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Accurate matrix conversion between Bernstein and h-Bernstein bases

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HIGHLIGHTS

- Explicit change-of-basis matrices between Bernstein and h-Bernstein bases are derived.
- Structural properties (symmetry and recurrences) of the conversion matrices are established.
- New h-binomial coefficients yield compact closed-form expressions for matrix entries.
- Recurrence-based algorithms compute the matrices with high relative accuracy (HRA).
- The approach avoids ill-conditioned collocation methods commonly used in CAGD practice.

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ABSTRACT

This paper investigates the matrix conversion between the classical Bernstein basis and its one-parameter generalization, the h-Bernstein basis. New h-analogues of the binomial coefficients are introduced, providing explicit and compact expressions for the entries of the corresponding change-of-basis matrices. Structural properties such as symmetry and recurrence relations are derived, offering both theoretical insight and practical computational advantages. The proposed recurrence formulations enable the generation of the conversion matrices with high relative accuracy, avoiding subtractive cancellations and the numerical instabilities associated with direct collocation-based approaches. These results ensure reliable computations even for very large degrees and establish a foundation for the development of accurate and efficient algorithms in geometric modeling and related numerical applications involving h-Bernstein polynomials. Numerical experiments confirm the theoretical findings and highlight the advantages of the proposed approach.

1. Introduction

Having optimal shape-preserving and stability properties, the Bernstein polynomial basis on the interval $[0,1]$ is the standard choice in CAGD for the parametric representation of curves and surfaces (Carnicer and Peña, 1993; Farouki and Goodman, 1996; Farouki, 2012; Delgado et al., 2023). The Bernstein basis of the space $\mathcal{P}^n$ of polynomials of degree not greater than n is $(B_0^n,\dots,B_n^n)$ with

$$B_k^n(t) := \binom{n}{k} t^k (1-t)^{n-k}, \quad t\in[0,1],\ k=0,\dots,n. \tag{1}$$

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In the context of industrial manufacturing, it is often desirable to have an extra degree of freedom that accounts for the inevitable fabrication tolerances (Goldman, 1985; Barry and Goldman, 1990, 1991). To this end, a popular one-parameter generalization that still preserves many of the good properties of Bernstein polynomials is given by the h-Bernstein basis on $[0,1]$ (Simeonov et al., 2011; Lamberti et al., 2020):

$$H_k^n(t) := \binom{n}{k} \frac{\prod_{i=0}^{k-1}(t+ih)\ \prod_{i=0}^{n-k-1}(1-t+ih)}{\prod_{i=0}^{n-1}(1+ih)}, \quad t \in [0,1],\ k=0,\dots,n, \tag{2}$$

which reduces to the classical Bernstein basis when $h=0$. Moreover, both bases (1) and (2) are related through a change-of-basis matrix $\widetilde{A}_n \in \mathbb{R}^{(n+1)\times(n+1)}$, such that

$$(H_0^n,\dots,H_n^n) = (B_0^n,\dots,B_n^n)\,\widetilde{A}_n. \tag{3}$$

The change-of-basis matrix in (3) makes it possible to obtain the control polygon of a polynomial parametric curve with respect to the h-Bernstein basis directly from the control polygon with respect to the classical Bernstein basis. In fact, from (3), the control polygons of a polynomial parametric curve:

$$\gamma(t) = \sum_{k=0}^{n} P_k B_k^n(t) = \sum_{k=0}^{n} Q_k H_k^n(t), \quad t \in [0,1],$$

are related by

$$(Q_0,\dots,Q_n)^t = \widetilde{A}_n (P_0,\dots,P_n)^t.$$

With the aim of simplifying the study of the matrix $\widetilde{A}_n$, satisfying (3), in this paper we introduce suitable auxiliary polynomials allowing a compact representation of this transformation. This framework leads naturally to the definition of a family of h-binomial coefficients and to a detailed analysis of the structural properties of the matrices involved.

An important aspect in the numerical treatment of structured matrices is the possibility of computing their entries with high relative accuracy (HRA). A numerical algorithm is said to achieve HRA if it delivers approximations $\hat{x}$ of exact values x satisfying

$$\frac{|\hat{x}-x|}{|x|} = \mathcal{O}(u),$$

where u denotes the unit roundoff or machine precision of the working floating-point representation. In contrast to standard backward stability, which guarantees small relative errors with respect to the problem data, HRA ensures that the relative error with respect to the computed quantity itself remains small. This property is particularly relevant when the exact value of an entry is very small or very large compared to others, since standard methods might suffer from catastrophic cancellation or loss of precision in such cases.

This feature is of special importance in applications of h-Bernstein bases, where an accurate representation of the conversion between bases is critical for preserving shape properties and numerical stability. The availability of a recurrence-based algorithm for constructing the change-of-basis matrices provides not only compact representations but also a computational tool that maintains accuracy even in challenging parameter regimes. From a numerical linear algebra perspective, basis transformations in polynomial spaces are closely related to structured matrices arising from collocation processes. It is well known that collocation matrices associated with Bernstein-type bases may be severely ill-conditioned, leading to significant loss of accuracy when solving the corresponding linear systems.

Recent work has shown that, by exploiting the intrinsic structure of these matrices, it is possible to design algorithms that achieve HRA in the computation of their entries and factorizations. This has been demonstrated, for instance, for collocation matrices associated with generalized Bernstein bases, where bidiagonal or recursive factorizations play a crucial role Marco et al. (2019); Marco and Martínez (2013); Marco et al. (2022).

The present work follows this philosophy. Rather than computing the change-of-basis matrix by solving a collocation-based linear system, we derive explicit recurrence relations that allow its construction entry by entry using only additions and multiplications of nonnegative quantities. This guarantees HRA and makes the approach particularly suitable for applications in Computer Aided Geometric Design.

The remainder of the paper is organized as follows. In Section 2 we study the change-of-basis problem between the classical Bernstein basis and the h-Bernstein basis. By introducing suitable auxiliary polynomial bases, we derive recurrence relations and explicit closed-form expressions for the entries of the associated change-of-basis matrices in terms of newly defined h-binomial coefficients, and analyze their main structural properties. Section 3 is devoted to the numerical aspects of the proposed approach. We present recurrence-based algorithms for the computation of the change-of-basis matrices and show that they achieve high relative accuracy. The numerical performance of the method is illustrated through several experiments and compared with collocation-based approaches based on the solution of linear systems. Finally, in Section 4 we summarize the main contributions of the paper and discuss directions for further research, with particular emphasis on the accurate treatment of Gram matrices associated with Bernstein-type bases and the development of HRA algorithms for least squares problems arising in Computer Aided Geometric Design.

2. Matrix conversion between h-Bernstein and Bernstein bases

With the aim of simplifying the study of the matrix $\tilde{A}_n$, satisfying (3), we define the auxiliary polynomials

$$b_k^n(t) := \frac{1}{\binom{n}{k}} B_k^n(t), \quad h_k^n(t) := \frac{\prod_{i=0}^{n-1}(1+ih)}{\binom{n}{k}} H_k^n(t), \quad t \in [0,1],\ k = 0,\dots,n. \tag{4}$$

Analogously, a change-of-basis matrix $A_n \in \mathbb{R}^{(n+1)\times(n+1)}$ can be defined with

$$(h_0^n, \dots, h_n^n) = (b_0^n, \dots, b_n^n) A_n. \tag{5}$$

It is easy to check that the entries of $\tilde{A}_n = (\tilde{a}_{i,j}^{(n)})_{i,j=1,\dots,n+1}$ and those of $A_n = (a_{i,j}^{(n)})_{i,j=1,\dots,n+1}$ satisfy

$$\tilde{a}_{i,j}^{(n)} = \frac{\binom{n}{j-1}}{\binom{n}{i-1}\prod_{k=0}^{n-1}(1+kh)} a_{i,j}^{(n)}, \quad i,j = 1,\dots,n+1. \tag{6}$$

Remark 2.1. Note that identities $b_i^n(t) = b_{n-i}^{(n)}(1-t)$ and $h_i^n(t) = h_{n-i}^{(n)}(1-t)$ are satisfied for $i = 0,\dots,n$. As a consequence, it is easy to see that the entries of the matrix of change of basis A_n have the following symmetry

$$a_{i,j}^{(n)} = a_{n+2-i,n+2-j}^{(n)}, \quad i,j = 1,\dots,n+1. \tag{7}$$

At this point it is useful to introduce the following result, which will be helpful in deducing the explicit form of A_n.

Proposition 2.2. *The entries of the matrix* $A_n = \left(a_{i,j}^{(n)}\right)_{i,j=1,\dots,n+1}$ *in* (5) *satisfy*

$$a_{i,j}^{(n)} = (1 + h(n-j)) a_{i,j}^{(n-1)} + h(n-j) a_{i-1,j}^{(n-1)}, \quad i = 1,\dots,n+1,\ j = 1,\dots,n, \tag{8}$$

$$a_{i,n+1}^{(n)} = (1 + h(n-1)) a_{i-1,n}^{(n-1)} + h(n-1) a_{i,n}^{(n-1)}, \quad i = 1,\dots,n+1, \tag{9}$$

for $n \geq 2$, *with the convention* $a_{0,j}^{(n-1)} := 0$, $a_{n+1,j}^{(n-1)} := 0$ *and* $A_1 \in \mathbb{R}^{2\times 2}$ *being the identity matrix.*

Proof. The coefficients of the matrix A_n satisfy

$$h_j^n(t) = \sum_{i=0}^{n} a_{i+1,j+1}^{(n)} b_i^n(t), \quad t \in [0,1],\ j = 0,\dots,n. \tag{10}$$

For $j = 0,\dots,n-1$, we can take out the last factor in (4), obtaining

$$\begin{aligned} h_j^n(t) &= (1 - t + (n-j-1)h) h_j^{n-1}(t) = (1 - t + (n-j-1)h(1-t+t)) \sum_{i=0}^{n-1} a_{i+1,j+1}^{(n-1)} b_i^{n-1}(t) \\ &= (1 + (n-j-1)h) \sum_{i=0}^{n-1} a_{i+1,j+1}^{(n-1)} b_i^n(t) + (n-j-1)h \sum_{i=1}^{n} a_{i,j+1}^{(n-1)} b_i^n(t), \quad t \in [0,1]. \end{aligned}$$

Using the convention $a_{0,j}^{(n-1)} := 0$ and $a_{n+1,j}^{(n-1)} := 0$, we can express h_j^n as

$$h_j^n(t) = \sum_{i=0}^{n} \left[(1 + (n-j-1)h) a_{i+1,j+1}^{(n-1)} + (n-j-1)h a_{i,j+1}^{(n-1)} \right] b_i^n(t), \quad t \in [0,1], \tag{11}$$

and, by comparing the coefficients of each $b_i^n(t)$ in (10) and (11), recurrence (8) is recovered.

Finally, for $j = n$, we can take advantage of the coefficient symmetry (7) to derive:

$$a_{i,n+1}^{(n)} = a_{n+2-i,1}^{(n)} = (1 + h(n-1)) a_{n+2-i,1}^{(n-1)} + h(n-1) a_{n+1-i,1}^{(n-1)} = (1 + h(n-1)) a_{i-1,n}^{(n-1)} + h(n-1) a_{i,n}^{(n-1)}.$$

□

In order to achieve a compact expression for the entries of the change of basis matrix A_n—and, consequently, for $\tilde{A}_n$—an h-analogue of the binomial coefficient is introduced.

Definition 2.3. For $n, m \in \mathbb{N}$, we define the h-binomial coefficient $\left\langle {n \atop m} \right\rangle_h$ as the coefficient of x^m in $\prod_{j=1}^n (jhx + (1 + jh))$, that is,

$$\prod_{j=1}^{n} (jhx + (1 + jh)) = \sum_{m=1}^{n} \left\langle {n \atop m} \right\rangle_h x^m, \tag{12}$$

satisfying

$$\left\langle {n \atop m} \right\rangle_h = \sum_{\substack{m_i \in \{0,1\}, i=1,\ldots,n \\ m_1+\cdots+m_n=m}} \prod_{j=1}^{n} (jh)^{m_j} (1 + jh)^{1-m_j}. \tag{13}$$

By construction, the h-binomial coefficient is a polynomial in h of order n, and all its coefficients in the monomial basis are nonnegative integers. It should be noted that (13) is not an extension of the standard binomial coefficients, but shares some key properties that favor the use of this notation. In this sense, the sum in (13) can be interpreted as going over every possible way of selecting m of the n powers m_j, therefore having $\binom{n}{m}$ summands. Noticeably, it is easy to see that the set $\{\left\langle {n \atop 0} \right\rangle_h, \ldots, \left\langle {n \atop n} \right\rangle_h\}$ is a basis of $P_n[h]$, since $\left\langle {n \atop m} \right\rangle_h$ lowest power is h^m.

The following sums admit particularly simple closed forms:

$$\sum_{m=1}^{n} \left\langle {n \atop m} \right\rangle_h = \prod_{j=1}^{n} (1 + 2jh), \quad \sum_{m=1}^{n} (-1)^m \left\langle {n \atop m} \right\rangle_h = 1,$$

which follow directly from (12). Moreover, (12) also yields an analogue of Pascal's identity,

$$\left\langle {n \atop m} \right\rangle_h = (1 + nh) \left\langle {n-1 \atop m} \right\rangle_h + nh \left\langle {n-1 \atop m-1} \right\rangle_h, \tag{14}$$

with $n, m \in \mathbb{N}$, and the convention $\left\langle {-1 \atop m} \right\rangle_h := \delta_{0,m}$.

Now, the matrix A_n in (5) can be explicitly formulated in terms of the h-binomial coefficients in a rather compact way.

Theorem 2.4. *The entries of the matrix $A_n = \left(a_{i,j}^{(n)}\right)_{i,j=1,\ldots,n+1}$ defined in (5) are given by*

$$a_{i,j}^{(n)} = \sum_{m=\max\{0,i-j\}}^{\min\{n+1-j,i-1\}} \left\langle {n-j \atop m} \right\rangle_h \left\langle {j-2 \atop m+j-i} \right\rangle_h. \tag{15}$$

Proof. We use induction on n. For $n = 1$, the matrix A_1 is the identity matrix and equality (15) is fulfilled. Now, let us assume the above relation (15) for n. Applying Proposition 2.2, for $j = 1, \ldots, n+1$, we have

$$\begin{aligned} a_{i,j}^{(n+1)} &= (1 + h(n+1-j)) a_{i,j}^{(n)} + h(n+1-j) a_{i-1,j}^{(n)} \\ &= \sum_{m=\max\{0,i-j\}}^{\min\{n+1-j,i-1\}} (1 + h(n+1-j)) \left\langle {n-j \atop m} \right\rangle_h \left\langle {j-2 \atop m+j-i} \right\rangle_h + \sum_{m=\max\{0,i-j-1\}}^{\min\{n+1-j,i-2\}} h(n+1-j) \left\langle {n-j \atop m} \right\rangle_h \left\langle {j-2 \atop m+j-i+1} \right\rangle_h. \end{aligned}$$

By shifting the index of the second sum, we derive

$$a_{i,j}^{(n+1)} = \sum_{m=\max\{0,i-j\}}^{\min\{n+1-j,i-1\}} (1 + h(n+1-j)) \left\langle {n-j \atop m} \right\rangle_h \left\langle {j-2 \atop m+j-i} \right\rangle_h + \sum_{m=\max\{1,i-j\}}^{\min\{n+2-j,i-1\}} h(n+1-j) \left\langle {n-j \atop m-1} \right\rangle_h \left\langle {j-2 \atop m+j-i} \right\rangle_h.$$

Since we can formally extend the sums to match both ranges—the added summands being zero—it follows that

$$a_{i,j}^{(n+1)} = \sum_{m=\max\{0,i-j\}}^{\min\{n+2-j,i-1\}} \left((1 + h(n+1-j)) \left\langle {n-j \atop m} \right\rangle_h + h(n+1-j) \left\langle {n-j \atop m-1} \right\rangle_h \right) \left\langle {j-2 \atop m+j-i} \right\rangle_h,$$

and applying Pascal's identity analogue (14), formula (15) for $n+1$ is obtained.

Finally, using (7) and (15) with $j = 1$, we obtain,

$$a_{i,n+2}^{(n+1)} = a_{n+3-i,1}^{(n+1)} = \sum_{m=\max\{0,n+2-i\}}^{\min\{n+1,n+2-i\}} \left\langle {n \atop m} \right\rangle_h \left\langle {-1 \atop m+i-n-2} \right\rangle_h = \left\langle {n \atop n+2-i} \right\rangle_h,$$

which coincides with (15) for $n+1$ and $j = n+2$. □

Let us give as a particular example of A_n for $n = 3$, which illustrates its structure and symmetry,

$$A_3 = \begin{pmatrix} (1+h)(1+2h) & 0 & 0 & 0 \\ h(1+2h)+(1+h)2h & 1+h & h & h(2h) \\ h(2h) & h & 1+h & h(1+2h)+(1+h)2h \\ 0 & 0 & 0 & (1+h)(1+2h) \end{pmatrix} = \begin{pmatrix} 2h^2+3h+1 & 0 & 0 & 0 \\ 4h^2+3 & 1+h & h & 2h^2 \\ 2h^2 & h & 1+h & 4h^2+3h \\ 0 & 0 & 0 & 2h^2+3h+1 \end{pmatrix}. \tag{16}$$

Finally, observe that the entries of $\tilde{A}_n$ can be recovered from those of A_n by means of (6), namely,

$$\tilde{a}^{(n)}_{i,j} = \frac{\binom{n}{j-1}}{\binom{n}{i-1}\prod_{k=0}^{n-1}(1+kh)} \sum_{m=\max\{0,i-j\}}^{\min\{n+1-j,i-1\}} \left\langle {n-j \atop m} \right\rangle_h \left\langle {j-2 \atop m+j-i} \right\rangle_h. \tag{17}$$

3. High relative accuracy computation of the change-of-basis matrix

In the present context, the recurrence relations derived for the coefficients of the change-of-basis matrices (see Proposition 2.2) are particularly well suited to ensure HRA (Demmel, 2000). Each new entry is computed as a short linear combination of previously generated entries with nonnegative coefficients, thereby avoiding subtractive cancellation. In addition, the recurrence involves only simple multiplicative updates by factors of the form $(1 + kh)$ or kh, whose relative errors remain bounded by machine precision. Consequently, the entire matrix can be generated entry by entry while maintaining HRA.

In order to evaluate the h-binomial coefficients, there exist several alternatives that ensure HRA. A simple approach, which only involves sums and products of positive quantities, consists in performing the sum over all combinations in (13); of course this procedure is far from being efficient, since the number of combinations quickly becomes prohibitive when n increases. It is much more convenient to take advantage of recurrence (14), since in this case one can compute any given coefficient $\left\langle {n \atop m} \right\rangle_h$ in $\mathcal{O}(nm)$ operations. The accurate computation of all h-binomial coefficients $\left\langle {n \atop m} \right\rangle_h$, $m = 0,\dots,n \le N$, can be achieved in order $\mathcal{O}(N^2)$ operations, which can be done simply by iteratively storing in memory the coefficients for each value of $n = 0,\dots,N$. This argument can be immediately extended to the cost of determining the entries of $A^{(n)}$: relying on the recurrences of Proposition 2.2, all the entries of the matrices $A^{(n)}$ for $n \le N$ can be computed in $3 \times (1 + 2^2 + \cdots + (N+1)^2) = \mathcal{O}(N^3)$ operations. Some efficiency can be gained if all the necessary h-binomial coefficients have been precomputed, in which case we can directly determine the coefficients of a given $A^{(n)}$ by means of the closed formula (15), which can also be performed in $\mathcal{O}(n^3)$ operations, but with a better prefactor—approximately an order of magnitude for large enough n. A pseudocode routine that computes both the required h-binomial coefficients and the entries of $A^{(n)}$ is detailed in Algorithm 1.

To illustrate the numerical advantages of the recurrence-based construction of the change-of-basis matrix described above, we compare it with an alternative procedure for different values of the matrix dimension n and the parameter h. In this way, the first method builds the matrix $A^{(n)}$ by means of formula (15), precomputing first all the needed h-binomial coefficients. As for the second method, it constructs the matrix numerically by evaluating both Bernstein and h-Bernstein bases at $t_j = j/n$, $j = 0,\dots,n$, then solving the linear system $BA = H$ for A (here B and H are the collocation matrices of the bases). The latter approach is a straightforward numerical implementation of the change-of-basis, but is known to be potentially ill-conditioned.

In Table 1 the maximum entrywise errors obtained by both approaches are shown. The results clearly indicate that solving the system $BA = H$ can produce entries whose relative difference with the exact values is large—often greater than the value itself. This is not surprising since the involved collocation matrices are known to be ill-conditioned, especially as n increases or h gets closer to zero. In contrast, the recurrence-based approach detailed in Algorithm 1, consisting only of products and sums of nonnegative numbers,

Algorithm 1 Evaluation to HRA of $A^{(n)}$ in terms of the h-binomial coefficients $\left\langle {n \atop m} \right\rangle_h$.

Require: n, h
  $\left\langle {-1 \atop 0} \right\rangle_h, \left\langle {0 \atop 0} \right\rangle_h \leftarrow 1$
  for $i = 1 : n$ **do**
    for $m = 0 : n$ **do**
      $\left\langle {i \atop m} \right\rangle_h \leftarrow (1+ih)\left\langle {i-1 \atop m} \right\rangle_h + ih\left\langle {i-1 \atop m-1} \right\rangle_h$
    end for
  end for
  for $i = 1 : n+1$ **do**
    for $j = 1 : n+1$ **do**
      $(A^{(n)})_{i,j} \leftarrow 0$
      for $k = \max(0, i-j) : \min(n+1-j, i-1)$ **do**
        $(A^{(n)})_{i,j} \leftarrow (A^{(n)})_{i,j} + \left\langle {n-j \atop k} \right\rangle_h \left\langle {j-2 \atop k+j-i} \right\rangle_h$
      end for
    end for
  end for

Table 1
Maximum entrywise relative errors in the recurrence-based matrix computation and in the direct numerical solve.

n	h	Algorithm 1	$BA = H$
20	0.01	$2.7e-16$	$4.4e-05$
20	0.1	$7.2e-16$	$4.8e-07$
20	1.0	$6.8e-17$	$5.7e-08$
20	5.0	$1.3e-16$	$3.7e-08$
100	0.01	$1.1e-15$	$1.0e+19$
100	0.1	$4.9e-15$	$2.0e+07$
100	1.0	$3.0e-16$	$5.4e+04$
100	5.0	$8.8e-17$	$3.0e+04$

keeps all entries relative errors at machine precision order. This underlines the practical importance of using the recurrence: it avoids the numerical instabilities inherent in direct evaluation/linear-solve strategies and therefore permits the change-of-basis matrix to be obtained with high relative accuracy (HRA), a property of particular value when some matrix entries are very small or when h takes small magnitudes.

4. Conclusion and further research

In this work we have studied in detail the change of basis between the classical Bernstein basis and the h-Bernstein polynomial basis. By introducing suitable auxiliary polynomials, the transformation problem has been reformulated in terms of newly defined h-binomial coefficients, leading to compact and explicit expressions for the entries of the corresponding change-of-basis matrices. Several structural properties of these matrices, including symmetry and recurrence relations, have been derived and analyzed.

Beyond the explicit formulas, a central contribution of this paper is the development of recurrence-based algorithms that allow the computation of the change-of-basis matrices with HRA. Instead of relying on collocation-based approaches that require solving ill-conditioned linear systems, the proposed method constructs the matrix entries directly through recurrences involving only additions and multiplications of nonnegative quantities. This guarantees relative accuracy even for large polynomial degrees or challenging parameter regimes, and places the results within the growing body of work on accurate numerical algorithms for structured matrices arising in Computer Aided Geometric Design.

From a numerical linear algebra perspective, these results are closely related to recent advances in the accurate treatment of collocation matrices associated with generalized Bernstein bases. It has been shown that, by exploiting structural properties such as total positivity and bidiagonal factorizations, algebraic problems involving these matrices can be solved with high relative accuracy. The present work extends this philosophy to the change-of-basis problem between Bernstein and h-Bernstein bases, showing that similar levels of numerical reliability can be achieved without explicitly forming or inverting collocation matrices.

The extension of our approach to two and higher dimensions is possible. Both the classical Bernstein basis and the h-Bernstein basis admit tensor-product generalizations on $[0,1]^d$. Therefore, if $A_n(h)$ denotes the univariate conversion matrix described in this paper, the bivariate change-of-basis matrix is

$$A_{n_1,n_2}(h_1,h_2) = A_{n_1}(h_1) \otimes A_{n_2}(h_2), \tag{18}$$

where $\otimes$ denotes the Kronecker product (Gasca and Martínez, 1990; Horn and Johnson, 1995). The validity of (18) follows directly from Lemma 4.2.10 in Horn and Johnson (1995), which states that the Kronecker product of two matrix products equals the product of the corresponding Kronecker products. Higher-dimensional cases follow analogously.

The compact representations and accurate computational techniques developed here provide a solid foundation for further research and applications. A particularly promising direction concerns the Gram matrices associated with h-Bernstein bases. The availability of HRA algorithms for the change-of-basis matrices is expected to play a key role in the accurate construction and factorization of these Gram matrices. This, in turn, will allow the reliable solution of least squares problems in h-Bernstein bases with HRA, which is of fundamental importance in approximation theory, data fitting, and geometric modeling.

Future work will therefore focus on extending the present techniques to the study of Gram matrices, their structured factorizations, and the design of HRA algorithms for least squares problems involving Bernstein and h-Bernstein bases. These developments are expected to further strengthen the role of h-Bernstein polynomials as a robust and reliable tool in Computer Aided Geometric Design and related numerical applications.

CRediT authorship contribution statement

Y. Khiar: Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization. **E. Mainar:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization. **J.M. Peña:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization. **E. Royo-Amondarain:** Writing – review &

editing, Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization.

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

Data will be made available on request.

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