

Yuansi Chen's result on the KLS conjecture

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Resumen

Estas notas constituyen un anexo del trabajo previo de los autores [AB2]. Aquí se incluye la demostración del reciente resultado de Yuansi Chen [Ch], en el que se mejoran sustancialmente los resultados incluidos en [AB2], y que fue simultaneo a su publicación. Se siguen las pautas de la exposición hecha por Bo'az Klartag en [K], pero manteniéndose dentro del esquema de demostración utilizado en el anterior trabajo [AB2].

Abstract

These notes constitute an annex to the previous work by the authors [AB2]. We include here the proof of the recent result by Yuansi Chen [Ch], in which the results included in [AB2] are substantially improved, and which was simultaneous to its publication. We follow the guidelines in the exposition made by Bo'az Klartag in [K], but remaining inside the scheme of the proof used in our previous work [AB2].

1 Introduction

The Kannan-Lovász-Simonovits (KLS) conjecture is a major open problem in asymptotic geometric analysis, which concerns Cheeger-type isoperimetric inequalities for log-concave probabilities, i.e., probabilities μ on \mathbb{R}^n of the form $d\mu(x) = e^{-V(x)}dx$, with $V : \mathbb{R}^n \rightarrow (-\infty, \infty]$ a convex function. It was posed in [KLS] and can be stated in the following way:

Conjecture 1.1 (KLS spectral gap conjecture). *There exists an absolute constant $C > 0$ such that, for any log-concave probability μ in \mathbb{R}^n*

$$(1) \quad \mu^+(A) \geq \frac{C}{\sqrt{\|Cov_\mu\|_{op}}} \min\{\mu(A), \mu(A^c)\}, \quad \text{for any Borel set } A \subset \mathbb{R}^n$$

where

$$\mu^+(A) = \liminf_{\varepsilon \rightarrow 0} \frac{\mu(A^\varepsilon) - \mu(A)}{\varepsilon},$$

being $A^\varepsilon = \{a + x : a \in A, |x| < \varepsilon\}$, and $\|Cov_\mu\|_{op}$ is the operator norm of the covariance matrix of μ .

Given a log-concave probability measure μ on \mathbb{R}^n , let us denote by ψ_μ the largest constant such that

$$\mu^+(A) \geq \frac{\psi_\mu}{\sqrt{\|Cov_\mu\|_{op}}} \min\{\mu(A), \mu(A^c)\} \quad \text{for any Borel set } A \subset \mathbb{R}^n.$$

Let us also denote by ψ_n , the infimum of the constants ψ_μ when μ runs over all log-concave probability measures on \mathbb{R}^n . That is,

$$\psi_n = \inf\{\psi_\mu : \mu \text{ is a log-concave probability on } \mathbb{R}^n\}.$$

Therefore, the KLS conjecture asks about the existence of a positive absolute constant $C > 0$ such that $\psi_n \geq C$ for every $n \in \mathbb{N}$. Let us point out that it is well known that there exists an absolute constant such that $\psi_\mu \leq C$ for every log-concave probability measure and that the KLS conjecture can be reduced to the setting of isotropic log-concave probabilities, which are centered log-concave probabilities whose covariance matrix Cov_μ is the identity matrix.

In a recent article [AB2] in this journal, the authors presented *Eldan's localization scheme* and proved, in a unified framework, the two best known estimates for ψ_n which had been proved by Eldan [E] and Lee & Vempala [LV]. It can be stated in the following way:

Theorem 1.1. *Let ψ_n be the best constant such that for any isotropic log-concave probability μ in \mathbb{R}^n the following isoperimetric inequality holds*

$$\mu^+(A) \geq \psi_n \min\{\mu(A), \mu(A^c)\} \quad \text{for any Borel set } A \subseteq \mathbb{R}^n.$$

Then, there exists an absolute constant $C > 0$ such that

$$\psi_n \geq \frac{C}{\min\{\sigma_n \log n, n^{1/4}\}},$$

where $\sigma_n = \sqrt{\sup \mathbb{E}_\mu [|X| - \sqrt{n}]^2}$ and the sup runs over all isotropic log-concave random vectors X in \mathbb{R}^n .

Simultaneously to the publication of [AB2], Yuansi Chen improved the best known estimate of ψ_n , by proving the following theorem:

Theorem 1.2 (Y. Chen [Ch]). *There exists absolute constants $c_1, c_2 > 0$ such that for any isotropic log-concave probability μ in \mathbb{R}^n the following isoperimetric inequality holds*

$$\mu^+(A) \geq c_1 \exp(-c_2 \sqrt{\log n \cdot \log \log n}) \min\{\mu(A), \mu(A^c)\}$$

for any Borel set $A \subseteq \mathbb{R}^n$.

This paper tries to be an Appendix to the aforementioned paper [AB2], in which we include the proof of Chen's estimate in the framework developed there. The proof that we present here follows the original idea appearing in Eldan's seminal paper [E]. The same idea was also used by Lee & Vempala in their approach [LV] and also in our previous paper [AB2]. However the proof by [Ch] presents some formal differences. Namely, he preferred to take expectations in the isoperimetric inequalities instead of control how the measure of the individual $1/2$ -Borel sets evolves throughout Eldan's stochastical localization scheme. In order to get Chen's result we mimic the method used by Klartag [K], which uses a stopping time instead of the original reiteration method by Chen.

2 Preliminary results

We will follow the framework developed in [AB2], which we recall here in order to improve the readability of this annex. Nevertheless, we refer the reader to [AB2] for more detailed explanations.

2.1 A first reduction

We already mentioned in the introduction that one can only consider isotropic log-concave probability measures. It was also showed in [AB2] that it is enough to prove Conjecture 1.1 for isotropic log-concave measures with compact support. This condition ensured the existence and uniqueness of solution on the system of stochastic differential equations considered in the proof (see (2) below). One can reduce even further the class of isotropic log-concave probability measures to consider, by also assuming that their support is contained in a Euclidean ball $r_n B_2^n$ of some large (but not “too large”) radius r_n . We state it in the following lemma:

Lemma 2.1. *There exists an absolute constant $C > 0$ such that if for every isotropic log-concave probability measure μ with $\text{supp } \mu \subseteq Cn^5 B_2^n$ we have that*

$$\psi_\mu \geq c_1 C_n,$$

where c_1 is an absolute constant, then, we have that

$$\psi_n \geq \tilde{c}_1 C_n,$$

where \tilde{c}_1 is another absolute constant.

Proof. Let $d\mu(x) = e^{-V(x)}dx$ be an isotropic log-concave probability measure on \mathbb{R}^n and let

$$d\mu_1 = \frac{e^{-V(x)} \chi_{n^5 B_2^n}(x) dx}{\int_{n^5 B_2^n} e^{-V(x)} dx}.$$

On the one hand, notice that, by Paouris’ inequality [BGVV, Theorem 5.2.1], there exists an absolute constant $c > 0$ such that

$$\int_{\mathbb{R}^n \setminus n^5 B_2^n} e^{-V(x)} dx = \mu\{x \in \mathbb{R}^n : |x| \geq n^5\} \leq e^{-cn^5}.$$

On the other hand, notice that for any $\theta \in S^{n-1}$, by Hölder’s inequality and by Borell’s

inequality, (see [BGVV, Theorem 2.4.6])

$$\int_{\mathbb{R}^n \setminus n^5 B_2^n} \langle x, \theta \rangle^2 e^{-V(x)} dx \leq \left(\int_{\mathbb{R}^n \setminus n^5 B_2^n} e^{-V(x)} dx \right)^{1/2} \left(\int_{\mathbb{R}^n} \langle x, \theta \rangle^4 e^{-V(x)} dx \right)^{1/2} \leq C_1 e^{-cn^5}.$$

Therefore, for every $\theta \in S^{n-1}$

$$A \leq 1 - C_1 e^{-cn^5} \leq \mathbb{E}_{\mu_1} \langle X, \theta \rangle^2 \leq \frac{1}{1 - e^{-cn^5}} \leq B,$$

where A, B are positive absolute constants and then $A \leq \|\text{Cov}_{\mu_1}\|_{op} \leq B$ and then, if we take $d\mu_2(x) = d\mu_1(Tx)$ such that μ_2 is isotropic, we have that $\text{supp } \mu_2 \subseteq Cn^5 B_2^n$, where C is an absolute constant. Assume that for this absolute constant C we can ensure that $\psi_{\mu_2} \geq c_1 C_n$ and then $\psi_{\mu_1} \geq c_1 C_n$.

For every integrable locally Lipschitz function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\mathbb{E}_\mu g = 0$ we have that

$$\begin{aligned} |\mathbb{E}_{\mu_1} g| \int_{n^5 B_2^n} e^{-V(x)} dx &= |\mathbb{E}_{\mu_1} g - \mathbb{E}_\mu g| \int_{n^5 B_2^n} e^{-V(x)} dx \\ &= \left| \left(\int_{n^5 B_2^n} g(x) e^{-V(x)} dx \right) \left(1 - \int_{n^5 B_2^n} e^{-V(x)} dx \right) - \left(\int_{\mathbb{R}^n \setminus n^5 B_2^n} g(x) e^{-V(x)} dx \right) \left(\int_{n^5 B_2^n} e^{-V(x)} dx \right) \right| \\ &\leq (\mathbb{E}_\mu g^2)^{1/2} \left[\left(\int_{n^5 B_2^n} e^{-V(x)} dx \right)^{1/2} \left(\int_{\mathbb{R}^n \setminus n^5 B_2^n} e^{-V(x)} dx \right) + \left(\int_{\mathbb{R}^n \setminus n^5 B_2^n} e^{-V(x)} dx \right)^{1/2} \left(\int_{n^5 B_2^n} e^{-V(x)} dx \right) \right] \\ &\leq (\mathbb{E}_\mu g^2)^{1/2} \left[e^{-cn^5} + e^{-\frac{c}{2}n^5} \right] \leq 2e^{-\frac{c}{2}n^5} (\mathbb{E}_\mu g^2)^{1/2} = 2e^{-\frac{c}{2}n^5} (\text{Var}_\mu g)^{1/2}. \end{aligned}$$

and

$$\int_{\mathbb{R}^n \setminus n^5 B_2^n} |g(x)| e^{-V(x)} dx \leq \left(\int_{\mathbb{R}^n \setminus n^5 B_2^n} e^{-V(x)} dx \right)^{1/2} (\text{Var}_\mu g)^{1/2} \leq e^{-cn^5} (\text{Var}_\mu g)^{1/2}.$$

Therefore, taking into account the relation between Cheeger-type isoperimetric inequalities and Poincaré-type inequalities, and the equivalence between the constants in different Poincaré-type inequalities (see, for instance, [AB1, Theorem 1.11]), we have that for every

integrable locally Lipschitz function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\mathbb{E}_\mu g = 0$

$$\begin{aligned}
\mathbb{E}_\mu |g| &= \mathbb{E}_{\mu_1} |g| \int_{n^5 B_2^n} e^{-V(x)} dx + \int_{\mathbb{R}^n \setminus n^5 B_2^n} |g(x)| e^{-V(x)} dx \\
&\leq \mathbb{E}_{\mu_1} |g - \mathbb{E}_{\mu_1} g| \int_{n^5 B_2^n} e^{-V(x)} dx + |\mathbb{E}_{\mu_1} g| \int_{n^5 B_2^n} e^{-V(x)} dx + \int_{\mathbb{R}^n \setminus n^5 B_2^n} |g(x)| e^{-V(x)} dx \\
&\leq \frac{c_2}{\psi_{\mu_1}} \mathbb{E}_{\mu_1} |\nabla g| \int_{n^5 B_2^n} e^{-V(x)} dx + 2e^{-\frac{c}{2}n^5} (\text{Var}_\mu g)^{1/2} + e^{-cn^5} (\text{Var}_\mu g)^{1/2} \\
&\leq \frac{c_2}{\psi_{\mu_1}} \mathbb{E}_{\mu_1} |\nabla g| \int_{n^5 B_2^n} e^{-V(x)} dx + 3e^{-\frac{c}{2}n^5} (\text{Var}_\mu g)^{1/2} \leq \frac{c_2}{\psi_{\mu_1}} \mathbb{E}_\mu |\nabla g| + \frac{c_3}{\psi_\mu} e^{-\frac{c}{2}n^5} (\mathbb{E}_\mu |\nabla g|^2)^{1/2}.
\end{aligned}$$

Since by Lee & Vempala's result and the fact that ψ_μ is bounded from above by an absolute constant, there exist absolute constants such that

$$\frac{c_3}{\psi_\mu} e^{-\frac{c}{2}n^5} \leq c_4 n^{1/4} e^{-\frac{c}{2}n^5} \leq c_5 \leq \frac{c_6}{\psi_{\mu_1}}$$

and then, for every integrable locally Lipschitz function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\mathbb{E}_\mu g = 0$,

$$\mathbb{E}_\mu |g| \leq \frac{c_7}{\psi_{\mu_1}} \|\nabla g\|_\infty.$$

Therefore, since for every integrable locally Lipschitz function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ we have that $g_1 = g - \mathbb{E}_\mu g$ verifies that $\mathbb{E}_\mu g_1 = 0$ and $\nabla g_1 = \nabla g$, we have that for every integrable locally Lipschitz function $g : \mathbb{R}^n \rightarrow \mathbb{R}$

$$\mathbb{E}_\mu |g - \mathbb{E}_\mu g| \leq \frac{c_7}{\psi_{\mu_1}} \|\nabla g\|_\infty.$$

Thus,

$$\psi_\mu \geq c_8 \psi_{\mu_1} \geq \tilde{c}_1 C_n.$$

□

Remark 2.2. Let us point out that the same proof would work with any power of n larger than $\frac{1}{2}$ instead of n^5 .

2.2 The general strategy

From now on, we will consider μ to be an isotropic log-concave probability measure such that $\text{supp } \mu \subseteq Cn^5 B_2^n$ where C is the absolute constant in Lemma 2.1. Let us recall that we consider Lee & Vempala's choice in the system of stochastic differential equations in Eldan's localization scheme. That is,

$$(2) \quad dc_t = b_t dt + dW_t, \quad c_0 = 0$$

where W_t a n -dimensional Wiener process and b_t is the barycenter of the density $f_t(x)$ given by

$$(3) \quad f_t(x) = \frac{e^{\langle c_t, x \rangle - \frac{t}{2}|x|^2} f(x)}{\int_{\mathbb{R}^n} e^{\langle c_t, x \rangle - \frac{t}{2}|x|^2} f(x) dx} \quad b_t = \int_{\mathbb{R}^n} x f_t(x) dx.$$

The probability measure with density $f_t(x)$ will be denoted by μ_t and its covariance matrix will be denoted by A_t .

Let us recall that, given μ an isotropic log-concave probability, our goal is to find two values $\Theta, C > 0$ such that for any Borel set $E \subseteq \mathbb{R}^n$ with $\mu(E) = 1/2$.

$$\mu(E^\Theta \setminus E) \geq C$$

in order to apply the following proposition:

Proposition 2.3 ([M]). *Let μ be a log-concave probability on \mathbb{R}^n . Assume that there exist two positive numbers $\Theta, C > 0$ such that*

$$\mu(E^\Theta \setminus E) \geq C$$

for any Borel set $E \in \mathbb{R}^n$ such that $\mu(E) = \frac{1}{2}$, where E^Θ is the Θ -dilation of E , i.e. $E^\Theta = \{e + x \in \mathbb{R}^n : e \in E, |x| < \Theta\}$. Then,

$$\mu^+(A) \geq \frac{C}{\Theta} \min\{\mu(A), \mu(A^c)\} \quad \text{for any Borel set } A \subset \mathbb{R}^n.$$

In order to control the probability of dilations of Borel sets, the following concentration results for *more log-concave than Gaussian* probabilities can be applied

Proposition 2.4. *Let ϕ be a convex function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ and let $t > 0$. Assume that*

$$d\mu(x) = e^{-\phi(x) - \frac{t}{2}|x|^2} dx,$$

is a centered probability on \mathbb{R}^n . Then for every Borel set $A \subset \mathbb{R}^n$ such that

$$\frac{1}{10} \leq \mu(A) \leq \frac{9}{10}$$

we have

$$\mu\left(A^{\frac{D}{\sqrt{t}}}\right) \geq \frac{95}{100},$$

where $D > 0$ is a suitably chosen absolute constant independent of every other parameter and $A^{D/\sqrt{t}}$ is the D/\sqrt{t} -dilation of A .

Remark 2.5. *Notice that, due to the fact that*

$$d\mu_t(x) = \frac{e^{\langle c_t, x \rangle - \frac{t}{2}|x|^2} f(x) dx}{\int_{\mathbb{R}^n} e^{\langle c_t, x \rangle - \frac{t}{2}|x|^2} f(x) dx}$$

is more log-concave than the Gaussian probability, as a trivial application of both propositions,

$$\mu_t^+(A) \geq c_1 \sqrt{t} \min\{\mu_t(A), \mu_t(A^c)\}, \quad \text{for any Borel set } A \subset \mathbb{R}^n,$$

where, a fortiori, $0 < c_1 < 1$ is an absolute constant.

In the sequel $d\mu(x) = f(x)dx$ is an isotropic log-concave probability on \mathbb{R}^n and E is a fixed Borel set in \mathbb{R}^n such that $\mu(E) = 1/2$. We introduce the stochastic process

$$g_E(t) = g(t) = \mu_t(E) = \int_E f_t(x) dx, \quad t \geq 0,$$

where μ_t and $f_t(x)$ are defined by the system of stochastic differential equations (2) and by (3). It is obvious that $g(0) = 1/2$, $\forall \omega \in \Omega$, since $f_0(x) = f(x)$ for every $\omega \in \Omega$. Besides, $(g(t))_{t \geq 0}$ is a martingale (see [AB2]) and for every $t \geq 0$ the expected value of $g(t)$ is $\mathbb{E}_{\mathbb{P}} g(t) = 1/2$.

Let $T > 0$ be a time to be precised later and notice that for any $\Theta > 0$, since also

$(g_{E^\Theta \setminus E}(t))_{t \geq 0}$ is a martingale,

$$\begin{aligned}\mu(E^\Theta \setminus E) &= \int_{E^\Theta \setminus E} f(x) dx = \int_{E^\Theta \setminus E} \mathbb{E}_{\mathbb{P}} f_T(x) dx \\ &= \mathbb{E}_{\mathbb{P}} \int_{E^\Theta \setminus E} f_T(x) dx = \mathbb{E}_{\mathbb{P}} \mu_T(E^\Theta \setminus E).\end{aligned}$$

In order to apply the preceding propositions we will consider the event

$$\mathcal{G} = \{\omega \in \Omega : |g(T) - 1/2| \leq 1/4\}.$$

By Proposition 2.4 and the way that the densities f_t are defined, we will have that there exists some absolute constant $D > 0$ such that for $\omega \in \mathcal{G}$ we will have $\mu_T(E^{D/\sqrt{T}}) \geq 0.95$ and therefore, by Markov's inequality,

$$\mu(E^{D/\sqrt{T}} \setminus E) = \mathbb{E}_{\mathbb{P}} \mu_T(E^{D/\sqrt{T}} \setminus E) \geq (0.95 - 0.5) \mathbb{P}(\mathcal{G}) = \frac{9}{20} \mathbb{P}(\mathcal{G}).$$

Hence, if we find $T, C_1 > 0$ independent of E such that $\mathbb{P}(\mathcal{G}) > C_1$ then we will get that

$$\mu^+(A) \geq \frac{C_1 \sqrt{T}}{D} \min\{\mu(A), \mu(A^c)\} \quad \forall A \text{ Borel set } \subset \mathbb{R}^n.$$

2.3 Estimating the probability of \mathcal{G}

In order to bound from below $\mathbb{P}\{|g(T) - 1/2| \leq 1/4\}$ by a positive absolute constant, for a particular choice of a time T , we recall that, as obtained in [AB2],

$$g(T) - \frac{1}{2} = g(T) - g(0) = \int_0^T dg(t) = \int_0^T \langle \eta_t, dW_t \rangle$$

where $\eta_t = \int_E f_t(x)(x - b_t) dx$, being f_t the density of the probability measure μ_t defined in (3).

The function $g(t)$ is a martingale and so, by Dambis, Dubins-Schwarz theorem, [AB2, Proposition 2.7], we have that in distribution

$$g(T) - g(0) = \bar{W}_{[g]_T},$$

where \bar{W}_s is a Wiener process and $[g]_T$ is the quadratic variation of g , which is,

$$[g]_T = \int_0^T |\eta_t|^2 dt.$$

Hence, for any $M > 0$,

$$\mathbb{P}\{|g(T) - 1/2| > 1/4\} = \mathbb{P}\{|\bar{W}_{[g]_T}| > 1/4\} \leq \mathbb{P}\{[g]_T > M\} + \mathbb{P}\left\{\max_{0 \leq t \leq M} |\bar{W}_t| > \frac{1}{4}\right\}.$$

We will bound both summands from above for an appropriate choice of M . Taking into account that for every $t \geq 0$

$$\begin{aligned} |\eta_t| &= \left\langle \eta_t, \frac{\eta_t}{|\eta_t|} \right\rangle = \int_E f_t(x) \left\langle (x - b_t), \frac{\eta_t}{|\eta_t|} \right\rangle dx \\ &\leq \sqrt{\mathbb{E}_{\mu_t} \left\langle (x - b_t), \frac{\eta_t}{|\eta_t|} \right\rangle^2} = \sqrt{\left\langle A_t \frac{\eta_t}{|\eta_t|}, \frac{\eta_t}{|\eta_t|} \right\rangle} \leq \sqrt{\|A_t\|_{\text{op}}}, \end{aligned}$$

we have that

$$[g]_T \leq \int_0^T \|A_t\|_{\text{op}} dt$$

and then

$$(4) \quad \mathbb{P}\{[g]_T > M\} \leq \mathbb{P}\left\{\int_0^T \|A_t\|_{\text{op}} dt > M\right\}.$$

On the other hand, $(-\bar{W}_t)_{t \geq 0}$ is also a Brownian motion and then, by the reflection principle, [AB2, Proposition 2.6], we have

$$\begin{aligned} (5) \quad \mathbb{P}\left\{\max_{0 \leq t \leq M} |\bar{W}_t| > \frac{1}{4}\right\} &\leq \mathbb{P}\left\{\max_{0 \leq t \leq M} \bar{W}_t > \frac{1}{4}\right\} + \mathbb{P}\left\{\max_{0 \leq t \leq M} -\bar{W}_t > \frac{1}{4}\right\} \\ &= 4\mathbb{P}\left\{\bar{W}_M > \frac{1}{4}\right\} \leq 4 \exp\left(-\frac{1}{32M}\right). \end{aligned}$$

Hence

$$\mathbb{P}\{|g(T) - 1/2| > 1/4\} \leq \mathbb{P}\left\{\int_0^T \|A_t\|_{\text{op}} dt > M\right\} + 4 \exp\left(-\frac{1}{32M}\right).$$

Our purpose is to find a suitable $M > 0$ and $T > 0$ such that we can ensure that the latter upper bound on the probability of \mathcal{G}^c is strictly smaller than 1. We will take $M = 1/256$. Then

$$\mathbb{P}\{|g(T) - 1/2| > 1/4\} \leq \mathbb{P}\left\{\int_0^T \|A_t\|_{\text{op}} dt > \frac{1}{256}\right\} + 4 \exp(-8)$$

and, by showing that for an appropriate choice of T the latter probability is bounded above by $\frac{1}{10}$, we will obtain the desired lower bound on the probability of \mathcal{G} .

3 Chen's estimate on $\|A_t\|_{\text{op}}$

Let us recall that the main result, which allowed to obtain Theorem 1.1 by following the described strategy, was the following:

Proposition 3.1. *[AB2, Proposition 5.1] Given the system of stochastic differential equations (2), let A_t be the covariance matrix of the measure μ_t defined by (3). Let $p \geq 2$ be an integer. Then*

$$d(\text{Tr}(A_t^p)) = \delta_t dt + \langle v_t, dW_t \rangle$$

where δ_t is an adapted, with bounded variation process, such that

$$\delta_t \leq \begin{cases} C p^2 \sigma_n^2 \log n \text{Tr}(A_t^p)^{1+\frac{1}{p}}, & \text{if } p \geq 3 \\ C \text{Tr}(A_t^2)^{3/2}, & \text{if } p = 2 \end{cases}$$

and

$$|v_t| \leq C p \text{Tr}(A_t^p)^{1+\frac{1}{2p}} \quad \forall p \geq 2,$$

where $C > 0$ is an absolute constant and $\sigma_n^2 = \sup \mathbb{E}|X| - \sqrt{n}|^2$ and the sup runs over all isotropic log-concave random vectors in \mathbb{R}^n .

In this section we present Chen's improvement on Proposition 3.1 and some consequences on the estimates of $\|A_t\|_{\text{op}}$.

Proposition 3.2. *Given the system of stochastic differential equations (2), let A_t be the covariance matrix of the measure μ_t defined by (3). Let $p \geq 3$ be an integer. Then*

$$d(\text{Tr}(A_t^p)) = \delta_t dt + \langle v_t, dW_t \rangle$$

where δ_t is an adapted, with bounded variation process, such that

$$\delta_t \leq Cp^2 \text{Tr}(A_t^p) \min \left\{ \frac{1}{t}, \frac{\|A_t\|_{op}}{\psi_n^2} \right\}, \quad |v_t| \leq Cp \text{Tr}(A_t^p)^{1+\frac{1}{2p}},$$

where $C > 0$ is an absolute constant and ψ_n is defined as in Theorem 1.1.

Proof. The estimate for $|v_t|$ is the same appearing in Proposition 3.1. We will follow the ideas appearing in [K] in order to estimate δ_t . Let us recall that, after Lee & Vempala's result collected in Theorem 1.1, we know that $\psi_n \geq \frac{C}{n^{1/4}}$ for some absolute constant C and that one trivially has $\psi_n \leq C$ for another absolute constant C .

According to the proof of Proposition 3.1,

$$\begin{aligned} \delta_t &\leq \frac{1}{2}p(p-1) \sum_{i=1}^n (\alpha_{ii})^{p-2} |\xi_{ii}|^2 + \sum_{\substack{i \neq j \\ 0 \leq k \leq p-2}} (\alpha_{ii})^k (\alpha_{jj})^{p-k-2} |\xi_{ij}|^2 \\ &\leq \frac{1}{2}p(p-1) \sum_{i=1}^n (\alpha_{ii})^{p-2} |\xi_{ii}|^2 + 2 \sum_{\substack{i < j \\ 0 \leq k \leq p-2}} (\alpha_{ii})^{p-2} |\xi_{ij}|^2 \\ &\leq p(p-1) \sum_{i=1}^n (\alpha_{ii})^{p-2} |\xi_{ii}|^2 + p(p-1) \sum_{i < j} (\alpha_{ii})^{p-2} |\xi_{ij}|^2 \\ &\leq p(p-1) \sum_{i,j=1}^n (\alpha_{ii})^{p-2} |\xi_{ij}|^2, \end{aligned}$$

where $(v_i)_{i=1}^n$ is an orthonormal basis of eigenvectors of the covariance matrix A_t , $\alpha_{ij} = \alpha_{ij}(t) = \langle A_t v_i, v_j \rangle$, ordered in such a way that $\alpha_{11} \geq \alpha_{22} \geq \dots \geq \alpha_{nn}$, and ξ_{ij} are the vectors $\xi_{ij} = \xi_{i,j}(t) = \mathbb{E}_{\mu_t} \langle x - b_t, v_i \rangle \langle x - b_t, v_j \rangle (x - b_t) \in \mathbb{R}^n$.

Let, for any $1 \leq i \leq n$, ξ_i be the symmetric matrix

$$\xi_i = \mathbb{E}_{\mu_t} \langle x - b_t, v_i \rangle (x - b_t) \otimes (x - b_t).$$

Then,

$$\text{Tr}(\xi_i^2) = \sum_{j,k=1}^n (\mathbb{E}_{\mu_t} \langle x - b_t, v_i \rangle \langle x - b_t, v_j \rangle \langle x - b_t, v_k \rangle)^2 = \sum_{j=1}^n |\xi_{ij}|^2.$$

Therefore,

$$\delta_t \leq p(p-1) \sum_{i=1}^n (\alpha_{ii})^{p-2} \text{Tr}(\xi_i^2).$$

Furthermore, since

$$\xi_i^2 = \xi_i \circ \mathbb{E}_{\mu_t} \langle x - b_t, v_i \rangle (x - b_t) \otimes (x - b_t) = \mathbb{E}_{\mu_t} \langle x - b_t, v_i \rangle (x - b_t) \otimes \xi_i(x - b_t)$$

and, for every $1 \leq i \leq n$, $\mathbb{E}_{\mu_t} \langle x - b_t, v_i \rangle = 0$ we have that

$$\begin{aligned} \text{Tr}(\xi_i^2) &= \mathbb{E}_{\mu_t} \langle x - b_t, v_i \rangle \langle x - b_t, \xi_i(x - b_t) \rangle \\ &= \mathbb{E}_{\mu_t} \langle x - b_t, v_i \rangle (\langle x - b_t, \xi_i(x - b_t) \rangle - \mathbb{E}_{\mu_t} \langle y - b_t, \xi_i(y - b_t) \rangle) \\ &\leq \text{(by Cauchy-Schwarz inequality)} \\ &\leq \sqrt{\alpha_{ii}} \sqrt{\text{Var}_{\mu_t} \langle x - b_t, \xi_i(x - b_t) \rangle}. \end{aligned}$$

Taking into account the well-known relationship of Cheeger-type isoperimetric inequalities with Poincaré's inequality (see, for instance, [AB1, Theorems 1.1 and 1.8]) we have that for any locally Lipschitz integrable function g and any log-concave probability μ in \mathbb{R}^n

$$\text{Var}_{\mu} g \leq \frac{C \|\text{Cov}_{\mu}\|_{\text{op}}}{\psi_n^2} \mathbb{E}_{\mu} |\nabla g|^2$$

for some absolute constant $C > 0$. Besides, by Remark 2.5, for any locally Lipschitz integrable function g

$$\text{Var}_{\mu_t}(g) \leq \frac{C}{t} \mathbb{E}_{\mu_t} |\nabla g|^2.$$

Therefore,

$$\text{Var}_{\mu_t} \langle x - b_t, \xi_i(x - b_t) \rangle \leq C \min \left\{ \frac{1}{t}, \frac{\|A_t\|_{\text{op}}}{\psi_n^2} \right\} \mathbb{E}_{\mu_t} |\nabla \langle x - b_t, \xi_i(x - b_t) \rangle|^2.$$

Since

$$\nabla \langle x - b_t, \xi_i(x - b_t) \rangle = (\xi_i + \xi_i^t)(x - b_t) = 2\xi_i(x - b_t),$$

we have that

$$\begin{aligned}\mathrm{Tr}(\xi_i^2) &\leq C\sqrt{\alpha_{ii}}\sqrt{\min\left\{\frac{1}{t}, \frac{\|A_t\|_{\mathrm{op}}}{\psi_n^2}\right\}}\sqrt{\mathbb{E}_{\mu_t}|\xi_i(x-b_t)|^2} \\ &\leq C\sqrt{\alpha_{ii}}\sqrt{\min\left\{\frac{1}{t}, \frac{\|A_t\|_{\mathrm{op}}}{\psi_n^2}\right\}}\sqrt{\mathrm{Tr}(A_t\xi_i^2)}.\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{i=1}^n(\alpha_{ii})^{p-2}\mathrm{Tr}(\xi_i^2) &\leq C\sqrt{\min\left\{\frac{1}{t}, \frac{\|A_t\|_{\mathrm{op}}}{\psi_n^2}\right\}}\sum_{i=1}^n\alpha_{ii}^{p-3/2}\sqrt{\mathrm{Tr}(A_t\xi_i^2)} \\ &\leq (\text{by Cauchy-Schwarz inequality}) \\ &\leq C\sqrt{\min\left\{\frac{1}{t}, \frac{\|A_t\|_{\mathrm{op}}}{\psi_n^2}\right\}}\sqrt{\sum_{i=1}^n\alpha_{ii}^p}\sqrt{\sum_{i=1}^n\alpha_{ii}^{p-3}\mathrm{Tr}(A_t\xi_i^2)} \\ &= C\sqrt{\min\left\{\frac{1}{t}, \frac{\|A_t\|_{\mathrm{op}}}{\psi_n^2}\right\}}\sqrt{\mathrm{Tr}(A_t^p)}\sqrt{\sum_{i=1}^n\alpha_{ii}^{p-3}\mathrm{Tr}(A_t\xi_i^2)}.\end{aligned}$$

Now, since the matrix A_t is diagonal in the basis $(v_j)_{j=1}^n$ and the entry $(\xi_i^2)_{jj}$ of the matrix ξ_i^2 is

$$(\xi_i^2)_{jj} = \sum_{k=1}^n (\mathbb{E}_{\mu_t}\langle x-b_t, v_i\rangle\langle x-b_t, v_j\rangle\langle x-b_t, v_k\rangle)^2 = |\xi_{ij}|^2,$$

we have that

$$\sum_{i=1}^n\alpha_{ii}^{p-3}\mathrm{Tr}(A_t\xi_i^2) = \sum_{i,j=1}^n\alpha_{ii}^{p-3}\alpha_{jj}(\xi_i^2)_{jj} = \sum_{i,j=1}^n\alpha_{ii}^{p-3}\alpha_{jj}|\xi_{ij}|^2.$$

For every $1 \leq i, j \leq n$ we have that $\alpha_{ii}^{p-3}\alpha_{jj} \leq \max\{\alpha_{ii}^{p-2}, \alpha_{jj}^{p-2}\} \leq \alpha_{ii}^{p-2} + \alpha_{jj}^{p-2}$. Thus,

$$\sum_{i=1}^n\alpha_{ii}^{p-3}\mathrm{Tr}(A_t\xi_i^2) \leq \sum_{i,j=1}^n\alpha_{ii}^{p-2}|\xi_{ij}|^2 + \sum_{i,j=1}^n\alpha_{jj}^{p-2}|\xi_{ij}|^2 = 2\sum_{i,j=1}^n\alpha_{ii}^{p-2}|\xi_{ij}|^2 = 2\sum_{i=1}^n\alpha_{ii}^{p-2}\mathrm{Tr}(\xi_i^2)$$

and then

$$\sum_{i=1}^n (\alpha_{ii})^{p-2} \text{Tr}(\xi_i^2) \leq C \sqrt{\min \left\{ \frac{1}{t}, \frac{\|A_t\|_{\text{op}}}{\psi_n^2} \right\}} \sqrt{\text{Tr}(A_t^p)} \sqrt{\sum_{i=1}^n \alpha_{ii}^{p-2} \text{Tr}(\xi_i^2)},$$

where C is an absolute constant. Therefore,

$$\sum_{i=1}^n (\alpha_{ii})^{p-2} \text{Tr}(\xi_i^2) \leq C \min \left\{ \frac{1}{t}, \frac{\|A_t\|_{\text{op}}}{\psi_n^2} \right\} \text{Tr}(A_t^p)$$

and

$$\delta_t \leq p(p-1) \sum_{i=1}^n (\alpha_{ii})^{p-2} \text{Tr}(\xi_i^2) \leq Cp^2 \min \left\{ \frac{1}{t}, \frac{\|A_t\|_{\text{op}}}{\psi_n^2} \right\} \text{Tr}(A_t^p),$$

where C is an absolute constant. □

Proposition 3.3. *There exists an absolute constant $c > 0$ and $n_0 \in \mathbb{N}$ such that if $T_0 := \frac{c\psi_n^2}{\log n}$, then for every $p \geq 3$ and every $n \geq n_0$, we have*

$$\max_{t \in [0, T_0]} \mathbb{E}_{\mathbb{P}} \|A_t\|_{\text{op}} \leq 3 \quad \text{and} \quad \max_{t \in [0, T_0]} \mathbb{E}_{\mathbb{P}} (\text{Tr}(A_t^p))^{1/p} \leq 3n^{1/p}.$$

Proof. First of all notice that, since ψ_n is bounded from above by an absolute constant, we can choose an absolute $0 < c_1 < 1/2$ and take $T_0 := \frac{c_1 \psi_n^2}{40^2 C \log n}$, where C is the constant appearing in Proposition 3.2.

Let us consider the stopping time $\tau(\omega) = \inf \{t > 0; \|A_t\|_{\text{op}} \geq 2\}$. It is clear that, by continuity, for $\omega \in \Omega$ and $t \leq \tau(\omega)$, $\|A_t\|_{\text{op}} \leq 2$. We define the following stochastic process $X_t(\omega) = \text{Tr}(A_{\min\{t, \tau(\omega)\}}^p)$. It is an Itô process. Indeed, $dX_t = \bar{\delta}_t dt + \langle \bar{v}_t, dW_t \rangle$ where

$$\bar{\delta}_t = \begin{cases} \delta_t & \text{if } t < \tau(\omega) \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \bar{v}_t = \begin{cases} v_t & \text{if } t < \tau(\omega) \\ 0, & \text{otherwise} \end{cases}.$$

If $0 \leq t < \tau(\omega)$,

$$\bar{\delta}_t = \delta_t \leq Cp^2 \frac{2}{\psi_n^2} \text{Tr}(A_t^p) = Cp^2 \frac{2}{\psi_n^2} X_t.$$

Since the latter inequality is trivially true if $t \geq \tau(\omega)$ we have that it holds for every $t > 0$.

It is clear that the stochastic process given by $Z_t = \int_0^t \langle \bar{v}_s, dW_s \rangle$ is a martingale and

then $\mathbb{E}_{\mathbb{P}} Z_t = 0$. Hence the deterministic function $\mathbb{E}_{\mathbb{P}} X_t$ verifies

$$\frac{d}{dt} \mathbb{E}_{\mathbb{P}} X_t \leq Cp^2 \frac{2}{\psi_n^2} \mathbb{E}_{\mathbb{P}} X_t, \quad \text{a.e. } t > 0.$$

Then, taking into account that $\mathbb{E}_{\mathbb{P}} X_0 = n$ and that $0 < c_1 < \frac{1}{2}$, we have that for every $t \in [0, T_0]$

$$\mathbb{E}_{\mathbb{P}} X_t \leq n \exp \left(\frac{2Cp^2}{\psi_n^2} T_0 \right) \leq n \exp \left(\frac{p^2}{40^2 \log n} \right)$$

Optimizing for $p = \lceil 40 \log n \rceil$ we obtain

$$\mathbb{E}_{\mathbb{P}} X_t \leq n \exp(\log n) = n^2 \quad \forall t \in [0, T_0].$$

Therefore, since for this value of p and $n \geq 3$

$$X_{\tau(\omega)} = \text{Tr}(A_{\tau(\omega)}^{[40 \log n]}) \geq \|A_{\tau(\omega)}\|_{\text{op}}^{30 \log n} = 2^{30 \log n}, \quad \forall t \in [0, T_0]$$

we achieve, using Markov's inequality, that for every $t \in [0, T_0]$

$$n^2 \geq \mathbb{E}_{\mathbb{P}} X_t \geq \int_{\{\omega: t > \tau(\omega)\}} X_t d\mathbb{P} = \int_{\{\omega: t > \tau(\omega)\}} X_{\tau(\omega)} d\mathbb{P} \geq 2^{30 \log n} \mathbb{P}\{\omega : t > \tau(\omega)\}$$

and then

$$\mathbb{P}\{\omega : t > \tau(\omega)\} \leq \frac{n^2}{n^{2^{30}}}, \quad \forall t \in [0, T_0].$$

Then, for every $n \geq n_0$, for some $n_0 \in \mathbb{N}$

$$\mathbb{E}_{\mathbb{P}} \|A_t\|_{\text{op}} \leq C_1 \frac{n^{12}}{n^{2^{30}}} + 2 \leq 3 \quad \forall t \in [0, T_0],$$

where we have used that

$$\|A_t\|_{\text{op}} = \sup_{\theta} \mathbb{E}_{\mu_t} \langle x - b_t, \theta \rangle^2 \leq 4(\text{diameter}(\text{supp } \mu_t))^2 = 4(\text{diameter}(\text{supp } \mu))^2 \leq C_1 n^{10}.$$

Besides, since for any $p \geq 3$,

$$\|A_t\|_{\text{op}} \leq (\text{Tr}(A_t^p))^{1/p} \leq \|A_t\|_{\text{op}} n^{1/p}$$

we have that

$$\mathbb{E}_{\mathbb{P}}(\text{Tr}(A_t^p))^{1/p} \leq 3n^{1/p} \quad \forall t \in [0, T_0].$$

□

Lemma 3.4. *Let $T_0 = c \frac{\psi_n^2}{\log n}$ as in Proposition 3.3. Let $T_1, T_2 > 0$ be such that $0 < T_1 \leq T_0$ and $0 < T_1 \leq T_2$. Then, for every $p \geq 3$ and $n \geq n_0$, for some $n_0 \in \mathbb{N}$, we have that*

$$\mathbb{P} \left\{ \int_0^{T_1} \|A_t\|_{\text{op}} dt \geq \frac{1}{512} \right\} \leq 1536 T_1$$

and

$$\mathbb{P} \left\{ \int_{T_1}^{T_2} \|A_t\|_{\text{op}} dt \geq \frac{1}{512} \right\} \leq 1536 n^{1/p} \left(\frac{T_2}{T_1} \right)^{Cp} T_2.$$

Proof. By Markov's inequality, since $T_1 < T_0$ and, as seen in Proposition 3.3, $\mathbb{E}_{\mathbb{P}} \|A_t\|_{\text{op}} \leq 3$ for every $t \in [0, T_0]$,

$$\mathbb{P} \left\{ \int_0^{T_1} \|A_t\|_{\text{op}} dt \geq \frac{1}{512} \right\} \leq 512 \mathbb{E}_{\mathbb{P}} \int_0^{T_1} \|A_t\|_{\text{op}} dt = 512 \int_0^{T_1} \mathbb{E}_{\mathbb{P}} \|A_t\|_{\text{op}} dt \leq 1536 T_1.$$

Consider the stochastic process $H_t = (\text{Tr}(A_t^p))^{1/p}$ which is an Itô process. Therefore, by Itô's formula,

$$dH_t = \frac{1}{p} (\text{Tr}(A_t^p))^{1/p-1} d(\text{Tr}(A_t^p)) + \frac{1}{2p} \left(\frac{1}{p} - 1 \right) (\text{Tr}(A_t^p))^{1/p-2} d[\text{Tr}(A_t^p)]_t = \eta_t dt + dM_t,$$

where, by Proposition 3.2, dM_t is a martingale with $M_0 = 0$ and η_t is an adapted process such that $\eta_t \leq C \frac{p}{t} H_t$. Taking expectation we have

$$\frac{d}{dt} \mathbb{E}_{\mathbb{P}} H_t \leq C \frac{p}{t} \mathbb{E}_{\mathbb{P}} H_t \Leftrightarrow \frac{d}{dt} \log (\mathbb{E}_{\mathbb{P}} H_t) \leq C \frac{p}{t}.$$

Integrating in the interval $[T_1, t] \subseteq [T_1, T_2]$ we deduce that for any $T_1 < t < T_2$

$$\mathbb{E}_{\mathbb{P}} H_t \leq \mathbb{E}_{\mathbb{P}} H_{T_1} \left(\frac{t}{T_1} \right)^{Cp}.$$

Taking into account that, by Proposition 3.3, $\mathbb{E}_{\mathbb{P}} H_{T_1} \leq 3n^{1/p}$, for any value of $p \geq 3$, we

obtain

$$\mathbb{E}_{\mathbb{P}} \int_{T_1}^{T_2} \|A_t\|_{\text{op}} dt \leq \mathbb{E}_{\mathbb{P}} \int_{T_1}^{T_2} H_t dt = \int_{T_1}^{T_2} \mathbb{E}_{\mathbb{P}} H_t dt \leq 3n^{1/p} \int_{T_1}^{T_2} \left(\frac{t}{T_1}\right)^{Cp} dt \leq 3n^{1/p} \left(\frac{T_2}{T_1}\right)^{Cp} T_2.$$

Hence, by Markov's inequality,

$$\mathbb{P} \left\{ \int_{T_0}^{T_1} \|A_t\|_{\text{op}} dt \geq \frac{1}{512} \right\} \leq 512 \mathbb{E}_{\mathbb{P}} \int_{T_1}^{T_2} \|A_t\|_{\text{op}} dt \leq 1536n^{1/p} \left(\frac{T_2}{T_1}\right)^{Cp} T_2.$$

□

4 Proof of Chen's Theorem

In this section we complete the proof of Chen's estimate of ψ_n .

Proof of Theorem 1.2. Let $T_0 = c \frac{\psi_n^2}{\log n}$ for some absolute constant c , as in Proposition 3.3

and let $T_1 = T_0$, $p = \sqrt{\frac{\log n}{\log \log n}}$, and $T_2 = \alpha n^{-\frac{1}{p(Cp+1)}} T_0^{\frac{Cp}{Cp+1}}$ for this fixed value of p and some $0 < \alpha < 1$ to be precised later.

There are two possibilities: either $T_2 \leq T_0$ or $T_2 > T_0$. Assume first that $T_2 \leq T_0$. According to Lemma 3.4, and taking into account that $\psi_n \leq C$ for some positive absolute constant C , we have that

$$\begin{aligned} \mathbb{P} \left\{ \int_0^{T_2} \|A_t\|_{\text{op}} dt \geq \frac{1}{256} \right\} &\leq \mathbb{P} \left\{ \int_0^{T_2} \|A_t\|_{\text{op}} dt \geq \frac{1}{512} \right\} \leq 1536 T_2 \\ &\leq 1536 T_0 \leq \frac{1536c\psi_n^2}{\log n} \leq \frac{1}{10} \end{aligned}$$

for every $n > n_0$, for some $n_0 \in \mathbb{N}$.

Assume now that $T_2 > T_0$. By Lemma 3.4, and taking into account that $1 + x \leq 2x$

whenever $x \geq 1$, we have

$$\begin{aligned} \mathbb{P} \left\{ \int_0^{T_2} \|A_t\|_{\text{op}} dt \geq \frac{1}{256} \right\} &\leq \mathbb{P} \left\{ \int_0^{T_0} \|A_t\|_{\text{op}} dt \geq \frac{1}{512} \right\} + \mathbb{P} \left\{ \int_{T_0}^{T_2} \|A_t\|_{\text{op}} dt \geq \frac{1}{512} \right\} \\ &\leq 1536 \left(T_0 + n^{1/p} \left(\frac{T_2}{T_0} \right)^{C_p} T_2, \right) = 1536 T_0 \left(1 + n^{1/p} \left(\frac{T_2}{T_0} \right)^{C_{p+1}} \right) \\ &\leq 3072 T_0 n^{1/p} \left(\frac{T_2}{T_0} \right)^{C_{p+1}} = 3072 \alpha^{C_{p+1}} < \frac{1}{10} \end{aligned}$$

whenever $n > n_1$, for some $n_1 \in \mathbb{N}$, by choosing $0 < \alpha < 1$ small enough.

Therefore, we can fix $0 < \alpha < 1$ such that there exists n_0 such that if $n \geq n_0$

$$\mathbb{P} \left\{ \int_0^{T_2} \|A_t\|_{\text{op}} dt \geq \frac{1}{256} \right\} \leq \frac{1}{10}.$$

By the arguments exposed in Section 2, we have that there exists an absolute constant $c_1 > 0$ such that

$$\psi_\mu \geq c_1 \sqrt{T_2} \geq c_2 \sqrt{\alpha n}^{-\frac{1}{2p(C_{p+1})}} T_0^{\frac{C_p}{2(C_{p+1})}} = c_3 \sqrt{\alpha n}^{-\frac{1}{2p(C_{p+1})}} \frac{\psi_n^{\frac{C_p}{C_{p+1}}}}{(\log n)^{\frac{C_p}{2(C_{p+1})}}}.$$

Since this inequality is true for any isotropic log-concave probability whose support is contained in $C_1 n^5 B_2^n$, where C_1 is the absolute constant in Lemma 2.1, we obtain that

$$\psi_n \geq c_4 \sqrt{\alpha n}^{-\frac{1}{2p(C_{p+1})}} \frac{\psi_n^{\frac{C_p}{C_{p+1}}}}{(\log n)^{\frac{C_p}{2(C_{p+1})}}}$$

and then

$$\psi_n \geq (c_4 \sqrt{\alpha})^{C_{p+1}} \frac{n^{-\frac{1}{2p}}}{(\log n)^{\frac{C_p}{2}}} = c_5 \exp \left(-c_6 \sqrt{\log n \cdot \log \log n} \right),$$

which finishes the proof. \square

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References

- [AB1] D. ALONSO-GUTIÉRREZ, J. BASTERO *Approaching the Kannan-Lovász-Simonovits and variance conjectures*. Lecture Notes in Mathematics, Springer. Monograph. **2131**, (2015).
- [AB2] D. ALONSO-GUTIÉRREZ, J. BASTERO *Approaching Eldan’s and Lee & Vampala’s bounds for the KLS conjecture in a unified method* Revista de la Real Academia de Ciencias de Zaragoza **75** no. 1 (2020), 85–110.
- [BGVV] S. BRAZITIKOS, A. GIANOPOULOS, P. VALETTAS, B. H. VRITSIOU, *Geometry of Isotropic Convex Bodies*, Mathematical Surveys and Monographs **196** (American Mathematical Society, Providence, RI., 2014)
- [Ch] Y. CHEN, *An almost constant lower bound of the isoperimetric coefficient in the KLS conjecture*. Geom. Funct. Anal. (GAFA) **31** (2021), 34–61.
- [E] R. ELKAN, *Thin shell implies spectral gap up to polylog via a stochastic localization scheme*. Geom. Funct. Anal. (GAFA) **23** (2013), 532–569
- [K] B. KLARTAG, Talk in the *Winter School on “The Interplay between High-Dimensional Geometry and Probability”*. January, 11th-15th, 2021. Hausdorff research Institute for Mathematics, <https://www.him.uni-bonn.de/programs/past-programs/past-trimester-programs/interplay-high-dimensional-geometry-probability/interplay-high-dimensional-geometry-probability-winterschool/>.
- [KLS] R. KANNAN, L. LOVÁSZ, M. SIMONOVITS *Isoperimetric problems for convex bodies and a localization lemma*. Discrete Comput. Geom. 13, no. 3-4, (1995), 541–559.
- [LV] Y. LEE, S. VEMPALA, *Eldan’s Stochastic Localization and the KLS hyperplane conjecture: an improved lower bound for expansion*, 58th Annual IEEE Symposim on Foundations of Computer Science-FOCS 2017. 998–1007, IEEE Computer Soc., 2017.
- [M] E. MILMAN, *Isoperimetric bounds on convex manifolds*, Proceedings of the Workshop on Concentration, Functional Inequalities and Isoperimetry, Contemporary Math. **545**, 195–208, (2011)