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Hodge Numbers of a Hypothetical Complex Structure on the Six Sphere

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Abstract. We prove that the terms $E_r^{p,q}(S^6)$ in the Frölicher spectral sequence associated to any hypothetical complex structure on S^6 would satisfy Serre duality. It is also shown that the vanishing of the Dolbeault cohomology group $H^{1,1}(S^6)$ ensures the existence of a holomorphic 2-form on S^6 living even in $E_2^{2,0}(S^6)$, which in particular implies the nondegeneration of Frölicher's sequence at the second level.

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1. Introduction

A. Gray has proved recently in [5] that, in spite of $H^1(S^6)$ being zero, the Hodge number $h^{0,1}(S^6)$ would not vanish for any hypothetical complex structure on S^6. As it is pointed out in [5], this property is interesting because it would imply the existence of a nonzero ($\bar{\partial}$-closed but non $\bar{\partial}$-exact) differential form $\alpha_{0,1}$ on S^6, which could be interpreted geometrically. Frölicher (or Hodge–de Rham) spectral sequence $\{E_r^{p,q}(S^6)\}_{r \geqslant 1}$ would also provide additional invariants of a complex structure on S^6 [3]. Terms $E_1^{p,q}$ in this sequence coincide with the usual Dolbeault cohomology groups $H^{p,q}$, and $\{E_r\}$ abuts onto the de Rham cohomology $H^*(S^6)$ of the six sphere. In this context, Gray's observation means that $E_1(S^6) \ncong E_\infty(S^6)$, that is, the sequence does not degenerate at the first level or, equivalently, Hodge decomposition $H^k = \oplus_{p+q=k} H^{p,q}$ fails in degree $k = 1$, which implies the well-known fact that S^6 would not be Kähler. It is also well-known that S^6 does not admit indefinite Kähler structure because its second Betti number $b_2(S^6) = 0$ and then it cannot be symplectic. However, the nonintegrable almost complex structure on S^6 coming from the identification of $\mathbb{R}^7$ with the purely imaginary Cayley numbers makes the six-dimensional sphere nearly Kähler, and therefore Einstein [4].

We show in Proposition 3.1 that a proof similar to the one given in [5] allows us to conclude that at least one between the Hodge numbers $h^{2,0}$ and $h^{1,1}$ is nonzero; thus, $H^k \neq \oplus_{p+q=k} H^{p,q}$ for $k = 1, 2, 4$ and 5.

Let us denote $h_r^{p,q} = \dim E_r^{p,q}(S^6)$ for $r \geqslant 1$. We shall refer to these dimensions as the Hodge numbers of a hypothetical complex structure on S^6 (for $r = 1$ they are the usual Hodge numbers $h^{p,q}$). To our knowledge it is not known whether for $r \geqslant 2$ the terms $E_r^{p,q}(M)$ and $E_r^{n-p,n-q}(M)$ are isomorphic for general complex n-dimensional manifolds M. In Theorem 3.2 we exhibit that the Hodge numbers of S^6 would satisfy Serre duality $h_r^{p,q} = h_r^{3-p,3-q}$ for all r.

Finally, Corollary 3.3 shows that the vanishing of the Dolbeault cohomology group $H^{1,1}$ (notice that $E_2^{1,1}$ always vanishes by Theorem 3.2) implies that Frölicher's spectral sequence does not collapse even at the second step. Therefore S^6 would be, as far as we know, the first example of a compact *simply-connected* manifold of the lower possible dimension for which $E_2 \ncong E_\infty$, i.e. providing a new solution to the question posed by Griffiths and Harris in [6, page 444]. So a better understanding of (simply-connected) compact complex manifolds having $E_2 \ncong E_\infty$ could provide an approximation to the study of the long standing problem on the existence of a complex structure on S^6. (Recent advances on this existence problem can be found in [8] and [9].) Simply-connected examples of dimension much larger than 3 with $E_2 \ncong E_\infty$ have been given by Pittie [10]. In the non simply-connected case, compact complex manifolds of dimension 3 have been found recently in [2].

2. Frölicher's Sequence of Complex Manifolds

In this section we prove that the sequence E_r degenerates at the step n for any compact complex n-dimensional manifold having no holomorphic n-forms.

Let us briefly recall that Frölicher's spectral sequence of a complex manifold M is the first spectral sequence associated to the double complex $(\Omega^{*,*}(M), \partial, \bar\partial)$, where $\Omega^{*,*}(M) = \oplus_{p,q}\Omega^{p,q}(M)$, $\Omega^{p,q}(M)$ being the space of complex-valued differential forms of bidegree (p, q) on M, and ∂, $\bar\partial$ are the differentials of type $(1, 0)$ and $(0, 1)$, respectively, in the usual decomposition $d = \partial + \bar\partial$ of the exterior derivative d. Thus, $E_1^{p,q}(M) \cong H^{p,q}(M) = \{\alpha_{p,q} \in \Omega^{p,q}(M) \mid \bar\partial\alpha_{p,q} = 0\}/\bar\partial(\Omega^{p,q-1}(M))$ and $\operatorname{Gr} H^k(M) \cong \oplus_{p+q=k} E_\infty^{p,q}(M)$ for any k, i.e. the terms $E_r^{p,q}(M)$ are complex invariants relating Dolbeault cohomology of M to de Rham cohomology of the manifold.

We shall be using the following well-known property: each term $E_{r+1}^{p,q}(M)$ can be obtained as the quotient $\operatorname{Ker} d_r/\operatorname{Im} d_r$ in a sequence of homomorphisms d_r

$$\cdots \longrightarrow E_r^{p-r,q+r-1}(M) \xrightarrow{d_r} E_r^{p,q}(M) \xrightarrow{d_r} E_r^{p+r,q-r+1}(M) \longrightarrow \cdots. \tag{1}$$

(For $r = 1$ the homomorphism $d_1 = \partial$, i.e.

$$d_1([\alpha_{p,q}]) = [\partial\alpha_{p,q}] \in H^{p+1,q}(M), \text{ for } [\alpha_{p,q}] \in H^{p,q}(M).)$$

Therefore, $h_r^{p,q} \geqslant h_{r+1}^{p,q}$ and $E_r^{p,q} = \{0\}$ implies $E_{r+s}^{p,q} = \{0\}$ for all $s \geqslant 1$.

In [5] it is proved that the Dolbeault cohomology groups $H^{0,3}(S^6)$ and $H^{3,0}(S^6)$ vanish for any possible complex structure on S^6. The argument given there still holds for any compact complex manifold M of complex dimension n with Betti number $b_n(M) = 0$; thus the Dolbeault cohomology groups $H^{n,0}(M)$ and $H^{0,n}(M)$ vanish for these manifolds M. In particular, the following lemma is valid for any such manifold.

LEMMA 2.1. *Let M be a compact complex manifold of complex dimension n. If there are no holomorphic n-forms on M then $E_n(M) \cong E_\infty(M)$.*

Proof: From (1), each term $E_{n+1}^{p,q}(M)$ is obtained as a quotient in

$$\cdots \longrightarrow E_n^{p-n,q+n-1}(M) \xrightarrow{d_n} E_n^{p,q}(M) \xrightarrow{d_n} E_n^{p+n,q-n+1}(M) \longrightarrow \cdots . \quad (2)$$

But for any complex n-dimensional manifold it is obvious that $E_r^{p,q}(M)$ is always zero if p or q are < 0 or $> n$. Therefore, the only terms $E_{n+1}^{p,q}(M)$ which could be non-isomorphic to $E_n^{p,q}(M)$ are those for which $(p, q) = (0, n-1), (0, n), (n, 0)$ or $(n, 1)$. Since the Dolbeault cohomology groups $H^{n,0}(M)$ and $H^{0,n}(M)$ vanish for M, $E_r^{n,0}(M) = E_r^{0,n}(M) = \{0\}$ for $r \geqslant 1$, which implies that all the homomorphisms d_n in (2) are zero, that is, $E_{n+1}^{p,q}(M) = E_n^{p,q}(M)$ for all (p, q). Analogously one concludes that $d_r \equiv 0$ for any $r > n$, and therefore $E_n^{p,q}(M) = E_r^{p,q}(M) = E_\infty^{p,q}(M)$ for all (p, q) and any $r > n$. □

3. Hodge Numbers of the Six Sphere

In this section we exhibit some properties of the Hodge numbers $h_r^{p,q}$ of any hypothetical complex structure on the six sphere.

PROPOSITION 3.1. *For any hypothetical complex structure on S^6 we have:*

$$h^{0,1} = h^{0,2} + 1 > 0 \quad \textit{and} \quad h^{2,0} + h^{1,1} = h^{1,0} + h^{1,2} + 1 > 0.$$

Proof: The Euler characteristic χ of a compact complex 3-dimensional manifold can be calculated by

$$\chi = \sum_{k=0}^{6} (-1)^k b_k = \sum_{p,q=0}^{3} (-1)^{p+q} h_r^{p,q},$$

for any $r \geqslant 1$ [3]. In particular, for $r = 1$ and using Serre duality we get

$$\chi = \sum_{p=0}^{3} \sum_{q=0}^{3} (-1)^{p+q} h^{p,q} = 2\left(\sum_{q=0}^{3} (-1)^q h^{0,q} - \sum_{q=0}^{3} (-1)^q h^{1,q} \right) = 2(\chi_0 - \chi_1),$$

where $\chi_p = \sum_{q=0}^{3} (-1)^q h^{p,q}$, for $p = 0, 1$. But χ_0 is precisely the arithmetic genus [7], i.e. the index of the elliptic complex $(\Omega^{0,*}, \bar{\partial})$, which in terms of the Chern numbers of the complex manifold is given by $\chi_0 = \frac{1}{2}c_1c_2$ [1].

If S^6 has a complex structure then $\chi_0 = 0$ because the Betti numbers b_2 and b_4 of S^6 are both zero. Therefore, since $\chi = 2$ for the six sphere, we get that $\chi_1 = -1$. Finally, since $h^{0,3} = 0$ and $h^{0,0} = 1$ we conclude that $0 = \chi_0 = 1 - h^{0,1} + h^{0,2}$ and $-1 = \chi_1 = h^{1,0} - h^{1,1} + h^{1,2} - h^{2,0}$. □

THEOREM 3.2. *For any hypothetical complex structure on* S^6 *we have that* $h_r^{p,q} = 0$, *for all* $(p, q) \neq (0, 0), (3, 3)$, *and* $h_r^{0,0} = h_r^{3,3} = 1$ *for any* $r \geqslant 3$. *Moreover:*

$$h_2^{0,0} = h_2^{3,3} = 1,$$

$$h_2^{1,0} = h_2^{1,1} = h_2^{3,0} = h_2^{0,3} = h_2^{2,2} = h_2^{2,3} = 0,$$

$$h_2^{0,1} = h_2^{2,0} = h_2^{1,3} = h_2^{3,2},$$

$$h_2^{0,2} = h_2^{2,1} = h_2^{1,2} = h_2^{3,1}.$$

Furthermore, $h_2^{0,1} = h^{1,2} - h^{1,1} + 1 \geqslant 0$.

Proof: Since $H^k(S^6) = \{0\}$ for $1 \leqslant k \leqslant 5$, from Lemma2.1 it follows that $\mathrm{Gr}\, H^k \cong \oplus_{p+q=k} E_\infty^{p,q} = \oplus_{p+q=k} E_r^{p,q}$ for any $r \geqslant 3$, which implies $h_r^{p,q} = 0$, for all $(p, q) \neq (0, 0), (3, 3)$, and $h_r^{0,0} = h_r^{3,3} = 1$ for any $r \geqslant 3$.

Let us prove that $E_2^{p,q} \cong E_2^{3-p,3-q}$ for all p and q. First of all, since $h^{0,3} = h^{3,0} = 0$ and $1 = h_3^{0,0} = h_3^{3,3} \leqslant h_2^{0,0}$, $h_2^{3,3} \leqslant h^{0,0} = h^{3,3} = 1$ we get

$$h_2^{0,0} = h_2^{3,3} = 1 \quad \text{and} \quad h_r^{0,3} = h_r^{3,0} = 0, \quad \text{for} \quad r \geqslant 1. \tag{3}$$

From (1) and (3) it follows that for $(p, q) = (1, 0), (1, 1), (2, 2)$ and $(2, 3)$ the sequences of homomorphisms d_2 reduce to $0 \longrightarrow E_2^{p,q} \longrightarrow 0$, and therefore $E_2^{p,q} = E_3^{p,q} = \{0\}$, that is,

$$h_2^{1,0} = h_2^{1,1} = h_2^{2,2} = h_2^{2,3} = 0. \tag{4}$$

The remaining sequences of homomorphisms d_2 are $0 \longrightarrow E_2^{p,q} \xrightarrow{d_2} E_2^{p+2,q-1} \longrightarrow 0$, for $(p, q) = (0, 1), (0, 2), (1, 2)$ and $(1, 3)$. Since

$$E_3^{p,q} \cong \mathrm{Ker}\, d_2 \quad \text{and} \quad E_3^{p+2,q-1} \cong E_2^{p+2,q-1}/\mathrm{Im}\, d_2,$$

we conclude that $h_2^{p,q} - h_2^{p+2,q-1} = h_3^{p,q} - h_3^{p+2,q-1} = 0$, for $(p, q) = (0, 1), (0, 2), (1, 2)$ and $(1, 3)$. This implies

$$h_2^{0,1} = h_2^{2,0}, \quad h_2^{0,2} = h_2^{2,1}, \quad h_2^{1,2} = h_2^{3,1}, \quad h_2^{1,3} = h_2^{3,2}. \tag{5}$$

Finally, the sequences of homomorphisms d_1 are

$$0 \longrightarrow E_1^{0,q} \xrightarrow{d_1} E_1^{1,q} \xrightarrow{d_1} E_1^{2,q} \xrightarrow{d_1} E_1^{3,q} \longrightarrow 0\,,$$

for $q = 0, 1, 2, 3$. Since $E_2^{p,q} \cong \operatorname{Ker} d_1/\operatorname{Im} d_1$ we have that

$$h^{0,q} - h^{1,q} + h^{2,q} - h^{3,q} = h_2^{0,q} - h_2^{1,q} + h_2^{2,q} - h_2^{3,q},$$

for $q = 0, 1, 2, 3$, and from (3) and (4) these equalities reduce to

$$\begin{aligned} &- h^{1,0} + h^{2,0} = h_2^{2,0}\,,\\ &h^{0,1} - h^{1,1} + h^{2,1} - h^{3,1} = h_2^{0,1} + h_2^{2,1} - h_2^{3,1}\,,\\ &h^{0,2} - h^{1,2} + h^{2,2} - h^{3,2} = h_2^{0,2} - h_2^{1,2} - h_2^{3,2}\,,\\ &- h^{1,3} + h^{2,3} = -h_2^{1,3}\,. \end{aligned} \tag{6}$$

Now Serre duality $h^{p,q} = h^{3-p,3-q}$ implies that $h_2^{2,0} = h_2^{1,3}$ and $h_2^{0,1} + h_2^{2,1} - h_2^{3,1} = -(h_2^{0,2} - h_2^{1,2} - h_2^{3,2})$. From (5) it follows

$$h_2^{0,1} = h_2^{2,0} = h_2^{1,3} = h_2^{3,2},$$

and then $h_2^{2,1} - h_2^{3,1} = -h_2^{0,2} + h_2^{1,2}$, which implies using again (5) that $2h_2^{0,2} = 2h_2^{1,2}$, i.e.

$$h_2^{2,1} = h_2^{0,2} = h_2^{1,2} = h_2^{3,1}.$$

Moreover, from (6) we obtain $h^{2,0} - h^{1,0} = h_2^{0,1} = h^{0,1} - h^{1,1} + h^{2,1} - h^{3,1} \geqslant 0$. Proposition 3.1 implies that $h_2^{0,1} = h^{1,2} - h^{1,1} + 1 \geqslant 0$. □

COROLLARY 3.3. *If S^6 had a complex structure then the Dolbeault cohomology group $H^{0,1}$ would be nonzero. Moreover, there would also be cohomology in degree 2 because either* (i) *$H^{1,1} \neq \{0\}$, or* (ii) *$H^{1,1} = \{0\}$ and then $H^{2,0} \neq \{0\}$ and $E_2^{2,0} \neq \{0\}$, i.e. $E_2(S^6) \ncong E_\infty(S^6)$.*

Remark 3.4. Since $E_2^{0,1} = \operatorname{Ker}\{\partial\colon H^{0,1} \longrightarrow H^{1,1}\}$ we have $h^{0,1} \geqslant h_2^{0,1} \geqslant h^{0,1} - h^{1,1}$, and from Theorem 3.2 we get $h^{1,1} \geqslant h^{1,2} - h^{0,2} \geqslant 0$. Therefore, in case (ii) all the Hodge numbers $h^{p,q}$ are determined by $h^{1,0}$ and $h^{0,2}$ (for example, $h^{1,2} = h^{0,2}$, $h^{0,1} = h^{0,2} + 1$, $h^{2,0} = h^{1,0} + h^{0,2} + 1, \ldots$); moreover, since $h_2^{2,0} = h_2^{0,1} = h^{0,1} = h^{2,0} - h^{1,0}$, the surprising fact $E_1(S^6) \ncong E_2(S^6) \ncong E_\infty(S^6)$ would be obtained if $H^{1,0} \neq \{0\}$.

Explicit descriptions of the terms $E_2^{p,q}$ and $E_3^{p,q}$ are [2]:

$$E_2^{p,q} \cong \frac{\{\alpha_{p,q} \in \Omega^{p,q} \mid 0 = \bar{\partial}\alpha_{p,q} = \partial\alpha_{p,q} + \bar{\partial}\alpha_{p+1,q-1}\}}{\{\bar{\partial}\beta_{p,q-1} + \partial\beta_{p-1,q} \in \Omega^{p,q} \mid 0 = \bar{\partial}\beta_{p-1,q}\}},$$

$$E_3^{p,q} \cong \frac{\{\alpha_{p,q} \in \Omega^{p,q} \mid 0 = \bar{\partial}\alpha_{p,q} = \partial\alpha_{p,q} + \bar{\partial}\alpha_{p+1,q-1} = \partial\alpha_{p+1,q-1} + \bar{\partial}\alpha_{p+2,q-2}\}}{\{\bar{\partial}\beta_{p,q-1} + \partial\beta_{p-1,q} \in \Omega^{p,q} \mid 0 = \bar{\partial}\beta_{p-1,q} + \partial\beta_{p-2,q+1} = \bar{\partial}\beta_{p-2,q+1}\}},$$

and the homomorphisms d_2 and d_3 in the sequence (1) are given by: $d_2([\alpha_{p,q}]) = [\partial\alpha_{p+1,q-1}]$ and $d_3([\alpha_{p,q}]) = [\partial\alpha_{p+2,q-2}]$. Using these descriptions, Corollary 3.3 and taking into account that $E_2^{1,1}(S^6) = \{0\}$ for any hypothetical complex structure on S^6, we get:

COROLLARY 3.5. *If S^6 has a complex structure then there exists a nonzero $\bar{\partial}$-closed differential form $\alpha_{0,1}$ on S^6 such that $\alpha_{0,1} \neq df$, for any function f on S^6, and with $\partial\alpha_{0,1} \neq 0$. Moreover, one of the following assertions holds:*

(i) *The Dolbeault cohomology group $H^{1,1}$ does not vanish and then there exists a nonzero $\bar{\partial}$–closed differential form $\alpha_{1,1}$ on S^6 such that $\alpha_{1,1} \notin \bar{\partial}(\Omega^{1,0}(S^6))$ and, either $\partial\alpha_{1,1} \notin \bar{\partial}(\Omega^{2,0}(S^6))$ or $\alpha_{1,1} = \bar{\partial}\beta_{1,0} + \partial\beta_{0,1}$ with $\bar{\partial}\beta_{0,1} = 0$, for some forms $\beta_{1,0}, \beta_{0,1}$ on S^6.*

(ii) *The Dolbeault cohomology group $H^{1,1} = \{0\}$ and then there exists a nonzero holomorphic 2-form $\alpha_{2,0}$ on S^6. Moreover, $\alpha_{2,0}$ is d-closed and therefore $\alpha_{2,0} \in d(\Omega^1_{\mathbb{C}}(S^6))$, although $\alpha_{2,0} \notin d(\Omega^{1,0}(S^6))$. On the other hand, since $E_2^{0,1} \neq \{0\}$ in this case, the form $\alpha_{0,1}$ given at the begining of this corollary satisfies that $\partial\alpha_{0,1} = \bar{\partial}\alpha_{1,0} \neq 0$ for some (non ∂-closed) form $\alpha_{1,0}$ on S^6.*

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References

1. Atiyah, M. F. and Singer, I. M.: The index of elliptic operators: III, *Ann. Math.* **87** (1968), 546–604.
2. Cordero, L. A., Fernández, M., Gray, A. and Ugarte, L.: A general description of the terms in the Frölicher spectral sequence, *Differential Geom. Appl.* **7** (1997), 75–84.
3. Frölicher, A.: Relations between the cohomology groups of Dolbeault and topological invariants, *Proc. Nat. Acad. Sci. U.S.A.* **41** (1955), 641–644.
4. Gray, A.: The structure of nearly Kähler manifolds, *Math. Ann.* **223** (1976), 233–248.

5. Gray, A.: A property of a hypothetical complex structure on the six sphere, *Bol. Un. Mat. Ital. B* (7) **11** (1997), Suppl. fasc. 2, 251–255.
6. Griffiths, P. and Harris, J.: *Principles of Algebraic Geometry*, Wiley, New York, 1978.
7. Hirzebruch, F.: *Topological Methods in Algebraic Geometry*, 3rd edn, Springer, Berlin, 1966.
8. Huckleberry, A., Kebekus, S. and Peternell, T.: Group actions on S^6 and complex structures on $\mathbb{P}_3$, *Duke Math. J.* (to appear).
9. LeBrun, C.: Orthogonal complex structures on S^6, *Proc. Amer. Math. Soc.* **101** (1) (1987), 136–138.
10. Pittie, H.: The nondegeneration of the Hodge-de Rham spectral sequence, *Bull. Amer. Math. Soc.* **20** (1989), 19–22.