ARS MATHEMATICA CONTEMPORANEA

ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA
https://doi.org/10.26493/1855-3974.3608.fc9
(Also available at http://amc-journal.eu)

Domination in cylindrical graphs

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Received 7 May 2025, accepted 1 October 2025

Abstract

The domination number $\gamma(C_m \square P_n)$ of the Cartesian product $C_m \square P_n$ of a cycle and a path has been computed when $m \equiv 0, 2 \pmod 5$. In the remaining cases $m \equiv 1, 3, 4 \pmod 5$, exact formulae for $\gamma(C_m \square P_n)$ have been determined when either $m \leq 30$ or $n \leq 22$. For the rest of the cases, only lower and upper bounds for $\gamma(C_m \square P_n)$ are known. In this paper, we study $\gamma(C_m \square P_n)$ when $m \equiv 1, 3, 4 \pmod 5$. In particular, we compute $\gamma(C_m \square P_n)$ if $m \equiv 1 \pmod 5$, $m \geq 30$ and $n \geq 22$, and we provide tighter lower and upper bounds for $\gamma(C_m \square P_n)$ if $m \equiv 3, 4 \pmod 5$.

Keywords: Domination in graphs, Cartesian product graphs, tropical matrix multiplication.

Math. Subj. Class. (2020): 05C76, 15B33, 68R10

*Supported by grant PID2021-123278OB-I00 funded by MICIU/AEI/10.13039/501100011033.

†Supported by grants PID2023-150725NB-I00 funded by MICIU/AEI/10.13039/501100011033PID2023-150725NB-I00 and Gen. Cat. DGR 2017SGR1336.

‡Corresponding author. Supported by grants PID2023-150725NB-I00 and PID2021-123278OB-I00 funded by MICIU/AEI/10.13039/501100011033.

§Supported by grant PID2023-150725NB-I00, funded by MICIU/AEI/10.13039/501100011033, and grant E41-23R, funded by Gobierno de Aragón.

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1 Introduction

Let $G = (V, E)$ be a simple graph. A *dominating set* of G is a subset $D \subseteq V$ such that every vertex not in D is adjacent to at least one vertex in D. The *domination number* of G, denoted by $\gamma(G)$, is defined as the minimum cardinality of a dominating set. Due to a lot of applications, domination in graphs has been widely studied in the literature. We refer the reader to [8] for an excellent book on this topic.

In this paper, we focus on studying the domination number in cylindrical graphs, or cylinders, $C_m \square P_n$, that is, the Cartesian product of the cycle C_m and the path P_n. In contrast to grid graphs (the Cartesian product of two paths), where the domination number was finally computed in [4] after almost thirty years of research, the problem of computing the domination number in cylindrical graphs is still open, and only partial results are known. In 1996, Klavžar and Žerovnik [9] used the notions of fasciagraph and rotagraph to develop a computational method that allows to obtain the domination number of such graphs in some particular cases. By running their algorithm, they computed the values of the domination number of cylindrical graphs when $2 \le n \le 5$ and $3 \le m \le 1000$. Later, Nandi, Parui and Adhikari [10] proposed a new method based on specific constructions to find the domination numbers of cylinders when $2 \le n \le 4$ and $m \ge 3$. This approach of considering cylinders with paths or cycles of bounded size was addressed again by Pavlič and Žerovnik [11], who computed the values of the domination number of cylinders $C_m \square P_n$ with $m \le 11$ or $n \le 7$. Later, Crevals [3] obtained exact values for $m \le 30$ or $n \le 22$.

The first result that computes the domination number of a family of cylinders with paths and cycles of unbounded size appeared in 2020. Carreño, Martínez and Puertas showed in [2] that

$$\gamma(C_m \square P_n) = \frac{mn + 2m}{5}$$

when $m \equiv 0 \pmod 5$, $m \ge 30$ and $n \ge 20$. Recently, Guichard [6] proved that

$$\gamma(C_m \square P_n) = \left\lceil \frac{mn + 2m}{5} + \frac{n - 13}{10} \right\rceil$$

when $m \equiv 2 \pmod 5$, $m \ge 32$ and $n \ge 27$. For the remaining cases, $m \equiv 1, 3, 4 \pmod 5$, only lower and upper bounds for $\gamma(C_m \square P_n)$ are known (see [5, 7, 11]).

Table 1 summarizes the best-known results up to date for m and n large enough. For small values of m or n, Crevals determined formulae for $\gamma(C_m \square P_n)$ when $m \le 30$ or $n \le 22$ are fixed parameters [3]. In this paper, we compute the exact value of $\gamma(C_m \square P_n)$ when $m \equiv 1 \pmod 5$ and we improve the known bounds when $m \equiv 3, 4 \pmod 5$. The results we have obtained are summarized in Table 2.

Note that we decrease the known upper bound by 4 for $m \equiv 3 \pmod 5$, and by 3 for $m \equiv 4 \pmod 5$. In the case of the lower bounds, we increase them by some fractions depending on n, roughly $\frac{n}{50}$ if $m \equiv 3 \pmod 5$, and $\frac{12n}{650}$ if $m \equiv 4 \pmod 5$. The upper bounds are obtained by building dominating sets in $C_m \square P_n$ that provide such upper bounds, while the lower bounds are obtained computationally following the ideas in [2, 7].

The paper is organized as follows. We present some definitions and terminology in Section 2. Section 3 is devoted to exhibit a dominating set in $C_m \square P_n$, when $m \equiv 1 \pmod 5$, $m \ge 11$ and $n \ge 5$. This dominating set provides an upper bound for $\gamma(C_m \square P_n)$. In Section 4, we show that this upper bound is also a lower bound for $\gamma(C_m \square P_n)$, when $m \equiv 1 \pmod 5$, $m \ge 30$ and $n \ge 22$. The bounds for the remaining cases are shown in Section 5. Finally, some conclusions and conjectures are given in Section 6.

$\gamma(C_m \square P_n)$			
$m \pmod 5$	Exact value	Lower bound [7]	Upper bound [11]
$m \equiv 0$	$\frac{mn+2m}{5}$ [2]	—	—
$m \equiv 1$	?	$\frac{mn+2m}{5} + \frac{6}{5}\left\lfloor \frac{n-20}{10} \right\rfloor$	$\frac{mn+2m}{5} + \frac{7(n+2)}{40}$
$m \equiv 2$	$\left\lceil \frac{mn+2m}{5} + \frac{n-13}{10} \right\rceil$ [6]	—	—
$m \equiv 3$	?	$\frac{mn+2m}{5} + \frac{9}{5}\left\lfloor \frac{n-20}{10} \right\rfloor$	$\frac{mn+2m}{5} + \frac{2(n+2)}{5}$
$m \equiv 4$	?	$\frac{mn+2m}{5} + \frac{6}{5}\left\lfloor \frac{n-20}{10} \right\rfloor$	$\frac{mn+2m}{5} + \frac{n+2}{5}$

Table 1: Known results about $\gamma(C_m \square P_n)$ for m and n large enough.

$\gamma(C_m \square P_n)$			
$m \pmod 5$	Exact value	Lower bound	Upper bound
$m \equiv 1$	$\left\lceil \frac{mn+2m}{5} + \frac{2n-26}{15} \right\rceil$	—	—
$m \equiv 3$	?	$\frac{mn+2m}{5} + \frac{n-19}{5}$	$\frac{mn+2m}{5} + \frac{2(n-8)}{5}$
$m \equiv 4$	?	$\frac{mn+2m}{5} + \frac{9n-152}{65}$	$\frac{mn+2m}{5} + \frac{n-13}{5}$

Table 2: Our results about $\gamma(C_m \square P_n)$ for m and n large enough.

2 Preliminaries

We first recall the definition of a cylindrical graph or cylinder. For integers $m \geq 3$ and $n \geq 2$, the *cylinder* $C_m \square P_n$ is the Cartesian product of the cycle C_m and the path P_n, whose vertex set is $V(C_m \square P_n) = \{u_{ij} : i \in \{1, \dots, n\}, j \in \{1, \dots, m\}\}$ and there is an edge between two different vertices u_{ij} and $u_{i'j'}$ if and only if one of the following properties is satisfied:

- $i = i'$ and $|j - j'| = 1$;
- $j = j'$ and $|i - i'| = 1$;
- $i = i'$ and $\{j, j'\} = \{1, m\}$.

We will always refer to the cycles as the *rows* of the cylinder and the paths as the *columns* of the cylinder. Hence, the cylinder $C_m \square P_n$ will consist of n rows (cycles) and m columns (paths). For instance, Figure 1 shows the cylinder $C_{11} \square P_3$, which consists of 3 rows (cycles) and 11 columns (paths). In most of the figures, the set of colored vertices in the cylinder defines a dominating set for that cylinder. Black vertices in Figure 1 form a dominating set in $C_{11} \square P_3$.

Given a pattern with colored vertices in a cylinder, as the one illustrated in the gray square of Figure 1, when saying that the pattern is *replicated* we mean that we enlarge the

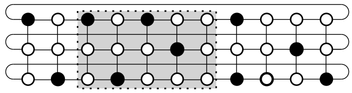


Figure 1: A pattern with colored vertices.

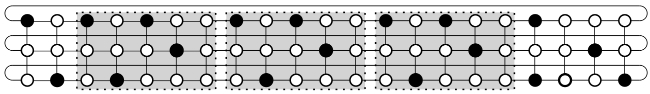


Figure 2: Replicating twice the pattern shown in the gray square of Figure 1.

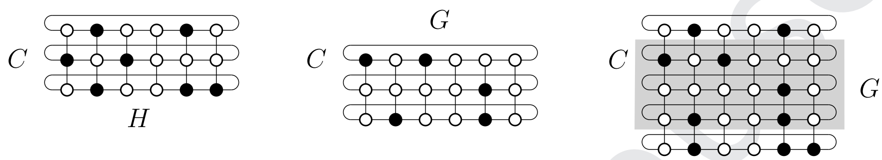


Figure 3: Gluing a cylinder G to a cycle C in a cylinder H.

cylinder by inserting columns to the right of the pattern so that the pattern can be replicated. Figure 2 illustrates the process of replicating twice the pattern in Figure 1: from the cylinder $C_{11}\Box P_3$ in Figure 1 and the set of colored vertices in that cylinder, we obtain the cylinder $C_{21}\Box P_3$ and a set of colored vertices. These colored vertices correspond with the initial colored vertices and the colored vertices in each one of the replications of the pattern. Note that the colored vertices in the last four columns of $C_{11}\Box P_3$ are now the colored vertices in the last four columns of $C_{21}\Box P_3$.

Similarly, given a cylinder G with some colored vertices, as the one illustrated in the middle of Figure 3, when saying that G is *glued* to a cycle C in a cylinder H with some colored vertices (where the pattern of colored vertices in C must coincide with the pattern of colored vertices in the first cycle in G), we mean that we insert G in H by identifying the first cycle (and the color of the vertices in the cycle) of G with C. The right part of Figure 3 shows the cylinder $C_6\Box P_5$ with some colored vertices, obtained after gluing the cylinder $C_6\Box P_3$ in the middle of Figure 3 to the second cycle of the cylinder $C_6\Box P_3$ in the left part of Figure 3.

3 Upper bound when $m \equiv 1 \pmod 5$

Crevals studied the domination number in cylinders [3], giving formulae for $\gamma(C_m\Box P_n)$ when $m \leq 30$ or $n \leq 22$ are fixed parameters. In particular, for $1 \leq k \leq 5$, Crevals showed that if $m = 1 + 5k$, then:

- $\gamma(C_{1+5\cdot 1}\Box P_n) = \lceil \frac{4n+2}{3} \rceil$, for $n \geq 6$;
- $\gamma(C_{1+5\cdot 2}\Box P_n) = \lceil \frac{7n+8}{3} \rceil$, for $n \geq 11$;
- $\gamma(C_{1+5\cdot 3}\Box P_n) = \lceil \frac{10n+14}{3} \rceil$, for $n \geq 16$;
- $\gamma(C_{1+5\cdot 4}\Box P_n) = \lceil \frac{13n+20}{3} \rceil$, for $n \geq 21$;

- $\gamma(C_{1+5\cdot 5}\square P_n) = \left\lceil \frac{16n+26}{3} \right\rceil$, for $n \geq 26$.

Previous values suggest the following closed formula, that we denote by $f(k,n)$, for the domination number of $C_{1+5k}\square P_n$, when $1 \leq k$ and n is large enough

$$f(k,n) = \left\lceil \frac{(1+3k)n + (6k-4)}{3} \right\rceil .$$

Notice that $f(k,n) = \gamma(C_{1+5\cdot k}\square P_n)$, for $k \in \{1,2,3,4,5\}$. In this section, we show that for every integer $k \geq 2$, there exists a dominating set of size $f(k,n)$ in $C_m\square P_n$, for $m = 1+5k$ and $n \geq 5$. After some algebraic manipulation, $f(k,n)$ can be rewritten as

$$\left\lceil \frac{mn+2m}{5} + \frac{2n-26}{15} \right\rceil .$$

Since the best known upper bound for $\gamma(C_m\square P_n)$ when $m \equiv 1 \pmod 5$ [11] is

$$\gamma(C_m\square P_n) \leq \frac{mn+2m}{5} + \frac{7(n+2)}{40}$$

we are providing a better upper bound for $\gamma(C_m\square P_n)$ in this case.

From the definition of $f(k,n)$, it is easy to check that $f(k,n+3) = f(k,n) + (1+3k)$ and $f(k,n) = f(k-1,n) + (n+2)$. In terms of dominating sets, this means that if we add three new rows to the cylinder $C_{1+5k}\square P_n$ to build the cylinder $C_{1+5k}\square P_{n+3}$, then we should add $(1+3k)$ new vertices to a dominating set of size $f(k,n)$ in $C_{1+5k}\square P_n$ to obtain a dominating set of size $f(k,n+3)$ in $C_{1+5k}\square P_{n+3}$. Similarly, if we add five new columns to the cylinder $C_{1+5(k-1)}\square P_n$ to build the cylinder $C_{1+5k}\square P_n$, then we should add $(n+2)$ new vertices to a dominating set of size $f(k-1,n)$ in $C_{1+5(k-1)}\square P_n$ to obtain a dominating set of size $f(k,n)$ in $C_{1+5k}\square P_n$.

3.1 Case $n = 5 + 3l$

For integers $k \geq 2$ and $l \geq 0$, we show in this section how to build a dominating set of size $f(k, 5+3l)$ in $C_{1+5k}\square P_{5+3l}$. Our starting point will be the dominating set D in $C_{1+5\cdot 2}\square P_5$ (so $k=2$ and $l=0$) defined by the 15 black vertices in the cylinder $C_{1+5\cdot 2}\square P_5$ shown in Figure 4. Note that $f(2,5)$ is 15.

Consider the pattern P given in the gray square of Figure 4. It consists of 25 vertices, seven of them black, in 5 consecutive columns of $C_{1+5\cdot 2}\square P_5$. The seven black vertices dominate all vertices in the pattern except for the two crossed vertices.

From D, we can obtain dominating sets in $C_{1+5k}\square P_5$ by replicating P. Figure 5 illustrates the cylinder $C_{1+5k}\square P_5$ and the set D' of $f(2,5) + 7(k-2)$ black vertices obtained after replicating $(k-2)$ times the pattern P. From the definition of P, it is easy to check that D' is a dominating set in $C_{1+5k}\square P_5$. Moreover, since $f(k,5) = f(k-1,5)+7$ (recall that $f(k,n) = f(k-1,n) + (n+2)$), we have $f(2,5) + 7(k-2) = f(k,5)$.

So far, we have built dominating sets for $k \geq 2$ and $n = 5$. Now, we show how to build dominating sets for $n = 5+3l$ and $l > 0$, by gluing cylinders to $C_{1+5k}\square P_5$, for a fixed k. We define the base cylinder shown in Figure 6. It consists of four rows, eleven columns, three black vertices, and seven gray vertices. From this cylinder, we can get a cylinder G with 4 rows and $1+5k$ columns, as the one depicted in Figure 7, by replicating $(k-2)$ times the pattern P' in the gray square of Figure 6. In this new cylinder G, the number of black and gray vertices is $(k+1)$ and $(1+3k)$, respectively. Note that the only vertices not

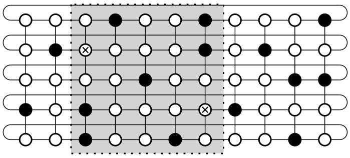


Figure 4: Black vertices define a dominating set D of size $15 = f(2,5)$ in $C_{1+5\cdot 2}\square P_5$.

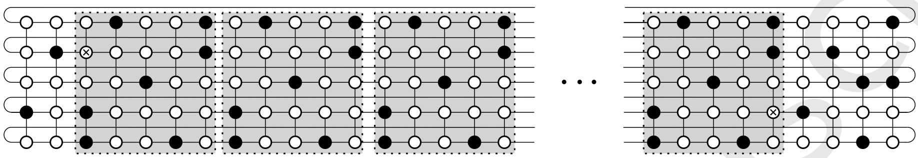


Figure 5: Black vertices define a dominating set D' in $C_{1+5k}\square P_5$ of size $f(2,5) + 7(k-2) = f(k,5)$, obtained after replicating $(k-2)$ times the pattern in the gray square of Figure 4.

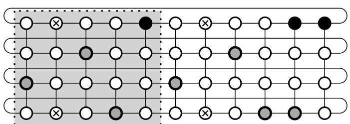


Figure 6: The base cylinder.

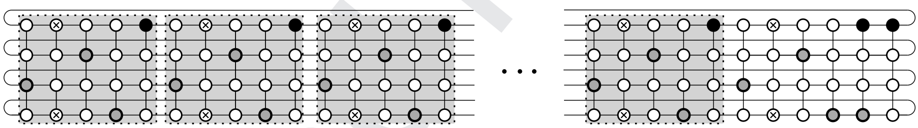


Figure 7: The cylinder G built by replicating $(k-2)$ times the pattern shown in the gray square of Figure 6.

dominated by this set of black and gray vertices are the crossed vertices in Figure 7. Also note that the relative positions of the black vertices in the first cycle and the gray vertices in the last cycle are the same (the distance between two consecutive black or gray vertices is 5, except for two of them that are adjacent). We remark that if $k = 2$, then G coincides with the base cylinder shown in Figure 6.

Observe now that the positions of the black vertices in the first cycle of G are the same as the positions of the black vertices in the third cycle of the cylinder $C_{1+5k}\square P_5$ shown in Figure 5. Thus, we can glue G to the third cycle of $C_{1+5k}\square P_5$, as illustrated in Figure 8. After gluing the cylinder, we move cyclically all black vertices in the last two cycles of $C_{1+5k}\square P_5$ one position to the left. In this way, we ensure that all non-dominated vertices in G (crossed vertices) are now dominated. From the definitions of D' and G, it is clear that the set D'' of black and gray vertices in Figure 8 forms a dominating set in $C_{1+5k}\square P_{5+3\cdot 1}$ of size $f(k,5) + (1+3k)$. Since $f(k, n+3) = f(k,n) + (1+3k)$, the size of D'' is $f(k, 5+3\cdot 1)$.

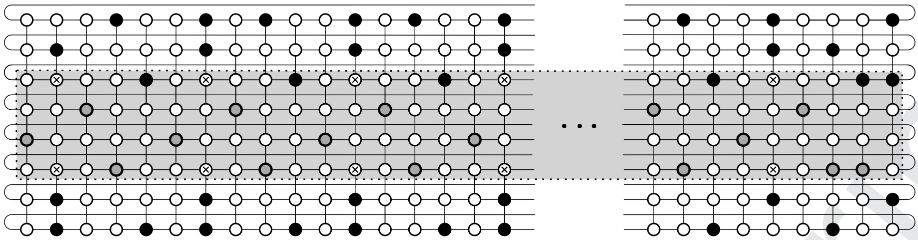


Figure 8: The set of black and gray vertices defines a dominating set in $C_{1+5k}\Box P_{5+3\cdot 1}$ of size $f(k,8)=f(k,5)+(1+3k)$.

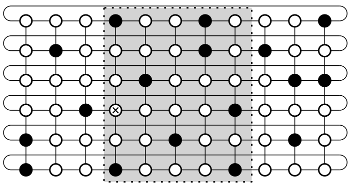


Figure 9: Black vertices define a dominating set of size $17=f(2,6)$ in $C_{1+5\cdot 2}\Box P_6$.

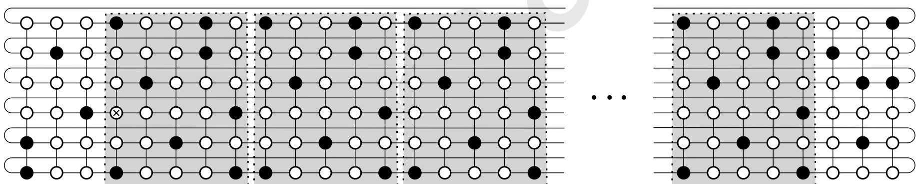


Figure 10: Black vertices define a dominating set in $C_{1+5k}\Box P_6$ of size $f(2,6)+8(k-2)=f(k,6)$, obtained after replicating $(k-2)$ the pattern in the gray square of Figure 9.

This process of gluing G can be done as many times as we want because the pattern of black vertices in the third cycle of the resulting cylinder remains (the distance between two consecutive black vertices is 5, except for two of them), regardless of the number of times that G is glued. To ensure that we obtain a dominating set each time we glue G, we have to move cyclically one position to the left all black and gray vertices in all rows below the last copy of G that is glued. As a consequence, since each time we glue G we are adding $(1+3k)$ new vertices to a dominating set, for $k\geq 2$ fixed and $n=5+3l$, we can obtain a dominating set of size $f(k,5)+l(1+3k)=f(k,5+3l)$ in $C_{1+5k}\Box P_{5+3l}$, by gluing l times the cylinder G from the cylinder $C_{1+5k}\Box P_5$ shown in Figure 5 (or Figure 4 if $k=2$).

3.2 Cases $n=6+3l$ and $n=7+3l$

In this section, we describe how to build dominating sets of sizes $f(k,6+3l)$ and $f(k,7+3l)$ in $C_{1+5k}\Box P_{6+3l}$ and $C_{1+5k}\Box P_{7+3l}$, respectively, for integers $k\geq 2$ and $l\geq 0$. Since the method for building the dominating sets is the same as the one explained in the previous section, we will not enter into the details.

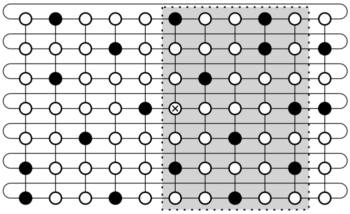


Figure 11: Black vertices define a dominating set of size $19 = f(2,7)$ in $C_{1+5\cdot 2}\Box P_7$.

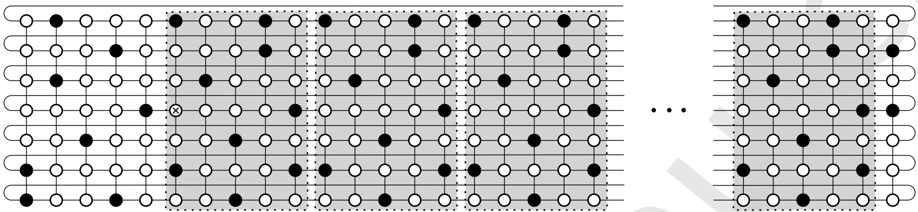


Figure 12: Black vertices define a dominating set in $C_{1+5k}\Box P_7$ of size $f(2,7)+9(k-2) = f(k,7)$, obtained after replicating $(k-2)$ the pattern in the gray square of Figure 11.

Figure 9 illustrates the base case when $n = 6+3l$. The set of $17 = f(2,6)$ black vertices in the figure forms a dominating set in $C_{1+5\cdot 2}\Box P_6$. To obtain a dominating set in $C_{1+5k}\Box P_6$, we replicate $(k-2)$ times the pattern shown in the gray square of Figure 9, as illustrated in Figure 10. By construction, the set of black vertices is clearly a dominating set in $C_{1+5k}\Box P_6$ of size $f(2,6)+8(k-2) = f(k,6)$.

Now, observe that the pattern of the black vertices in the third cycle of the cylinder $C_{1+5k}\Box P_6$ of Figure 10 (and also in Figure 9) is the same as the pattern of the black vertices in the first cycle of the cylinder G (see Figures 6 and 7). Thus, using this third cycle, for $l \geq 0$ we can get a dominating set of size $f(k, 6+3l)$ for $C_{1+5k}\Box P_{6+3l}$, by gluing l times the cylinder G, as described in the previous section.

When $n = 7+3l$, Figures 11 and 12 show a dominating set of size $19 = f(2,7)$ in $C_{1+5\cdot 2}\Box C_7$, the base case, and a dominating set of size $f(2,7)+9(k-2) = f(k,7)$ in $C_{1+5k}\Box C_7$, obtained after replicating $(k-2)$ times the pattern in the gray square of Figure 11. Now, the black vertices in the fourth cycle of Figures 11 and 12 follow the pattern of the black vertices in the first cycle of G. Therefore, for $l \geq 0$, we can use this cycle to glue l times the cylinder G and obtain a dominating set of size $f(k, 7+3l)$ in $C_{1+5k}\Box P_{7+3l}$.

From the discussion in the previous sections, the following theorem holds.

Theorem 3.1. *If* $m \equiv 1 \pmod 5$, $m \geq 11$ *and* $n \geq 5$, *then*

$$\gamma(C_m \Box P_n) \leq \left\lceil \frac{mn+2m}{5} + \frac{2n-26}{15} \right\rceil.$$

4 Lower bound when m ≡ 1 (mod 5)

In this section, we prove that the upper bound for $\gamma(C_m \Box P_n)$ given in Theorem 3.1 is also a lower bound for this parameter. As a consequence, that upper bound is the value of $\gamma(C_m \Box P_n)$, when $m \equiv 1$ (mod 5).

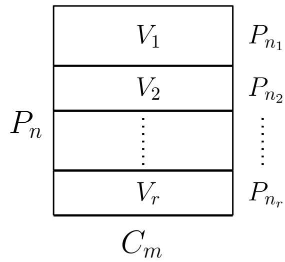


Figure 13: A partition of the vertex set of the cylinder $C_m \square P_n$.

Our approach to computing the lower bound follows the ideas in [2, 7] and deepens the technique of the so-called wasted domination [5]. It is known that a lower bound L for the minimum wasted domination of $C_m \square P_n$ gives a lower bound for $\gamma(C_m \square P_n)$. Besides, the larger L, the better the lower bound for $\gamma(C_m \square P_n)$. Hence, our main goal will be finding the largest possible value of L. To this end, $C_m \square P_n$ is partitioned into a set of r small cylinders, as described in [2, 7]. Every partition of $C_m \square P_n$ provides a lower bound L for the minimum wasted domination of $C_m \square P_n$. Therefore, our main task will be computing the minimum wasted domination in small cylinders and finding a partition of $C_m \square P_n$ maximizing L.

We start with the definition of wasted domination. Recall that the closed neighborhood of a vertex set S consists of the vertices in S and all their neighbors.

Definition 4.1 ([5]). Let $S \subseteq V(C_m \square P_n)$ and denote by $N[S]$ the closed neighborhood of S. The wasted domination of S is $\omega(S) = 5|S| - |N[S]|$.

The wasted domination provides a measure of how much the neighborhoods of the vertices of a subset overlap. In particular, in the case of dominating sets, the wasted domination provides the following lower bound for $\gamma(C_m \square P_n)$, that can be found in [2, 7].

Proposition 4.2 ([2, 7]). *If $L \leq \min\{\omega(D) \colon D$ is a dominating set of $C_m \square P_n\}$ then,*

$$\frac{mn + L}{5} \leq \gamma(C_m \square P_n).$$

This proposition implies that to provide a good lower bound for $\gamma(C_m \square P_n)$ we have to find a value of L as large as possible. To find such L, we follow the strategy proposed in [7], which consists of dividing the cylinder into several smaller cylinders and computing the minimum wasted domination of these small cylinders.

Let $V_1, \ldots, V_r$ be a partition of the set of vertices of $C_m \square P_n$ such that the subgraph induced by V_i is a cylinder $C_m \square P_{n_i}$ (see Figure 13). For any vertex subset $S \subseteq V(C_m \square P_n)$, we denote $S_i = S \cap V_i$, for $i = 1, \ldots, r$.

The wasted domination of a vertex subset S is related to the wasted domination of the partition subsets, as the following lemma shows.

Lemma 4.3 ([2, 7]). *Let $S \subseteq V(C_m \square P_n)$. Then, $\omega(S) \geq \sum_{i=1}^{r} \omega(S_i)$.*

These kinds of partitions can be used to obtain a lower bound for the wasted domination of dominating sets of the cylinder.

Lemma 4.4. *Let* $\omega(i,m) = \min\{\omega(D_i)\colon D_i = D \cap V_i, D$ *is a dominating set of* $C_m \square P_n\}$. *Then,* $\sum_{i=1}^{r} \omega(i,m) \leq \min\{\omega(D)\colon D$ *is a dominating set of* $C_m \square P_n\}$.

Proof. Let D be a dominating set of $C_m \square P_n$. Then, $\omega(D) \geq \sum_{i=1}^{r} \omega(D_i) \geq \sum_{i=1}^{r} \omega(i,m)$. Therefore, $\min\{\omega(D)\colon D$ is a dominating set of $C_m \square P_n\} \geq \sum_{i=1}^{r} \omega(i,m)$. □

Since every partition of the vertex set provides a lower bound for the wasted domination of dominating sets, the challenge is to find the partition that gives the largest lower bound for the wasted domination. To this end, we study how to compute $\omega(i,m)$ distinguishing whether $C_m \square P_{n_i}$ (the cylinder induced by V_i) is in the interior of $C_m \square P_n$ or not, that is, whether $C_m \square P_{n_i}$ contains or not the first or last row of $C_m \square P_n$.

4.1 Interior subcylinders

Suppose that V_i induces an interior subcylinder $C_m \square P_t$ in $C_m \square P_n$ (see Figure 14a). Our goal is to compute $\omega(i,m)$. Since this value depends only on the order of the cycle C_m and the path P_t, we denote it by $\omega_t(m)$.

In order to compute $\omega_t(m)$, we adapt the algorithm given by Carreño, Martínez and Puertas [2]. For the sake of completeness, we include a full description of the adapted algorithm, although quite a few of the details are also described in [2].

We remark that Guichard [7] computed $\omega_t(m)$ for interior cylinders with $t \leq 10$ using a dynamic programming algorithm. However, since these values of t are not enough for our purposes, we propose a different approach.

The main tool used to formulate our algorithm is a theorem from [1], which we quote from [9], related to the $(\min, +)$ matrix product. Let $\mathcal{P} = (\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$ be the semiring of tropical numbers in the min convention (see for instance [12]), so-called a path algebra in [9]. The $(\min, +)$ matrix multiplication of two matrices A and B, denoted by $A \boxtimes B$, is a matrix $C = A \boxtimes B$, where $c_{ij} = \min_k (a_{ik} + b_{kj})$, for all i, j.

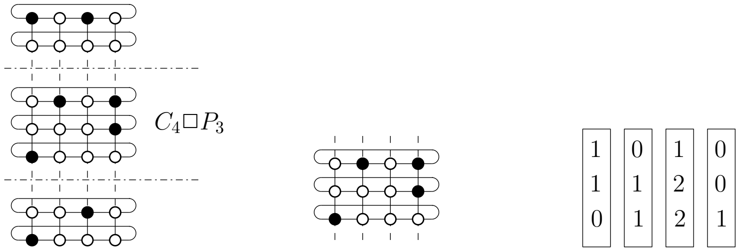
Let $\mathcal{H}$ be a directed graph with vertex set $V(\mathcal{H}) = \{v_1, v_2, \ldots, v_z\}$ and with a weight function ℓ that assigns an element of the semiring $\mathcal{P}$ to every arc of $\mathcal{H}$. A *path of length* k is a sequence of k consecutive arcs $Q = (v_{i_0}, v_{i_1})(v_{i_1}, v_{i_2}) \ldots (v_{i_{k-1}}, v_{i_k})$ and the path Q is *closed* if $v_{i_0} = v_{i_k}$. The arc weight function ℓ can be easily extended to paths: $\ell(Q) = \ell(v_{i_0}, v_{i_1}) + \ell(v_{i_1}, v_{i_2}) + \cdots + \ell(v_{i_{k-1}}, v_{i_k})$.

Theorem 4.5 ([1, 9]). *Let* $\mathcal{P}_{ij}^k$ *be the set of all paths of length* k *from* v_i *to* v_j *in* $\mathcal{H}$ *and let* $A(\mathcal{H})$ *be the matrix defined by*

$$A(\mathcal{H})_{ij} = \begin{cases} \ell(v_i, v_j) & \text{if } (v_i, v_j) \text{ is an arc of } \mathcal{H}, \\ \infty & \text{otherwise.} \end{cases}$$

If $A(\mathcal{H})^k$ *is the* k*-th* $(\min, +)$*-power of* $A(\mathcal{H})$ *then,* $(A(\mathcal{H})^k)_{ij} = \min\{\ell(Q)\colon Q \in \mathcal{P}_{ij}^k\}$.

Our purpose is to define a weighted digraph to compute $w_t(m)$ applying Theorem 4.5. To this end, we first define the following concept.



(a) A dominating set of $C_4\square P_r$ and the subgraph $C_4\square P_3$.

(b) Black vertices define an internal almost dominating set of $C_4\square P_3$.

(c) Labeling corresponding to the black vertices of (b).

Figure 14: Black vertices dominate the cylinder.

Definition 4.6. A vertex subset $B \subseteq V(C_m\square P_t)$ is an *internal almost dominating set* if B dominates all vertices in $V(C_m\square P_t)$, except possibly the vertices in the first row and the last row (see Figure 14b).

Observe that if D is a dominating set for $C_m\square P_n$, then $B = D \cap V(C_m\square P_t)$ is an internal almost dominating set. As a consequence, when partitioning $C_m\square P_n$ into a set of smaller cylinders, D induces a set of internal almost dominating sets (for different values of t). Besides, to compute

$$\omega_t(m) = \min\{\omega(D_i)\colon D_i = D \cap V(C_m\square P_t), D \text{ is a dominating set of } C_m\square P_n\}$$

it is enough to focus on sets D_i that are internal almost dominating sets, so

$$\omega_t(m) = \min\{\omega(B)\colon B \text{ is a internal almost dominating set of } C_m\square P_t\}\cdot$$

We can label the vertices in $C_m\square P_t$ as follows.

Definition 4.7. Let $V = V(C_m\square P_t)$ and $B \subseteq V$. The *label* of $v \in V$ associated to B is:

(i) 0 if $v \in B$;

(ii) 1 if $v \in V \setminus B$ and v has at least one neighbor in its column or in the previous one, that belongs to B;

(iii) 2 if $v \in V \setminus B$ and v has no neighbors in its column or in the previous one, that belongs to B.

Thus, given any $B \subseteq V$, the vertex set $V(C_m\square P_t)$ can be identified as a sequence of m columns (words) $\mathbf{p}^1, \dots, \mathbf{p}^m$ of length t, each of them having entries in $\{0, 1, 2\}$ (see Figure 14c).

In the particular case that B is an internal almost dominating set, the sequence of words satisfies some special properties, for which the following definitions are needed.

Definition 4.8. A word $\mathbf{p} = p_1 \dots p_t$ of length t in the alphabet $\{0, 1, 2\}$ is called *suitable* if it does not contain the sequences 02 or 20.

Definition 4.9. We say that a suitable word $\mathbf{p} = p_1 \dots p_t$ of length t can *follow* another suitable word $\mathbf{q} = q_1 \dots q_t$ if the following conditions hold.

(i) for $i \in \{1, \dots, t\}$: if $q_i = 0$ then, $p_i \neq 2$;

(ii) for $i \in \{2, \dots, t-1\}$: if $q_i = 1, p_i = 1$ then $p_{i-1} = 0$ or $p_{i+1} = 0$; if $q_i = 2$ then, $p_i = 0$;

(iii) for $i = 1$: if $q_i \in \{1, 2\}, p_i = 1$ then, $p_{i+1} = 0$;

(iv) for $i = t$: if $q_i \in \{1, 2\}, p_i = 1$ then, $p_{i-1} = 0$.

With the above definitions, it is not difficult to check that there is a one-to-one correspondence between internal almost dominating sets B of $C_m \Box P_t$ and sequences $\mathbf{p}^1 \dots \mathbf{p}^m$ of m suitable words of length t such that $\mathbf{p}^{i+1}$ can follow $\mathbf{p}^i$, for $i \in \{1, \dots, m\}$ (indices are taken module m).

We use this fact to construct the following digraph $\mathcal{H}$, which relates internal almost dominating sets and closed paths of length m.

Definition 4.10. $\mathcal{H}$ is the digraph whose vertex set is the set of all suitable words of length t in the alphabet $\{0, 1, 2\}$ and there is an arc from $\mathbf{q}$ to $\mathbf{p}$ if $\mathbf{p}$ can follow $\mathbf{q}$.

Proposition 4.11. *There is a bijective correspondence between the internal almost dominating sets of $C_m \Box P_t$ and the closed paths of length m in the digraph $\mathcal{H}$.*

Proof. Let B be an internal almost dominating set of $C_m \Box P_t$. By construction, we know that B can be identified with a sequence $\mathbf{p}^1 \dots \mathbf{p}^m$ of m suitable words of length t, such that $\mathbf{p}^{i+1}$ can follow $\mathbf{p}^i$, for $i \in \{1, \dots m\}$ (module m). So $\mathbf{p}^1 \dots \mathbf{p}^m$ is a closed path of length m in $\mathcal{H}$.

Conversely, let $Q = \mathbf{p}^1\mathbf{p}^2 \dots \mathbf{p}^m$ be a closed path of length m in $\mathcal{H}$. So each $\mathbf{p}^i$ is a suitable word of length t, and $\mathbf{p}^{i+1}$ can follow $\mathbf{p}^i$, for $i \in \{1, \dots, m\}$ (module m). As the word $\mathbf{p}^i$ can be seen as the $i - th$ column of $C_m \Box P_t$, we define the vertex subset B as those vertices with label 0. By using that words are suitable and the condition that every word can follow the previous one, B is an internal almost dominating set of $C_m \Box P_t$. □

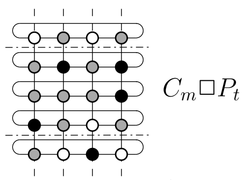
Since we want to compute the minimum wasted domination among the internal almost dominating sets, the weight function of $\mathcal{H}$ must be related to that parameter. In particular, we have to compute the number of vertices in the closed neighborhood of internal almost dominating sets.

Algorithm 1 provides a way of computing it. If $\mathbf{p}^1 \dots \mathbf{p}^m$ is the sequence of suitable words associated with an internal almost dominating set B, the algorithm computes the contribution $n(\mathbf{p}^{i-1}\mathbf{p}^i)$ of a column $\mathbf{p}^i$ to $N[B]$, in relation to the previous column $\mathbf{p}^{i-1}$ (mod m). Using the labels of $\mathbf{p}^i$ and $\mathbf{p}^{i-1}$, the algorithm adds all vertices of column i in $C_m \Box P_t$ to $N[B]$ (line 1), adds vertices outside $C_m \Box P_t$ that are dominated by the first or the last vertex of the column (lines 2-7) and removes vertices in column $i - 1$ that are not dominated (lines 8-13). Clearly, the addition of all contributions gives the number of vertices in $N[B]$, because $N[B]$ consists of the vertices of $V(C_m \Box P_t)$, minus the vertices with label 2 having no neighbor in B (if any, they must be in the first or the last row of

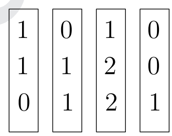
Algorithm 1 Computation of the number of vertices in the closed neighborhood, for each column

Require: $t > 0$ an integer, $\mathbf{q} = q_1 \dots q_t$, $\mathbf{p} = p_1 \dots p_t$, suitable words of length t such that $\mathbf{p}$ can follow $\mathbf{q}$
Ensure: $n(\mathbf{qp})$

```
n(qp) ← t;
if p_1 == 0 then
    n(qp) ← n(qp) + 1;
end if
if p_t == 0 then
    n(qp) ← n(qp) + 1;
end if
if q_1 == 2 and p_1 ≠ 0 then
    n(qp) ← n(qp) − 1;
end if
if q_t == 2 and p_t ≠ 0 then
    n(qp) ← n(qp) − 1;
end if
```



(a) Black vertices are in B and black and gray vertices are in its closed neighborhood.



(b) The sequence $\mathbf{p}^1 \dots \mathbf{p}^m$.

Figure 15: Illustrating Algorithm 1. For $i = 1, \dots 4$, the contributions $n(\mathbf{p}^{i-1}\mathbf{p}^i)$ are $4, 4, 3$ and 3. The addition of the contributions gives the number of vertices in $N[B]$.

$C_m \square P_t$), plus the number of vertices with label 0 in the first and the last row (each of these vertices add an extra neighbor which is in $C_m \square P_n$ but outside $C_m \square P_t$). See Figure 15.

Using the previous contributions, we can now define the appropriate weight function for $\mathcal{H}$, which allows us to compute the wasted domination of an internal almost dominating set (Proposition 4.13) and the internal almost dominating set minimizing the wasted domination (Lemma 4.14).

Definition 4.12. Let $\mathbf{qp}$ an arc of $\mathcal{H}$ and consider the parameter $n(\mathbf{qp})$ computed by Algorithm 1. We define $\ell(\mathbf{qp}) = 5|\{j : p_j = 0 \text{ and } 1 \le j \le t\}| - n(\mathbf{qp})$.

Proposition 4.13. *Let B be an internal almost dominating set of $C_m \square P_t$ and denote by $Q = \mathbf{p}^1\mathbf{p}^2 \dots \mathbf{p}^m$ its associated closed path of length m in $\mathcal{H}$. Then, $\omega(B) = \ell(Q)$, where $\ell(Q) = \sum_{i=1}^m \ell(\mathbf{p}^i\mathbf{p}^{i+1})$.*

Proof. On the one hand, if $Q = \mathbf{p}^1\mathbf{p}^2\ldots\mathbf{p}^m$ is the closed path in $\mathcal{H}$ associated with B then, by Proposition 4.11, it is clear that $\sum_{i=1}^m |\{j\colon \mathbf{p}^i_j = 0 \text{ and } 1 \le j \le t\}| = |B|$.

On the other hand, since B is an internal almost dominating set, we have $|N[B]| = \sum_{i=1}^m n(\mathbf{p}^i\mathbf{p}^{i+1})$ (indices are taken module m). Hence,

$$\begin{aligned}\ell(Q) &= \sum_{i=1}^m \ell(\mathbf{p}^i\mathbf{p}^{i+1}) = \sum_{i=1}^m \left(5|\{j\colon \mathbf{p}^i_j = 0, 1 \le j \le t\}| - n(\mathbf{p}^i\mathbf{p}^{i+1})\right)\\ &= 5|B| - |N[B]| = \omega(B).\end{aligned}$$

□

Lemma 4.14. *Let $\mathcal{H}$ be the digraph with the weight function ℓ that we have constructed above. Define the matrix*

$$A(\mathcal{H})_{\mathbf{qp}} = \begin{cases} \ell(\mathbf{qp}) & \textit{if } (\mathbf{qp}) \textit{ is an arc of } \mathcal{H},\\ \infty & \textit{otherwise.}\end{cases}$$

Then, $\min_{\mathbf{p}\in V(\mathcal{H})}(A(\mathcal{H})^m)_{\mathbf{pp}} = \min\{\omega(B)\colon B$ *internal dominating set of* $C_m \Box P_t\}$.

Proof. Theorem 4.5 and Proposition 4.13 give that

$$\begin{aligned}(A(\mathcal{H})^m)_{\mathbf{pp}} &= \min\{\ell(Q)\colon Q \text{ closed path in } \mathcal{H}, \text{ with length } m, \text{ from } \mathbf{p} \text{ to } \mathbf{p}\}\\ &= \min\{\omega(B)\colon B \text{ internal almost almost dominating set with first column } \mathbf{p}\}.\end{aligned}$$

Therefore,

$$\begin{aligned}\min_{\mathbf{p}\in V(\mathcal{H})}(A(\mathcal{H})^m)_{\mathbf{pp}} &= \min_{\mathbf{p}}\{\omega(B)\colon B \text{ internal almost almost dominating set with first column } \mathbf{p}\}\\ &= \min\{\omega(B)\colon B \text{ internal almost dominating set of } C_m \Box P_t\} = \omega_t(m).\end{aligned}$$

□

For a fixed value of t and m, it is enough to apply Lemma 4.14 to find $\omega_t(m)$. But, to simplify the computations when m is large, we will use the following well-known property of the $(\min, +)$ matrix product.

Lemma 4.15. *Let A be a square matrix. Suppose that there exist natural numbers m_0, a, b such that $A^{m_0+a} = b \boxtimes A^{m_0}$. Then, $A^{m+a} = b \boxtimes A^m$, for every $m \ge m_0$.*

As a consequence of the previous lemma, we have the following result.

Lemma 4.16. *If there exist natural numbers m_0, a, b such that $A(\mathcal{H})^{m_0+a} = b \boxtimes A(\mathcal{H})^{m_0}$, then $\omega_t(m+a) - \omega_t(m) = b$, for every $m \ge m_0$.*

Proof. By Lemma 4.15, we obtain that $A(\mathcal{H})^{m+a} = b \boxtimes A(\mathcal{H})^m$, for every $m \ge m_0$. Therefore, by Lemma 4.14,

$$\begin{aligned}\omega_t(m+a) &= \min_{\mathbf{p}}(A(\mathcal{H})^{m+a})_{\mathbf{pp}} = \min_{\mathbf{p}}(b \boxtimes A(\mathcal{H})^m)_{\mathbf{pp}}\\ &= b + \min_{\mathbf{p}}(A(\mathcal{H})^m)_{\mathbf{pp}} = b + \omega_t(m)\cdot\end{aligned}$$

□

Algorithm 2 Computation of the matrix $A(\mathcal{H})$ and its powers

Require: $t > 0$ an integer
Ensure: $A(\mathcal{H})^k$ and $\min_{\mathbf{p}\in V(\mathcal{H})}(A(\mathcal{H})^k)_{\mathbf{pp}}$ for k large enough
1: Compute the suitable words of length t ▷ Definition 4.8
2: Compute the matrix $A(\mathcal{H})$ ▷ Definitions 4.10 and 4.12, and Lemma 4.14
3: Compute $A(\mathcal{H})^k$, for k large enough ▷ $(\min, +)$ matrix product
4: Compute $\min_{\mathbf{p}}(A(\mathcal{H})^k)_{\mathbf{pp}}$, for each k

To find the parameters a, b, m_0 needed to pose the finite difference equation appearing in Lemma 4.16, we run Algorithm 2.

In Step 1 of Algorithm 2, we first obtain the set of suitable words by computing the t-element variations with repetition of the elements in the alphabet $\{0, 1, 2\}$ and keeping those of them that satisfy the conditions given in Definition 4.8.

In Step 2, we compute $\mathcal{H}$ using Definition 4.10, the weights of the arcs of $\mathcal{H}$ using Definition 4.12, and the matrix $A(\mathcal{H})$ as described in Lemma 4.14.

In Step 3, we compute successive $(\min, +)$ powers of the matrix $A(\mathcal{H})$. We compare them to each other until we obtain $A(\mathcal{H})^{m_0+a} = b \boxtimes A(\mathcal{H})^{m_0}$. The results we have obtained, for different values of t, are in Table 3.

t	suitable words	m_0	a	b
5	99	17	5	0
6	239	18	5	0
7	577	18	5	0
8	1393	20	5	0
9	3363	23	5	0
10	8128	24	5	0
11	19616	26	5	0
12	47321	28	5	0
13	114243	30	5	0

Table 3: Values obtained by Algorithm 2.

In Step 4, the minimum of the main diagonal of some powers of $A(\mathcal{H})$ is obtained. In particular, we have computed $\omega_t(m) = \min_{\mathbf{p}}(A(\mathcal{H})^m)_{\mathbf{pp}}$ for $m_0 \leq m \leq m_0 + a - 1$, which are the boundary values of the finite difference equation given in Lemma 4.16

$$\omega_t(m + a) - \omega_t(m) = b, \text{ for every } m \geq m_0.$$

With these data, we have solved such an equation to obtain $\omega_t(m)$ for the different values of m. Table 4 shows such values, for different values of t, which are valid for $m \geq 30$ in all cases.

We have run Algorithm 2 on an NVIDIA Tesla V100, leveraging its advanced computational capabilities. Less demanding parts of the algorithm (steps 1, 2, and 4) run sequentially, while the more intensive matrix tropical multiplication (step 3) utilizes the Tesla V100's architecture with 640 Tensor Cores and 5,120 CUDA Cores for parallel processing.

	$\omega_t(m)$ (congruence module 5), for $m \geq 30$				
t	$m \equiv 0$	$m \equiv 1$	$m \equiv 2$	$m \equiv 3$	$m \equiv 4$
5	0	2	2	2	1
6	0	2	3	4	2
7	0	4	3	6	3
8	0	4	4	6	4
9	0	6	4	7	5
10	0	6	5	9	6
11	0	6	5	10	7
12	0	8	6	11	8
13	0	8	6	13	9

Table 4: Values of $\omega_t(m)$ for internal almost dominating sets.

The card's 32 GB memory is efficiently utilized, significantly speeding up data processing. This enhances efficiency, particularly for computationally intensive tasks like matrix multiplication, critical for optimizing algorithm performance.

t	Matrix Size (MB)	Execution Time (seconds)
5	-	-
6	0.393	0.229
7	2.217	0.253
8	11.895	0.528
9	69.033	2.769
10	396.387	24.816
11	2308.725	285.537
12	13439.637	3812.484
13	119764.667	56933.480

Table 5: Memory usage and execution times for Algorithm 2 indexed by t.

Table 5 shows memory usage and execution times for Algorithm 2. As t increases from 5 to 13, both matrix size and execution time grow significantly. For instance, at $t = 13$, the matrix size reaches around 120 GB, and execution time peaks at 16 hours. This exponential growth indicates increased demands on computational resources and time. In CUDA programming, managing matrix memory efficiently, especially for operations like matrix tropic multiplication, is crucial on the NVIDIA Tesla V100 card, limited to 32 GB of memory. Case $t = 13$, which exceeds such size, employs a matrix multiplication method dividing matrices into 8 submatrices due to this memory constraint. The source code, in programming language C, of both Algorithm 1 and Algorithm 2 can be found online in the repository `https://github.com/hpcjmart/DominationCylinders`.

4.2 Border subcylinders

When partitioning $C_m \square P_n$ into small cylinders, the subgraphs $C_m \square P_{n_1}$ and $C_m \square P_{n_r}$ are located in both borders of $C_m \square P_n$. The procedure to compute the minimum wasted domination in these cases, which uses similar techniques to the ones described in Section 4.1, can be found in [2] and takes into account that there are no more vertices above the first row of $C_m \Box P_n$ and below the last row.

We just consider the case $C_m \Box P_{n_1}$ (the other case is symmetrical) and we call this subgraph $C_m \Box P_s$, for simplicity. We have applied the techniques described in [2] to compute the minimum wasted domination for several values of s, as we show in Table 6.

	$\omega_s(m)$ (congruence module 5), for $m \geq 30$				
s	$m \equiv 0$	$m \equiv 1$	$m \equiv 2$	$m \equiv 3$	$m \equiv 4$
11	m	$m+3$	$m+2$	$m+3$	$m+3$
12	m	$m+3$	$m+3$	$m+5$	$m+4$
13	m	$m+5$	$m+3$	$m+5$	$m+6$

Table 6: Values of $\omega_s(m)$ for both borders.

As it can be seen in Tables 4 and 6, for a cylinder $C_m \Box P_n$ the wasted domination in the interior strips ($\omega_t(m)$) depends on the length of the path (t) and the parity of the length of the cycle (m) module 5. Meanwhile, the wasted domination in both borders ($\omega_s(m)$) explicitly depends on the length of the path (s) and the length of the cycle (m).

The reason why this happens is that for a cylinder in the interior, the stacking of the vertices, which is measured by the wasted domination, does not occur either in the first or the last row of the cylinder, because those vertices can stay not dominated. However, on the top border (symmetrically on the bottom border), the first row of the border cylinder must be dominated and this forces the vertices to stack throughout that entire row.

4.3 Computation of the lower bound

In Section 3, we have obtained an upper bound for $\gamma(C_m \Box P_n)$ when $m \equiv 1 \pmod 5$. Our target is to obtain a lower bound that equals the upper one, by choosing a suitable partition of $C_m \Box P_n$. The following theorem shows how to find such a partition.

Theorem 4.17. *If* $m \equiv 1 \pmod 5$, $m \geq 30$ *and* $n \geq 22$, *then*

$$\left\lceil \frac{mn+2m}{5} + \frac{2n-26}{15} \right\rceil \leq \gamma(C_m \Box P_n).$$

Proof. On the one hand, by Proposition 4.2, $\dfrac{mn+L}{5} \leq \gamma(C_m \Box P_n)$ for every value of L such that $L \leq \min\{\omega(D)\colon D \text{ dominating set of } C_m \Box P_n\}$.

On the other hand, if $V_1, \dots, V_r$ is a partition of $V(C_m \Box P_n)$ into subcylinders, we know by Lemma 4.4 that

$$L = \sum_{i=1}^{r} \omega(i,m) \leq \min\{\omega(D)\colon D \text{ dominating set of } C_m \Box P_n\}$$

where

$$\begin{aligned}\omega(i,m) &= \min\{\omega(D_i)\colon D_i = S \cap V_i, D \text{ dominating set of } C_m \Box P_n\} \\ &= \min\{\omega(B)\colon B \text{ is a internal almost dominating set of } C_m \Box P_{n_i}\}\end{aligned}$$

Moreover, Lemma 4.16 gives that each $\omega(i,m), 2 \le i \le r-1$ (interior), is the solution of a finite difference equation, involving the parameters a, b, m_0 given by Algorithm 2. Such solutions can be found in Table 4, for $5 \le n_i \le 13$. The values of $\omega(1,m)$ and $\omega(r,m)$ (borders), for $n_1, n_r = 11, 12, 13$, are in Table 6.

We denote the cycle length as $m = 1+5k$. We decompose the cylinder into $r = h+2$ parts, with both borders of length 11, that is, $n = n_1 + n_2 + \cdots + n_{r=h+2}$ with $n_1 = n_{h+2} = 11$. Regarding the interior part, we consider h parts of different lengths n_i, depending on the parity of $(n-22)$ module 9, so we have to analyze nine different cases $n = 22+9h+\alpha, 0 \le \alpha \le 8$. We remark that the reason to choose module 9 is that $n_i = 9$ is the smallest size for the interior parts, for which the lower bound computed by our method equals the upper bound. In the nine cases, we will show that

$$\frac{mn+L}{5} = \left\lceil \frac{mn+2m}{5} + \frac{2n-26}{15} \right\rceil .$$

We show the first two cases in detail.

- $n = 22+9h, h \ge 0$. In this case, we write $n = 11+9h+11$, that is, $n_1 = n_{h+2} = 11$ (borders) and $h = \frac{n-22}{9}$ interior subgraphs with $n_i = 9$. From Table 4, we know that $\omega_9(m) = 6$ (interior), and from Table 6, we have that $\omega_{11}(m) = m+3$ (border). Thus,

$$L = (m+3) + 6\frac{n-22}{9} + (m+3),$$
$$\frac{mn+L}{5} = \frac{mn+2m}{5} + \frac{2n-26}{15}.$$

- $n = 22+9h+1, h \ge 1$. We rewrite n as $n = 11+9(h-1)+10+11$, that is, $n_1 = n_{h+2} = 11$ (borders), $h-1 = \frac{n-32}{9}$ interior subgraphs with $n_i = 9$, and one interior subgraph with $n_{h+1} = 10$. In this case, we have (remember that $m = 1+5k$)

$$L = (m+3) + 6\frac{n-32}{9} + 6 + (m+3),$$
$$\frac{mn+L}{5} = \frac{mn+2m}{5} + \frac{2n-28}{15} = 9kh + 25k + 3h + \frac{31}{5}.$$

Moreover, $\dfrac{mn+2m}{5} + \dfrac{2n-26}{15} = 9kh + 25k + 3h + \dfrac{19}{3}$.

In this case, the expressions in terms of k and h show that

$$\left\lceil \frac{mn+L}{5} \right\rceil = \left\lceil \frac{mn+2m}{5} + \frac{2n-26}{15} \right\rceil .$$

In the two preceding cases, the values $n = n_1 + \cdots + n_{h+2}$ providing the partition that gives the desired lower bound are:

- $n = 22+9h = 11+9h+11$, valid for $h \ge 0$;
- $n = 22+9h+1 = 11+9(h-1)+10+11$, valid for $h \ge 1$.

We have also found appropriate partitions $n = n_1 + n_2 + \cdots + n_{r=h+2}$ for the seven remaining cases, $n = 22+9h+\alpha, 2 \le \alpha \le 8$, which are the following:

- $n = 22+9h+2 = 11+9(h-1)+11+11$, valid for $h \ge 1$;

- $n = 22 + 9h + 3 = 11 + 9(h-1) + 12 + 11$, valid for $h \geq 1$;
- $n = 22 + 9h + 4 = 11 + 9(h-2) + 10 + 12 + 11$, valid for $h \geq 2$;
- $n = 22 + 9h + 5 = 11 + 9(h-2) + 11 + 12 + 11$, valid for $h \geq 2$;
- $n = 22 + 9h + 6 = 11 + 9(h-2) + 12 + 12 + 11$, valid for $h \geq 2$;
- $n = 22 + 9h + 7 = 11 + 9(h-3) + 10 + 12 + 12 + 11$, valid for $h \geq 3$;
- $n = 22 + 9h + 8 = 11 + 9(h-3) + 11 + 12 + 12 + 11$, valid for $h \geq 3$.

Using the partitions shown in the previous list, it can be proven in a completely analogous way to that shown in the first two cases that

$$\left\lceil \frac{mn+L}{5} \right\rceil = \left\lceil \frac{mn+2m}{5} + \frac{2n-26}{15} \right\rceil$$

in the seven remaining cases.

Note that except for $\alpha = 0$, we have considered $h > 0$ for the rest of the cases, so some values of $n \geq 22$ have not been studied yet. Next, we give suitable partitions for such values.

- $n = 22 + 9h$. No remaining cases.
- $n = 22 + 9h + 1, h = 0$. Then, $n = 23 = n_1 + n_2 = 11 + 12$, (borders $n_1 = 11$ and $n_2 = 12$). The detailed computations in this first case are (remember that $m = 1 + 5k$):

  $L = (m+3) + (m+3)$,

  $\dfrac{mn+L}{5} = \dfrac{mn+2m}{5} + \dfrac{6}{5} = 25k + \dfrac{31}{5}$,

  $\dfrac{mn+2m}{5} + \dfrac{2n-26}{15} = 25k + \dfrac{19}{3}$.
- $n = 22 + 9h + 2, h = 0$. Then, $n = 24 = n_1 + n_2 = 12 + 12$.
- $n = 22 + 9h + 3, h = 0$. Then, $n = 25 = n_1 + n_2 = 12 + 13$.
- $n = 22 + 9h + 4$.
  - $h = 0$. Then, $n = 26 = n_1 + n_2 = 13 + 13$.
  - $h = 1$. Then, $n = 35 = n_1 + n_2 + n_3 = 12 + 11 + 12$ (borders $n_1 = n_3 = 12$ and one interior $n_2 = 11$).
- $n = 22 + 9h + 5$.
  - $h = 0$. Then, $n = 27 = 11 + 5 + 11$.
  - $h = 1$. Then, $n = 36 = 12 + 12 + 12$.
- $n = 22 + 9h + 6$.
  - $h = 0$. Then, $n = 28 = 11 + 6 + 11$.
  - $h = 1$. Then, $n = 37 = 12 + 13 + 12$.
- $n = 22 + 9h + 7$.

- $h = 0$. Then, $n = 29 = 11 + 7 + 11$.
- $h = 1$. Then, $n = 38 = 11 + 7 + 9 + 11$, (borders $n_1 = n_4 = 11$ and two interior $n_2 = 7, n_3 = 9$).
- $h = 2$. Then, $n = 47 = 11 + 7 + 9 + 9 + 11$ (borders $n_1 = n_5 = 11$ and three interior $n_2 = 7, n_3 = n_4 = 9$).

• $n = 22 + 9h + 8$.

- $h = 0$. Then, $n = 30 = 11 + 8 + 11$.
- $h = 1$. Then, $n = 39 = 11 + 8 + 9 + 11$.
- $h = 2$. Then, $n = 48 = 11 + 8 + 9 + 9 + 11$.

For all values of n and h in the previous list, basic algebraic manipulations give

$$\left\lceil \frac{mn + L}{5} \right\rceil = \left\lceil \frac{mn + 2m}{5} + \frac{2n - 26}{15} \right\rceil .$$

This concludes the proof of the theorem. □

As a consequence of both Theorem 3.1 and Theorem 4.17, we obtain the desired exact value.

Theorem 4.18. *If* $m \equiv 1 \pmod 5$*,* $m \geq 30$ *and* $n \geq 22$*, then*

$$\gamma(C_m \Box P_n) = \left\lceil \frac{mn + 2m}{5} + \frac{2n - 26}{15} \right\rceil .$$

5 Bounds when m $\equiv$ 3, 4 (mod 5)

We provide in this section new upper and lower bounds for $\gamma(C_m \Box P_n)$ when $m \equiv 3, 4 \pmod 5$, that improve the bounds given in [7, 11].

5.1 Bounds when m $\equiv$ 3 (mod 5)

Theorem 5.1. *If* $m \equiv 3 \pmod 5$*,* $m \geq 30$ *and* $n \geq 22$*, then*

$$\frac{mn + 2m}{5} + \frac{n - 19}{5} \leq \gamma(C_m \Box P_n) \leq \frac{mn + 2m}{5} + \frac{2(n - 8)}{5}.$$

Proof. To give a lower bound for $\gamma(C_m \Box P_n)$ when $m \equiv 3 \pmod 5$, we argue as in Section 4.3 and search for partitions of $C_m \Box P_n$, trying to maximize the lower bound L for the wasted domination. We consider $26 \leq n = 13h + \alpha$, with $0 \leq \alpha \leq 12$, and analyze 13 cases. If $\alpha = 0$, we choose $n_1 = n_r = 13$ (the sizes of the border parts), and $h - 2$ internal parts of size 13. For $1 \leq \alpha \leq 9$, we choose $n_1 = n_r = 11$, $h - 2$ internal parts of size 13, and an internal part of size $4 + \alpha$. For $\alpha = 10, 11, 12$, we increase the sizes of the border parts to 12 or 13 and choose $h - 1$ internal parts of size 13. Using the values in Tables 4 and 6 to compute L, the worst case is $\alpha = 1$, for which we obtain $L = 2m + n - 19$.

For $n = 22, 23, 24, 25$, we only choose two parts, which are border parts, of sizes 11, 12, or 13. In any of the four cases, we have that L is greater than $2m+n-19$. For instance, if $n = 22$, we choose $n_1 = n_r = 11$, so $L = m+3+m+3 = 2m+n-16 > 2m+n-19$. As a consequence, when $n \geq 22$, we have the following lower bound for $\gamma(C_m \Box P_n)$

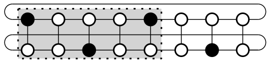


Figure 16: A dominating set of size $g(1, 2) = 4$ for $C_{3+5\cdot 1}\Box P_2$.

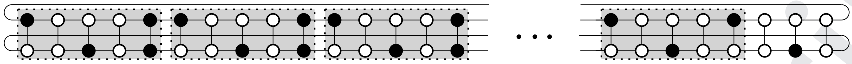


Figure 17: A dominating set of size $g(1, 2) + (k - 1)4 = g(k, 2)$ in $C_{3+5k}\Box P_2$, obtained after replicating $(k - 1)$ times the pattern in the gray square in Figure 16 (and adding an extra black vertex in the bottom rightmost corner of the pattern).

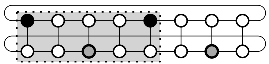


Figure 18: The base cylinder.

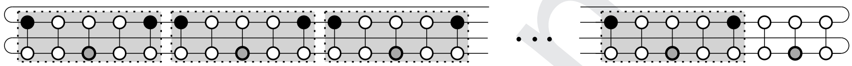


Figure 19: The cylinder G built by replicating $(k - 1)$ times the pattern shown in the gray square of Figure 18. G contains $(1 + k)$ gray vertices.

$$\frac{mn + 2m}{5} + \frac{n - 19}{5}\cdot$$

To give an upper bound for $\gamma(C_m\Box P_n)$ if $m \equiv 3 \pmod 5$, we will build a dominating set of size $g(k, n)$ in $C_{3+5k}\Box P_n$, for $k \geq 1$ and $n \geq 2$, where $g(k, n)$ is defined as

$$g(k, n) = (1 + k)n + 2k - 2\cdot$$

Note that if $m = 3 + 5k$, then $g(k, n)$ can be rewritten as

$$\frac{mn + 2m}{5} + \frac{2(n - 8)}{5}\cdot$$

From the definition of $g(k, n)$, it is easy to check that $g(k, n + 1) = g(k, n) + (1 + k)$ and that $g(k, n) = g(k - 1, n) + (n + 2)$. This means that when adding a new row to $C_{3+5k}\Box P_n$, we have to add $(1 + k)$ new vertices to a dominating set in $C_{3+5k}\Box P_n$, and when adding five columns to $C_{3+5(k-1)}\Box P_n$, we have to add $(n + 2)$ new vertices to a dominating set in $C_{3+5k}\Box P_n$.

The method to obtain dominating sets is the same as the one explained in Section 3. For this reason, we only sketch the construction. The base case is shown in Figure 16. We replicate $(k-1)$ times the pattern in the gray square of Figure 16, assuming that the bottom rightmost vertex in that square is also black. This way, the set of black vertices defines a dominating set in $C_{3+5k}\Box P_2$ of size $g(1, 2) + 4(k - 1) = g(k, 2)$ (see Figure 17).

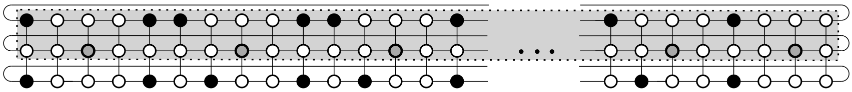


Figure 20: Black and gray vertices form a dominating set of size $g(k,2)+(1+k) = g(k,3)$ in $C_{3+5k}\Box P_3$, after gluing the cylinder G to the first cycle of the cylinder $C_{3+5k}\Box P_2$ of Figure 17. Black vertices in the last row are moved cyclically two positions to the right.

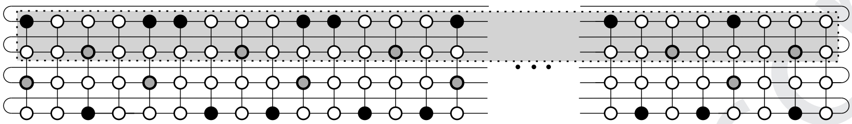


Figure 21: Black and gray vertices form a dominating set of size $g(k,2) + 2(1 + k) = g(k,4)$ in $C_{3+5k}\Box P_4$, after gluing twice the cylinder G.

The base cylinder is depicted in Figure 18, and the cylinder G to be glued in Figure 19. G is obtained by replicating $(k-1)$ times the pattern in the gray square of Figure 18, and contains $(1+k)$ gray vertices. G is glued to the first cycle of the cylinder $C_{3+5k}\Box P_2$ shown in Figure 17. After gluing G, the black vertices in the last cycle are moved cyclically two positions to the right (see Figure 20). G can be glued as many times as required, always using the first cycle of the resulting cylinder, and moving cyclically all black/gray vertices in the rows below G two positions to the right. Figure 21 shows a dominating set in $C_{3+5k}\Box P_4$ of size $g(k,2) + 2(1+k) = g(k,4)$, after gluing twice G.

From the previous discussion, if $m \equiv 3 \pmod 5$, $m \geq 30$ and $n \geq 22$, then

$$\frac{mn+2m}{5} + \frac{n-19}{5} \leq \gamma(C_m\Box P_n) \leq \frac{mn+2m}{5} + \frac{2(n-8)}{5}$$

and the theorem follows. □

5.2 Bounds when m ≡ 4 (mod 5)

Theorem 5.2. *If* $m \equiv 4 \pmod 5$, $m \geq 30$ *and* $n \geq 22$, *then*

$$\frac{mn+2m}{5} + \frac{9n-152}{65} \leq \gamma(C_m\Box P_n) \leq \frac{mn+2m}{5} + \frac{n-13}{5}.$$

Proof. In this case, to give a lower bound for $\gamma(C_m\Box P_n)$ if $m \equiv 4 \pmod 5$ and $26 \leq n = 13h + \alpha$, with $0 \leq \alpha \leq 12$, we choose the same partitions as described in Section 5.1: If $\alpha = 0$, we choose $n_1 = n_r = 13$ (the sizes of the border parts) and $h-2$ internal parts of size 13, for $1 \leq \alpha \leq 9$ we choose $n_1 = n_r = 11$, $h-2$ internal parts of size 13 and an internal part of size $4+\alpha$, and for $\alpha = 10, 11, 12$ we increase the sizes of the border parts to 12 or 13 and choose $h-1$ internal parts of size 13. For $n = 22, 23, 24, 25$, we choose two border parts of size $11, 12$, or 13.

Using the values in Tables 4 and 6 to compute L, the worst case is $\alpha = 1$, for which we obtain $L = 2m + \frac{9n-152}{13}$. Therefore, if $m \equiv 4 \pmod 5$, $m \geq 30$ and $n \geq 22$, we have the following lower bound for $\gamma(C_m\Box P_n)$

$$\frac{mn+2m}{5} + \frac{9n-152}{65}.$$

(a) A dominating set of size $g'(2,3) = 12$ in $C_{4+5\cdot 2}\square P_3$.

(b) The base cylinder.

Figure 22: Base cases for n odd.

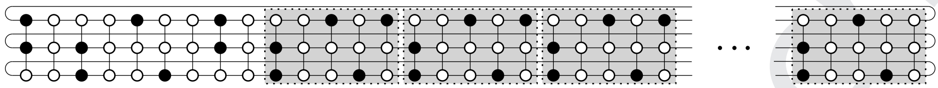


Figure 23: A dominating set of size $g'(2,3) + (k-2)5 = g'(k,3)$ in $C_{4+5k}\square P_3$, obtained after replicating $(k-2)$ times the pattern in the gray square in Figure 22a (and adding an extra black vertex in the top rightmost corner of the pattern).

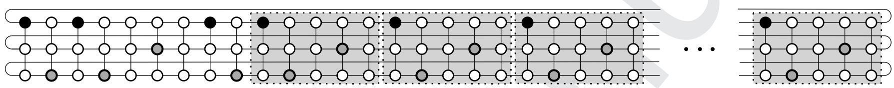


Figure 24: The cylinder G built by replicating $(k-2)$ times the pattern shown in the gray square of Figure 22b. G contains $2(k-2)+6 = 2(1+k)$ gray vertices.

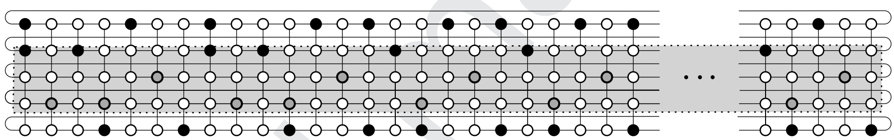


Figure 25: Black and gray vertices form a dominating set of size $g'(k,3) + 2(k+1) = g'(k,5)$ in $C_{4+5k}\square P_5$, after gluing the cylinder G to the second cycle in $C_{4+5k}\square P_3$ of Figure 23. Black vertices in the last row are moved cyclically one position to the right.

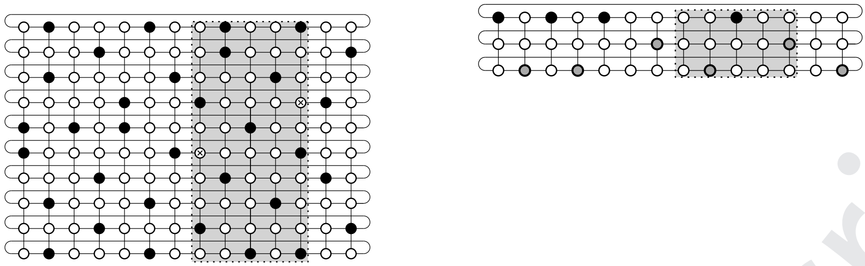
For an upper bound for $\gamma(C_m\square P_n)$ when $m \equiv 4 \pmod 5$, we define $g'(k,n)$ as

$$g'(k,n) = (k+1)n + 2k - 1$$

for $k \geq 1$ and $n \geq 2$, and we show how to build a dominating set of size $g'(k,n)$ in $C_{4+5k}\square P_n$ when $k \geq 2$ and $n \geq 3$, if n is odd, and when $k \geq 2$ and $n \geq 10$, if n is even. Note that $g'(k,n) = g'(k-1,n) + (n+2)$ and $g'(k,n+2) = g'(k,n) + 2(1+k)$, so when adding two new rows to $C_{4+5k}\square P_n$, we have to add $2(1+k)$ new vertices to a dominating set in $C_{4+5k}\square P_n$, and when adding five columns to $C_{4+5(k-1)}\square P_n$, we have to add $(n+2)$ new vertices to a dominating set in . Besides, if $m = 4+5k$, then $g'(k,n)$ can be rewritten as

$$\frac{mn+2m}{5} + \frac{n-13}{5}.$$

Since the method to build dominating sets is the same as described in the rest of the sections, we only sketch the method using figures. Figures 22-25 summarize the construction when n is odd, and Figures 26-29 when n is even.



(a) A dominating set of size $g'(2, 10) = 33$ in $C_{4+5\cdot 2}\Box P_{10}$.

(b) The base cylinder.

Figure 26: Base cases for n even.

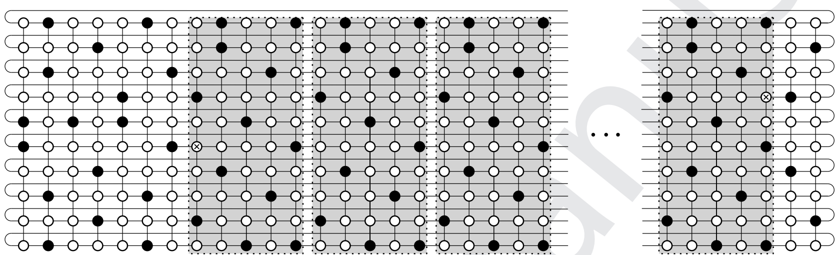


Figure 27: A dominating set in $C_{4+5k}\Box P_{10}$ of size $g'(2, 10) + 12(k - 2) = g'(k, 10)$, obtained after replicating $(k-2)$ times the pattern shown in the gray square of Figure 26a.

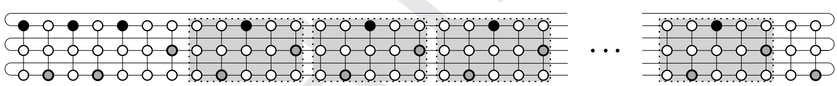


Figure 28: The cylinder G, after replicating $(k - 2)$ times the pattern shown in the gray square of Figure 26b. The number of gray vertices is $6 + 2(k - 2) = 2(1 + k)$.

Therefore, if $m \equiv 4 \pmod 5$, $m \geq 30$ and $n \geq 22$, we have

$$\frac{mn + 2m}{5} + \frac{9n - 152}{65} \leq \gamma(C_m\Box P_n) \leq \frac{mn + 2m}{5} + \frac{n - 13}{5}$$

and the theorem holds. □

6 Conclusions

In this paper, we have computed $\gamma(C_m\Box P_n)$ when $m \equiv 1 \pmod 5$, $m \geq 30$ and $n \geq 22$, and we have provided new lower and upper bounds for $\gamma(C_m\Box P_n)$ when $m \equiv 3, 4 \pmod 5$, $m \geq 30$ and $n \geq 22$, which are tighter than the ones known so far. The upper bounds are obtained by building specific dominating sets, while the lower bounds are obtained computationally.

Crevals proved in [3] that $\gamma(C_{4+5\cdot 2}\Box P_n) = 3n + 3$, for $n \geq 14$, $\gamma(C_{4+5\cdot 3}\Box P_n) = 4n + 5$, for $n \geq 19$, $\gamma(C_{4+5\cdot 4}\Box P_n) = 5n + 7$, for $n \geq 24$ and $\gamma(C_{4+5\cdot 5}\Box P_n) = 6n + 9$, for $n \geq 29$. Observe that $g'(2, n) = 3n + 3$, $g'(3, n) = 4n + 5$, $g'(4, n) = 5n + 7$ and $g'(5, n) = 6n + 9$, so we conjecture that if $m \equiv 4 \pmod 5$, the construction provided in Theorem 5.2 to obtain the upper bound gives the exact value of $C_m\Box P_n$.

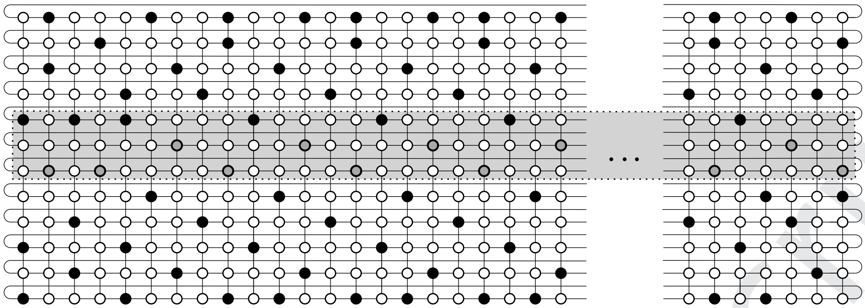


Figure 29: Black and gray vertices form a dominating set in $C_{4+5k}\Box P_{12}$ of size $g'(k,10)+2(1+k) = g'(k,12)$, after gluing G to the fifth cycle in the cylinder $C_{4+5k}\Box P_{10}$ of Figure 27. Black vertices in the rows below G are moved cyclically one position to the left.

Conjecture 6.1. *For $k \geq 2$, $m = 4+5k$ and $n \geq 14$, then*

$$\gamma(C_{4+5k}\Box P_n) = g'(k,n) = \frac{mn+2m}{5} + \frac{n-13}{5}.$$

To prove the conjecture using our approach, it would be necessary to increase the lower bound

$$\frac{mn+2m}{5} + \frac{9n-152}{65}$$

given in Theorem 5.2. To this end, if we want to use the same method implemented in Algorithm 2, we should compute the wasted domination of internal almost dominating sets of size $t \geq 14$, because the best lower bound we can achieve using internal subgraphs of size at most 13 is the previous one. From a computational point of view, the number of suitable words in case $t = 14$ is 275808 and the matrix size is expected to increase substantially, possibly reaching around 1 TB or more. Execution time may also extend into days or weeks, given the exponential growth trend observed from previous cases. This reflects the escalating computational demands with increasing problem size, highlighting the need for efficient resource management in CUDA programming on high-performance GPUs like the NVIDIA Tesla V100. We should approach the problem using another parallel programming method that uses multiple GPUs simultaneously, or even a heterogeneous programming method that uses both GPUs and high-performance generic processors simultaneously.

Finally, Crevals also showed in [3] that

- $\gamma(C_{3+5\cdot 1}\Box P_n) = \lceil \frac{18n+10}{10} \rceil$, for $n \geq 8$;
- $\gamma(C_{3+5\cdot 2}\Box P_n) = \lceil \frac{45n+29}{16} \rceil$, for $n \geq 13$;
- $\gamma(C_{3+5\cdot 3}\Box P_n) = \lceil \frac{84n+92}{22} \rceil$, for $n \geq 18$;
- $\gamma(C_{3+5\cdot 4}\Box P_n) = \lceil \frac{135n+171}{28} \rceil$, for $n \geq 23$;
- $\gamma(C_{3+5\cdot 5}\Box P_n) = \lceil \frac{198n+274}{34} \rceil$, for $n \geq 28$.

Depending on the parity of n, we have to add one unit to the previous formulae. From them, a closed formula for $\gamma(C_{3+5k}\Box P_n)$ is unclear. The ratios $\frac{18}{10}$, $\frac{45}{16}$, $\frac{84}{22}$, $\frac{135}{28}$ and $\frac{198}{34}$ can be expressed as

$$\frac{3+9k+6k^2}{6k+4}, 1 \leq k \leq 5.$$

However, we have not found any common expression for $\frac{10}{10}$, $\frac{29}{16}$, $\frac{92}{22}$, $\frac{117}{28}$ and $\frac{274}{34}$. This leads us to pose the following conjecture.

Conjecture 6.2. *For $k \geq 2$, $m = 3+5k$ and $n \geq 13$, then*

$$\gamma(C_{3+5k} \square P_n) = \left\lceil \frac{(3+9k+6k^2)n}{6k+4} \right\rceil + O(k)\cdot$$

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