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Bidiagonal Decompositions and Accurate Computations for the Ballot Table and the Fibonacci Matrix

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ABSTRACT

Riordan arrays include many important examples of matrices. Here we consider the ballot table and the Fibonacci matrix. For finite truncations of these Riordan arrays, we obtain bidiagonal decompositions. Using them, algorithms to solve key linear algebra problems for ballot tables and Fibonacci matrices with high relative accuracy are derived. We include numerical experiments showing the accuracy of our method.

1 | Introduction

Riordan arrays [1] have important applications in combinatorics, where they play an important role in lattice paths enumeration and walk problems, recurrence relations or combinatorial identities (cf. [2, 3]). One property satisfied by some Riordan arrays is their total positivity (see Section 2). Total positivity is an old subject (cf. [4–7]) with applications to many fields such as mechanics, approximation theory, combinatorics, numerical analysis, finance, statistics, computer-aided geometric design, or differential equations and that has shown its great vitality in the past decades.

One recent application of total positivity in numerical analysis deals with the construction of high relative accurate algorithms. It provides examples of matrices for which such algorithms have been found through the bidiagonal decomposition of the matrix (see Section 2). In fact, for some matrices related to Riordan arrays these algorithms have been obtained (cf. [8]). This paper contributes to this field by providing important examples of finite matrices derived from the truncation of Riordan arrays with high relative accurate algorithms for linear algebra problems.

We first consider in the paper the Riordan array given by the ballot table with the ballot numbers (cf. [9]). This matrix arises from counting the number of ways candidate A can stay strictly ahead of candidate B during a sequential vote tally, as described in classical voting problems (cf. [10, 11]). More details are shown in Section 3. We also consider the Riordan array of the Fibonacci matrix, which is closely related with the inverse of the previous one. This matrix is a reorganization of Pascal's triangle such that each row sums to a Fibonacci number, capturing combinatorial structures (cf. [12]). A formal description of its applications is detailed in Section 3.

The paper is organized as follows. Section 2 presents totally positive matrices and their bidiagonal decompositions, which can be theoretically obtained through an elimination procedure called Neville elimination. Section 3 presents Riordan arrays and some examples that will be considered in this paper. Section 4 provides an accurate bidiagonal decomposition of the ballot table and of the Fibonacci matrix. It can be used to derive high relative accurate algorithms to calculate their inverses, all their singular values and the solution of some associated linear systems. Numerical experiments of Section 5 confirm the theoretical results.

2 | Total Positivity, Neville Elimination, and Bidiagonal Decompositions

This section presents some basic concepts and tools used in the paper. We start with concepts related to total positivity and we continue with Neville elimination, which is very useful in this framework.

A matrix is totally positive (TP) if all its minors are nonnegative. TP matrices are also called totally nonnegative matrices. This class of matrices have applications in many fields: see the surveys [4, 13], the classical book [6], and the recent books [5, 7].

Neville elimination (NE) is an alternative procedure to Gaussian elimination that produces zeros in a column of a matrix by adding to each row an appropriate multiple of the previous one. This elimination procedure is very useful when dealing with some classes of matrices such as TP matrices. For nonsingular TP matrices, it is always possible to perform NE without row exchanges (for more details on NE, see [14, 15]). Given a nonsingular matrix $A = (a_{ij})_{0\leq i,j\leq n}$, the Neville elimination procedure without row exchanges consists of n steps and leads to the following sequence of matrices:

$$A =: A^{(0)} \to A^{(1)} \to A^{(2)} \to \cdots \to A^{(n)} = U, \tag{1}$$

where U is an upper triangular matrix.

The entries of $A^{(k+1)} = (a_{ij}^{(k+1)})_{0\leq i,j\leq n}$ can be obtained from $A^{(k)}$ using the formula

$$a_{ij}^{(k+1)} = \begin{cases} a_{ij}^{(k)} - \frac{a_{ik}^{(k)}}{a_{i-1,k}^{(k)}} a_{i-1,j}^{(k)}, & \text{if } k+1 \leq i \leq n, k \leq j \leq n \text{ and } a_{i-1,k}^{(k)} \neq 0, \\ a_{ij}^{(k)}, & \text{otherwise,} \end{cases}$$

for $k = 0, \ldots, n-1$.

The (i,j) pivot of the NE of A is given by $p_{ij} = a_{ij}^{(j)}$, for $0 \leq j \leq i \leq n$. If $i = j$, we say that p_{ii} is a diagonal pivot. The (i,j) multiplier of the NE of A with $1 \leq j < i \leq n$, is defined as

$$m_{ij} = \begin{cases} \frac{a_{ij}^{(j)}}{a_{i-1,j}^{(j)}} = \frac{p_{ij}}{p_{i-1,j}}, & \text{if } a_{i-1,j}^{(j)} \neq 0, \\ 0, & \text{if } a_{i-1,j}^{(j)} = 0. \end{cases}$$

The multipliers satisfy that $m_{ij} = 0 \Rightarrow m_{hj} = 0$, for all $h > i$. The (i,j) multiplier of the Neville Elimination of A^T is denoted by $\widetilde{m}_{ij}$.

The following theorem is a consequence of Theorem 4.2 of [15] and characterizes nonsingular TP matrices by their bidiagonal decompositions.

Theorem 1. *A nonsingular matrix $A = (a_{ij})_{0\leq i,j\leq n}$ is TP if and only if it admits the following decomposition*:

$$A = F_{n-1}F_{n-2}\cdots F_0 D G_0 \cdots G_{n-2}G_{n-1}, \tag{2}$$

where D is the diagonal matrix $\mathrm{diag}(p_{00}, p_{11}, \dots, p_{nn})$ *with positive diagonal entries and* F_i, G_i *are the nonnegative bidiagonal matrices given by*

$$F_i = \begin{pmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ & \ddots & \ddots & & & & \\ & & 0 & 1 & & & \\ & & & m_{i+1,0} & 1 & & \\ & & & & \ddots & \ddots & \\ & & & & & m_{n,n-i-1} & 1 \end{pmatrix}, \quad G_i = \begin{pmatrix} 1 & 0 & & & & & \\ & 1 & \ddots & & & & \\ & & \ddots & 0 & & & \\ & & & 1 & \widetilde{m}_{i+1,0} & & \\ & & & & 1 & \ddots & \\ & & & & & \ddots & \widetilde{m}_{n,n-i-1} \\ & & & & & & 1 \end{pmatrix}, \tag{3}$$

for all $i \in \{0, 1, \dots, n-1\}$. *In addition, if the entries* m_{ij} *and* $\widetilde{m}_{ij}$ *satisfy*

$$\begin{aligned} m_{ij} = 0 \Rightarrow m_{hj} = 0 \quad \forall h > i, \\ \widetilde{m}_{ij} = 0 \Rightarrow \widetilde{m}_{hj} = 0 \quad \forall h > i, \end{aligned} \tag{4}$$

then the decomposition is unique.

In the bidiagonal decomposition given by (2–4), the entries m_{ij} and p_{ii} are the multipliers and diagonal pivots, respectively, corresponding to the NE of A, and the entries $\widetilde{m}_{ij}$ are the multipliers of the NE of A^T (see pp. 116 and 120 of [15]).

Bidiagonal decomposition can be used to factorize more classes of matrices (cf. [16]). If we consider the factorization given by (2) and (3) without any further requirement than the nonsingularity of D, the matrix notation $\mathcal{BD}(A)$ can be used to represent a bidiagonal decomposition (2) of a nonsingular matrix,

$$(\mathcal{BD}(A))_{ij} = \begin{cases} m_{ij}, & \text{if } i > j, \\ \widetilde{m}_{ji}, & \text{if } i < j, \\ p_{ii}, & \text{if } i = j, \end{cases} \tag{5}$$

for $0 \le i, j \le n$.

Let us recall that an algorithm can be performed with high relative accuracy (HRA) if it does not include subtractions (except of initial data), that is, if it only includes products, divisions, sums (of numbers of the same sign) and subtractions of initial data (cf. [17, 18]).

The entries of the matrices in $\mathcal{BD}(A)$ (see (5)) have been considered natural parameters associated with A in many recent references ([19–21]). In many cases, we know them with high relative accuracy. If we assume this for a nonsingular totally positive matrix A, then algorithms with high relative accuracy can be applied (see [20]) to compute the singular values and the eigenvalues of A, the inverse A^{-1} or solving certain linear systems $Ax = b$ (those where b has alternating signs). In fact, for many subclasses of nonsingular TP matrices it has been possible to obtain the bidiagonal decomposition $\mathcal{BD}(A)$ of their matrices A with HRA, so that the mentioned linear algebra computations can also be solved with HRA (see [21–23] for some examples).

3 | Riordan Arrays

Riordan arrays were introduced in 1991 in [1] and play an important role in combinatorics, where they are related to recurrence relations, walk problems and combinatorial identities, among others fields (cf. [3]).

The concept of generating function is important for the definition of a Riordan array. Before defining it, note that although the following definitions talk about commutative rings, in general it will suffice for us that the commutative ring be $\mathbb{R}$.

Definition 1. Let R be a commutative ring, and let $(a_n)_{n\geq 0}$ be a sequence with coefficients in R. The generating function (GF) for $(a_n)_{n\geq 0}$ is the formal power series

$$a(t) = \sum_{n=0}^{\infty} a_n t^n = a_0 + a_1 t + a_2 t^2 + \cdots ,$$

where t is an auxiliary variable.

Observe that GFs are algebraically defined as formal power series rather than real-valued functions. Therefore, the requirement for series convergence is not necessary for the existence of GFs.

Let us introduce a formal definition of a Riordan array and explain its two characterizations (by two formal power series and by two sequences).

Definition 2. Let R be a commutative ring, and let $f(t) = \sum_{n=0}^{\infty} f_n t^n$ and $g(t) = \sum_{n=1}^{\infty} g_n t^n$ be formal power series with coefficients in R. The Riordan array associated to the pair (f, g) is the infinite lower triangular matrix $\mathcal{R}(f, g) = (r_{nk})_{n,k\geq 0}$ defined by

$$r_{nk} = [t^n] f(t) g(t)^k, \quad n \geq k,$$

where $[t^n]$ denotes the operator for extracting the n-th coefficient of a generating function.

That is, $\mathcal{R}(f, g) = (r_{nk})_{n,k\geq 0}$ is a Riordan array if the GF for the sequence $(r_{nk})_{n\geq 0}$ of numbers in the k-th column of $\mathcal{R}(f, g)$ is $f(t)g(t)^k$, for all $k \geq 0$. From now on, we will use the notations $\mathcal{R} = \mathcal{R}(f, g) = (f(t), g(t))$. Note that g has zero constant term.

Remark 1. By Section n 1 of [1], the inverse of a Riordan array $\mathcal{R}(f, g)$ is given by

$$(f(t), g(t))^{-1} = (1/f(\overline{g}(t)), \overline{g}(t)), \tag{6}$$

where $\overline{g}$ is the composition inverse of g (so it satisfies $\overline{g}(g(t)) = g(\overline{g}(t)) = t$). Furthermore, the Riordan array of the product $(f(t), g(t)) \cdot (h(t), l(t))$ is given by

$$(f(t), g(t)) \cdot (h(t), l(t)) = (f(t)h(g(t)), l(g(t))). \tag{7}$$

The characterization of a Riordan array $\mathcal{R} = (r_{nk})_{n,k\geq 0}$ by two sequences $(a_n)_{n\geq 0}$ and $(z_n)_{n\geq 0}$ and an element of the commutative ring $r \in R$ is described by the following recurrence (see [24])

$$r_{0,0} = r \qquad r_{n+1,0} = \sum_{j\geq 0} z_j r_{n,j} \qquad r_{n+1,k+1} = \sum_{j\geq 0} a_j r_{n+k,j},$$

for $n, k \geq 0$. $(a_n)_{n\geq 0}$ and $(z_n)_{n\geq 0}$ are called the A- and Z-sequences of $\mathcal{R}$ respectively.

It is very common that $r = 1$, which induces that Riordan arrays can be characterized only by two sequences $A = (a_n)_{n\geq 0}$ and $Z = (z_n)_{n\geq 0}$. In fact, this will be the case for all the examples considered in this paper.

Now we present some examples of Riordan arrays that appear in several references [1, 9, 25] and are considered in this paper.

Example 2. Catalan numbers are related with Riordan arrays in many ways, in particular with the ballot table. The n-Catalan number is $C_n := \frac{1}{n+1}\binom{2n}{n}$, and the ballot numbers (see [9]) are $b_{nk} := \frac{k+1}{n+1}\binom{2n-k}{n}$, for $n \geq k$.

These numbers form the ballot table $B = (b_{nk})_{n,k\geq 0}$, which is a Riordan array, and can be defined as

$$B = \left(\frac{1-\sqrt{1-4t}}{2t}, \frac{1-\sqrt{1-4t}}{2}\right) = \begin{pmatrix} 1 \\ 1 & 1 \\ 2 & 2 & 1 \\ 5 & 5 & 3 & 1 \\ 14 & 14 & 9 & 4 & 1 \\ \vdots & & & & & \ddots \end{pmatrix}. \quad (8)$$

It is also characterized by the sequences $A = Z = (1, 1, \dots)$. Note that its first column is formed by the Catalan numbers.

The ballot table arises directly from the classical Bertrand's Ballot Theorem (cf. [11]), which is enunciated as follows:

"In an election where candidate A receives i votes and candidate B receives j votes with $i > j$, in how many different ways can the votes be ordered so that candidate A is ahead of B throughout the entire counting process?". The entry $(i, i-j)$ of the ballot matrix gives exactly this number: the number of different ways to order the votes so that candidate A (with i votes) is always ahead of candidate B (with j votes).

In combinatorial terms, it counts lattice paths from $(0,0)$ to (i,j), using steps $(1,1)$ and $(1,-1)$, that never passes below the x-axis. The resulting matrix forms a lower-triangular Riordan array that encodes these constrained paths, known as ballot paths or Dyck paths. These paths have been used historically in a wide range of contexts, including stochastic processes such as gambler's ruin problems, electrostatics via the method of images, and non-parametric statistical inference (cf. [10]).

Example 3. Pascal's triangle needs no introduction: it is a very important concept in probability theory, combinatorics and many other fields. The Riordan array $A = (a_{nk})_{n,k\geq 0}$ associated with this triangle is the lower triangular matrix with entries $a_{nk} = \binom{n}{k}$, for $n \geq k$. As a Riordan array, the Pascal matrix is

$$A = \left(\frac{1}{1-t}, \frac{t}{1-t}\right) = \begin{pmatrix} 1 \\ 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 & 1 \\ \vdots & & & & & \ddots \end{pmatrix}. \quad (9)$$

It is also characterized by the sequences $A = (1, 1, 0, 0, \dots)$ and $Z = (1, 0, 0, \dots)$.

Example 4. One of the several versions of the Pascal matrix is given by the following example. In [12], it was defined as the Fibonacci matrix because its row sums are the ordered Fibonacci numbers. Let $P = (p_{nk})_{n,k\geq 0}$ be the lower triangular matrix where $p_{nk} = \binom{k}{n-k}$, for $n \geq k$. It can be written as a Riordan array in the following way:

$$P = \left(1, t + t^2\right) = \begin{pmatrix} 1 \\ 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 & 1 \\ \vdots & & & & & \ddots \end{pmatrix}. \quad (10)$$

It is also characterized by the sequences $A = (1, C_0, -C_1, C_2, -C_3, \dots)$ and $Z = (0, 0, \dots)$, where C_n are the Catalan numbers.

Using the property of the inverse matrix of a Riordan array (see Remark 1), we have that the inverse of the ballot table is $B^{-1} = \left(\frac{1-\sqrt{1-4t}}{2t}, \frac{1-\sqrt{1-4t}}{2}\right)^{-1} = (1-t, t-t^2)$.

Now, let us consider the diagonal matrix $J = \mathrm{diag}(1, -1, 1, -1, \dots)$. It is clear that J is a Riordan array $J = (1, -t)$.

Using the rule of the multiplication of Riordan arrays (see Remark 1) we have that

$$JB^{-1}J = (1, -t)(1-t, t-t^2)(1, -t) = (1+t, t+t^2).$$

Given a Riordan array $(f(t), g(t))$, we can define the formal power series $h(t) = \frac{f(t)g(t)}{t}$. By the definition of Riordan array, the matrix $(h(t), g(t))$ is exactly equal to $(f(t), g(t))$ but removing its first row and column. So the Riordan arrays $JB^{-1}J = (1+t, t+t^2)$ and $P = (1, t+t^2)$ are related by the following formula:

$$P = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & JB^{-1}J \end{array}\right). \tag{11}$$

4 | Bidiagonal Decomposition and HRA Computations for Ballot Table and Fibonacci Matrix

The total positivity of the ballot table (8) and the Fibonacci matrix (10) has already been proved in the literature (cf. [26]). The following result presents the bidiagonal decomposition of the truncated ballot table and simultaneously gives a new proof of its total positivity.

Theorem 5. *Suppose that $B_n = (b_{ij})_{0\le i,j\le n}$ is the matrix formed by the first $n+1$ rows and columns of the matrix (8). Then, B_n is TP and its bidiagonal decomposition is given by*

$$(\mathcal{BD}(B_n))_{ij} = \begin{cases} 1, & \text{if } i = j, \\ \frac{4(i-j)-2}{i+1}, & \text{if } i > j \text{ and } j \text{ even}, \\ 0, & \text{otherwise}. \end{cases}$$

Therefore, the entries of $\mathcal{BD}(B_n)$ can be obtained with HRA.

Proof. First, B_n is a lower triangular matrix with all of its diagonal entries equal to one, so it is also nonsingular. Then its bidiagonal decomposition (2) is formed only by the lower triangular matrices, and the diagonal matrix is the identity matrix. Let $B^{(r)} = (b_{ij}^{(r)})_{i,j\ge 0}$ be the lower triangular matrix obtained in (1) through the NE of the matrix $B^{(0)} := B_n$, for $0 \le r < n$. In fact, we have $b_{ij}^{(r)} = 0$ for $i < j$ and $b_{ii}^{(r)} = 1$.

The rest of the proof consists of an induction on the entries $b_{ij}^{(r)}$ of the Neville elimination matrices and the entries m_{ij} of the bidiagonal decomposition (5) of $B^{(0)}$.

Let us prove by induction on $k \ge 0$ than

$$m_{ik} = \begin{cases} \frac{4(i-k)-2}{i+1}, & \text{if } k \text{ even}, \\ 0, & \text{if } k \text{ odd}, \end{cases} \tag{12}$$

where $k < i \le n$, and also that

$$b_{ij}^{(k+1)} = \begin{cases} b_{ij}\frac{j(j-1)\cdots(j-(k+1))}{(2i-j)(2i-j-1)\cdots(2i-j-(k+1))}, & \text{if } k+1 \text{ odd}, \\ b_{ij}^{(k)}, & \text{if } k+1 \text{ even}, \end{cases} \tag{13}$$

where $k+1 \le j < i \le n$.

Considering 0 as even, first let us prove the cases $k = 0$ and $k = 1$. The case $k = 0$ corresponds to the first step of the NE. Since $B^{(0)} = B_n$, let us calculate the multipliers of the NE of this first step and the entries of $B^{(1)}$. For $0 < i \leq n$, we have that

$$m_{i0} = \frac{b_{i0}^{(0)}}{b_{i-1,0}^{(0)}} = \frac{\frac{1}{i+1}\binom{2i}{i}}{\frac{1}{i}\binom{2i-2}{i-1}} = \frac{4i-2}{i+1} \tag{14}$$

because $b_{ij}^{(0)} = b_{ij} = \frac{j+1}{i+1}\binom{2i-j}{i}$. For $1 \leq j < i \leq n$ by (14), we get

$$b_{ij}^{(1)} = b_{ij}^{(0)} - m_{i0} b_{i-1,j}^{(0)} = b_{ij}\frac{j(j-1)}{(2i-j)(2i-j-1)},$$

which are the formulas (12) and (13) respectively for $k = 0$. For the second step, since $b_{i1}^{(1)} = b_{i1}\frac{1(1-1)}{(2i-1)(2i-2)} = 0$ for $i \geq 2$, then we deduce that

$$m_{i1} = 0, \text{ for } 1 < i \leq n, \quad \text{and} \quad b_{ij}^{(2)} = b_{ij}^{(1)}, \text{ for } 2 \leq j < i \leq n.$$

For the induction step, assume that (12) and (13) hold until some odd $k \geq 1$, and now we must prove that they are also valid for $k+1$ (even) and $k+2$ (odd).

Let us start with $k+1$ even. For $k+1 < i \leq n$, by the induction hypothesis we get

$$\begin{aligned} m_{i,k+1} &= \frac{b_{i,k+1}^{(k+1)}}{b_{i-1,k+1}^{(k+1)}} = \frac{b_{i,k+1}^{(k)}}{b_{i-1,k+1}^{(k)}} = \frac{b_{i,k+1}\frac{(k+1)k\cdots(k+1-k)}{(2i-k-1)(2i-k-2)\cdots(2i-k-1-k)}}{b_{i-1,k+1}\frac{(k+1)k\cdots(k+1-k)}{(2i-k-3)(2i-k-4)\cdots(2i-k-3-k)}} \\ &= \frac{b_{ik}(2i-k-3-(k-1))(2i-k-2-k)}{b_{ik}(2i-k-1)(2i-k-2)} = \frac{4(i-(k+1))-2}{i+1}. \end{aligned}$$

By this formula, and taking into account that, by the induction hypothesis,

$$b_{ij}^{(k+1)} = b_{ij}^{(k)} = b_{ij}\frac{j(j-1)\cdots(j-k)}{(2i-j)(2i-j-1)\cdots(2i-j-k)},$$

we have

$$\begin{aligned} b_{ij}^{(k+2)} &= b_{ij}^{(k+1)} - m_{i,k+1} b_{i-1,j}^{(k+1)} = b_{ij}^{(k)} - m_{i,k+1} b_{i-1,j}^{(k)} \\ &= b_{ij}\frac{j(j-1)\cdots(j-(k+2))}{(2i-j)(2i-j-1)\cdots(2i-j-(k+2))}, \end{aligned}$$

for $k+2 \leq j < i \leq n$. These are the formulas (12) and (13), respectively, for $k+1$.

For $j = k+2$, we have that

$$b_{i,k+2}^{(k+2)} = b_{i,k+2}\frac{(k+2)(k+1)\cdots(k+2-(k+2))}{(2i-(k+2))(2i-(k+1))\cdots(2i-(k+2)-(k+2))} = 0,$$

for $k+2 < i \leq n$, and then

$$\begin{aligned} m_{i,k+2} &= 0, \text{ for } k+2 < i \leq n, \\ b_{ij}^{(k+3)} = b_{ij}^{(k+2)} - m_{i,k+2} b_{i-1,j}^{(k+2)} &= b_{ij}^{(k+2)}, \text{ for } k+3 \leq j < i \leq n. \end{aligned}$$

These are formulas (12) and (13), respectively, for $k+2$. Hence the induction follows.

Taking into account that, for the construction of the bidiagonal factorization of B_n, it has been used only products, quotients, sums of nonnegative numbers and substractions of integers, it can be obtained with HRA. Also, since the diagonal pivots are positive and the multipliers $\frac{4(i-j)-2}{i+1} > 0$ for $i > j$, then the matrix B_n is TP by Theorem 1. ◻

Taking into account the results of Koev in [20] and recalled in Section 2, we derive the following corollary.

Corollary 1. *Let $B_n = (b_{ij})_{0\le i,j\le n}$ be the matrix formed by the first $n+1$ rows and columns of the matrix defined in (8). Then the computations of the singular values, the inverse and the solution of lineal systems $B_n x = b$, where b has an alternating pattern of signs, can be solved with HRA.*

We now obtain the bidiagonal decomposition of the truncated Fibonacci matrix given in Example 4.

Corollary 2. *Let $P_n = (p_{ij})_{0\le i,j\le n}$ be the matrix formed by the first $n+1$ rows and columns of the Fibonacci matrix (10). Then P_n is TP and a bidiagonal decomposition is given by*

$$(\mathcal{BD}(P_n))_{ij} = \begin{cases} 1, & \text{if } i = j, \\ \frac{2(2j-1)}{i}, & \text{if } i-j>0 \text{ is odd and } j \ge 1, \\ 0, & \text{otherwise.} \end{cases} \tag{15}$$

Therefore, the entries of $\mathcal{BD}(P_n)$ can be obtained with HRA, as well as the singular values, the inverse and the solution of lineal systems $P_n x = b$ where b has an alternating pattern of signs.

Proof. In Theorem 5, it has been provided the bidiagonal decomposition of the ballot table, so $\mathcal{BD}(B_n) =: (m_{ij}^{(B)})_{0\le i,j\le n}$ is known.

By Theorem 2.2. of [27], given the entries $m_{ij}^{(B)}$ of the bidiagonal decomposition of B_n, we know a bidiagonal decomposition of B_n^{-1}. Furthermore, since B_n is a lower triangular matrix, B_n^{-1} must be lower triangular too. Taking into account the previous arguments, the decomposition can be written as follows:

$$B_n^{-1} = \widetilde{F}_{n-1}\widetilde{F}_{n-2}\cdots\widetilde{F}_1\widetilde{F}_0 \tag{16}$$

where for $0 \le i < n$,

$$\widetilde{F}_i = \begin{pmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & 0 & 1 & & & & \\ & & & -m_{i+1,i}^{(B)} & 1 & & & \\ & & & & -m_{i+2,i}^{(B)} & 1 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & -m_{n,i}^{(B)} & 1 \end{pmatrix}.$$

This is a bidiagonal factorization of B_n^{-1} satisfying (2) and (3). Since $m_{ij}^{(B)} = 0$ if j is odd, the entries of $\mathcal{BD}(B_n^{-1})$ corresponding to (16) are given by

$$\begin{aligned}(\mathcal{BD}(B_n^{-1}))_{ij} &= \begin{cases} 1, & \text{if } i = j, \\ -m_{i,i-j-1}^{(B)}, & \text{if } i > j, \\ 0, & \text{otherwise,} \end{cases} \\ &= \begin{cases} 1, & \text{if } i = j, \\ -m_{i,i-j-1}^{(B)}, & \text{if } i-j>0 \text{ is odd,} \\ 0, & \text{otherwise.} \end{cases}\end{aligned} \tag{17}$$

Given a matrix $A = (a_{ij})_{0\le i,j\le n}$, we denote $|A| := (|a_{ij}|)_{0\le i,j\le n}$. Considering $J = \text{diag}(1,-1,1,-1,\dots)$, since B_n is TP, then $|B_n^{-1}| = JB_n^{-1}J$ is also TP by Theorem 3.3 of [4]. Therefore, we have the following bidiagonal decomposition of $|B_n^{-1}|$

$$\begin{aligned}|B_n^{-1}| &= JB_n^{-1}J = J\widetilde{F}_{n-1}\widetilde{F}_{n-2}\cdots\widetilde{F}_0 J \\ &= J\widetilde{F}_{n-1}J\cdot J\widetilde{F}_{n-2}J\cdots J\widetilde{F}_0 J \\ &= |\widetilde{F}_{n-1}||\widetilde{F}_{n-2}|\cdots|\widetilde{F}_0|,\end{aligned}$$

and so we can take

$$(\mathcal{BD}(|B_n^{-1}|))_{ij} = |(\mathcal{BD}(B_n^{-1}))_{ij}|. \tag{18}$$

Finally, using the relation between the ballot table and the Fibonacci matrix in (11), it can be deduced that

$$(\mathcal{BD}(P_n))_{ij} = \begin{cases} 1, & \text{if } i = j = 0, \\ 0, & \text{if } i = 0, j \neq 0 \text{ or } i \neq 0, j = 0, \\ (\mathcal{BD}(|B_n^{-1}|))_{i-1,j-1}, & \text{otherwise.} \end{cases}$$

By (17) and (18), the previous formula can be written as

$$(\mathcal{BD}(P_n))_{ij} = \begin{cases} 1, & \text{if } i = j = 0, \\ m^{(B)}_{i-1,i-j-1}, & \text{if } i - j > 0 \text{ is odd and } j \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

From this formula, using Theorem 5, formula (15) is obtained.

Taking into account that for the construction of the bidiagonal factorization of P_n it has been used only products, quotients, sums of nonnegative numbers and substractions of indices, it can be obtained with HRA. Besides, since the diagonal pivots are positive and the multipliers $\frac{2(2j-1)}{i} > 0$ for $1 \leq j < i$, then the matrix P_n is TP by Theorem 1.

The results of [20] guarantee that the mentioned algebraic computations can be carried out with HRA. □

5 | Numerical Experiments

In [20], starting from the bidiagonal factorization, $\mathcal{BD}(A)$, of a nonsingular TP matrix A, Koev developed algorithms to compute the eigenvalues, the singular values of A and the solution of linear systems of equations $Ax = b$. In addition, if $B = \mathcal{BD}(A)$ is known to HRA, the mentioned linear algebra problems are solved to HRA by using Koev's algorithms (for the case of the linear system of equations when b has an alternating sign pattern). Assuming the same hypotheses, Marco and Martínez developed in [27] an algorithm to compute the inverse of A, A^{-1}, with HRA. A Matlab implementation of the four algorithms is available in the software library TNTool, which can be downloaded in [28]. The functions for solving these algebra problems have been implemented in the following Matlab functions:

- `TNEigenValues(B)` for the eigenvalues of A.
- `TNSingularValues(B)` for the singular values of A.
- `TNSolve(B,b)` for the solution of the system $Ax = b$.
- `TNInverseExpand(B)` for the inverse A^{-1}.

The following two sections provide numerical experiments illustrating the high accuracy of the new approach for computing the singular values of a ballot table and a Fibonacci matrix, the inverses of them and the solution of related linear systems of equations. Taking into account that both matrices, the ballot table and the Fibonacci matrix, are triangular matrices its eigenvalues are its corresponding diagonal entries. So, no experiment for the eigenvalues is carried out.

5.1 | Ballot Table

In Theorem 5, an explicit formula is given for the parameters of the bidiagonal decomposition of the Ballot table B_n, $\mathcal{BD}(B_n)$. Algorithm 1 provides the pseudocode to compute the parameters of $\mathcal{BD}(B_n)$ to HRA. We have implemented the algorithm in Matlab function `TNBDBallot`.

By Corollary 1, using $\mathcal{BD}(B_n)$ provided by Algorithm 1, the singular values and the inverse of B_n, and the solution of linear systems $B_nx = b$, where b has an alternating pattern of signs, can be solved with HRA using the corresponding functions of TNTool mentioned above.

ALGORITHM 1 | Computation of $\mathcal{BD}(B_n)$.

```
Require: n
Ensure: M = BD(B_n) bidiagonal decomposition of B_n
  M = eye(n + 1)
  for j = 0:2:n do
    for i = j+1:n do
      M(i + 1, j + 1) = (4(i−j)−2)/(i+1)
    end for
  end for
```

Let us recall that the entries b_{nk} of a ballot table count how many ways a candidate A can stay strictly ahead of candidate B in a sequence of votes. Their singular values provide important information about this voting matrix. The ballot table grows quickly in both rows and columns. The first singular value σ_1 captures the bulk of this growth, reflecting the dominant combinatorial behavior, i.e., the increasing advantage of candidate A as their vote count increases. So σ_1 reflects the main axis of combinatorial complexity, this is the dominant growth direction of the ballot numbers. The steep drop-off after σ_1 indicates that the matrix has low effective rank. That means most of its information lies in just a few components. The ballot table behaves like a system with few dominant features. The remaining smaller singular values reflect finer structural variations. These values capture secondary patterns in the data like how the advantage evolves when vote counts are more balanced. In addition, each singular value corresponds to an orthogonal component in the space of Dyck paths.

First, the singular values of B_{39} have been computed with Mathematica using a 100-digits precision. Then, the singular values have also been obtained in Matlab with both the usual function `svd` and the function `TNSingularValues` using the bidiagonal decomposition $\mathcal{BD}(B_{39})$ provided by `TNBDBallot` to HRA. Then, the relative errors of the approximation obtained by both algorithms with Matlab have been calculated considering the singular values obtained with Mathematica as exact. To show the relative errors, we have considered the singular values in decreasing order: $\sigma_1 > \sigma_2 > \cdots > \sigma_{40} > 0$. In Figure 1, the relative errors for the approximation to the singular values σ_i, $i = 1, \ldots, 40$, of the matrix B_{39} can be seen for both methods. From the figure it is concluded that `TNSingularValues` joint with `TNBDBallot` provide more accurate results than these obtained by using the Matlab command `svd`. In Table 1, it can be observed the average and the maximum relative errors for both methods, `svd` and `TNSingularValues`.

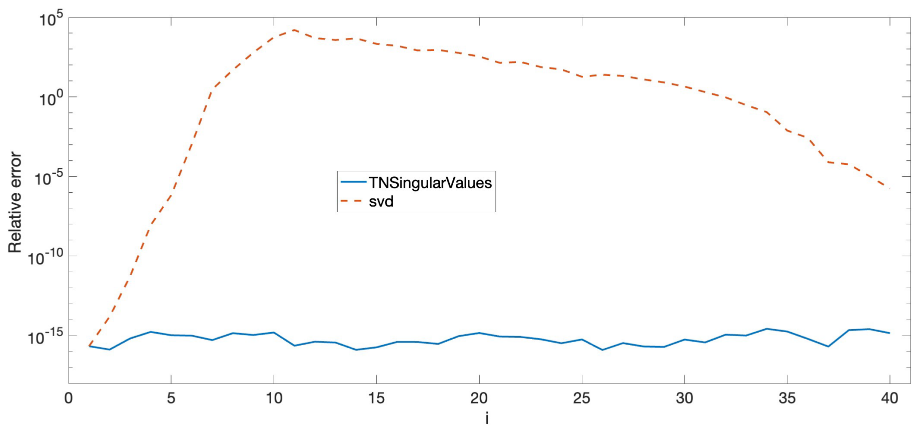

FIGURE 1 | Relative errors when computing the singular values σ_i, $i = 1, \ldots, 40$, of B_{39} with Matlab.

TABLE 1 | Average and maximum relative errors when computing the singular values σ_i, $i = 1, \ldots, 40$, of B_{39}.

	`TNSingularValues`	`svd`
Average rel. error	8.28691218927224e-16	1.05712146808601e+03
Maximum rel. error	2.70130446293661e-15	1.57820559244306e+04

Now let us consider the linear system of equations $B_{39}x = b$, where b has an alternating pattern of signs and the absolute values of its entries have been randomly generated as integers in the interval $[1, 1000]$. The exact solution x has been obtained with Mathematica. Then, approximations to the exact solution of the system have been calculated with the usual Matlab command \ and with `TNSolve` using $\mathcal{BD}(B_{39})$ to HRA. Finally, componentwise relative errors have been computed:

$$\frac{|x_i - \hat{x}_i|}{|x_i|}, \quad i = 1, \dots, 40.$$

Figure 2 shows these componentwise relative errors. The breaks in both lines of the figure correspond to components of the solution calculated exactly with the corresponding method. Table 2 shows the average and the maximum componentwise relative errors for both methods, \ and with `TNSolve`.

Finally, the computation of $(B_{39})^{-1}$ has been considered. Approximations to it have been obtained with Matlab, first using the usual `inv` function, and second by using `TNInverseExpand` joint with the bidiagonal decomposition of B_{39} to HRA. Then, the componentwise relative errors have been obtained from the exact inverse computed with Mathematica. Table 3 shows the average and the maximum relative errors for both methods, `TNInverseExpand` and `inv`.

5.2 | Fibonacci Matrix

By Corollary 2 the matrix P_n is TP, the bidiagonal decomposition given by (15) can be calculated with HRA and some algebra problems can be solved to HRA. Algorithm 2 provides the pseudocode to compute the parameters of $\mathcal{BD}(P_n)$ to HRA. We have implemented the algorithm in the Matlab function `TNBDFibo`.

The same numerical tests carried out for the Ballot table B_{39} have been performed for the Fibonacci matrix P_{39}. Figure 3 and Table 4 show the relative errors for the singular values.

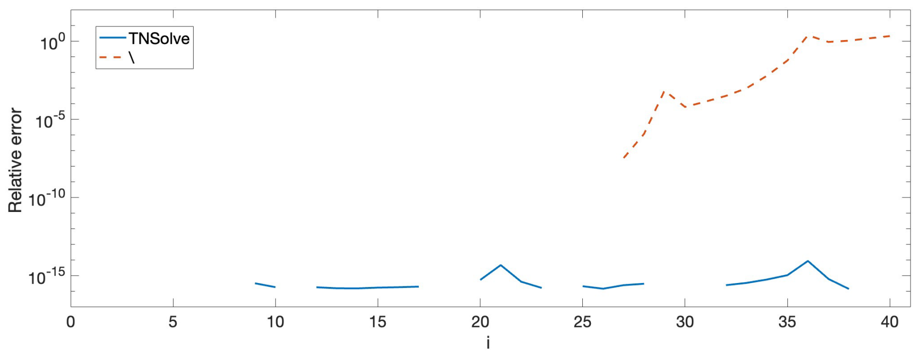

FIGURE 2 | Componentwise relative errors when solving $B_{39}x = b$.

TABLE 2 | Average and maximum componentwise relative errors when solving $B_{39}x = b$.

	`TNSolve`	\
Average rel. error	4.94057553928929e-16	1.94331891164663e-01
Maximum rel. error	8.40973259394895e-15	2.29128441614927e+00

TABLE 3 | Relative errors when computing $(B_{39})^{-1}$.

	`TNInverseExpand`	`inv`
Average rel. error	9.39122374559444e-17	9.32294993682554e-01
Maximum rel. error	5.96259638400084e-16	4.1000000000000e+02

ALGORITHM 2 | Computation of $\mathcal{BD}(P_n)$.

Require: n
Ensure: $M = \mathcal{BD}(P_n)$ bidiagonal decomposition of P_n

```
M = eye(n + 1)
for j = 1:n do
    for i = j+1:2:n do
        M(i + 1, j + 1) = 2(2j−1)/i
    end for
end for
```

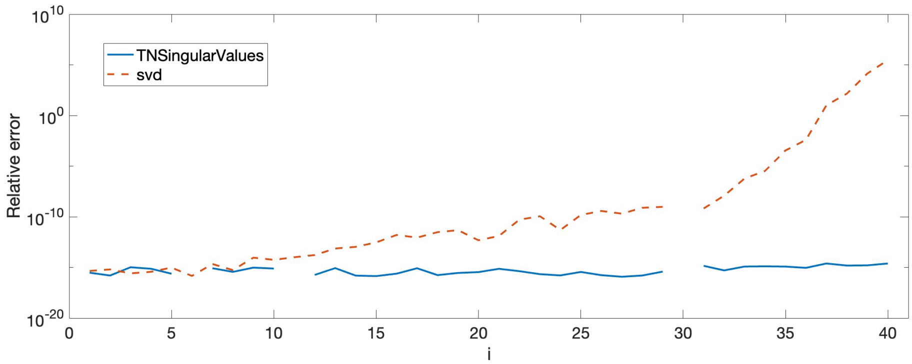

FIGURE 3 | Relative errors when computing the singular values σ_i, $i = 1, \dots, 40$, of P_{39} with Matlab.

TABLE 4 | Average and maximum relative errors when computing the singular values σ_i, $i = 1, \dots, 40$, of P_{39}.

	`TNSingularValues`	`svd`
Average rel. error	6.53339934168188e-16	7.825590752204e+03
Maximum rel. error	2.46666176350777e-15	2.98975019848274e+05

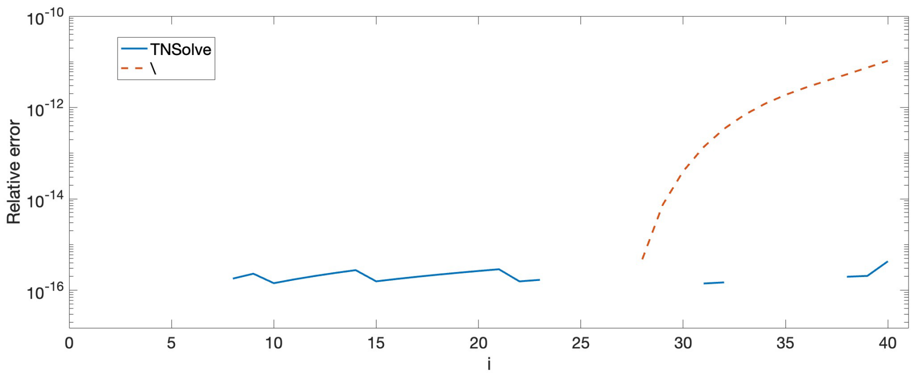

FIGURE 4 | Componentwise relative errors when solving $P_{39}x = b$ with Matlab.

TABLE 5 | Average and maximum componentwise relative errors when solving $P_{39}x = b$.

	`TNSolve`	`\`
Average rel. error	1.10946724308344e-16	8.57308988112928e-13
Maximum rel. error	4.29929271387725e-16	1.05367065831704e-11

TABLE 6 | Relative errors when computing $(P_{39})^{-1}$.

	`TNInverseExpand`	`inv`
Average rel. error	1.135800768754562e-16	2.304971702859434e-13
Maximum rel. error	6.849613794230046e-16	1.484215377084401e-11

Figure 4 and Table 5 show the componentwise relative errors when solving a linear system of equations $P_{39}x = b$, where b has an alternating pattern of signs.

In Table 6, it can be seen the average and the maximum relative errors for both methods with the Fibonacci matrix P_{39}, `TNInverseExpand` and `inv`.

The numerical examples illustrate that the new HRA algorithms outperform the classical methods for the considered linear algebra problems.

6 | Conclusions

Finite truncations of some Riordan arrays admit bidiagonal decompositions. They have been obtained for the particular triangular matrices of ballot tables and Fibonacci matrices. These decompositions have been used to prove the total positivity of Fibonacci matrices and to provide a new proof of the total positivity of ballot tables. Since these bidiagonal decompositions can be obtained with high relative accuracy, they can be also used to compute with high relative accuracy all the singular values and inverses of these matrices, as well as the solution of some linear systems associated to them. Numerical examples show that high relative accuracy algorithms derived from the proposed methods outperform the classical methods for these linear algebra problems.

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Conflicts of Interest

The authors declare no conflicts of interest.

Data Availability Statement

The authors have nothing to report.

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