

A recursive formula for the motivic Milnor fiber of a plane curve

Manuel González Villa, Gary Kennedy, and Lee J. McEwan

Abstract We find a recursive formula for the motivic Milnor fiber of an irreducible plane curve, using the notions of a truncation and derived curve. We then apply natural transformations to obtain a similar recursion for the Hodge-theoretic spectrum.

Dedicated to Antonio Campillo on the occasion of his 65th birthday

1 Introduction

In this paper we develop a recursive formula for the motivic Milnor fiber of a singular point of a plane curve, expressed in terms of the essential exponents in its Puiseux expansion. This subject has been treated in [5], and later in [4], but the novel feature here is that we develop a recursion at the level of virtual motives, adapting the topological techniques of [6]. Alternative approaches to the computation of the motivic Milnor fibre of a singular point of a plane curve proposed in the literature are to use an embedded or log resolution of the singular point [3], the Newton process [1], or splicing [2].

After developing the recursion, we apply natural transformations to obtain similar recursions for the Hodge-theoretic spectrum and the topological monodromy. We conclude with an extended example in which, using our

Manuel González Villa

Centro de Investigación en Matemáticas, A.C., Jalisco S/N, Col. Valenciana CP: 36023 Guanajuato, Gto, México e-mail: manuel.gonzalez@cimat.mx

Gary Kennedy

Ohio State University at Mansfield, 1760 University Drive, Mansfield, Ohio 44906, USA e-mail: kennedy@math.ohio-state.edu

Lee J. McEwan

Ohio State University at Mansfield, 1760 University Drive, Mansfield, Ohio 44906, USA e-mail: kennedy@math.ohio-state.edu

recursion, we write out the virtual motive of a specific curve and compute its spectrum; for comparison we compute the same virtual motive using the original construction of Denef and Loeser [3].

2 The theorem

Consider an irreducible germ at the origin of a complex plane curve C with Puiseux expansion

$$\zeta = \sum c_\lambda x^\lambda$$

in which the essential exponents are $\lambda_1 < \lambda_2 < \dots < \lambda_e$. Let $\lambda_1 = n/m$, where m and n are relatively prime. If necessary we will change coordinates so that none of the exponents are integers and so that $c_{\lambda_1} = 1$. Then C is defined by the vanishing of a function which can be written both as a product and a sum:

$$f(x, y) = \prod (y - \zeta) = (y^m - x^n)^{d'} + \dots \quad (1)$$

Here d' is some positive integer, and the unnamed terms have higher order with respect to the weighting $\text{wt}(x) = m$, $\text{wt}(y) = n$; the product is taken over all possible conjugates, the number of which is $d = md'$.

As in [6], we define two associated curves. The *truncation* C_1 is the curve with Puiseux expansion $\zeta_1 = x^{\lambda_1}$ and defining function $f_1(x, y) = y^m - x^n$; it has m conjugates. The *derived curve* C' has Puiseux expansion

$$\zeta' = \sum_{\lambda > \lambda_1} c_\lambda x^\lambda, \quad (2)$$

where $\lambda' = m(\lambda - \lambda_1 + n)$. It has d' conjugates, and its defining function is

$$f'(x, y) = \prod (y - \zeta') = y^{d'} + \dots, \quad (3)$$

a product using each conjugate once. Its essential exponents are

$$\lambda'_1, \lambda'_2, \dots, \lambda'_{e-1}$$

where

$$\lambda'_i = m(\lambda_{i+1} - \lambda_1 + n).$$

Denote by μ_N the group of roots of the unity of order N and by $\hat{\mu}$ the inverse limit of the groups of roots of unity (with respect to the morphisms $\mu_M \rightarrow \mu_N$ given by $x \mapsto x^{M/N}$, whenever M is a multiple of N). An action of the group μ_N in a variety is called good if each orbit is contained in an affine subvariety. A good action of $\hat{\mu}$ on a variety is an action of $\hat{\mu}$ which factors through a good μ_N -action for some N . Following Denef and Loeser [3, Subsection 2.4], let $K_0^{\hat{\mu}}(\text{Var}_{\mathbb{C}})$ denote the monodromic Grothendieck ring

of varieties with good $\hat{\mu}$ -action, and let $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ be the localization obtained by inverting \mathbb{L} , the class of the affine line. The class of a variety X in this ring will be denoted $[X]$ and will be called a *virtual motive*.

In the space $\mathcal{L}_N(\mathbb{C}_{x,y}^2)_{\mathbf{0}}$ of N -jets based at the origin, let $\mathcal{X}_{N,1}(f)$ be the subspace consisting of jets φ for which

$$(f \circ \varphi)(t) = t^N + \text{higher order terms.}$$

Notice that the group μ_N acts on $\mathcal{X}_{N,1}(f)$ by $a \cdot \varphi(t) \mapsto \varphi(at)$.

The *local motivic zeta function* of f at the origin is

$$Z_{\mathbf{0}}(f) = \sum_{N=1}^{\infty} [\mathcal{X}_{N,1}(f)] \mathbb{L}^{-2N} T^N.$$

The *motivic Milnor fiber* of f at the origin is the limit

$$S_{\mathbf{0}}(f) = - \lim_{T \rightarrow \infty} Z_{\mathbf{0}}(f).$$

(In fact $Z_{\mathbf{0}}(f)$ is known to be a rational function of degree ≤ 0 , so that it makes sense to consider its limit at $T = +\infty$.)

In Section 3 we prove the following recursive formula.

Theorem 1. $S_{\mathbf{0}}(f) = S_{\mathbf{0}}((f_1)^{d'}) + S_{\mathbf{0}}(f') - [\mu_{d'}]$.

In this formula $[\mu_{d'}]$ denotes the virtual motive of roots of unity of order d' . The complex multiplication in $\mu_{d'}$ induces a good $\hat{\mu}$ -action on $\mu_{d'}$.

To make the base case of this recursion more explicit, we introduce the following notation: $[(f_1)^{d'} - 1]$ represents the virtual motive associated to the variety $\{(x, y) \in \mathbb{C}^2 \mid (f_1(x, y))^{d'} = 1\}$.

Theorem 2. $S_{\mathbf{0}}((f_1)^{d'}) = [(f_1)^{d'} - 1] - [\mu_{d'}](\mathbb{L} - 1)$.

This formula is implicit in the results of Section 3 of Guibert [5], but we prefer to give a short self-contained proof in Section 4 below. Combining Theorems 1 and 2, we obtain the following formula:

$$S_{\mathbf{0}}(f) = [(f_1)^{d'} - 1] - [\mu_{d'}]\mathbb{L} + S_{\mathbf{0}}(f'). \quad (4)$$

As a consequence of (4), the motivic Milnor fiber of f at the origin $S_{\mathbf{0}}(f)$ is determined by the essential exponents of the complex plane curve C .

3 Proof of Theorem 1

We begin with two basic properties of the motivic Milnor fiber.

Lemma 1. *For the coordinate function y , we have $S_{\mathbf{0}}(y^{d'}) = [\mu_{d'}]$.*

Proof. This follows from a direct calculation.

Lemma 2. *If the quotient g/h is a unit in the local ring of the complex plane at the origin, then $[\mathcal{X}_{N,1}(g)] = [\mathcal{X}_{N,1}(h)]$ for each N .*

Proof. Suppose that $g = uh$. Choose λ to be an N th root of $1/u(0,0)$. Given an arc φ , map it to the arc ψ , where

$$\psi(t) = \varphi(\lambda t).$$

This sets up an equivariant identification between $\mathcal{X}_{N,1}(g)$ and $\mathcal{X}_{N,1}(h)$.

To prove Theorem 1, we will use a pair of maps $\mathbb{C}_{v,w}^2 \rightarrow \mathbb{C}_{x,y}^2$ defined as follows:

$$\begin{aligned}\pi(v, w) &= (v^m, v^n(1+w)), \\ \pi'(v, w) &= (v, v^{mn}w),\end{aligned}$$

and we will work with jets in $\mathcal{L}_N(\mathbb{C}_{v,w}^2)\mathbf{o}$.

Lemma 3. *The quotient $\frac{f' \circ \pi'}{f \circ \pi}$ is a unit in the local ring of the complex plane at the origin. Thus $[\mathcal{X}_{N,1}(f \circ \pi)] = [\mathcal{X}_{N,1}(f' \circ \pi')]$ for each N .*

Proof. Looking at the product in (1) and composing with π , we observe that we may write

$$\begin{aligned}y - \zeta &= v^n(1+w) - \left((v^m)^{n/m} + \sum_{\lambda > \lambda_1} c_\lambda (v^m)^\lambda \right) \\ &= v^n \left(1 + w - 1^{n/m} - \sum_{\lambda > \lambda_1} c_\lambda v^{m\lambda - n} \right).\end{aligned}$$

The interpretation of $1^{n/m}$ depends on the choice of conjugate; each possible m th root of unity occurs equally often. Thus in d' of the factors the interpretation is that $1^{n/m} = 1$. In the remaining factors $1^{n/m}$ is some other m th root of unity, and within each of these factors the subfactor within parentheses is a unit. The full product $f \circ \pi(v, w)$ is thus a unit times

$$v^{nd'} \prod_{\lambda > \lambda_1} (w - \sum c_\lambda v^{m\lambda - n}),$$

where the product is taken over all d' conjugates.

Now looking at the product in (3) and composing with π' , we see that

$$\begin{aligned}f' \circ \pi'(v, w) &= \prod (v^{mn}w - \sum_{\lambda > \lambda_1} c_\lambda v^{\lambda'}) \\ &= v^{mnd'} \prod (w - \sum_{\lambda > \lambda_1} c_\lambda v^{\lambda' - mn}),\end{aligned}$$

precisely the same product over all d' conjugates.

Given a jet $\rho \in \mathcal{L}_N(\mathbb{C}_{v,w}^2)_{\mathbf{0}}$, its *pushforward* by π is the jet $\pi \circ \rho \in \mathcal{L}_N(\mathbb{C}_{x,y}^2)_{\mathbf{0}}$. Note that the pushforward of a jet in $\mathcal{X}_{N,1}(f \circ \pi)$ is a jet in $\mathcal{X}_{N,1}(f)$. The pushforward jets form a subspace which we denote by $\mathcal{X}_{N,1}^\pi(f)$. In the same way, we define three similar subspaces:

- $\mathcal{X}_{N,1}^\pi((f_1)^{d'})$ inside $\mathcal{X}_{N,1}((f_1)^{d'})$,
- $\mathcal{X}_{N,1}^{\pi'}(f')$ inside $\mathcal{X}_{N,1}(f')$,
- $\mathcal{X}_{N,1}^{\pi'}(y^{d'})$ inside $\mathcal{X}_{N,1}(y^{d'})$.

Lemma 4. *Each jet in $\mathcal{X}_{N,1}^\pi(f)$ is the pushforward of a unique jet in $\mathcal{X}_{N,1}(f \circ \pi)$. Thus the map π identifies the space $\mathcal{X}_{N,1}(f \circ \pi)$ with the space $\mathcal{X}_{N,1}^\pi(f)$. It likewise identifies $\mathcal{X}_{N,1}((f_1)^{d'} \circ \pi)$ with $\mathcal{X}_{N,1}^\pi((f_1)^{d'})$. The jet spaces $\mathcal{X}_{N,1}(f) \setminus \mathcal{X}_{N,1}^\pi(f)$ and $\mathcal{X}_{N,1}((f_1)^{d'}) \setminus \mathcal{X}_{N,1}^\pi((f_1)^{d'})$ obtained by set-theoretic difference are equal.*

Proof. We claim that $\mathcal{X}_{N,1}^\pi(f)$ consists of all jets in $\mathcal{X}_{N,1}(f)$ of the form

$$\varphi(t) = (x_0 t^{mp} + \dots, y_0 t^{np} + \dots), \quad (5)$$

where p is a positive integer, with x_0 and y_0 being nonzero numbers satisfying

$$(y_0)^m = (x_0)^n. \quad (6)$$

Indeed, consider a jet ρ in $\mathcal{X}_{N,1}(f \circ \pi)$; suppose that $\rho(t) = (v(t), w(t))$, where $v(t) = v_0 t^p + \dots$, with $v_0 \neq 0$. Then

$$\pi \circ \rho(t) = ((v_0)^m t^{mp} + \dots, (v_0)^n t^{np} + \dots).$$

Conversely, given

$$\varphi(t) = (x(t), y(t)) = (x_0 t^{mp} + \dots, y_0 t^{np} + \dots)$$

with $(y_0)^m = (x_0)^n$, note that there is a unique number v_0 for which $x_0 = (v_0)^m$ and $y_0 = (v_0)^n$. Explicitly, letting r and s be the smallest positive integers for which

$$\det \begin{bmatrix} m & n \\ r & s \end{bmatrix} = 1,$$

we have $v_0 = (x_0)^s / (y_0)^r$. Let $v(t)$ be the m th root of $x(t)$ with this leading coefficient, i.e.,

$$v(t) = v_0 t^p + \dots \quad \text{and} \quad (v(t))^m = x(t).$$

Let $w(t) = y(t) / (v(t))^n - 1$. Then $\rho(t) = (v(t), w(t))$ gives the unique jet whose pushforward is φ .

The same argument applies with $(f_1)^{d'}$ in place of f , showing that $\mathcal{X}_{N,1}^\pi((f_1)^{d'})$ consists of all jets in $\mathcal{X}_{N,1}((f_1)^{d'})$ of the form specified by by

equations (5) and (6), and that each such jet is a pushforward in a unique way.

Again consider a jet $\varphi \in \mathcal{X}_{N,1}^\pi(f)$. Equations (5) and (6) imply a cancellation: there is no term of degree $mnpd'$, and thus we must have $N > mnpd'$. The same remark applies to a jet in $\mathcal{X}_{N,1}^\pi((f_1)^{d'})$. For a jet $\varphi \in \mathcal{X}_{N,1}(f) \setminus \mathcal{X}_{N,1}^\pi(f)$, however, the cancellation just described does not occur. This means that $(f \circ \varphi)(t)$ and $((f_1)^{d'} \circ \varphi)(t)$ have the same lowest-order term. Thus the jet spaces $\mathcal{X}_{N,1}(f) \setminus \mathcal{X}_{N,1}^\pi(f)$ and $\mathcal{X}_{N,1}((f_1)^{d'}) \setminus \mathcal{X}_{N,1}^\pi((f_1)^{d'})$ coincide.

Lemma 5. *Each jet in $\mathcal{X}_{N,1}^{\pi'}(f')$ is the pushforward of a unique jet in $\mathcal{X}_{N,1}(f' \circ \pi')$. Thus the map π' identifies the space $\mathcal{X}_{N,1}(f' \circ \pi')$ with the space $\mathcal{X}_{N,1}^{\pi'}(f')$. It likewise identifies $\mathcal{X}_{N,1}(y^{d'} \circ \pi')$ with $\mathcal{X}_{N,1}^{\pi'}(y^{d'})$. The $\mathcal{X}_{N,1}(f') \setminus \mathcal{X}_{N,1}^{\pi'}(f')$ and $\mathcal{X}_{N,1}(y^{d'}) \setminus \mathcal{X}_{N,1}^{\pi'}(y^{d'})$ are equal.*

Proof. We claim that $\mathcal{X}_{N,1}^{\pi'}(f')$ consists of all jets in $\mathcal{X}_{N,1}(f')$ of the form

$$\varphi(t) = (x(t), y(t)) = (x_0 t^p + \dots, y_0 t^q + \dots), \quad (7)$$

(with $x_0 \neq 0$ and $y_0 \neq 0$), where $q > mnp$, together with all jets in which $y(t)$ is identically zero. Indeed, consider a jet $\rho(t) = (v(t), w(t))$ in $\mathcal{X}_{N,1}(f' \circ \pi')$. Note that $v(t)$ can't be identically zero, and write it as $v(t) = x_0 t^p + \dots$. Then its pushforward has the required form. Conversely, given a jet $\varphi(t) = (x(t), y(t))$ of this form, compose with the birational inverse of π' , i.e., let $v(t) = x(t)$ and $w(t) = y(t)/(x(t))^{mn}$, to obtain the unique jet ρ whose pushforward is φ . The same argument applies with $y^{d'}$ in place of f' .

Thus a jet φ in $\mathcal{X}_{N,1}(f') \setminus \mathcal{X}_{N,1}^{\pi'}(f')$ has the form shown in equation (7), with $q \leq mnp$, and likewise for a jet in $\mathcal{X}_{N,1}(y^{d'}) \setminus \mathcal{X}_{N,1}^{\pi'}(y^{d'})$. If we expand the product in (3) and then compose each term with such a jet, the leading term $y^{d'}$ has order $d'q$. Observe that each exponent λ' appearing in (2) is greater than mn . Thus each other term in the expansion will have order greater than $d'q$. Thus $f \circ \varphi$ and $y^{d'} \circ \varphi$ have the same leading term.

Recall that Lemma 3 states this equation of virtual motives:

- $[\mathcal{X}_{N,1}(f \circ \pi)] = [\mathcal{X}_{N,1}(f' \circ \pi')]$

Lemma 4 implies the following equations:

- $[\mathcal{X}_{N,1}^\pi(f)] = [\mathcal{X}_{N,1}(f \circ \pi)]$
- $[\mathcal{X}_{N,1}^\pi((f_1)^{d'})] = [\mathcal{X}_{N,1}((f_1)^{d'} \circ \pi)]$
- $[\mathcal{X}_{N,1}(f)] - [\mathcal{X}_{N,1}^\pi(f)] = [\mathcal{X}_{N,1}((f_1)^{d'})] - [\mathcal{X}_{N,1}^\pi((f_1)^{d'})]$

Lemma 5 gives us the following equations:

- $[\mathcal{X}_{N,1}^{\pi'}(f')] = [\mathcal{X}_{N,1}(f' \circ \pi')]$
- $[\mathcal{X}_{N,1}^{\pi'}(y^{d'})] = [\mathcal{X}_{N,1}(y^{d'} \circ \pi')]$
- $[\mathcal{X}_{N,1}(f')] - [\mathcal{X}_{N,1}^{\pi'}(f')] = [\mathcal{X}_{N,1}(y^{d'})] - [\mathcal{X}_{N,1}^{\pi'}(y^{d'})]$

Thus

$$\begin{aligned} -[\mathcal{X}_{N,1}(f)] &= -[\mathcal{X}_{N,1}((f_1)^{d'})] - [\mathcal{X}_{N,1}(f')] \\ &\quad + [\mathcal{X}_{N,1}((f_1)^{d'} \circ \pi)] + [\mathcal{X}_{N,1}(y^{d'})] - [\mathcal{X}_{N,1}(y^{d'} \circ \pi')]. \end{aligned}$$

Using each term as coefficient in a power series $\sum a_N \mathbb{L}^{-2N} T^N$ and taking the limit as T approaches infinity, we obtain this equation:

$$\begin{aligned} S_{\mathbf{0}}(f) &= S_{\mathbf{0}}((f_1)^{d'}) + S_{\mathbf{0}}(f') \\ &\quad - S_{\mathbf{0}}((f_1)^{d'} \circ \pi) - S_{\mathbf{0}}(y^{d'}) + S_{\mathbf{0}}(y^{d'} \circ \pi') \end{aligned} \quad (8)$$

Both $(f_1)^{d'} \circ \pi(v, w)$ and $y^{d'} \circ \pi'(v, w)$ are $v^{mnd'} w$ times a unit; Lemma 2 tells us that $S_{\mathbf{0}}((f_1)^{d'} \circ \pi) = S_{\mathbf{0}}(y^{d'} \circ \pi)$. Thus by applying Lemma 1, we obtain the statement of Theorem 1.

4 Proof of Theorem 2

To prove Theorem 2, we will again use the decomposition of the jet space $\mathcal{X}_{N,1}((f_1)^{d'})$ into $\mathcal{X}_{N,1}^{\pi}((f_1)^{d'})$ plus its complement. The theorem is an immediate consequence of these two equations:

$$- \lim_{T \rightarrow \infty} \sum_{N=1}^{\infty} [\mathcal{X}_{N,1}^{\pi}((f_1)^{d'})] \mathbb{L}^{-2N}(f) T^N = -[\mu_{d'}](\mathbb{L} - 1) \quad (9)$$

$$- \lim_{T \rightarrow \infty} \sum_{N=1}^{\infty} [\mathcal{X}_{N,1}((f_1)^{d'}) \setminus \mathcal{X}_{N,1}^{\pi}((f_1)^{d'})] \mathbb{L}^{-2N}(f) T^N = [(f_1)^{d'} - 1] \quad (10)$$

To prove equation (9), we first invoke Lemma 4 to replace $\mathcal{X}_{N,1}^{\pi}((f_1)^{d'})$ by $\mathcal{X}_{N,1}((f_1)^{d'} \circ \pi)$, and remark that

$$(f_1)^{d'} \circ \pi(u, v) = (v^{mn}(mw + \dots))^{d'}.$$

Consider a jet ρ in $[\mathcal{X}_{N,1}((f_1)^{d'} \circ \pi)]$, and write it as

$$\rho(t) = (v_p t^p + \dots, w_q t^q + \dots),$$

with nonzero v_p and w_q . Then $N = d'(mnp + q)$ and $(mv_p w_q)^{d'} = 1$. Thus

$$\begin{aligned}
\sum_{N=1}^{\infty} [\mathcal{X}_{N,1}((f_1)^{d'} \circ \pi)] \mathbb{L}^{-2N} T^N &= \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} [\mu'_d] (\mathbb{L} - 1) \mathbb{L}^{N-p} \mathbb{L}^{N-q} \mathbb{L}^{-2N} T^N \\
&= [\mu'_d] (\mathbb{L} - 1) \sum_{p=1}^{\infty} \mathbb{L}^{-p} T^{d' m n p} \sum_{q=1}^{\infty} \mathbb{L}^{-q} T^{d' q} \\
&= [\mu'_d] (\mathbb{L} - 1) \cdot \frac{\mathbb{L}^{-1} T^{d' m n}}{1 - \mathbb{L}^{-1} T^{d' m n}} \cdot \frac{\mathbb{L}^{-1} T^{d'}}{1 - \mathbb{L}^{-1} T^{d'}},
\end{aligned}$$

which yields the desired formula.

To prove equation (10), we examine a jet

$$\varphi \in \mathcal{X}_{N,1}((f_1)^{d'}) \setminus \mathcal{X}_{N,1}^{\pi}((f_1)^{d'}).$$

As we remarked in the proof of Lemma 4, for such a jet we do not see the cancellation of terms implied by equations (5) and (6).

Suppose that the order of vanishing of $(f_1)^{d'} \circ \varphi$ is divisible by md' but not mnd' . Then φ must have the following form:

$$\varphi(t) = (x(t), y(t)) = (x_0 t^p + \dots, y_0 t^q + \dots),$$

where the following conditions are satisfied:

- q is not a multiple of n ,
- $(y_0)^{md'} = 1$,
- $p = \lceil \frac{mq}{n} \rceil$.

Note that x_0 may vanish; in fact $x(t)$ may even be identically zero. The order of vanishing of $(f_1)^{d'} \circ \varphi$ is $N = m q d'$. The virtual motive associated to all such jets is

$$[\mu_{md'}] \mathbb{L}^{2mqd' - p - q + 1},$$

and thus its contribution to the sum in (10) is

$$[\mu_{md'}] \mathbb{L}^{-p - q + 1} T^{mqd'}.$$

Observe that if we increase q by n , then p increases by m and the order of vanishing increases by mnd' . Thus the contributions of these jet spaces to the sum in (10) may be packaged as $n - 1$ geometric series with common ratio $r = \mathbb{L}^{-(m+n)} T^{mnd'}$. For each series, the leading term is determined by a value of q that is less than n , and thus the order of vanishing is less than mnd' . Hence the contribution of this series to the limit as $T \rightarrow \infty$ is zero. A similar argument shows that there is no contribution from jets for which the order of vanishing of $(f_1)^{d'} \circ \varphi$ is divisible by nd' but not mnd' .

It remains to analyze those jets for which the order of vanishing N of $(f_1)^{d'} \circ \varphi$ is divisible by mnd' . Such a jet has the form given in equation (5), where

$$N = mnpd'$$

$$\text{and } ((y_0)^m - (x_0)^n)^{d'} = 1;$$

note that x_0 or y_0 may vanish. The virtual motive associated to such jets is

$$[(f_1)^{d'} - 1] \mathbb{L}^{2mnpd' - (m+n)p},$$

and thus the total contribution of these jets to the sum in equation (10) is

$$[(f_1)^{d'} - 1] \sum_{p=1}^{\infty} \mathbb{L}^{-(m+n)p} T^{mnpd'} = [(f_1)^{d'} - 1] \frac{\mathbb{L}^{-(m+n)T^{mnd'}}}{1 - \mathbb{L}^{-(m+n)T^{mnd'}}}.$$

Taking the limit as T goes to infinity gives us $-[(f_1)^{d'} - 1]$.

5 A recursive formula for the spectrum

Steenbrink [9, (2.1)] and [10, Section 1] and Saito [7, Section 2] associate to a plane curve singularity a multiset of rational numbers called its *spectrum* — it can be regarded as an element of the group ring $\mathbb{Z}[t^{\mathbb{Q}}]$. As Steenbrink [9] says, it “gathers the information about the eigenvalues of the monodromy operator . . . and about the Hodge filtration on the vanishing cohomology.” We briefly explain this construction.

We have defined the motivic Milnor fiber $S_0(f)$ as an element of $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$; the class $1 - S_0(f)$ is called the *motivic vanishing cycle*. There is a natural map χ_h^{mon} , called the *Hodge characteristic*, from $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ to $K_0(HS^{\text{mon}})$, the Grothendieck ring of mixed Hodge structures with a finite automorphism. For a variety X with good $\hat{\mu}$ -action, we associate the sum

$$\sum_i (-1)^i [H_c^i(X, \mathbb{Q})] \in K_0(HS^{\text{mon}}).$$

Here $H_c^i(X, \mathbb{Q})$ indicates the i th cohomology group with compact supports, which is naturally endowed with a mixed Hodge structure and a finite automorphism. This map passes to the localization; thus we have a linear map

$$\chi_h^{\text{mon}} : \mathcal{M}_{\mathbb{C}}^{\hat{\mu}} \rightarrow K_0(HS^{\text{mon}}).$$

Steenbrink [8] and Varchenko [11] showed that one can define a natural mixed Hodge structure $\text{MHS}(f)$ on the cohomology of the classical Milnor fiber, which in turn determines an element of $K_0(HS^{\text{mon}})$. In Theorem 3.5.5 of [3], Denef and Loeser prove that the Hodge characteristic of the motivic Milnor fiber $S_0(f)$ is $\text{MHS}(f)$.

There is also a natural linear map from $K_0(HS^{\text{mon}})$ to $\mathbb{Z}[t^{\mathbb{Q}}]$, called the *Hodge spectrum*. For a mixed Hodge structure H endowed with a finite automorphism, let $H_{\alpha}^{p,q}$ denote the eigenspace of $H^{p,q}$ associated to the eigenvalue $\exp(2\pi i\alpha)$. We define

$$\text{hsp}(H) = \sum_{\alpha \in \mathbb{Q} \cap (0,1)} t^{\alpha} \sum t^p \dim(H_{\alpha}^{p,q}).$$

Composing with the Hodge characteristic, we obtain a map

$$\text{Sp} : \mathcal{M}_{\mathbb{C}}^{\hat{\mu}} \rightarrow \mathbb{Z}[t^{\mathbb{Q}}].$$

Denef and Loeser [3] show that the spectrum of f (as defined by Steenbrink [10] and Saito [7]) can be computed using

$$\text{spectrum}(f) = \text{Sp}(1 - S_0(f)).$$

Applying Sp to the formula of Theorem 1, we obtain this recursive formula for the spectrum:

$$\text{spectrum}(f) = \text{spectrum}((f_1)^{d'}) + \text{spectrum}(f') + \text{Sp}([\mu_{d'}] - 1). \quad (11)$$

As an alternative, here is the result of applying Sp to formula (4):

$$\text{spectrum}(f) = -\text{Sp}([(f_1)^{d'} - 1]) + \text{Sp}([\mu_{d'}]\mathbb{L}) + \text{spectrum}(f'). \quad (12)$$

To make this a usable recursion, we supplement formula (12) with results from Guibert [5]. In Lemme 3.4.2(ii), he tells us that

$$\text{Sp}([f_1 - 1]) = t - \frac{t^{1/m} - t}{1 - t^{1/m}} \cdot \frac{t^{1/n} - t}{1 - t^{1/n}}.$$

Thus, as a multiset, it consists of $(m-1)(n-1)/2$ numbers in the interval $(0, 1)$ and an equal number in the interval $(1, 2)$ (all of these counting negatively), together with the number 1. Furthermore we remark that the multiset is symmetric with respect to reflection across the number 1. We may write $\text{Sp}([f_1 - 1])$ as a sum $\text{Sp}^{(0)}([f_1 - 1]) + \text{Sp}^{(1)}([f_1 - 1])$, using first the terms with exponents between 0 and 1, and then those with exponents greater than or equal to 1. Guibert's Lemme 3.4.3 says that

$$\begin{aligned} \text{Sp}([(f_1)^{d'} - 1])(t) &= \frac{1 - t}{1 - t^{1/d'}} \cdot \\ &\left(\text{Sp}^{(0)}([f_1 - 1])(t^{1/d'}) + t^{1-1/d'} \text{Sp}^{(1)}([f_1 - 1])(t^{1/d'}) \right). \end{aligned} \quad (13)$$

We claim that the term $\text{Sp}([\mu_{d'}]\mathbb{L})$ in formula (12) is

$$\frac{1-t}{1-t^{1/d}} \cdot t.$$

Indeed, the action on the product $\mu_{d'} \times \mathbb{C}$ has as eigenvalues the d' -th roots of unity, and the cohomology lives in $H^{1,1}$. Now note that this term cancels some of the contributions to formula (13); this has the same effect as if the initial term t had been omitted from $\mathrm{Sp}^{(1)}([f_1 - 1])$. Thus again the spectrum has reflectional symmetry. The upshot is that the contribution of the first two terms on the left of (12) to the spectrum of f is obtained from the spectrum of f_1 by the following process:

- Discard the spectral numbers greater than 1.
- Compress by a factor of $1/d'$, i.e., multiply each spectral number by this value.
- Take the union of d' copies: use the first copy unaltered, add $1/d'$ to each spectral number to get the second copy, etc.
- To obtain the remaining contributions in the interval $(1, 2)$, reflect across 1.

Finally we remark that formula (11) and the auxiliary formulas imply formula (6) of Theorem 2.3 in [6], a recursion for the monodromy:

$$\mathbf{H}(t) = \frac{\mathbf{H}_1(t^{d'}) \cdot \mathbf{H}'(t)}{t^{d'} - 1}. \quad (14)$$

To see this, observe that formula (13) implies that the eigenvalues of the monodromy for $(f_1)^{d'}$ are the d' th roots of the eigenvalues of the monodromy for f_1 . Combining this with formula (11) yields the result.

6 An example

Let us consider Example 2.2 from [6]. We begin with the curve whose Puiseux expansion is

$$\zeta = x^{3/2} + x^{7/4} + x^{11/6}. \quad (15)$$

Then its truncation $f_1 = y^2 - x^3$ is parametrized by $\zeta_1 = x^{3/2}$, and its derived curve is parametrized by

$$\zeta' = x^{13/2} + x^{20/3}.$$

Repeating this process, we obtain a truncation $f'_1 = y^2 - x^{13}$, parametrized by $\zeta'_1 = x^{13/2}$ and a second derived curve $f'' = y^3 - x^{79}$ with parametrization

$$\zeta'' = x^{79/3}.$$

The number of conjugates of the curve, its truncation, the derived curve, the second truncation, and the second derived curve are $d = 1$, $d_1 = 2$, $d' = 6$, $d'_1 = 2$, and $d'' = 3$.

Applying our recursive formula (4) repeatedly, the motivic Milnor fiber $S_0(f)$ equals

$$[(y^2 - x^3)^6 - 1] + [(y^2 - x^{13})^3 - 1] + [y^3 - x^{79} - 1] - [\mu_6]\mathbb{L} - [\mu_3]\mathbb{L} - \mathbb{L} + 1.$$

Before computing the spectrum, we remark that formula (14) implies a recursion for the Euler characteristic of the Milnor fiber (formula (4) of the same cited theorem). Applying that formula here, we infer that the Milnor number is 204; thus our recursion will yield 204 spectral numbers.

To compute the spectrum of f , we first apply the process described in the previous section to the two spectral numbers $\frac{5}{6}$ and $\frac{7}{6}$ of $f_1 = y^2 - x^3$. Using just the spectral number $\frac{5}{6}$, compression and copying gives these six spectral numbers in the interval $(0, 1)$:

$$\frac{5}{36}, \frac{11}{36}, \frac{17}{36}, \frac{23}{36}, \frac{29}{36}, \frac{35}{36}.$$

Similarly, the spectral numbers $\frac{15}{26}, \frac{17}{26}, \frac{19}{26}, \frac{21}{26}, \frac{23}{26}, \frac{25}{26}$ of $f'_1 = y^2 - x^{13}$ give us eighteen additional spectral numbers of f :

$$\frac{15}{78}, \dots, \frac{25}{78}, \quad \frac{41}{78}, \dots, \frac{51}{78}, \quad \frac{67}{78}, \dots, \frac{77}{78}.$$

Likewise there are contributions coming from the 78 spectral numbers of $y^3 - x^{79}$ which lie in the interval $(0, 1)$; these contributions are

$$\frac{82}{237}, \frac{85}{237}, \frac{88}{237}, \frac{91}{237}, \dots, \frac{235}{237}$$

and

$$\frac{161}{237}, \frac{164}{237}, \frac{167}{237}, \frac{170}{237}, \dots, \frac{236}{237}.$$

In total this gives 102 spectral numbers less than 1; after reflection we obtain an additional 102 values, which together account for the Milnor number.

We now compare our computation with Denef and Loeser's computation in terms of an embedded resolution [3], continuing to work with the function f whose Puiseux expansion is specified in (15). Let $h : Y \rightarrow \mathbb{C}^2$ be a function achieving this embedded resolution. Using index set J , denote by E_i the irreducible components of $h^{-1}(f^{-1}(0))$. For each $i \in J$, denote by N_i the multiplicity of E_i in the divisor of $f \circ h$ on Y . For $i, j \in J$, we consider the non singular varieties $E_i^\circ := E_i \setminus \cup_{j \neq i} E_j$ and $E_{i,j}^\circ := E_i \cap E_j$. We let $m_{i,j} := \gcd(N_i, N_j)$.

Denef and Loeser proved the following formula:

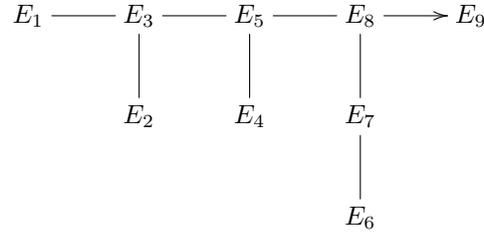
$$S_0(f) = \sum_{i \in J} -[\tilde{E}_i^\circ \cap h^{-1}(0)] + \sum_{i,j \in J, E_i \cap E_j \neq \emptyset} [\tilde{E}_{i,j}^\circ \cap h^{-1}(0)](\mathbb{L} - 1),$$

where $E_{i,j}^\circ = \mu_{m_{i,j}}$ and \tilde{E}_i° is an unramified Galois cover of E_i° with Galois group μ_{N_i} defined as follows. Let U be a Zariski open subset of Y such that $f \circ h = uv^{N_i}$, with u a unit on U and v a morphism from U to \mathbb{C} . The restriction of \tilde{E}_i° above the set $E^\circ \cap U$ is defined by

$$\{(z, y) \in \mathbb{C} \times (E_i^\circ \cap U) : z^{N_i} = u^{-1}\}.$$

Considering an open covering of E° and gluing the corresponding restrictions we get the cover \tilde{E}_i° which has a natural μ_{N_i} -action obtained by multiplying the z -coordinate by the elements of μ_{N_i} .

To obtain the minimal embedded resolution of the curve whose Puiseux expansion is shown in (15), we perform 8 point blow-ups. The corresponding irreducible components E_i , which have been labeled according to their order of appearance, intersect according to the following dual resolution graph:



The irreducible component E_9 corresponds to the strict transform of the curve. The multiplicities N_i are given in the following table

	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8	E_9
N_i	12	18	36	39	78	79	158	237	1

We now use this resolution to compute $S_{\mathbf{0}}(f)$ in $K_0^{\mu}(\text{Var}_{\mathbb{C}})$. First, observe that $E_i^\circ \cap h^{-1}(0) = E_i^\circ$ except when $i = 9$; in the latter case, $E_9^\circ \cap h^{-1}(0)$ is the point $E_8 \cap E_9$. Because $E_1^\circ, E_2^\circ, E_4^\circ$, and E_6° are each isomorphic to \mathbb{C} , we have

$$[\tilde{E}_1^\circ] = [\mu_{12}]\mathbb{L}, \quad [\tilde{E}_2^\circ] = [\mu_{18}]\mathbb{L}, \quad [\tilde{E}_4^\circ] = [\mu_{39}]\mathbb{L}, \quad [\tilde{E}_6^\circ] = [\mu_{78}]\mathbb{L}.$$

Each $E_{i,j}^\circ$ is a point; thus

$$\begin{aligned}
 [\tilde{E}_{1,3}^\circ] &= [\mu_{12}], [\tilde{E}_{2,3}^\circ] = [\mu_{18}], [\tilde{E}_{3,5}^\circ] = [\mu_6], [\tilde{E}_{4,5}^\circ] = [\mu_{39}], \\
 [\tilde{E}_{5,8}^\circ] &= [\mu_3], [\tilde{E}_{6,7}^\circ] = [\mu_{79}], [\tilde{E}_{7,8}^\circ] = [\mu_{79}], [\tilde{E}_{8,9}^\circ] = 1,
 \end{aligned}$$

where we have used the relevant $m_{i,j}$'s (for example $m_{1,3} = 12$). Since $\gcd(N_6, N_7, N_8) = 79$, and E_7° is isomorphic to \mathbb{C}^* , it follows that

$$[\tilde{E}_7^\circ] = [\mu_{79}](\mathbb{L} - 1).$$

To analyze E_3 , we choose an affine coordinate w so that the intersection with E_5 is the point at infinity, and so that the intersections with E_1 and E_2 are at 0 and 1, respectively. Thus we must consider

$$\{(z, w) \in \mathbb{C} \times E_3^\circ : z^{N_3} = w^{m_{1,3}}(1-w)^{m_{3,5}}, \text{ i.e., } z^{36} = w^{12}(1-w)^6\}.$$

There is an isomorphism between this locus and

$$\{(x, y) \in (\mathbb{C}^*)^2 : (y^2 - x^3)^6 = 1\}$$

given by

$$z = \frac{x}{y}, w = \frac{x^3}{y^2} \quad x = \frac{w}{z^2}, y = \frac{w}{z^3}.$$

It identifies $[\tilde{E}_3^\circ]$ with $[\{(x, y) \in (\mathbb{C}^*)^2 : (y^2 - x^3)^6 = 1\}]$ (together with the usual quasi-homogeneous action) in $K_0^{\hat{\mu}}(\text{Var}_{\mathbb{C}})$.

Similarly, the isomorphism between $\{(z, w) \in \mathbb{C} \times E_3^\circ : z^{78} = w^{36}(1-w)^3\}$ and $\{(x, y) \in (\mathbb{C}^*)^2 : (y^2 - x^{13})^3 = 1\}$, given by

$$z = \frac{x^6}{y}, w = \frac{x^{13}}{y^2} \quad x = \frac{w}{z^2}, y = \frac{w^6}{z^{13}},$$

identifies $[\tilde{E}_5^\circ]$ with $[\{(x, y) \in (\mathbb{C}^*)^2 : (y^2 - x^{13})^3 = 1\}]$. In the same way, the isomorphism between $\{(z, w) \in \mathbb{C} \times E_8^\circ : z^{237} = w^{79}(1-w)\}$ and $\{(x, y) \in (\mathbb{C}^*)^2 : (y^2 - x^{13})^3 = 1\}$, given by

$$z = \frac{x^{26}}{y}, w = \frac{x^{79}}{y^3} \quad x = \frac{w}{z^3}, y = \frac{w^{26}}{z^{79}},$$

identifies $[\tilde{E}_8^\circ]$ with $[\{(x, y) \in (\mathbb{C}^*)^2 : y^3 - x^{79} = 1\}]$.

Assembling these equations, we see that

- $[\tilde{E}_1^\circ] + [\tilde{E}_2^\circ] + [\tilde{E}_3^\circ] - [\tilde{E}_{1,3}^\circ](\mathbb{L} - 1) - [\tilde{E}_{2,3}^\circ](\mathbb{L} - 1) = [(y^2 - x^3)^6 - 1];$
- $[\tilde{E}_4^\circ] + [\tilde{E}_5^\circ] - [\tilde{E}_{3,5}^\circ](\mathbb{L} - 1) - [\tilde{E}_{4,5}^\circ](\mathbb{L} - 1) = [(y^2 - x^{13})^3 - 1] - [\mu_6]\mathbb{L};$
- $[\tilde{E}_6^\circ] + [\tilde{E}_7^\circ] + [\tilde{E}_8^\circ] - [\tilde{E}_{5,8}^\circ](\mathbb{L} - 1) - [\tilde{E}_{6,7}^\circ](\mathbb{L} - 1) - [\tilde{E}_{7,8}^\circ](\mathbb{L} - 1) = [y^3 - x^{79} - 1] - [\mu_3]\mathbb{L};$

in $K_0^{\hat{\mu}}(\text{Var}_{\mathbb{C}})$. These equalities, together with $[\tilde{E}_{8,9}^\circ](\mathbb{L} - 1) = (\mathbb{L} - 1)$, give us the expression for $S_0(f)$ computed with our recursive formula.

Acknowledgements This paper benefited greatly from extensive conversations with Mirel Caibăr, and Alejandro Melle Hernández; we also received helpful advice from François Loeser. The first author was supported by Spanish Ministerio de Ciencia y Tecnología Grant no. MTM2016-76868-C2-1-P. The second author was supported by a Collaboration Grant from the Simons Foundation, and worked on this project while in residence at the Fields Institute.

References

1. E. Artal Bartolo, Pi. Cassou-Nogus, I. Luengo, A. Melle Hernández, *Quasi-ordinary power series and their zeta functions*. Mem. Amer. Math. Soc. **178** (2005), no. 841, vi+85 pp.
2. T. Cauwbergs, *Splicing motivic zeta functions*. Rev. Mat. Complut. **29** (2016), no. 2, 455–483.
3. J. Denef and F. Loeser, *Geometry on arc spaces of algebraic varieties*, European Congress of Mathematics, Vol. I (Barcelona, 2000), Progr. Math., vol. 201, Birkhäuser, Basel, 2001, pp. 327–348.
4. P. D. González Pérez and M. González Villa, *Motivic Milnor fiber of a quasi-ordinary hypersurface*. J. Reine Angew. Math. **687** (2014), 159–205.
5. G. Guibert, *Espaces d'arcs et invariants d'Alexander*, Comment. Math. Helv. **77** (2002), no. 4, 783–820.
6. G. Kennedy and L. McEwan, *Monodromy of plane curves and quasi-ordinary surfaces*, J. Singul. **1** (2010), 146–168.
7. M. Saito, *On Steenbrink's conjecture*, Math. Ann. **289** (1991), no. 4, 703–716.
8. J. H. M. Steenbrink, *Mixed Hodge structure on the vanishing cohomology*, Real and complex singularities, Sijthoff and Noordhoff, Alphen aan den Rijn, 1977, pp. 525–563.
9. J. H. M. Steenbrink, *Semicontinuity of the singularity spectrum*, Invent. math. **79** (1985), 557–565.
10. J. H. M. Steenbrink, *The spectrum of hypersurface singularities*, Astérisque (1989), no. 179-180, 11, 163–184.
11. A. N. Varchenko, *Asymptotic Hodge structure on vanishing cohomology*, Izv. Akad. Nauk SSSR Ser. Mat. **45** (1981), no. 3, 540–591, 688.