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Distributed Control of Flexible Chained Multiagent Formations

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***Abstract*—This letter presents a novel distributed approach for the control of flexible multiagent formations. We propose a formulation based on affine formation control in which, instead of considering a single nominal configuration as in standard formulations, we consider multiple nominal configurations. This has the advantage of providing higher flexibility to adapt to different task conditions. In our approach, the agents are arranged in a chained structure; specifically, we group them in chained sets and propose a control law based on orthogonal projections defined for each of these sets. The resulting strategy is distributed, as it uses local interactions, and it can be implemented using position measurements expressed in the agents' local reference frames. We support the proposed approach via formal analysis and illustrate it with simulations.**

***Index Terms*—Cooperative control, distributed control, autonomous systems, robotics.**

I. Introduction

FORMATION control is a fundamental competence for teams of mobile agents in diverse types of missions. Numerous existing formation control schemes consider a geometric reference (e.g., a nominal configuration of the multiagent team) and design a motion strategy aimed at achieving and maintaining that reference, allowing for specific degrees of freedom to enhance flexibility. Translations, rigid transformations, and shape-preserving transformations are common examples of allowed degrees of freedom. Many schemes along these lines have been proposed, with formulations based on inter-agent relative positions [1]–[3], distances [4], [5], or bearings [6]. Affine formation control, which allows for more general types of transformations, has been studied in a significant number of recent works, such as [7]–[12]. In these studies, the affine formations are defined with respect to a single nominal configuration. This letter presents a novel approach that defines affine formations with respect to multiple nominal configurations, considered simultaneously. By using multiple geometric references, this approach makes a team capable of adapting its shape more flexibly to mission requirements (e.g., avoiding obstacles, reacting to threats, or enclosing mobile targets) while still exhibiting the behavior of a formation.

The proposed approach is based on grouping the agents in chained sets. We adopt this grouping because it has practical interest and it allows us to obtain formal guarantees. The agents in every set use motion vectors based on orthogonal projections to move toward partial affine formations, defined with respect to multiple nominal configurations. Every pair of consecutive sets along the considered chain share a number of agents that ensures that the parameters of the achieved partial formations are the same for both sets. As a result, the team as a whole achieves a consistent formation. This approach has interesting features: it is distributed, since it is based on local agent interactions, and it accommodates the use of measurements expressed in local reference frames. The type of chained formations we define is particularly useful when specific physical vicinities between the agents need to be kept; e.g., to form a contour enclosing a region, or to transport a deformable object.

Recent works [13]–[15] used multiagent formations designed to fit environmental boundaries or to form low-frequency closed curves. In comparison with these approaches, our formulation is more general, as it encompasses different types of formations, and it does not require global information or communications-based estimators. Other related studies proposed placing robots at discrete samples of a circle [16], [17] or of other virtual parametric curves [18] or surfaces [19]. The method we propose provides higher flexibility in terms of the shapes that can be achieved. The approach in [20] extends affine formation control to so-called linear formation control, enhancing flexibility by defining a nominal configuration in a space of higher dimension than the one where the agents move. Our strategy is different, as it is based on multiple nominal configurations, and it uses neither estimators nor pre-defined design matrices. We support and illustrate the benefits of our approach with formal analysis and numerical simulation.

A. Notation and Preliminary Definitions

$\mathbb{R}$, $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ denote the set of real numbers, real n-dimensional column vectors, and real matrices with m rows and n columns, respectively. I_n denotes the $n \times n$ identity matrix, and $\otimes$ denotes the Kronecker product. For a given matrix $A \in \mathbb{R}^{m\times n}$, $A^+ \in \mathbb{R}^{n\times m}$ denotes its Moore-Penrose inverse. The column space of A is the set of all possible vectors $v \in \mathbb{R}^m$ that are a linear combination of A's columns. If A is a square matrix (i.e., $m = n$), it is positive semidefinite if it is symmetric and $q^\intercal A q \geqslant 0$ $\forall q \in \mathbb{R}^n$. AA^+ is a symmetric and idempotent matrix such that AA^+w for $w \in \mathbb{R}^m$ is the orthogonal projection of w onto the column space of A [21, ch. III]. $row_i(A)$ for $i \in \{1, 2, \dots, m\}$ with $row_i^\intercal(A) \in \mathbb{R}^n$, and $col_i(A) \in \mathbb{R}^m$ for $i \in \{1, 2, \dots, n\}$ denote, respectively, the i-th row and i-th column of matrix A. For a vector $a \in \mathbb{R}^n$, $\|a\|$ denotes its Euclidean norm. $SO(D)$ denotes the rotation group of D dimensions.

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II. Problem Statement

Consider a team of N mobile agents, each having a different identifying index in the set $\mathcal{N} = \{1, 2, \ldots, N\}$. Agent index values are always interpreted modulo N in this letter, with values outside $\mathcal{N}$ being wrapped around into $\mathcal{N}$. D denotes the number (2 or 3) of spatial dimensions. Positions in D-dimensional Euclidean space are expressed in a fixed global Cartesian coordinate frame. The position of agent i is denoted by $p_i \in \mathbb{R}^D$. The stack vector of the agents' positions, defined as $p = [p_1^\intercal, \ldots, p_N^\intercal]^\intercal \in \mathbb{R}^{D\cdot N}$, is called the configuration of the team. We assume the agents have single-integrator dynamics, i.e., $\dot{p}_i(t) = u_i(t)\ \forall i \in \mathcal{N}$, where $u_i(t)$ is the control input and $t \in \mathbb{R}_{\geqslant 0}$ denotes time.

A. Affine Formations for a Nominal Configuration

Our work is based on the framework of affine multiagent formation control. We use the same concept of what an affine formation is as in the related literature [7]–[12]. Defining a constant nominal configuration $c = [c_1^\intercal, \ldots, c_N^\intercal]^\intercal \in \mathbb{R}^{D\cdot N}$ with position $c_i \in \mathbb{R}^D$ corresponding to agent i, the multiagent team is in an affine formation with respect to the nominal configuration c if there exist $A \in \mathbb{R}^{D\times D}$ and $r \in \mathbb{R}^D$ such that

$$p_i = Ac_i + r \quad \forall i \in \mathcal{N}. \tag{1}$$

A and r represent, respectively, a general linear transformation and a general translation that are applied to the positions in the nominal configuration for all the agents. This specification can allow for flexible configurations p which still preserve certain geometric characteristics of the nominal configuration c, such as collinearity and parallelism in the agents' relative positions, making affine formations useful in diverse tasks. We can express (1) compactly as

$$p = (\widetilde{C} \otimes I_D)\widetilde{v}, \tag{2}$$

where we define

$$\widetilde{C} = \begin{bmatrix} c_1^\intercal & 1 \\ \vdots & \vdots \\ c_N^\intercal & 1 \end{bmatrix} \in \mathbb{R}^{N\times(D+1)}, \ \widetilde{v} = \begin{bmatrix} col_1(A) \\ \vdots \\ col_D(A) \\ r \end{bmatrix} \in \mathbb{R}^{D\cdot(D+1)}. \tag{3}$$

Note that $\widetilde{C}$ contains the information of the constant nominal configuration. On the other hand, $\widetilde{v}$ is a parameter vector containing the elements of A and r that characterizes, for the particular team configuration p, the specific affine formation with respect to the nominal configuration c. The concept of affine image [7], [8] can be used to define the set of the configurations p that satisfy (1) for a given c.

B. Affine Formations for Multiple Nominal Configurations

To further enhance flexibility, in this letter we propose using multiple distinct nominal configurations, each of them representing a certain geometric pattern of agent positions. Suppose we have L distinct nominal configurations, with $L \geqslant 1$, denoted $c_{(l)} = [c_{l,1}^\intercal, \ldots, c_{l,N}^\intercal]^\intercal \in \mathbb{R}^{D\cdot N}\ \forall l \in \{1, 2, \ldots, L\}$. $c_{l,i} \in \mathbb{R}^D$ denotes the position for agent i in the nominal configuration $c_{(l)}$. Our control goal will be the achievement of a sum of affine formations, in the sense of (1), with respect to these nominal configurations. Therefore, the condition is now that there exist $A_l \in \mathbb{R}^{D\times D}$ and $r_l \in \mathbb{R}^D\ \forall l \in \{1, 2, \ldots, L\}$, such that

$$p_i = \sum_{l=1}^{L}(A_l c_{l,i} + r_l) = \sum_{l=1}^{L} A_l c_{l,i} + \sum_{l=1}^{L} r_l \quad \forall i \in \mathcal{N}. \tag{4}$$

We can write an expression analogous to (2), with a parameter vector $v \in \mathbb{R}^{D\cdot(D\cdot L+1)}$, and $C \in \mathbb{R}^{N\times(D\cdot L+1)}$, as

$$p = (C \otimes I_D)v, \tag{5}$$

$$\text{with } C = \begin{bmatrix} c_{1,1}^\intercal & \cdots & c_{L,1}^\intercal & 1 \\ \vdots & \ddots & \vdots & \vdots \\ c_{1,N}^\intercal & \cdots & c_{L,N}^\intercal & 1 \end{bmatrix}, \ v = \begin{bmatrix} col_1(A_1) \\ \vdots \\ col_D(A_1) \\ \vdots \\ col_1(A_L) \\ \vdots \\ col_D(A_L) \\ \sum_{l=1}^{L} r_l \end{bmatrix}. \tag{6}$$

Note that with $L = 1$, one has the case in Section II-A. Based on (5), we now present our definition for the formation task.

Definition 1. *The multiagent team is in an affine formation with respect to the nominal configurations $c_{(1)}, c_{(2)}, \ldots, c_{(L)}$ if there exists $v \in \mathbb{R}^{D\cdot(D\cdot L+1)}$ such that $p = (C \otimes I_D)v$.*

This letter tackles the problem of designing a control approach for achieving multiagent formations in the sense of Definition 1. Next, we describe our proposed solution.

III. Definition of Chained Team Structure

In various multiagent tasks, it is beneficial for the agents' interactions to form a chain; e.g., when specific agent vicinities are to be kept, such as in cooperative transport or target enclosing. Here, we propose grouping the agents in chained sets. Consecutive sets along the chain are interlaced, i.e., there are several agents in the intersection set of every two consecutive sets. We choose the number of agents in these intersection sets to be equal to the number of columns of C: $D \cdot L + 1$. This way, the agent positions in each intersection provide the same number ($D \cdot (D \cdot L + 1)$) of degrees of freedom as the parameter vector v; and, therefore, the positions of intersection agents can uniquely determine v, allowing us to obtain formation convergence guarantees.

Then, we choose the number of agents per set, which we call M, to be one more than the number of agents in the intersections. Therefore, we have $M = D \cdot L + 2$. We choose all sets to have the same number of agents, and to form a chain that may be open or closed. For an open chain, we define $N - M + 1$ chained sets of agents as follows:

$$\mathcal{S}_1 = \{1, \ldots, M\}, \ldots, \mathcal{S}_{N-M+1} = \{N-M+1, \ldots, N\}. \tag{7}$$

On the other hand, for a closed chain we define the N sets

$$\mathcal{S}_1 = \{1, \ldots, M\}, \ldots, \mathcal{S}_N = \{N, \ldots, N + M - 1\}. \tag{8}$$

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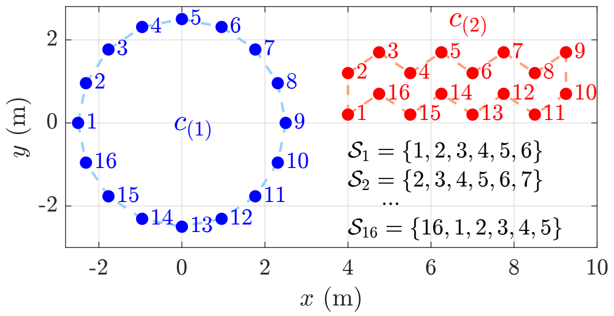

Fig. 1. Example with two nominal configurations $c_{(1)}$ and $c_{(2)}$ in two dimensions (i.e., $M = 6$) for a team of $N = 16$ agents. The positions (circles), agent indices, and sets $\mathcal{S}_j$ for a closed chain are depicted.

Figure 1 shows an illustrative example. Let N_s denote the number of chained sets, i.e., $N_s = N - M + 1$ for an open chain, and $N_s = N$ for a closed chain, and let us define $\mathcal{N}_s = \{1, 2, \ldots, N_s\}$. For readability, we will use j as the index for sets in $\mathcal{N}_s$. Our choice of using pre-defined sets $\mathcal{S}_j$ which all have the minimum possible cardinality is made for the sake of sparsity and uniformity, Still, note that other designs would be possible. To define the proposed structure based on multiple chained sets, the number of agents, N, must satisfy $N > M$. Note that N can be arbitrarily large. Next, we define several matrices used in our control formulation. We first define for every set $\mathcal{S}_j$, i.e., $\forall j \in \mathcal{N}_s$,

$$C_j = \begin{bmatrix} row_j(C) \\ row_{j+1}(C) \\ \vdots \\ row_{j+M-1}(C) \end{bmatrix} \in \mathbb{R}^{M \times (M-1)}, \tag{9}$$

$$\bar{C}_j = C_j \otimes I_D \in \mathbb{R}^{(D \cdot M) \times (D \cdot (M-1))}. \tag{10}$$

These constant matrices collect the part of the nominal configurations including the M agents in $\mathcal{S}_j$. We also define

$$G_j = \begin{bmatrix} row_j(C) \\ row_{j+1}(C) \\ \vdots \\ row_{j+M-2}(C) \end{bmatrix} \in \mathbb{R}^{(M-1) \times (M-1)} \quad \forall j \in \mathcal{N}_s. \tag{11}$$

A useful property, due to the chained structure, is that if $\mathcal{S}_{j-1}$ and $\mathcal{S}_j$ are consecutive sets in the chain, then

$$\bar{C}_j = \begin{bmatrix} G_j \otimes I_D \\ row_M(C_j) \otimes I_D \end{bmatrix}, \; \bar{C}_{j-1} = \begin{bmatrix} row_1(C_{j-1}) \otimes I_D \\ G_j \otimes I_D \end{bmatrix}. \tag{12}$$

Next, we make an assumption about these matrices.

Assumption 1. *G_j is nonsingular $\forall j \in \mathcal{N}_s$.*

Assumption 1 is not satisfied if the geometry of the nominal configurations makes some G_j singular. For example, this can happen when these configurations include several collinear points. This letter focuses on the nonsingular cases.

IV. Control Strategy

From Definition 1, achieving an affine formation implies that p is in the column space of the matrix $C \otimes I_D$. Based on this fact, our control strategy consists in:

1) Defining, for the agents in $\mathcal{S}_j$ $\forall j \in \mathcal{N}_s$, motion vectors toward the column space of the matrix $\bar{C}_j$. This produces a motion toward a partial (i.e., encompassing only $\mathcal{S}_j$) affine formation in the sense of Definition 1.
2) Adding up these motion vectors for all sets $\mathcal{S}_j$.

To accomplish 1), we use the orthogonal projection of the current positions of the agents $i \in \mathcal{S}_j$, i.e., from $i = j$ to $i = j + M - 1$, onto the column space of $\bar{C}_j$, given by:

$$\bar{C}_j \bar{C}_j^+ \begin{bmatrix} p_j(t) \\ \vdots \\ p_{j+M-1}(t) \end{bmatrix} \in \mathbb{R}^{D \cdot M}. \tag{13}$$

Note that if Assumption 1 is satisfied, then $\bar{C}_j$ has full rank equal to $D \cdot (M-1)$, which means that $\bar{C}_j^+ = (\bar{C}_j^\intercal \bar{C}_j)^{-1} \bar{C}_j^\intercal$.

Let $d_{j,i}(t) \in \mathbb{R}^D$ denote the vector from the current position, $p_i(t)$, of agent $i \in \mathcal{S}_j$ to the component corresponding to agent i of the orthogonal projection onto the column space of $\bar{C}_j$ in (13). Note the vectors $d_{j,i}(t)$ are only defined for agents i from $i = j$ to $i = j + M - 1$. They are defined as

$$\begin{bmatrix} d_{j,j}(t) \\ \vdots \\ d_{j,j+M-1}(t) \end{bmatrix} = -(I_{D \cdot M} - \bar{C}_j \bar{C}_j^+) \begin{bmatrix} p_j(t) \\ \vdots \\ p_{j+M-1}(t) \end{bmatrix}. \tag{14}$$

Then, according to 2), for every agent $i \in \mathcal{N}$, the control law is the sum of the vectors for the sets that contain i:

$$u_i(t) = \dot{p}_i(t) = \sum_{j \in \mathcal{N}_s \,|\, i \in \mathcal{S}_j} d_{j,i}(t) \quad \forall i \in \mathcal{N}. \tag{15}$$

Let us now express this control law in stacked form, which will be useful for analysis. We define for every $j \in \mathcal{N}_s$ a selector matrix $Q_j = [q_{j_{m,n}}] \in \mathbb{R}^{M \times N}$ with $q_{j_{1,j}} = 1$, $q_{j_{2,j+1}} = 1, \ldots$, $q_{j_{M,j+M-1}} = 1$, and zeros in all other positions. Then, the matrix $\bar{Q}_j = Q_j \otimes I_D \in \mathbb{R}^{(D \cdot M) \times (D \cdot N)}$ selects the elements of $p(t)$ corresponding to the set $\mathcal{S}_j$; i.e.,

$$\bar{Q}_j p(t) = \begin{bmatrix} p_j(t) \\ \vdots \\ p_{j+M-1}(t) \end{bmatrix}. \tag{16}$$

We can, therefore, express the control law motion vectors as

$$\begin{bmatrix} d_{j,j}(t) \\ \vdots \\ d_{j,j+M-1}(t) \end{bmatrix} = -(I_{D \cdot M} - \bar{C}_j \bar{C}_j^+) \bar{Q}_j p(t). \tag{17}$$

We now define $d_j(t) \in \mathbb{R}^{D \cdot N}$ which is the motion vector, for the full team, due to the set $\mathcal{S}_j$. This vector simply includes the motion vectors from (17) for the agents that are in $\mathcal{S}_j$, and it has zeros in all other positions. It is defined by

$$d_j(t) = \bar{Q}_j^\intercal \begin{bmatrix} d_{j,j}(t) \\ \vdots \\ d_{j,j+M-1}(t) \end{bmatrix} = -\bar{Q}_j^\intercal (I_{D \cdot M} - \bar{C}_j \bar{C}_j^+) \bar{Q}_j p(t). \tag{18}$$

The expression of the control law in stacked form is, hence,

$$\dot{p}(t) = \sum_{j \in \mathcal{N}_s} d_j(t) = -\sum_{j \in \mathcal{N}_s} \bar{Q}_j^\intercal (I_{D \cdot M} - \bar{C}_j \bar{C}_j^+) \bar{Q}_j p(t). \tag{19}$$

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Example 1. *Consider a team of $N = 7$ agents with two nominal configurations, i.e., $L = 2$, in two dimensions, i.e., $D = 2$. Therefore, $M = 6$. Suppose the chain being used is open, i.e., $\mathcal{N}_s = \{1, 2\}$ with sets $\mathcal{S}_1 = \{1, 2, 3, 4, 5, 6\}$ and $\mathcal{S}_2 = \{2, 3, 4, 5, 6, 7\}$. The control law* (19) *is, using* (18)*,*

$$\begin{bmatrix} \dot{p}_1(t) \\ \vdots \\ \dot{p}_7(t) \end{bmatrix} = d_1(t) + d_2(t) = \begin{bmatrix} d_{1,1}(t) \\ \vdots \\ d_{1,6}(t) \\ [0,0]^\intercal \end{bmatrix} + \begin{bmatrix} [0,0]^\intercal \\ d_{2,2}(t) \\ \vdots \\ d_{2,7}(t) \end{bmatrix}. \quad (20)$$

Notice that this corresponds to the control law expression for every individual agent $i = 1, 2, \ldots, 7$ given in (15)*.*

A. Control Strategy Implementation

To compute (15), an agent $i \in \mathcal{N}$ needs to know the identities, nominal configuration positions, and positions at time t, of the agents that are in common sets with i. It does not need any information about the agents that are not in common sets; e.g., agent 1 in Example 1 needs no information about agent 7, and vice versa. The maximum possible number of agents that a given agent has to interact with is limited to $2 \cdot (M - 1)$ even if N grows indefinitely. Hence, the control strategy is distributed. C_j, C_j^+ are constant matrices and can be pre-stored offline. The online (i.e., at time t) information used in this approach consists of measurements of positions. Importantly, it is possible for every agent to use measurements expressed in its local reference frame; e.g., the relative positions of the other agents, measured with an onboard sensor. Let the superscript (i) denote that a variable is expressed in agent i's local frame. Assume that, instead of p_k, what agent i measures is $p_k^{(i)} = R_i p_k + w_i$, where the rotation $R_i \in SO(D)$ and translation $w_i \in \mathbb{R}^D$ express the transformation between frames. We omit t in this section, for compactness, but R_i and w_i can be time-varying. As mentioned, a common particular case is $p_k^{(i)} = R_i(p_k - p_i)$, i.e., $w_i = -R_i p_i$, which means that agent i measures relative positions in its local frame. Clearly, if $u_i^{(i)} = R_i u_i$, the agent's motion is the same as when computed in the global frame. Next, we show that this condition is indeed satisfied.

Due to Kronecker product properties, the matrix $I_{D \cdot M} - \bar{C}_j \bar{C}_j^+$ for any $j \in \mathcal{N}_s$ has the form $(I_M - C_j C_j^+) \otimes I_D$; therefore, it consists of blocks $\Omega_{j_{m,n}} = \alpha_{j_{m,n}} I_D$, with $\alpha_{j_{m,n}} \in \mathbb{R}$. Expressing $I_{D \cdot M} - \bar{C}_j \bar{C}_j^+$ in terms of these blocks, we can write from (14), for a j such that $i \in \mathcal{S}_j$,

$$\begin{bmatrix} d_{j,j}^{(i)} \\ \vdots \\ d_{j,j+M-1}^{(i)} \end{bmatrix} = -\begin{bmatrix} \Omega_{j_{1,1}} & \cdots & \Omega_{j_{1,M}} \\ \vdots & \ddots & \vdots \\ \Omega_{j_{M,1}} & \cdots & \Omega_{j_{M,M}} \end{bmatrix} \left(\begin{bmatrix} R_i p_j \\ \vdots \\ R_i p_{j+M-1} \end{bmatrix} + \begin{bmatrix} w_i \\ \vdots \\ w_i \end{bmatrix} \right). \quad (21)$$

Let us define $\widetilde{w}_i = [I_D, \ldots, I_D]^\intercal w_i \in \mathbb{R}^{D \cdot M}$, which is the rightmost vector of (21). Note the last D columns of $\bar{C}_j$ are $[I_D, \ldots, I_D]^\intercal$; therefore, $\widetilde{w}_i$ is in the column space of $\bar{C}_j$, i.e., $(I_{D \cdot M} - \bar{C}_j \bar{C}_j^+)\widetilde{w}_i = 0$. In addition, $\Omega_{j_{m,n}} R_i = R_i \Omega_{j_{m,n}}$ for every block $\Omega_{j_{m,n}}$. Comparing (14) and (21), we conclude that $d_{j,i}^{(i)} = R_i d_{j,i}$, which holds for every addend in the control law (15) of agent i. Consequently, $u_i^{(i)} = R_i u_i$.

B. Control Law Analysis

Our formal result is presented in the following.

Theorem 1. *If Assumption 1 is satisfied, under control law* (15)*, $p(t)$ converges exponentially fast to a static configuration for which the team is in an affine formation with respect to the nominal configurations $c_{(1)}, c_{(2)}, \ldots, c_{(L)}$. The parameter vector of this formation is unique and can be computed from the positions of sets of $M - 1$ agents.*

Proof. In this proof, we use several elementary properties of positive semidefinite matrices. Further details in this respect are widely available; e.g., in textbooks such as [22]. From the control law expression (19), we have the system

$$\dot{p}(t) = -B\,p(t), \quad (22)$$

where we define

$$B = \sum_{j \in \mathcal{N}_s} \bar{Q}_j^\intercal (I_{D \cdot M} - \bar{C}_j \bar{C}_j^+) \bar{Q}_j \ \in \mathbb{R}^{(D \cdot N) \times (D \cdot N)}. \quad (23)$$

Here, all $\bar{C}_j \bar{C}_j^+$ and $I_{D \cdot M} - \bar{C}_j \bar{C}_j^+$ for all sets j are symmetric and idempotent and, hence, positive semidefinite. This implies the matrices $\bar{Q}_j^\intercal (I_{D \cdot M} - \bar{C}_j \bar{C}_j^+) \bar{Q}_j$ are all positive semidefinite, and their sum (i.e., B) is positive semidefinite. Therefore, from standard results in the theory of linear dynamical systems, $\|p(t)\|$ remains upper-bounded over time and $p(t)$ converges, exponentially fast, to a static configuration which is the orthogonal projection of $p(0)$ onto the nullspace of B. Next, we examine the static configuration that is reached asymptotically. For this, we define the notation $\breve{p}_i = p_i(t \to \infty)$ and $\breve{p} = p(t \to \infty)$.

As $\breve{p}$ is in the nullspace of B, we have

$$B\breve{p} = 0. \quad (24)$$

We can then write

$$\breve{p}^\intercal B \breve{p} = 0 \quad (25)$$

and, substituting the expression of B, and using $\breve{p}^\intercal \bar{Q}_j^\intercal = (\bar{Q}_j \breve{p})^\intercal$, we get

$$\sum_{j \in \mathcal{N}_s} (\bar{Q}_j \breve{p})^\intercal (I_{D \cdot M} - \bar{C}_j \bar{C}_j^+) \bar{Q}_j \breve{p} = 0. \quad (26)$$

As noted above, all matrices $I_{D \cdot M} - \bar{C}_j \bar{C}_j^+$ are positive semidefinite. Therefore, we infer from (26) that

$$(\bar{Q}_j \breve{p})^\intercal (I_{D \cdot M} - \bar{C}_j \bar{C}_j^+) \bar{Q}_j \breve{p} = 0 \quad \forall j \in \mathcal{N}_s, \quad (27)$$

and (see, e.g., [22, Obs. 7.1.6]) that

$$(I_{D \cdot M} - \bar{C}_j \bar{C}_j^+) \bar{Q}_j \breve{p} = 0 \quad \forall j \in \mathcal{N}_s. \quad (28)$$

Recalling (16), let us arrange (28) as follows:

$$\begin{bmatrix} \breve{p}_j \\ \vdots \\ \breve{p}_{j+M-1} \end{bmatrix} = \bar{C}_j \breve{v}_j \quad \forall j \in \mathcal{N}_s, \quad (29)$$

where we defined

$$\breve{v}_j = \bar{C}_j^+ \begin{bmatrix} \breve{p}_j \\ \vdots \\ \breve{p}_{j+M-1} \end{bmatrix} \in \mathbb{R}^{D \cdot (M-1)} \quad \forall j \in \mathcal{N}_s. \quad (30)$$

From (29), every one of the N_s systems of linear equations

$$\begin{bmatrix} \breve{p}_j \\ \vdots \\ \breve{p}_{j+M-1} \end{bmatrix} = \bar{C}_j g_j \quad \forall j \in \mathcal{N}_s, \tag{31}$$

where $g_j \in \mathbb{R}^{D\cdot(M-1)}$ is the unknown vector, has $g_j = \breve{v}_j$ as a solution. Moreover, due to Assumption 1, this solution is unique, since every $\bar{C}_j$ has full column rank. In addition, if we apply Definition 1 to the set of agents in $\mathcal{S}_j$ only, (29) implies that these agents are in an affine formation (which can be understood as a partial affine formation) with respect to the corresponding partial nominal configurations. Hence, we have established that every set of agents $\mathcal{S}_j$ converges as $t \to \infty$ to a partial affine formation with a parameter vector $\breve{v}_j$ uniquely determined by the positions $\breve{p}_i$ for $i \in \mathcal{S}_j$. Next, we show that $\breve{v}_j$ is the same for all the sets $\mathcal{S}_j$.

Let us take an arbitrary integer k such that $1 < k \leqslant N_s$. For the cases $j = k$ and $j = k-1$ in (29), we have

$$\begin{bmatrix} \breve{p}_k \\ \vdots \\ \breve{p}_{k+M-1} \end{bmatrix} = \bar{C}_k \breve{v}_k, \begin{bmatrix} \breve{p}_{k-1} \\ \vdots \\ \breve{p}_{k+M-2} \end{bmatrix} = \bar{C}_{k-1} \breve{v}_{k-1}. \tag{32}$$

From (12), we know that the matrix G_k satisfies

$$\bar{C}_k = \begin{bmatrix} G_k \otimes I_D \\ row_M(C_k) \otimes I_D \end{bmatrix}, \bar{C}_{k-1} = \begin{bmatrix} row_1(C_{k-1}) \otimes I_D \\ G_k \otimes I_D \end{bmatrix}. \tag{33}$$

Therefore, from (32) and (33), we have

$$\begin{bmatrix} \breve{p}_k \\ \vdots \\ \breve{p}_{k+M-2} \end{bmatrix} = (G_k \otimes I_D) \cdot \breve{v}_k = (G_k \otimes I_D) \cdot \breve{v}_{k-1}, \tag{34}$$

i.e.,

$$(G_k \otimes I_D) \cdot (\breve{v}_k - \breve{v}_{k-1}) = 0. \tag{35}$$

Recall that G_k is a nonsingular matrix, due to Assumption 1. Hence, $G_k \otimes I_D$ is nonsingular too. Therefore, the only solution to (35) is $\breve{v}_k - \breve{v}_{k-1} = 0$, and we thus have $\breve{v}_k = \breve{v}_{k-1}$. It is direct to see that this reasoning can be extended to every set $\mathcal{S}_j$ $\forall j \in \mathcal{N}_s$, due to their chained structure. Hence, there exists a unique $\breve{v}$ such that $\breve{v}_j = \breve{v}$ $\forall j \in \mathcal{N}_s$. Considering (29) and the form of the $\bar{C}_j$ matrices according to (9), (10), we can then write

$$\begin{aligned}
\breve{p}_1 &= (row_1(C_1) \otimes I_D)\breve{v} = (row_1(C) \otimes I_D)\breve{v} \\
\breve{p}_2 &= (row_2(C_1) \otimes I_D)\breve{v} = (row_2(C) \otimes I_D)\breve{v} \\
&\vdots \\
\breve{p}_M &= (row_M(C_1) \otimes I_D)\breve{v} = (row_M(C) \otimes I_D)\breve{v} \\
\breve{p}_{M+1} &= (row_M(C_2) \otimes I_D)\breve{v} = (row_{M+1}(C) \otimes I_D)\breve{v} \\
&\vdots \\
\breve{p}_N &= (row_M(C_{N-M+1}) \otimes I_D)\breve{v} = (row_N(C) \otimes I_D)\breve{v}.
\end{aligned} \tag{36}$$

Therefore, $\breve{p} = (C \otimes I_D)\breve{v}$; i.e., according to Definition 1, the multiagent team converges as $t \to \infty$ to an affine formation with respect to the nominal configurations $c_{(1)}, c_{(2)}, \dots, c_{(L)}$. The unique parameter vector $\breve{v}$ of this formation can be computed from sets of $M-1$ agent positions: using $\breve{p} = (C \otimes I_D)\breve{v}$ for agents j to $j+M-2$,

$$\breve{v} = (G_j \otimes I_D)^{-1} \begin{bmatrix} \breve{p}_j \\ \vdots \\ \breve{p}_{j+M-2} \end{bmatrix} \quad \forall j \in \mathcal{N}_s, \tag{37}$$

which concludes the proof. □

Our control strategy makes every set $\mathcal{S}_j$ reach a partial affine formation of M agents as $t \to \infty$. This is expressed by (29). Due to Assumption 1, $M-1$ consecutive agents in each set uniquely determine the parameter vector $\breve{v}_j$ for the M agents in the set. Then, as two chained sets have $M-1$ consecutive agents in common (e.g., sets $\mathcal{S}_1$ and $\mathcal{S}_2$ have the agents $2, \dots, M$ in common), they have the same parameter vector (i.e., $\breve{v}_1 = \breve{v}_2$). Hence, the partial formations *fit together*: they all have the same parameter vector, $\breve{v}_j = \breve{v}$ $\forall j \in \mathcal{N}_s$. As a result, a full formation for the N agents is achieved. As in [2], [3], [6], [7], [15], we focus on the problem of achieving a formation. We do not control the parameters, $\breve{v}$, of the achieved formation, which depend on the initial state $p(0)$ and the system dynamics.

V. Simulation Results

We illustrate our approach via numerical simulation in MATLAB. The task is for the agents to form a low-frequency discretized 2D closed curve [14], [15]. By avoiding high frequencies, such a curve avoids sharp local variations and preserves physical agent vicinities, while having a highly flexible shape. These are useful features for, e.g., enclosing a phenomenon taking place in the interior of the curve. We use a discrete Fourier-based representation of planar closed curves with two low-frequency components. The nominal configurations are Fourier basis vectors at these frequencies. We build C in (6) as follows, where the arguments have the form $k\frac{2\pi(i-1)}{N}$ with k indexing the frequency and i the agent:

$$C = \begin{bmatrix} \cos(1\frac{2\pi\cdot 0}{N}) & \cos(1\frac{2\pi\cdot 1}{N}) & \cdots & \cos(1\frac{2\pi\cdot(N-1)}{N}) \\ \sin(1\frac{2\pi\cdot 0}{N}) & \sin(1\frac{2\pi\cdot 1}{N}) & \cdots & \sin(1\frac{2\pi\cdot(N-1)}{N}) \\ \cos(2\frac{2\pi\cdot 0}{N}) & \cos(2\frac{2\pi\cdot 1}{N}) & \cdots & \cos(2\frac{2\pi\cdot(N-1)}{N}) \\ \sin(2\frac{2\pi\cdot 0}{N}) & \sin(2\frac{2\pi\cdot 1}{N}) & \cdots & \sin(2\frac{2\pi\cdot(N-1)}{N}) \\ 1 & 1 & \cdots & 1 \end{bmatrix}^{\intercal}. \tag{38}$$

Note that we have $L = 2$ and $M = D \cdot L + 2 = 6$. We take $N = 15$ and use a closed chain structure with $N_s = N$ sets $\mathcal{S}_j$. For every G_j, the determinant is $8.26 \cdot 10^{-3}$ and the condition number is $3.15 \cdot 10^2$. Therefore, Assumption 1 is satisfied. To assess formation achievement, we use the error metric $e(t) = \|(I_{D\cdot N} - (C \otimes I_D)(C \otimes I_D)^+)p(t)\|$ which expresses the distance from $p(t)$ to the column space of $C \otimes I_D$. We simulate our control law, using the time step 0.01 s. Figure 2 illustrates the convergence of the multiagent team toward a low-frequency discretized closed curve. We run again the same test incorporating a multiplicative control gain equal to 10 for every agent and saturation of every $\|u_i(t)\|$ at 1 m/s. As seen in the plots, this improves the convergence behavior,

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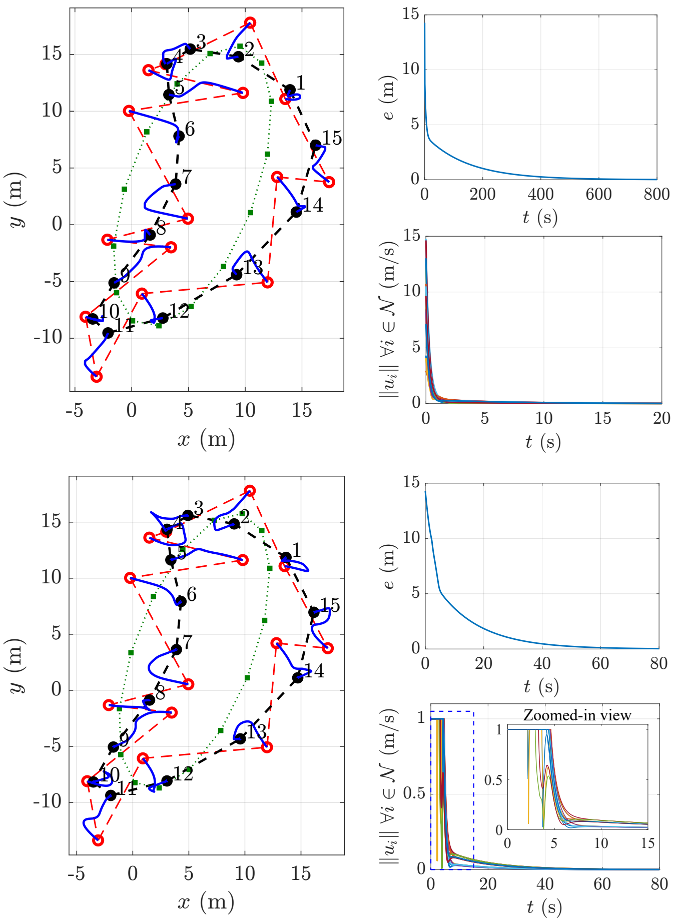

Fig. 2. The three plots at the top show the results with the original control law. The three at the bottom show the results with saturation and higher gain. Left plots: initial agent positions (red hollow circles), final agent positions (black solid circles), agent paths (blue solid lines). Agent indices and dashed lines joining consecutive agents are also shown. The final positions if using only one nominal configuration are shown as small green squares. Right plots: error metric (top) and velocity norms (bottom) over time.

which is influenced by matrix conditioning. The configuration $p(t \to \infty)$ depends on the initial state $p(0)$ and the system dynamics, and is not controlled in our approach.

Taking only the frequency index $k = 1$, i.e., a single nominal configuration with the geometry of a discretized circle, the achievable final configurations would be limited to discretized ellipses, as seen in the figure. This is because an ellipse is an affine transformation of a circle. By using two nominal configurations, our approach allows the team to acquire more complex shapes than an ellipse while still forming a low-frequency closed curve. This increases flexibility during enclosing. Using different nominal configurations (not Fourier-based) would offer different properties, while retaining our core advantage: allowing a wider range of achievable shapes than affine formation control in its standard form (i.e., with a single and constant nominal configuration).

VI. Conclusion

In this letter, we have introduced a novel approach for flexible multiagent formation control where multiple nominal configurations are used simultaneously. We considered a chained team structure, which is interesting in practical tasks and allowed us to develop a distributed and formally supported approach. Future directions include (i) controlling the formation parameter vector via a subset of leader agents using the results in Theorem 1, and (ii) a specific study of achievable shapes for various nominal configurations.

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