconvexity of the cost function when $b \neq 0$. However, in other optimization problems (see, for example Levi et al. 2015), it is important to investigate whether this random variable excluding the point mass at 0 has a log-concave pdf. This is a difficult problem when M_1 is random, as noted in Badía et al. (2021), Section 5, and positive answers are known only for certain distributions (see Ninh and Prekopa 2013). It would be interesting to study this problem in more detail to obtain specific conditions on the summands, so that this property holds (or to relax the condition on the number of summands).

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