Mathematical Methods in the Applied Sciences

WILEY

RESEARCH ARTICLE OPEN ACCESS

Approximation of Discontinuous Functions by Positive Linear Operators. A Probabilistic Approach

J.A. Adell[1] | P. Garrancho[2] | F.J. Martínez-Sánchez[2]

[1]Departamento de Métodos Estadísticos, Universidad de Zaragoza, Zaragoza, Spain | [2]Departamento de Matemáticas, Universidad de Jaén, Jaén, Spain

Correspondence: P. Garrancho (pgarran@ujaen.es)

Received: 9 May 2025 | **Revised:** 22 September 2025 | **Accepted:** 8 November 2025

Keywords: Bernstein polynomial | discontinuous function | positive linear operator | rate of convergence | total variation

ABSTRACT

We obtain approximation results for general positive linear operators satisfying mild conditions, when acting on discontinuous functions and absolutely continuous functions having discontinuous derivatives. The upper bounds, given in terms of a local first modulus of continuity, are best possible, in the sense that we can construct particular sequences of operators attaining them. When applied to functions of bounded variation or absolutely continuous functions having derivatives of bounded variation, these upper bounds are better and simpler to compute than the usual total variation bounds. The particular case of the Bernstein polynomials is thoroughly discussed. We use a probabilistic approach based on representations of such operators in terms of expectations of random variables.

MSC2020 Classification: 41A36, 41A10, 41A25, 26A15

1 | Introduction

Denote by $\mathbb{N}$ the set of positive integers and by I a real interval. It is known (c.f. [1]. and [2].) that many sequences L_n of positive linear operators allow for a probabilistic representation of the form

$$L_n(f, x) = \mathbb{E}\, f\left(x + \frac{Z_n(x)}{\sqrt{n}} \right), \quad x \in I, \; n \in \mathbb{N}, \tag{1}$$

where $\mathbb{E}$ stands for mathematical expectation, $Z_n(x)$ is a random variable such that $x + Z_n(x)/\sqrt{n}$ takes values in I, and $f : I \to \mathbb{R}$ is any measurable function for which the expectations in (1) are finite.

Consider, for instance, the Bernstein polynomials B_n defined by

$$B_n(f, x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}, \quad x \in [0, 1], \; n \in \mathbb{N}. \tag{2}$$

Let $x \in [0,1]$ and $n \in \mathbb{N}$. To give a probabilistic representation for B_n, let $(U_k)_{k\geq 1}$ be a sequence of independent copies of a random variable U having the uniform distribution on $[0,1]$. Define

$$S_n(x) = \sum_{k=1}^{n} 1_{[0,x]}(U_k), \tag{3}$$

where 1_A denotes the indicator function of the set A. Clearly, $S_n(x)$ has the binomial law with parameters n and x, that is,

$$P(S_n(x) = k) = \binom{n}{k} x^k (1-x)^{n-k}, \quad k = 0, 1, \ldots, n. \tag{4}$$

We therefore have from (2)

$$B_n(f,x) = \mathbb{E} f\left(\frac{S_n(x)}{n}\right) = \mathbb{E}\, f\left(x + \frac{Z_n(x)}{\sqrt{n}}\right),$$

where

$$Z_n(x) = \frac{S_n(x) - nx}{\sqrt{n}}. \tag{5}$$

The averaging operators considered by Khan [3], which include the Bernstein, Szász-Mirakyan, Baskakov, Gamma, and Weierstrass operators as particular cases, can be represented as in (1). Another interesting example (see the comments before Theorem 2) is the following Bézier variant of the Baskakov-Kantorovich operator introduced by Abel and Gupta [4]. Let $n \in \mathbb{N}$ and $x \geq 0$. Consider the negative binomial probabilities

$$v_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-n-k}, \quad k \in \mathbb{N} \cup \{0\},$$

and define the positive linear operator

$$V_{n,\alpha}^{*}(f,x) = n \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x) \int_{k/n}^{(k+1)/n} f(t)dt,$$

where f is locally integrable on $(0,\infty)$ and $f(t) = O(t^r)$, as $t \to \infty$, for some positive integer r, and

$$Q_{n,k}^{(\alpha)}(x) = \left(\sum_{j=k}^{\infty} v_{n,j}(x)\right)^{\alpha} - \left(\sum_{j=k+1}^{\infty} v_{n,j}(x)\right)^{\alpha}, \; k \in \mathbb{N} \cup \{0\}, \; \alpha \geq 1.$$

Let U and $T_{n,\alpha}(x)$ be two independent random variables such that U is uniformly distributed on $[0,1]$ and $T_{n,\alpha}(x)$ has probability law

$$P\big(T_{n,\alpha}(x) = k\big) = Q_{n,k}^{(\alpha)}(x), \quad k \in \mathbb{N} \cup \{0\}.$$

Then it can be checked that the operator $V_{n,\alpha}^{*}$ is represented in probabilistic terms as

$$V_{n,\alpha}^{*}(f,x) = \mathbb{E} f\left(\frac{T_{n,\alpha}(x) + U}{n}\right) = \mathbb{E} f\left(x + \frac{1}{\sqrt{n}} \frac{T_{n,\alpha}(x) + U - nx}{\sqrt{n}}\right). \tag{6}$$

Fix $x \in \mathring{I}$, the interior set of I. Let $B(I)$ be the set of all measurable bounded real functions defined on I whose right and left limits at x, respectively denoted by $f(x+)$ and $f(x-)$, exist. Note that both limits are finite, since f is a bounded function. Also, denote by $DB(I)$ the set of absolutely continuous functions ϕ such that

$$\phi(y) = \phi(x) + \int_{x}^{y} f(u)\, du, \quad y \in I, \; f \in B(I). \tag{7}$$

The aim of this paper is to give estimates for both

$$L_n(f,x) - \frac{f(x+) + f(x-)}{2} \quad \text{and} \quad L_n(\phi,x) - \phi(x),$$

where L_n is a positive linear operator of the form (1) satisfying the following two properties

$$P(|Z_n(x)| \geq \theta) \leq C_n(\beta) \exp(-\beta\theta), \quad \theta \geq 0, \tag{8}$$

for some positive constants β and $C_n(\beta)$, possibly depending upon x, and

$$\lim_{n\to\infty} P(Z_n(x) > 0) = \lim_{n\to\infty} P(Z_n(x) < 0) = \frac{1}{2}. \tag{9}$$

Assumptions (8) and (9) are fulfilled in many usual cases. For instance, suppose that $\mathbb{E}\exp(\beta|Z_n(x)|) < \infty$, for some $\beta > 0$. Then, Markov's inequality gives us

$$P(|Z_n| \geq \theta) \leq \mathbb{E}\exp(\beta|Z_n(x)|)\exp(-\beta\theta), \quad \theta \geq 0. \tag{10}$$

In the same way, if $Z_n(x)$ satisfies the central limit theorem, then property (9) holds. This is the case for the random variables $Z_n(x)$ defined in (5).

Many authors have considered the problems posed here for specific sequences of positive linear operators, such as Bernstein and Szász-Mirakyan operators (see, for example, Cheng [5], Zeng and Cheng [6], and Bustamante et al. [7].). Important subsets of $B(I)$ and $DB(I)$ are, respectively, the set of functions having bounded variation on I, denoted by $BV(I)$, and the set of absolutely continuous functions whose Radon-Nikodym derivative has bounded variation on I, denoted by $DBV(I)$. There is a huge literature on approximation results for functions belonging to $BV(I)$ and $DBV(I)$ for particular sequences of operators such as Bernstein,Szász-Mirakyan, Meyer-König and Zeller, Bézier variants of Kantorovich operators, King-type or exponential-type operators (cf. Cheng [8], Bojanic and Cheng [9], Guo [10], Zeng and Piriou [11], Gupta [12, 13], Gupta and Ispir [14], Özarslan et al. [15], and Abel and Gupta [4, 16], among many others). Khan [3]. (see also Guo and Khan [17].) obtained approximation results for functions in $BV(I)$ for general univariate and multivariate positive linear operators.

Usually, the results referring to the sets $BV(I)$ and $DBV(I)$ take on the following form. If $f \in BV(I)$, denote by $V(f, J)$ the total variation of f on the subset $J \subset I$ and define the function

$$\begin{aligned} f_x(y) = (f(y) - f(x+))1_{I\cap(x,\infty)}(y) \\ + (f(y) - f(x-))1_{I\cap(-\infty,x)}(y), \qquad y \in I. \end{aligned} \tag{11}$$

For the Bernstein polynomials, Zeng and Piriou [11]. (see also Bustamante et al. [7, Th.17]) showed that

$$\begin{aligned} \left| B_n(f,x) - \frac{f(x+) + f(x-)}{2} \right| \leq \frac{3}{nx(1-x)+1} \sum_{k=0}^{n-1} V\left(f_x, J(x,k)\right) \\ + \frac{2}{\sqrt{nx(1-x)+1}} \left(|f(x+) - f(x-)| + e_n(x)|f(x) - f(x-)| \right), \end{aligned} \tag{12}$$

where

$$J(x,k) = \left[x - \frac{x}{\sqrt{k+1}}, x + \frac{1-x}{\sqrt{k+1}} \right] \text{ and } e_n(x) = \sum_{k=1}^{n-1} 1_{\{k/n\}}(x).$$

For other results concerning functions in $DBV(I)$, see Section 5.

In this paper, we give approximation results for functions in $B(I)$ and $DB(I)$. The corresponding upper bounds are given in terms of a local first modulus of continuity of the functions under consideration (see Section 3, particularly Definition 3.1). When applied to the subsets $BV(I)$ and $DBV(I)$, such bounds are better and simpler to compute than the total variation term on the right-hand side in (12). This is one of the main features of this paper. In fact, we show that our upper bounds are best possible, in the sense that we can construct a sequence of positive linear operators attaining them when acting on large sets of symmetric functions (see Section 4). The main results are given in Section 3. Such results are illustrated in the case of the Bernstein polynomials in Section 5, including a comparative discussion with other known results in the literature.

We finally point out that the approach given in this paper could be generalized at the price of introducing more involved notations. Specifically, the set $B(I)$ could be enlarged to include unbounded functions. In this respect, Khan [3]. obtained approximation results for functions having finite local variation and satisfying appropriate integrability conditions. On the other hand, assumptions (8) and (9) could be weakened (see, for instance, Theorem 4 in Section 3 regarding assumption (9)).

2 | Preliminary Results

Let $\mathcal{L}$ be the set of nondecreasing functions $\psi : [0,\infty] \to [0,\infty)$ such that

$$\psi(0) = \psi(0+) = 0, \qquad 0 < \psi(\infty) := \lim_{\theta\to\infty}\psi(\theta) < \infty. \tag{13}$$

Given $\psi \in \mathcal{L}$, we define

$$\Psi(y) = \int_0^y \psi(\theta)\, d\theta, \quad y \geq 0. \tag{14}$$

On the other hand, let X_β be a random variable having the exponential distribution with parameter β, that is,

$$F_\beta(y) := P(X_\beta \leq y) = \int_0^y \beta\, \exp(-\beta\theta)\, d\theta = 1 - \exp(-\beta y), \quad y \geq 0. \tag{15}$$

A crucial quantity in this paper is the following

$$K_n(\beta) := \mathbb{E}\, \psi\left(\frac{X_\beta}{\sqrt{n}}\right), \quad n \in \mathbb{N},\ \psi \in \mathcal{L}. \tag{16}$$

By (13) and the dominated convergence theorem, we see that $K_n(\beta) = o(1)$, as $n \to \infty$. Upper and lower bounds for $K_n(\beta)$ of the same order of magnitude are given in the following result.

Lemma 1. *Let $\psi \in \mathcal{L}$ and let Ψ be as in (14). For any $m \in \mathbb{N}$, with $m \geq 2$, we have*

$$\frac{1}{m}\sum_{k=1}^{m-1}\psi\left(\frac{1}{\beta\sqrt{n}}\log\frac{m}{k}\right) \leq \mathbb{E}\,\psi\left(\frac{X_\beta}{\sqrt{n}}\right) \leq \frac{1}{m}\sum_{k=1}^{m-1}\psi\left(\frac{1}{\beta\sqrt{n}}\log\frac{m}{k}\right) + \frac{1}{m}\psi(\infty). \tag{17}$$

In addition,

$$\mathbb{E}\,\Psi\left(\frac{X_\beta}{\sqrt{n}}\right) = \frac{1}{\beta\sqrt{n}}\mathbb{E}\,\psi\left(\frac{X_\beta}{\sqrt{n}}\right). \tag{18}$$

Proof. Let $0 = y_0 < y_1 < \ldots < y_{m-1} < y_m = \infty$ be a finite sequence such that $F_\beta(y_j) = j/m$ or, equivalently, $P(y_{j-1} \leq X_\beta \leq y_j) = 1/m,\ j = 1, \ldots, m$. Note that

$$y_j = \frac{1}{\beta}\log\frac{m}{m-j}, \quad j = 1, \ldots, m-1.$$

Since ψ is nondecreasing, we have from (16)

$$\mathbb{E}\,\psi\left(\frac{X_\beta}{\sqrt{n}}\right) = \sum_{j=1}^{m}\int_{y_{j-1}}^{y_j}\psi\left(\frac{\theta}{\sqrt{n}}\right)\beta\,\exp(-\beta\theta)\,d\theta \leq \frac{1}{m}\sum_{j=1}^{m-1}\psi\left(\frac{y_j}{\sqrt{n}}\right) + \frac{1}{m}\psi(\infty),$$

which shows the upper bound in (17). The lower bound is proved in a similar way.

On the other hand, we have from (14), (15), and Fubini's theorem

$$\mathbb{E}\,\Psi\left(\frac{X_\beta}{\sqrt{n}}\right) = \mathbb{E}\int_0^{X_\beta/\sqrt{n}} \psi(\theta)\,d\theta = \mathbb{E}\int_0^\infty \psi(\theta)1_{\{\theta\le X_\beta/\sqrt{n}\}}\,d\theta$$
$$= \int_0^\infty \psi(\theta)P(X_\beta \ge \theta\sqrt{n})d\theta$$
$$= \frac{1}{\beta}\int_0^\infty \psi(\theta)\beta\exp(-\beta\theta\sqrt{n})d\theta = \frac{1}{\beta\sqrt{n}}\mathbb{E}\psi\left(\frac{X_\beta}{\sqrt{n}}\right),$$

thanks to the change $u = \theta\sqrt{n}$. This shows (18) and completes the proof. □

Since $K_n(\beta) = o(1)$, as $n \to \infty$, we see from (18) that

$$\mathbb{E}\Psi\left(\frac{X_\beta}{\sqrt{n}}\right) = \frac{1}{\sqrt{n}}o(1), \quad \text{as } n \to \infty.$$

Let X be a nonnegative random variable whose tail probabilities satisfy

$$P(X \ge y) \le C(\beta)\ \exp(-\beta y), \quad y \ge 0, \tag{19}$$

for some positive constant $C(\beta)$.

Lemma 2. *Let $\psi \in \mathcal{L}$ and let X be as in (19). Then,*

$$\mathbb{E}\psi\left(\frac{X}{\sqrt{n}}\right) \le C(\beta)\mathbb{E}\ \psi\left(\frac{X_\beta}{\sqrt{n}}\right)$$

and

$$\mathbb{E}\Psi\left(\frac{X}{\sqrt{n}}\right) \le \frac{C(\beta)}{\beta\sqrt{n}}\mathbb{E}\ \psi\left(\frac{X_\beta}{\sqrt{n}}\right).$$

Proof. Define

$$\tilde{\psi}(y) := \lim_{\theta\downarrow y}\psi(\theta), \quad y \ge 0.$$

The function $\tilde{\psi} \in \mathcal{L}$ is right-continuous and satisfies

$$\psi(y) \le \tilde{\psi}(y), \quad y \ge 0, \qquad \tilde{\psi}(\infty) = \psi(\infty).$$

Let W be a random variable whose distribution function is given by

$$F(y) = P(W \le y) = \frac{\tilde{\psi}(y)}{\psi(\infty)}, \quad y \ge 0. \tag{20}$$

Without loss of generality, we can assume that W is independent of X and X_β. From (19)-(20), we infer that

$$\mathbb{E}\psi\left(\frac{X}{\sqrt{n}}\right) \le \mathbb{E}\ \tilde{\psi}\left(\frac{X}{\sqrt{n}}\right) = \psi(\infty)P(W \le X/\sqrt{n}) \le \psi(\infty)C(\beta)\mathbb{E}\ \exp(-\beta W\sqrt{n}). \tag{21}$$

We repeat the same procedure replacing X by X_β. Since ψ and $\tilde{\psi}$ differ in a countable set at most and $P(X_\beta \ge y) = \exp(-\beta y)$, $y \ge 0$, we get

$$\mathbb{E}\psi\left(\frac{X_\beta}{\sqrt{n}}\right) = \mathbb{E}\tilde{\psi}\left(\frac{X_\beta}{\sqrt{n}}\right) = \psi(\infty)P(W \le X_\beta/\sqrt{n}) = \psi(\infty)\mathbb{E}\exp(-\beta W\sqrt{n}).$$

This and (21) show the first inequality in Lemma 2. The proof of the second one is analogous to that of (18) and therefore we omit it. □

3 | Main Results

The upper bounds in the approximation results in this paper will be given in terms of the local first modulus of continuity defined as follows.

Definition 1. *Let $x \in \mathring{I}$ and $h \in B(I)$. The first modulus of continuity of h at x is defined as*

$$\omega_x(h,\theta) = \sup\{|h(x+u) - h(x)| : \ x+u \in I,\ |u| \le \theta\}, \quad \theta \ge 0. \tag{22}$$

From now on, we fix $x \in \mathring{I}$. For this reason, we simply write $\omega_x(h,\cdot) = \omega(h,\cdot)$. Also, if $f \in B(I)$, its associated function f_x, as defined in (11), is simply denoted by $g = f_x$. Observe that g is continuous at x and $g(x) = 0$. It therefore follows from (22) that

$$\omega(g,\theta) = \sup\{|g(x+u)| : \ x+u \in I,\ |u| \le \theta\}, \quad \theta \ge 0. \tag{23}$$

Observe that $\omega(g,\cdot) \in \mathcal{L}$. By distinguishing the cases $y < x$, $y = x$, and $y > x$, it can be checked that

$$\begin{aligned} f(y) - g(y) - \frac{1}{2}(f(x+) + f(x-)) &= (f(x) - f(x-))1_{\{x\}}(y) \\ &\quad + (f(x+) - f(x-))\left(1_{I\cap(x,\infty)}(y) - \frac{1}{2}\right), \qquad y \in I. \end{aligned} \tag{24}$$

We give the following result for the operators L_n considered in (1).

Theorem 1. *Let $f \in B(I)$ and let $Z_n(x)$ be as in (1). Then,*

$$\begin{aligned} \left| L_n(f,x) - \frac{f(x+) + f(x-)}{2} \right| &\le |f(x) - f(x-)| P(Z_n(x) = 0) \\ &\quad + |f(x+) - f(x-)| \left| P(Z_n(x) > 0) - \frac{1}{2} \right| + C_n(\beta)\mathbb{E}\omega\left(g, \frac{X_\beta}{\sqrt{n}}\right), \end{aligned} \tag{25}$$

where $C_n(\beta)$ is defined in (8).

Proof. Replacing y by the random variable $x + Z_n(x)/\sqrt{n}$ in (24) and then taking expectations, we obtain

$$\begin{aligned} L_n(f,x) - \frac{f(x+) + f(x-)}{2} &= (f(x) - f(x-))P(Z_n(x) = 0) \\ &\quad + (f(x+) - f(x-))\left(P(Z_n(x) > 0) - \frac{1}{2}\right) + L_n(g,x). \end{aligned} \tag{26}$$

On the other hand, we have from (8) and Lemma 2 applied to the function $\omega(g,\cdot) \in \mathcal{L}$, as defined in (23),

$$|L_n(g,x)| = \left| \mathbb{E}g\left(x + \frac{Z_n(x)}{\sqrt{n}}\right)\right| \le \mathbb{E}\omega\left(g, \frac{|Z_n(x)|}{\sqrt{n}}\right) \le C_n(\beta)\mathbb{E}\omega\left(g, \frac{X_\beta}{\sqrt{n}}\right),$$

which, in conjunction with (26), shows the result. □

For the operators defined in (6), we have (see Abel and Gupta [4].)

$$P(Z_n(x) > 0) \to p(x), \quad P(Z_n(x) < 0) \to q(x), \quad n \to \infty, \tag{27}$$

where $p(x) + q(x) = 1$, $p(x) \neq 1/2$. The same happens for the operators considered in Gupta [13], Gupta and Ispir [14], and Mowzer [18]. Under condition (27), Theorem 1 can be reformulated as follows. We replace identity (24) by

$$\begin{aligned} f(y) - g(y) - (f(x+)p(x) + f(x-)q(x)) \\ = (f(x+) - f(x))(1_{I\cap(x,\infty)}(y) - p(x)) \\ + (f(x-) - f(x))(1_{I\cap(-\infty,x)}(y) - q(x)), \quad y \in I. \end{aligned}$$

Following the same steps as in the proof of Theorem 1, we can show the following result.

Theorem 2. *Let $f \in B(I)$ and let $Z_n(x)$ be as in (1). Under assumption (27), we have*

$$|L_n(f,x) - (f(x+)p(x) + f(x-)q(x))| \leq |f(x+) - f(x)||P(Z_n(x) > 0) - p(x))| + |f(x-) - f(x)||P(Z_n(x) < 0) - q(x))| + C_n(\beta)\mathbb{E}\omega\left(g, \frac{X_\beta}{\sqrt{n}}\right).$$

Let $\phi \in DB(I)$ as in (7) and let $g := f_x \in B(I)$ be the function associated to the Radon-Nikodym derivative $f \in B(I)$ of ϕ. By (24), we can write

$$\begin{aligned}\phi(y) - \phi(x) &= \int_x^y (f(u) - g(u))\, du + \int_x^y g(u)\, du \\ &= \frac{1}{2}(f(x+) + f(x-))(y-x) + \frac{1}{2}(f(x+) - f(x-))|y-x| \\ &+ \int_0^{y-x} g(x+u)\, du, \quad y \in I.\end{aligned} \tag{28}$$

For functions in $DB(I)$, we give the following result.

Theorem 3. *Let $\phi \in DB(I)$ and let $Z_n(x)$ be as in (1). Then,*

$$\left| L_n(\phi,x) - \phi(x) - \frac{1}{2\sqrt{n}}(f(x+) + f(x-))\mathbb{E}Z_n(x) - \frac{1}{2\sqrt{n}}(f(x+) - f(x-))\mathbb{E}|Z_n(x)| \right| \leq \frac{C_n(\beta)}{\beta\sqrt{n}}\mathbb{E}\omega\left(g, \frac{X_\beta}{\sqrt{n}}\right).$$

Proof. Replacing y by the random variable $x + Z_n(x)/\sqrt{n}$ in (28) and then taking expectations, we get

$$\begin{aligned}xL_n(\phi,x) - \phi(x) &= \frac{1}{2\sqrt{n}}(f(x+) + f(x-))\mathbb{E}Z_n(x) \\ &+ \frac{1}{2\sqrt{n}}(f(x+) - f(x-))\mathbb{E}|Z_n(x)| + \mathbb{E}\int_0^{Z_n(x)/\sqrt{n}} g(x+u)\, du.\end{aligned} \tag{29}$$

From (23), we see that

$$\left|\int_0^y g(x+u)\, du\right| \leq \int_0^{|y|} \omega(g,u)\, du, \ \ y \in \mathbb{R}.$$

Applying Lemma (14) with $\psi = \omega(g,\cdot)$, this implies that

$$\begin{aligned}\left|\mathbb{E}\int_0^{Z_n(x)/\sqrt{n}} g(x+u)\, du\right| &\leq \mathbb{E}\int_0^{|Z_n(x)|/\sqrt{n}} \omega(g,u)\, du = \mathbb{E}\Psi\left(\frac{|Z_n(x)|}{\sqrt{n}}\right) \\ &\leq \frac{C_n(\beta)}{\beta\sqrt{n}}\mathbb{E}\omega\left(g, \frac{X_\beta}{\sqrt{n}}\right).\end{aligned}$$

This and (29) show the result. □

Observe that Theorems 1 and 3 are meaningful, since

$$\mathbb{E}\omega\left(g, \frac{X_\beta}{\sqrt{n}}\right) = o(1), \quad n \to \infty,$$

as asserted after definition (16). On the other hand, such theorems can be respectively applied to functions in the sets $BV(I)$ and $DBV(I)$ without any modification. Suppose, for instance, that $f \in BV(I)$. It follows from (23) that

$$\omega(g, \theta) \leq V(g, [x-\theta, x+\theta] \cap I), \quad \theta \geq 0. \tag{30}$$

Therefore, there is no need to express the upper bounds in Theorems 1 and 3 in terms of the total variation of g on appropriate subintervals of I, as done in (12). More details will be given in Section 5.

4 | The Convolution Operators L_n^β

The purpose of this section is to construct a sequence L_n^β of positive linear operators attaining the upper bounds in Theorems 1 and 3 when acting on certain subsets of symmetric functions.

Let $\beta > 0$. Denote by Y_β a random variable having the symmetric exponential density

$$\rho_\beta(\theta) = \frac{\beta}{2} \exp(-\beta|\theta|), \quad \theta \in \mathbb{R}.$$

Note that

$$\mathbb{E}h(Y_\beta) = \frac{\beta}{2} \int_0^\infty (h(\theta) + h(-\theta)) \, \exp(-\beta\theta) \, d\theta = \frac{1}{2}\mathbb{E}(h(X_\beta) + h(-X_\beta)), \tag{31}$$

for any measurable function $h : \mathbb{R} \to \mathbb{R}$ for which the preceding integral makes sense, X_β being the random variable defined in (15).

We consider the sequence L_n^β of convolution operators defined as

$$L_n^\beta(f, x) = \mathbb{E}f\left(x + \frac{Y_\beta}{\sqrt{n}}\right), \quad x \in \mathbb{R}, \ n \in \mathbb{N}, \tag{32}$$

acting on measurable functions $f : \mathbb{R} \to \mathbb{R}$ for which the preceding expectations exist. As follow from (31) and (32), we can write

$$L_n^\beta(f, x) = \frac{1}{2}\left(\mathbb{E}f\left(x + \frac{X_\beta}{\sqrt{n}}\right) + \mathbb{E}f\left(x - \frac{X_\beta}{\sqrt{n}}\right)\right), \quad x \in \mathbb{R}, \quad n \in \mathbb{N}. \tag{33}$$

Fix $x \in \mathbb{R}$. Denote by $\mathcal{S} \subset B(\mathbb{R})$ the set of functions f which are symmetric around x and nondecreasing in $[x, \infty)$. Note that

$$a := f(x+) = f(x-), \qquad f \in \mathcal{S}. \tag{34}$$

Also, it follows from (33) that

$$L_n^\beta(f, x) = \mathbb{E}f\left(x + \frac{X_\beta}{\sqrt{n}}\right), \qquad f \in \mathcal{S}. \tag{35}$$

Finally, denote by $\widetilde{\mathcal{S}}$ the set of symmetric functions $\tilde{\phi}$ having the form

$$\tilde{\phi}(y) = \int_0^{|y-x|} f(x+u) \, du, \quad y \in \mathbb{R}, \quad f \in \mathcal{S}. \tag{36}$$

We state the main result of this section.

Theorem 4. *Let a be as in (34). For the operators L_n^β defined in (32), we have*

a. *If $f \in \mathcal{S}$, there is equality in Theorem 1. Specifically,*

$$L_n^\beta(f, x) - a = \mathbb{E}\omega\left(g, \frac{X_\beta}{\sqrt{n}}\right).$$

b. *If $\tilde{\phi} \in \widetilde{\mathcal{S}}$, there is equality in Theorem 3. More precisely,*

$$L_n^\beta(\tilde{\phi}, x) - \frac{a}{\beta\sqrt{n}} = \frac{1}{\beta\sqrt{n}}\mathbb{E}\omega\left(g, \frac{X_\beta}{\sqrt{n}}\right).$$

Proof. (a) In the first place, since $Y_\beta = Z_n(x)$ is an absolutely continuous and symmetric random variable, we have

$$P(Y_\beta = 0) = 0, \quad P(Y_\beta > 0) = \frac{1}{2}, \quad C_n(\beta) = 1.$$

By (11) and (34), the function g associated to f is given by

$$g(y) = (f(y) - a)1_{\mathbb{R}\setminus\{x\}}(y), \quad y \in \mathbb{R}.$$

Since $f \in \mathcal{S}$, this implies, by virtue of (34), that

$$\omega(g, \theta) = f(x + \theta) - a, \quad \theta \geq 0. \tag{37}$$

In turn, this implies that

$$\mathbb{E}\omega\left(g, \frac{X_\beta}{\sqrt{n}}\right) = \mathbb{E}f\left(x + \frac{X_\beta}{\sqrt{n}}\right) - a = L_n^\beta(f, x) - a,$$

where the last equality follows from (35). This shows part (a).

(b) Taking $h(y) = y$ and $h(y) = |y|$ in (31), we see that

$$\mathbb{E}Y_\beta = 0, \quad \mathbb{E}|Y_\beta| = \mathbb{E}X_\beta = \frac{1}{\beta}.$$

Since the function $\tilde{\phi}$ defined in (36) is symmetric around x, we have from (33) and (37)

$$L_n^\beta(\tilde{\phi}, x) = \mathbb{E}\tilde{\phi}\left(x + \frac{X_\beta}{\sqrt{n}}\right) = \mathbb{E}\int_0^{X_\beta/\sqrt{n}} f(x+u)\, du$$
$$= \frac{a}{\sqrt{n}}\mathbb{E}X_\beta + \mathbb{E}\int_0^{X_\beta/\sqrt{n}} \omega(g, u)\, du = \frac{a}{\beta\sqrt{n}} + \frac{1}{\beta\sqrt{n}}\mathbb{E}\omega\left(g, \frac{X_\beta}{\sqrt{n}}\right),$$

where the last equality follows from (18) for $\psi = \omega(g, \cdot)$. This shows part (b) and completes the proof. □

5 | Bernstein Polynomials

We illustrate Theorems 1 y 3 in the case of the Bernstein polynomials. To give explicit approximation results, some auxiliary lemmas will be needed. Recall that Stirling's approximation states that

$$\sqrt{2\pi k}\ \exp\left(\frac{1}{12k+1}\right) \leq \frac{k!}{k^k}\ \exp(k) \leq \sqrt{2\pi k}\ \exp\left(\frac{1}{12k}\right),\ k \in \mathbb{N}. \tag{38}$$

Lemma 3. *Let $Z_n(x)$ be as in (5). Then,*

$$P(Z_n(x) = 0) \leq a_n(x) := \frac{1}{\sqrt{2\pi n x(1-x)}}1_{A_n}(x),\ A_n = \left\{\frac{k}{n} : k = 1, \ldots, n-1\right\}.$$

Proof. By (4) and (5), we have

$$P(Z_n(x)=0)=P(S_n(x)=nx)=\sum_{k=1}^{n-1}P(S_n(x)=k)1_{\{k/n\}}(x). \tag{39}$$

Differentiating with respect to x, it can be checked that

$$P(S_n(x)=k)\leq P(S_n(k/n)=k),\quad k=1,\ldots,n-1.$$

By (38) and the fact that $1/(12n)-1/(12k+1)-1/(12(n-k)+1)<0$, this implies that

$$P(S_n(x)=k)\leq\frac{1}{\sqrt{2\pi}}\sqrt{\frac{n}{k(n-k)}},\quad k=1,\ldots,n-1.$$

This and (39) show the result. □

Let Z be a random variable having the standard normal density and let $(X_k)_{k\geq 1}$ be a sequence of independent copies of a random variable X such that $\mathbb{E}X=\mu,\ 0<\mathbb{E}(X-\mu)^2=\sigma^2$. and $\mathbb{E}|X-\mu|^3=\gamma<\infty$. The Berry-Esseen bounds establish that

$$\sup_{y\in\mathbb{R}}\left|P\left(\frac{X_1+\ldots+X_n-n\mu}{\sigma\sqrt{n}}\leq y\right)-P(Z\leq y)\right|\leq C\frac{\gamma}{\sigma^3}\frac{1}{\sqrt{n}}, \tag{40}$$

for some positive constant C. According to Essen [19], C cannot be less than $(3+\sqrt{10})/(\sigma\sqrt{2\pi})=0.4097\cdots$. Shevtsova [20]. (see also Korolev and Shevtsova [21].) showed that $C\leq 0.4690$.

Lemma 4. *Let $Z_n(x)$ be as in (5). Then,*

$$\left|P(Z_n(x)>0)-\frac{1}{2}\right|\leq b_n(x):=\left|\frac{1}{2}-(1-x)^n\right|1_{\left(0,\frac{1}{n}\right)}(x)$$
$$+\left|\frac{1}{2}-x^n\right|1_{\left[\frac{n-1}{n},1\right)}(x)+0.4690\frac{x^2+(1-x)^2}{\sqrt{nx(1-x)}}1_{\left[\frac{1}{n},\frac{n-1}{n}\right)}(x).$$

Proof. If $0<x<1/n$ or $(n-1)/n\leq x<1$, the upper bound $b_n(x)$ is easily checked. Suppose that $1/n\leq x<(n-1)/n$. We apply the Berry-Esseen bounds in (40) to the random variable $S_n(x)$ defined in (3). In this regard, note that

$$\sigma^2=\mathbb{E}|S_1(x)-x|^2=x(1-x),$$

and

$$\gamma=\mathbb{E}|S_1(x)-x|^3=x(1-x)(x^2+(1-x)^2). \tag{41}$$

Since

$$\left|P(Z_n(x)>0)-\frac{1}{2}\right|=\left|P\left(\frac{S_n(x)-nx}{\sqrt{nx(1-x)}}\leq 0\right)-P(Z\leq 0)\right|,$$

the result follows from (40), (41), and the comments following (40). □

To estimate the constant $C_n(\beta)$ in (8) referring to the Bernstein polynomials, we give the following result.

Lemma 5. *Let $Z_n(x)$ be as in (5). For any $\beta>0$, we have*

$$\mathbb{E}\exp(\beta|Z_n(x)|)\leq 2\ \exp\left(\frac{x(1-x)}{2}\left(\beta^2+\frac{x^2+(1-x)^2}{3\sqrt{n}}\beta^3\ \exp\left(\frac{\beta}{\sqrt{n}}\right)\right)\right).$$

Proof. The result will follow as soon as we show that

$$\mathbb{E}\exp(\beta|S_n(x)-nx|) \le 2\ \exp\left(\frac{nx(1-x)}{2}\left(\beta^2+\frac{x^2+(1-x)^2}{3}\beta^3\ \exp(\beta)\right)\right), \tag{42}$$

since, by (5), it will suffice to replace β by $\beta/\sqrt{n}$ in (42).

Let $\beta > 0$. We claim that

$$\begin{aligned}&\max\left\{\mathbb{E}\exp\big(\beta(S_n(x)-nx)\big),\ \mathbb{E}\exp\big(-\beta(S_n(x)-nx)\big)\right\}\\ &\le \exp\left(nx(1-x)\left(\frac{\beta^2}{2}+\left(x^2+(1-x)^2\right)\right)\left(\exp(\beta)-1-\beta-\frac{\beta^2}{2}\right)\right).\end{aligned} \tag{43}$$

Indeed, for $k = 3, 4, \ldots$, we have

$$|\mathbb{E}(S_1(x)-x)^k| = x(1-x)|(1-x)^{k-1}+(-1)^k x^{k-1}| \le x(1-x)(x^2+(1-x)^2).$$

Therefore,

$$\begin{aligned}\mathbb{E}\exp\big(\beta(S_1(x)-x)\big) &\le 1+\frac{x(1-x)}{2}\beta^2+\sum_{k=3}^{\infty}\frac{\beta^k}{k!}|\mathbb{E}(S_1(x)-x)^k|\\ &\le 1+x(1-x)\left(\frac{\beta^2}{2}+(x^2+(1-x)^2)\sum_{k=3}^{\infty}\frac{\beta^k}{k!}\right)\\ &\le \exp\left(x(1-x)\left(\frac{\beta^2}{2}+(x^2+(1-x)^2)\left(\exp(\beta)-1-\beta-\frac{\beta^2}{2}\right)\right)\right).\end{aligned}$$

This, together with the fact that

$$\mathbb{E}\exp\big(\beta(S_n(x)-nx)\big) = \big(\mathbb{E}\exp(\beta(S_1(x)-x))\big)^n,$$

shows the claim for the first term on the left-hand side in (43). For the second term, the proof is similar.

Since $\exp(\beta)-1-\beta-\beta^2/2 \le \beta^3\exp(\beta)/6$, we have from (43)

$$\begin{aligned}\mathbb{E}\exp(\beta|S_n(x)-nx|) &\le \mathbb{E}\exp(\beta(S_n(x)-nx))+\mathbb{E}\exp(-\beta(S_n(x)-nx))\\ &\le 2\exp\left(\frac{nx(1-x)}{2}\left(\beta^2+\frac{x^2+(1-x)^2}{3}\beta^3\exp(\beta)\right)\right).\end{aligned}$$

This shows (5.5) and completes the proof. □

Recall that a random variable X is said to be subgaussian with variance proxy $\sigma^2 > 0$ if $\mathbb{E}X = 0$ and

$$\mathbb{E}\exp(\beta X) \le \exp\left(\sigma^2\frac{\beta^2}{2}\right),\ \ \beta \in \mathbb{R}.$$

It is well known that $\sigma^2 \ge Var(X)$. Inequality (43) says that, for values of β near the origin, the random variable $S_n(x)-nx$ is essentially subgaussian with optimal variance proxy $\sigma^2 = nx(1-x) = Var(S_n(x)-nx)$.

We are in a position to apply Theorems 1 and 3 to the case of the Bernstein polynomials.

Corollary 1. *Let* $f \in B([0,1])$ *and* $x \in (0,1)$. *Then,*

$$\begin{aligned}\left|B_n(f,x)-\frac{f(x+)+f(x-)}{2}\right| &\le a_n(x)|f(x)-f(x-)|+b_n(x)|f(x+)-f(x-)|\\ &+2\exp\left(\frac{x(1-x)}{2}\left(1+\frac{x^2+(1-x)^2}{3\sqrt{n}}\ \exp\left(\frac{1}{\sqrt{n}}\right)\right)\right)\\ &\times\frac{1}{n}\sum_{k=0}^{n-1}\omega\left(g,\frac{1}{\sqrt{n}}\log\frac{n}{k}\right),\end{aligned} \tag{44}$$

where $a_n(x)$ and $b_n(x)$ are defined in Lemmas 3 and 4, respectively.

Proof. Starting from Theorem 1, use Lemmas 3 and 4 to bound the terms multiplying the quantities $|f(x) - f(x-)|$ and $|f(x+) - f(x-)|$, respectively. To bound the last term on the right-hand side in (25), use (10) and Lemma 5 with $\beta = 1$ to estimate $C_n(\beta)$, and inequality (17) with $\psi = \omega(g, \cdot)$, $m = n$, and $\beta = 1$ to bound the term

$$\mathbb{E}\omega\left(g, \frac{X_\beta}{\sqrt{n}}\right).$$

This completes the proof. □

The comparison between Corollary 1 and formula (12) reveals the following. The terms $a_n(x)$ and $b_n(x)$ in (44) are smaller than the corresponding ones in expression (12). However, the main difference lies in the last term on the right-hand side in (44). On the one hand, it follows from (30) that

$$\sum_{k=0}^{n-1}\omega\left(g, \frac{1}{\sqrt{n}}\log\frac{n}{k}\right) \leq \sum_{k=0}^{n-1}V\left(g, \left[x - \frac{1}{\sqrt{n}}\log\frac{n}{k}, x + \frac{1}{\sqrt{n}}\log\frac{n}{k}\right] \cap [0,1]\right), \tag{45}$$

and the term on the left is easier to compute than term on the right in (45). On the other hand, excepting the case $k = 0$, the lengths of the intervals in (45) are asymptotically much shorter than those in (12). For instance, the lengths of the $(n-1)th$ intervals in (12) and (45) are, respectively,

$$\frac{1}{\sqrt{n}} \text{ and } \frac{2}{\sqrt{n}}\log\frac{n}{n-1} \sim \frac{2}{n\sqrt{n}}, \quad n \to \infty.$$

Denote by $\lfloor y \rfloor$ and $\lceil y \rceil$ the floor and the ceiling of $y \in \mathbb{R}$, respectively.

Corollary 2. *Let $\phi \in DB([0,1])$ and $x \in (0,1)$. As in (28), let $g = f_x \in B(I)$ be the function associated to the Radon-Nikodym derivative $f \in B(I)$ of ϕ. Then,*

$$\begin{aligned}
&|B_n(\phi, x) - \phi(x) - x(1-x)P(S_{n-1}(x) = \lfloor nx \rfloor)(f(x+) - f(x-))| \\
&\leq 2\exp\left(\frac{x(1-x)}{2}\left(1 + \frac{x^2 + (1-x)^2}{3\sqrt{n}}\exp\left(\frac{1}{\sqrt{n}}\right)\right)\right) \\
&\times \frac{1}{n}\sum_{k=0}^{\lceil\sqrt{n}\rceil - 1}\omega\left(g, \frac{1}{\sqrt{n}}\log\frac{n}{k}\right).
\end{aligned} \tag{46}$$

Proof. It is known (cf. Johnson and Kotz [22 , p. 52].) that

$$\mathbb{E}\left|\frac{S_n(x) - nx}{n}\right| = 2x(1-x)P(S_{n-1}(x) = \lfloor nx \rfloor). \tag{47}$$

Hence, starting from Theorem 3 and recalling (5), we see that

$$\mathbb{E}Z_n(x) = 0, \quad \frac{1}{2\sqrt{n}}\mathbb{E}|Z_n(x)| = x(1-x)P(S_{n-1}(x) = \lfloor nx \rfloor).$$

As in the proof of Corollary 1, the upper bound is (46) is obtained by choosing $\beta = 1$ in Lemma 5, and $\beta = 1$ and $m = \lceil\sqrt{n}\rceil$ in (17). This completes the proof. □

Bojanic and Cheng [9, Th. 2], showed the following result. In the setting of Corollary 2, assume further that $n \geq x(1-x)$. Then,

$$\left| B_n(\phi, x) - \phi(x) - \frac{1}{\sqrt{n}}\left(\frac{x(1-x)}{2\pi}\right)^{1/2}(f(x+) - f(x-)) \right| \leq \frac{M}{2n\sqrt{x(1-x)}}|f(x+) - f(x-)| + \frac{2}{n}\sum_{k=0}^{\lfloor\sqrt{n}-1\rfloor} V\left(g, \left[x - \frac{x}{\sqrt{k+1}}, x + \frac{1-x}{\sqrt{k+1}}\right]\right), \tag{48}$$

where M is an explicit constant (cf.[9, Th.1]). We point out that there is a misprint in [9, Th 2], concerning the main term of the approximation.

Observe that there is no restriction on n in Corollary 2. On the other hand, the main term of the approximation in (46) and (48) is essentially the same. Indeed, we have from (47) and the central limit theorem for $S_n(x)$

$$x(1-x)P(S_{n-1}(x) = \lfloor nx \rfloor) = \frac{\sqrt{x(1-x)}}{2\sqrt{n}}\mathbb{E}\left|\frac{S_n(x) - nx}{\sqrt{nx(1-x)}}\right| \sim \frac{\sqrt{x(1-x)}}{2\sqrt{n}}\mathbb{E}|Z| = \frac{1}{\sqrt{n}}\left(\frac{x(1-x)}{2\pi}\right)^{1/2}, \quad n \to \infty,$$

where Z is a standard normal random variable. Finally, the asymptotic value of the constant in the upper bound in (46) is $2\ \exp(x(1-x)/2)$, which is slightly worse than 2. However, the total variation term in (48) is worse than the upper bound in (46), as explained in the comments following Corollary 1.

6 | Conclusions

The approximation of discontinuous functions or absolutely continuous functions with discontinuous derivatives can be attacked using sequences of linear operators. These operators must have an expression in probabilistic terms. In this way, estimates of the rate of convergence of the process are obtained in terms of a local continuity modulus.

Author Contributions

J.A. Adell: conceptualization, formal analysis, methodology, investigation, supervision, writing – review and editing, writing – original draft. **P. Garrancho:** conceptualization, formal analysis, writing – review and editing, methodology, investigation, supervision, writing – original draft. **F.J. Martínez-Sánchez:** conceptualization, methodology, investigation, formal analysis, writing – original draft, writing – review and editing, supervision.

Acknowledgments

The authors are very grateful to the referees for their careful reading of the manuscript and for their comments and suggestions, which greatly improved the final outcome.

The first author is supported by Research Project DGA (E48_23R). The second and third authors are supported by Junta de Andalucía Research Group FQM-0178.

Funding

This study was supported by Junta de Andalucía Research Group FQM-0178 and Research Project DGA (E48_23R).

Conflict of Interest

The authors declare that they have no competing interests.

References

1. J. A. Adell and J. de la Cal, “Bernstein-Durrmeyer operators,” *Computers & Mathematcs with Applications* 30 (1995): 1–14, https://doi.org/10.1016/0898-1221(95)00081-X.

2. J. A. Adell and J. de la Cal, “Bernstein-type Operators Diminish the ψ−variation,” *Constructive Approximation* 12, no. 4 (1996): 489–507, https://doi.org/10.1007/s003659900027.

3. M. K. Khan, “Approximation at Discontinuity,” *Rendiconti del Circolo Matematico di Palermo* 2, no. Suppl. 68 (2002): 539–553.

4. U. Abel and V. Gupta, “An Estimate of the Rate of Convergence of a Bézier Variant of the Baskakov-Kantorovich Operators for Bounded Variation Functions,” *Demonstratio Mathematica* 36, no. 1 (2003): 123–136, https://doi.org/10.1515/dema.2003.36.1.123.

5. F. Cheng, “On the Rate of Convergence of Bernstein Polynomials of Functions of Bounded Variation,” *Journal of Approximation Theory* 39, no. 3 (1983): 259–274, https://doi.org/10.1016/0021-9045(83)90098-9.

6. X. Zeng and F. Cheng, “On the Rates of Approximation of Bernstein Type Operators,” *Journal of Approximation Theory* 109, no. 2 (2001): 242–256, https://doi.org/10.1006/jath.2000.3538.

7. J. Bustamante, M. A. García-Muoz, and J. M. Quesada, “Bernstein Polynomials and Discontinuous Functions,” *Journal of Mathematical Analysis and Applications* 411, no. 2 (2014): 829–837, https://doi.org/10.1016/j.jmaa.2013.10.024.

8. F. Cheng, “On the Rate of Convergence of the Szàsz–Mirakyan Operator for Functions of Bounded Variation,” *Journal of Approximation Theory* 40 (1984): 226–241, https://doi.org/10.1016/0021-9045(84)90064-9.

9. R. Bojanic and F. Cheng, “Rate of Convergence of Bernstein Polynomials for Functions With Derivatives of Bounded Variation,” *Journal of Mathematical Analysis and Applications* 141, no. 1 (1989): 136–151, https://doi.org/10.1016/0022-247X(89)90211-4.

10. S. Guo, “On the Rate of Convergence of the Integrated Meyer-Knig and Zeller Operators for Functions of Bounded Variation,” *Journal of Approximation Theory* 56, no. 3 (1989): 245–255, https://doi.org/10.1016/0021-9045(89)90114-7.

11. X. Zeng and A. Piriou, “On the Rates of Convergence of two Bernstein–Bezier Type Operators for Bounded Variation Functions,” *Journal of Approximation Theory* 95, no. 3 (1998): 369–387, https://doi.org/10.1006/jath.1997.3227.

12. Gupta, V.: Rate of Approximation by a new Sequence of Linear Positive Operators. *Computers & Mathematcs With Applications* 45(12), 1895–1904 (2003). https://doi.org/10.1016/S0898–1221(03)90010–1.

13. V. Gupta, “Degree of Approximation to Function of Bounded Variation by Bézier Variant of MKZ Operators,” *Journal of Mathematical Analysis and Applications* 289, no. 1 (2004): 292–300, https://doi.org/10.1016/j.jmaa.2003.09.059.

14. V. Gupta and N. Ispir, “On the Bézier Variant of Generalized Kantorovich Type Balazs Operators,” *Applied Mathematics Letters* 18, no. 9 (2003): 1053–1061, https://doi.org/10.1016/j.aml.2004.11.002.

15. M. A. Zarslan, O. Duman, and C. Kaanoglu, “Rates of Convergence of Certain King-Type Operators for Functions With Derivative of Bounded Variation,” *Mathematical and Computer Modelling* 52 (2010): 334–345.

16. Abel, U., Gupta, V.: “Rate of convergence of exponential type operators related to $p(x) = 2x^{3/2}$ for functions of bounded variation.” RACSAM 114–188 (2020). https://doi.org/10.1007/s13398-020-00919-y.

17. S. Guo and M. K. Khan, “On the Rate of Convergence of Some Operators on Functions of Bounded Variation,” *Journal of Approximation Theory* 58, no. 1 (1989): 90–101, https://doi.org/10.1016/0021-9045(89)90011-7.

18. Mowzer, M. T.: Convergence in Distribution of the Bernstein-Durrmeyer Kernel and Pointwise Convergence of a Generalized Operator for Functions of Bounded Variation. arXiv:2303.11061v2 [math.CA]. https://doi.org/10.48550/arXiv.2303.11061.

19. C. G. Esseen, “A Moment Inequality With an Application to the Central Limit Theorem,” *Skand. Aktuarietidskr.* 39 (1956): 160–170, https://doi.org/10.1080/03461238.1956.10414946.

20. I. G. Shevtsova, “On Absolute Constants in Inequalities of Berry-Esseen Type,” *Doklady Mathematics* 89, no. 3 (2014): 378–381, https://doi.org/10.1134/s1064562414030338.

21. V. Y. Korolev and I. G. Shevtosva, “On the Upper Bound for the Absolute Constant in the BerryEsseen Inequality,” *Theory of Probability and Its Applications* 54, no. 4 (2010): 638–658, https://doi.org/10.1137/S0040585X97984449.

22. N. L. Johnson and S. Kotz, *Discrete Distributions* (Wiley, 1969).