Characterizations and accurate computations for tridiagonal Toeplitz matrices

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ABSTRACT
Tridiagonal Toeplitz P-matrices, M-matrices and totally positive matrices are characterized. For some classes of tridiagonal matrices and tridiagonal Toeplitz matrices it is shown that many algebraic computations can be performed with high relative accuracy.

KEYWORDS
Toeplitz matrices; Tridiagonal matrices; High relative accuracy

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1. Introduction

Toeplitz matrices arise in many important applications, but they provide an example of an structured class of matrices for which it is not possible to perform some elementary algebraic computations with high relative accuracy (HRA). In fact, in [1] it was proved that the determinant of a general square Toeplitz matrix cannot be calculated with HRA. In contrast, for other classes of structured matrices, algorithms with HRA for many algebraic computations, in addition to the determinant, have been found. In this paper, we prove that, for some classes of tridiagonal Toeplitz matrices, many algebraic computations can be performed with HRA. Tridiagonal Toeplitz matrices arise in important applications, such as the solution of ordinary and partial differential equations, time series analysis or as regularization matrices in Tikhonov regularization for the solution of discrete ill-posed problems (see [2–7]). Recent results on the total positivity of some Toeplitz matrices and algorithms for determinants of tridiagonal periodic Toeplitz matrices can be seen in [8,9], respectively.

Let us now recall some concepts and notations used in this paper. Let A be a real matrix. We say that A is a nonnegative (positive) matrix and write $A \geq 0$ ($A > 0$) when all the entries of A are nonnegative (positive). A square matrix is a *P-matrix* if all its principal minors are positive. Let us recall that in a Linear Complementarity Problem, very important in the field of Optimization, there exists always a unique solution if

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and only if the associated matrix is a P-matrix. Some subclasses of P-matrices are very important in many applications. For instance, nonsingular TP matrices. A matrix A is said to be *totally positive* (TP) if all its minors are nonnegative. If all its minors are positive, then A is called *strictly totally positive* (STP). TP and STP matrices arise in many applications in Approximation Theory, Statistics, Economy, Biology and Computer Aided Geometric Design, among other fields (see [10–12]). A real matrix A is a Z-matrix if all its off-diagonal entries are nonpositive. The matrix A is called an *M-matrix* if it can be expressed in the form $A = sI - B$, where I is the identity matrix, $B \geq 0$ and $s \geq \rho(B)$, where $\rho(B)$ is the spectral radius of B. If $s > \rho(B)$, then A is a nonsingular M-matrix. Equivalently, a Z-matrix A is a nonsingular M-matrix if and only if its inverse is nonnegative (see characterization (N_{38}) in Theorem (2.3) of [13, Ch. 6]). Nonsingular M-matrices arise in the discretization of partial differential equations and in many applications to Dynamic Systems, Economy and Optimization (see [13]). We call a square real matrix $A = (a_{ij})_{1 \leq i,j \leq n}$ *sign symmetric* (*sign skew-symmetric*, respectively) if $a_{ij}a_{ji} \geq 0$ (≤ 0, respectively) whenever $i \neq j$ and A is tridiagonal if $a_{ij} = 0$ whenever $|i - j| > 1$. Given a matrix $A = (a_{ij})_{1 \leq i,j \leq n}$, $|A| := (b_{ij})_{1 \leq i,j \leq n}$ denotes the matrix such that $b_{ij} := |a_{ij}|$ for all $1 \leq i, j \leq n$. An algorithm can be performed with HRA (independently of the conditioning of the problem) if all the included subtractions are of initial data, that is, if it only includes products, divisions, sums of numbers of the same sign and subtractions of the initial data (cf. [1,14,15]). A first step to obtain HRA algorithms for a class of matrices is an adequate parametrization of the matrices. Up to now, HRA algorithms for algebraic computations have been obtained for some subclasses of P-matrices, in particular for diagonally dominant M-matrices and for some subclasses of TP matrices (see, for instance, [14,16–22]). This paper shows that some classes of tridiagonal Toeplitz matrices can be added to the previous list.

The paper is organized as follows. Section 2 includes some auxiliary results and presents the Neville elimination and the bidiagonal factorization, which provide the parametrization of nonsingular TP matrices that can be used to apply the HRA algorithms of Koev (see [15,23,24]) for nonsingular TP matrices. With these algorithms and the mentioned parametrization, one can perform the following algebraic calculations with HRA: inverse, all singular values, all eigenvalues and the solution of some linear systems. These algorithms will be used in this paper to obtain HRA computations with some tridiagonal Toeplitz matrices. In Section 3, we introduce Toeplitz matrices and characterize tridiagonal Toeplitz TP matrices, tridiagonal Toeplitz M-matrices and tridiagonal Toeplitz P-matrices. Section 4 deals with sign skew-symmetric tridiagonal matrices with positive diagonal entries. It is shown that their leading principal minors and all minors of their inverses can be computed with HRA. In Section 5, a condition is provided to calculate the bidiagonal decomposition of sign symmetric tridiagonal Toeplitz P-matrices with HRA, and so their eigenvalues and singular values, as illustrated also with numerical experiments in Section 6. As shown in Figure 1, our results outperform those obtained with the usual MATLAB functions. Let us also recall that the eigenvalues of a tridiagonal Toeplitz matrix are already known (cf. page 59 of [7]), in contrast to the singular values. Section 5 also provides the bidiagonal factorization of the inverse of a tridiagonal Toeplitz M-matrix.

2. Auxiliary results

Let us denote by $Q_{k,n}$ the set of strictly increasing sequences of k integers chosen from $\{1,\ldots,n\}$. Let $\alpha = (\alpha_1,\ldots,\alpha_k)$, $\beta = (\beta_1,\ldots,\beta_k)$ be two sequences of $Q_{k,n}$. Then $A[\alpha|\beta]$ denotes the $k\times k$ submatrix of A formed using the rows numbered by $\alpha_1,\ldots,\alpha_k$ and the columns numbered by $\beta_1,\ldots,\beta_k$. Whenever $\alpha=\beta$, the submatrix $A[\alpha|\alpha]$ is called a principal submatrix and it is denoted by $A[\alpha]$, and $\det A[1,\ldots,k]$ is called a leading principal minor of A. For each $\alpha\in Q_{k,n}$, the dispersion number $d(\alpha)$ is defined by

$$d(\alpha) := \alpha_k - \alpha_1 - (k-1). \tag{1}$$

So, α consists of consecutive integers if and only if $d(\alpha)=0$. Let $D=(d_{ij})_{1\le i,j\le n}$ be a diagonal matrix, which can be denoted by $D=\mathrm{diag}(d_1,\ldots,d_n)$, where $d_i := d_{ii}$ for $i=1,\ldots,n$. Let us denote by $E_i(x)$, with $i=2,\ldots,n$, the $n\times n$ lower elementary bidiagonal matrix whose $(i,i-1)$ entry is x:

$$E_i(x) = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & x & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}. \tag{2}$$

In particular, $E_i(x)$ can be identified by its 2×2 principal submatrix using the rows and columns with indices $i-1$ and i. This submatrix will be denoted by

$$\overline{E}_i(x) := (E_i(x))\,[i-1,i], \quad i=2,\ldots,n. \tag{3}$$

The matrix $E_i^T(x) := (E_i(x))^T$ is called upper elementary bidiagonal matrix.

The following two results will allow us to characterize tridiagonal Toeplitz P-matrices. The next proposition characterizes a P-matrix in terms of the positivity of the real eigenvalues of its principal submatrices.

Proposition 2.1. *(cf. 2.5.6.5 in p. 120 of [25]) An $n\times n$ matrix A is a P-matrix if and only if every real eigenvalue of every principal submatrix of A is positive.*

The following theorem provides a sufficient condition for the total positivity of an $n\times n$ nonnegative tridiagonal matrix using only the positivity of $n-1$ minors.

Theorem 2.2. *(Theorem 7 of [26]) Let A be an $n\times n$ ($n\ge 3$) tridiagonal nonnegative matrix. If* $\det A[1,\ldots,k]>0$ *for $k\le n-2$ and* $\det A>0$*, then A is TP.*

Neville elimination (NE) has been very useful to characterize TP matrices and for parallel computations (cf. [26,27]). Neville elimination is an alternative procedure to Gaussian elimination that produces zeros in a column of a matrix by adding to each row an appropiate multiple of the previous one. Given a nonsingular matrix $A=(a_{ij})_{1\le i,j\le n}$, the NE procedure consists of $n-1$ steps and leads to the following sequence of matrices:

$$A =: A^{(1)} \to \widetilde{A}^{(1)} \to A^{(2)} \to \widetilde{A}^{(2)} \to \cdots \to A^{(n)} = \widetilde{A}^{(n)} = U, \tag{4}$$

where U is an upper triangular matrix.

The matrix $\widetilde{A}^{(k)} = (\widetilde{a}_{ij}^{(k)})_{1\le i,j\le n}$ is obtained from the matrix $A^{(k)} = (a_{ij}^{(k)})_{1\le i,j\le n}$ by a row permutation that moves to the bottom the rows with a zero entry in column k below the main diagonal. For nonsingular TP matrices, it is always possible to perform NE without row exchanges (see [28]). If a row permutation is not necessary at the kth step, we have that $\widetilde{A}^{(k)} = A^{(k)}$. The entries of $A^{(k+1)} = (a_{ij}^{(k+1)})_{1\le i,j\le n}$ can be obtained from $\widetilde{A}^{(k)} = (\widetilde{a}_{ij}^{(k)})_{1\le i,j\le n}$ using the formula:

$$a_{ij}^{(k+1)} = \begin{cases} \widetilde{a}_{ij}^{(k)} - \dfrac{\widetilde{a}_{ik}^{(k)}}{\widetilde{a}_{i-1,k}^{(k)}}\widetilde{a}_{i-1,j}^{(k)}, & \text{if } k \le j < i \le n \text{ and } \widetilde{a}_{i-1,k}^{(k)} \ne 0, \\ \widetilde{a}_{ij}^{(k)}, & \text{otherwise}, \end{cases} \tag{5}$$

for $k = 1, \ldots, n-1$. The (i,j) *pivot* of the NE of A is given by

$$p_{ij} = \widetilde{a}_{ij}^{(j)}, \quad 1 \le j \le i \le n.$$

If $i = j$ we say that p_{ii} is a *diagonal pivot*. The (i,j) *multiplier* of the NE of A, with $1 \le j \le i \le n$, is defined as

$$m_{ij} = \begin{cases} \dfrac{\widetilde{a}_{ij}^{(j)}}{\widetilde{a}_{i-1,j}^{(j)}} = \dfrac{p_{ij}}{p_{i-1,j}}, & \text{if } \widetilde{a}_{i-1,j}^{(j)} \ne 0, \\ 0, & \text{if } \widetilde{a}_{i-1,j}^{(j)} = 0. \end{cases}$$

The multipliers satisfy that

$$m_{ij} = 0 \Rightarrow m_{hj} = 0 \quad \forall h > i.$$

Nonsingular TP matrices can be expressed as a product of nonnegative bidiagonal matrices. The following theorem (see Theorem 4.2 and p. 120 of [29]) introduces this representation, which is called the *bidiagonal decomposition*.

Theorem 2.3. *(cf. Theorem 4.2 of [29]) Let $A = (a_{ij})_{1\le i,j\le n}$ be a nonsingular TP matrix. Then A admits the following representation:*

$$A = F_{n-1}F_{n-2}\cdots F_1 D G_1 \cdots G_{n-2}G_{n-1}, \tag{6}$$

where D is the diagonal matrix $\operatorname{diag}(p_{11}, \ldots, p_{nn})$ with positive diagonal entries and F_i, G_i are the nonnegative bidiagonal matrices given by

$$F_i = \begin{pmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ & \ddots & \ddots & & & & \\ & & 0 & 1 & & & \\ & & & m_{i+1,1} & 1 & & \\ & & & & \ddots & \ddots & \\ & & & & & m_{n,n-i} & 1 \end{pmatrix}, \tag{7}$$

$$G_i = \begin{pmatrix} 1 & 0 & & & & & \\ & 1 & \ddots & & & & \\ & & \ddots & 0 & & & \\ & & & 1 & \widetilde{m}_{i+1,1} & & \\ & & & & 1 & \ddots & \\ & & & & & \ddots & \widetilde{m}_{n,n-i} \\ & & & & & & 1 \end{pmatrix}, \tag{8}$$

for all $i \in \{1, \ldots, n-1\}$. If, in addition, the entries m_{ij} and $\widetilde{m}_{ij}$ satisfy

$$\begin{aligned} m_{ij} = 0 &\Rightarrow m_{hj} = 0 \quad \forall h > i, \\ \widetilde{m}_{ij} = 0 &\Rightarrow \widetilde{m}_{hj} = 0 \quad \forall h > i, \end{aligned} \tag{9}$$

then the decomposition is unique.

In the bidiagonal decomposition given by (6), (7) and (8), the entries m_{ij} and p_{ii} are the multipliers and diagonal pivots, respectively, corresponding to the NE of A (see Theorem 4.2 of [29] and the comment below it) and the entries $\widetilde{m}_{ij}$ are the multipliers of the NE of A^T (see p. 116 of [29]). In general, more classes of matrices can be represented as a product of bidiagonal matrices. The following remark shows which hypotheses of Theorem 2.3 are sufficient for the uniqueness of a representation following (6).

Remark 2.4. *If we consider the factorization given by* (6)-(9) *without any further requirement than the nonsingularity of D, by Proposition 2.2 of [30] the uniqueness of* (6) *holds.*

In [15] the following matrix notation $\mathcal{BD}(A)$ was introduced to represent the bidiagonal decomposition of a nonsingular TP matrix

$$(\mathcal{BD}(A))_{ij} = \begin{cases} m_{ij}, & \text{if } i > j, \\ \widetilde{m}_{ji}, & \text{if } i < j, \\ p_{ii}, & \text{if } i = j. \end{cases} \tag{10}$$

Throughout this paper, $\mathcal{BD}(A)$ will denote the bidiagonal decomposition of a matrix under the hypotheses of Remark 2.4.

3. Characterizations of tridiagonal Toeplitz *P*-matrices

An $n \times n$ *Toeplitz matrix* $A = (a_{ij})_{1\le i,j\le n}$ is a real matrix such that all its diagonals are constant. These matrices can be defined through a sequence of $2n-1$ real numbers $\{\alpha_k\}_{-n+1}^{n-1}$ with

$$a_{ij} := \alpha_{i-j}, \quad 1 \le i, j \le n. \tag{11}$$

If an $n \times n$ Toeplitz matrix is also tridiagonal, it can be uniquely represented with 3 parameters,

$$T_n(a,b,c) := \begin{pmatrix} a & c & & & \\ b & a & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & c \\ & & & b & a \end{pmatrix}. \tag{12}$$

Given a positive matrix $A = (a_{ij})_{1\le i,j\le n}$, the following condition is sufficient for its total positivity (see [31] or section 2.6 of [12]):

$$a_{ij}a_{i+1,j+1} \ge 4\cos^2\left(\frac{\pi}{n+1}\right) a_{i,j+1}a_{i+1,j},$$

with $i, j = 1, \dots, n-1$. If all these inequalities are strict, then A is STP. In particular, given an $n \times n$ Toeplitz matrix (11) with $\alpha_i > 0$ for $i = -n+1, \dots, 0, \dots, n-1$, the sufficient condition for a positive matrix A to be TP presents the following form:

$$\alpha_i^2 \ge 4\cos^2\left(\frac{\pi}{n+1}\right) \alpha_{i-1}\alpha_{i+1},$$

with $i = -n+2, \dots, 0, \dots, n-2$. This condition requires the positivity of all the entries of the matrix. Nevertheless, we are going to prove that a similar condition (jointly with the nonnegativity of the parameters) is sufficient and also necessary for a tridiagonal Toeplitz matrix to be TP.

Proposition 3.1. *Let $A = T_n(a,b,c)$ be the tridiagonal Toeplitz matrix given by* (12)*. Then A is TP if and only if*

$$a, b, c \ge 0, \quad a \ge 2\sqrt{bc}\cos\left(\frac{\pi}{n+1}\right). \tag{13}$$

Proof. It is known (see page 59 of [7]) that the eigenvalues of the $n \times n$ tridiagonal

Toeplitz matrix $T_n(a,b,c)$ are given by

$$\lambda_k = a + 2\sqrt{bc}\cos\left(\frac{k\pi}{n+1}\right), \quad k = 1,\dots,n. \tag{14}$$

Let us suppose that A is a TP matrix. Then $a,b,c \geq 0$ and its eigenvalues are real and nonnegative (see Corollary 5.5 of [12]). Moreover, since we know that the eigenvalues satisfy (14), it is sufficient to guarantee that the smallest eigenvalue, λ_n, is nonnegative:

$$\lambda_n = a + 2\sqrt{bc}\cos\left(\frac{n\pi}{n+1}\right) = a - 2\sqrt{bc}\cos\left(\frac{\pi}{n+1}\right) \geq 0,$$

or equivalently,

$$a \geq 2\sqrt{bc}\cos\left(\frac{\pi}{n+1}\right),$$

which is precisely (14) for $k = n$.

Let us now suppose that conditions (13) hold. We start with the case where the second inequality of (13) is strict. By Theorem 2.2, in order to prove that A is a TP matrix it is sufficient to check that its leading principal minors of order h are positive for $h = 1,\dots,n-2$ and that its determinant is also positive. Due to the structure of A we have that $A[1,\dots,h] = T_h(a,b,c)$ for $h = 1,\dots,n$. So, let us check the positivity of the minors by studying the positivity of the eigenvalues of the matrices $T_h(a,b,c)$ for $h = 1,\dots,n-2$ and for $h = n$. We can include the case $h = n-1$. Then the set of eigenvalues to check is given by $\lambda_{k,h} := a + 2\sqrt{bc}\cos\left(\frac{k\pi}{h+1}\right)$ with $1 \leq h \leq n$ and $k = 1,\dots,h$, where h represents the size of the $h \times h$ matrix whose eigenvalues are given by $\lambda_{k,h}$.

Since all the eigenvalues are real, it suffices to check that the smallest eigenvalue is positive in order to assure that $\lambda_{k,h} > 0$ for all $h = 1,\dots,n$ and for all $k = 1,\dots,h$:

$$\min_{k,h} \lambda_{k,h} = \lambda_{n,n} = a + 2\sqrt{bc}\cos\left(\frac{n\pi}{n+1}\right) = a - 2\sqrt{bc}\cos\left(\frac{\pi}{n+1}\right) > 0,$$

which is true by hypothesis. The value of the $h \times h$ leading principal minor of A is equal to the product of $\lambda_{1,h},\dots,\lambda_{h,h}$, and so it is positive. By Theorem 2.2 A is TP, and so the case where the strict inequality holds is proven.

Let us finally consider the case where the second inequality of (13) holds as an equality, $a = 2\sqrt{bc}\cos\left(\frac{\pi}{n+1}\right)$, which corresponds to the singular case. Let us define the set of matrices $T_n(a+\epsilon,b,c)$ with $\epsilon > 0$. These matrices satisfy that $a+\epsilon > 2\sqrt{bc}\cos\left(\frac{\pi}{n+1}\right)$, and so they are TP because of the previous case where the second inequality of (13) was strict. Moreover, this set of matrices satisfies that $\lim_{\epsilon\to 0} T_n(a+\epsilon,b,c) = T_n(a,b,c)$, and so $T_n(a,b,c)$ is TP because the set of TP matrices is closed (let us recall that this fact is a direct consequence of the continuity of the determinant as a function of the matrix entries). □

If we consider parameters b and c with nonpositive sign, we can deduce an analogous characterization for M-matrices of the form $T_n(a,b,c)$.

Corollary 3.2. *Let $A = T_n(a, b, c)$ be the tridiagonal Toeplitz matrix given by* (12)*. Then A is an M-matrix if and only if $a \geq 2\sqrt{bc}\cos\left(\frac{\pi}{n+1}\right)$ and $b, c \leq 0$.*

Proof. Since M-matrices are Z-matrices, the condition $b, c \leq 0$ is mandatory. Let us recall that a Z-matrix A is an M-matrix if and only if every real eigenvalue of A is nonnegative (see characterization (C_8) of Theorem (4.6) of [13, Ch. 6]). Since A is a tridiagonal Toeplitz matrix we know (see page 59 of [7]) that its eigenvalues are real, distinct and that they are given by (14). Then we only need to check that the smallest eigenvalue, λ_n, is nonnegative:

$$\lambda_n = a + 2\sqrt{bc}\cos\left(\frac{n\pi}{n+1}\right) = a - 2\sqrt{bc}\cos\left(\frac{\pi}{n+1}\right) \geq 0,$$

which is true if and only if $a \geq 2\sqrt{bc}\cos\left(\frac{\pi}{n+1}\right)$. □

We now consider a third case of tridiagonal Toeplitz matrices $T_n(a, b, c)$ where the parameters satisfy $a > 0$ and $bc \leq 0$. This particular case, where the off-diagonal entries have opposite sign, verifies that $T_n(a, b, c)$ is a P-matrix without any further requirement. Moreover, the following result proves that all tridiagonal matrices with positive diagonal and with an analogous sign pattern are P-matrices.

Proposition 3.3. *Let $A = (a_{ij})_{1\leq i,j\leq n}$ be a tridiagonal matrix. If $a_{ii} > 0$ for $i = 1, \ldots, n$ and $a_{i+1,i}a_{i,i+1} \leq 0$ for $i = 1, \ldots, n-1$, then A is a P-matrix.*

Proof. Let us first prove by induction that the leading principal minors of A, $\theta_k := \det A[1, \ldots, k]$ for $k = 1, \ldots, n$, are positive. It is straightforward to see that $\theta_1 = a_{11} > 0$ and that $\theta_2 = a_{11}a_{22} - a_{21}a_{12} > 0$. Let us suppose that $\theta_{k-1}, \theta_{k-2} > 0$ for some $k \in \{3, \ldots, n\}$ and let us prove that $\theta_k > 0$. Since A is a tridiagonal matrix, using the Laplace expansion of a determinant we can write θ_k as

$$\theta_k = a_{kk}\theta_{k-1} - a_{k,k-1}a_{k-1,k}\theta_{k-2}, \tag{15}$$

and so $\theta_k > 0$ by the induction hypothesis. Now let us prove that all principal minors using consecutive rows and columns are positive. These minors are of the form $\det A[\alpha]$ with $\alpha = (s, \ldots, r)$, $d(\alpha) = 0$ (see (1)) and $1 \leq s < r \leq n$. Given an index $1 \leq s \leq n$ we consider the principal submatrix $A_s := A[s, \ldots, n]$. The matrix A_s is a tridiagonal matrix that satisfies the hypotheses of this proposition. Hence, we can apply the previous case to A_s and deduce that its leading principal minors are positive. These minors can be written as $\det A_s[1, \ldots, p]$, with $1 \leq p \leq n-s+1$, and, since A_s is a submatrix of A, these minors satisfy that $\det A_s[1, \ldots, p] = \det A[s, \ldots, p+s-1] > 0$. Then we have as a direct consequence the positivity of all the principal minors using consecutive rows and columns. Finally, it only remains to study the principal minors $\det A[\alpha]$ such that $d(\alpha) > 0$. Given $\alpha \in Q_{k,n}$ with $d(\alpha) > 0$, let us consider the decomposition $\alpha = (\beta_1, \ldots, \beta_r)$, with $|\beta_i| \geq 1$ and $d(\beta_i) = 0$ for $i = 1, \ldots, r$, such that $d(\beta_j, \beta_{j+1}) > 0$ for all $j = 1, \ldots, r-1$. Then $A[\alpha]$ is a block diagonal matrix such that the determinant of its ith block $A[\beta_i]$ is a principal minor of A using consecutive rows and columns, and hence, it is positive. So we conclude that $\det A[\alpha] = \det A[\beta_1] \cdots \det A[\beta_r] > 0$. □

Observe that the previous result can be stated in the following way. A tridiagonal sign skew-symmetric matrix with positive diagonal entries is a P-matrix. We now characterize tridiagonal Toeplitz P-matrices.

Theorem 3.4. *Let $A = T_n(a, b, c)$ be the tridiagonal Toeplitz matrix given by* (12)*. Then A is a P-matrix if and only if one of the following two conditions holds:*

(i) $bc \leq 0$ and $a > 0$.
(ii) $bc \geq 0$ and $a > 2\sqrt{bc}\cos\left(\frac{\pi}{n+1}\right)$.

Proof. If (i) holds, then by Proposition 3.3 A is a P-matrix. Let us now suppose that condition (ii) holds. If $b, c \geq 0$, by Proposition 3.1, A is a nonsingular TP matrix, and hence, a P-matrix because, by Theorem 11.3 of [12], nonsingular TP matrices are P-matrices. If $b, c \leq 0$, by Corollary 3.2, A is a nonsingular M-matrix and so a P-matrix because, by characterization (A_1) of Theorem (2.3) of [13, Ch. 6], nonsingular M-matrices are P-matrices.

Assume now that A is a P-matrix. We have to see that if (i) does not hold, then (ii) holds. Since by definition $A[1, 1] = a > 0$, it is sufficient to consider parameters b, c such that $bc \geq 0$. By Proposition 2.1, the real eigenvalues of all the principal submatrices of A are positive. Given $\alpha \in Q_{h,n}$, $A[\alpha] = T_h(a, b, c)$ whenever $d(\alpha) = 0$. If $d(\alpha) > 0$, then we can consider the decomposition $\alpha = (\beta_1, \ldots, \beta_r)$, with $|\beta_i| \geq 1$ and with $d(\beta_i) = 0$ for $i = 1, \ldots, r$, such that $d(\beta_j, \beta_{j+1}) > 0$ for all $j = 1, \ldots, r-1$. Then $A[\alpha]$ is a block diagonal matrix such that its ith block $A[\beta_i]$ is the tridiagonal Toeplitz matrix $T_{|\beta_i|}(a, b, c)$. In either case, the eigenvalues of $A[\alpha]$, $\alpha \in Q_{h,n}$, are included in the set $\lambda_{r,h} := a + 2\sqrt{bc}\cos\left(\frac{r\pi}{h+1}\right)$ with $1 \leq h \leq n$ and with $r = 1, \ldots, h$, where h represents the size of the $h \times h$ matrix whose eigenvalues are given by $\lambda_{r,h}$ (see page 59 of [7]). Therefore, $\lambda_{r,h} > 0$ for all $h = 1, \ldots, n$ and $r = 1 \ldots, h$. In particular, $\min_{r,h} \lambda_{r,h} = \lambda_{n,n} > 0$, and hence, $a > 2\sqrt{bc}\cos\left(\frac{\pi}{n+1}\right)$ and the result holds. $\square$

Remark 3.5. *From Proposition 3.1 and Theorem 3.4, we deduce that a tridiagonal Toeplitz matrix $T_n(a, b, c)$ is a nonsingular TP matrix if and only if $a > 2\sqrt{bc}\cos\left(\frac{\pi}{n+1}\right)$ and $b, c \geq 0$. Analogously, from Corollary 3.2 and Theorem 3.4, we deduce that a tridiagonal Toeplitz matrix $T_n(a, b, c)$ is a nonsingular M-matrix if and only if $a > 2\sqrt{bc}\cos\left(\frac{\pi}{n+1}\right)$ and $b, c \leq 0$. Then, by Theorem 3.4 a sign symmetric tridiagonal Toeplitz P-matrix is either a nonsingular TP matrix or a nonsingular M-matrix. Besides, taking into account that a tridiagonal Toeplitz matrix is either sign symmetric or sign skew-symmetric, we can reformulate Theorem 3.4 in the following way. A tridiagonal Toeplitz matrix $A = T_n(a, b, c)$ is a P-matrix if and only if $a > 0$ and, if A is sign symmetric, then $a > 2\sqrt{bc}\cos\left(\frac{\pi}{n+1}\right)$.*

In Theorem 3.4 (ii), the condition $a > 2\sqrt{bc}\cos\left(\frac{\pi}{n+1}\right)$ (or analogously, $a^2 > 4bc\cos^2\left(\frac{\pi}{n+1}\right)$) has been used to characterize tridiagonal Toeplitz P-matrices. If this condition is satisfied independently of n, we obtain the new condition $a^2 > 4bc$. In fact, this inequality will play a key role in Section 5 since it is used in order to assure HRA for some computations with the matrices $T_n(a, b, c)$. In fact, the positive number $a^2 - 4bc$ will be an additional natural parameter to assure the HRA. The case (i) of Theorem 3.4 will be considered in a more general framework in the following section.

4. Computing with HRA the minors of sign skew-symmetric tridiagonal matrices with positive diagonal entries

Whenever a tridiagonal matrix A satisfies the hypotheses of Proposition 3.3 (sign skew-symmetric with positive diagonal entries), it is possible to compute its bidiagonal decomposition accurately. Moreover, the bidiagonal decomposition allows us to compute all its minors and its inverse with HRA. The following result provides the $\mathcal{BD}(A)$ for such A.

Proposition 4.1. *Let $A = (a_{ij})_{1\le i,j\le n}$ be a tridiagonal matrix such that $a_{ii} > 0$ for $i = 1,\dots,n$ and $a_{i+1,i}a_{i,i+1} \le 0$ for $i = 1,\dots,n-1$. Then*

$$\mathcal{BD}(A) = \begin{pmatrix} \delta_1 & \frac{a_{12}}{\delta_1} & & & \\ \frac{a_{21}}{\delta_1} & \delta_2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \frac{a_{n-1,n}}{\delta_{n-1}} \\ & & & \frac{a_{n,n-1}}{\delta_{n-1}} & \delta_n \end{pmatrix}, \tag{16}$$

where δ_i are the diagonal pivots associated to the NE of A. The diagonal pivots satisfy the following recurrence relation:

$$\delta_1 = a_{11}, \quad \delta_i = a_{ii} - \frac{a_{i,i-1}a_{i-1,i}}{\delta_{i-1}} \quad i = 2,\dots,n. \tag{17}$$

If we know the entries of A with HRA then we can compute $\mathcal{BD}(A)$ (16) to HRA, and hence, the leading principal minors of A to HRA.

Proof. Clearly, for tridiagonal P-matrices, no row exchanges are needed in Neville elimination and Gauss elimination, which coincide. Hence, by Proposition 3.3, $\delta_1,\dots,\delta_n$ are also the pivots of the Gauss elimination of A and it is well known that they satisfy that

$$\delta_k = \frac{\theta_k}{\theta_{k-1}}, \quad k = 1,\dots,n, \tag{18}$$

with $\theta_0 := 1$ and $\theta_k := A[1,\dots,k]$ for $k = 1,\dots,n$. From (15) and (18) we deduce (17). Since $a_{i+1,i}a_{i,i+1} \le 0$ for $i = 1,\dots,n-1$, the diagonal pivots can be computed by (17) without performing any subtraction. As a consequence, all pivots δ_k are computed to HRA. The leading principal minors can be obtained with HRA through the computation $\theta_k = \delta_1 \cdots \delta_k$, for $k = 1,\dots,n$. □

Proposition 4.1 allows us to prove that some computations can be performed with HRA, as the following result shows.

Theorem 4.2. *Let $A = (a_{ij})_{1\le i,j\le n}$ be a tridiagonal matrix such that $a_{ii} > 0$ for $i = 1,\dots,n$ and $a_{i+1,i}a_{i,i+1} \le 0$ for $i = 1,\dots,n-1$. Then all the minors and the inverse of A can be computed to HRA.*

Proof. By Proposition 4.1, we can compute the leading principal minors of A to HRA. Following the proof of Proposition 3.3, it can be deduced that all the principal minors of A can be obtained without subtractions, and so, with HRA. Given $\alpha = (i_1, \ldots, i_k), \beta = (j_1, \ldots, j_k) \in Q_{k,n}$, if $|i_r - j_r| \geq 2$ for any $r = 1, \ldots, k$ then $\det A[i_1, \ldots, i_k|j_1, \ldots, j_k] = 0$ and if $|i_s - j_s| = 1$ for an index $s = 1, \ldots, k$ then $A[\alpha|\beta] = A[i_1, \ldots, i_{s-1}|j_1, \ldots, j_{s-1}]\ a_{i_s,j_s} A[i_{s+1}, \ldots, i_k|j_{s+1}, \ldots, j_k]$. Hence, any nonzero minor of a tridiagonal matrix can be written as a product of off-diagonal entries and principal minors using consecutive rows and columns. Then all the entries of A^{-1} can be computed to HRA as a consequence. For example, by using formula (1.33) of [10], corresponding to the well-known expression of the entries of the inverse in terms of determinants. □

An alternative HRA method to obtain A^{-1} is presented in the following remark.

Remark 4.3. *Let $A = (a_{ij})_{1\leq i,j\leq n}$ be a nonsingular tridiagonal matrix. Then, by (47) of [32], we can give the following explicit expression of the entries of $A^{-1} := (b_{ij})_{1\leq i,j\leq n}$ in terms of principal minors of A using consecutive rows and columns. In fact,*

$$b_{ij} = \begin{cases} \dfrac{\theta_{i-1}\widehat{\theta}_{n-j}}{\theta_n} \prod_{l=i}^{j-1} -a_{l,l+1}, & \text{for } i < j, \\ \dfrac{\theta_{i-1}\widehat{\theta}_{n-i}}{\theta_n}, & \text{for } i = j, \\ \dfrac{\theta_{j-1}\widehat{\theta}_{n-i}}{\theta_n} \prod_{l=j}^{i-1} -a_{l+1,l}, & \text{for } i > j, \end{cases} \tag{19}$$

where $\widehat{\theta}_k := \det A[n-k+1, \ldots, n]$ for $k = 1, \ldots, n$. If $a_{ii} > 0$ for $i = 1, \ldots, n$ and $a_{i+1,i}a_{i,i+1} \leq 0$ for $i = 1, \ldots, n-1$, then A^{-1} can also be computed to HRA by (19).

5. Computations with sign symmetric tridiagonal Toeplitz *P*-matrices with HRA

In this section, we guarantee the HRA for the bidiagonal decomposition, and so for many other algebraic computations, in the case of sign symmetric tridiagonal Toeplitz P-matrices with the additional parameter $a^2 - 4bc$ commented at the end of Section 3. By Theorem 3.4 and Remark 3.5, the P-matrices corresponding to this case are either nonsingular M-matrices or nonsingular TP matrices.

From now on, we assume that the parameters a, b, c are always positive:

$$a, b, c > 0.$$

Let us recall that the inverse of a nonsingular tridiagonal M-matrix is TP (see [33]). We are going to obtain the bidiagonal decomposition of an M-matrix $A = T_n(a, -b, -c)$. From the $\mathcal{BD}(A)$ obtained in Theorem 5.1, in Theorem 5.5 we shall deduce $\mathcal{BD}(A^{-1})$. Besides, by Remark 5.4, if we know $\mathcal{BD}(A)$ to HRA, then we can also perform many algebraic computations with A to HRA.

It is well known (see p. 99 of [12]) that the principal minors of a tridiagonal matrix $A = (a_{ij})_{1\leq i,j\leq n}$ satisfy:

$$\begin{aligned} \det A = &\det A[1, \ldots, i] \det A[i+1, \ldots, n] \\ &- a_{i,i+1}a_{i+1,i} \det A[1, \ldots, i-1] \det A[i+2, \ldots, n]. \end{aligned} \tag{20}$$

From (20) we deduce that the leading principal minors of a tridiagonal Toeplitz matrix A, $\theta_j := \det A[1,\ldots,j]$ with $j = 1,\ldots,n$, satisfy the following relation:

$$\theta_n = \theta_j\theta_{n-j} - bc\theta_{j-1}\theta_{n-j-1}, \text{ with } \theta_{-1} = 0,\ \theta_0 = 1, \quad j = 1,\ldots,n. \tag{21}$$

Theorem 5.1. *Let $A = T_n(a, -b, -c)$ be a nonsingular M-matrix given by* (12)*. Then*

$$\mathcal{BD}(A) = \begin{pmatrix} \delta_1 & -\frac{c}{\delta_1} & & & \\ -\frac{b}{\delta_1} & \delta_2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -\frac{c}{\delta_{n-1}} \\ & & & -\frac{b}{\delta_{n-1}} & \delta_n \end{pmatrix}, \tag{22}$$

where δ_i are the diagonal pivots associated to the NE of A and are given by:

$$\delta_1 = a, \quad \delta_i = a - \frac{bc}{\delta_{i-1}} \quad \text{with } i = 2,\ldots,n. \tag{23}$$

Moreover, if we know a, b, c with HRA and $a^2 - 4bc$ is a positive number known with HRA, then we can compute $\mathcal{BD}(A)$ (22) to HRA.

Proof. Since nonsingular M-matrices are P-matrices (see characterization (A_1) of Theorem 2.3 of [13, Ch. 6]), the principal minors of A are positive, and so, $\theta_i > 0$ for $i = 1,\ldots,n$. Since A is a tridiagonal Toeplitz matrix, its leading principal minors satisfy

$$\theta_i = a\theta_{i-1} - bc\theta_{i-2}, \text{ with } \theta_{-1} = 0,\ \theta_0 = 1, \quad i = 1,\ldots,n \tag{24}$$

by (15). Moreover, (23) is a consequence of (24) and (18). There is an explicit expression for the leading principal minors of A (see p. 15 of [34]):

$$\theta_i = (\sqrt{bc})^i U_i\left(\frac{a}{2\sqrt{bc}}\right), \tag{25}$$

where $U_i(x)$ is the ith Chebyshev polynomial of the second kind. We can evaluate $U_i(x)$ through (see Section 3 of [34]):

$$\begin{aligned} &U_i(x) = \frac{r_+^{i+1}(x) - r_-^{i+1}(x)}{r_+(x) - r_-(x)}, \\ &\text{with } r_+(x) := x + \sqrt{x^2-1} \text{ and } r_-(x) := x - \sqrt{x^2-1}. \end{aligned} \tag{26}$$

Let us denote $s_+ := r_+\big(\frac{a}{2\sqrt{bc}}\big)$ and $s_- := r_-\big(\frac{a}{2\sqrt{bc}}\big)$. By (25) and (26), we can write the pivots δ_i as:

$$\delta_i = \frac{\theta_i}{\theta_{i-1}} = \sqrt{bc}\frac{s_+^{i+1} - s_-^{i+1}}{s_+^i - s_-^i} = \sqrt{bc}\ s_+ \frac{1 + \frac{s_-}{s_+} + \ldots + \frac{s_-^i}{s_+^i}}{1 + \frac{s_-}{s_+} + \ldots + \frac{s_-^{i-1}}{s_+^{i-1}}}.$$

If we obtain s_+ and $\frac{s_-}{s_+}$ with HRA, then we can compute δ_i for $i = 1, \dots, n$ to HRA, and as a direct consequence, $\mathcal{BD}(A)$ to HRA. We can compute s_+ by

$$s_+ = \frac{a}{2\sqrt{bc}} + \sqrt{\frac{a^2}{4bc} - 1} = \frac{a + \sqrt{a^2 - 4bc}}{2\sqrt{bc}},$$

and the quotient $\frac{s_-}{s_+}$ by

$$\frac{s_-}{s_+} = \frac{s_- s_+}{s_+^2} = \frac{4bc}{2a^2 - 4bc + 2a\sqrt{a^2 - 4bc}} = \frac{4bc}{a^2 + (a^2 - 4bc) + 2a\sqrt{a^2 - 4bc}}.$$

Since $a^2 - 4bc$ is known with HRA by hypothesis, $\mathcal{BD}(A)$ can be obtained with HRA.

□

Remark 5.2. *The computational cost of obtaining $\mathcal{BD}(A)$ following Theorem 5.1 is of $6n$ elementary operations, as can be checked from its proof.*

Corollary 5.3. *Let $A := T_n(a, -b, -c)$ be a nonsingular M-matrix. If we know a, b, c and $a - 2\max\{b, c\}$ with HRA and $a - 2\max\{b, c\} \geq 0$, then we can compute $\mathcal{BD}(A)$ (22) to HRA.*

Proof. Without loss of generality, let us suppose that $b \geq c$. Then we can write the quantity $a^2 - 4bc$ as

$$a^2 - 4bc = (a - 2b)(a + 2c) + 2a(b - c). \tag{27}$$

Taking into account that, by hypothesis, $a - 2b$ is known to HRA and that $b - c$ is a subtraction of initial data, $a^2 - 4bc$ can also be computed to HRA. As a consequence, s_+ and $\frac{s_-}{s_+}$ can be obtained with HRA. Finally, following the proof of Theorem 5.1 we can compute $\mathcal{BD}(A)$ to HRA. □

The bidiagonal decomposition of a nonsingular M-matrix is unique by Remark 2.4. If A is a tridiagonal M-matrix, then $\mathcal{BD}(A)$ allows us to perform some algebraic computations with A to HRA.

Remark 5.4. *Let A be a tridiagonal Toeplitz M-matrix such that we know $\mathcal{BD}(A)$ to HRA. In this case, we also know the bidiagonal decomposition to HRA of $|A| = J_n A J_n$, where $J_n = diag(1, -1, \dots, (-1)^{n-1})$. Since $|A|$ is TP by Proposition 3.1, we can apply the HRA algorithms for TP matrices to $\mathcal{BD}(|A|) = |\mathcal{BD}(A)|$. For instance, in Section 6 we comment how to compute the singular values and eigenvalues of A to HRA.*

The following result provides the bidiagonal decomposition of the inverse of a nonsingular tridiagonal Toeplitz M-matrix.

Theorem 5.5. *Let $A = T_n(a, -b, -c)$ be a nonsingular M-matrix. Then A^{-1} is a TP matrix and*

$$\mathcal{BD}(A^{-1}) = \begin{pmatrix} 1/\delta_n & c/\delta_{n-1} & c/\delta_{n-2} & \cdots & c/\delta_1 \\ b/\delta_{n-1} & 1/\delta_{n-1} & 0 & \cdots & 0 \\ b/\delta_{n-2} & 0 & 1/\delta_{n-2} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ b/\delta_1 & 0 & \dots & 0 & 1/\delta_1 \end{pmatrix}, \tag{28}$$

where δ_i are the diagonal pivots associated to the NE of A for $i = 1, \dots, n$.

Proof. A^{-1} is TP because it is the inverse of a tridiagonal M-matrix (see Theorem 2.2 of [33]). Let us define $D := \operatorname{diag}(\delta_1, \dots, \delta_n)$. By Theorem 5.1, we can write A as

$$A = E_2\left(\tfrac{-b}{\delta_1}\right) \cdots E_n\left(\tfrac{-b}{\delta_{n-1}}\right) D E_n^T\left(\tfrac{-c}{\delta_{n-1}}\right) \cdots E_2^T\left(\tfrac{-c}{\delta_1}\right),$$

and so

$$A^{-1} = E_2^T\left(\tfrac{c}{\delta_1}\right) \cdots E_n^T\left(\tfrac{c}{\delta_{n-1}}\right) D^{-1} E_n\left(\tfrac{b}{\delta_{n-1}}\right) \cdots E_2\left(\tfrac{b}{\delta_1}\right). \tag{29}$$

The factorization (29) is different from the bidiagonal decomposition (6). In order to obtain $\mathcal{BD}(A^{-1})$ from (29), we first need to rewrite $E_n^T\left(\tfrac{c}{\delta_{n-1}}\right) D^{-1} E_n\left(\tfrac{b}{\delta_{n-1}}\right)$ as the product of a lower elementary bidiagonal matrix $E_n(\alpha)$, a diagonal matrix and an upper elementary bidiagonal matrix $E_n^T(\beta)$, with $\alpha, \beta \in \mathbb{R}$.

Let us start by computing the following product:

$$E_n^T\left(\frac{c}{\delta_{n-1}}\right) D^{-1} E_n\left(\frac{b}{\delta_{n-1}}\right) = \begin{pmatrix} \frac{1}{\delta_1} & & & & \\ & \ddots & & & \\ & & \frac{1}{\delta_{n-2}} & & \\ & & & \frac{1}{\delta_{n-1}} + \frac{bc}{\delta_{n-1}^2 \delta_n} & \frac{c}{\delta_{n-1}\delta_n} \\ & & & \frac{b}{\delta_{n-1}\delta_n} & \frac{1}{\delta_n} \end{pmatrix}. \tag{30}$$

By (23) for $i = 1, n$, the $(n-1, n-1)$ entry of (30) can be written as:

$$\frac{1}{\delta_{n-1}} + \frac{bc}{\delta_{n-1}^2 \delta_n} = \frac{1}{\delta_{n-1}\delta_n}\left(\delta_n + \frac{bc}{\delta_{n-1}}\right) = \frac{1}{\delta_{n-1}\delta_n}\left(a - \frac{bc}{\delta_{n-1}} + \frac{bc}{\delta_{n-1}}\right) = \frac{\delta_1}{\delta_{n-1}\delta_n}.$$

The effect of the matrices $E_n(\alpha), E_n^T(\beta)$ over D is restricted to the submatrix $D^{-1}[n-1, n]$. Hence, using the notation (3), we can decompose the principal submatrix of (30) using the $n-1, n$ rows as

$$\begin{pmatrix} \frac{\delta_1}{\delta_{n-1}\delta_n} & \frac{c}{\delta_{n-1}\delta_n} \\ \frac{b}{\delta_{n-1}\delta_n} & \frac{1}{\delta_n} \end{pmatrix} = \overline{E}_n\left(\frac{b}{\delta_1}\right) \begin{pmatrix} \frac{\delta_1}{\delta_{n-1}\delta_n} & \\ & \frac{1}{\delta_n} - \frac{cb}{\delta_{n-1}\delta_n\delta_1} \end{pmatrix} \overline{E}_n^T\left(\frac{c}{\delta_1}\right). \tag{31}$$

Then by (31) we have that the required elementary matrices are $E_n\left(\tfrac{b}{\delta_1}\right)$ and $E_n^T\left(\tfrac{c}{\delta_1}\right)$. Moreover, using again (23) we can write the last entry of the diagonal matrix in (31) as:

$$\frac{1}{\delta_n} - \frac{cb}{\delta_{n-1}\delta_n\delta_1} = \frac{1}{\delta_1\delta_n}\left(\delta_1 - \frac{bc}{\delta_{n-1}}\right) = \frac{\delta_n}{\delta_1\delta_n} = \frac{1}{\delta_1}.$$

If we denote by $D^{(2)} := \operatorname{diag}(\delta_1^{-1}, \ldots, \delta_{n-2}^{-1}, \frac{\delta_1}{\delta_n\delta_{n-1}}, \delta_1^{-1})$ then, by (31), we have that

$$E_n\left(\tfrac{b}{\delta_1}\right) D^{(2)} E_n^T\left(\tfrac{c}{\delta_1}\right) = E_n^T\left(\tfrac{c}{\delta_{n-1}}\right) D^{-1} E_n\left(\tfrac{b}{\delta_{n-1}}\right),$$

and so we have achieved our first goal. Let us now express A^{-1} as the following matrix product

$$A^{-1} = E_2^T\left(\tfrac{c}{\delta_1}\right)\cdots E_{n-1}^T\left(\tfrac{c}{\delta_{n-2}}\right) E_n\left(\tfrac{b}{\delta_1}\right) D^{(2)} E_n^T\left(\tfrac{c}{\delta_1}\right) E_{n-1}\left(\tfrac{b}{\delta_{n-2}}\right)\cdots E_2\left(\tfrac{b}{\delta_1}\right). \quad (32)$$

Since the elementary bidiagonal matrices satisfy that $E_j(\alpha_j)E_n^T(\alpha_n) = E_n^T(\alpha_n)E_j(\alpha_j)$ whenever $j < n$, we can reorder the matrices in (32) and deduce that

$$A^{-1} = E_n\left(\tfrac{b}{\delta_1}\right) E_2^T\left(\tfrac{c}{\delta_1}\right)\cdots E_{n-1}^T\left(\tfrac{c}{\delta_{n-2}}\right) D^{(2)} E_{n-1}\left(\tfrac{b}{\delta_{n-2}}\right)\cdots E_2\left(\tfrac{b}{\delta_1}\right) E_n^T\left(\tfrac{c}{\delta_1}\right). \quad (33)$$

After rearranging the matrices we arrive at an analogous problem to (29). Hence, our aim is now expressing $E_{n-1}^T\left(\frac{c}{\delta_{n-2}}\right) D^{(2)} E_{n-1}\left(\frac{b}{\delta_{n-2}}\right)$ as the product of a matrix $E_{n-1}(\alpha)$, a diagonal matrix that will be denoted by $D^{(3)}$ and a matrix $E_{n-1}^T(\beta)$. Then we could rearrange again the elementary bidiagonal matrices as we did in (33). In general, after performing this procedure $k-1$ times we would obtain the following factorization:

$$\begin{aligned} A^{-1} = E_n\left(\tfrac{b}{\delta_1}\right)\cdots E_{n-k+2}\left(\tfrac{b}{\delta_{k-1}}\right) E_2^T\left(\tfrac{c}{\delta_1}\right)\cdots E_{n-k+1}^T\left(\tfrac{c}{\delta_{n-k}}\right) D^{(k)} \\ \cdot E_{n-k+1}\left(\tfrac{b}{\delta_{n-k}}\right)\cdots E_2\left(\tfrac{b}{\delta_1}\right) E_{n-k+2}^T\left(\tfrac{c}{\delta_{k-1}}\right)\cdots E_n^T\left(\tfrac{c}{\delta_1}\right), \end{aligned} \quad (34)$$

where $D^{(k)} = \operatorname{diag}(\delta_1^{-1}, \ldots, \delta_{n-k}^{-1}, \frac{\theta_{k-1}\theta_{n-k}}{\theta_n}, \delta_{k-1}^{-1}, \ldots, \delta_1^{-1})$.

When $k = n$, (34) coincides with the decomposition (28). Therefore, let us prove that (34) holds by induction on $k \in \{2, \ldots, n\}$. We have already checked the first step, $k = 2$. So, let us assume that (34) holds for $k \in \{2, \ldots, n-1\}$ and let us prove that it also holds for $k+1$. Let us first see that

$$E_{n-k+1}^T\left(\tfrac{c}{\delta_{n-k}}\right) D^{(k)} E_{n-k+1}\left(\tfrac{b}{\delta_{n-k}}\right) = E_{n-k+1}\left(\tfrac{b}{\delta_k}\right) D^{(k+1)} E_{n-k+1}^T\left(\tfrac{c}{\delta_{n-k}}\right).$$

In general, the effect of the matrices $E_{n-k+1}(\alpha), E_{n-k+1}^T(\beta)$ is restricted to the submatrix $D^{(k)}[n-k, n-k+1]$. Using the notation (3) we deduce that

$$\overline{E}_{n-k+1}^T\left(\frac{c}{\delta_{n-k}}\right)\begin{pmatrix} \frac{1}{\delta_{n-k}} & \\ & \frac{\theta_{k-1}\theta_{n-k}}{\theta_n} \end{pmatrix}\overline{E}_{n-k+1}\left(\frac{b}{\delta_{n-k}}\right)$$
$$= \begin{pmatrix} \frac{\theta_{n-k-1}}{\theta_{n-k}} + \frac{bc\theta_{k-1}\theta_{n-k-1}^2}{\theta_n\theta_{n-k}} & \frac{\theta_{k-1}\theta_{n-k-1}}{\theta_n}c \\ \frac{\theta_{k-1}\theta_{n-k-1}}{\theta_n}b & \frac{\theta_{k-1}\theta_{n-k}}{\theta_n} \end{pmatrix}. \qquad (35)$$

By (21) with $j = k$, the first diagonal entry of (35) can be written as

$$\frac{\theta_{n-k-1}\theta_n + bc\theta_{k-1}\theta_{n-k-1}^2}{\theta_n\theta_{n-k}} = \frac{\theta_{n-k-1}^2}{\theta_n\theta_{n-k}}\left(\frac{\theta_n}{\theta_{n-k+1}} + bc\theta_{k-1}\right)$$
$$= \frac{\theta_{n-k-1}^2}{\theta_n\theta_{n-k}}\left(\theta_k\frac{\theta_{n-k}}{\theta_{n-k-1}} - bc\theta_{k-1} + bc\theta_{k-1}\right) = \frac{\theta_k\theta_{n-k-1}}{\theta_n}.$$

Applying Gauss elimination to the submatrix (35) we obtain, by (18), the following multiplier

$$\frac{\theta_{k-1}\theta_{n-k-1}\theta_n}{\theta_{n-k-1}\theta_k\theta_n}b = \frac{\theta_{k-1}}{\theta_k}b = \frac{b}{\delta_k}.$$

Analogously, applying Gauss elimination to the transpose of that submatrix we obtain the multiplier $\frac{c}{\delta_k}$. Hence, we can decompose (35) as

$$\overline{E}_{n-k+1}\left(\frac{b}{\delta_k}\right)\begin{pmatrix} \frac{\theta_k\theta_{n-k-1}}{\theta_n} & \\ & \frac{\theta_{k-1}\theta_{n-k}}{\theta_n} - bc\frac{\theta_{k-1}\theta_{n-k-1}}{\theta_n\delta_k} \end{pmatrix}\overline{E}_{n-k+1}^T\left(\frac{c}{\delta_k}\right). \qquad (36)$$

Using (21), we express the last entry of the diagonal matrix in (36) in terms of the diagonal pivots

$$\frac{\theta_{k-1}}{\theta_n\delta_k}\left(\theta_{n-k}\frac{\theta_k}{\theta_{k-1}} - bc\theta_{n-k-1}\right) = \frac{\theta_n}{\theta_n\delta_k} = \frac{1}{\delta_k}.$$

Then we have deduced that $D^{(k+1)} = diag(\delta_1^{-1}, \ldots, \delta_{n-k-1}^{-1}, \frac{\theta_k\theta_{n-k-1}}{\theta_n}, \delta_k^{-1}, \ldots, \delta_1^{-1})$, and so we can factorize A^{-1} as

$$A^{-1} = E_n\left(\tfrac{b}{\delta_1}\right)\cdots E_{n-k+2}\left(\tfrac{b}{\delta_{k-1}}\right)E_2^T\left(\tfrac{c}{\delta_1}\right)\cdots E_{n-k}^T\left(\tfrac{c}{\delta_{n-k-1}}\right)E_{n-k+1}\left(\tfrac{b}{\delta_k}\right)$$
$$\cdot D^{(k+1)}E_{n-k+1}^T\left(\tfrac{c}{\delta_k}\right)E_{n-k}\left(\tfrac{b}{\delta_{n-k-1}}\right)\cdots E_2\left(\tfrac{b}{\delta_1}\right)E_{n-k+2}^T\left(\tfrac{c}{\delta_{k-1}}\right)\cdots E_n^T\left(\tfrac{c}{\delta_1}\right). \qquad (37)$$

Finally, reordering the elementary bidiagonal matrices of (37) we deduce (34) for $k+1$. Therefore, (34) holds for $k = 2, \ldots, n$, and, taking $k = n$ in (34), we deduce that

$$A^{-1} = E_n\left(\tfrac{b}{\delta_1}\right)\cdots E_2\left(\tfrac{b}{\delta_{n-1}}\right)D^{(n)}E_2^T\left(\tfrac{c}{\delta_{n-1}}\right)\cdots E_n^T\left(\tfrac{c}{\delta_1}\right)$$

with $D^{(n)} = \mathrm{diag}(\delta_n^{-1}, \ldots, \delta_1^{-1})$, which is precisely $\mathcal{BD}(A^{-1})$. □

6. Numerical experiments

In [15,23], assuming that the parameterization $\mathcal{BD}(A)$ of an square TP matrix A is known with HRA, Plamen Koev presented algorithms to solve some algebraic problems for A to HRA. Let us focus on the computation of the eigenvalues and the singular values. Koev implemented these algorithms in order to be used with Matlab and Octave in the software library *TNTool* available in [24]. The corresponding functions are `TNEigenValues` and `TNSingularValues`, respectively. The functions require as input argument the data determining the bidiagonal decomposition (6) of A, $\mathcal{BD}(A)$ given by (10), to HRA.

Let

$$A = T_n(a, -b, -c), \quad a, b, c > 0,$$

be a tridiagonal Toeplitz matrix satisfying $a^2 \geq 4bc\cos^2\left(\frac{\pi}{n+1}\right)$. Let us denote by J_n the $n \times n$ matrix $\mathrm{diag}(1, -1, \ldots, (-1)^{n-1})$. Then, by Proposition 3.1, the matrix $J_nAJ_n = |A|$ is TP. In addition, taking into account that $J_n^{-1} = J_n$, the matrix A is similar to the TP matrix $|A| = J_nAJ_n$. Thus, A and $|A|$ have the same eigenvalues and, since J_n is unitary, also the same singular values. In Algorithm 1, the pseudocode for the computation of $\mathcal{BD}(A)$ to HRA can be seen. Taking into account that $\mathcal{BD}(|A|) = |\mathcal{BD}(A)|$, the eigenvalues and singular values of A can be computed to HRA by using Koev's algorithms and Algorithm 1 if $a^2 - 4bc$ is known to HRA.

Algorithm 1 Computation of the bidiagonal decomposition of A to HRA

Require: n, $a > 0$, $b, c < 0$ such that $m = a^2 - 4bc > 0$, and m known to HRA
Ensure: The $n \times n$ $\mathcal{BD}(A)$ of $A = T_n(a, -b, -c)$ to HRA
  $\frac{s_-}{s_+} = \frac{4bc}{a^2+m+2a\sqrt{m}}$
  $s_+ = \frac{a+\sqrt{m}}{2\sqrt{bc}}$
  $num = 1 + \frac{s_-}{s_+}$
  $den = 1$
  for $i = 0 : n$ **do**
    $\delta_i = \sqrt{bc}\, s_+ \frac{num}{den}$
    $den = num$
    $num = num\frac{s_-}{s_+} + 1$
  end for
  $(\mathcal{BD}(A))_{ij} = 0$ for $1 \leq i, j \leq n$
  $(\mathcal{BD}(A))_{11} = \delta_1$
  $(\mathcal{BD}(A))_{12} = -\frac{c}{\delta_1}$
  for i=2:n-1 **do**
    $(\mathcal{BD}(A))_{i,i-1} = -\frac{b}{\delta_{i-1}}$
    $(\mathcal{BD}(A))_{ii} = \delta_i$
    $(\mathcal{BD}(A))_{i+1,i} = -\frac{c}{\delta_i}$
  end for
  $(\mathcal{BD}(A))_{n,n-1} = -\frac{b}{\delta_{n-1}}$
  $(\mathcal{BD}(A))_{nn} = \delta_n$

In order to illustrate the accuracy of `TNEigenValues` and `TNSingularValues` with Algorithm 1, the sequence of matrices $A_5, A_{10}, \ldots, A_{100}$, given by $A_n =$

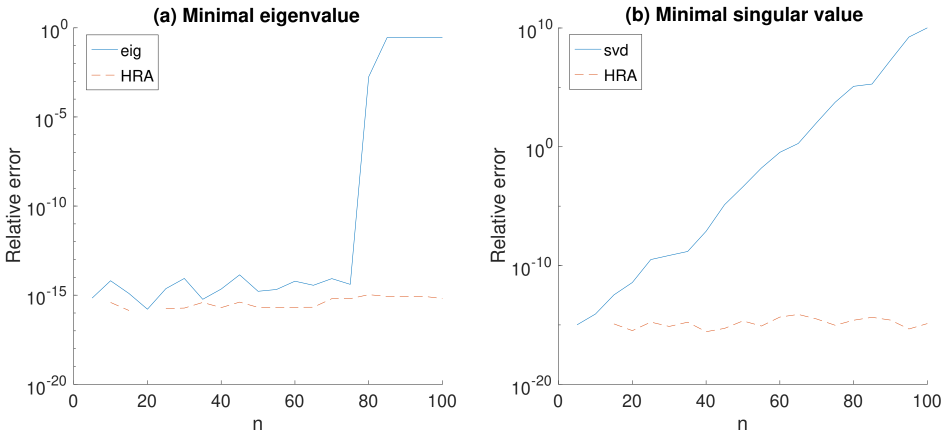

Figure 1. Relative error for the minimal eigenvalues and singular values of $A_5, A_{10}, \ldots, A_{100}$

$T_n(4, -1/4, -15)$, has been considered. First, we have computed the eigenvalues and the singular values of these matrices with Mathematica using a precision of 100 digits. We have also computed approximations to the eigenvalues of those matrices in Matlab with `eig` and also with `TNEigenValues` using the absolute value of the bidiagonal decomposition provided by Algorithm 1. Then we have computed the relative errors of the approximations obtained considering the eigenvalues obtained with Mathematica as exact computations.

In Figure 1 (a) we can see the relative error for the minimal eigenvalue of each matrix $A_5, A_{10}, \ldots, A_{100}$ for both `eig` and `TNEigenValues`.

We have also computed approximations to the singular values of the matrices $A_5, \ldots, A_{100}$ in Matlab with `svd` and also with `TNSingularValues` using the absolute value of the bidiagonal decomposition provided by Algorithm 1. Then we have computed the relative errors of the approximations obtained considering the singular values obtained with Mathematica as exact computations. In Figure 1 (b) we can see the relative error for the minimal singular value of each matrix $A_5, A_{10}, \ldots, A_{100}$ for both `svd` and `TNSingularValues`.

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