PROSPECTIVE SECONDARY MATHEMATICS TEACHERS READ CLAIRAUT. PROFESSIONAL KNOWLEDGE AND ORIGINAL SOURCES

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Abstract. The use of original sources is a useful resource not only to be used with secondary school students but also with prospective mathematics teachers. In this work, we designed a series of tasks based on a fragment excerpted from Clairaut's *Éléments de Géométrie* to be carried out with 24 participants enrolled on a Masters' Degree in Secondary School Mathematics Teaching. This fragment was chosen both due to its content and to its narrative structure and our main goal was to determine which elements of professional knowledge were used by prospective secondary mathematics teachers when reading this fragment. In order to do so, we used the MKT model as an analytical tool and we also assessed some aspects related to literacy skills. The prospective teachers were able to recognize mathematical and pedagogical components within the source that relate to their future practice. In addition, the participant's literacy skills seem to play a role in the richness of their reading.

Keywords. Mathematics teacher training, professional competence, original sources, MKT, literacy skills, Clairaut.

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INTRODUCTION

The use of history of mathematics in the context of Mathematics Education is not a new idea and it can be traced back, at least, to the final years of the 19th century. However, it has been during the last 50 years that it has become an intensive worldwide area of research (Clark, Kjeldsen, Schorcht, & Tzanakis, 2018).

There are many reasons and many ways to introduce a historical dimension in Mathematics Education (Jankvist, 2009), and even participants themselves seem to demand a wider use of it when confronted with historic texts for the first time (Chorlay, 2018, p. 125). In the case of teacher training, the use of history not only promotes cultural understanding, but it is also useful in order to provide a meaning to mathematical objects through experiencing historical moments of their construction (Furinghetti, 2007). Some authors even suggest that "the reading of original sources should become an obligatory part of mathematics teacher education at all levels" (Jahnke, Arcavi, Barbin, Bekken, Furinghetti, El Idrissi, Silva da Silva, & Weeks, 2000, p. 299). However, as it is the case with other resources like ICT (information and communication technologies), reading original sources can be a difficult task that must be trained, especially if we want the future teachers to be able to integrate this resource in their future practices (Pugalee & Robinson, 1998).

This being said, our work is related to an active and fruitful research area both in the contexts of teaching (Chorlay, 2016; de Vittori, 2018; Romero Vallhonesta & Massa Esteve, 2019) and of teaching training (Jankvist, Clark & Mosvold, 2020; Schorcht & Buchholtz, 2019) which explores and assesses the use of original sources in the context of teacher training. In fact, we are particularly interested in the possible interplay between the use of original sources and the use and development of prospective teachers' teaching skills and professional competence. Thus, our main research question is: Does the reading of original historical sources contribute to the development of the professional knowledge of prospective secondary mathematics teachers?

In order to answer this question, we establish the following specific objectives:

1. To design an activity based on the reading of a historical source and implement it with prospective secondary mathematics teachers.
2. To determine elements of professional knowledge that are used by prospective secondary mathematics teachers when they read an original source.

THEORETICAL FRAMEWORK

In his seminal work, Shulman (1986) reflected on the type of knowledge, with special emphasis on content, required by a teacher in order to be proficient in his job. He distinguished between subject matter content knowledge, pedagogical content knowledge and curricular knowledge. Soon after, Shulman (1987) added four more categories: general pedagogical knowledge, knowledge of learners, knowledge of pedagogical context and knowledge of educational purposes, values and their philosophical and historical grounds. In the case of mathematics, Shulman's ideas were developed and adapted by Ball, Thames, and Phelps (2008) leading to the so-called Mathematical Knowledge for Teaching (MKT) model (Figure 1).

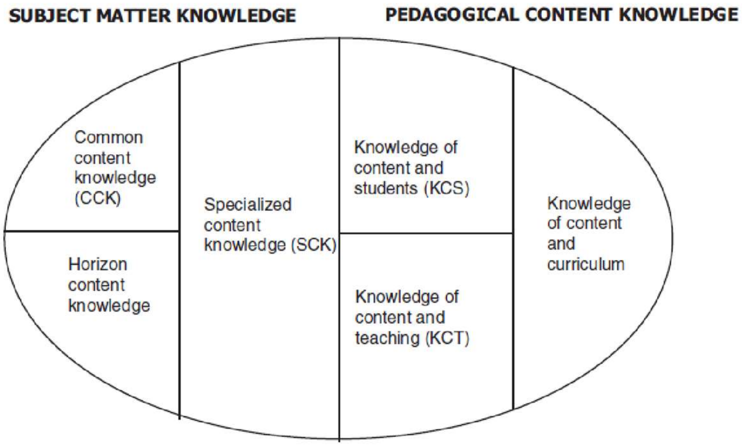


Figure 1. MKT model (Ball, Thames, & Phelps, 2008)

The MKT model considers that the mathematical demand for teaching can be divided into two domains: Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK). Each of these domains can be further divided into three subdomains:

- Common Content Knowledge (CCK), defined as the mathematical knowledge used in settings other than teaching.
- Specialized Content Knowledge (SCK), which is the mathematical knowledge unique to teaching.
- Horizon Content Knowledge (HCK), an awareness of how mathematical topics are related over the span of mathematics included in the curriculum.
- Knowledge of Content and Students (KCS), which implies knowing how participants interact with the discipline.
- Knowledge of Content and Teaching (KCT), directly related to the design of instruction.
- Knowledge of Content and Curriculum (KCC), a knowledge of materials and programs.

As we see, this model somehow identifies teachers' proficiency (Schoenfeld& Kilpatrick, 2008) with certain knowledge possessed (or not) by them. Moreover, since MKT (and, in particular, PCK) focuses so heavily on the teachers side, it neglects in some sense the fact that teaching practices are intended to foster learning by the students through the design and implementation of tasks. Graeber and Tirosh (2008) provide a good account of the main criticisms and challenges faced by the PCK construct.

In this sense, it must be noted that there are more recent trends in the field of mathematics teacher training which suggest that, in order to assess or to develop teachers' proficiency, the focus should be shifted from just teachers' knowledge to the outcome of their teaching practice; namely, to more specific (and practical) competencies related, for instance, to the design, lead and assessment of classroom interventions.

One example of this latter approach would be the work of Godino and collaborators (Godino, 2013; Godino, Giacomone, Batanero & Font, 2017) who develop the concept of *idoneidad didáctica* (didactical appropriateness) of the process of study, together with a series of specific didactical competencies that a proficient teacher should possess. Another example would be the re-elaboration of John Mason's concept of professional noticing (Mason, 2002) by Llinares and collaborators (Llinares, 2012; 2013; Sánchez-Matamoros, Fernández & Llinares, 2015). For these authors, one of the main components of teachers' proficiency is that of using their knowledge to resolve professional tasks. In particular, they focus on the way in which teachers analyze students' productions and on how they may reconstruct and infer the students' understanding from their analyses.

Leaving aside these various approaches, Jankvist, Mosvold, Fauskanger, and Jakobsen (2015) point out that "there is a dual relationship between the history of mathematics, including its use in mathematics education and the framework of MKT" (pp. 504-505). In fact, these authors show that the MKT model is useful in order to analyze history-based teacher training activities and to communicate the results of such research to other areas of Mathematics Education. Conversely, Mosvold, Jakobsen, and Jankvist (2014) provide several explicit examples of how all of the six MKT subdomains can profit from the study of history of mathematics. The particular case of HCK is addressed by Smestad, Jankvist, and Clark (2014) comparing examples from Denmark, Norway and the US. In a more empirical approach, Youchu (2016) shows how a history-based course for pre-service teachers had a strong influence over SCK, HCK, KCT and KCS.

One way to introduce history in the classroom is by the use of original sources. Barnett, Lodder, and Pengelley (2014) propose the so-called "guided reading modules" which consist of excerpts from relevant original sources together with a series of tasks aimed at developing the participants' own understanding of the underlying material. This approach has proved fruitful even in a context of higher education (Barnett, Lodder, Pengelley, Pivkina, &Ranjan, 2011). A noteworthy effect achieved by the use of original sources is the so-called *dépaysement épistémologique* (Barbin, 1997) described as "the astonishment of the learner facing a posture, a framework, a process or a particular argument, far from those of today" (Guille , 2012); even in combination with the use of ICT (Massa-Esteve, 2012) or incorporating software like GeoGebra (Chorlay, 2015; Zengin, 2018). However, Jankvist (2014b) stated that "not much emphasis has been put on the use of original sources in teacher education" (p. 903).

Even if the use of original sources can be an important didactical tool, incorporating them into the classroom usually involves several issues of concern. As Jahnke et al. (2000, p. 317) point out "reading a source is quite different from reading a normal text of mathematics" and hence different obstacles and difficulties may arise. On the one hand, we can find "logistical obstacles" (Pengelley, 2011), which have been clearly pointed out by Siu (2007), and that could be mostly related to difficulties foreseen, or actually encountered, by teachers in the process of design and orchestration of history-based tasks (lack of experience or materials, time issues, etc.). On the other hand, we can also find obstacles associated to the fact that reading historical mathematical texts is an inherently demanding activity for many reasons (Wardhaugh, 2010).

First, we could mention the lack of historical knowledge (both from the mathematical and cultural point of view) which could lead to whig interpretations that should be avoided (Fried, 2014). Another important obstacle is related to the ignorance of the original language, and even typography (Métin, 2019). Even if this can be partially avoided by translating or adapting the material, it remains the danger of translating unclear expressions or using anachronistic terms and notations. Furthermore, we must also consider the issue of the power of authority (Amit & Fried, 2005). In reading original sources, the reader might be overwhelmed by the fame of the author or by the historical importance of the document and this could affect his or her reading of the text.

We have just mentioned the difficulty related to the ignorance of the original language. However, Jahnke et al. (2000, p. 299) also advert that "even if an original source is given in the native language of the participants its interpretation presupposes a considerable linguistic competence". Consequently, when reading a historical text, some elements of literacy must also be taken into account (Chorlay, 2019). In our case, we mostly rely on the concept of content literacy.

McKenna, and Robinson (1990, p. 184) define content literacy as "the ability to use reading and writing for the acquisition of new content in a given discipline" and they distinguish three components of this ability: general literacy skills, content-specific literacy skills and prior knowledge of content. Kintsch and van Dijk (1978) point out that comprehension and summarization, among others, are important general skills. These authors also give an account of actions that individuals may perform when producing a summary, such as reproduction (the simplest operation involved in text production), reconstruction (the subject reconstructs information using the available material), and metastatement (the subject adds comments, opinions, etc.). When it comes to the reading of mathematical texts, Österholm (2005) has shown that in the comprehension of mathematical texts without symbols the main component is the general literacy skills, while for mathematical texts with symbols the component that plays the main role is the content-specific literacy skills. In addition, Niss (2006) mentions linguistic competence and, in particular, "the ability to read and decode different sorts of texts" as one of the required elements to be a competent mathematics teacher (p. 45).

METHOD

The experiment was carried out with 24 participants (13 men and 11 women) of the Masters' Degree in Secondary School Teaching (see Table 1). In particular, it took place within the course "Design, organization and development of activities for the learning of Mathematics" during the academic year 2018-2019.

	Degree			
	Mathematics	Engineering	Sciences	Total
age≤25	5	0	0	5
25<age ≤30	8	1	1	10
30<age ≤35	0	2	0	2
age≥35	1	5	1	7
Total	14	8	2	24

Table 1. Contextual variables

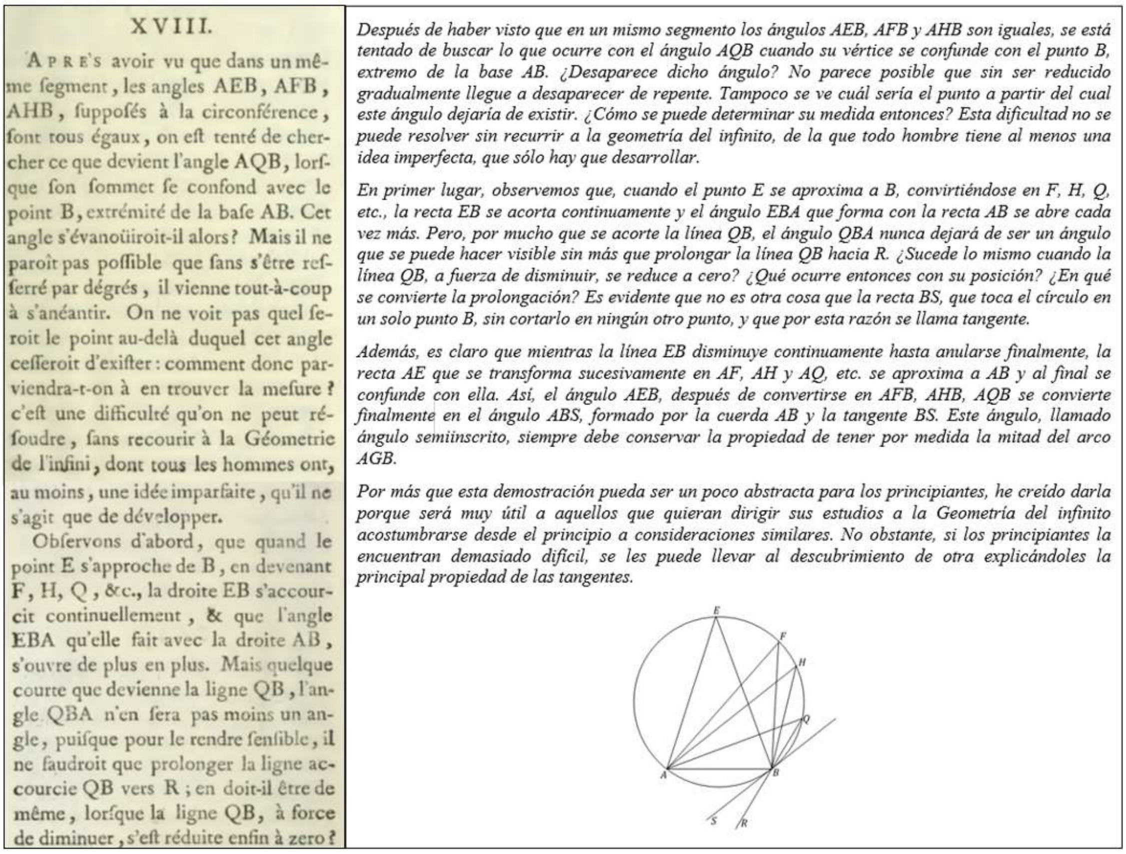
Considering the age, there were two clearly different groups: participants aged 30 and younger and participants older than 30. It is noteworthy the fact that only one of the older participants graduated in Mathematics. On the other hand, only two of the younger participants did not graduate in Mathematics. The academical background of the participants who did not graduate in mathematics, included between 30 and 50 ECTS credits (750-1000 hours of student work). None of the participants had received any undergraduate course about the history of mathematics.

The activity was proposed to the participants through the web platform of the course. The participants worked individually, and they had five days to read the text, complete the required tasks and send them back to the teacher. Part of a subsequent class session (about 30 minutes) was also devoted to a group discussion about the participants' responses. This group discussion was recorded, but in this work we will focus on the participants' written productions.

The activity essentially consisted of a guided reading (Barnett, Lodder, & Pengelley, 2014) of a fragment excerpted from Clairaut's *Éléments de Géométrie*. In a guided reading context, as described by Jankvist (2014a, p. 122), "the reading of the original text is 'interrupted' by explanatory comments, tasks, etc.". In our setting, explanatory comments were provided during the group discussion while, rather than to 'interrupt', the designed tasks were meant to make the participants revisit and reread Clairaut's text several times.

In particular, we focused on the fragment (Clairaut, 1741, pp. 125-127) in which Clairaut introduces the tangent to a circle at a point and he proves what we call nowadays "alternate segment theorem" which states that, in any circle, an angle between a tangent and a chord through the point of contact is equal to the angle in the alternate segment. As a corollary, Clairaut also proves that the angle between the former angle is half of the arc defined by the chord. The definition given by Clairaut can be seen as a *procedural* definition while the classical one would be a *structural* definition (Zaslavsky & Shir, 2005, p. 322).

The interest of this fragment was already pointed out by Chorlay (2015) in a context of combining historical sources and ICT around the concept of derivative. In fact, Clairaut's excerpt present features that makes its reading interesting not only mathematically, but also from a pedagogical point of view (Barbin, 1991). It involves several mathematical concepts both elementary and relatively advanced (tangents, triangles, infinitesimals, limits, etc.), but it also deals with pedagogical elements like discovery, proof-related processes, difficulty of a reasoning, etc. Thus, we think that this fragment is suitable to foster the use of both SMK and PCK by the participants. In addition, the fact that the fragment provides a written description of a dynamical process makes it suitable to assess elements related to linguistic competence and (content-specific) literacy skills.



XVIII.

APRE'S avoir vu que dans un même segment, les angles AEB, AFB, AHB, suppofés à la circonférence, font tous égaux, on eft tenté de chercher ce que devient l'angle AQB, lorfque fon fommet fe confond avec le point B, extrémité de la bafe AB. Cet angle s'évanoüiroit-il alors? Mais il ne paroît pas poffible que fans s'être reflerré par dégrés, il vienne tout-à-coup à s'anéantir. On ne voit pas quel feroit le point au-delà duquel cet angle cefferoit d'exifter : comment donc parviendra-t-on à en trouver la mefure? c'eft une difficulté qu'on ne peut réfoudre, fans recourir à la Géometrie de l'infini, dont tous les hommes ont, au moins, une idée imparfaite, qu'il ne s'agit que de développer.

Obfervons d'abord, que quand le point E s'approche de B, en devenant F, H, Q, &c., la droite EB s'accourcit continuellement, & que l'angle EBA qu'elle fait avec la droite AB, s'ouvre de plus en plus. Mais quelque courte que devienne la ligne QB, l'angle QBA n'en fera pas moins un angle, puifque pour le rendre fenfible, il ne faudroit que prolonger la ligne accourcie QB vers R ; en doit-il être de même, lorfque la ligne QB, à force de diminuer, s'eft réduite enfin à zero?

Después de haber visto que en un mismo segmento los ángulos AEB, AFB y AHB son iguales, se está tentado de buscar lo que ocurre con el ángulo AQB cuando su vértice se confunde con el punto B, extremo de la base AB. ¿Desaparece dicho ángulo? No parece posible que sin ser reducido gradualmente llegue a desaparecer de repente. Tampoco se ve cuál sería el punto a partir del cual este ángulo dejaría de existir. ¿Cómo se puede determinar su medida entonces? Esta dificultad no se puede resolver sin recurrir a la geometría del infinito, de la que todo hombre tiene al menos una idea imperfecta, que sólo hay que desarrollar.

En primer lugar, observemos que, cuando el punto E se aproxima a B, convirtiéndose en F, H, Q, etc., la recta EB se acorta continuamente y el ángulo EBA que forma con la recta AB se abre cada vez más. Pero, por mucho que se acorte la línea QB, el ángulo QBA nunca dejará de ser un ángulo que se puede hacer visible sin más que prolongar la línea QB hacia R. ¿Sucede lo mismo cuando la línea QB, a fuerza de disminuir, se reduce a cero? ¿Qué ocurre entonces con su posición? ¿En qué se convierte la prolongación? Es evidente que no es otra cosa que la recta BS, que toca el círculo en un solo punto B, sin cortarlo en ningún otro punto, y que por esta razón se llama tangente.

Además, es claro que mientras la línea EB disminuye continuamente hasta anularse finalmente, la recta AE que se transforma sucesivamente en AF, AH y AQ, etc. se aproxima a AB y al final se confunde con ella. Así, el ángulo AEB, después de convertirse en AFB, AHB, AQB se convierte finalmente en el ángulo ABS, formado por la cuerda AB y la tangente BS. Este ángulo, llamado ángulo semiinscrito, siempre debe conservar la propiedad de tener por medida la mitad del arco AGB.

Por más que esta demostración pueda ser un poco abstracta para los principiantes, he creído darla porque será muy útil a aquellos que quieran dirigir sus estudios a la Geometría del infinito acostumbrarse desde el principio a consideraciones similares. No obstante, si los principiantes la encuentran demasiado difícil, se les puede llevar al descubrimiento de otra explicándoles la principal propiedad de las tangentes.

Figure 2. Part of the original text (left) and the text as it was provided to the participants[1] (right)

The content of the excerpt closely relates to the course in which the experiment took place. For example, some of the learning outcomes of the course are "to explain the specific characteristics and difficulties of learning mathematics" or "to describe and assess activities and resources for learning mathematics". Moreover, Clairaut's fragment clearly resonates with ideas the participants have dealt with in other courses involving the use of GeoGebra (moving points, limit cases, etc.). Consequently, the participants were able to easily place the content of the excerpt in a mathematics teaching and learning context.

In the worksheet that was provided to the participants the text was translated into Spanish and the original figure was remade (but maintaining the exact same original appearance). We edited out three marginal notes from the original text in which Clairaut introduced both the definition of tangent to a circle and the statement of the theorem. We did so because identifying these elements

[1]Find the translation at the appendix.

was considered to be part of the designed tasks. No further changes were made to the original apart from explicitly dividing the text into paragraphs in order to organize the upcoming tasks (see Figure 2).

In the design of the tasks we took into account the work by Suzuka, Sleep, Ball, Bass, Lewis, and Thames (2009). For these authors, a task is well suited for developing MKT if, among other features, it creates opportunities to unpack, make explicit, and develop a flexible understanding of mathematical ideas, it opens opportunities to build connections among mathematical ideas or it provides opportunities to engage in mathematical practices central to teaching.

The participants were asked to complete the following five tasks. Each of the first four tasks corresponded to one of the paragraphs in which the text was divided, while the last one asked for a general personal comment about the whole text.

1. Which mathematical contents appear implicitly on the first paragraph?
2. Which procedure is Clairaut using on the second paragraph in order to define the tangent to a circle at a point? Which is that definition?
3. What is Clairaut doing on the third paragraph? Explain in your own words.
4. Analyze with a personal perspective the final paragraph of the text.
5. Write your personal opinion about the text.

In the following table (Table 2) we explicitly state the researchers' expectations for each task and we also present the domains of the MKT model (Ball, Thames, & Phelps, 2008) that we think the participants may put into practice when completing each task.

Task	The participants were expected to	Main related MKT domain (subdomain)
1	Identify relevant mathematical objects	SMK (CCK)
2	Understand a verbal description of a dynamic procedure and identify the underlying limit process	SMK (SCK)
3	Restate the described process in modern terms	SMK (SCK), PKC (KCT)
4	Discuss on the appropriateness or difficulty of the process	All
5	Provide personal opinions, interpretations, etc.	All

Table 2. Description of the tasks

The first task mostly relates to CCK because it involves only the identification of mathematical objects. The second task is mostly related to SCK because it deals with the verbal (and informal) description of a limit process which is frequent in a teaching environment. The third task relates to SCK for the same reason as the second one, but it also relates to KCT because restating a mathematical process is directly related to instruction. Finally, since the last two tasks are rather open questions, they might eventually involve all the MKT subdomains. Also observe that content literacy is present throughout all the tasks and particularly in tasks two to five because the participants have to read and "decode" the text, extract relevant information, re-elaborate and reconstruct its contents and provide opinions and comments about it.

The study that we carried out is exploratory and descriptive. Information was acquired by two means: the written productions of the participants and the recording of the class session. Data analysis combined quantitative and qualitative methods and it was performed with the aid of the software Atlas.ti (Smit, 2002). The tasks were analyzed independently, Table 4 provides details about the variables that we have used for the analysis in each of them.

Task	Variables	Categories	Informs about

1	Number of concepts	n/a	CCK
	Mathematical concept	Emerging	
2	Type of response	Reproduction / Reconstruction / Metastatement (Kintsch, & van Dijk, 1978)	Content literacy
	Use of word limit	Yes / No	SCK
3	Type of response	Reproduction / Reconstruction / Metastatement (Kintsch, & van Dijk, 1978)	Content literacy
	Content	Emerging	SCK / KCT
4	Type of response	Reproduction / Reconstruction / Metastatement (Kintsch, & van Dijk, 1978)	Content literacy
	Theme	Emerging using a thematic analysis (Clarke, & Braun, 2016)	MKT
5	Length of the response	n/a	Content literacy
	Content knowledge	MKT subdomains (Ball, Thames, &Phelps, 2008)	MKT

Table 3. Variables for the analysis

The variables studied in the first task (see Table 3) are the number of concepts used by the prospective teachers in their responses and the mathematical concepts identified on them. Both of them inform about CCK since the ability to identify abstract mathematical objects in the text is directly related to the general mathematical knowledge of the participants. The type of response (reproduction, reconstruction or metastatement), which is considered in tasks two to four, obviously informs about the participants' content literacy skills. In the second task, the use of the word 'limit' is a variable that informs about the SCK subdomain, because explicitly identifying the concept of limit underlying Clairaut's excerpt is closely related to the specific mathematical knowledge that a teacher needs for its practice. In the third task, the content of the response informs about the SCK and KCT subdomains through four emerging categories (process, definition, result and figure). In the fourth task, the themes emerging from the participant's responses are analyzed through a thematic analysis finding four categories (difficulty, fit, target and intention) that ultimately inform about different subdomains of the MKT model. Finally, in the fifth task, the length of the response informs about content literacy aspects since it is related to the ability to use reading and writing for the learning of contents. In this task, the variable 'content knowledge' is used to analyze the comments in terms of the MKT subdomains.

RESULTS

Task 1

In the first Task, the participants were asked to identify the mathematical contents contained in the first paragraph of the text. All the participants completed this task. We focus on two variables: 'number of concepts identified' and 'mathematical concept'.

Regarding the number of concepts, we analyzed 75 sentences for a total amount of 96 identified concepts. When more than one sentence dealt with the same concept, it was counted as many times as it appeared. The number of different concepts mentioned by each participant ranged between one and eight. The mean, mode and median were 4.00 and the standard deviation was 1.96 (see Figure 3)

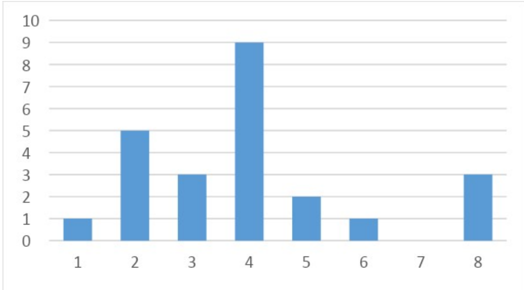


Figure 3. Number of different concepts mentioned by participants

For example, at the top of Figure 4 we show a response providing two concepts (triangle and inscribed angle) while in the response shown at the bottom of the figure, we find five concepts (segment, angle, vertex, base, geometry and limit).

Fragmento del texto	Contenidos
"Ángulos AEB, AFB…", "extremo de la base AB"	**Triángulos**
"Ángulos AEB, AFB…",	**Ángulo inscrito**

Fragmento del texto	Contenidos
Segmento los ángulos,…vértice,….base.	– Segmento – ángulos. – Vértice – Base
Esta dificultad no se puede resolver sin recurrir a la geometría del infinito, de la que todo hombre tiene al menos una idea imperfecta, que sólo hay que desarrollar	– Geometría del infinito, límites (cuando se aproxima a la recta AB)

Figure 4. Responses of two participants to Task 1

In order to analyze the concepts identified by participants, we initially carried out a word-count analysis. After a more refined analysis, we were able to classify the 96 identified concepts into 20 different categories. However, only eight of them were mentioned by at least four participants (see Table 4).

Limit	21	Arc	7
Angle	17	Tangent definition	5
Continuity	9	Segment	4
Inscribed angles property	8	Triangle	4

Table 4. Most mentioned categories

It is no surprise that the most mentioned concepts were 'limit' and 'angle'. Moreover, as this fact suggests, it is interesting to observe that there are two different types of concepts in the previous table: dynamic concepts related to Calculus (limit and continuity) and static concepts related to Geometry. It is remarkable that specific concepts (limit, triangle, etc.) were more frequently identified in the text than properties or mathematical results. In fact, as we see in Table 4, only the inscribed angles property was among the most identified categories.

Task 2

In the second Task, the participants had to identify the procedure used by the author to define the tangent to a circle at a point. Only one participant did not complete the task, for a total of 23 responses. We pay attention to two variables: 'type of response' and 'use of the word limit'.

According to the type of response, most of the participants (14) just reproduced the original information with a varying degree of re-elaboration. In fact, some participants even quoted fragments of the text verbatim. The remaining nine participants reconstructed the procedure rather than re-writing it; the following transcription is one such example:

[The author defines tangent line] through the explanation of the continuous transformation process of a triangle by modifying the position of one of its vertices (but keeping always the circumscribed circumference of the original triangle). The author suggests (using the limit case where this vertex gets one of the other two) to consider the line passing through the segment that links these vertices. By doing so, the participants could observe that, while the segment disappears, the line remains passing through only one point of the triangle (and the circumferences). At this point, he calls this concept as "tangent line".

Only half of the participants used explicitly the word limit in their responses. In the following table we show the relation between the type of response and the use of the word 'limit'.

		Use of the word limit	
		Yes	No
Type of response	Reproduction	4	10
	Reconstruction	8	1

Table 5. Use of the word 'limit' vs. type of response

As we see in Table 5, participants reproducing the text seem to use the word limit less frequently than those reconstructing it. In fact, if we perform a χ^2 test (with Yates correction) we can conclude that the relation between both variables is statistically significant at 95%.

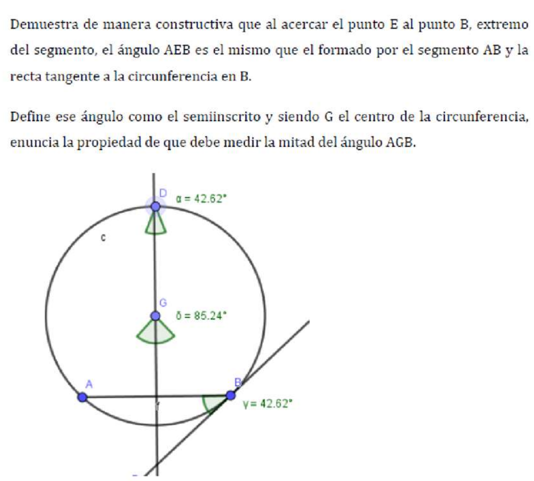
Task 3

In the third task, the participants were asked to explain in their own words the contents of the third paragraph. All the participants completed this task. We focus on two variables: 'type of response' and 'content'.

			Content			
		Participants	Process	Definition	Result	Figure
Type of response	Reproduction	11	11	7	8	3
	Reconstruction	9	7	4	4	1
	Mixed	4	4	1	3	2
	Total	24	22	12	15	6

Table 6. Type of response vs. Type of content

In this case, 11 participants reproduced the original information, nine reconstructed it and the remaining four gave a response combining both features (see Table 6). Regarding the content included in the participants' responses, almost all of them focused on the underlying process, while about only one half of the participants explicitly pointed out the appearance of a definition (12/24) or a result (15/24). Only six out of the 24 participants used a figure to support their discourse.



"He proves in a constructive way that, when point E gets closer to point B (one of the endpoints of the segment) the angle AEB is congruent with the angle formed by the segment AB and the tangent line to the circumference at B. He defines this angle as the semi-inscribed and, being G the center of the circumference, he states that its measure is half of the angle AGB."

Figure 5. Using a GeoGebra figure to support an explanation

Obviously, the participants' responses often cover more than one type of content. However, only eight of them included in their responses the three main aspects of the original text (process, definition and result). Moreover, four of these participants also included a figure (see Figure 5). On the other hand, five of them mentioned the process without relating it to any definition or result (see Figure 6).

A medida que el vértice móvil del triángulo se va acercando a uno de los vértices fijos, el segmento que los une se va transformando de secante a tangente a la circunferencia.

Figure 6. "As the mobile vertex gets closer to a fixed vertex, the segment joining them gets transformed from secant to tangent"

Task 4

In the fourth Task, the participants were asked to analyze with a personal perspective the content of the last paragraph. This task was completed by 19 participants. Their responses can be classified according to the actions performed by the participants (see Table 7). We focus on two variables: 'type of response' and 'theme'.

Reproduction	Reconstruction	Metastatement
6	5	13

Table 7. Actions present in the responses

Five participants included both some kind of reconstruction of the original test and metastatements (personal opinions or comments) in their responses. Eight participants only included metastatements in their responses while the remaining six participants just reproduced the original information of the paragraph, all of their comments being just a rewriting of Clairaut's words.

We are particularly interested in the 'metastatement' category, where the participant's opinions and attitudes towards the text are more likely to be shown. We have identified four different themes emerging among the participants' comments. In Table 8 we present the definition and frequency of these themes and we provide an example for each of them.

Theme	Definition	N	Example
Difficulty	Comments about the easiness of the mathematical content of the text.	6	*"This proof is more abstract*[2] *and can be difficult."*
Fit	Comments about the adequacy of the text for its purpose.	5	*"This is quite a visual proof."*
Target	Comments about the addressees of the text.	5	*"The text is addressed to Geometry students."*
Intention	Comments about the aim of the author when writing the text.	6	*"I think he is calming those readers who have not understood the previous explanation."*

Table 8. Themes present in the metastatements

Note that while the themes 'Difficulty' and 'Fit' provide information about the content of the text, the themes 'Target' and 'Intention' mostly deal with contextual aspects of the text.

Now, we try to relate these themes to domains of the MKT. Regarding the comments classified under the theme 'Difficulty', we find that participants elaborate on the reasons of the difficulties or the easiness of the content. These difficulties are related either to mathematics as a science or

[2] The comparison is made with the classical proof from Euclid's Elements.

to the specific mathematics needed for teaching. These two types of comments could correspond, respectively to CCK and SCK, two of the components of SMK. On the other hand, if we focus on the comments classified under the theme ‘Fit’, we find that participants elaborate on the relationship between the text and its addressees, either participants or teachers. These two types of comments could correspond, respectively, to KCS and KCT which are two of the components of PCK. We think that he two remaining themes cannot be related to any MKT domain

Task 5

In the fifth Task, the participants were asked to give their personal opinion about the text. Again, this task was completed by every participant. We focus on two variables: ‘length of response’ and ‘content knowledge’.

Regarding the length of the responses, we can see in Table 10 that most of them have less than 150 words. In fact, the median was 82 words with a shortest response of 32 words a longest one of 381 words.

After analyzing all the participants’ responses, we observed that about half of the participants made comments only about the SMK domain, while only three participants included comments about both the SMK and the PCK domains. Table 10 shows that there seems to be no relationship between the participants’ educational background and the content of their comments.

	SMK	PCK	Both	None
Mathematics	7	6	0	1
Engineering	4	1	3	0
Sciences	0	2	0	0

Table 10. Background of the participants vs. content of their comments

In Figure 7 we give an example of a participant writing only about the SMK domain.

Debido a la época en la que el texto fue redactado, detecto una falta de formalismo en el lenguaje matemático, el cual aún no estaba consolidad tal y como se conoce en la actualidad.

Por otro lado, y aunque el autor ya lo viene advirtiendo, me parece un texto algo abstracto, el cual hay que leerlo más de una vez para poder llegar a comprender todo lo que te está contando y lo que hay detrás de ello.

Due to the time when the text was written, I can detect a lack of formalism in the mathematical language, which had not been strengthened as it is known nowadays. On the other hand, although the author had warned about that, it seems to me quite an abstract text, which has to be read more than once to grasp all of its content and all of its background.

Figure 7.Example of a response involving only SMK comments.

On the other hand, the response from Figure 8 contains only comments about the PCK domain.

Esta forma de presentar la recta tangente se apoya en conceptos geométricos (circunferencia, rectas, ángulos) que se introducen en la educación secundaria obligatoria. Esto permitiría adelantar de una manera relativamente sencilla la definición de este objeto matemático sin necesidad de saber derivar, requisito necesario tal y como se trabaja en el currículo oficial. El enfoque geométrico de un concepto que generalmente se aborda desde el Análisis es, en mi opinión, uno de los puntos fuertes de este texto y por lo que sería interesante utilizarlo en el aula.

This way of introducing the tangent line is based on geometrical concepts (circumference, lines, and angles) that are introduced during Secondary education. This would allow to easily anticipate the definition of this mathematical object without the use of derivatives as it is done in the official curriculum. This geometrical view of a concept usually approached through Analysis is, in my opinion, one of the strong points of the text and the reason why it would be worth using in the classroom.

Figure 8.Example of a response involving only PCK comments.

Refining the analysis, we found 44 statements that could be related to some of the MKT subdomains (Table 11). As we can see, the number of comments related to each MKT domains is quite similar with a slight unbalance in favor of SMK (25 comments related to SMK and 19 related to PCK). Comments about the SCK and KCT subdomains are the most frequent. On the other hand, HCK and KCC are the least mentioned subdomains.

SMK			PCK		
CCK	SCK	HCK	KCS	KCT	KCC
8	13	4	8	9	2

Table 11. MKT subdomains identified in the comments

We now provide some examples of comments corresponding to each of the MKT subdomains[3]:

- CCK: “The text analyzes the concept and properties of semi-inscribed angles”.
- SCK: “I detect a lack of formalism”.
- HCK: “It is a purely geometrical text, in which the author introduces analytical concepts”.
- KCS: “We must be aware of the difficulties it may cause to the participants”.
- KCT: “Showing this type of proof can help to develop logical and mathematical thinking”.
- KCC: “In the school it is not usual to learn the concept of tangent through a geometric construction as the one given by Clairaut”.

Finally, regarding the richness of the responses, most of the participants (15) commented on two of the MKT subdomains (Table 12). Five participants commented on only one subdomain and three participants commented about three subdomains (CCK-SCK-KCS, CCK-SCK-KCS and CCK-SCK-HCK). The response of the remaining participant could not be categorized within the MKT model.

	SCK	HCK	KCT	KCC
CCK	4	1		
SCK			1	
HCK	2			
KCS			5	1
KCT				1

Table 12. MKT subdomains identified in the comments combining two MKT subdomains

DISCUSSION AND CONCLUSION

With this work, we have tried to contribute to Jankvist proposal (2014) “if positive empirical results (quantitative and qualitative) could be produced in support of these roles of primary original sources, then surely this would assist in the use of original sources gaining impact in mathematics education in general” (pp. 903-904). In fact, we illustrate the two possible interactions between history and the MKT model that are described in the literature (history as a tool to develop MKT and MKT as a tool to analyze history-based activities). More specifically, with our first objective (tasks one to four) we have shown a guided reading module that might contribute to the development of MKT among prospective teachers. On the other hand, the second objective (tasks four and five) shows the usefulness of the MKT model in order to analyze the productions of the participants.

In task one, we were focused on the development of the CCK subdomain. The number and frequency of the different concepts that appeared while answering this task indicate that it helped to make our participants read the text carefully and to make explicit the mathematical ideas underlying the text (Suzuka et al., 2009). In addition, it was useful in order to introduce the

[3] We give the translations (by the authors) of the participant’s actual statements.

following tasks. Thus, we think that starting the work by exploring the Common Content Knowledge was a good decision, possibly because, due to their educational background, it is the type of content they are most comfortable working with.

Task two dealt with SCK subdomain and the results were not as good as in task one. Specifically, the number of participants explicitly mentioning the word "limit" substantially decreased from task one to task two. Due to their background, we may assume that the participants are familiar with concepts such as limit, continuity, etc. Furthermore, they easily identified them in task one (recall Table 4). Consequently, we think that the main reason for this fact might be that task two involved the understanding and comprehension of Clairaut's text. As Österholm (2005) points out the main component in the comprehension of a mathematical text without symbols is the general literacy skills. Our results support this idea since, as we have seen in Table 2, the participants that reconstructed the text (showing a better comprehension) used the word limit in task two more frequently than the participants that only reproduced the text.

Task three was related to the SCK and KCT subdomains in the sense that participants were asked to explain with their own words the content of the third paragraph. This is a classical type of activity in teaching practice which is related to what Niss (2006) calls communication competency. We have seen that most of the participants focused on the process. However, only about half of the participants included ideas related to definitions and results. The question included the verb "to do", which is an action verb. Consequently, it seems that some of our participants do not perceive results and definitions as the outcome of a process, but rather they see them as static pre-existent mathematical objects. In this regard, Martín-Molina, González-Regaña and Gavilán-Izquierdo (2018) have shown that professional mathematicians follow a process to construct a definition for a new object, while this is not the case for students and teachers. Moreover, Zaslavsky and Shir (2005) showed how 12th grade students didn't accept definitions based on guidelines for constructing an object. The same authors (Shir & Zaslavsky, 2001) had previously found similar results with teachers. This could be seen as a Platonist view of mathematics that sees the teacher as an explainer and learning as the reception of knowledge (Ernest, 1988). This view is probably still common in higher education mathematics teaching (Mura, 1993; Viirman, 2015), and working with original sources might help to overcome this view (Jankvist, 2009). In our case, Clairaut's text provides an example of a mathematical discourse quite different from the axiomatic presentation which is usually adopted in higher education. In addition, the structure of the third paragraph is, somehow, inverted with respect to the traditional mathematical discourse. Usually the statement of a result precedes its proof and the definition of a concept does not usually arise as the result of a process. Part of the group discussion was devoted to these features of the text which might constitute an example of "*dépaysement épistémologique*" (Barbin, 1997)

We have also seen in task three that some participants turn to GeoGebra to support their discourse or even to better understand it. In fact, facilitating understanding and visualization of concepts are benefits of using GeoGebra in combination with history of mathematics (Zengin, 2018).

Task four is the first one in which metastatements appear, probably because the participants were asked to analyze the text with a personal perspective. We identified four different themes among these answers. Two of them dealt with elements that are more explicitly present in the text (such as the target of the text or the author's intention), while the others (that are MKT-related) implied a somehow deeper reading of the text. It is possible that the presence of non-MKT related themes could be related with the participants' lack of experience reading original sources, and the corresponding difficulties (Jahnke et al., 2000). Anyway, these themes can also be of interest because they involve the human dimension of mathematical activity (Tzanakis et al., 2000).

The educational background of teachers seems to have an impact over both Pedagogical Content Knowledge and Content Knowledge (Krauss et al., 2008). As we have seen in our analysis of Task five (Table 10), there seems to have been little difference between the participants' responses regarding their background. Consequently, working with original sources might be beneficial to develop MKT in a context where the participants have different backgrounds. A possible explanation is that, regardless the educational backgrounds, all of the participants are equally inexperienced in the use of original sources, so they are able to develop their competences in a similar degree.

In Task five, there is a balance between the number of participants mentioning PCK subdomains and those mentioning SMK subdomains. However, these roles seem to be mutually exclusive because only three participants combined both of them. From a teacher training point of view, it would be desirable to promote PCK profiles or, even better, mixed profiles.

The most frequently mentioned MKT subdomains were SCK, KCT and KCS. This partially agrees with Youchu (2016) with the exception of HCK. Hence, it could be interesting to integrate this activity into a wider sequence in order to develop the remaining MKT subdomains and, in particular, HCK that "is the domain of MKT which has the most to benefit from the study of history of mathematics" (Smestad, Jankvist, & Clark, 2014, p. 170).

As we have seen, the way in which the questions are posed seems to influence the type of responses given by the participants. This implies that we have to be careful when designing the tasks in order to achieve the desired goals. In particular, the use of certain expressions like "analyze with a personal perspective" and "your own words" promotes the appearance of metastatements which is particularly useful in order to trigger group discussions in the classroom.

We have found evidences of some of the obstacles and difficulties associated to the use of original sources that were described earlier in this paper. They have been found especially in the participants' answers to Task five, in which they were asked for their personal opinion. In fact, one participant bluntly expressed her lack of familiarity with historical texts: "I am not used to read this kind of texts". Another participant expressed his concerns about the fact that the text is just a fragment: "I think it is difficult to assess the text, since we lack information about what was previously covered in the same work". Related to this, some participants seem to have read other parts of Clairaut's work on their own: "It is a very interesting text [...] the selected fragment (as well as many of the sections of the third part of Éléments de Géométrie) contains an understandable mathematical discourse…". We do not know, however, if the participant was seeking for more context or if he was just interested in the work of Clairaut.

Some issues with the language were identified. In particular, the term 'geometry of the infinite' caused some problems. One of the participants, for instance, stated that "The text contains some confusing parts, like when talking about the geometry of the infinite, which I am not sure whether it refers to differential geometry or if it is talking about limits". As we see, this participant tries to assign a modern category to the confusing term. Finally, regarding the possible influence of authority, one of the participants wrote: "The text is difficult to understand, little educational [...] I do value the brightness of the author (obviously a genius at his time) ...". It is interesting to point out that, even if this participant clearly states his criticism, he somehow needs to balance it praising the author.

Most of the research investigating the use of history of mathematics in an MKT context use original sources with an emphasis on the mathematical content (Jankvist, Mosvold, Fauskanger, & Jakobsen, 2015). However, Clairaut's text also contains a very important pedagogical element (the fourth paragraph). Working with this paragraph has been particularly fruitful for our participants since it has shown them not only historical mathematics but also the way in which these concepts were taught in the past, leading them to a sort of "*dépaysement pédagogique*". We think that working with original sources that involve pedagogical content (like the fourth paragraph in Clairaut's text) might be an interesting line of research in the context of mathematics teacher training, that does not seem to have been explored yet.

Finally, even if the MKT model has been very useful in our setting in order to analyze the participants' written productions, it has some weaknesses if we were trying to assess the possible benefits of the use of original sources in terms of the development of more specific teachers' professional competencies. As we pointed out earlier in the paper, the MKT model has been criticized for possibly being too teacher-directed and for considering that teacher competencies are more related to possessing certain knowledge than to having certain practice-oriented competencies (Graeber & Tirosh, 2008). As a consequence, our results are limited since we restricted our analysis to the participants verbal reports and written productions rather than to their actions in actual didactic situations.

We are not aware of any published work trying to determine or to assess the possible impact of the use of original sources on teachers' professional practice and not only on their knowledge (either mathematical or pedagogical). Thus, it would be interesting to determine, for example, if the work with original sources can develop teachers' professional noticing (Llinares, 2013) in some sense; or if it improves the appropriateness (Godino, 2013) of teachers' classroom interventions. In any case, we think that this is a gap in the literature that would be worth exploring.

APPENDIX

In this appendix we provide the English translation of Clairaut's fragment. It is slightly adapted from (Chorlay, 2015, p. 490):

"*Since we saw that the angles on the perimeter AEB, AFB, AHB are all equal, one wonders what becomes of angle AQB as its vertex Q coincides with point B, the extremity of its base. Would this angle then vanish? It does not seem possible that it suddenly vanishes without gradually decreasing. Also, one cannot see after which point this angle would cease to exist; how, then, could we measure this angle? The only way out of this conundrum is to resort to the geometry of the infinite; a geometry of which all men have some (maybe imperfect) grasp, and which we aim at improving.*

Let us first observe that, as point E approaches point B, thus becoming F, H, Q etc., line EB gradually decreases, as the angle EBA which it makes with line AB increases ever more. But, however short line QB may become, the angle QBA will not cease to be an angle, since, to make it perceptible, we only need to extend line QB to point R. Will the same hold for line QB once it has decreased to the point of vanishing? What has then become of its position? What about its extension QR? It is obvious that it becomes no other than the line BS which touches the circle only at B, without cutting it at any other points; for this reason, this line is called the tangent.

Moreover, it is clear that as line EB continuously decreases and eventually vanishes, the line AE, which successively becomes AF, AH and AQ etc., comes ever closer to AB, and eventually coincides with it: hence the angle AEB subtended at the perimeter, after becoming AFB, AHB and AQB, eventually becomes the angle ABS between chord AB and tangent BS; and this angle, which is called the alternate-segment angle, must retain the property of being half of the measure of arc AGB.

In spite of the fact that this proof may be a little abstract for the beginner, I thought fit to include it, since it will be very useful for those who will further their study into the geometry of the infinite to have been accustomed to these considerations fairly early on. However, if beginners find it too difficult, they can be led to the discovery of another one explaining them the main property of tangents."

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