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New continuous bivariate distributions developed based on general shock models

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ABSTRACT

In this study, we develop novel classes of continuous bivariate distributions based on general shock models. One class is that of absolutely continuous bivariate distributions, whereas the other one is that of non-absolutely continuous bivariate distributions. These classes are versatile in the sense that they can generate numerous families of distributions. We explore the distributional characteristics of the proposed classes, examining the bivariate ageing property and the dependence structure. Under some conditions, the proposed class of distributions satisfy certain kind of dependence property, called conditional PQD. Our result also reveals that a well-defined subclass of the proposed class satisfies the bivariate lack of memory property. Finally, we generate particular distribution families and apply them to two real-world datasets to illustrate their usefulness.

1. Introduction

Various shock models for reliability or survival analysis have been intensively explored and studied in the literature (see, for example, Al-Hameed and Proschan [1], Montoro-Cazorla and Pérez-Ocón [2–4], Finkelstein [5,6], Eryilmaz [7], Eryilmaz and Tekin [8], Chadjiconstantinidis [9], Goyal et al. [10]). Generally, these models are categorized into two types: the cumulative shock model and the extreme shock model. In extreme shock models, a system is exposed to an external shock process, where each shock may either cause the system to fail with a certain probability or have no effect on the system with the complementary probability. In contrast, cumulative shock models involve the accumulation of shock impacts, with system failure occurring once the accumulated effect reaches a specific threshold.

In the early stages of studies on bivariate or multivariate distributions, many researchers attempted to extend the exponential distribution to the bivariate or multivariate case. Marshall and Olkin [11] introduced an important bivariate exponential distribution based on an extreme shock model, demonstrating how reliability shock models could be effectively applied to construct dependent distributions. After the work of Marshall and Olkin [11], other shock models also contributed to the development of various bivariate or multivariate distributions. In Arnold [12], a class of multivariate exponential distribution is suggested, where the model is based on a concept of hierarchical successive damage caused by shock process. In Raftery [13], a continuous multivariate exponential distribution is introduced which can model a full range of correlation structures and attains the Fréchet bounds in the bivariate case, which arises as a model for reliability and failure due to shocks. O'cinneide and Raftery [14] establishes a continuous multivariate

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exponential distribution that is also a multivariate phase type (MPH) distribution, by extending a reliability model for a bivariate exponential distribution to the multivariate case. To extend the bivariate exponential distributions by Marshall and Olkin [11] and Freund [26] and to derive bivariate extensions of the Weibull distribution, Lu [15] assumed that the failure rate of one component changes upon the failure of the other and Poisson fatal shocks cause simultaneous failures of both components. Wang and Li [16] introduces the bivariate homogeneous shock model to represent a two-component series system that is subjected to fatal shocks. Shocks can affect component 1, component 2, or both components simultaneously. The paper then studies several dependency measures for this model, including Kendall's tau, Yanagimoto and Okamoto's grade correlation coefficient, and Pearson's correlation coefficient. Cha and Badía [30] stochastically modeled positively dependent multivariate reliability distributions based on stochastically dependent dynamic shock model and suggested explicit parametric forms for the multivariate reliability functions. Furthermore, multivariate ageing properties and dependence structures of the class were discussed as well. Recently, Lee and Cha [17] proposed a new general bivariate distribution based on a shock process. The model assumes that a system consists of two components, both of which are subject to a nonhomogeneous Poisson process (NHPP) of external shock. The first shock randomly destroys one of the components, while the second shock destroys the remaining component. Under this shock model, in Lee and Cha [17], the joint distribution of the lifetimes of the two components was obtained.

In this paper, we develop a new general class of bivariate distributions based on a novel shock model. We assume that the system with two components is subject to a generalized Polya process (GPP) of external shocks. For two positive integers $n_1 < n_2$, it is assumed that the n_1-th shock causes the failure of one of the components randomly, and the n_2-th shock destroys the remaining component. We derive both the joint distribution and the marginal distributions of the lifetimes of the two components. By specifying the baseline intensity function of the GPP, several useful specific parametric families of distributions are generated. Additionally, by incorporating an extra external shock process that leads to the simultaneous failure of both components, we develop a non-absolutely continuous bivariate distribution that allows common (or tied) observations of the two involved random variables.

This paper is organized as follows. Section 2 provides some basic concepts which are used to develop a new bivariate distribution model. Section 3 introduces the shock model used to develop an absolutely continuous bivariate distribution, and based on it, a new class of bivariate distributions is constructed. By specifying parametric baseline intensity functions in the model, several specific families of distributions are generated. Section 4 demonstrates that the bivariate distribution developed in this paper exhibits a kind of positive dependency and satisfies the bivariate lack of memory property. In Section 5, we explore a shock model that allows for common failures, and based on it, a non-absolutely continuous bivariate distribution is derived. Finally, in Section 6, the bivariate distribution developed in this paper is applied to fit real data sets.

2. Preliminaries

First, we need to introduce the GPP [18], which is used to develop a new bivariate distribution in this paper. Cha [18] defined the GPP based on the concept of stochastic intensity. For an orderly counting process $\{N(t), t \geq 0\}$ and its past history $H_{t-} \equiv \{N(u), 0 \leq u < t\}$, the stochastic intensity is defined by (see also the works of Aven and Jensen [19], Finkelstein and Cha [20])

$$\lambda_t = \lim_{\Delta t \to 0} \frac{P(N(t, t+\Delta t) = 1 | H_{t-})}{\Delta t} = \lim_{\Delta t \to 0} \frac{E[N(t, t+\Delta t) | H_{t-}]}{\Delta t},$$

where $N(t_1, t_2), t_1 < t_2$ is the number of events in $[t_1, t_2)$.

Definition 1. **(Generalized Polya Process (GPP))**

For an orderly counting process $\{N(t), t \geq 0\}$, if

i. $N(t) = 0$,
ii. $\lambda_t = (\alpha N(t-) + \beta)\lambda(t)$

then it is called the GPP with the corresponding parameter set $\{\lambda(t), \alpha, \beta\}$, $\alpha \geq 0, \beta > 0$.

As stated in the work of Cha, the GPP with $(\lambda(t), \alpha = 0, \beta = 1)$ reduces to the NHPP with intensity function $\lambda(t)$. Thus, the GPP is a generalization of the NHPP. It is obvious that the GPP with $\alpha > 0$ does not possess the independent increments property.

Remark 1. Note that the GPP with the parameter set $\{\lambda(t), \alpha, \beta\}$ in Definition 1 is not identifiable in estimation procedure, although the results in Cha [18] have been obtained with the definition for the convenience of the characterization of the GPP. Thus, when developing the bivariate distribution in this paper, we will set it as $\beta = 1$, and the GPP with the parameter set $\{\lambda(t), \alpha, 1\}$ will be used for the development of the distribution. Due to this reason, in the following, the properties of the GPP will be stated for the GPP with the parameter set $\{\lambda(t), \alpha, 1\}$

Definition 2. (Restarting property, Cha [18]) Let $t > 0$ be an arbitrary time point. If the conditional future stochastic process from t, given the history unit time t, follows the same type of stochastic process with possibly different set of process parameters, then the process is called to possess the restarting property. A stochastic process with the restarting property is called the restarting process.

From the Definition 2, it is clear that the GPP has the restarting property and it is stated in detail in the following proposition (see Cha [18]).

Proposition 1. *Let $\{N(t), t \geq 0\}$ be the GPP with the set of parameters $\{\lambda(t), \alpha, 1\}$. At an arbitrary time $u > 0$, given $N(u-) = n$ and the arrival times $T_1 = t_1, ..., T_n = t_n$, the conditional future process $\{N_u(t), t \geq 0\}$, where $N_u(t) = N(t+u) - N(u)$, is also the GPP with the set of parameters $(\lambda(u+t), \alpha, 1+n\alpha), t > 0$.*

The following propositions provide the distribution of the number of events and the conditional distribution of the arrival times in the GPP, which will be used to derive important results in this paper. For convenience, throughout this paper, we define $\Lambda(t) \equiv \int_0^t \lambda(u)du$.

Proposition 2. *Let $t > 0$ and $0 \equiv u_0 < u_1 < \cdots < u_m$. For the GPP with the parameter set $\{\lambda(t), \alpha, 1\}$,*

(i) $$P(N(t) = n) = \frac{\Gamma\left(\frac{1}{\alpha}+n\right)}{\Gamma\left(\frac{1}{\alpha}\right)n!}(1-\exp\{-\alpha\Lambda(t)\})^n(\exp\{-\alpha\Lambda(t)\})^{1/\alpha}.$$

(ii) $$P(N(u_i) - N(u_{i-1}) = n_i, i = 1,2,...,m) = \prod_{i=1}^{m}\left[\frac{\Gamma\left(1/\alpha+\sum_{k=1}^{i}n_k\right)}{\Gamma\left(1/\alpha+\sum_{k=1}^{i-1}n_k\right)n_i!}(1-\exp\{-\alpha[\Lambda(u_i)-\Lambda(u_{i-1})]\})^{n_i}\right.$$
$$\left.\times\exp\{-\alpha[\Lambda(u_i)-\Lambda(u_{i-1})]\}^{\sum_{k=1}^{i-1}n_k+1/\alpha}\right]$$

(iii) $$P(N(u_2) - N(u_1) = n_2 | N(u_1) = n_1) = \frac{\Gamma(1/\alpha+n_1+n_2)}{\Gamma(1/\alpha+n_1)n_2!}(1-\exp\{-\alpha[\Lambda(u_2)-\Lambda(u_1)]\})^{n_2}$$
$$\times(1-\exp\{-\alpha[\Lambda(u_2)-\Lambda(u_1)]\})^{n_1+1/\alpha}$$

From Proposition 2, it can be seen that the distribution of $N(t)$ is a negative binomial distribution with two parameters $(1/\alpha, \exp\{-\alpha\Lambda(t)\})$, and thus $E[N(t)]$ is given by

$$E[N(t)] = (1/\alpha)(\exp\{\alpha\Lambda(t)\}-1).$$

Proposition 3. *For the GPP with the parameter set $\{\lambda(t), \alpha, 1\}$, the pdf of S_n, denoted by $f_n(t)$ is given by*

$$f_n(t) = (\alpha(n-1)+1)\lambda(t)\frac{\Gamma\left(\frac{1}{\alpha}+n-1\right)}{\Gamma\left(\frac{1}{\alpha}\right)(n-1)!}(1-\exp\{-\alpha\Lambda(t)\})^{n-1}(\exp\{-\alpha\Lambda(t)\})^{\frac{1}{\alpha}}, t \geq 0.$$

Proof.

Observe that the event $\{S_n > t\}$ is equivalent to $\{N(t) < n\}$. Thus,

$$P(S_n > t) = \sum_{k=0}^{n-1}P(N(t) = k) = (\exp\{-\alpha\Lambda(t)\})^{\frac{1}{\alpha}}\sum_{k=0}^{n-1}\frac{\Gamma\left(\frac{1}{\alpha}+k\right)}{\Gamma\left(\frac{1}{\alpha}\right)k!}(1-\exp\{-\alpha\Lambda(t)\})^k.$$

$$= (\exp\{-\alpha\Lambda(t)\})^{\frac{1}{\alpha}}\sum_{k=0}^{n-1}\frac{\Gamma\left(\frac{1}{\alpha}+k\right)}{\Gamma\left(\frac{1}{\alpha}\right)k!}(1-\exp\{-\alpha\Lambda(t)\})^k.\sum_{k=0}^{n-1}\frac{\Gamma\left(\frac{1}{\alpha}+k\right)}{\Gamma\left(\frac{1}{\alpha}\right)k!}(1-\exp\{-\alpha\Lambda(t)\})^k.$$

Thus, the result can be obtained by $f_n(t) = \sum_{k=0}^{n-1}\frac{d}{dt}P(N(t) = k)$. ■

The result in Proposition 3 can also be interpreted/constructed as follows:

$$f_n(t)\Delta t \approx P(t < S_n \leq t+\Delta t) = P(t < S_n \leq t+\Delta t, N(t-) = n-1)$$
$$= P(t < S_n \leq t+\Delta t | N(t-) = n-1)P(N(t-) = n-1)$$
$$= (\alpha(n-1)+1)\lambda(t)\frac{\Gamma\left(\frac{1}{\alpha}+n-1\right)}{\Gamma\left(\frac{1}{\alpha}\right)(n-1)!}(1-\exp\{-\alpha\Lambda(t)\})^{n-1}(\exp\{-\alpha\Lambda(t)\})^{\frac{1}{\alpha}}.$$

3. Development of continuous bivariate distribution

A system is composed of two components and is subject to a Generalized Polya process (GPP) $\{N(t), t \geq 0\}$ of external shocks with the set of parameters $\{\lambda(t), \alpha, 1\}$. For two positive integers $n_1 < n_2$, it is assumed that the n_1-*th* shock destroys the first component with probability θ and it destroys the second component with probability $(1-\theta)$. The n_2-*th* shock destroys the remaining component.

Denote by X_i the lifetime of component i, $i = 1,2$. Let S_i be the time from 0 to the occurrence of *i-th* shock, $i = 1,2,3,\ldots$.

First, we derive the joint probability density function of (X_1, X_2). For convenience, throughout this paper, we define $I = 1$, if the n_1-*th* shock destroys the first component, $I = 0$, otherwise.

Proposition 4. *The joint probability density function of* (X_1, X_2) *is given by*

$$\begin{aligned} & f_{(X_1,X_2)}(x_1,x_2) \\ &= \theta \frac{\Gamma(1/\alpha + n_1 - 1)}{\Gamma(1/\alpha)(n_1-1)!}(1-\exp\{-\alpha\Lambda(x_1)\})^{n_1-1}(\exp\{-\alpha\Lambda(x_1)\})^{1/\alpha}\{(n_1-1)\alpha+1\}\lambda(x_1) \\ &\times \frac{\Gamma(1/\alpha+n_2-1)}{\Gamma(1/\alpha+n_1)(n_2-n_1-1)!}(1-\exp\{-\alpha[\Lambda(x_2)-\Lambda(x_1)]\})^{n_2-n_1-1}\exp\{-\alpha[\Lambda(x_2)-\Lambda(x_1)]\}^{n_1+1/\alpha} \\ &\times\{(n_2-1)\alpha+1\}\lambda(x_2), x_1 \leq x_2. \end{aligned}$$

$$\begin{aligned} & f_{(X_1,X_2)}(x_1,x_2) \\ &= (1-\theta) \frac{\Gamma(1/\alpha + n_1 - 1)}{\Gamma(1/\alpha)(n_1-1)!}(1-\exp\{-\alpha\Lambda(x_2)\})^{n_1-1}(\exp\{-\alpha\Lambda(x_2)\})^{1/\alpha}\{(n_1-1)\alpha+1\}\lambda(x_2) \\ &\times \frac{\Gamma(1/\alpha+n_2-1)}{\Gamma(1/\alpha+n_1)(n_2-n_1-1)!}(1-\exp\{-\alpha[\Lambda(x_1)-\Lambda(x_2)]\})^{n_2-n_1-1}\exp\{-\alpha[\Lambda(x_1)-\Lambda(x_2)]\}^{n_1+1/\alpha} \\ &\times\{(n_2-1)\alpha+1\}\lambda(x_1), x_1 > x_2. \end{aligned}$$

Proof.

We first consider the case of $I = 1$ in which it follows that $X_1 = S_{n_1}, X_2 = S_{n_2}$ with probability θ. Then the joint density function of (X_1, X_2) is given by

$$\begin{aligned} f_{(X_1,X_2)}(x_1,x_2) &= \theta P(N(x_1) = n_1 - 1)\{(n_1-1)\alpha+1\}\lambda(x_1) \\ &\times P(N(x_2) - N(x_1) = n_2 - n_1 - 1 | N(x_1) = n_1)\{(n_2-1)\alpha+1\}\lambda(x_2) \\ &= \theta \frac{\Gamma(1/\alpha + n_1 - 1)}{\Gamma(1/\alpha)(n_1-1)!}(1-\exp\{-\alpha\Lambda(x_1)\})^{n_1-1}(\exp\{-\alpha\Lambda(x_1)\})^{1/\alpha}\{(n_1-1)+1\}\lambda(x_1) \\ &\times \frac{\Gamma(1/\alpha+n_2-1)}{\Gamma(1/\alpha+n_1)(n_2-n_1-1)!}(1-\exp\{-\alpha[\Lambda(x_2)-\Lambda(x_1)]\})^{n_2-n_1-1}\exp\{-\alpha[\Lambda(x_2)-\Lambda(x_1)]\}^{n_1+1/\alpha} \\ &\times\{(n_2-1)\alpha+1\}\lambda(x_2), x_1 \leq x_2. \end{aligned}$$

In the case of $I = 0$, it follows that $X_1 = S_{n_2}, X_2 = S_{n_1}$ with probability $1-\theta$. Then, symmetrically, the joint density function of (X_1, X_2) is given by

$$\begin{aligned} f_{(X_1,X_2)}(x_1,x_2) &= (1-\theta) \frac{\Gamma(1/\alpha + n_1 - 1)}{\Gamma(1/\alpha)(n_1-1)!}(1-\exp\{-\alpha\Lambda(x_2)\})^{n_1-1}(\exp\{-\alpha\Lambda(x_2)\})^{1/\alpha}\{(n_1-1)\alpha+1\}\lambda(x_2) \\ &\times \frac{\Gamma(1/\alpha+n_2-1)}{\Gamma(1/\alpha+n_1)(n_2-n_1-1)!}(1-\exp\{-\alpha[\Lambda(x_1)-\Lambda(x_2)]\})^{n_2-n_1-1}\exp\{-\alpha[\Lambda(x_1)-\Lambda(x_2)]\}^{n_1+1/\alpha} \\ &\times\{(n_2-1)\alpha+1\}\lambda(x_1), x_1 > x_2. \end{aligned}$$

Combining the two cases, the desired result is obtained as follows. ■

Now, we derive the joint survival function of (X_1, X_2).

Proposition 5. *The bivariate joint survival function of* (X_1, X_2) *is given by*

$$S_{(X_1,X_2)}(x_1,x_2)$$
$$= \theta \sum_{i=0}^{n_1-1} \sum_{j=0}^{(n_2-1)-i} \left[\frac{\Gamma(1/\alpha+i)}{\Gamma(1/\alpha)i!}(1-\exp\{-\alpha\Lambda(x_1)\})^i \exp\{-\alpha\Lambda(x_1)\}^{1/\alpha} \right.$$
$$\left. \times \frac{\Gamma(1/\alpha+i+j)}{\Gamma(1/\alpha+i)j!}(1-\exp\{-\alpha[\Lambda(x_2)-\Lambda(x_1)]\})^j \exp\{-\alpha[\Lambda(x_2)-\Lambda(x_1)]\}^{i+1/\alpha} \right]$$
$$+ (1-\theta) \sum_{i=0}^{n_1-1} \frac{\Gamma(1/\alpha+i)}{\Gamma(1/\alpha)i!}(1-\exp\{-\alpha\Lambda(x_2)\})\exp\{-\alpha\Lambda(x_2)\}^{1/\alpha},\ x_1 \le x_2,$$

$$S_{(X_1,X_2)}(x_1,x_2)$$
$$= \theta \sum_{i=0}^{n_1-1} \frac{\Gamma(1/\alpha+i)}{\Gamma(1/\alpha)i!}(1-\exp\{-\alpha\Lambda(x_1)\})\exp\{-\alpha\Lambda(x_1)\}^{1/\alpha}$$
$$+(1-\theta) \sum_{i=0}^{n_1-1} \sum_{j=0}^{(n_2-1)-i} \left[\frac{\Gamma(1/\alpha+i)}{\Gamma(1/\alpha)i!}(1-\exp\{-\alpha\Lambda(x_2)\})^i \exp\{-\alpha\Lambda(x_2)\}^{1/\alpha} \right.$$
$$\left. \times \frac{\Gamma(1/\alpha+i+j)}{\Gamma(1/\alpha+i)j!}(1-\exp\{-\alpha[\Lambda(x_1)-\Lambda(x_2)]\})^j \exp\{-\alpha[\Lambda(x_1)-\Lambda(x_2)]\}^{i+1/\alpha} \right], x_1 > x_2.$$

Proof.

For $x_1 \le x_2$,

$$P(X_1 > x_1, X_2 > x_2) = \theta P(S_{n_1} > x_1, S_{n_2} > x_2) + (1-\theta) P(S_{n_2} > x_1, S_{n_1} > x_2)$$
$$= \theta P(S_{n_1} > x_1, S_{n_2} > x_2) + (1-\theta) P(S_{n_1} > x_2)$$
$$= \theta P(N(x_1) \le n_1 - 1, N(x_2) \le n_2 - 1) + (1-\theta) P(N(x_2) \le n_1 - 1)$$
$$= \theta \sum_{i=0}^{n_1-1} P(N(x_1) = i, N(x_2) - N(x_1) \le (n_2-1) - i) + (1-\theta) P(N(x_2) \le n_1 - 1)$$
$$= \theta \sum_{i=0}^{n_1-1} \sum_{j=0}^{(n_2-1)-i} \left[\frac{\Gamma(1/\alpha+i)}{\Gamma(1/\alpha)i!}(1-\exp\{-\alpha\Lambda(x_1)\})^i \exp\{-\alpha\Lambda(x_1)\}^{1/\alpha} \right.$$
$$\left. \times \frac{\Gamma(1/\alpha+i+j)}{\Gamma(1/\alpha+i)j!}(1-\exp\{-\alpha[\Lambda(x_2)-\Lambda(x_1)]\})^j \exp\{-\alpha[\Lambda(x_2)-\Lambda(x_1)]\}^{i+1/\alpha} \right]$$
$$+(1-\theta) \sum_{i=0}^{n_1-1} \frac{\Gamma(1/\alpha+i)}{\Gamma(1/\alpha)i!}(1-\exp\{-\alpha\Lambda(x_2)\})\exp\{-\alpha\Lambda(x_2)\}^{1/\alpha}$$

For $x_1 > x_2$,

$$P(X_1 > x_1, X_2 > x_2) = \theta P(S_{n_1} > x_1, S_{n_2} > x_2) + (1-\theta) P(S_{n_2} > x_1, S_{n_1} > x_2)$$
$$= \theta P(S_{n_1} > x_1) + (1-\theta) P(S_{n_2} > x_1, S_{n_1} > x_2)$$
$$= \theta P(N(x_1) \le n_1 - 1) + (1-\theta) P(N(x_2) \le n_1 - 1, N(x_1) \le n_2 - 1)$$
$$= \theta \sum_{i=0}^{n_1-1} P(N(x_1) = i) + (1-\theta) \sum_{i=0}^{n_1-1} \sum_{j=0}^{(n_2-1)-i} P(N(x_2) = i, N(x_1) - N(x_2) = j)$$
$$= \theta \sum_{i=0}^{n_1-1} \frac{\Gamma(1/\alpha+i)}{\Gamma(1/\alpha)i!}(1-\exp\{-\alpha\Lambda(x_1)\})\exp\{-\alpha\Lambda(x_1)\}^{1/\alpha}$$
$$+(1-\theta) \sum_{i=0}^{n_1-1} \sum_{j=0}^{(n_2-1)-i} \left[\frac{\Gamma(1/\alpha+i)}{\Gamma(1/\alpha)i!}(1-\exp\{-\alpha\Lambda(x_2)\})^i \exp\{-\alpha\Lambda(x_2)\}^{1/\alpha} \right.$$

$$\times\frac{\Gamma(1/\alpha+i+j)}{\Gamma(1/\alpha+i)j!}(1-\exp\{-\alpha[\Lambda(x_1)-\Lambda(x_2)]\})^j\exp\{-\alpha[\Lambda(x_1)-\Lambda(x_2)]\}^{i+1/\alpha}\Bigg]$$

■

From Theorem 2, we can obtain the marginal survival functions as follows.

Corollary 1. *The marginal survival functions of X_1 and X_2 are given by*

$$P(X_1>x_1)=\theta\sum_{i=0}^{n_1-1}\frac{\Gamma(1/\alpha+i)}{\Gamma(1/\alpha)i!}(1-\exp\{-\alpha\Lambda(x_1)\})\exp\{-\alpha\Lambda(x_1)\}^{1/\alpha}$$

$$+(1-\theta)\frac{\Gamma\left(\frac{1}{\alpha}+n_2-1\right)}{\Gamma(1/\alpha)(n_2-1)!}(1-\exp\{-\alpha[\Lambda(x_1)]\})^{n_2-1}\exp\{-\alpha[\Lambda(x_1)]\}^{n_2-1+1/\alpha}$$

$$P(X_2>x_2)=(1-\theta)\sum_{i=0}^{n_1-1}\frac{\Gamma(1/\alpha+i)}{\Gamma(1/\alpha)i!}(1-\exp\{-\alpha\Lambda(x_2)\})\exp\{-\alpha\Lambda(x_2)\}^{1/\alpha}$$

$$+\theta\frac{\Gamma\left(\frac{1}{\alpha}+n_2-1\right)}{\Gamma(1/\alpha)(n_2-1)!}(1-\exp\{-\alpha[\Lambda(x_2)]\})^{n_2-1}\exp\{-\alpha[\Lambda(x_2)]\}^{n_2-1+1/\alpha}$$

■

[Some Specific Families]

The class of bivariate distributions studied in this paper is very general in the sense that, just by specifying $\lambda(t)$, numerous specific families of distributions can be generated. Some useful families of distributions are listed below.

Model 1. $\lambda(t)=\lambda$

$$f_{(X_1,X_2)}(x_1,x_2)$$

$$=\theta\frac{\Gamma(1/\alpha+n_1-1)}{\Gamma(1/\alpha)(n_1-1)!}(1-\exp\{-\alpha\lambda x_1\})^{n_1-1}(\exp\{-\alpha\lambda x_1\})^{1/\alpha}\{(n_1-1)\alpha+1\}\lambda$$

$$\times\frac{\Gamma(1/\alpha+n_2-1)}{\Gamma(1/\alpha+n_1)(n_2-n_1-1)!}(1-\exp\{-\alpha\lambda(x_2-x_1)\})^{n_2-n_1-1}\exp\{-\alpha\lambda(x_2-x_1)\}^{n_1+1/\alpha}$$

$$\times\{(n_2-1)\alpha+1\}\lambda,\ x_1\le x_2.$$

$$f_{(X_1,X_2)}(x_1,x_2)$$

$$=(1-\theta)\frac{\Gamma(1/\alpha+n_1-1)}{\Gamma(1/\alpha)(n_1-1)!}(1-\exp\{-\alpha\lambda x_2\})^{n_1-1}(\exp\{-\alpha\lambda x_2\})^{1/\alpha}\{(n_1-1)\alpha+1\}\lambda$$

$$\times\frac{\Gamma(1/\alpha+n_2-1)}{\Gamma(1/\alpha+n_1)(n_2-n_1-1)!}(1-\exp\{-\alpha\lambda(x_1-x_2)\})^{n_2-n_1-1}\exp\{-\alpha\lambda(x_1-x_2)\}^{n_1+1/\alpha}$$

$$\times\{(n_2-1)\alpha+1\}\lambda,\ x_1>x_2$$

Model 2. $\lambda(t)=\lambda\delta t^{\delta-1}$

$$f_{(X_1,X_2)}(x_1,x_2)$$

$$=\theta\frac{\Gamma(1/\alpha+n_1-1)}{\Gamma(1/\alpha)(n_1-1)!}\left(1-\exp\{-\alpha\lambda x_1^\delta\}\right)^{n_1-1}\left(\exp\{-\alpha\lambda x_1^\delta\}\right)^{1/\alpha}\{(n_1-1)\alpha+1\}\lambda\delta x_1^{\delta-1}$$

$$\times\frac{\Gamma\left(\frac{1}{\alpha}+n_2-1\right)}{\Gamma\left(\frac{1}{\alpha}+n_1\right)(n_2-n_1-1)!}\left(1-\exp\left\{-\alpha\lambda\left(x_2^\delta-x_1^\delta\right)\right\}\right)^{n_2-n_1-1}\exp\left\{-\alpha\lambda\left(x_2^\delta-x_1^\delta\right)\right\}^{n_1+1/\alpha}$$

$$\times\ \{(n_2-1)\alpha+1\}\lambda\delta x_2^{\delta-1},x_1\le x_2$$

$$f_{(X_1,X_2)}(x_1,x_2)$$
$$= (1-\theta)\frac{\Gamma(1/\alpha+n_1-1)}{\Gamma(1/\alpha)(n_1-1)!}\left(1-\exp\{-\alpha\lambda x_2^{\delta}\}\right)^{n_1-1}\left(\exp\{-\alpha\lambda x_2^{\delta}\}\right)^{1/\alpha}\{(n_1-1)\alpha+1\}\lambda\delta x_2^{\delta-1}$$
$$\times\frac{\Gamma\left(\frac{1}{\alpha}+n_2-1\right)}{\Gamma\left(\frac{1}{\alpha}+n_1\right)(n_2-n_1-1)!}\left(1-\exp\{-\alpha\lambda(x_1^{\delta}-x_2^{\delta})\}\right)^{n_2-n_1-1}\exp\{-\alpha\lambda(x_1^{\delta}-x_2^{\delta})\}^{n_1+1/\alpha}$$
$$\times\{(n_2-1)\alpha+1\}\lambda\delta x_1^{\delta-1}, x_1>x_2$$

Model 3. $\lambda(t)=\lambda(1+\delta t)$

$$f_{(X_1,X_2)}(x_1,x_2)$$
$$=\theta\frac{\Gamma\left(\frac{1}{\alpha}+n_1-1\right)}{\Gamma\left(\frac{1}{\alpha}\right)(n_1-1)!}\left(1-\exp\left\{-\alpha\lambda\left(x_1+\frac{1}{2}\delta x_1^2\right)\right\}\right)^{n_1-1}\left(\exp\left\{-\alpha\lambda\left(x_1+\frac{1}{2}\delta x_1^2\right)\right\}\right)^{1/\alpha}\{(n_1-1)\alpha+1\}\lambda(1+\delta x_1)$$
$$\times\frac{\Gamma\left(\frac{1}{\alpha}+n_2-1\right)}{\Gamma\left(\frac{1}{\alpha}+n_1\right)(n_2-n_1-1)!}\left(1-\exp\left\{-\alpha\lambda\left[x_2-x_1+\frac{1}{2}\delta(x_2^2-x_1^2)\right]\right\}\right)^{n_2-n_1-1}$$
$$\times\exp\left\{-\alpha\lambda\left[x_2-x_1+\frac{1}{2}\delta(x_2^2-x_1^2)\right]\right\}^{n_1+1/\alpha}\{(n_2-1)\alpha+1\}\lambda(1+\delta x_2), x_1\le x_2$$

$$f_{(X_1,X_2)}(x_1,x_2)$$
$$=\theta\frac{\Gamma\left(\frac{1}{\alpha}+n_1-1\right)}{\Gamma\left(\frac{1}{\alpha}\right)(n_1-1)!}\left(1-\exp\left\{-\alpha\lambda\left(x_2+\frac{1}{2}\delta x_2^2\right)\right\}\right)^{n_1-1}\left(\exp\left\{-\alpha\lambda\left(x_2+\frac{1}{2}\delta x_2^2\right)\right\}\right)^{1/\alpha}\{(n_1-1)\alpha+1\}\lambda(1+\delta x_2)$$
$$\times\frac{\Gamma\left(\frac{1}{\alpha}+n_2-1\right)}{\Gamma\left(\frac{1}{\alpha}+n_1\right)(n_2-n_1-1)!}\left(1-\exp\left\{-\alpha\lambda\left[x_1-x_2+\frac{1}{2}\delta(x_1^2-x_2^2)\right]\right\}\right)^{n_2-n_1-1}$$
$$\times\exp\left\{-\alpha\lambda\left[x_1-x_2+\frac{1}{2}\delta(x_1^2-x_2^2)\right]\right\}^{n_1+1/\alpha}\{(n_2-1)\alpha+1\}\lambda(1+\delta x_1), x_1>x_2$$

Remark 2. In this section, considering a system having two components, a class of continuous bivariate distributions has been developed. Similar approach could be applied to generate multivariate distributions. For example, to generate tri-variate distributions, consider a system having three components. We can assume that these three components probabilistically fail at one of the combinations of the n_1*-th,* n_2*-th* and n_3*-th* shock, where total 6 combinations exist. Then, applying similar procedure as in the bivariate case, the joint distribution could be obtained without difficulty.

4. Dependence structure and bivariate ageing property

We now introduce a concept of conditional dependence, referred to as *conditional positive quadrant dependence*, which was originally defined by Brady and Singpurwalla [21].

Definition 3. The random variables (Z_1,Z_2) are said to be conditionally positive quadrant dependent with respect to random variable W, if

$$P(Z_1>z_1,Z_2>z_2|W=w)\ge P(Z_1>z_1|W=w)P(Z_2>z_2|W=w),$$

for all z_1,z_2,w.

The following theorem shows that for the case $n_1=1,n_2=2$ of the proposed model, the random variables (X_1,X_2) are conditionally positive quadrant dependent with respect to random variable I. To this end, the following lemma will be useful whose proof can be found in Joe [22], Cuadras [23].

Lemma 1. *Let X be a random variable with its cdf $F(x)$, and let $g(x)$, $h(x)$ be real- valued functions. If both $g(x)$ and $h(x)$ are increasing, or if both are decreasing, then $E[g(X)h(X)] \geq E[g(X)]E[h(X)]$.*

Proposition 6. *(Cha and Finkelstein [24],) Let $(\{N^*(t), t \geq 0\}|Z = z) \sim NHPP(zv(t))$, where $v(t) = \lambda(t)exp\ \{\alpha\Lambda(t)\}$ and the pdf of Z is given by*

$$f_Z(z) = \frac{b^a z^{a-1}}{\Gamma(a)}\exp\{-bz\},\ z \geq 0,\ \text{with } a = \frac{\beta}{\alpha},\ b = \frac{1}{\alpha}, \tag{1}$$

and let $\{N(t), t \geq 0\}$ be the GPP with the set of parameters $\{\lambda(t), \alpha, \beta\}$. Then $\{N(t), t \geq 0\}$ and $\{N^*(t), t \geq 0\}$ share the same stochastic properties.

Theorem 1. *The random variables (X_1, X_2) with $n_1 = 1, n_2 = 2$ are conditionally positive quadrant dependent with respect to random variable I, that is,*

$$P(X_1 > x_1, X_2 > x_2 | I = i) \geq P(X_1 > x_1 | I = i)P(X_2 > x_2 | I = i),$$

for all x_1, x_2 and $i = 0, 1$.

Proof.
Let $(\{N^*(t), t \geq 0\}|Z = z) \sim NHPP(zv(t))$ with the pdf of Z given in (1). For $x_1 \leq x_2$,

$$P(X_1 > x_1, X_2 > x_2 | I = 1) = P(S_1 > x_1, S_2 > x_2)$$

$$= P(N(x_1) = 0, N(x_2) - N(x_1) \leq 1)$$

$$= P(N^*(x_1) = 0, N^*(x_2) - N^*(x_1) \leq 1)$$

$$= E[P(N^*(x_1) = 0, N^*(x_2) - N^*(x_1) \leq 1 | Z)]$$

Observe that

$$\begin{aligned} &P(N^*(x_1) = 0, N^*(x_2) - N^*(x_1) \leq 1 | Z = z) \\ &= P(N^*(x_1) = 0 | Z = z)P(N^*(x_2) - N^*(x_1) \leq 1 | Z = z) \\ &= P(N^*(x_1) \leq 0 | Z = z)P(N^*(x_2) - N^*(x_1) \leq 1 | Z = z) \\ &\geq P(N^*(x_1) \leq 0 | Z = z)P(N^*(x_2) \leq 1 | Z = z). \end{aligned} \tag{2}$$

Now, taking expectation of both sides of (2) with respect to Z yields

$$E[P(N^*(x_1) = 0, N^*(x_2) - N^*(x_1) \leq 1 | Z)] \geq E[P(N^*(x_1) \leq 0 | Z)P(N^*(x_2) \leq 1 | Z)].$$

Thus,

$$P(X_1 > x_1, X_2 > x_2 | I = 1) = E[P(N^*(x_1) = 0, N^*(x_2) - N^*(x_1) \leq 1 | Z)]$$

$$\geq E[P(N^*(x_1) \leq 0 | Z)P(N^*(x_2) \leq 1 | Z)].$$

Note that both $P(N^*(x_1) \leq 0 | Z = z)$ and $P(N^*(x_2) \leq 1 | Z = z)$ are decreasing functions in z. Then, by applying Lemma 1 (letting $g(Z) \equiv P(N^*(x_1) \leq 0 | Z)$, $h(Z) \equiv P(N^*(x_2) \leq 1 | Z)$)

$$E[P(N^*(x_1) \leq 0 | Z)P(N^*(x_2) \leq 1 | Z)]$$

$$\geq E[P(N^*(x_1) \leq 0 | Z)]E[P(N^*(x_2) \leq 1 | Z)]$$

$$= P(N(x_1) \leq 0)P(N(x_2) \leq 1)$$

$$= P(S_1 > x_1)P(S_2 > x_2)$$

$$= P(X_1 > x_1 | I = 1)P(X_2 > x_2 | I = 1).$$

For $x_1 > x_2$, a similar argument gives

$$P(X_1 > x_1, X_2 > x_2 | I = 1) = P(S_1 > x_1, S_2 > x_2) = P(S_1 > x_1) = P(X_1 > x_1 | I = 1)$$

$$\geq P(X_1 > x_1 | I = 1)P(X > x_2 | I = 1).$$

The other case with $I = 0$ can be shown symmetrically. ■

Now, we will show that a well-defined subclass of the proposed class satisfies the bivariate lack of memory property (BLMP). In the case of continuous random variables (X, Y), the joint probability distribution function $f(X, Y)$ is said to have the bivariate lack of memory property (BLMP) iff (see Block and Basu [25])

$$P(X > t+s_1, Y > t+s_2 | X > t, Y > t) = P(X > s_1, Y > s_2), \text{for all } t, s_1,\ s_2 \geq 0.$$

Interpreting (1), if both of the two items are alive at t, then the joint distribution of their remaining lifetimes is the original joint distribution. Block and Basu [25] have shown that Freund's bivariate exponential distribution (see Freund [26]) is absolutely continuous and that it possesses the BLMP. It has been known that Freund's bivariate exponential distribution (see Freund [26]) is 'one of the few absolutely continuous bivariate continuous distributions' which possess the BLMP.

The following theorem presents that for the case $n_1 = 1, n_2 = 2$, all the members belonging to well-defined subclasses of the proposed class possess BLMP.

Theorem 2. *Suppose that the shock process is a GPP with the set of parameters $(\alpha,\ \beta = 1,\ \lambda(t))$, where the intensity function $\lambda(t)$ is given by a constant $\lambda(t) = \lambda$. Then the joint probability distribution of (X_1, X_2) possesses the discrete BLMP.*

Proof.
For $x_1 \leq x_2$, observe that

$$P(X_1 > t+x_1, X_2 > t+x_2)$$

$$= \theta P(N(t+x_1) \leq 0, N(t+x_2) \leq 1) + (1-\theta) P(N(t+x_2) \leq 0)$$

$$= \theta P(N(t+x_2) = 0) + \theta P(N(t+x_1) = 0,\ N(t+x_2) - N(t+x_1) = 1)$$

$$+ (1-\theta) P(N(t+x_2) = 0)$$

$$= P(N(t+x_2) = 0) + \theta P(N(t+x_1) = 0,\ N(t+x_2) - N(t+x_1) = 1),$$

and

$$P(X_1 > t, X_2 > t) = \theta P(N(t) \leq 0) + (1-\theta) P(N(t) \leq 0) = P(N(t) = 0).$$

Thus, it can be shown that

$$P(X_1 > t+x_1, X_2 > t+x_2 | X_1 \rangle t, X_2 > t)$$

$$= \frac{P(N(t+x_2) = 0)}{P(N(t) = 0)} + \theta \frac{P(N(t+x_1) = 0,\ N(t+x_2) - N(t+x_1) = 1)}{P(N(t) = 0)}$$

$$= P(N(x_2) = 0) + \theta P(N(x_1) = 0, N(x_2) - N(x_1) = 1\ )$$

$$= \theta P(N(x_2) = 0) + \theta P(N(x_1) = 0, N(x_2) - N(x_1) = 1\ ) + (1-\theta) P(N(x_2) \leq 0)$$

$$= \theta P(N(x_1) = 0, N(x_2) - N(x_1) \leq 1\ ) + (1-\theta) P(N(x_2) \leq 0)$$

$$= P(X_1 > x_1, X_2 > x_2)$$

The case for $x_1 > x_2$ can also be shown symmetrically. ■

5. A model allowing common failures

In this section, we consider a shock model allowing common failures. Thus, in addition to the assumption stated in Section 2, we assume that the system is additionally subject to another Generalized Polya process $\{M(t), t \geq 0\}$ (with parameter set $\{\nu(t), \alpha', \beta' = 1\}$) of external shocks which causes common failures of the components on the occurrence of the shock. Let T_1 be the time from 0 to the occurrence of the first shock in $\{M(t), t \geq 0\}$. We further assume that $\{M(t), t \geq 0\}$ and $\{N(t), t \geq 0\}$ are independent. Under this model, denote by X_i^* the lifetime of component i, $i = 1, 2$. Then, we obtain the joint survival function of (X_1^*, X_2^*) in the following theorem.

Proposition 7. *The bivariate joint survival function of (X_1^*, X_2^*) is given by*

$$\mathrm{P}(X_1^* > x_1, X_2^* > x_2)$$

$$= \theta \sum_{i=0}^{n_1-1} \sum_{j=0}^{(n_2-1)-i} \left[\frac{\Gamma\left(\frac{1}{\alpha} + i\right)}{\Gamma\left(\frac{1}{\alpha}\right) i!} (1 - \exp\{-\alpha\Lambda(x_1)\})^i \exp\{-\alpha\Lambda(x_1)\}^{\frac{1}{\alpha}} \right.$$

$$\left.\times\frac{\Gamma\left(\frac{1}{\alpha}+i+j\right)}{\Gamma\left(\frac{1}{\alpha}+i\right)j!}(1-\exp\{-\alpha[\Lambda(x_2)-\Lambda(x_1)]\})^j\exp\{-\alpha[\Lambda(x_2)-\Lambda(x_1)]\}^{i+\frac{1}{\alpha}}\right]\cdot\exp\{-\alpha' V(x_2)\}^{\frac{1}{\alpha'}}$$

$$+(1-\theta)\sum_{i=0}^{n_1-1}\frac{\Gamma\left(\frac{1}{\alpha}+i\right)}{\Gamma\left(\frac{1}{\alpha}\right)i!}(1-\exp\{-\alpha\Lambda(x_2)\})\exp\{-\alpha\Lambda(x_2)\}^{1/\alpha}\cdot\exp\{-\alpha' V(x_2)\}^{1/\alpha'},$$

$x_1 \le x_2$,

$$\mathrm{P}\left(X_1^* > x_1, X_2^* > x_2\right)$$

$$=\theta\sum_{i=0}^{n_1-1}\frac{\Gamma\left(\frac{1}{\alpha}+i\right)}{\Gamma\left(\frac{1}{\alpha}\right)i!}(1-\exp\{-\alpha\Lambda(x_1)\})\exp\{-\alpha\Lambda(x_1)\}^{1/\alpha}\cdot\exp\{-\alpha' V(x_1)\}^{1/\alpha'}$$

$$+(1-\theta)\sum_{i=0}^{n_1-1}\sum_{j=0}^{(n_2-1)-i}\left[\frac{\Gamma\left(\frac{1}{\alpha}+i\right)}{\Gamma\left(\frac{1}{\alpha}\right)i!}(1-\exp\{-\alpha\Lambda(x_2)\})^i\exp\{-\alpha\Lambda(x_2)\}^{\frac{1}{\alpha}}\right.$$

$$\left.\times\frac{\Gamma\left(\frac{1}{\alpha}+i+j\right)}{\Gamma\left(\frac{1}{\alpha}+i\right)j!}(1-\exp\{-\alpha[\Lambda(x_1)-\Lambda(x_2)]\})^j\exp\{-\alpha[\Lambda(x_1)-\Lambda(x_2)]\}^{i+\frac{1}{\alpha}}\right]\cdot\exp\{-\alpha' V(x_1)\}^{\frac{1}{\alpha'}},$$

$x_1 > x_2$.

Proof.

For $x_1 \le x_2$,

$$P\left(X_1^* > x_1, X_2^* > x_2\right) = \theta P(S_{n_1} > x_1, S_{n_2} > x_2, T_1 > x_2) + (1-\theta)P(S_{n_2} > x_1, S_{n_1} > x_2, T_1 > x_2)$$

$$= \theta P(S_{n_1} > x_1, S_{n_2} > x_2)P(T_1 > x_2) + (1-\theta)P(S_{n_1} > x_2)P(T_1 > x_2),$$

due to the independence of the two shock processes. Then, the result can be directly obtained by using Theorem 2.

The result for the case of $x_1 > x_2$ can be derived symmetrically. ■

Proposition 8. *The joint probability density function of* $\left(X_1^*, X_2^*\right)$ *is given by*

$$f_{\left(X_1^*, X_2^*\right)}(x_1, x_2)$$

$$=\theta\frac{\Gamma\left(\frac{1}{\alpha}+n_1-1\right)}{\Gamma\left(\frac{1}{\alpha}\right)(n_1-1)!}(1-\exp\{-\alpha\Lambda(x_1)\})^{n_1-1}\exp\{-\alpha\Lambda(x_1)\}^{1/\alpha}\{(n_1-1)\alpha+1\}\lambda(x_1)$$

$$\times\frac{\Gamma\left(\frac{1}{\alpha}+n_2-1\right)}{\Gamma\left(\frac{1}{\alpha}+n_1\right)(n_2-n_1-1)!}(1-\exp\{-\alpha[\Lambda(x_2)-\Lambda(x_1)]\})^{n_2-n_1-1}\exp\{-\alpha[\Lambda(x_2)-\Lambda(x_1)]\}^{n_1+\frac{1}{\alpha}}$$

$$\times\ \{(n_2-1)\alpha+1\}\lambda(x_2)\ \cdot\exp\{-\alpha' V(x_2)\}^{1/\alpha'},\ x_1 < x_2$$

$$f_{\left(X_1^*, X_2^*\right)}(x_1, x_2)$$

$$= (1-\theta)\frac{\Gamma\left(\frac{1}{\alpha}+n_1-1\right)}{\Gamma\left(\frac{1}{\alpha}\right)(n_1-1)!}(1-\exp\{-\alpha\Lambda(x_2)\})^{n_1-1}\exp\{-\alpha\Lambda(x_2)\}^{1/\alpha}\{(n_1-1)\alpha+1\}\lambda(x_2)$$

$$\times\frac{\Gamma\left(\frac{1}{\alpha}+n_2-1\right)}{\Gamma\left(\frac{1}{\alpha}+n_1\right)(n_2-n_1-1)!}(1-\exp\{-\alpha[\Lambda(x_1)-\Lambda(x_2)]\})^{n_2-n_1-1}\exp\{-\alpha[\Lambda(x_1)-\Lambda(x_2)]\}^{n_1+\frac{1}{\alpha}}$$

$$\times\ \{(n_2-1)\alpha+1\}\lambda(x_1)\ \cdot\exp\{-\alpha' V(x_1)\}^{1/\alpha'},\ x_1>x_2,$$

$$f_{(X_1^*,X_2^*)}(x_1,x_2)$$

$$= v(x)\exp\{-\alpha' V(x)\}^{1/\alpha'} exp\ \{-\alpha\Lambda(x)\}^{1/\alpha},\ x_1=x_2=x$$

Proof.

In the cases for $x_1 < x_2$ and $x_1 > x_2$, the result can be obtained similarly to the proof of Theorem 1, additionally considering the probability that there should not be a common shock until $max\ \{x_1, x_2\}$. As $S_{n_1} \neq S_{n_2}$ with probability 1, the only way to have a common failures at time x is that there should be the first shock from $\{M(t), t \geq 0\}$ at time x and there should no shock from $\{N(t), t \geq 0\}$ until time x. This yields the result. ■

6. Application to a real data set

In this section, we apply the model developed in Section 3 to analyze two real bivariate datasets. The first dataset was obtained from a life test on 18 identical parallel systems (Reliability Edge [27]). Each parallel system is composed of two motors, Motor A and Motor B. The lifetimes X_1 (Motor A) and X_2 (Motor B) were measured in days. The parallel system is in a redundant configuration, that is, the system properly operates if at least one of the two motors functions. Table 1 presents the failure times (x_{i1}, x_{i2}) of the two motors in the i-th system $i = 1, 2, \ldots, 18$ and Fig. 1 shows that there seems to be a strong (positive) dependency between X_1 and X_2 having the sample correlation coefficient $\rho = 0.669$.

This data set has been analyzed by applying the bivariate models (Model 1–3) generated in Section 3. For comparative purposes, we also have considered the bivariate distribution models (Model A-D) proposed by Lee and Cha [28,17] to analyze the data set. For all considered models, parameters were obtained numerically by the maximum likelihood estimation method.

Table 2 summarizes the maximum likelihood (ML) estimates of the parameters of each model, along with the corresponding estimated log-likelihood $\widehat{l}$, as well as the values of the Akaike information criterion (AIC) and the Bayesian information criterion (BIC). From Table 2, we can see that Model 1 outperforms the other models, as indicated by its lowest AIC values. Additionally, Model 1–3 show similar estimation performance, despite having a greater number of parameters compared to Model A (Model 1 in Lee and Cha [17]), Model B (Model 2 in Lee and Cha [17]), and Model C (Model 2 in Lee and Cha [28]).

Fig. 2 illustrates two representations of the fitted joint pdf of Model 1, highlighting the shape and behavior of the proposed distribution. The left panel displays the fitted joint pdf of Model 1 over the range $0 \leq x_1, x_2 \leq 400$. It can be seen that higher density regions appear along the diagonal line $x_1 \approx x_2$, indicating positive dependence between the two components. The right panel presents the normalized contour plot of the fitted joint pdf, scaled between 0 and 1 to emphasize the relative density structure. The elliptical and elongated contour levels clearly show the strength and direction of dependency. The diagonal ridge corresponding to the major axis of the density contours reflects the positive dependence modeled by Model 1.

We now can perform the chi-squared goodness-of-fit test for Model 1 with the null hypothesis H_0: the data originate from Model 1. For this, we separated the support $\{(x_1, x_2) | 0 \leq x_1 < \infty, 0 \leq x_2 < \infty\}$ into the total 10 non-overlapping regions (9 rectangles and the remaining region), so that each region can include a balanced number of observations. Then the test statistics is given by $\chi^2 = 1.6159$

Table 1
Failure times (in days) of motors A and B.

System	Motor A (X_1)	Motor B (X_2)	System	Motor A (X_1)	Motor B (X_2)
1	102	65	10	207	214
2	84	148	11	250	212
3	88	202	12	212	220
4	156	121	13	213	265
5	148	123	14	220	275
6	139	150	15	243	300
7	245	156	16	300	248
8	235	172	17	257	330
9	220	192	18	263	350

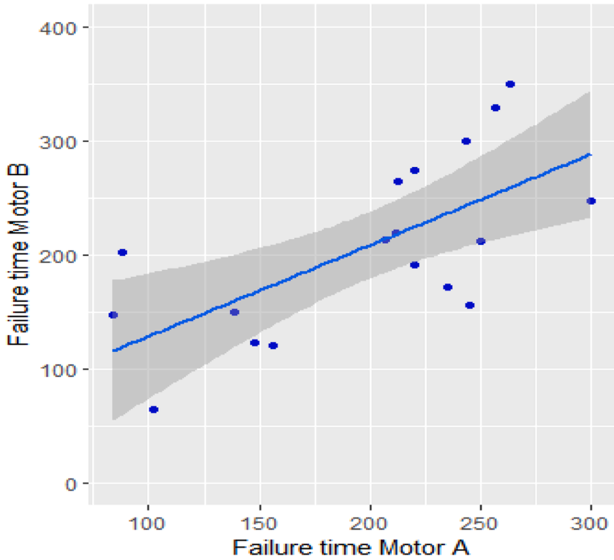

Fig. 1. Scatter plot for the motor data.

Table 2
Estimation result for the motor data.

Model	parameter	estimate	$\hat{l}$	AIC	BIC
Model 1	λ	0.0385	−198.5585	407.117	411.5689
	α	0.0061			
	θ	0.5555			
	n_1	7			
	n_2	9			
Model 2	λ	0.0027	−198.2264	408.4528	413.795
	δ	1.3764			
	α	0.1010			
	θ	0.5555			
	n_1	4			
	n_2	6			
Model 3	λ	0.0194	−198.0407	408.0814	413.4236
	δ	0.0036			
	α	0.0317			
	θ	0.5556			
	n_1	5			
	n_2	7			
Model A	λ	0.0088	−218.9210	441.842	443.6227
	θ	0.5558			
Model B	λ	2.982×10^{-7}	−200.8051	407.6102	410.2813
	β	2.8565			
	θ	0.5555			
Model C	λ	0.0042	−217.7069	441.4138	444.0849
	β	0.0088			
	θ	0.5554			

with the degree of freedom 4, which is obtained by 10 (number of regions) – 5(number of estimated parameters) – 1, which yields the test statistics $\chi^2 = 1.6159 < \chi^2(\alpha,4)$, $\alpha = 0.10,\ 0.05,\ 0.01$, where $\chi^2(\alpha, 4)$ is the critical value of the chi-squared distribution with the degree of freedom 4 under the significance level α. This result implies that the null hypothesis H_0 should be accepted, and that consequently the data set may have significantly arisen from Model 1.

For additional model diagnostics, we compare probability integral transform (PIT) histograms in Fig. 3. Although a comparison of all models in Table 3 is possible, the remaining models (Models 2 and 3) exhibit very similar estimation performance and produce PIT patterns that are nearly indistinguishable from those of Model 1. For this reason, we report PIT results only for the best-performing model (Model 1) and the poorest-performing model (Model A), which most clearly illustrates the contrast in model fit. For Model A, almost all PIT values lie above 0.6, indicating a strong right-skew in the PIT values. This pattern occurs because the fitted model places too little joint probability mass in the region where the observations lie, making the joint CDF values at those points too large and leading to PIT values concentrated near the upper range. In contrast, the PIT values of Model 1 are more evenly spread over $[0, 1]$

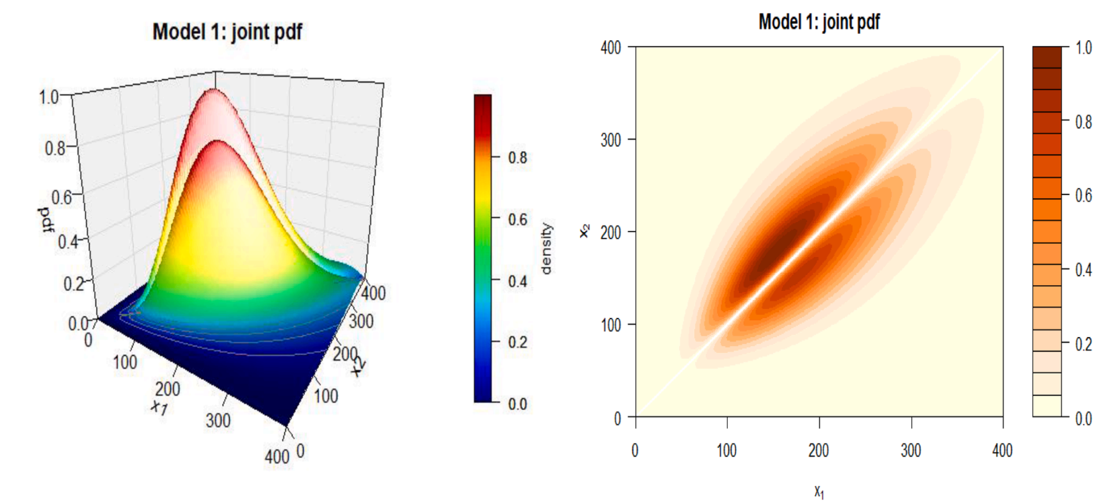

Fig. 2. Joint pdf of Model 1.

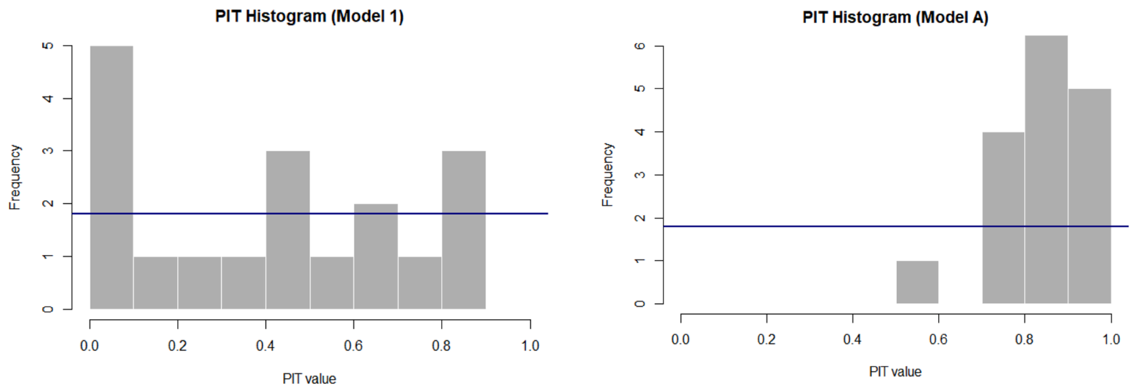

Fig. 3. PIT histograms for Model 1 and Model A.

and closer to the uniform reference, suggesting a better calibrated fit despite the small sample size ($n = 18$). These graphical diagonostics support the conclusion based on AIC/BIC that Model 1 provides a superior fit of the data compared with Model A.

Next, we will analyze the data from a large insurance company in Canada. The dataset is based on information of 14,889 contracts in force with a larger Canadian insurer over the period December 29, 1988 through December 31, 1993. These contracts are joint and last-survivor annuities that were in the payout status during the observation period (see details in Frees et al. [29]), where one annuitant is male and the other is female. The data are available in limited form in the R package CASdatasets as 'canlifins'. In this analysis we use the entry ages of the male and female annuitants for each contract, denoted by X_1 and X_2, respectively. Fig. 4 shows that there seems to be a strong (positive) dependency between X_1 and X_2 having the sample correlation coefficient $\rho = 0.718$ ($n =$ 100).

The bivariate models (Models 1–-3) developed in Section 3 were applied to analyze this dataset. For comparative purposes, we also have considered the bivariate models (Model A-C) proposed by Lee and Cha [17] to analyze the data set. For all considered models, parameters were obtained numerically by the maximum likelihood estimation method.

Table 3 summarizes the maximum likelihood (ML) estimates of the parameters of each model, along with the estimated log-likelihood $\widehat{l}$ and the values of the Akaike information criterion (AIC) and the Bayesian information criterion (BIC). From Table 3, it is shown that Model 2 outperforms the other models with the minimum AIC and BIC values. We should note that Model 2 has the best fitting performance even though they have more number of parameters compared with the other models.

Fig. 5 presents the fitted joint pdf of Model 2 based on the sample from 'canlifins' dataset. The left panel show the 3D surface, where the peak is concentrated in a narrow region because the observed ages of males and females lie within a relatively restricted range. The right panel displays the corresponding contour plot, highlighting the elongated and positively sloped ridge that reflects the positive dependence between X_1 and X_2 implied by Model 2.

We now conduct the chi-squared goodness-of-fit test for Model 2 with the null hypothesis H_0: the data originate from Model 2. To do this, we divide the support $\{(x_1, x_2) | 0 \le x_1 < \infty, 0 \le x_2 < \infty\}$ into the total 10 non-overlapping regions (9 rectangles and the

Table 3
Estimation result for the entry age data.

Model	parameter	estimate	$\hat{l}$	AIC	BIC
Model 1	λ	0.3281	- 652.0981	1314.196	1327.222
	α	0.0067			
	θ	0.2400			
	n_1	23			
	n_2	24			
Model 2	λ	2.511×10^{-8}	- 604.4426	1220.885	1236.516
	δ	4.5127			
	α	0.0060			
	θ	0.2451			
	n_1	4			
	n_2	5			
Model 3	λ	0.0015	- 627.7850	1267.570	1283.201
	δ	2.7034			
	α	0.0062			
	θ	0.0246			
	n_1	1			
	n_2	9			
Model A	λ	0.0291	- 966.735	1937.47	1942.68
	θ	0.2800			
Model B	λ	4.851×10^{-15}	- 651.0067	1308.013	1315.829
	β	7.8993			
	θ	0.3380			
Model C	λ	1.833×10^{-5}	- 837.0322	1680.064	1687.88
	β	45.778			
	θ	0.2802			

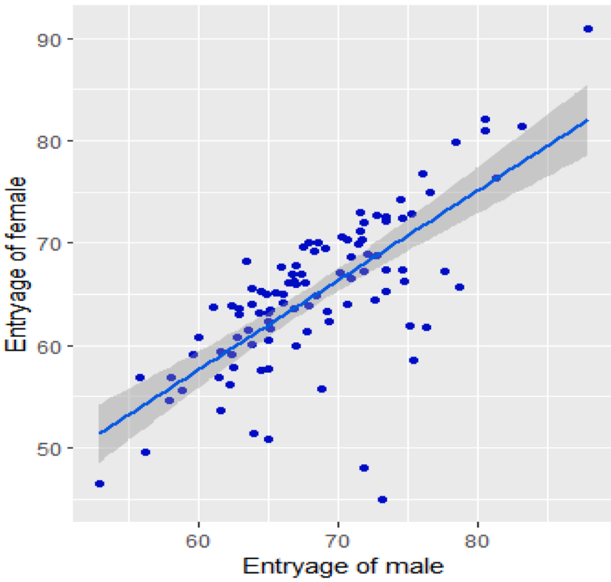

Fig. 4. Scatter plot for the entry age data.

remaining region), ensuring that each region contains a balanced number of observations. Then the test statistics is given by $\chi^2 = 7.509$ with the degree of freedom 3, which is obtained by 10 (number of regions) – 6(number of estimated parameters) – 1, which yields the test statistics $\chi^2 = 7.509 < \chi^2(\alpha, 3)$, $\alpha = 0.05, 0.01$, where $\chi^2(\alpha, 3)$ is the critical value of the chi-squared distribution with the degree of freedom 3 at the significance level α. This result suggests that we should accept the null hypothesis H_0, implying that the data set may indeed have originated from Model 2.

For further diagnostics, Fig. 6 compares the PIT histograms of Model 2 and Model A, the weakest fitting model. Model 2 yields PIT values closer to a uniform distribution, indicating a better overall fit. By contrast, Model A exhibits noticeable deviations from uniformity, with PIT values concentrated in the middle range, indicating that the model underestimates joint probability mass around the observed data, which in turn leads to PIT values concentrated in the middle range. These diagnostics confirm that Model 2 provides a superior fit compared with Model A.

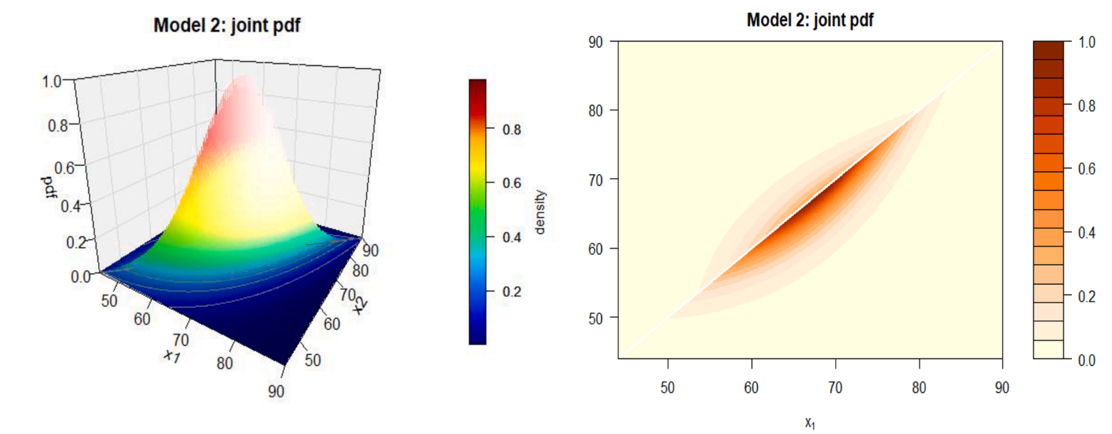

Fig. 5. Joint pdf of Model 2.

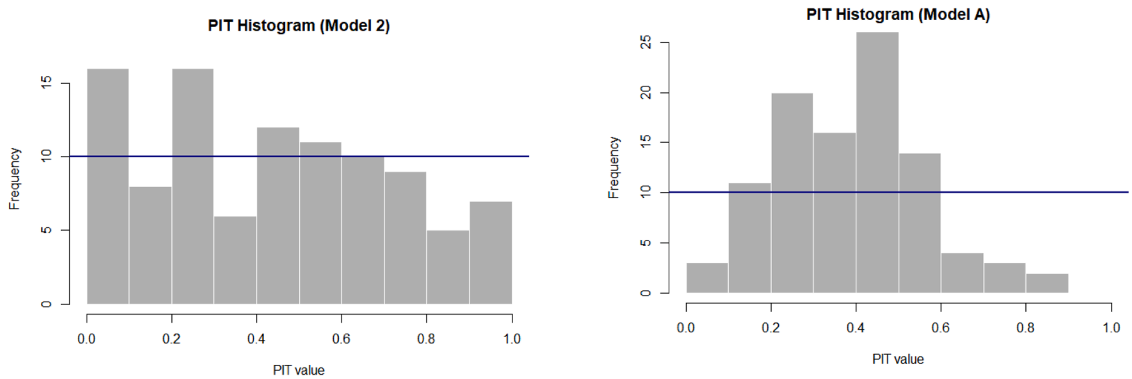

Fig. 6. PIT histograms for Model 2 and Model A.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

The data set on a life test on 18 identical parallel systems in Table 1 is given in Reliability Edge [27]. The data set on the entry ages of the male and female annuitants for insurance contract is available in the R package CASdatasets as 'canlifins'.

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