A novel approach to pessimistic bilevel problems. An application to the rank pricing problem with ties

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ABSTRACT
This paper introduces a novel method to address the pessimistic approach to the bilevel problem. It consists of considering a lexicographic biobjective optimisation problem at the lower level. To emphasise the significance of this approach, we implement it in the context of the Rank Pricing Problem with Ties. This problem can be formulated as a bilevel problem that inherently demands the use of the pessimistic approach. Considering the properties of the lexicographic biobjective problem involved, we formulate this problem as a single level mixed integer optimisation problem, deriving also valid values for the big-Ms involved and valid inequalities for this formulation. The computational experiment carried out confirms the relevance of the proposed method.

KEYWORDS
Bilevel optimisation; pessimistic approach; lexicographic biobjective optimisation; rank pricing problems.
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1. Addressing pessimistic bilevel problems through a novel approach

Bilevel optimisation models involve two decision-makers within a hierarchical framework. Each of these decision makers manages a subset of variables and seeks to optimise his/her respective objective functions while fulfilling certain constraints. The lower level (LL) decision maker, or follower, performs optimisation with full awareness of the values assigned to the variables controlled by the upper level (UL) decision maker, or leader. The UL decision-maker, having complete information about the LL decision-maker's reactions, selects variable values to optimise his/her own objective function. Bilevel optimisation can be computationally challenging due to the interdependence between the levels and the need to find optimal solutions for both levels

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simultaneously. But, at the same time, this interaction between both levels makes bilevel optimisation relevant in many real-world applications, such as supply chain management, transportation planning, and pricing strategies, since it can effectively represent systems involving hierarchical decision-making.

In general, a bilevel optimisation model can be formulated as:

$$
\begin{array}{ll}
\text{“}\min_{x}\text{”} & F(x,y) \\
\text{subject to} & \\
 & G_j(x,y) \leqslant 0, \;\; j=1,\dots,q \\
\multicolumn{2}{l}{\text{where, for every } x \text{ fixed, } y \text{ solves}} \\
\min_{y} & f(x,y) \\
\text{subject to} & \\
 & g_h(x,y) \leqslant 0, \;\; h=1,\dots,p
\end{array} \tag{1}
$$

where $x \in \mathbb{R}^n$ are the UL variables controlled by the leader, and $y \in \mathbb{R}^m$ are the lower level variables controlled by the follower; $F, f : \mathbb{R}^{n+m} \longrightarrow \mathbb{R}$ are the UL and LL objective functions, respectively; and $G_j(x,y) \leqslant 0$, $j = 1,\dots,q$, and $g_h(x,y) \leqslant 0$, $h = 1,\dots,p$, indicate, in their respective cases, the constraints associated with UL and LL the upper and lower levels.

Bilevel problems are difficult to manage and solve because of their nonconvex nature. In addition, complications arise when the set of optimal solutions of the LL problem, called $M(x)$, is not a singleton for certain values of x. If the UL objective function is sensitive to different values of $y \in M(x)$, it becomes necessary to establish a rule for selecting $y^* \in M(x)$ in order to evaluate F. Quotation marks have been employed to convey the uncertainty in defining the bilevel problem when lower-level optimal solutions are not uniquely determined. Several assumptions have been proposed in the literature, with the most common being the optimistic or weak approach. This approach assumes that the UL decision maker has the possibility to influence the LL decision maker so that the latter selects y^* that yields the best possible value of F. In this case, the UL objective function is minimised with respect to both x and y. The pessimistic or strong approach, which assumes that the LL decision maker always selects the optimal solution which provides the worst value of F, is notably more challenging to handle than the optimistic one. The main findings concerning optimality of bilevel problems, algorithms for solving them, applications and related topics can be found in [1–7] and the references therein.

This paper proposes a novel approach to deal with pessimistic bilevel problems.

Assuming that the bilevel problem has an optimal solution, to ensure the pessimistic approach we propose to consider a lexicographic biobjective problem associated to the LL decision maker. Therefore, the pessimistic approach to the bilevel problem (1) presented in this paper consists of reformulating it as:

$$\begin{array}{ll} \min\limits_{x,y} & F(x,y) \\ \text{subject to} & \\ & G_j(x,y) \leqslant 0, \;\; j=1,\ldots,q \\ \multicolumn{2}{l}{\text{where, for every } x \text{ fixed, } y \text{ solves}} \\ \operatorname{lex}\min\limits_{y} & (f(x,y),\, -F(x,y)) \\ \text{subject to} & \\ & g_h(x,y) \leqslant 0, \;\; h=1,\ldots,p \end{array} \tag{2}$$

Lexicographic optimisation [8] is based on the assumption that objectives are ranked in order of importance, and the objective functions are optimised one at a time in a prioritised manner. Therefore, for given values of the variables x, $f(x,y)$ is minimised first. Then, $F(x,y)$ is maximised while ensuring that the optimum objective function value is achieved with respect to the first criterion. To accomplish this, an additional constraint can be added to guarantee the optimal value of $f(x,y)$. In cases where the LL problem (first criterion) has multiple optimal solutions, choosing the optimal solution for the second criterion guarantees the selection of the worst solution for the UL decision maker, so applying the pessimistic approach. Also, it is worth noting that the UL objective function now minimises over x and y. Bearing in mind that the pessimistic approach is ensured by the lexicographic approach on the LL problem, minimising over y in the bilevel problem occurs when, for any given value of x, there are multiple values of y that yield the same minimum value of $f(x,y)$ and the same maximum value of $F(x,y)$ (as required by the lexicographic approach). Hence, these feasible bilevel solutions provide the same value of the UL objective function. In such scenarios, the selection of the feasible bilevel solution can be left to the UL decision maker, as it does not affect the value of any of the objective functions. While the reformulation proposed in problem (2) can be applied to any pessimistic bilevel problem, its effectiveness will, of course, hinge upon the characteristics and favourable properties of the resulting problem. To demonstrate its efficacy, we have selected a problem from the literature whose characteristics make it particularly well-suited for being reformulated as a single level problem using this approach. On the other hand, it is worth pointing out that this approach can also be effectively employed in the development of metaheuristic algorithms, as it facilitates the generation of feasible bilevel solutions from upper-level variable values through solving the lexicographic optimisation problem at the lower level.

To highlight the relevance of this approach, we apply it to the Rank Pricing Problem with Ties (RPPT) [9], an optimisation problem which admits a bilevel formulation that inherently requires the pessimistic approach. Moreover, the properties of the resulting LL lexicographic problem allow for a compact formulation of the RPPT as a single level optimisation problem. The aim of Rank Pricing Problems (RPP), as introduced by Rusmevichientong [10] and Rusmevichientong, Van Roy, and Glynn [11], is to determine the prices of a range of products aiming to maximise a company's revenue. They assume that each customer has a budget and a ranked list of products of interest, and is looking to acquire at most one unit of a single product. Setting a lower price can result in a loss of income if customers would have been willing to pay a higher price, but it can also make the product accessible to a larger number of customers. In contrast, a higher price can generate greater revenue, but customers may be prevented from buying it if the price is too high. As a consequence, these problems exhibit a clear hierarchical structure with two interdependent levels. On the one hand, the company, at the upper level, must make decisions regarding the prices, taking into account the response that those prices will elicit in the customer purchase decisions. On the other hand, in the LL problem, each customer decides the product to purchase. The RPP [12] assumes that each customer establishes strict preferences over products, meaning that no two or more products are equally preferred. Then, for each set of prices given by the UL decision maker, the LL problem has unique optimal solution i.e. the customer selects the most preferred product he/she can afford. In this case, the bilevel problem is well-posed. In the RPPT [9] ties are allowed, i.e., for each customer there can be several products with the same preference and therefore equally favoured by the customer. In this case, following what is usually general human behaviour, the customer chooses the cheapest among those whose preference is the highest and can afford to purchase. Therefore, the nature of this bilevel problem calls for addressing it with the pessimistic approach, as the company cannot compel the customer to purchase the most expensive product when the customer derives maximum satisfaction from several products.

The key contributions of this paper are the following:

- We propose to address the pessimistic approach to a bilevel problem by reformulating the LL problem as a lexicographic biobjective optimization model. Bearing in mind model (2), in which each level of decision making minimises its objective function, the highest priority objective function is the LL objective function and the second highest priority is the UL objective function reversed in sign.

- We introduce the formulation of the RPPT as a bilevel problem, which requires the pessimistic approach to be solved, and formally reformulate it as a single level mixed integer optimisation problem, based on the properties of the lexicographic biobjective problem involved in the LL problem.
- We derive valid values for some big Ms involved in the single level reformulated model as well as some valid inequalities for this model.

The remainder of this paper is structured as follows. Section 2 describes the RPPT. Section 3 presents the pessimistic bilevel formulation of the RPPT. Taking into account the properties of the lexicographic biobjective LL problem, section 4 reformulates the problem as a single level mixed integer optimisation model. In section 5 valid values for the big Ms involved and valid inequalities are derived. Section 6 presents the results of the extensive computational experiments conducted, evaluating several variants of the reformulated single level model. The outcomes yielded by this approach indicate that it is competitive with the previous formulations of the RPPT proposed in the literature.

2. Rank pricing problems

The RPP [10–14] involves determining the price of multiple products assuming that each customer has a budget, wants to purchase a unit of a single product, and possesses his/her own ranking of the available products, resulting in incomplete preference lists. Moreover, preferences are assumed to be strict, i.e. no ties are allowed and there is an unlimited supply of products. The flexibility of the ranked-based model lies in its ability to incorporate a variety of product characteristics, apart from price, into the customer's decision-making process. The use of preferences is also frequent, for instance, in location problems. Hanjoul and Peeters [15] and Cánovas et al. [16] assume that customers have ranked the facilities based on preferences influenced by their personal characteristics and the attributes of the sites and trips to those sites. Hansen et al. [17], Vasilyev and Klimentova [18] and Vasilyev et al. [19], assuming that the preference ranking is strictly ordered, propose a bilevel formulation of that problem in which the upper level decision maker selects the facilities, while in the lower level problem, customers are allocated aiming to minimise the sum of preferences (the lower the value, the higher the preference). Camacho-Vallejo et al. [20] introduce an evolutionary algorithm for addressing the bilevel model. Finally, Calvete et al. [21] extend that model by introducing a capacity constraint on the number of customers assigned to each facility.

Calvete et al. [12] propose two different formulations of the RPP. The first one, following the intuitive notion provided by the two levels of decision-making, formulates the RPP as a bilevel multi-follower optimisation model with independent followers. Since the RPP does not allow for ties in customer preferences, the second-level problem has a unique optimal solution, ensuring that the bilevel problem is well-posed. The second formulation is based on the fact that each customer purchases the product he/she prefers the most among the products he/she can afford, resulting in a single-level non-linear optimisation model. Both formulations are transformed into binary linear optimisation models.

This paper focuses on the bilevel approach to the generalisation of the RPP known as the RPPT, in which customers are allowed to have indifference among candidate products, and ties are permitted in their preference lists. This problem was introduced by Dominguez et al. [9]. Although they mention that there is an implicit bilevel framework associated to the problem, they did not address it as a bilevel problem. Instead, they proposed a mathematical formulation involving three indices and developed two distinct resolution methods. One formulation is based on projecting out the customer decision variables, thus resulting in a streamlined formulation. The second one adopts a Benders decomposition approach which takes advantage of the separability of the problem. Both approaches were strengthened with valid inequalities. Considering the bilevel structure of the RPPT, when ties exist in customer preferences it cannot be guaranteed that a unique optimal solution exists to the LL problem. In this case, i.e. if there are several products that a customer can afford and that satisfy him the most, he/she will choose the cheapest among them, thereby providing the least revenue to the company. Therefore, to address the bilevel optimisation problem, a pessimistic approach needs to be considered. From now on, we denote this model as the Pessimistic Bilevel Rank Pricing Problem with Ties (PB-RPPT). As far as we are aware, the PB-RPPT remains unexplored in the existing literature.

Regarding other papers related to the RPP, Domínguez et al. [22] propose another extension of the problem in which they assume that the amount of available products is limited. After comparing the envy-free allocation of products with the envy approach, they focus on the second one and propose two formulations of the problem as mixed integer linear optimisation models, deriving also valid inequalities. The computational study shows the performance of the formulations. Additionally, Ansari [23] presents a bilevel model for an extension of the RPP which involves both customer utility and rank, implying that customers make decisions considering their preferences and potential savings. To address this issue, they reformulate the problem as a single-level problem and devise two algorithms: one based on Scatter Search and the other on price perturbation.

3. Bilevel formulation of the RPPT: The PB-RPPT

As mentioned above, the RPPT involves determining the price of multiple products assuming that there is an unlimited supply of products and that each customer has a budget and intends to purchase a unit of a single product. In order to formulate the PB-RPPT, we introduce in Table 1 the notations used.

Table 1. Notations used to formulate the PB-RPPT

Sets	
K	Set of customers. $K = \{1, \dots, \lvert K\rvert\}$.
I	Set of products. $I = \{1, \dots, \lvert I\rvert\}$.
$S^k \subseteq I$	Subset of products in which customer $k \in K$ is interested.
Indices	
$k \in K$	Index of customer.
$i, j \in I$	Index of product.
Parameters	
$b^k > 0$	Budget of customer $k \in K$.
$s_i^k > 0$	Value of the preference assigned by customer $k \in K$ to product $i \in S^k$.
UL variables	
$p_i \geqslant 0$	Price of product $i \in I$.
LL variables	
$x_i^k \in \{0, 1\}$	If customer $k \in K$ decides to buy product $i \in S^k$, $x_i^k = 1$. Otherwise, $x_i^k = 0$.

Depending on his/her personal interests, as well as the features of the products, each customer $k \in K$ has ranked the products, S^k, he/she is interested in from worst to best, i.e., has a set of predefined nonnegative preferences s_i^k, $i \in S^k$. It is assumed that the greater the number, the greater the preference, i.e. $s_i^k > s_j^k$ implies that customer k prefers product i over product j, where $i, j \in S^k$. If $s_i^k = s_j^k$ customer k has an equal preference for both products. Hence, the customer k selects one product from S^k with the highest preference that he/she can afford. Note that customers are not required to establish strict preferences for the products, which means they may have equal preferences for two or more of them. In the event of a tie, i.e., if two or more products are equivalent in terms of their appeal to a customer, he/she, as it is common for individuals, chooses the cheapest product among those he/she equally prefers. If a customer cannot afford any product, he/she does not make a purchase. We also assume that each customer is interested in some product, i.e. $S^k \neq \emptyset$, $k \in K$. Otherwise, the customer may be eliminated from the study. Similarly, we assume that all products are on some customer's preference list, i.e. for every $i \in I$, there exists $k \in K$ such that $i \in S^k$. Otherwise, the product could be eliminated from the study.

In the bilevel approach of the RPPT proposed in this paper, the UL decision maker decides on product pricing, i.e., on the value of variables $\{p_i\}_{i \in I}$, while each of the LL decision makers decides on product purchasing, i.e. on the value of variables $\{x_i^k\}_{k \in K, i \in S^k}$. Notice that there are as many LL decision makers as customers. Hence, the RPPT can be formulated as the following bilinear-linear bilevel mixed integer optimisation problem with multiple independent followers:

$$\text{“}\max_p\text{”} \quad \sum_{k \in K} \sum_{i \in S^k} p_i x_i^k \tag{3a}$$

subject to

$$p_i \geqslant 0, \qquad i \in I \tag{3b}$$

where, for each customer $k \in K$, the variables $\{x_i^k\}_{i \in S^k}$ solve

$$\max_x \quad \sum_{i \in S^k} s_i^k x_i^k \tag{3c}$$

subject to

$$\sum_{i \in S^k} x_i^k \leqslant 1 \tag{3d}$$

$$\sum_{i \in S^k} p_i x_i^k \leqslant b^k \tag{3e}$$

$$x_i^k \in \{0, 1\} \qquad i \in S^k \tag{3f}$$

The UL objective function (3a) maximises the revenue of the company. Constraints (3b) ensure the requirements of the price variables. The LL problem corresponding to the customer $k \in K$ is defined by (3c)-(3f). The LL objective function (3c) maximises the preference of the product chosen by the customer k. Constraint (3d) guarantees that the customer k chooses at most one product from his/her preference list. Constraint (3e) enforces customer k to purchase only among the products on his/her preference list that cost less than or equal to the available budget. Constraints (3f) ensure that the variables x_i^k, $k \in K$, $i \in S^k$ are binary. Notice that, due to the unlimited supply assumption, the LL problems are independent in the sense that each of them involves only the UL variables and the LL decision variables of the corresponding customer [24].

When there are no ties, as assumed in the RPP studied in [12], each of the LL problems has a unique optimal solution. This property allows us to ensure that the bilevel problem is well-posed. However, the RPPT allows ties, which means that each LL problem can have multiple optimal solutions, for given values of the prices. This issue raises concerns in bilevel optimisation as it can lead to an ill-posed model. The optimistic approach would result in the customer choosing the most expensive among several products with the same (and higher) preference that he/she can afford. However, this contradicts the typical behaviour of customers who tend to seek the best product at the lowest possible price. Therefore, the appropriate approach to solving the RPPT is the pessimistic approach where customers act in a manner that is contrary to the interests of the company and instead prioritise their own benefit.

To illustrate these issues, Table 2 displays an instance of the RPPT with 12 customers and 8 products. The inner part of the table shows the preferences assigned by each customer to every product, while the budget of each customer is shown in the last column. When the problem is solved using the optimistic approach, the prices assigned to the products in the optimal solution are shown in the second to last row of the table. The products purchased by customers are identified with a blue dot in the inner part of the table. Note that customer k_1 is forced to choose product i_3 whereas his/her choice would be product i_4 which has the same priority and is cheaper. Hence, this is not an achievable solution. The optimal prices of the products when the pessimistic approach is applied are shown in the last row of the table and the products purchased are identified with a red dot. Notice that customers k_1 to k_{10} purchase the cheapest product among those they prefer the most and can afford within their budget. Moreover, with these prices, customers k_{11} and k_{12} cannot buy any product.

As pointed out in section 1, the novel approach to handle pessimistic bilevel problems consists of considering the k-th LL problem, $k \in K$, as a lexicographic biobjective optimisation problem, i.e., in the problem (3c)-(3f), the objective (3c) should be sub-

Table 2. An instance of the RPPT with 12 customers and 8 products.

	Products								Budget
Customers	i_1	i_2	i_3	i_4	i_5	i_6	i_7	i_8	
k_1	7	5	🔵 8	🔴 8	7	6	-	7	99
k_2	6	🔵 🔴 8	5	3	7	-	6	4	78
k_3	🔵 🔴 8	-	-	8	-	8	7	-	70
k_4	🔵 🔴 8	4	5	6	3	2	7	6	70
k_5	5	6	6	🔵 8	🔴 7	7	-	5	63
k_6	-	🔴 8	-	🔵 8	-	-	-	-	57
k_7	-	4	6	5	7	4	🔵 🔴 8	7	43
k_8	-	6	8	8	🔵 7	-	-	🔴 6	42
k_9	4	1	5	3	2	6	7	🔵 🔴 8	37
k_{10}	7	5	-	4	🔵 8	-	🔴 6	-	30
k_{11}	6	5	7	8	3	🔵 4	-	-	27
k_{12}	7	7	-	8	6	-	6	🔵 8	20
	Product prices in the optimal solution								
Optimistic 🔵	70	78	99	57	30	27	43	20	
Pessimistic 🔴	70	57	99	99	63	99	30	37	

stituted by

$$\operatorname*{lex\,max}_{x} \left(\sum_{j \in S^k} s_j^k x_j^k, \qquad -\sum_{j \in S^k} p_j x_j^k \right) \tag{4}$$

The first objective aims to maximise the preference of the k-th customer, while the second objective aims to minimise the price paid for the product.

Rusmevichientong et al. [11] state that there is an optimal solution of the RPP such that only distinct budgets need to be considered for price selection. This result can be directly extended to the RPPT [9]. Based on these findings, and bearing in mind that each product has at most one price, prices can be expressed in terms of budgets, as done in [12]. For this purpose, let us assume that budgets are sorted in increasing order and let L be the set of indices of different budgets, $L = \{1, \ldots, |L|\}$. The indices in L are consistent with that order, i.e., $l_1 < l_2$ if $b^{l_1} < b^{l_2}$. We also define the function $\sigma\colon K \longrightarrow L$ which maps the customer k to the index l corresponding to the position that his/her budget occupies in the ordered set of distinct budgets. With this definition, from now on, $b^{\sigma(k)}$ refers to the budget of customer k. Let v_i^l be a binary variable which takes value 1 if product i has price b^l; otherwise v_i^l is equal to 0. Using these variables, the price p_i can be written as:

$$p_i = \sum_{l=1}^{|L|} b^l v_i^l, \qquad \text{where} \qquad \sum_{l=1}^{|L|} v_i^l \leqslant 1, \qquad i \in I$$

On the other hand, customer k can only buy a product that is on his/her preference list and whose price is less than or equal to his/her budget. Hence,

$$x_i^k \leqslant \sum_{l=1}^{\sigma(k)} v_i^l, \qquad i \in S^k \tag{5}$$

Given the above considerations, the PB-RPPT can be formulated as the following bilinear linear bilevel optimisation problem with binary variables, multiple independent followers and a lexicographic biobjective function at the LL of decision-making:

$$\max_{v,x} \qquad \sum_{k \in K} \sum_{j \in S^k} \left(\sum_{l=1}^{\sigma(k)} b^l v_j^l \right) x_j^k \tag{6a}$$

subject to

$$\sum_{l=1}^{|L|} v_i^l \leqslant 1, \qquad i \in I \tag{6b}$$

$$v_i^l \in \{0,1\}, \qquad i \in I, \quad l \in L \tag{6c}$$

where, for each customer $k \in K$, the variables $\{x_i^k\}_{i \in S^k}$ solve

$$\operatorname{lex}\max_{x} \qquad \left(\sum_{j \in S^k} s_j^k x_j^k, \qquad - \sum_{j \in S^k} \left(\sum_{l=1}^{\sigma(k)} b^l v_j^l \right) x_j^k \right) \tag{6d}$$

subject to

$$\sum_{j \in S^k} x_j^k \leqslant 1 \tag{6e}$$

$$x_i^k \leqslant \sum_{l=1}^{\sigma(k)} v_i^l, \qquad i \in S^k \tag{6f}$$

$$x_i^k \in \{0,1\}, \qquad i \in S^k \tag{6g}$$

Note that the UL objective function (6a) maximises over v and x, as indicated when formulating problem (1). Bearing in mind that the pessimistic approach is guaranteed by (6d), maximisation over x only occurs when there are multiple products that the customer can buy having the same priority, which is the highest, and having the same price, which is the lowest. In other words, the problem LL has multiple optima with several of these products having the lowest price. In such cases, the control over the product purchased by the customer can be left to the UL decision maker, as it does not affect the value of any of the objective functions.

In order to exactly solve the PB-RPPT, in the next section we reformulate it as a single level optimisation problem by transforming the LL problems corresponding to each customer.

4. A single level reformulation of the PB-RPPT

Let us consider the LL problem (6d)-(6g) corresponding to the k-th customer, for a given value of the UL variables $\{v_i^l\}_{i\in I, l\in L}$. Based on these values, product prices can be determined and it is possible to know which products each customer can afford. Let $I(k) \subseteq S^k$ be the subset of products that customer k is interested in and can purchase with his/her budget, $I(k) = \{i \in S^k \colon \sum_{l=1}^{\sigma(k)} v_i^l = 1\}$. Note that $x_i^k = 0$ for $i \in S^k \setminus I(k)$, since the price of these products is greater than the k-th customer's budget. Hence, the LL problem corresponding to the k-th customer can be written as:

$$\operatorname*{lex\,max}_{x} \quad \left(\sum_{j\in I(k)} s_j^k x_j^k, \qquad - \sum_{j\in I(k)} \left(\sum_{l=1}^{\sigma(k)} b^l v_j^l \right) x_j^k \right) \tag{7a}$$

subject to

$$\sum_{j\in I(k)} x_j^k \leqslant 1 \tag{7b}$$

$$x_i^k \in \{0,1\}, \qquad i \in I(k) \tag{7c}$$

Since lexicographic optimisation takes into consideration one objective at a time, let us consider the maximisation of problem (7a)-(7c) with respect to the first objective function:

$$\max_{x} \quad \sum_{j\in I(k)} s_j^k x_j^k \tag{8a}$$

subject to

$$\sum_{j\in I(k)} x_j^k \leqslant 1 \tag{8b}$$

$$x_i^k \in \{0,1\} \qquad i \in I(k) \tag{8c}$$

Its optimal solution is the product $i \in I(k)$ with the largest preference (or any of the products tied if there are several products with the same largest preference), i.e., such that $\sum_{j\in I(k)} s_j^k x_j^k \geqslant s_i^k$.

Let us consider the linear relaxation of problem (8):

$$\max_{x} \quad \sum_{j\in I(k)} s_j^k x_j^k \tag{9a}$$

subject to

$$\sum_{j\in I(k)} x_j^k \leqslant 1 \tag{9b}$$

$$x_i^k \geqslant 0 \qquad i \in I(k) \tag{9c}$$

Taking into account the characteristics of the constraint (9b), the linear optimisation problem (9) has an optimal solution which takes integer values, and so solves problem (8). The lexicographic approach then maximises with respect to the second objective, which, basically, selects the most economical product from the tied ones among the affordable and most preferred by the customer k. As pointing out above, in the event that there are still tied products among that most economical ones, we have left it up to the UL decision maker to select from among these products the one that the customer k purchases since neither the UL objective function nor the LL one have their optimal values modified. Based on previous statements, problem (10a)-(10d) includes among its possible alternative optima an optimal solution of problem (7a)-(7c):

$$\min_{x} \quad \sum_{j\in I(k)} \left(\sum_{l=1}^{\sigma(k)} b^l v_j^l \right) x_j^k \tag{10a}$$

subject to

$$\sum_{j\in I(k)} x_j^k \leqslant 1 \tag{10b}$$

$$\sum_{j\in I(k)} s_j^k x_j^k \geqslant s_i^k, \quad i \in I(k) \tag{10c}$$

$$x_i^k \geqslant 0, \quad i \in I(k) \tag{10d}$$

where constraints (10c) guarantee the optimality of the solution with respect to the first objective function.

The dual of problem (10) is:

$$\max_{y,w} \quad -y^k + \sum_{j\in I(k)} s_j^k w_j^k \tag{11a}$$

subject to

$$-y^k + s_i^k \sum_{j\in I(k)} w_j^k \leqslant \sum_{l=1}^{\sigma(k)} b^l v_i^l, \qquad i \in I(k) \tag{11b}$$

$$y^k \geqslant 0 \tag{11c}$$

$$w_i^k \geqslant 0, \qquad i \in I(k) \tag{11d}$$

where y^k is the dual variable associated with the constraint (10b) and $\{w_i^k\}_{i\in I(k)}$ are the dual variables associated with constraints (10c). As problem (10) possesses an optimal solution, the dual problem also has an optimal solution, and the optimal objective function values for both problems coincide. Hence, by applying duality theory, $\{x_i^k\}_{i\in I(k)}$ and $\{y^k, w_i^k\}_{i\in I(k)}$ are optimal solutions to their respective problems if and only if:

$$\sum_{j\in I(k)} \left(\sum_{l=1}^{\sigma(k)} b^l v_j^l\right) x_j^k = -y^k + \sum_{j\in I(k)} s_j^k w_j^k \tag{12a}$$

$$\sum_{j\in I(k)} x_j^k \leqslant 1 \tag{12b}$$

$$\sum_{j\in I(k)} s_j^k x_j^k \geqslant s_i^k, \qquad i \in I(k) \tag{12c}$$

$$-y^k + s_i^k \sum_{j\in I(k)} w_j^k \leqslant \sum_{l=1}^{\sigma(k)} b^l v_i^l \qquad i \in I(k) \tag{12d}$$

$$x_i^k \geqslant 0, \qquad i \in I(k) \tag{12e}$$

$$w_i^k \geqslant 0, \qquad i \in I(k) \tag{12f}$$

$$y^k \geqslant 0 \tag{12g}$$

Substituting the value of the variable y^k obtained from equation (12a) in constraints (12d) and (12g), we obtain:

$$\sum_{j\in I(k)} x_j^k \leqslant 1 \tag{13a}$$

$$\sum_{j\in I(k)} s_j^k x_j^k \geqslant s_i^k, \qquad i \in I(k) \tag{13b}$$

$$\sum_{j\in I(k)} \left(\sum_{l=1}^{\sigma(k)} b^l v_j^l\right) x_j^k - \sum_{j\in I(k)} s_j^k w_j^k + s_i^k \sum_{j\in I(k)} w_j^k \leqslant \sum_{l=1}^{\sigma(k)} b^l v_i^l, \qquad i \in I(k) \tag{13c}$$

$$\sum_{j\in I(k)} s_j^k w_j^k - \sum_{j\in I(k)} \left(\sum_{l=1}^{\sigma(k)} b^l v_j^l \right) x_j^k \geqslant 0 \tag{13d}$$

$$x_i^k \geqslant 0, \qquad i \in I\,(k) \tag{13e}$$

$$w_i^k \geqslant 0, \qquad i \in I\,(k) \tag{13f}$$

These constraints have been derived by assuming that a value of the UL variables is known and thus only the LL variables corresponding to products in $I(k)$ are needed, since the remaining variables are equal to zero due to constraints (6f). Therefore, to replace the LL problem of the k-th customer by constraints (13) in problem PB-RPPT, we need to include these constraints and handle them when writing the summation in S^k, guaranteeing that they apply when $i \in I(k)$ and do not impose any additional condition when $i \in S^k \setminus I(k)$.

For this purpose, assuming that constraints (6f) are included, next we analyse each constraint in (13):

- Variables w_i^k have been defined only for $i \in I(k)$. We introduce the variables w_i^k for $i \in S^k \setminus I(k)$ and impose that they are equal to zero. Thus, constraints (13f) are replaced by $w_i^k \geqslant 0$, $i \in S^k$, and $w_i^k \leqslant M_i^k \sum_{l=1}^{\sigma(k)} v_i^l$, $i \in S^k$, where M_i^k is a constant big enough to guarantee that these constraints are only restrictive when $i \in S^k \setminus I(k)$. In the following section, we propose values for M_i^k, $i \in S^k, k \in K$.
- Constraint (13a) can be replaced by $\sum_{j\in S^k} x_j^k \leqslant 1$, as only terms equal to zero are added.
- Constraints (13b) can be replaced by $\sum_{j\in S^k} s_j^k x_j^k \geqslant s_i^k \sum_{l=1}^{\sigma(k)} v_i^l$, $i \in S^k$ as $\sum_{j\in S^k} s_j^k x_j^k \geqslant 0$, and $\sum_{l=1}^{\sigma(k)} v_i^l$ is equal to 1 when $i \in I(k)$, and is equal to 0 otherwise.
- In constraints (13c), the summation can be extended over S^k as only zeros are added. Thus, they can be replaced by

$$\sum_{j\in S^k} \left(\sum_{l=1}^{\sigma(k)} b^l v_j^l \right) x_j^k - \sum_{j\in S^k} s_j^k w_j^k + s_i^k \sum_{j\in S^k} w_j^k \leqslant \sum_{l=1}^{\sigma(k)} b^l v_i^l + \widetilde{M}_i^k \left(1 - \sum_{l=1}^{\sigma(k)} v_i^l \right), \; i \in S^k$$

  where $\widetilde{M}_i^k$ are constants big enough to guarantee that the constraints are only restrictive when $i \in I(k)$. In the following section, we propose values for $\widetilde{M}_i^k$, $i \in S^k, k \in K$.
- Constraint (13d) can be substituted by $\sum_{j\in S^k} s_j^k w_j^k - \sum_{j\in S^k} \left(\sum_{l=1}^{\sigma(k)} b^l v_j^l \right) x_j^k \geqslant 0$

as only terms equal to zero are added.

Hence, the PB-RPPT can be reformulated as the following single level mixed integer bilinear optimisation problem:

$$\max_{v,x,w} \quad \sum_{k \in K} \sum_{j \in S^k} \left(\sum_{l=1}^{\sigma(k)} b^l v_j^l \right) x_j^k \tag{14a}$$

subject to

$$\sum_{l=1}^{|L|} v_i^l \leqslant 1, \qquad i \in I \tag{14b}$$

$$\sum_{j \in S^k} x_j^k \leqslant 1, \qquad k \in K \tag{14c}$$

$$x_i^k \leqslant \sum_{l=1}^{\sigma(k)} v_i^l, \qquad k \in K, \quad i \in S^k \tag{14d}$$

$$\sum_{j \in S^k} s_j^k x_j^k \geqslant s_i^k \sum_{l=1}^{\sigma(k)} v_i^l, \qquad k \in K, \quad i \in S^k \tag{14e}$$

$$w_i^k \leqslant M_i^k \sum_{l=1}^{\sigma(k)} v_i^l, \qquad k \in K, \quad i \in S^k \tag{14f}$$

$$\sum_{j \in S^k} s_j^k w_j^k - \sum_{j \in S^k} \left(\sum_{l=1}^{\sigma(k)} b^l v_j^l \right) x_j^k \geqslant 0, \qquad k \in K \tag{14g}$$

$$\begin{aligned} \sum_{j \in S^k} \left(\sum_{l=1}^{\sigma(k)} b^l v_j^l \right) x_j^k - \sum_{j \in S^k} s_j^k w_j^k + s_i^k \sum_{j \in S^k} w_j^k \leqslant \\ \sum_{l=1}^{\sigma(k)} b^l v_i^l + \widetilde{M}_i^k \left(1 - \sum_{l=1}^{\sigma(k)} v_i^l \right), \qquad k \in K, \quad i \in S^k \end{aligned} \tag{14h}$$

$$v_i^l \in \{0,1\}, \qquad i \in I, \quad l \in L \tag{14i}$$

$$x_i^k \in \{0,1\}, \qquad k \in K, \quad i \in S^k \tag{14j}$$

$$w_i^k \geqslant 0, \qquad k \in K, \quad i \in S^k \tag{14k}$$

In order to linearise problem (14), following the approach proposed in [12], we introduce the non-negative variables $\{z_i^k\}_{i \in I, k \in K}$ defined as the profit obtained by the company due to customer k purchasing product i, i.e., $z_i^k = \left(\sum_{l=1}^{\sigma(k)} b^l v_i^l \right) x_i^k$. Since the problem maximises, in order to guarantee this equality the following two sets of additional constraints must be added:

$$z_i^k \leqslant \sum_{l=1}^{\sigma(k)} b^l v_i^l, \qquad k \in K, \quad i \in S^k$$

$$z_i^k \leqslant b^{\sigma(k)} x_i^k, \qquad k \in K, \quad i \in S^k$$

The first one ensures that the profit cannot exceed the price set for the product. The second one imposes that the profit obtained from customer k must be zero if customer k does not purchase product i, either because he/she cannot afford it or because he/she decides to purchase another product.

Therefore, finally the PB-RPPT can be stated as the following single level linear mixed integer optimisation problem:

$$\max_{z,v,x,w} \quad \sum_{k \in K} \sum_{j \in S^k} z_j^k \tag{15a}$$

subject to

$$\sum_{l=1}^{|L|} v_i^l \leqslant 1, \qquad i \in I \tag{15b}$$

$$\sum_{j \in S^k} x_j^k \leqslant 1, \qquad k \in K \tag{15c}$$

$$x_i^k \leqslant \sum_{l=1}^{\sigma(k)} v_i^l, \qquad k \in K, \quad i \in S^k \tag{15d}$$

$$z_i^k \leqslant \sum_{l=1}^{\sigma(k)} b^l v_i^l, \qquad k \in K, \quad i \in S^k \tag{15e}$$

$$z_i^k \leqslant b^{\sigma(k)} x_i^k, \qquad k \in K, \quad i \in S^k \tag{15f}$$

$$\sum_{j \in S^k} s_j^k x_j^k \geqslant s_i^k \sum_{l=1}^{\sigma(k)} v_i^l, \qquad k \in K, \quad i \in S^k \tag{15g}$$

$$w_i^k \leqslant M_i^k \sum_{l=1}^{\sigma(k)} v_i^l, \qquad k \in K, \quad i \in S^k \tag{15h}$$

$$\sum_{j \in S^k} s_j^k w_j^k - \sum_{j \in S^k} z_j^k \geqslant 0, \qquad k \in K \tag{15i}$$

$$\sum_{j \in S^k} z_j^k - \sum_{j \in S^k} s_j^k w_j^k + s_i^k \sum_{j \in S^k} w_j^k \leqslant$$

$$\sum_{l=1}^{\sigma(k)} b^l v_i^l + \widetilde{M}_i^k \left(1 - \sum_{l=1}^{\sigma(k)} v_i^l\right), \qquad k \in K, \quad i \in S^k \tag{15j}$$

$$v_i^l \in \{0,1\}, \qquad i \in I, \quad l \in L \tag{15k}$$

$$x_i^k \in \{0,1\}, \qquad k \in K, \quad i \in S^k \tag{15l}$$

$$w_i^k \geqslant 0, \qquad k \in K, \quad i \in S^k \tag{15m}$$

$$z_i^k \geqslant 0, \qquad k \in K, \quad i \in S^k \tag{15n}$$

Note that the final model has as many variables as $|I|\,|L| + 3\sum_{k\in K} \left|S^k\right|$ and as many constraints as $|I| + 2\,|K| + 6\sum_{k\in K} \left|S^k\right|$. Nevertheless, the formal simplicity of the model allows it to be implemented in most commercial software. In the following section we will complete the model formulation by suggesting suitable values for the big-Ms.

5. Deriving values for M_i^k and $\widetilde{M}_i^k$ and valid inequalities for problem (15)

The computation of bilevel-correct big-Ms needed when working with the reformulation of the bilevel problem using duality properties is a problem that Kleinert et al. [25] have shown to be as complex, in general, as solving the original bilevel problem. Additionally, they suggest the need to explore problem-specific bounds when employing this approach to tackle bilevel problems. Furthermore, Pineda and Morales [26] highlight the issues encountered in solving bilevel problems when M is set either too small or too large. However, in certain specific bilevel problems, as the PB-RPPT, it is possible to ensure the computation of suitable constants by taking advantage of the characteristics of the model.

Note that, according to constraints (15h), M_i^k must be an upper bound of the value of the variable w_i^k, $i \in S^k$, $k \in K$. On the other hand, taking into account constraints (15i), it follows that the left part of constraints (15j) is:

$$\sum_{j\in S^k} z_j^k - \sum_{j\in S^k} s_j^k w_j^k + s_i^k \sum_{j\in S^k} w_j^k \leqslant s_i^k \sum_{j\in S^k} w_j^k \tag{16}$$

Therefore, the task of determining $\widetilde{M}_i^k$ becomes finding an upper bound for $\sum_{j\in S^k} w_j^k$, $k \in K$.

The strategy proposed below to find valid bounds consists of constructively finding an optimal solution of the dual problem. Let us recall that the dual variables were introduced when formulating the dual problem of (10) that consisted of minimising the price paid by the customer guaranteeing that the product with the highest preference among those accessible according to his/her budget is chosen. We denote $s^k_{max} = \max\{s^k_i : i \in I(k)\}$, $I(k)^+ = \{i \in I(k) : s^k_i = s^k_{max}\}$ and $p^k_{min} = \min\{p_i : i \in I(k)^+\}$. Notice that customer k will purchase a product $h \in I(k)^+$, such that $p_h = p^k_{min}$.

Theorem 5.1. *For a given value of the UL variables $\{p_i\}_{i\in I}$ and a customer $k \in K$, let $h \in I(k)^+$, such that $p_h = p^k_{min}$. Then,*

$$y^k = -p^k_{\min} + s^k_h w^k_h \tag{17a}$$

$$w^k_h = \max\left\{ \max_{i\in I(k)} \left\{ \frac{p^k_{min} - p_i}{s^k_h - s^k_i} : \quad s^k_i < s^k_{max}, \quad p_i < p^k_{min} \right\}, \quad \frac{p^k_{min}}{s^k_h} \right\} \tag{17b}$$

$$w^k_i = 0 \qquad \text{if} \quad i \neq h, \quad i \in I(k) \tag{17c}$$

is an optimal solution of the dual problem (11).

Proof. From (17), $w^k_i = 0$ for $i \in I(k)$, $i \neq h$. In order to guarantee that it is a feasible solution, together with $y^k \geqslant 0$ and $w^k_h \geqslant 0$, it is necessary to ensure that

$$-y^k + s^k_i w^k_h \leqslant \sum_{l=1}^{\sigma(k)} b^l v^l_i = p_i \tag{18}$$

Moreover, in order to be optimal, the objective function value of this solution must be equal to the optimal objective function value of the primal problem (10). Hence $-y^k + s^k_h w^k_h = p^k_{min}$, providing the value of y^k in (17a). To guarantee that $y^k \geqslant 0$, it must be ensured that $w^k_h \geqslant \frac{p^k_{min}}{s^k_h}$.

Finally, bearing in mind the value of y^k, constraint (18) is trivially met for $i \in I(k)^+$. Otherwise, to ensure the constraint (18), it must be met that:

$$w^k_h \geqslant \frac{p^k_{min} - p_i}{s^k_h - s^k_i} \qquad i \in I(k) \setminus I(k)^+$$

As a consequence,

$$w^k_h = \max\left\{ \max_{i\in I(k)} \left\{ \frac{p^k_{min} - p_i}{s^k_h - s^k_i} : \quad s^k_i < s^k_{max}, \quad p_i < p^k_{min} \right\}, \quad \frac{p^k_{min}}{s^k_h} \right\}$$

and the proof is complete. □

Corollary 5.2. *For every product $i \in S^k$ and customer $k \in K$,*

$$\begin{aligned} M_i^k &= \max\left\{ b^{\sigma(k)} - b^1,\, \frac{b^{\sigma(k)}}{s_i^k} \right\} \\ \widetilde{M}_i^k &= s_i^k \max_{j \in S^k} M_j^k \end{aligned} \tag{19}$$

are valid constants in constraints (15h) and (15j), respectively.

Proof. According to theorem (5.1) and taking into account that $b^1 \leqslant p_i \leqslant b^{\sigma(k)}$ for $i \in I(k)$,

$$w_i^k \leqslant \max\left\{ b^{\sigma(k)} - b^1,\, \frac{b^{\sigma(k)}}{s_i^k} \right\}$$

Moreover, $w_i^k = 0$ for $i \in S^k \setminus I(k)$. Thus, M_i^k is a valid constant in constraint (15h).

In addition, $\sum_{j \in S^k} w_j^k$ is equal to the value of the dual variable whose price is the lowest among those that provide the greatest customer satisfaction for customer k, therefore, it is less than or equal to $\max_{j \in S^k} M_j^k$. Thus, $\widetilde{M}_i^k$ is a valid constant in constraint (15j). □

Corollary 5.3. *For every product $i \in S^k$ and customer $k \in K$, the following set of constraints which relate the variables of the LL problem and its dual:*

$$w_i^k \leqslant M_i^k x_i^k, \qquad i \in S^k, \quad k \in K \tag{20}$$

are valid inequalities for problem (15).

Proof. Note that when $x_i^k = 1$, the constraint does not impose any restriction on the value of w_i^k. On the other hand, if $x_i^k = 0$, according to theorem (5.1) an optimal solution of the dual problem (11) exists for which $w_i^k = 0$. □

Theorem 5.4. *The following inequalities are valid for problem (15):*

$$z_i^k \leqslant b^r x_i^k + \sum_{l=r+1}^{\sigma(k)} \left(b^l - b^r \right) v_i^l, \qquad k \in K, \quad i \in S^k, \quad r = 1, \ldots, \sigma(k) - 1 \tag{21}$$

Proof. See Proposition 4.3 in [12] and Propositions 5.2 and 5.3 in [22]. □

6. Computational study

In this section we present and discuss the results of the computational experiments carried out. The numerical experiments were performed on a PC 13th Gen Intel Core i9-13900F at 2.0 GHz $\times$ 32 having 64.0 GB of RAM, and Windows 11 64-bit as the operating system, using Gurobi 10.0.3 with 6 threads. The absolute MIP optimality gap was set at 0.999 and the relative MIP optimality gap was set at 1e-5. The stopping criterion was set at 3600 seconds.

The performance of the algorithm was tested on the set of RPPT benchmark problems described in [9], which are available at `https://github.com/cdomsa/RPPT/` and are themselves based on the instances proposed in [12]. In this set of instances, $|K| \in \{50, 100, 150\}$, $|I| \in \{0.1\,|K|\,, 0.5\,|K|\,, |K|\}$ and $\left|S^k\right| \in \{\lceil 0.2\,|I|\rceil\,, \lceil 0.5\,|I|\rceil\,, |I|\}$. The number of ties can be 1, 2, 3, 5 or 10, depending on the combination of $|K|$, $|I|$ and $\left|S^k\right|$. Typically, for each combination of these parameters, there are three distinct possible values for the number of ties, except for instances with the smallest number of customers and products. This information can be observed in Table 4 in the first, second and third (corresponding to $|K| = 50$), tenth, eleventh and twelfth (corresponding to $|K| = 100$), and nineteenth, twentieth and twenty-first (corresponding to $|K| = 150$) columns. Moreover, for each combination of $|K|$, $|I|$, $\left|S^k\right|$ and number of ties, 5 instances were generated, making a total of 365 instances.

In order to reduce the size of the preference list and consequently the instance size before solving it, the preprocessing technique introduced in [9], based on the one developed in [12], is applied. The purpose of preprocessing is to reduce the preference list of each customer by removing some products (or, equivalently, to fix some of the x-variables and v-variables to 0) since it is guaranteed that an optimal solution exists so that the customer does not purchase the removed products. The theoretical results supporting the aforementioned assertions can be found in [12]. Next, we briefly describe how the preprocessing technique operates. Assuming that the customers have been sorted by budget from highest to lowest (in case of a tie in the budget, the customers are arbitrarily ordered), the first customer on the list (the richest) is selected and assigned the set of products he/she prefers the most. The preference value of the products in this set is established as the score of this customer. Then, in descending order according to budget, each customer is assigned either his/her most preferred set of products not contained in the union of the sets previously assigned to richer customers, and the corresponding score or, in case all of them are contained in such union, his/her least preferred set of products which determines his/her score. At the end of the process, for each customer, all products whose preference is lower than the customer's score are removed from his/her preference list. Table 3 shows how the preprocessing works with the illustrative example presented in Table 2. The preference values of the products eligible for removal are highlighted in grey. The corresponding products are removed from the customer's preference list.

Table 3. Preprocessing of the illustrative example shown in Table 2.

Customers	Products								Budget	Score
	i_1	i_2	i_3	i_4	i_5	i_6	i_7	i_8		
k_1	7	5	8	8	7	6	-	7	99	8
k_2	6	8	5	3	7	-	6	4	78	8
k_3	8	-	-	8	-	8	7	-	70	8
k_4	8	4	5	6	3	2	7	6	70	7
k_5	5	6	6	8	7	7	-	5	63	7
k_6	-	8	-	8	-	-	-	-	57	8
k_7	-	4	6	5	7	4	8	7	43	7
k_8	-	6	8	8	7	-	-	6	42	6
k_9	4	1	5	3	2	6	7	8	37	1
k_{10}	7	5	-	4	8	-	6	-	30	4
k_{11}	6	5	7	8	3	4	-	-	27	3
k_{12}	7	7	-	8	6	-	6	8	20	6

We have evaluated the formulation presented in problem (15), called variant V_0, and three more variants of this model which involve the use of the valid inequalities proposed in section 5. These variants are: i) $V_1 = V_0$ plus the set of inequalities (20); ii) $V_2 = V_0$ plus the set of inequalities (21); and iii) $V_3 = V_0$ plus both sets of inequalities. In all cases, the values of M_i^k and $\widetilde{M}_i^k$ developed in (19) are applied. It is worth mentioning that there is an important difference in the way each set of inequalities is introduced into the models. On the one hand, inequalities (21) are valid cuts that help tighten the relaxation of a MIP by removing fractional solutions. Since it is not mandatory to satisfy all the constraints simultaneously and the number of them does not increase exponentially, these inequalities have been defined as user cuts in Gurobi. They act as a pool of inequalities the solver can use when they are needed. On the other hand, the purpose of the inequalities (20) is to achieve a solution with the characteristics of the Theorem 5.1, and so they must all be fulfilled simultaneously. Therefore, they are introduced directly as constraints in the model.

Table 4 displays the number of instances solved to optimality for each variant proposed in this paper, V_0, $V_1 = V_0 + (20)$, $V_2 = V_0 + (21)$ and $V_3 = V_0 + (20) + (21)$, as well as the number of problems solved by the best models in [9]. Models V_R and V_B correspond to the models referred to in this paper as (RM) + VIs + prepro and (BM) + VIs + prepro, respectively. According to [9], both formulations were implemented by means of Mosel version 4.0.3 of Xpress-MP, Optimizer version 29.01.10, running on a Dell PowerEdge T110 II Server (Intel Xeon E3-1270, 3.40 GHz) with 16 GB of RAM. In the Table 4, there are three blocks, which refer to the number of customers (50, 100 and 150). In each block, the three first columns refer to the number of products, the size of the list of preferences and the number of ties. The remaining six columns indicate the number of problems solved to optimality. Each cell groups the 5 instances generated with the corresponding characteristics. The green colour means that every instance in the group has been solved. A colour shading from green to red indicates the transition from 5 to 0 solved problems. The last row in the table displays the total number of instances solved by each model.

Table 4. Number of solved problems by variant. Variant V_0 refers to model (15), $V_1 = V_0$ plus the set of inequalities (20), $V_2 = V_0$ plus the set of inequalities (21), $V_3 = V_0$ plus both sets of inequalities. Variant V_R is model (RM) + VIs + prepro and variant V_B refers to model (BM) + VIs + prepro in [9].

$|K| = 50$ (# of instances = 115)

			Proposed models				Best in [9]	
$\lvert I\rvert$	$\lvert S^k\rvert$	Ties	V_0	V_1	V_2	V_3	V_R	V_B
5	2	1	5	5	5	5	5	5
5	3	1	5	5	5	5	5	5
5	5	1	5	5	5	5	5	5
5	5	2	5	5	5	5	5	5
5	5	3	5	5	5	5	5	5
25	5	1	5	5	5	5	5	5
25	5	2	5	5	5	5	5	5
25	5	3	5	5	5	5	5	5
25	13	1	5	5	5	5	5	5
25	13	3	5	5	5	5	5	5
25	13	5	5	5	5	5	5	5
25	25	3	5	5	5	5	5	5
25	25	5	5	5	5	5	5	5
25	25	10	5	5	5	5	5	5
50	10	1	5	5	5	5	5	5
50	10	3	5	5	5	5	5	5
50	10	5	5	5	5	5	5	5
50	25	3	5	5	5	5	5	5
50	25	5	5	5	5	5	5	5
50	25	10	5	5	5	5	5	5
50	50	3	5	5	5	5	5	5
50	50	5	5	5	5	5	5	5
50	50	10	5	5	5	5	5	5
Total			**115**	**115**	**115**	**115**	**115**	**115**

$|K| = 100$ (# of instances = 125)

			Proposed models				Best in [9]	
$\lvert I\rvert$	$\lvert S^k\rvert$	Ties	V_0	V_1	V_2	V_3	V_R	V_B
10	2	1	5	5	5	5	5	5
10	5	1	5	5	5	5	5	5
10	5	2	5	5	5	5	5	5
10	5	3	5	5	5	5	5	5
10	10	1	5	5	5	5	5	4
10	10	3	5	5	5	5	5	5
10	10	5	5	5	5	5	5	3
50	10	1	5	5	5	5	5	5
50	10	3	5	5	5	5	5	5
50	10	5	5	5	5	5	5	5
50	25	3	5	5	5	5	5	5
50	25	5	3	5	5	5	5	5
50	25	10	0	2	2	3	4	4
50	50	3	2	1	5	5	5	5
50	50	5	2	0	5	5	4	3
50	50	10	0	0	4	3	3	2
100	20	1	5	5	5	5	5	5
100	20	3	5	5	5	5	5	5
100	20	5	5	5	5	5	5	5
100	50	3	5	5	5	5	5	5
100	50	5	5	5	5	5	5	5
100	50	10	5	5	5	5	5	5
100	100	3	5	5	5	5	5	5
100	100	5	5	5	5	5	5	5
100	100	10	5	5	5	5	5	5
Total			107	108	**121**	**121**	**121**	116

$|K| = 150$ (# of instances = 125)

			Proposed models				Best in [9]	
$\lvert I\rvert$	$\lvert S^k\rvert$	Ties	V_0	V_1	V_2	V_3	V_R	V_B
15	3	1	5	5	5	5	5	5
15	8	1	5	5	5	5	5	4
15	8	2	5	5	5	5	5	3
15	8	3	5	5	5	5	2	2
15	15	1	0	4	5	3	1	0
15	15	3	0	1	5	2	1	0
15	15	5	0	1	5	1	0	0
75	15	1	5	5	5	5	5	5
75	15	3	4	4	5	5	5	5
75	15	5	1	3	5	5	5	5
75	38	3	0	1	5	5	3	3
75	38	5	0	0	3	3	2	2
75	38	10	0	0	1	0	2	1
75	75	3	0	0	1	1	0	0
75	75	5	0	0	1	1	0	0
75	75	10	0	0	0	0	0	0
150	30	3	5	5	5	5	5	5
150	30	5	5	5	5	5	5	5
150	30	10	5	5	5	5	5	5
150	75	3	5	5	5	5	5	5
150	75	5	5	5	5	5	5	5
150	75	10	5	5	5	5	5	5
150	150	3	5	5	5	5	5	5
150	150	5	5	5	5	5	5	5
150	150	10	5	5	5	5	5	5
Total			75	84	**106**	96	86	80

Summarising the results, every variant solves all instances with 50 customers. Regarding the 125 instances with 100 customers, variants V_2 and V_3 show the best performance since they solve 121 instances, as variant V_R. Finally, for the 125 instances with 150 customers, variants V_2 clearly outperform the remaining ones, solving 106 instances. Globally, V_2 solves 342 (93.7%) instances, while V_R and V_B solve 322 (88.2%) and 311 (85.2%) instances, respectively. Concerning the computational times of solved instances, Table 5 presents several statistical measures which reinforce the quality of the variant V_2. This variant has the lowest mean time, 167.25 seconds, and 75% of the solved instances are processed in less than 48.50 seconds.

Table 5. Numerical summary of the computational time of solved instances by variant, in seconds.

Variant	# of instances	Mean	CV	Min	P_{25}	P_{50}	P_{75}	Max
V_0	297	173.76	2.62	0.07	1.04	4.79	54.98	2817.10
V_1	307	236.45	2.65	0.05	0.99	3.39	70.94	3385.25
V_2	342	167.25	2.78	0.04	1.13	4.03	48.50	3230.00
V_3	332	217.36	2.82	0.04	1.33	3.93	65.40	3537.44

In addition, Table 6 displays various statistical measures related to the MIPGap provided by Gurobi (multiplied by 100) for instances that were not solved to optimality. It is worth noting that variant V_2 consistently yields the best results. The number of unsolved problems is the lowest, 23 instances, as well as the mean of the MIPGap, 0.0057. Moreover, 75% of these instances have a MIPGap less than or equal to 0.0082. Figure 1 shows the corresponding box plots. A box plot has been drawn to illustrate the MIPGap of instances that were not solved to optimality in each of the variants, with outliers represented by a black dot. The mean value is represented by a grey circle. It can be observed that variant V_2 is the only one with no outliers, meaning there are no problems where the MIPGap value is considerably higher than in the rest of the problems. In addition, variant V_2 shows much less dispersion than the other variants. From all the previous remarks, we conclude that variant V_2, which exploits the use of the valid inequalities (21), outperforms the other variants analysed.

Table 6. Numerical summary of the MIPGap (multiplied by 100) of non-solved instances by variant.

Variant	# of instances	Mean	CV	Min	P_{25}	P_{50}	P_{75}	Max
V_0	68	1.83	0.56	0.15	1.10	1.88	2.31	6.19
V_1	58	1.62	0.58	0.06	1.01	1.59	2.36	4.57
V_2	23	0.57	0.61	0.04	0.36	0.57	0.82	1.36
V_3	33	2.12	1.50	0.09	0.42	0.80	1.55	12.34

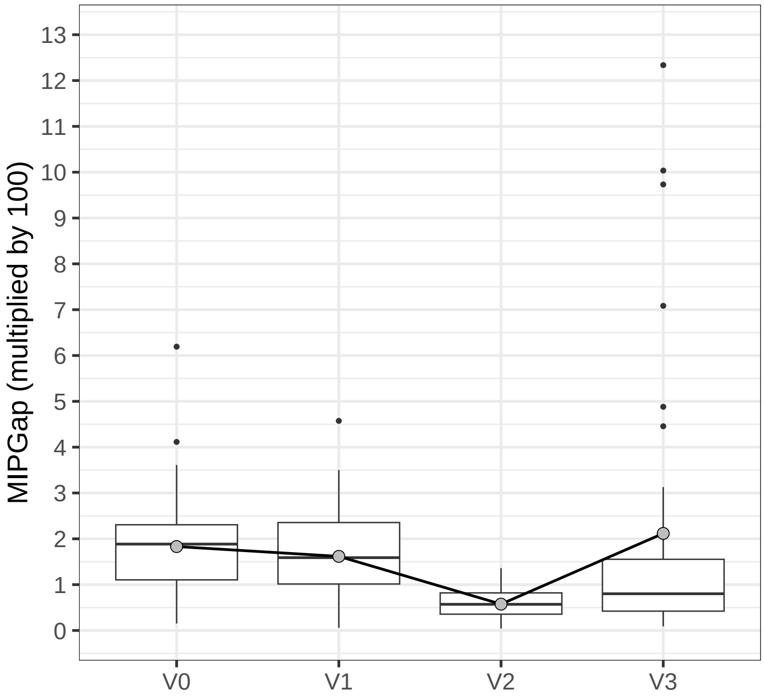

Figure 1. Box plots of the MIPGap (multiplied by 100) of non-solved instances by variant.

In [9], two additional randomly generated large-scale instances were solved to optimality. For both instances, $|K| = 350$, $|I| = 10$, $\left|S^k\right| = 5$ and the number of ties is 1, with all customers having distinct budgets. Table 7 shows the optimal value of the objective function, Z, as well as the computation time, T, required to obtain the optimal solution. Note that, again, the V_2 variant provides the best results. In fact, comparing the computation times required by variants V_R and V_B versus those required by the variant V_2, these are 34 and 66 times lower, respectively, for the first instance and 51 and 17 times lower, respectively, for the second instance.

Table 7. Additional instances solved to optimality. Z refers to the optimal objective function value, T means computational time involved. Variant V_0 refers to model (15), $V_1 = V_0$ plus the set of inequalities (20), $V_2 = V_0$ plus the set of inequalities (21), $V_3 = V_0$ plus both sets of inequalities. Variant V_R is model (RM) + VIs + prepro and variant V_B refers to model (BM) + VIs + prepro in [9].

$\|K\|$	$\|I\|$	$\left\|S^k\right\|$	Ties	Inst.	Z	T					
						V_0	V_1	V_2	V_3	V_R	V_B
350	10	5	1	1	148414	852.5	1171.9	354.9	679.6	12109.9	23644.0
350	10	5	1	2	143469	1931.5	1479.8	1452.6	1649.1	77737.9	25292.9

7. Conclusions and further research

This paper introduces a novel and competitive approach to tackle bilevel problems from a pessimistic standpoint. The proposed method entails reformulating the bilevel problem using a lexicographic biobjective approach associated with the LL decision-maker. It is illustrated by its successful application to the RPPT. The RPPT is presented as a bilevel optimisation problem where the company, as the UL decision-maker, must set prices considering customer purchasing decisions at the lower level. When ties exist in customer preferences, the existence of a unique optimal solution to the LL problem cannot be guaranteed. In this scenario, if customers have several products within their budget that equally satisfy their preferences, they will opt for the least expensive among them, thereby providing the least revenue to the company. Therefore, the RPPT is an inherently hierarchical problem aligned with the pessimistic approach.

After formulating the RPPT as a pessimistic bilevel problem, the paper goes on to develop the theory that allows us to reformulate it as a single-level problem. The idea is to replace the lower-level lexicographic optimisation problem associated to each customer with a set of constraints by applying the relationships between primal and dual problems. In addition, it is necessary in some constraints to use big-Ms, for which valid values are proposed. The paper also includes the derivation of valid inequalities whose possible interest is analysed in the computational study.

The computational experiment aims to solve the benchmark instances existing in the literature for the RPPT. The quality of four variants of the model is checked, obtaining that the best one is the variant that adds to the original model formulation the valid inequalities on the z variables which are used to linearise the pricing problem. The results achieved improve on the results presented in the literature.

The purpose of further research will be to explore other bilevel problems where the pessimistic approach is applicable and there exists a structure which allows making use of the approach proposed in this paper. On the other hand, focusing on the RPPT, it will be also of interest to establish relationships between the formulation RM proposed in [9] and the one proposed in this paper, as well as to study the inclusion of a limitation in the number of available products as proposed in [22].

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References

[1] Camacho-Vallejo J, Corpus C, Villegas J. Metaheuristics for bilevel optimization: A comprehensive review. Computers & Operations Research. 2023;161(106410). Available from: https://doi.org/10.1016/j.cor.2023.106410.

[2] Colson B, Marcotte P, Savard G. An overview of bilevel programming. Annals of Operations Research. 2007;153:235–256.

[3] Dempe S. Foundations of bilevel programming. Dordrecht, Boston, London: Kluwer Academic Publishers; 2002.

[4] Dempe S. Bilevel optimization: theory, algorithms and applications; 2018. Preprint 2018-11, Fakultät für Mathematik und Informatik, TU Bergakademie Freiberg; Available from: http://www.optimization-online.org/DB_HTML/2018/08/6773.html.

[5] Dempe S, Zemkoho A. Bilevel optimization. advances and next challenges. Berlin: Springer; 2020.

[6] Liu J, Fan Y, Chen Z, et al. Pessimistic bilevel optimization: A survey. International Journal of Computational Intelligence Systems. 2018;11:725–736.

[7] Sinha A, Malo P, Deb K. A review on bilevel optimization: From classical to evolutionary approaches and applications. IEEE Transactions on Evolutionary Computation. 2018; 22:276–295.

[8] Ehrgott M. Multicriteria optimization. 2nd ed. Berlin, Heildeberg: Springer; 2005.

[9] Domínguez C, Labbé M, Marín A. The rank pricing problem with ties. European Journal of Operational Research. 2021;294:492–506.

[10] Rusmevichientong P. A non-parametric approach to multi-product pricing: Theory and application [dissertation]. Standford University; 2003.

[11] Rusmevichientong P, Roy BV, Glynn P. A nonparametric approach to multiproduct pricing. Operations Research. 2006;54:82–98.

[12] Calvete H, Domínguez C, Galé C, et al. The rank pricing problem: Models and branch-and-cut algorithms. Computers and Operations Research. 2019;105:12–31.

[13] Aggarwal G, Feder T, Motwani R, et al. Algorithms for multi-product pricing. In: Díaz J, Karhumäki J, Lepistö A, et al., editors. Automata, Languages and Programming. ICALP 2004. Lecture Notes in Computer Science, vol 3142. Springer, Berlin, Heildeberg; 2004. p. 72–83.

[14] Briest P, Krysta P. Buying cheap is expensive: Approximability of combinatorial pricing problems. SIAM Journal on Computing. 2011;40(6):1554–1586.

[15] Hanjoul P, Peeters D. A facility location problem with clients' preference orderings. Regional Science and Urban Economics. 1987;17:451–473.

[16] Cánovas L, García S, Labbé M, et al. A strengthened formulation for the simple plant location problem with order. Operations Research Letters. 2007;35:141–150.

[17] Hansen P, Kochetov Y, Mladenović N. Lower bounds for the uncapacitated facility location problem with user preferences. Les Cahiers du GERAD. 2004;G-2004-24.

[18] Vasilyev I, Klimentova K. The branch and cut method for the facility location problem with client's preferences. Journal of Applied and Industrial Mathematics. 2010;4(3):441–454.
[19] Vasilyev I, Klimentova K, Kochetov Y. New lower bounds for the facility location problem with client's preferences. Computational Mathematics and Mathematical Physics. 2009; 49(6):1010–1020.
[20] Camacho-Vallejo J, Cordero-Franco A, González-Ramírez R. Solving the bilevel facility location problem under preferences by a Stackelberg-evolutionary algorithm. Mathematical Problems in Engineering. 2014;2014(ID 430243):1–15.
[21] Calvete H, Galé C, Iranzo JA, et al. A matheuristic for solving the bilevel approach of the facility location problem with cardinality constraints and preferences. Computers and Operations Research. 2020;124(105066):1–15.
[22] Domínguez C, Labbé M, Marín A. Mixed-integer formulations for the capacitated rank pricing problem with envy. Computers and Operations Researc. 2022;140(105664):1–12.
[23] Ansari A. A bilevel product pricing problem with ranks and utilities: Models and algorithms Thesis for the degree of master of applied science (industrial engineering). Concordia University, Montréal, Québec, Canada; 2022.
[24] Calvete H, Galé C. Linear bilevel multi-follower programming with independent followers. Journal of Global Optimization. 2007;39:409–417.
[25] Kleinert T, Labbé M, Plein F, et al. There's no free lunch: On the hardness of choosing a correct big-M in bilevel optimization. Operations Research. 2020;68:1716–1721.
[26] Pineda S, Morales J. Solving linear bilevel problems using big-Ms: Not all that glitters is gold. IEEE Transactions on Power Systems. 2019;34(3):2469–2471.

8. Appendix

In order to facilitate future comparisons with other algorithms developed to solve the RPPT, this appendix makes available all relevant information provided by Gurobi on the application of the V_2 variant to the instances. Tables 8 to 10 provide for each instance the characteristics which define it, together with the best objective function value provided by Gurobi, Z, the computational time invested, T, and the MIPGap when applicable, G.

Table 8. Results using variant 2 for each instance of size $|K| = 50$, problem by problem. Inst. refers to the number of the instance, Z refers to the best objective function value provided by Gurobi, T means computational time and G refers to the MIPGap. Stopping criterion 3600 seconds.

Inst.	$\|I\|$	$\|S^k\|$	Ties	Z	T	G
1	5	2	1	1551	0.345	-
2	5	2	1	1802	0.414	-
3	5	2	1	1460	0.266	-
4	5	2	1	1545	0.283	-
5	5	2	1	1375	0.289	-
6	5	3	1	1796	0.466	-
7	5	3	1	1928	0.545	-
8	5	3	1	1641	0.486	-
9	5	3	1	1709	0.583	-
10	5	3	1	1519	0.518	-
11	5	5	1	1901	1.141	-
12	5	5	1	2023	0.903	-
13	5	5	1	1825	1.099	-
14	5	5	1	1867	0.856	-
15	5	5	1	1717	1.227	-
16	5	5	2	1867	1.080	-
17	5	5	2	1944	1.178	-
18	5	5	2	1773	0.944	-
19	5	5	2	1736	1.301	-
20	5	5	2	1583	0.989	-
21	5	5	3	1780	1.128	-
22	5	5	3	1866	0.990	-
23	5	5	3	1635	1.684	-
24	5	5	3	1727	1.188	-
25	5	5	3	1524	1.576	-
26	25	5	1	2142	0.348	-
27	25	5	1	2211	0.849	-
28	25	5	1	2047	0.403	-
29	25	5	1	2310	0.217	-
30	25	5	1	2575	0.417	-
31	25	5	2	2053	0.948	-
32	25	5	2	2099	1.390	-
33	25	5	2	2028	0.395	-
34	25	5	2	2212	0.906	-
35	25	5	2	2446	1.023	-
36	25	5	3	2013	2.482	-
37	25	5	3	2059	8.023	-
38	25	5	3	1813	6.896	-
39	25	5	3	2093	5.183	-
40	25	5	3	2312	7.581	-
41	25	13	1	2245	2.094	-
42	25	13	1	2336	1.397	-
43	25	13	1	2149	1.451	-
44	25	13	1	2433	1.478	-
45	25	13	1	2661	0.434	-
46	25	13	3	2203	2.398	-
47	25	13	3	2329	1.788	-
48	25	13	3	2092	2.430	-
49	25	13	3	2402	2.034	-
50	25	13	3	2605	2.689	-
51	25	13	5	2207	3.035	-
52	25	13	5	2259	11.090	-
53	25	13	5	2047	4.764	-
54	25	13	5	2348	4.078	-
55	25	13	5	2565	12.549	-
56	25	25	3	2266	6.470	-
57	25	25	3	2365	6.788	-
58	25	25	3	2096	9.183	-
59	25	25	3	2419	9.602	-
60	25	25	3	2704	5.704	-

Inst.	$\|I\|$	$\|S^k\|$	Ties	Z	T	G
61	25	25	5	2274	8.756	-
62	25	25	5	2365	8.006	-
63	25	25	5	2104	13.113	-
64	25	25	5	2442	7.554	-
65	25	25	5	2662	10.226	-
66	25	25	10	2230	8.721	-
67	25	25	10	2271	45.460	-
68	25	25	10	2075	47.199	-
69	25	25	10	2369	27.623	-
70	25	25	10	2583	59.075	-
71	50	10	1	2473	0.070	-
72	50	10	1	3035	0.148	-
73	50	10	1	2610	0.037	-
74	50	10	1	2652	0.108	-
75	50	10	1	2552	0.092	-
76	50	10	3	2423	0.473	-
77	50	10	3	3019	0.388	-
78	50	10	3	2556	0.461	-
79	50	10	3	2619	0.287	-
80	50	10	3	2547	0.190	-
81	50	10	5	2349	1.749	-
82	50	10	5	2962	2.278	-
83	50	10	5	2515	2.234	-
84	50	10	5	2585	0.983	-
85	50	10	5	2476	1.372	-
86	50	25	3	2481	0.123	-
87	50	25	3	3050	0.269	-
88	50	25	3	2625	0.055	-
89	50	25	3	2648	0.337	-
90	50	25	3	2559	0.286	-
91	50	25	5	2470	1.026	-
92	50	25	5	3044	0.393	-
93	50	25	5	2617	0.119	-
94	50	25	5	2656	0.131	-
95	50	25	5	2557	0.252	-
96	50	25	10	2438	2.574	-
97	50	25	10	3016	3.637	-
98	50	25	10	2586	1.909	-
99	50	25	10	2626	2.182	-
100	50	25	10	2533	1.110	-
101	50	50	3	2484	0.270	-
102	50	50	3	3061	0.097	-
103	50	50	3	2627	0.092	-
104	50	50	3	2657	0.090	-
105	50	50	3	2564	0.097	-
106	50	50	5	2484	0.285	-
107	50	50	5	3061	0.096	-
108	50	50	5	2627	0.074	-
109	50	50	5	2657	0.089	-
110	50	50	5	2563	0.212	-
111	50	50	10	2472	2.672	-
112	50	50	10	3047	1.835	-
113	50	50	10	2622	0.558	-
114	50	50	10	2647	2.141	-
115	50	50	10	2560	0.610	-

Table 9. Results using variant 2 for each instance of size $|K| = 100$, problem by problem. Inst. refers to the number of the instance, Z refers to the best objective function value provided by Gurobi, T means computational time and G refers to the MIPGap. Stopping criterion 3600 seconds.

Inst.	$\lvert I \rvert$	$\lvert S^k \rvert$	Ties	Z	T	G
116	10	2	1	7089	0.773	-
117	10	2	1	6258	0.784	-
118	10	2	1	5763	0.738	-
119	10	2	1	5759	0.630	-
120	10	2	1	5657	1.832	-
121	10	5	1	7977	3.317	-
122	10	5	1	7125	7.103	-
123	10	5	1	6903	5.230	-
124	10	5	1	6700	4.356	-
125	10	5	1	6580	3.973	-
126	10	5	2	7777	7.608	-
127	10	5	2	6927	27.599	-
128	10	5	2	6472	17.253	-
129	10	5	2	6577	5.583	-
130	10	5	2	6409	7.051	-
131	10	5	3	7372	30.737	-
132	10	5	3	6780	13.647	-
133	10	5	3	6292	28.963	-
134	10	5	3	6025	23.908	-
135	10	5	3	6316	29.083	-
136	10	10	1	8579	11.382	-
137	10	10	1	7851	12.240	-
138	10	10	1	7295	11.761	-
139	10	10	1	7091	19.571	-
140	10	10	1	7093	32.026	-
141	10	10	3	8305	39.740	-
142	10	10	3	7763	30.883	-
143	10	10	3	7149	42.351	-
144	10	10	3	6931	48.928	-
145	10	10	3	6912	41.877	-
146	10	10	5	7979	62.847	-
147	10	10	5	7553	64.036	-
148	10	10	5	6756	65.826	-
149	10	10	5	6614	87.780	-
150	10	10	5	6747	67.652	-
151	50	10	1	9674	2.384	-
152	50	10	1	10586	3.817	-
153	50	10	1	9185	3.438	-
154	50	10	1	8549	3.149	-
155	50	10	1	9418	2.947	-
156	50	10	3	9369	30.575	-
157	50	10	3	10356	50.562	-
158	50	10	3	9086	23.127	-
159	50	10	3	8421	4.906	-
160	50	10	3	9261	111.701	-
161	50	10	5	8963	1637.883	-
162	50	10	5	10051	468.380	-
163	50	10	5	8778	177.615	-
164	50	10	5	8042	149.137	-
165	50	10	5	8890	809.246	-
166	50	25	3	9820	29.554	-
167	50	25	3	10798	54.806	-
168	50	25	3	9391	18.478	-
169	50	25	3	8628	124.791	-
170	50	25	3	9619	86.881	-
171	50	25	5	9689	192.435	-
172	50	25	5	10628	841.358	-
173	50	25	5	9309	129.788	-
174	50	25	5	8633	228.770	-
175	50	25	5	9568	279.139	-
176	50	25	10	9561	637.457	-
177	50	25	10	10338	-	0.007
178	50	25	10	9130	1049.079	-
179	50	25	10	8464	-	0.004
180	50	25	10	9298	-	0.001
181	50	50	3	9949	92.465	-
182	50	50	3	10896	476.565	-
183	50	50	3	9443	149.015	-
184	50	50	3	8853	490.241	-
185	50	50	3	9773	147.031	-
186	50	50	5	9912	146.681	-
187	50	50	5	10824	1394.719	-
188	50	50	5	9478	149.510	-
189	50	50	5	8782	726.056	-
190	50	50	5	9765	190.495	-
191	50	50	10	9904	202.346	-
192	50	50	10	10696	-	0.006
193	50	50	10	9444	442.149	-
194	50	50	10	8760	1533.954	-
195	50	50	10	9675	847.401	-
196	100	20	1	12055	0.680	-
197	100	20	1	10897	0.640	-
198	100	20	1	10215	0.441	-
199	100	20	1	9557	0.366	-
200	100	20	1	10070	0.413	-
201	100	20	3	12028	2.341	-
202	100	20	3	10894	0.995	-
203	100	20	3	10214	0.731	-
204	100	20	3	9534	0.839	-
205	100	20	3	10055	0.891	-
206	100	20	5	11994	2.388	-
207	100	20	5	10807	4.300	-
208	100	20	5	10180	1.953	-
209	100	20	5	9467	2.642	-
210	100	20	5	10033	2.237	-
211	100	50	3	12060	0.720	-
212	100	50	3	10922	1.228	-
213	100	50	3	10221	0.930	-
214	100	50	3	9597	1.059	-
215	100	50	3	10071	0.532	-
216	100	50	5	12048	2.642	-
217	100	50	5	10915	1.986	-
218	100	50	5	10218	1.105	-
219	100	50	5	9596	1.720	-
220	100	50	5	10063	1.300	-
221	100	50	10	12017	8.730	-
222	100	50	10	10901	3.131	-
223	100	50	10	10175	12.010	-
224	100	50	10	9583	2.535	-
225	100	50	10	10035	5.598	-
226	100	100	3	12062	1.494	-
227	100	100	3	10926	0.529	-
228	100	100	3	10221	1.459	-
229	100	100	3	9603	0.369	-
230	100	100	3	10073	0.583	-
231	100	100	5	12057	1.898	-
232	100	100	5	10923	2.293	-
233	100	100	5	10223	0.481	-
234	100	100	5	9602	0.576	-
235	100	100	5	10069	2.061	-
236	100	100	10	12038	13.888	-
237	100	100	10	10915	5.148	-
238	100	100	10	10218	9.477	-
239	100	100	10	9598	1.736	-
240	100	100	10	10066	2.702	-

Table 10. Results using variant 2 for each instance of size $|K| = 150$, problem by problem. Inst. refers to the number of the instance, Z refers to the best objective function value provided by Gurobi, T means computational time and G refers to the MIPGap. Stopping criterion 3600 seconds.

Inst.	$\|I\|$	$\|S^k\|$	Ties	Z	T	G
241	15	3	1	14671	16.383	-
242	15	3	1	14327	18.660	-
243	15	3	1	15024	4.793	-
244	15	3	1	16261	6.238	-
245	15	3	1	17610	13.507	-
246	15	8	1	17128	88.737	-
247	15	8	1	16176	107.276	-
248	15	8	1	16822	71.120	-
249	15	8	1	18189	31.350	-
250	15	8	1	19541	161.105	-
251	15	8	2	16570	727.047	-
252	15	8	2	16050	148.430	-
253	15	8	2	16709	86.859	-
254	15	8	2	17982	81.615	-
255	15	8	2	19229	149.653	-
256	15	8	3	16461	480.098	-
257	15	8	3	15587	542.917	-
258	15	8	3	16249	472.055	-
259	15	8	3	17720	143.755	-
260	15	8	3	18847	458.212	-
261	15	15	1	17768	422.776	-
262	15	15	1	17119	373.875	-
263	15	15	1	17527	329.278	-
264	15	15	1	18824	262.455	-
265	15	15	1	20586	1122.884	-
266	15	15	3	17415	919.347	-
267	15	15	3	16739	1080.546	-
268	15	15	3	17483	311.436	-
269	15	15	3	18515	416.604	-
270	15	15	3	20452	1547.155	-
271	15	15	5	17517	974.659	-
272	15	15	5	16663	974.958	-
273	15	15	5	17257	547.615	-
274	15	15	5	18244	1355.965	-
275	15	15	5	19909	2239.446	-
276	75	15	1	21180	23.232	-
277	75	15	1	22619	20.832	-
278	75	15	1	22343	65.087	-
279	75	15	1	20016	36.622	-
280	75	15	1	19966	11.078	-
281	75	15	3	21142	44.784	-
282	75	15	3	22395	30.613	-
283	75	15	3	21946	672.658	-
284	75	15	3	19745	59.135	-
285	75	15	3	19803	23.339	-
286	75	15	5	20818	388.586	-
287	75	15	5	21871	2515.510	-
288	75	15	5	21892	807.789	-
289	75	15	5	19594	206.893	-
290	75	15	5	19145	1512.165	-
291	75	38	3	21515	1283.728	-
292	75	38	3	23034	86.470	-
293	75	38	3	22580	1981.924	-
294	75	38	3	20367	3229.996	-
295	75	38	3	20226	455.895	-
296	75	38	5	21494	1129.141	-
297	75	38	5	22933	208.843	-
298	75	38	5	22438	-	0.005
299	75	38	5	20273	-	0.004
300	75	38	5	20021	1406.413	-
301	75	38	10	21345	-	0.008
302	75	38	10	22775	2942.789	-
303	75	38	10	22270	-	0.014
304	75	38	10	20009	-	0.011
305	75	38	10	20008	-	0.007
306	75	75	3	21785	2650.917	-
307	75	75	3	23067	-	0.000
308	75	75	3	22850	-	0.004
309	75	75	3	20467	-	0.008
310	75	75	3	20337	-	0.004
311	75	75	5	21804	-	0.000
312	75	75	5	23095	2924.169	-
313	75	75	5	22857	-	0.006
314	75	75	5	20519	-	0.008
315	75	75	5	20352	-	0.001
316	75	75	10	21739	-	0.003
317	75	75	10	22809	-	0.010
318	75	75	10	22712	-	0.009
319	75	75	10	20473	-	0.008
320	75	75	10	20342	-	0.004
321	150	30	3	23434	1.953	-
322	150	30	3	23294	2.359	-
323	150	30	3	21260	2.453	-
324	150	30	3	21121	2.500	-
325	150	30	3	22856	1.438	-
326	150	30	5	23426	3.001	-
327	150	30	5	23245	8.110	-
328	150	30	5	21270	2.016	-
329	150	30	5	21118	2.125	-
330	150	30	5	22832	3.203	-
331	150	30	10	23319	15.236	-
332	150	30	10	23144	18.379	-
333	150	30	10	21088	19.096	-
334	150	30	10	21006	13.860	-
335	150	30	10	22712	23.750	-
336	150	75	3	23441	2.172	-
337	150	75	3	23301	3.376	-
338	150	75	3	21287	3.032	-
339	150	75	3	21140	1.265	-
340	150	75	3	22875	2.187	-
341	150	75	5	23436	3.359	-
342	150	75	5	23303	2.204	-
343	150	75	5	21293	1.845	-
344	150	75	5	21134	2.846	-
345	150	75	5	22872	3.532	-
346	150	75	10	23432	3.984	-
347	150	75	10	23281	10.296	-
348	150	75	10	21278	5.640	-
349	150	75	10	21097	13.593	-
350	150	75	10	22874	3.641	-
351	150	150	3	23443	3.267	-
352	150	150	3	23303	4.984	-
353	150	150	3	21296	1.407	-
354	150	150	3	21140	1.343	-
355	150	150	3	22878	1.032	-
356	150	150	5	23444	2.281	-
357	150	150	5	23304	3.234	-
358	150	150	5	21296	2.328	-
359	150	150	5	21139	2.640	-
360	150	150	5	22876	5.047	-
361	150	150	10	23439	5.126	-
362	150	150	10	23297	10.408	-
363	150	150	10	21278	25.944	-
364	150	150	10	21137	4.266	-
365	150	150	10	22875	5.595	-