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Keywords (separated by '-')	Algorithmic randomness - Algorithmic information theory - Effective dimension - Point to set principle - Borel normality	



A Point to Set Principle for Finite-State Dimension

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Abstract. Effective dimension has proven very useful in geometric measure theory through the point-to-set principle [9] that characterizes Hausdorff dimension by relativized effective dimension. Finite-state dimension is the least demanding effectivization in this context [3] that among other results can be used to characterize Borel normality [2].

In this paper we prove a characterization of finite-state dimension in terms of information content of a real number at a certain precision. We then use this characterization to give a robust concept of relativized normality and prove a finite-state dimension point-to-set principle. We finish with an open question on the equidistribution properties of relativized normality.

[AQ1](#)[AQ2](#)

Keywords: Algorithmic randomness · Algorithmic information theory · Effective dimension · Point to set principle · Borel normality

1 Introduction

Effective dimension was introduced in [7, 8] as an effectivization of Hausdorff dimension. One of its generalizations is finite-state dimension [3] that is a robust notion that interacts with compression and characterizes Borel normality [2].

In [9] Lutz and Lutz proved a point-to-set principle that characterizes Hausdorff dimension in terms of relativized effective dimension. This principle has already produced a number of interesting results in geometric fractal theory through computability based proofs (see [6, 10] and a number of more recent results such as [5]).

In this paper we provide a characterization of finite-state dimension on Euclidean space based on the finite-state information content of a real number at a certain precision, which also provides an alternative characterization of Borel normality. This new characterization gives rise to a natural and robust relativization of finite state dimension with the strong property of a finite-state dimension point-to-set principle. We finish with open questions on the equidistribution properties of the corresponding relativized normality.

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2 Preliminaries

Let Σ be a finite alphabet. We write Σ^* for the set of all (finite) strings over Σ and Σ^∞ for the set of all (infinite) sequences over Σ . We write $|x|$ for the length of a string or sequence x , and we write λ for the empty string, the string of length 0. For $x \in \Sigma^* \cup \Sigma^\infty$ and $0 \leq n < |x|$, we write $x \upharpoonright n = x[0..n-1]$. For $w \in \Sigma^*$ and $x \in \Sigma^* \cup \Sigma^\infty$, we say that w is a prefix of x , and we write $w \sqsubseteq x$, if $x \upharpoonright |w| = w$.

A Σ finite-state transducer (Σ -FST) is a 4-tuple $T = (Q, \delta, \nu, q_0)$, where

- Q is a nonempty, finite set of states,
- $\delta : Q \times \Sigma \rightarrow Q$ is the transition function,
- $\nu : Q \times \Sigma \rightarrow \Sigma^*$ is the output function, and
- $q_0 \in Q$ is the initial state.

For $q \in Q$ and $w \in \Sigma^*$, we define the output from state q on input w to be the string $\nu(q, w)$ defined by the recursion

$$\begin{aligned}\nu(q, \lambda) &= \lambda, \\ \nu(q, wa) &= \nu(q, w)\nu(\delta(q, w), a)\end{aligned}$$

for all $w \in \Sigma^*$ and $a \in \Sigma$. We then define the output of T on input $w \in \Sigma^*$ to be the string $T(w) = \nu(q_0, w)$.

For each integer $b \geq 1$ we let $\Sigma_b = \{0, 1, \dots, b-1\}$ be the alphabet of base- b digits. We use infinite sequences over Σ_b to represent real numbers in $[0, 1)$. Each $S \in \Sigma_b^\infty$ is associated the real number $\text{real}_b(S) = \sum_{i=1}^{\infty} S[i-1]b^{-i}$ and for each $x \in [0, 1)$, $\text{seq}_b(x)$ is the infinite sequence S that does not finish with infinitely many $b-1$ and such that $x = \text{real}_b(S)$.

A set of real numbers $A \subseteq [0, 1)$ is represented by the set

$$\text{seq}_b(A) = \{\text{seq}_b(\alpha) \mid \alpha \in A\}$$

of sequences. If $X \subseteq \Sigma_b^\infty$ then

$$\text{real}_b(X) = \{\text{real}_b(x) \mid x \in X\}.$$

We will denote with \mathcal{D}_b the set of rational numbers that have finite representation in base b , that is,

$$\mathcal{D}_b = \{q \mid \text{seq}_b(q) = w0^\infty, w \in \Sigma_b^*\}.$$

We will write $\text{real}_b(w)$ for $\text{real}_b(w0^\infty)$ when $w \in \Sigma_b^*$.

For the sake of completeness we include the definition of Borel normality.

Definition 1 (Borel [1]). A real number α is normal in base $b \in \mathbb{N}$ ($b \geq 2$) if, for every $m \geq 1$ and every length- m sequence w of base- b digits, the asymptotic, empirical frequency of w in the base- b expansion of α is b^{-m} .

α is absolutely normal if it is normal in every base $b \geq 2$.

3 An Euclidean Characterization of Finite-State Dimension and Borel Normality

Finite-state dimension was introduced in [3] on the space of infinite sequences over a finite alphabet. The original definition in terms of gambling was proven robust by several characterizations in terms of information lossless compression [3] and several versions of entropy [2].

Here we present an alternative definition on the Euclidean space and then prove its equivalence with [3].

Definition 2. Let T be a Σ -FST and let $w \in \Sigma^*$. The T -information content of w is

$$K^T(w) = \min \{ |\pi| \mid T(\pi) = w \}.$$

Definition 3. Let T be a Σ_b -FST, $\delta > 0$ and $x \in [0, 1)$. The base- b T -information content of x at precision δ is

$$K_\delta^T(x) = \min \{ K^T(w) \mid |\text{real}_b(w) - x| < \delta \}.$$

We next define the finite-state dimension of points and sets.

Definition 4. Let $b \geq 1$. Let $x \in [0, 1)$ and $A \subseteq [0, 1)$. The base- b finite-state dimension of x is

$$\dim_{\text{FS}}^b(x) = \inf_{T \Sigma_b\text{-FST}} \liminf_{\delta > 0} \frac{K_\delta^T(x)}{\log_b(1/\delta)},$$

the base- b finite-state dimension of A is

$$\dim_{\text{FS}}^b(A) = \inf_{T \Sigma_b\text{-FST}} \sup_{x \in A} \liminf_{\delta > 0} \frac{K_\delta^T(x)}{\log_b(1/\delta)}.$$

Observation 1. $\dim_{\text{FS}}^b(x) = \inf_{T \Sigma_b\text{-FST}} \liminf_n \frac{K_{b^{-n}}^T(x)}{n}$.

The definition of finite-state dimension from [3] is usually done in a space of infinite sequences, while identifying $[0, 1)$ and Σ_b^∞ through seq_b or base- b representation.

Doty and Moser [4] proved that finite dimension on sequences can be characterized in terms of finite-state transducers.

Theorem 2 ([4]). Let $S \in \Sigma^\infty$,

$$\dim_{\text{FS}}(S) = \inf_{T \Sigma\text{-FST}} \liminf_n \frac{K^T(S \upharpoonright n)}{n}.$$

We next show that the notion of information content at a certain precision characterizes finite-state dimension.

Theorem 3. For each $b \geq 1$, $x \in [0, 1)$, and $A \subseteq [0, 1)$

$$\dim_{\text{FS}}^b(x) = \dim_{\text{FS}}(\text{seq}_b(x)),$$

$$\dim_{\text{FS}}^b(A) = \dim_{\text{FS}}(\text{seq}_b(A)).$$

Proof. Let $x \in [0, 1)$, let $S = \text{seq}_b(x)$. Then for every $n \in \mathbb{N}$ and $T \in \Sigma_b\text{-FST}$, $K_{b^{-n}}^T(x) \leq K^T(S \upharpoonright (n+1))$ and therefore $\dim_{\text{FS}}^b(x) \leq \dim_{\text{FS}}(S)$.

For each $w \in \Sigma_b^* \cup \Sigma_b^\infty$, let $\text{comp}(w)$ be the complement of w , that is, $\text{comp}(w)[i] = b - 1 - w[i]$ for $0 \leq i < |w|$.

Claim. $\dim_{\text{FS}}(S) = \dim_{\text{FS}}(\text{comp}(S))$. $\dim_{\text{FS}}^b(x) = \dim_{\text{FS}}^b(\text{real}_b(\text{comp}(\text{seq}_b(x))))$.

Claim. $\dim_{\text{FS}}(S) \leq \dim_{\text{FS}}^b(x)$.

To prove this claim, notice that $\dim_{\text{FS}}^b(x)$ needs to be witnessed either by approximations from above or for approximations from below and that tighter approximations can be delayed.

That is, for every FST T there exist and infinitely many n_i such that

$$\lim_i \frac{K_{b^{-n_i}}^T(x)}{n_i} = \liminf_n \frac{K_{b^{-n}}^T(x)}{n}.$$

For each i let w_i be such that $K^T(w_i) = K_{b^{-n_i}}^T(x)$ and $|x - \text{real}_b(w_i)| < b^{-n_i}$. Let m_i be such that $b^{-m_i-1} \leq |x - \text{real}_b(w_i)| < b^{-m_i} \leq b^{-n_i}$. Then $\frac{K_{b^{-m_i}}^T(x)}{m_i} \leq \frac{K_{b^{-n_i}}^T(x)}{n_i}$, and for $T'(\pi) = \text{comp}(T(\pi))$, either

$$\liminf_n \frac{K^T(S \upharpoonright n)}{n} \leq \lim_i \frac{K_{b^{-m_i}}^T(x)}{m_i}$$

or

$$\liminf_n \frac{K^{T'}(\text{comp}(S) \upharpoonright n)}{n} \leq \lim_i \frac{K_{b^{-m_i}}^T(x)}{m_i}.$$

Therefore either $\dim_{\text{FS}}(S) \leq \dim_{\text{FS}}^b(x)$ or $\dim_{\text{FS}}(\text{comp}(S)) \leq \dim_{\text{FS}}^b(x)$ and the claim follows.

Since finite-state dimension in the space of sequences characterizes Borel normality [2], we have an alternative characterization of normality in terms of finite-state dimension in the Euclidean space.

Corollary 1. Let $b \geq 1$, $x \in [0, 1)$. x is b -normal if and only if $\dim_{\text{FS}}^b(x) = 1$, that is,

$$\inf_{T \in \Sigma_b\text{-FST}} \liminf_{\delta > 0} \frac{K_\delta^T(x)}{\log_b(1/\delta)} = 1.$$

4 Point to Set Principle for Finite-State Dimension

We denote as *separator* a set $S \subseteq [0, 1)$ such that S is countable and dense in $[0, 1]$.

Definition 5. A *separator enumerator (SE)* is a function $f : \Sigma^* \rightarrow [0, 1)$ such that $\text{Im}(f)$ is a separator.

For each separator enumerator f we can define information content in $[0, 1)$ relative to f .

Definition 6. Let $f : \Sigma^* \rightarrow [0, 1)$ be a SE. Let T be a Σ -FST, $\delta > 0$ and $x \in [0, 1)$. The f - T -information content of x at precision δ is

$$K_{\delta}^{T,f}(x) = \min \{ K^T(w) \mid |f(w) - x| < \delta \}.$$

Definition 7. Let $f : \Sigma^* \rightarrow [0, 1)$ be a SE. Let $x \in [0, 1)$ and $A \subseteq [0, 1)$. The f -enumerator finite-state dimension of x is

$$\dim_{\text{FS}}^f(x) = \inf_{T \Sigma\text{-FST}} \liminf_{\delta > 0} \frac{K_{\delta}^{T,f}(x)}{\log_{|\Sigma|}(1/\delta)},$$

the f -enumerator finite-state dimension of A is

$$\dim_{\text{FS}}^f(A) = \inf_{T \Sigma\text{-FST}} \sup_{x \in A} \liminf_{\delta > 0} \frac{K_{\delta}^{T,f}(x)}{\log_{|\Sigma|}(1/\delta)}.$$

We can generalize Borel normality through the same relativization.

Definition 8. Let $f : \Sigma^* \rightarrow [0, 1)$ be a SE, let $x \in [0, 1)$. x is f -normal if $\dim_{\text{FS}}^f(x) = 1$.

Given this natural relativization of finite-state dimension we next prove a point-to-set principle stating that for every set A there exists an SE f such that classical Hausdorff dimensional of A is exactly f -finite-state dimension. This implies that classical geometrical measure theory results can be obtained using only finite-state dimension.

Theorem 4. Let $A \subseteq [0, 1)$.

$$\dim_{\text{H}}(A) = \min_{f \text{ SE}} \dim_{\text{FS}}^f(A),$$

$$\dim_{\text{H}}(A) = \min_{f: \{0,1\}^* \rightarrow \mathcal{D}_2} \dim_{\text{FS}}^f(A).$$

Proof. Let C be such that $\dim_{\text{H}}(A) = \dim^C(A)$ from the point-to-set principle in [9]. Let U be the universal oracle Turing Machine used in the definition of Kolmogorov Complexity for effective dimension \dim . Let $h : \{0, 1\}^* \rightarrow \{0, 1\}^*$ be such that $h(w) = U^C(w)$ when $U^C(w)$ is defined, and $U^C(w) = 0$ otherwise. Then $f(w) = \text{real}_2(h(w))$ is the required SE.

Notice that the previous theorem holds even when fixing a particular countable dense set. In terms of Borel normality, it shows that reordering the set \mathcal{D}_b of base- b finite representation numbers is enough to obtain normality for any other base.

5 Conclusions and Open Questions

We expect that our main theorem will prove new lower bounds on Hausdorff dimension in different settings. Notice that the result can be directly translated into any separable metric space and any reasonable gauge family.

Our result helps clarify the oracle role in the point to set principles. The next step would be to classify the different enumerations of a countable dense set.

We believe that the notion of f -normal sequence can be of independent interest with robustness properties inherited from those of the original concept, for instance from the fact that x is b -normal exactly when the sequence $(b^n x)_n$ is equidistributed modulo 1.

Open Question. Let $f : \Sigma^* \rightarrow [0, 1)$ be a SE, let $x \in [0, 1)$. For each $n \in \mathbb{N}$, let $a_n(x) = f(w)$ for $|w| \leq n$ such that $f(w) \leq x$ and $x - f(w)$ is minimum. Can we characterize f -normality in terms of the equidistribution properties of $(|\Sigma|^n a_n(x))$?

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