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ORIGINAL PAPER

J-ternary algebras, structurable algebras, and Lie superalgebras

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Abstract

A Lie superalgebra is attached to any finite-dimensional J-ternary algebra over an algebraically closed field of characteristic 3, using a process of semisimplification via tensor categories. Some of the exceptional simple Lie algebras, specific of this characteristic, are obtained in this way from J-ternary algebras coming from structurable algebras and, in particular, a new magic square of Lie superalgebras is constructed, with entries depending on a pair of composition algebras.

Keywords J-ternary algebra · Structurable · Lie algebra · Lie superalgebra · Exceptional

Mathematics Subject Classification Primary 17B60; Secondary 17B25 · 17B50 · 17A30

1 Introduction

In a recent work [25], Arun Kannan showed how to obtain a Lie superalgebra starting with a Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic 3 endowed with a nilpotent derivation d such that $d^3 = 0$, using a semisimplification process of symmetric tensor categories, taking advantage that the semisimplification of the category of representations of $\mathbb{F}[X]/(X^3)$ is the Verlinde category Ver_3, which is equivalent to the category of finite-dimensional vector superspaces over $\mathbb{F}$. In characteristic $p > 3$ still a suitable subcategory of the Verlinde category is equivalent to the category of vector superspaces, and this allows Kannan to get interesting exceptional Lie superalgebras also in characteristic 5 from classical Lie algebras endowed with a derivation d such that $d^5 = 0$.

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A natural class of Lie algebras over a field of characteristic 3 equipped with nilpotent derivations of degree 3 is given by the Lie algebras with a subalgebra isomorphic to $\mathfrak{sl}_2$ and such that, as a module for this subalgebra, they are a sum of copies of the adjoint, the natural and the trivial modules. In this case, the adjoint action of the element $F = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \in \mathfrak{sl}_2$ is a nilpotent derivation and $\mathrm{ad}_F^3 = 0$. These are the Lie algebras with a *short* $\mathbf{SL}_2$*-structure* in the sense of Vinberg [30]. These Lie algebras, over fields of characteristic $\neq 2, 3$, are coordinatized by the so called J-ternary algebras, introduced by Hein [21] and Allison [1]. However, only minor changes are needed in characteristic 3.

Therefore, giving a J-ternary algebra $\mathcal{T}$ over a field of characteristic 3, we can attach canonically a Lie algebra $\mathcal{L}(\mathcal{T})$ with a short $\mathbf{SL}_2$-structure (see (2.18)), which can be viewed as a Lie algebra in the category of representations of $\mathbb{F}[X]/(X^3)$, and then we can apply the semisimplification process to obtain a Lie superalgebra $\mathcal{L}^{\mathrm{ss}}(\mathcal{T})$ (Corollary 4.3).

The goal of this paper is to study the main examples of simple finite-dimensional J-ternary algebras over an algebraically closed field of characteristic 3, apply the process above, and check which Lie superalgebras appear.

The outcome is that the 'prototypical' J-ternary algebras give classical Lie superalgebras (§5.1), the J-ternary algebras obtained from simple structurable algebras of skew-dimension one give the new exceptional Lie superalgebras in [14, Theorem 3.2], which are specific of characteristic 3. For the remaining class of simple J-ternary algebras, i.e., those obtained from structurable algebras which are tensor products of a Cayley algebra and another unital composition algebra, we get two of the exceptional Lie superalgebras in the magic square of superalgebras in [10], as well as the exceptional Lie superalgebra $\mathfrak{el}(5; 3)$ (notation as in [9]).

The paper is organized as follows. Section 2 reviews the connection of J-ternary algebras and Lie algebras with a short $\mathbf{SL}_2$-structure, with the modifications needed to include characteristic 3. In particular, given any J-ternary algebra $\mathcal{T}$ over a field of characteristic $\neq 2$, a Lie algebra $\mathcal{L}(\mathcal{T})$ with a short $\mathbf{SL}_2$-structure will be canonically defined in Theorem 2.14.

Section 3 will show how to construct a J-ternary algebra out of a structurable algebra containing a skew-symmetric element s with invertible left multiplication L_s (Theorem 3.8). In the process, it will be proved that the Kantor Lie algebra $\mathcal{K}(\mathcal{A}, -)$ attached to a structurable algebra ([3]) makes sense too in characteristic 3 (Proposition 3.4), which is a result of independent interest.

Section 4 is devoted to review the transition from a Lie algebra with a nilpotent derivation of degree 3 over a field of characteristic 3 to a Lie superalgebra via tensor categories, following the ideas in [12, 17, 25]. Then this will be applied to the Lie algebra $\mathcal{L}(\mathcal{T})$ attached in Sect. 2 to a J-ternary algebra and its derivation ad_F above, thus obtaining a Lie superalgebra $\mathcal{L}^{\mathrm{ss}}(\mathcal{T})$.

In Sect. 5 it will be shown that if $\mathcal{T}$ is a simple J-ternary algebra of 'prototypical type', then the Lie superalgebra $\mathcal{L}^{\mathrm{ss}}(\mathcal{T})$ is either a projective special linear Lie superalgebra or an orthosymplectic Lie superalgebra (Theorem 5.1). The situation for the simple J-ternary algebras obtained from simple structurable algebras other than the tensor products of a Cayley algebra and a unital composition algebra will be dealt with too.

Finally, Sect. 6 will treat the case missed in Sect. 5. Here three exceptional simple Lie superalgebras specific of characteristic 3 appear: $\mathfrak{g}(3, 3)$, $\mathfrak{el}(5; 3)$ and $\mathfrak{g}(6, 6)$. Moreover, putting together the Lie superalgebras constructed from J-ternary algebras obtained from the structurable algebras given by a tensor product of two unital composition algebras, we get a new 'magic square' of Lie superalgebras in §6.4.

Immediately after posting our results in arXiv, we learned that Michiel Smet had arrived to the same results independently. The main differences with his work [28] are that our treatment of the 'prototypical type' in Sect. 5 is a bit more concrete, dealing directly with classical Lie algebras instead of suitable algebras of matrices over a composition algebra; and that in our treatment of the J-ternary algebras coming from a structurable algebra obtained as the tensor product of two composition algebras in Sect. 6, the invertible skew-symmetric element s has no extra restrictions. We thank Michiel Smet for sharing his results with us even before posting them to arXiv.

All the algebras considered will be finite-dimensional and defined over a ground field $\mathbb{F}$ *of characteristic not* 2*, unless otherwise stated.*

2 J-ternary algebras and short SL_2-structures on Lie algebras

This section is devoted to review the two different related definitions of J-ternary algebras in the literature and their role as 'coordinate algebras' of certain Lie algebras.

Definition 2.1 ([21, 22]) Let $\mathcal{T}$ be a triple system with ternary multiplication xyz. Then $\mathcal{T}$ is said to be a *J-ternary algebra* if the following conditions hold:

$$xy(uvz) = (xyu)vz + u(yxv)z + uv(xyz), \tag{2.1a}$$

$$xyz - zyx = zxy - xzy, \tag{2.1b}$$

for any $x, y, z, u, v \in \mathcal{T}$.

Definition 2.2 ([1, 5]) Let $\mathcal{J}$ be a unital Jordan algebra with multiplication $a \cdot b$, for $a, b \in \mathcal{J}$. Let $\mathcal{T}$ be a unital special Jordan module for $\mathcal{J}$ with action $a \bullet x$ for $a \in \mathcal{J}$ and $x \in \mathcal{T}$, so that

$$(a \cdot b) \bullet x = \frac{1}{2}\Big(a \bullet (b \bullet x) + b \bullet (a \bullet x)\Big),$$

for any $a, b \in \mathcal{J}$ and $x \in \mathcal{T}$. Assume $\langle . \mid . \rangle : \mathcal{T} \times \mathcal{T} \to \mathcal{J}$ is a skew-symmetric bilinear map and $(.,.,.) : \mathcal{T} \times \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ is a trilinear product on $\mathcal{T}$. Then $\mathcal{T}$ is called a *$\mathcal{J}$-ternary algebra* if the following axioms hold for any $a \in \mathcal{J}$ and $x, y, z, w, v \in \mathcal{T}$:

$$a \cdot \langle x \mid y \rangle = \frac{1}{2}\big(\langle a \bullet x \mid y \rangle + \langle x \mid a \bullet y \rangle\big), \tag{2.2a}$$

$$a \bullet (x, y, z) = (a \bullet x, y, z) - (x, a \bullet y, z) + (x, y, a \bullet z), \tag{2.2b}$$

$$(x, y, z) = (z, y, x) - \langle x \mid z \rangle \bullet y, \tag{2.2c}$$

$$(x, y, z) = (y, x, z) + \langle x \mid y \rangle \bullet z, \tag{2.2d}$$

$$\langle (x, y, z) \mid w \rangle + \langle z \mid (x, y, w) \rangle = \langle x \mid \langle z \mid w \rangle \bullet y \rangle, \tag{2.2e}$$

$$(x, y, (z, w, v)) = ((x, y, z), w, v) + (z, (y, x, w), v) + (z, w, (x, y, v)). \tag{2.2f}$$

These two definitions are intimately related, as we will see in Theorem 2.8. In the proof, a new type of triple systems will be used.

Actually, Yamaguti and Ono [31] defined a wide class of triple systems: the (ϵ, δ) Freudenthal-Kantor triple systems, which extend the classical Freudenthal triple systems and are useful tools in the construction of Lie algebras and superalgebras.

Definition 2.3 ([31]) An (ϵ, δ) *Freudenthal-Kantor triple system* (ϵ, δ are either 1 or -1) is a triple system U, with multiplication xyz, such that, if $L(x, y), K(x, y) \in \operatorname{End}_{\mathbb{F}}(U)$ are defined by

$$\begin{cases} L(x, y)z = xyz \\ K(x, y)z = xzy - \delta yzx, \end{cases} \tag{2.3}$$

then

$$[L(u, v), L(x, y)] = L\big(L(u, v)x, y\big) + \epsilon L\big(x, L(v, u)y\big), \tag{2.4a}$$

$$K\big(K(u, v)x, y\big) = L(y, x)K(u, v) - \epsilon K(u, v)L(x, y), \tag{2.4b}$$

hold for any $x, y, u, v \in U$.

For $\epsilon = -1$ and $\delta = 1$, these are the so called *Kantor triple systems* (or generalized Jordan triple systems of second order [26]).

Let us show the close relationship between the J-ternary algebras and some particular $(1, 1)$ Freudenthal-Kantor triple systems.

As in [19] we get the following properties.

Lemma 2.4 *Let U be a triple system satisfying equation (2.4a), with $\epsilon \in \{\pm 1\}$, and define the endomorphisms $S(x, y)$ and $T(x, y) \in \operatorname{End}_{\mathbb{F}}(U)$ by*

$$\begin{aligned} S(x, y) &= L(x, y) + \epsilon L(y, x) \\ T(x, y) &= L(y, x) - \epsilon L(x, y). \end{aligned} \tag{2.5}$$

Then for any $u, v \in U$, $S(u, v)$ is a derivation of the triple system (U, xyz), while $T(u, v)$ satisfies

$$T(u, v)\big(xyz\big) = \big(T(u, v)x\big)yz - x\big(T(u, v)y\big)z + xy\big(T(u, v)z\big)$$

for any $x, y, z \in U$. As a consequence, the following equations hold:

$$[S(u, v), L(x, y)] = L\big(S(u, v)x, y\big) + L\big(x, S(u, v)y\big) \tag{2.6a}$$

$$[T(u, v), L(x, y)] = L\big(T(u, v)x, y\big) - L\big(x, T(u, v)y\big) \tag{2.6b}$$

$$[S(u, v), T(x, y)] = T\big(S(u, v)x, y\big) + T\big(x, S(u, v)y\big) \tag{2.6c}$$

$$[T(u, v), S(x, y)] = -\epsilon T\big(T(u, v)x, y\big) + \epsilon T\big(x, T(u, v)y\big) \tag{2.6d}$$

$$[T(u, v), T(x, y)] = -\epsilon S\big(T(u, v)x, y\big) + \epsilon S\big(x, T(u, v)y\big) \tag{2.6e}$$

Proof The fact that $S(u, v)$ is a derivation and that $T(u, v)$ satisfies the equation above when applied to a product xyz follow at once from (2.4a). These conditions are equivalent to equations (2.6a) and (2.6b). The other equations are obtained from these. □

Remark 2.5 Let U be a triple system satisfying equation (2.4a), with $\epsilon \in \{\pm 1\}$, and denote by $L(U, U)$ (respectively $S(U, U)$, $T(U, U)$) the linear span of the operators $L(x, y)$ (respectively $S(x, y)$, $T(x, y)$) for $x, y \in U$. Lemma 2.4 shows that $S(U, U)$ is a subalgebra of $L(U, U)$, and that the conditions $L(U, U) = S(U, U) + T(U, U)$, $[S(U, U), T(U, U)] \subseteq T(U, U)$ and $[T(U, U), T(U, U)] \subseteq S(U, U)$ hold. In particular, this shows that $S(U, U) \cap T(U, U)$ is an ideal of $L(U, U)$, and modulo this ideal we get a $\mathbb{Z}/2\mathbb{Z}$-graded Lie algebra.

Definition 2.6 Let U be an (ϵ, δ) Freudenthal-Kantor triple system. Then U is said to be *special* in case

$$K(x, y) = \epsilon\delta L(y, x) - \epsilon L(x, y) \tag{2.7}$$

holds for any $x, y \in U$.

Proposition 2.7 *The J-ternary algebras are exactly the special* $(1, 1)$ *Freudenthal-Kantor triple systems.*

Proof First note that, for $\epsilon = 1$, (2.1a) coincides with (2.4a), while for $\epsilon = \delta = 1$ (2.1b) can be rewritten as

$$K(x, z) = T(x, z)$$

due to (2.3) and (2.5). Also, for $\epsilon = \delta = 1$, a Freudenthal-Kantor triple system is special if and only if $K(x, y) = T(x, y)$ for any x, y, because of (2.7).

Therefore, if (U, xyz) is a special $(1, 1)$ Freudenthal-Kantor triple system, (2.1a) follows from (2.4a), while (2.1b) follows from (2.7).

Conversely, if $\mathcal{T}$ is a J-ternary algebra then, with $\epsilon = \delta = 1$, (2.4a) is equivalent to (2.1a), (2.7) is equivalent to (2.1b). Besides, (2.1a) implies

$$T(u, v)(xyz) = \big(T(u, v)x\big)yz - x\big(T(u, v)y\big)z + xy\big(T(u, v)z\big).$$

Interchanging x and z above and subtracting we get

$$T(u, v)K(x, z) = K\big(T(u, v)x, z\big) - K(x, z)T(u, v) + K\big(x, T(u, v)z\big),$$

which, because $K(x, z) = T(x, z)$ by (2.1b), becomes

$$K(u, v)K(x, z) + K(x, z)K(u, v) = K\big(K(u, v)x, z\big) + K\big(x, K(u, v)z\big), \tag{2.8}$$

for all $x, y, z, u, v \in \mathcal{T}$. Using that

$$L(x, y) = \frac{1}{2}\big(S(x, y) - T(x, y)\big) = \frac{1}{2}\big(S(x, y) - K(x, y)\big),$$

we compute, as in [19, Proof of Proposition 3.5]:

$$\begin{aligned} &L(y, x)K(u, v) - K(u, v)L(x, y) \\ &\quad = \frac{1}{2}\Big(S(y, x) - K(y, x)\Big)K(u, v) - \frac{1}{2}K(u, v)\Big(S(x, y) - K(x, y)\Big) \\ &\quad = \frac{1}{2}[S(x, y), K(u, v)] + \frac{1}{2}\Big(K(x, y)K(u, v) + K(u, v)K(x, y)\Big) \\ &\quad = -\frac{1}{2}[K(u, v), S(x, y)] + \frac{1}{2}\Big(K(x, y)K(u, v) + K(u, v)K(x, y)\Big) \\ &\quad = K\big(K(u, v)x, y\big) \quad \text{because of (2.6d) and (2.8)}, \end{aligned}$$

thus obtaining that $\mathcal{T}$ also satisfies (2.4b). □

Recall that given an associative algebra $\mathcal{A}$, its associated Jordan algebra $\mathcal{A}^{(+)}$ is the algebra defined on the vector space $\mathcal{A}$ with multiplication $a \cdot b = \frac{1}{2}(ab + ba)$ for $a, b \in \mathcal{A}$.

Theorem 2.8 *Let $\mathcal{J}$ be a unital Jordan algebra and $\mathcal{T}$ a $\mathcal{J}$-ternary algebra (Definition 2.2). Then $\mathcal{T}$ is a J-ternary algebra (Definition 2.1).*

Conversely, let $\mathcal{T}$ be a J-ternary algebra, then the subspace $\mathcal{J} := \mathbb{F}\mathrm{id} + K(\mathcal{T}, \mathcal{T})$ is a Jordan subalgebra of $\mathrm{End}_{\mathbb{F}}(\mathcal{T})^{(+)}$*, where $K(x, y)z = xzy - yzx$ for $x, y \in \mathcal{T}$ and $K(\mathcal{T}, \mathcal{T}) =$* $\mathrm{span}\,\{K(x, y) \mid x, y \in \mathcal{T}\}$*, and $\mathcal{T}$ becomes a $\mathcal{J}$-ternary algebra with the natural action of J on $\mathcal{T}$: $a \bullet x = a(x)$, and the multilinear maps:*

$$(.,.,.) : \mathcal{T} \times \mathcal{T} \times \mathcal{T} \to \mathcal{T}, \quad (x, y, z) = xyz,$$
$$\langle . \mid . \rangle : \mathcal{T} \times \mathcal{T} \to \mathcal{J}, \quad \langle x \mid y \rangle = -K(x, y).$$

Proof If $\mathcal{T}$ is a $\mathcal{J}$-ternary algebra for a Jordan algebra $\mathcal{J}$, then with $xyz = (x, y, z)$, equation (2.2f) becomes (2.1a), while equations (2.2c) and (2.2d) give:

$$xyz - zyx = -\langle x \mid z \rangle \bullet y = zxy - xzy,$$

which gives (2.1b).

Conversely, let $\mathcal{T}$ be a J-ternary algebra. Then the subspace $\mathcal{J} = \mathbb{F}\mathrm{id} + K(\mathcal{T}, \mathcal{T})$ is a Jordan subalgebra of $\mathrm{End}_{\mathbb{F}}(\mathcal{T})^{(+)}$ because of (2.8), with multiplication $f \cdot g = \frac{1}{2}(fg + gf)$, and $\mathcal{T}$ becomes a special Jordan module with $a \bullet x = a(x)$ for $a \in \mathcal{J} \subseteq \mathrm{End}_{\mathbb{F}}(\mathcal{T})$ and $x \in \mathcal{T}$. Now, with $(x, y, z) = xyz$, and $\langle x \mid y \rangle = -K(x, y) \in \mathcal{J}$, (2.2f) is just (2.1a), (2.2c) is the definition of $\langle x \mid z \rangle$ and (2.2d) follows from (2.1b). Also, (2.8) gives (2.2a), and equation (2.6b) gives (2.2b) because $K(x, y) = T(x, y)$ for any x, y, as $\mathcal{T}$ is a special $(1, 1)$ Freudenthal-Kantor triple system. Finally, (2.1a) gives, for any x, y, z, w, u:

$$\begin{aligned} L(x, y)K(z, w)u &= L(x, y)\big(zuw - wuz\big) \\ &= K\big(L(x, y)z, w\big)u + K\big(z, L(x, y)w\big)u + K(z, w)L(y, x)u, \end{aligned}$$

so that

$$K\big(L(x, y)z, w\big) + K\big(z, L(x, y)w\big) = L(x, y)K(z, w) - K(z, w)L(y, x).$$

Hence, (2.4b) (with $\epsilon = 1$) can be written as follows:

$$K\big(K(z, w)y, x\big) = K\big(L(x, y)z, w\big) + K\big(z, L(x, y)w\big),$$

which gives (2.2e), because of the skew-symmetry of K. □

It is time to provide the most important class of examples of J-ternary algebras (see [8, Proposition 4.1] and references there in).

Example 2.9 *(Prototypical example)* Let $(\mathcal{A}, *)$ be a unital associative algebra with involution and let $\mathcal{T}$ be a left $\mathcal{A}$-module endowed with a skew-hermitian form $\mathrm{h} : \mathcal{T} \times \mathcal{T} \to \mathcal{A}$. Then $\mathcal{T}$ is a $\mathcal{J}$-ternary algebra, where $\mathcal{J} = \mathcal{H}(\mathcal{A}, *)$ is the Jordan algebra of symmetric elements of $\mathcal{A}$ relative to the involution $*$, with the following operations:

- $a \cdot b = \frac{1}{2}(ab + ba)$ for any $a, b \in \mathcal{J} = \mathcal{H}(\mathcal{A}, *)$,
- $a \bullet x = ax$ for any $a \in \mathcal{J}$ and $x \in \mathcal{T}$,
- $\langle x \mid y \rangle = \mathrm{h}(x, y) - \mathrm{h}(y, x)$ for $x, y \in \mathcal{T}$, and
- $(x, y, z) = \mathrm{h}(x, y)z + \mathrm{h}(z, x)y + \mathrm{h}(z, y)x$ for $x, y, z \in \mathcal{T}$.

The J-ternary algebras are tightly related with the Lie algebras with short $\mathbf{SL}_2$-structures. This connection has been dealt with in [8], and references there in, assuming the characteristic of $\mathbb{F}$ is not 2 nor 3. However, due to [8, Remark 2.3], most of the arguments are valid too in characteristic 3.

First the necessary definitions, following [8]:

Definition 2.10 *([30, Definition 0.1])* Let $\mathbf{S}$ be a reductive algebraic group and let $\mathcal{L}$ be a Lie algebra. An $\mathbf{S}$-*structure* on $\mathcal{L}$ is a homomorphism $\Phi : \mathbf{S} \to \mathbf{Aut}(\mathcal{L})$ from $\mathbf{S}$ into the algebraic group of automorphisms of $\mathcal{L}$.

The $\mathbf{S}$-structure $\Phi : \mathbf{S} \to \mathbf{Aut}(\mathcal{L})$ on the Lie algebra $\mathcal{L}$ is said to be *inner* if there is a one-to-one Lie algebra homomorphism $\iota : \mathfrak{s} \hookrightarrow \mathcal{L}$, where $\mathfrak{s}$ is the Lie algebra of $\mathbf{S}$, such that the following diagram commutes, where $\mathrm{d}\Phi$ denotes the differential of Φ:

$$\begin{array}{ccc} \mathfrak{s} & \xrightarrow{\ \iota\ } & \mathcal{L} \\ & {}_{\mathrm{d}\Phi}\searrow & \big\downarrow \mathrm{ad} \\ & & \mathrm{Der}(\mathcal{L}) \end{array} \tag{2.9}$$

Note that if $\Phi : \mathbf{S} \to \mathbf{Aut}(\mathcal{L})$ is a non-inner $\mathbf{S}$-structure on the Lie algebra $\mathcal{L}$, then we can take the split extension $\tilde{\mathcal{L}} = \mathcal{L} \oplus \mathfrak{s}$ as in [24, p. 18], where $\mathfrak{s}$ acts on $\mathcal{L}$ through $\mathrm{d}\Phi$, and this is endowed with a natural $\mathbf{S}$-structure. Hence, it is not harmful to restrict to inner $\mathbf{S}$-structures.

Definition 2.11 An $\mathbf{SL}_2$-structure $\Phi : \mathbf{SL}_2 \to \mathbf{Aut}(\mathcal{L})$ on a Lie algebra $\mathcal{L}$ is said to be *short* if $\mathcal{L}$ decomposes, as a module for $\mathbf{SL}_2$ via Φ, into a direct sum of copies of the adjoint, natural, and trivial modules.

Therefore, the isotypic decomposition of $\mathcal{L}$ allows us to describe $\mathcal{L}$ as follows:

$$\mathcal{L} = \big(\mathfrak{sl}(V) \otimes \mathcal{J}\big) \oplus \big(V \otimes \mathcal{T}\big) \oplus \mathcal{D}, \tag{2.10}$$

for vector spaces $\mathcal{J}$, $\mathcal{T}$, and $\mathcal{D}$, where V is the natural two-dimensional representation of $\mathbf{SL}_2 \simeq \mathbf{SL}(V)$. The action of $\mathbf{SL}_2$ is given by the adjoint action of $\mathbf{SL}_2$ on $\mathfrak{sl}(V)$, its natural action on V, and the trivial action on $\mathcal{J}$, $\mathcal{T}$ and $\mathcal{D}$. The subspace $\mathcal{D}$, being the subspace of fixed elements by $\mathbf{SL}_2$, is a subalgebra of $\mathcal{L}$.

Lemma 2.12 *Let V be a two-dimensional vector space.*

(i) *The space* $\mathrm{Hom}_{\mathfrak{sl}(V)}\big(\mathfrak{sl}(V) \otimes \mathfrak{sl}(V), \mathfrak{sl}(V)\big)$ *of* $\mathfrak{sl}(V)$*-invariant linear maps* $\mathfrak{sl}(V) \otimes \mathfrak{sl}(V) \to \mathfrak{sl}(V)$ *is spanned by the (skew-symmetric) Lie bracket:*

$$f \otimes g \mapsto [f, g] = fg - gf.$$

(ii) *The space* $\mathrm{Hom}_{\mathfrak{sl}(V)}\big(\mathfrak{sl}(V) \otimes \mathfrak{sl}(V), \mathbb{F}\big)$ *is spanned by the trace map:*

$$f \otimes g \mapsto \mathrm{tr}(fg).$$

(iii) *The space* $\mathrm{Hom}_{\mathfrak{sl}(V)}\big(\mathfrak{sl}(V) \otimes V, V\big)$ *is spanned by the natural action:*

$$f \otimes v \mapsto f(v).$$

(iv) *The space* $\mathrm{Hom}_{\mathfrak{sl}(V)}\big(V \otimes V, \mathbb{F}\big)$ *is one-dimensional. Its nonzero elements are of the form*

$$u \otimes v \mapsto (u \mid v),$$

for a nonzero skew-symmetric bilinear form $(. \mid .)$ *on* V.

(v) *The space* $\mathrm{Hom}_{\mathfrak{sl}(V)}\big(V \otimes V, \mathfrak{sl}(V)\big)$ *is one-dimensional. Once a nonzero skew-symmetric bilinear form* $(. \mid .)$ *is fixed on* V*, this subspace is spanned by the following symmetric map:*

$$u \otimes v \mapsto \gamma_{u,v}\big(: w \mapsto (u \mid w)v + (v \mid w)u\big).$$

(vi) *The spaces* $\mathrm{Hom}_{\mathfrak{sl}(V)}\big(\mathfrak{sl}(V) \otimes \mathfrak{sl}(V), V\big)$, $\mathrm{Hom}_{\mathfrak{sl}(V)}\big(\mathfrak{sl}(V) \otimes V, \mathfrak{sl}(V)\big)$, $\mathrm{Hom}_{\mathfrak{sl}(V)}\big(\mathfrak{sl}(V) \otimes V, \mathbb{F}\big)$*, and* $\mathrm{Hom}_{\mathfrak{sl}(V)}\big(V \otimes V, V\big)$ *are all trivial.*

Moreover, $\mathrm{Hom}_{\mathfrak{sl}(V)}$ *may be replaced by* $\mathrm{Hom}_{\mathbf{SL}(V)}$ *all over.*

Proof This is well known if the characteristic of $\mathbb{F}$ is $\neq 2, 3$ (see, e.g., [19, Lemma 2.1]), but it remains valid in characteristic 3 by [8, Remark 2.3]. □

Let $\mathcal{L}$ be a Lie algebra with an inner $\mathbf{SL}_2$-structure and isotypic decomposition as in (2.10). The $\mathbf{SL}_2$-structure being inner forces $\mathcal{J}$ to contain a distinguished element 1, such that $\mathfrak{sl}(V)\otimes 1$ is the image of ι in (2.9).

The $\mathbf{SL}_2$-invariance or, equivalently, the $\mathfrak{sl}(V)$-invariance, of the Lie bracket in our Lie algebra $\mathcal{L}$ gives, for any $f, g \in \mathfrak{sl}(V)$, $u, v \in V$, $a, b \in \mathcal{J}$, $x, y \in \mathcal{T}$, and $D \in \mathcal{D}$, the following conditions:

$$\begin{aligned}
[f\otimes a, g\otimes b] &= [f,g]\otimes a\cdot b + 2\mathrm{tr}(fg)D_{a,b},\\
[f\otimes a, u\otimes x] &= f(u)\otimes a\bullet x,\\
[u\otimes x, v\otimes y] &= \gamma_{u,v}\otimes\langle x\mid y\rangle + \big(u\mid v\big)d_{x,y},\\
[D, f\otimes a] &= f\otimes D(a),\\
[D, u\otimes x] &= u\otimes D(x),
\end{aligned}\tag{2.11}$$

for suitable $\mathcal{D}$-invariant bilinear maps

$$\begin{aligned}
\mathcal{J}\times\mathcal{J}\to\mathcal{J}: &\quad (a,b)\mapsto a\cdot b \quad \text{(symmetric)},\\
\mathcal{J}\times\mathcal{J}\to\mathcal{D}: &\quad (a,b)\mapsto D_{a,b} \quad \text{(skew-symmetric)},\\
\mathcal{J}\times\mathcal{T}\to\mathcal{T}: &\quad (a,x)\mapsto a\bullet x,\\
\mathcal{T}\times\mathcal{T}\to\mathcal{J}: &\quad (x,y)\mapsto \langle x\mid y\rangle \quad \text{(skew-symmetric)},\\
\mathcal{T}\times\mathcal{T}\to\mathcal{D}: &\quad (x,y)\mapsto d_{x,y} \quad \text{(symmetric)},\\
\mathcal{D}\times\mathcal{J}\to\mathcal{J}: &\quad (D,a)\mapsto D(a),\\
\mathcal{D}\times\mathcal{T}\to\mathcal{T}: &\quad (D,x)\mapsto D(x),
\end{aligned}\tag{2.12}$$

such that

$$1\cdot a = a,\quad D_{1,a} = 0,\quad \text{and}\quad 1\bullet x = x,\tag{2.13}$$

for any $a \in \mathcal{J}$ and $x \in \mathcal{T}$.

The Jacobi identity on $\mathcal{L}$ also shows that all these maps are invariant under the action of the Lie subalgebra $\mathcal{D}$. The next result summarizes the properties of these maps:

Theorem 2.13 *A Lie algebra* $\mathcal{L}$ *is endowed with an inner short* $\mathbf{SL}_2$*-structure if and only if there is a two-dimensional vector space* V *such that* $\mathcal{L}$ *is, up to isomorphism, the Lie algebra in* (2.10)*, with Lie bracket given in* (2.11)*, for suitable bilinear maps given in* (2.12)*, satisfying the following conditions:*

- $\mathcal{J}$ *is a unital commutative algebra with the multiplication* $a\cdot b$.
- *The following equations hold for* $a, b \in \mathcal{J}$ *and* $x \in \mathcal{T}$*:*

$$1\bullet x = x,\quad (a\cdot b)\bullet x = \frac{1}{2}\big(a\bullet(b\bullet x) + b\bullet(a\bullet x)\big).\tag{2.14}$$

- *For any $a, b, c \in \mathfrak{J}$ and $x, y, z \in \mathfrak{T}$, the following identities hold:*

$$D_{a,b}(c) = a \cdot (b \cdot c) - b \cdot (a \cdot c), \tag{2.15a}$$
$$D_{a \cdot b,c} + D_{b \cdot c,a} + D_{c \cdot a,b} = 0, \tag{2.15b}$$
$$4D_{a,b}(x) = a \bullet (b \bullet x) - b \bullet (a \bullet x), \tag{2.15c}$$
$$4D_{a,\langle x|y\rangle} = -d_{a\bullet x,y} + d_{x,a\bullet y}, \tag{2.15d}$$
$$2a \cdot \langle x \mid y\rangle = \langle a \bullet x \mid y\rangle + \langle x \mid a \bullet y\rangle, \tag{2.15e}$$
$$d_{x,y}(a) = \langle a \bullet x \mid y\rangle - \langle x \mid a \bullet y\rangle, \tag{2.15f}$$
$$d_{x,y}(z) - d_{z,y}(x) = \langle x \mid y\rangle \bullet z - \langle z \mid y\rangle \bullet x + 2\langle x \mid z\rangle \bullet y. \tag{2.15g}$$

- *For any $D \in \mathfrak{D}$, the linear endomorphism of $\mathfrak{J} \oplus \mathfrak{T}$, given by $a + x \mapsto D(a) + D(x)$, for $a \in \mathfrak{J}$ and $x \in \mathfrak{T}$, is an even derivation of the $\mathbb{Z}/2\mathbb{Z}$-graded algebra with even part $\mathfrak{J}$, odd part $\mathfrak{T}$, and multiplication given by the formula:*

$$(a + x) \diamond (b + y) = \big(a \cdot b + \langle x \mid y\rangle\big) + \big(a \bullet y + b \bullet x\big), \tag{2.16}$$

for $a, b \in \mathfrak{J}$ and $x, y \in \mathfrak{T}$. □

Proof The proof in [19, Theorem 2.2], where the characteristic is assumed to be $\neq 2, 3$, works word for word in characteristic 3 too, with the only difference that (2.15b) does not imply the algebra $\mathfrak{J}$ to be a Jordan algebra.

Note that if the characteristic is $\neq 2, 3$, (2.15b) is equivalent to the condition $D_{a\cdot a,a} = 0$ for any $a \in \mathfrak{J}$, and hence (2.15a), with $b = a^{\cdot 2}$, gives $a^{\cdot 2} \cdot (c \cdot a) = (a^{\cdot 2} \cdot c) \cdot a$, due to the commutativity of $a \cdot b$. It follows that $\mathfrak{J}$ is a Jordan algebra in this case. In characteristic 3, we get that $\mathfrak{J}$ only satisfies the complete linearization of the Jordan identity, but not necessarily the Jordan identity. □

Theorem 2.14 *Let $\mathfrak{L}$ be a Lie algebra endowed with an inner short $\mathbf{SL}_2$-structure, with isotypic decomposition as in (2.10) and bracket in (2.11). Then $\mathfrak{T}$ is a J-ternary algebra with the triple product*

$$(x, y, z) := \frac{1}{2}\big(d_{x,y}(z) - \langle x \mid y\rangle \bullet z\big) \tag{2.17}$$

for $x, y, z \in \mathfrak{T}$.

Conversely, let $\mathfrak{T}$ be a J-ternary algebra (that is, a special (1,1) Freudenthal-Kantor triple system) and consider the Jordan algebra $\mathfrak{J} = \mathbb{F}\mathrm{id} + K(\mathfrak{T}, \mathfrak{T})$ as in Theorem 2.8, then the direct sum

$$\mathfrak{L}(\mathfrak{T}) := \big(\mathfrak{sl}(V) \otimes \mathfrak{J}) \oplus \big(V \otimes \mathfrak{T}\big) \oplus S(\mathfrak{T}, \mathfrak{T}) \tag{2.18}$$

is a Lie algebra, with an inner $\mathbf{SL}_2$-structure, with the Lie bracket given by the bracket in

$$S(\mathfrak{T}, \mathfrak{T}) = \mathrm{span}\,\{S(x, y) = L(x, y) + L(y, x) \mid x, y \in \mathfrak{T}\} \leq \mathfrak{gl}(\mathfrak{T}) \tag{2.19}$$

and the following equations:

$$\begin{aligned}
[f \otimes a, g \otimes b] &= [f, g] \otimes \frac{1}{2}(ab + ba) + \frac{1}{2}\mathrm{tr}(fg)[a, b],\\
[f \otimes a, u \otimes x] &= f(u) \otimes a(x),\\
[u \otimes x, v \otimes y] &= \gamma_{u,v} \otimes K(x, y) + (u|v)S(x, y)\\
[\varphi, f \otimes a] &= f \otimes [\varphi, a],\\
[\varphi, u \otimes x] &= u \otimes \varphi(x),
\end{aligned} \tag{2.20}$$

for any $f, g \in \mathfrak{sl}(V)$, $u, v \in V$, $a, b \in \mathfrak{J}$, $x, y \in \mathfrak{T}$ and $\varphi \in S(\mathfrak{T}, \mathfrak{T})$.

Proof If $\mathcal{L}$ is a Lie algebra endowed with an inner short $\mathbf{SL}_2$-structure and with isotypic decomposition as in (2.10), and we consider the triple product on $\mathcal{T}$ given by (2.17), then for any $x, y, z \in \mathcal{T}$ we get

$$\begin{aligned}(x, y, z) - (z, y, x) &= \frac{1}{2}\big(d_{x,y}(z) - d_{z,y}(x) - \langle x \mid y\rangle \bullet z + \langle z \mid y\rangle \bullet x\big)\\ &= \langle x \mid z\rangle \bullet y\end{aligned}$$

because of (2.15g), and also

$$\begin{aligned}(z, x, y) - (x, z, y) &= \frac{1}{2}\big(d_{z,x}(y) - d_{x,z}(y) - \langle z \mid x\rangle \bullet y + \langle x \mid z\rangle \bullet y\big)\\ &= \langle x \mid z\rangle \bullet y\end{aligned}$$

by the symmetry of $d_{x,y}$ and the skew-symmetry of $\langle x \mid y\rangle$. Hence, we obtain

$$(x, y, z) - (z, y, x) = \langle x \mid z\rangle \bullet y = (z, x, y) - (x, z, y),$$

which implies (2.1b). Now, for $a \in \mathcal{J}$ and $x, y, z \in \mathcal{T}$ we get:

$$\begin{aligned}&a\bullet(x, y, z) - (x, y, a \bullet z)\\ &= \frac{1}{2}\Big(a \bullet d_{x,y}(z) - d_{x,y}(a \bullet z) - a \bullet \big(\langle x \mid y\rangle \bullet z\big) + \langle x \mid y\rangle \bullet (a \bullet z)\Big)\\ &= \frac{1}{2}\Big(-d_{x,y}(a) \bullet z - 4D_{a,\langle x|y\rangle}(z)\Big)\\ &\qquad \text{(using (2.15c) and the fact that } d_{x,y} \text{ is a derivation)}\\ &= \frac{-1}{2}\Big(\langle a \bullet x \mid y\rangle \bullet z - \langle x \mid a \bullet y\rangle \bullet z - d_{a\bullet x,y}(z) + d_{x,a\bullet y}(z)\Big)\\ &\qquad \text{(because of (2.15d) and (2.15f))}\\ &= (a \bullet x, y, z) - (x, a \bullet y, z),\end{aligned}$$

thus obtaining the validity of (2.2b). The validity of (2.1a) follows from this, together with the invariance of (x, y, z) under the action of $\mathcal{D}$:

$$\begin{aligned}(x, y, (u, v, z)) &= \frac{1}{2}d_{x,y}\big((u, v, z)\big) - \frac{1}{2}\langle x \mid y\rangle \bullet (u, v, z)\\ &= \frac{1}{2}\Big((d_{x,y}(u), v, z) + (u, d_{x,y}(v), z) + (u, v, d_{x,y}(z))\\ &\qquad - (\langle x \mid y\rangle \bullet u, v, z) + (u, \langle x \mid y\rangle \bullet v, z) - (u, v, \langle x \mid y\rangle \bullet z)\Big)\\ &= ((x, y, u), v, z) + (u, (y, x, v), z) + (u, v, (x, y, z)).\end{aligned}$$

Conversely, let $\mathcal{T}$ be a J-ternary algebra and consider the Jordan algebra $\mathcal{J} = \mathbb{F}\mathrm{id} + K(\mathcal{T}, \mathcal{T})$. Define the anticommutative algebra $\mathcal{L}(T)$ as in (2.20). Note first that this is well defined because for $a, b \in \mathcal{J} \leq \mathrm{End}_{\mathbb{F}}(\mathcal{T})^{(+)}$, the bracket $[a, b]$ lies in $S(\mathcal{T}, \mathcal{T})$ by (2.6e), as $T(x, y) = K(x, y)$ for any $x, y \in \mathcal{T}$ (see the proof of Proposition 2.7). The maps in (2.11) are the following:

$$a \cdot b = \frac{1}{2}(ab + ba), \quad a \bullet x = a(x), \quad \langle x \mid y\rangle = K(x, y),$$
$$D_{a,b} = \frac{1}{4}[a, b], \quad d_{x,y} = S(x, y),$$

for $a, b \in \mathfrak{J}$ and $x, y \in \mathfrak{T}$. All these maps are $S(\mathfrak{T}, \mathfrak{T})$-invariant by (2.6a).

Let us check that the conditions in Theorem 2.13 hold. Clearly $\mathfrak{J}$ is a unital commutative algebra, as it is a Jordan subalgebra of $\mathrm{End}_{\mathbb{F}}(\mathfrak{T})^{(+)}$, and (2.14) holds. Also $S(\mathfrak{T}, \mathfrak{T})$ acts by derivations of the algebra in (2.16).

Equation (2.15a) is clear, because for any $a, b, c \in \mathrm{End}_{\mathbb{F}}(\mathfrak{T})$ we have

$$\begin{aligned} a \cdot (b \cdot c) - b \cdot (a \cdot c) &= \frac{1}{4}\big(a(bc + cb) + (bc + cb)a - b(ac + ca) - (ac + ca)b\big) \\ &= \frac{1}{4}\big(abc + cba - bac - cab\big) \\ &= \frac{1}{4}\big([a, b]c - c[a, b]\big) = \frac{1}{4}[[a, b], c]. \end{aligned}$$

Equation (2.15b) is a consequence then of the Jacobi identity on $\mathrm{End}_{\mathbb{F}}(\mathfrak{T})$. Also, for $a, b \in \mathfrak{J}$ and $x \in \mathfrak{T}$ we get

$$a \bullet (b \bullet x) - b \bullet (a \bullet x) = a(b(x)) - b(a(x)) = [a, b](x),$$

thus obtaining (2.15c). Equations (2.15d), (2.15e) and (2.15f) follow, respectively from equations (2.6e), (2.8) and (2.6d). Finally, for $x, y, z \in \mathfrak{T}$ we compute:

$$\begin{aligned} \langle x \mid y\rangle \bullet z - \langle z \mid y\rangle \bullet x + 2\langle x \mid z\rangle \bullet y &= K(x, y)z - K(z, y)x + 2K(x, z)y \\ &= xzy - yzx - zxy + yxz + 2xyz - 2zyx \\ &= (xzy - zxy) + (xyz + yxz) - (zyx + yzx) + (xyz - zyx) \\ &= (xyz + yxz) - (zyx + yzx) \quad \text{(because of (2.1b))} \\ &= S(x, y)z - S(z, y)x = d_{x,y}(z) - d_{z,y}(x), \end{aligned}$$

thus getting (2.15g). □

3 Structurable algebras and J-ternary algebras

This section is devoted to study the connection of structurable algebras with some J-ternary algebras.

Structurable algebras were defined in [2] over fields of characteristic $\neq 2, 3$. The definition over arbitrary fields, or even commutative rings, requires an extra condition [6, §5].

Definition 3.1 A unital algebra with involution $(\mathcal{A}, -)$ is a *structurable algebra* if the following two conditions hold:

$$[V_{a,b}, V_{c,d}] = V_{V_{a,b}(c),d} - V_{c,V_{b,a}(d)} \tag{3.1a}$$

$$(a - \bar{a}, b, c) = (b, \bar{a} - a, c) \tag{3.1b}$$

for any $a, b, c, d \in \mathcal{A}$, where $(a, b, c) := (ab)c - a(bc)$ is the associator of a, b, c, and $V_{a,b}(c) = (a\bar{b})c + (c\bar{b})a - (c\bar{a})b$.

Note that (3.1a) shows that any structurable algebra satisfies (2.4a) with $\epsilon = -1$ for the triple product

$$\{a, b, c\} := V_{a,b}(c) = (a\bar{b})c + (c\bar{b})a - (c\bar{a})b. \tag{3.2}$$

Also, (3.1b) may be written as $(s, x, y) = -(x, s, y)$ for any $s \in \mathcal{S} = \{x \in \mathcal{A} \mid \bar{x} = -x\}$ and $x, y \in \mathcal{A}$. Applying the involution we also have $(x, y, s) = -(x, s, y)$. These equations will be referred to as the skew-alternativity of $(\mathcal{A}, -)$.

Our first aim in this section is to show that the construction of the Lie algebra $\mathcal{K}(\mathcal{A}, -)$ in [3], defined over fields of characteristic $\neq 2, 3$, makes sense too in characteristic 3.

Recall that a Lie triple system is a vector space $\mathcal{T}$ endowed with a trilinear product $[a, b, c]$ satisfying the following equations:

$$[u, u, v] = 0, \tag{3.3a}$$

$$[u, v, w] + [v, w, u] + [w, u, v] = 0, \tag{3.3b}$$

$$[a, b, [u, v, w]] = [[a, b, u], v, w] + [u, [a, b, v], w] + [u, v, [a, b, w]], \tag{3.3c}$$

for any $a, b, u, v, w \in \mathcal{T}$. Also, for any Lie triple system $\mathcal{T}$, its *standard embedding* is the ($\mathbb{Z}/2\mathbb{Z}$-graded) Lie algebra $L(\mathcal{T}, \mathcal{T}) \oplus \mathcal{T}$, where $L(x, y)(z) = [x, y, z]$, $L(\mathcal{T}, \mathcal{T})$ is the span of the operators $L(x, y)$ (a Lie subalgebra of $\mathrm{End}_{\mathbb{F}}(\mathcal{T})$ because of (3.3c)), with the Lie bracket given by

$$[A + x, B + y] = \big([A, B] + L(x, y)\big) + \big(A(y) - B(x)\big)$$

for $x, y \in \mathcal{T}$ and $A, B \in L(\mathcal{T}, \mathcal{T})$.

As in [20], given a structurable algebra, consider the triple product $[.,.,.]$ defined on the direct sum of two copies of $\mathcal{A}$: $KT(\mathcal{A}) := \mathcal{A}_+ \oplus \mathcal{A}_-$, as follows:

$$\begin{aligned} [a_\delta, b_{-\delta}, c_\delta] &= \{a, b, c\}_\delta, \\ [a_\delta, b_\delta, c_\delta] &= 0, \\ [a_{-\delta}, b_\delta, c_\delta] &= -\{b, a, c\}_\delta, \\ [a_\delta, b_\delta, c_{-\delta}] &= \{a, c, b\}_\delta - \{b, c, a\}_\delta, \end{aligned} \tag{3.4}$$

for $a, b, c \in \mathcal{A}$, $\delta = \pm$.

Proposition 3.2 *Let $(\mathcal{A}, -)$ be a structurable algebra. Then, with the triple product in (3.4), $KT(\mathcal{A})$ is a Lie triple system.*

Proof This is a consequence of [20, Lemma 1.8 and Corollary 1.4]. □

Remark 3.3 Let $(\mathcal{A}, -)$ be a structurable algebra. Consider the triple product $\{a, b, c\} = V_{a,b}(c)$ in (3.2) and, as in (2.3), the operators $K_{a,b} : c \mapsto \{a, c, b\} - \{b, c, a\}$. A simple computation gives:

$$\begin{aligned} K_{a,b}(c) &= \{a, c, b\} - \{b, c, a\} = V_{a,c}(b) - V_{b,c}(a) \\ &= (a\bar{c})b + (b\bar{c})a - (b\bar{a})c - (b\bar{c})a - (a\bar{c})b + (a\bar{b})c \\ &= (a\bar{b} - b\bar{a})c = L_{a\bar{b} - b\bar{a}}(c), \end{aligned} \tag{3.5}$$

for any $a, b, c \in \mathcal{A}$, where L_x denotes the left multiplication by x. Hence, $K(\mathcal{A}, \mathcal{A})$ (the linear span of the operators $K_{a,b}$) is just $L_{\mathcal{S}}$, where $\mathcal{S} = \{a \in \mathcal{A} \mid \bar{a} = -a\} = \mathrm{span}\{x - \bar{x} \mid x \in \mathcal{A}\}$ is the subspace of skew-symmetric elements for the involution.

[20, Corollary 1.4] shows that the following equation, which is equation (STS2) in [20], holds:

$$K_{a,b}V_{u,v} + V_{v,u}K_{a,b} = K_{K_{a,b}(u),v} \tag{3.6}$$

for any $a, b, u, v \in \mathcal{A}$. Equations (3.1a) and (3.6) show that $\mathcal{A}$ is a $(-1, 1)$ Freudenthal-Kantor triple system, i.e., a Kantor triple system.

Given a structurable algebra $(\mathcal{A}, -)$, consider the Lie triple system $\mathcal{T} = KT(\mathcal{A}) = \mathcal{A}_+ \oplus \mathcal{A}_-$ in (3.4). This Lie triple system $\mathcal{T}$ is graded by $\mathbb{Z}$ with $\mathcal{T}_1 = \mathcal{A}_+$ and $\mathcal{T}_{-1} = \mathcal{A}_-$. As a consequence, its standard embedding $\mathcal{L} = L(\mathcal{T}, \mathcal{T}) \oplus \mathcal{T}$ is $\mathbb{Z}$-graded with:

$$\mathcal{L}_{\pm 1} = \mathcal{T}_{\pm 1}, \quad \mathcal{L}_2 = L(\mathcal{T}_1, \mathcal{T}_1), \quad \mathcal{L}_{-2} = L(\mathcal{T}_{-1}, \mathcal{T}_{-1}), \quad \mathcal{L}_0 = L(\mathcal{T}_1, \mathcal{T}_{-1}).$$

In what follows we will identify $\mathrm{End}_{\mathbb{F}}(\mathcal{T})$ with the matrix algebra $\mathrm{Mat}_2\big(\mathrm{End}_{\mathbb{F}}(\mathcal{A})\big)$. With this identification, we compute elements in $L(\mathcal{T}, \mathcal{T})$:

- For $a, b, c \in \mathcal{A}$ we have $[a_+, b_-, c_+] = \{a, b, c\}_+ = V_{a,b}(c)_+$ and $[a_+, b_-, c_-] = -[b_-, a_+, c_-] = -V_{b,a}(c)_-$. Hence, the operator $L(a_+, b_-)$, when considered as an element in $\mathrm{Mat}_2\big(\mathrm{End}_{\mathbb{F}}(\mathcal{A})\big)$, is

$$L(a_+, b_-) = \begin{pmatrix} V_{a,b} & 0 \\ 0 & -V_{b,a} \end{pmatrix}.$$

We may consider, as in [2], the linear endomorphism ε on $\mathrm{End}_{\mathbb{F}}(\mathcal{A})$ given by

$$T^{\varepsilon} = T - L_{T(1)+\overline{T(1)}}.$$

For $x, y, z \in \mathcal{A}$ we have

$$\begin{aligned} V_{x,y}(z) + V_{y,x}(z) &= (x\overline{y})z + (z\overline{y})x - (z\overline{x})y + (y\overline{x})z + (z\overline{x})y - (z\overline{y})x \\ &= (x\overline{y} + y\overline{x})z = L_{x\overline{y}+y\overline{x}}(z). \end{aligned}$$

but $V_{x,y}(1) = x\overline{y} + \overline{y}x - \overline{x}y$, $\overline{V_{x,y}(1)} = y\overline{x} + \overline{x}y - \overline{y}x$, so $x\overline{y} + y\overline{x} = V_{x,y}(1) + \overline{V_{x,y}(1)}$, and hence we get $V_{x,y} + V_{y,x} = L_{V_{x,y}(1)+\overline{V_{x,y}(1)}}$, or

$$V^{\varepsilon}_{x,y} = -V_{y,x}.$$

Therefore, ε restricts to a linear automorphism of order 2 of the Lie algebra (*inner structure Lie algebra*) $\mathfrak{instrl}(\mathcal{A}, -) = \mathrm{span}\left\{V_{a,b} \mid a, b \in \mathcal{A}\right\}$. Equation (3.1a) shows that ε becomes an order two automorphism of the Lie algebra $\mathfrak{instrl}(\mathcal{A}, -)$.
With this notation, we get

$$L(a_+, b_-) = \begin{pmatrix} V_{a,b} & 0 \\ 0 & V^{\varepsilon}_{a,b} \end{pmatrix}, \tag{3.7}$$

and hence $\mathcal{L}_0 = L(\mathcal{T}_1, \mathcal{T}_{-1})$ is isomorphic to $\mathfrak{instrl}(\mathcal{A}, -)$, by means of the map $T \mapsto \begin{pmatrix} T & 0 \\ 0 & T^{\varepsilon} \end{pmatrix}$.

- Again, for $a, b, c \in \mathcal{A}$, we have $[a_+, b_+, c_-] = \{a, c, b\}_+ - \{b, c, a\}_+ = \big(L_{a\bar{b}-b\bar{a}}\big)(c)$ because of (3.5), which shows

$$L(a_+, b_+) = \begin{pmatrix} 0 & L_{a\bar{b}-b\bar{a}} \\ 0 & 0 \end{pmatrix}, \tag{3.8}$$

so that $\mathcal{L}_2$ is just $\begin{pmatrix} 0 & L_{\mathcal{S}} \\ 0 & 0 \end{pmatrix}$, which can be identified with $\mathcal{S}$ by means of the map $s \mapsto \begin{pmatrix} 0 & L_s \\ 0 & 0 \end{pmatrix}$.

- In the same vein,

$$L(a_-, b_-) = \begin{pmatrix} 0 & 0 \\ L_{a\bar{b}-b\bar{a}} & 0 \end{pmatrix}, \tag{3.9}$$

and $\mathcal{L}_{-2}$ can be identified too with $\mathcal{S}$: $s \mapsto \begin{pmatrix} 0 & 0 \\ L_s & 0 \end{pmatrix}$.

Our next result shows that the Lie algebra $\mathcal{K}(\mathcal{A}, -)$ constructed in [3] makes sense too assuming only that the characteristic is not 2. It must be remarked that in characteristic 3 it is no longer true that $\mathfrak{instrl}(\mathcal{A}, -)$ is the direct sum of $V_{\mathcal{A},1}$ and $\mathrm{Der}(\mathcal{A}, -)$, a result which is crucial in [3] and that we cannot use here.

Proposition 3.4 *Let $(\mathcal{A}, -)$ be a structurable algebra. Consider the Lie triple system $\mathcal{T} = KT(\mathcal{A})$ in (3.4) and let $\mathcal{L}$ be its standard embedding. Then $\mathcal{L}$ is isomorphic, as a $\mathbb{Z}$-graded Lie algebra, to the algebra defined on the vector space*

$$\mathcal{K}(\mathcal{A}, -) = \mathcal{N}^{\sim} \oplus \mathfrak{instrl}(\mathcal{A}, -) \oplus \mathcal{N},$$

where $\mathcal{N} = \mathcal{A} \times \mathcal{S} = \{(x, s) \mid x \in \mathcal{A},\ s \in \mathcal{S}\}$, $\mathcal{N}^{\sim}$ is a copy of $\mathcal{N}$, and the bracket is given by imposing that $\mathfrak{instrl}(\mathcal{A}, -)$ is a subalgebra, together with the following formulas:

$$\begin{aligned} [T, (x, s)] &= \big(T(x), T^{\delta}(s)\big), \\ [T, (x, s)^{\sim}] &= \big(T^{\varepsilon}(x), T^{\varepsilon\delta}(s)\big)^{\sim}, \\ [(x, s), (y, t)] &= (0, x\bar{y} - y\bar{x}), \\ [(x, s)^{\sim}, (y, t)^{\sim}] &= (0, x\bar{y} - y\bar{x})^{\sim}, \\ [(x, s), (y, t)^{\sim}] &= -(tx, 0)^{\sim} + \big(V_{x,y} + L_s L_t\big) + (sy, 0), \end{aligned} \tag{3.10}$$

for $x, y \in \mathcal{A}$, $s, t \in \mathcal{S}$, $T \in \mathfrak{instrl}(\mathcal{A}, -)$, where $T^{\delta} = T + R_{\overline{T(1)}}$, that is, $T^{\delta}(x) = T(x) + x\overline{T(1)}$ for any $T \in \mathfrak{instrl}(\mathcal{A}, -)$ and $x \in \mathcal{A}$.

Note that for $s, t \in \mathcal{S}$ and $x \in \mathcal{A}$, an easy computation gives

$$\begin{aligned} V_{st,1}(x) - V_{s,t}(x) &= (st)x + x(st) - x(\overline{st}) - (s\bar{t})x - (x\bar{t})s + (x\bar{s})t \\ &= (st)x + x(st) - x(ts) + (st)x + (xt)s - (xs)t \\ &= 2(st)x - (x, s, t) + (x, t, s) \\ &= 2\big((st)x - (s, t, x)\big) \quad \text{by skew-alternativity} \\ &= 2s(tx) = 2L_s L_t(x), \end{aligned}$$

so that $L_s L_t = \frac{1}{2}(V_{st,1} - V_{s,t})$ is indeed in $\mathfrak{instrl}(\mathcal{A}, -)$, and the bracket above is well defined. The $\mathbb{Z}$-grading on $\mathcal{K} = \mathcal{K}(\mathcal{A}, -)$ is given by $\mathcal{K}_2 = \{(0, s) \mid s \in \mathcal{S}\}$, $\mathcal{K}_1 = \{(x, 0) \mid x \in \mathcal{A}\}$, $\mathcal{K}_0 = \mathfrak{instrl}(\mathcal{A}, -)$, $\mathcal{K}_{-1} = \{(x, 0)^{\sim} \mid x \in \mathcal{A}\}$, and $\mathcal{K}_{-2} = \{(0, s)^{\sim} \mid s \in \mathcal{S}\}$.

Proof Define the degree-preserving linear isomorphism $\varphi : \mathcal{L} \to \mathcal{K}$ as follows:

- $\varphi(x_+) = (x, 0)$ and $\varphi(x_-) = (x, 0)^{\sim}$, for $x \in \mathcal{A}$,
- $\varphi\left(\begin{pmatrix} T & 0 \\ 0 & T^{\varepsilon} \end{pmatrix}\right) = T$, for $T \in \mathfrak{instrl}(\mathcal{A}, -)$,

- $\varphi\left(\begin{pmatrix} 0 & L_s \\ 0 & 0 \end{pmatrix}\right) = (0, s)$ and $\varphi\left(\begin{pmatrix} 0 & 0 \\ L_s & 0 \end{pmatrix}\right) = (0, s)^{\sim}$, for $s \in \mathcal{S}$.

Equations (3.7), (3.8), and (3.9) prove that $\varphi([X, Y]) = [\varphi(X), \varphi(Y)]$ for $X, Y \in \mathcal{L}_1 \cup \mathcal{L}_{-1}$. This is also trivially true for $X \in \mathcal{L}_1$ and $Y \in \mathcal{L}_{-2}$, or $X \in \mathcal{L}_{-1}$ and $Y \in \mathcal{L}_2$.

For $s \in \mathcal{S}$, (3.5) gives $K_{s,1}(x) = (s\overline{1} - 1\overline{s})x = 2sx$, so we get $K_{s,1} = 2L_s$. Equations (3.6) and (3.5) give

$$L_s V_{x,y} + V_{y,x} L_s = K_{sx,y} = L_{(sx)\overline{y} - y(\overline{sx})}.$$

Hence, for $x, y \in \mathcal{A}$ and $s \in \mathcal{S}$ we get the following bracket in $\mathcal{L}$:

$$\left[\begin{pmatrix} V_{x,y} & 0 \\ 0 & -V_{y,x} \end{pmatrix}, \begin{pmatrix} 0 & L_s \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & V_{x,y}L_s + L_s V_{y,x} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & L_{(sy)\overline{x} - x(\overline{sy})} \\ 0 & 0 \end{pmatrix},$$

but, since $\overline{s} = -s$, we compute

$$\begin{aligned}
(sy)\overline{x} - x(\overline{sy}) &= (sy)\overline{x} + x(\overline{y}s) \\
&= (s, y, \overline{x}) + s(y\overline{x}) - (x, \overline{y}, s) + (x\overline{y})s \\
&= V_{x,y}(s) - (s, \overline{y}, x) - s(\overline{y}x) + (s, \overline{x}, y) + s(\overline{x}y) \\
&\quad + (s, y, \overline{x}) + s(y\overline{x}) - (s, x, \overline{y}) \\
&= V_{x,y}(s) + s(y\overline{x} + \overline{x}y - \overline{y}x) \\
&\quad + (s, \overline{x}, y) + (s, y, \overline{x}) - (s, \overline{y}, x) - (s, x, \overline{y}) \\
&= V_{x,y}(s) + s(\overline{x\overline{y} + \overline{y}x - \overline{x}y}) \\
&= V_{x,y}(s) + s\overline{V_{x,y}(1)} = V^{\delta}_{x,y}(s),
\end{aligned}$$

because

$$\begin{aligned}
&(s, \overline{x}, y) + (s, y, \overline{x}) - (s, \overline{y}, x) - (s, x, \overline{y}) \\
&\quad = (s, \overline{x} - x, y) + (s, y, \overline{x} - x) - (s, \overline{y} - y, x) - (s, x, \overline{y} - y) = 0,
\end{aligned}$$

by skew-alternativity. Thus, for $x, y \in \mathcal{A}$ and $s \in \mathcal{S}$, we obtain the following bracket in $\mathcal{L}$:

$$\left[\begin{pmatrix} V_{x,y} & 0 \\ 0 & -V_{y,x} \end{pmatrix}, \begin{pmatrix} 0 & L_s \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & V^{\delta}_{x,y}(s) \\ 0 & 0 \end{pmatrix},$$

which shows that $\varphi([X, Y]) = [\varphi(X), \varphi(Y)]$ also for $X \in \mathcal{L}_0$ and $Y \in \mathcal{L}_2$.

In the same vein we compute, for $x, y \in \mathcal{A}$ and $s \in \mathcal{S}$,

$$\begin{aligned}
&\left[\begin{pmatrix} V_{x,y} & 0 \\ 0 & -V_{y,x} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ L_s & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 \\ -V_{y,x}L_s - L_s V_{x,y} & 0 \end{pmatrix} \\
&\quad = \begin{pmatrix} 0 & 0 \\ -L_{V_{y,x}(s) + s\overline{V_{y,x}(1)}} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ L_{V^{\varepsilon}_{x,y}(s) + s\overline{V^{\varepsilon}_{x,y}(1)}} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ L_{V^{\varepsilon\delta}_{x,y}(s)} & 0 \end{pmatrix}
\end{aligned}$$

and this shows that $\varphi([X, Y]) = [\varphi(X), \varphi(Y)]$ also for $X \in \mathcal{L}_0$ and $Y \in \mathcal{L}_{-2}$.

Finally, for $s, t \in \mathcal{S}$ we get the bracket

$$\left[\begin{pmatrix} 0 & L_s \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ L_t & 0 \end{pmatrix}\right] = \begin{pmatrix} L_s L_t & 0 \\ 0 & -L_t L_s \end{pmatrix}.$$

Also, for any $x, y \in \mathcal{A}$, $V_{x,1}(y) = xy + yx - y\overline{x} = V_{1,\overline{x}}(y)$, and hence

$$(L_s L_t)^{\varepsilon} = \frac{1}{2}(V_{st,1} - V_{s,t})^{\varepsilon} = \frac{1}{2}(-V_{1,st} + V_{t,s}) = \frac{1}{2}(-V_{ts,1} + V_{t,s}) = -L_t L_s.$$

We conclude that $\varphi([X,Y]) = [\varphi(X),\varphi(Y)]$ also for $X \in \mathcal{L}_2$ and $Y \in \mathcal{L}_{-2}$. ◻

Remark 3.5 A slightly different version of $\mathcal{K}(\mathcal{A},-)$ is given in [5, §6.4], the bracket in (3.10) is substituted by

$$\begin{aligned}
[T,(x,s)] &= \big(T(x), T^{\delta}(s)\big),\\
[T,(x,s)^{\sim}] &= \big(T^{\varepsilon}(x), T^{\varepsilon\delta}(s)\big)^{\sim},\\
[(x,s),(y,t)] &= \big(0, 2(x\overline{y}-y\overline{x})\big),\\
[(x,s)^{\sim},(y,t)^{\sim}] &= \big(0, 2(x\overline{y}-y\overline{x})\big)^{\sim},\\
[(x,s),(y,t)^{\sim}] &= -(tx,0)^{\sim} + \big(2V_{x,y} + L_sL_t\big) + (sy,0).
\end{aligned} \tag{3.11}$$

As remarked in [5, §6.4], the map

$$(y,t)^{\sim} + T + (x,s) \mapsto (y, \frac{1}{2}s)^{\sim} + T + (2x, 2s)$$

gives an isomorphism from the Lie algebra in (3.11) to the Lie algebra in (3.10).

The next results are taken from [8, §4.2], but one has to be a bit careful as the characteristic of the ground field here is just $\neq 2$. First, an easy lemma.

Lemma 3.6 *Let $\mathcal{L}$ be a 5-graded Lie algebra: $\mathcal{L} = \mathcal{L}_{-2}\oplus\mathcal{L}_{-1}\oplus\mathcal{L}_0\oplus\mathcal{L}_1\oplus\mathcal{L}_2$, and assume that there are elements $E \in \mathcal{L}_2$ and $F \in \mathcal{L}_{-2}$, such that the linear span of E, F, and $H = [E,F]$ is a subalgebra isomorphic to $\mathfrak{sl}_2$, with $\mathcal{L}_i = \{X \in \mathcal{L} \mid [H,X] = iX\}$, for $i = -2,-1,0,1,2$. (In particular $[H,E] = 2E$ and $[H,F] = -2F$.) Then the following properties hold:*

(i) *The linear map $\mathrm{ad}_F|_{\mathcal{L}_i}$ is one-to-one for $i = 1,2$. Actually, the map $\mathcal{L}_1 \to \mathcal{L}_{-1}$, $X \mapsto [F,X]$, is bijective with inverse $\mathcal{L}_{-1} \to \mathcal{L}_1$, $Y \mapsto [E,Y]$; and the map $\mathcal{L}_2 \to \mathcal{L}_{-2}$, $X \mapsto [F,[F,X]]$, is bijective with inverse $\mathcal{L}_{-2} \to \mathcal{L}_2$, $Y \mapsto \frac{1}{4}[E,[E,Y]]$.*

(ii) *Denote by $\mathfrak{sl}_2$ the subalgebra spanned by H, E, and F. Then we have*

$$\mathrm{Cent}_{\mathcal{L}}(\mathfrak{sl}_2) = \{X \in \mathcal{L}_0 \mid [F,X] = 0\} = \{X \in \mathcal{L}_0 \mid [E,X] = 0\},$$

where $\mathrm{Cent}_{\mathcal{L}}(\mathfrak{sl}_2)$ denotes the centralizer in $\mathcal{L}$ of the subalgebra $\mathfrak{sl}_2$.

(iii) *The subalgebra $\mathcal{L}_0$ splits as*

$$\mathcal{L}_0 = \mathrm{Cent}_{\mathcal{L}}(\mathfrak{sl}_2) \oplus [F,\mathcal{L}_2].$$

(iv) *The Lie algebra $\mathcal{L}$ splits as*

$$\mathcal{L} = \big(\mathcal{L}_2 \oplus [F,\mathcal{L}_2] \oplus \mathcal{L}_{-2}\big) \oplus \big(\mathcal{L}_1 \oplus \mathcal{L}_{-1}\big) \oplus \mathrm{Cent}_{\mathcal{L}}(\mathfrak{sl}_2),$$

and the subspace $\mathcal{L}_2 \oplus [F,\mathcal{L}_2] \oplus \mathcal{L}_{-2} = \sum_{X\in\mathcal{L}_2} \mathrm{span}\{X,[F,X],[F,[F,X]]\}$ is a sum of copies of the adjoint module for $\mathfrak{sl}_2$, the subspace $\mathcal{L}_1 \oplus \mathcal{L}_{-1} = \sum_{X\in\mathcal{L}_1} \mathrm{span}\{X,[F,X]\}$ is a sum of copies of its natural two-dimensional module, and $\mathrm{Cent}_{\mathcal{L}}(\mathfrak{sl}_2)$ is a sum of copies of the one-dimensional trivial module.
In particular the dimension of $\mathcal{L}$ coincides with $3\dim\mathcal{L}_2 + 2\dim\mathcal{L}_1 + \dim\mathrm{Cent}_{\mathcal{L}}(\mathfrak{sl}_2)$.

(v) *If the conditions $\mathcal{L}_2 = [\mathcal{L}_1,\mathcal{L}_1]$ and $\mathcal{L}_0 = [\mathcal{L}_1,\mathcal{L}_{-1}]$ hold, then we get*

$$\begin{aligned}
\mathrm{Cent}_{\mathcal{L}}(\mathfrak{sl}_2) &= \mathrm{span}\{[X,[F,Y]] + [Y,[F,X]] \mid X,Y \in \mathcal{L}_1\},\\
[F,\mathcal{L}_2] &= \mathrm{span}\{[X,[F,Y]] - [Y,[F,X]] \mid X,Y \in \mathcal{L}_1\}.
\end{aligned}$$

Proof If $X \in \mathcal{L}_i$, $i = 1, 2$, and $[F, X] = 0$, then we get $0 = [E, [F, X]] = [[E, F], X] = [H, X] = iX$, because $[E, X]$ belongs to $\mathcal{L}_{i+2} = 0$. For $X \in \mathcal{L}_1$, this argument, together with the analogous argument with F changed by E and i by $-i$, shows that the linear maps $\mathrm{ad}_F : \mathcal{L}_1 \to \mathcal{L}_{-1}$ and $\mathrm{ad}_E : \mathcal{L}_{-1} \to \mathcal{L}_1$ are bijective and one is the inverse of the other. Also, for $X \in \mathcal{L}_2$, we get

$$\begin{aligned}[E, [E, [F, [F, X]]]] &= [E, [F, [E, [F, X]]]]\\ &= [E, [F, [H, X]]] = 2[E, [F, X]] = 2[H, X] = 4X,\end{aligned}$$

where we have used that $[F, X]$ lies in $\mathcal{L}_0 = \{Z \in \mathcal{L} \mid [H, Z] = 0\}$ and that $[E, X] = 0$. Assertion (i) follows.

It is clear that $\mathrm{Cent}_{\mathcal{L}}(\mathfrak{sl}_2)$ lies in the centralizer of F, or of E. On the other hand, $\mathrm{Cent}_{\mathcal{L}}(\mathfrak{sl}_2)$ lies in the centralizer of H, which is $\mathcal{L}_0$. Now, for $X \in \mathcal{L}_0$ with $[F, X] = 0$ we get $0 = [E, [F, X]] = [F, [E, X]]$, but $\mathrm{ad}_F|_{\mathcal{L}_2}$ is one-to-one, so $[E, X] = 0$ follows. This proves (ii).

For (iii), if X lies in the intersection $\mathrm{Cent}_{\mathcal{L}}(\mathfrak{sl}_2) \cap [F, \mathcal{L}_2]$, then there is an element $Y \in \mathcal{L}_2$ such that $X = [F, Y]$ and $[E, X] = 0$, so that $0 = [E, X] = [E, [F, Y]] = [H, Y] = 2Y$, and hence $Y = 0$, and also $X = 0$. On the other hand, for $X \in \mathcal{L}_0$, an easy computation gives:

$$\begin{aligned}[F, [F, [E, X]]] &= -[F, [H, X]] + [F, [E, [F, X]]]\\ &= [F, [E, [F, X]]] \quad \text{because } [H, X] = 0,\\ &= -[H, [F, X]] + [E, [F, [F, X]]]\\ &= 2[F, X] \quad \text{because } [F, X] \in \mathcal{L}_{-2} \text{ and } [F, [F, X]] \in \mathcal{L}_{-4} = 0.\end{aligned}$$

Hence, we have

$$X = \left(X - \frac{1}{2}[F, [E, X]]\right) + \frac{1}{2}[F, [E, X]] \in \mathrm{Cent}_{\mathcal{L}}(\mathfrak{sl}_2) + [F, \mathcal{L}_2],$$

and (iii) follows.

The assertion in (iv) follows at once from the previous ones.

Finally, in case $\mathcal{L}_2 = [\mathcal{L}_1, \mathcal{L}_1]$ and $\mathcal{L}_0 = [\mathcal{L}_1, \mathcal{L}_{-1}]$, we get

$$\mathcal{L}_0 = [\mathcal{L}_1, \mathcal{L}_{-1}] = [\mathcal{L}_1, [F, \mathcal{L}_1]] = \mathrm{span}\,\{[X, [F, Y]] \mid X, Y \in \mathcal{L}_1\},$$

but for $X, Y \in \mathcal{L}_1$ we have

$$[X, [F, Y]] - [Y, [F, X]] = [[F, X], Y] + [X, [F, Y]] = [F, [X, Y]] \in [F, \mathcal{L}_2],$$

and

$$\Big[F, [X, [F, Y]] + [Y, [F, X]]\Big] = [[F, X], [F, Y]] + [[F, Y], [F, X]] = 0,$$

because $[F, [F, \mathcal{L}_1]] \in \mathcal{L}_{-3} = 0$. This means that $[X, [F, Y]] + [Y, [F, X]]$ belongs to $\mathrm{Cent}_{\mathcal{L}}(\mathfrak{sl}_2)$. The assertion in (v) follows at once from (iii) because $[X, [F, Y]] = \frac{1}{2}([X, [F, Y]] + [Y, [F, X]]) + \frac{1}{2}([X, [F, Y]] - [Y, [F, X]])$. □

Proposition 3.7 *Let $\mathcal{L}$ be a 5-graded Lie algebra: $\mathcal{L} = \mathcal{L}_{-2} \oplus \mathcal{L}_{-1} \oplus \mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2$, and assume that there are elements $E \in \mathcal{L}_2$ and $F \in \mathcal{L}_{-2}$, such that* $\mathrm{span}\,\{E, F, H = [E, F]\}$ *is a subalgebra isomorphic to $\mathfrak{sl}_2$, with $\mathcal{L}_i = \{X \in \mathcal{L} \mid [H, X] = iX\}$, for $i = -2, -1, 0, 1, 2$. Then $\mathcal{L}_1$ is a J-ternary algebra with the triple product*

$$(X, Y, Z) = \frac{1}{2}[[X, [F, Y]], Z]$$

for $X, Y, Z \in \mathcal{L}_1$.

Proof $\mathcal{L}$ splits, as a module for the Lie subalgebra $\mathfrak{sl}_2 = \operatorname{span}\{E, F, H\}$, as a direct sum of copies of the adjoint module, the natural module, and the trivial module, as shown in Lemma 3.6. Actually, for any $X \in \mathcal{L}_2$, $\operatorname{span}\{X, [F, X], [F, [F, X]]\}$ is (isomorphic to) the adjoint module, while for any $X \in \mathcal{L}_1$, $\operatorname{span}\{X, [F, X]\}$ is (isomorphic to) the natural module. The sum of the trivial modules, that is, the centralizer of $\mathfrak{sl}_2$, is contained in $\mathcal{L}_0$. Therefore, $\mathcal{L}$ is endowed with a short $\mathbf{SL}_2$-structure and can be written as in (2.10), with $V = \mathbb{F}p \oplus \mathbb{F}q$, $(p \mid q) = 1$, $H, E, F \in \mathfrak{sl}(V)$ with coordinate matrices $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, respectively, in the basis $\{p, q\}$. In particular $\mathcal{T}$ can be identified with $\mathcal{L}_1 = p \otimes \mathcal{T}$.

Now, $\mathcal{T}$ is a J-ternary algebra by Theorem 2.14, with triple product in (2.17), and for $X = p \otimes x$, $Y = p \otimes y$, and $Z = p \otimes z$ in $\mathcal{L}_1$, we use (2.11) to get

$$\begin{aligned} [[X, [F, Y]], Z] &= [[p \otimes x, [F, p \otimes y]], p \otimes z] \\ &= [[p \otimes x, q \otimes y], p \otimes z] \\ &= [\gamma_{p,q} \otimes \langle x \mid y \rangle + d_{x,y}, p \otimes z] \\ &= \gamma_{p,q}(p) \otimes \langle x \mid y \rangle \bullet z + p \otimes d_{x,y}(z) \\ &= p \otimes \Big(d_{x,y}(z) - \langle x \mid y \rangle \bullet z\Big) \\ &= p \otimes 2(x, y, z), \end{aligned}$$

thus proving our result. □

Theorem 3.8 *Let* $(\mathcal{A}, -)$ *be a structurable algebra and let* $s \in \mathcal{S}$ *be a skew-symmetric element such that the left multiplication* L_s *is bijective. Then* $\mathcal{A}$ *is a* J*-ternary algebra with the triple product*

$$(x, y, z) = V_{x,sy}(z)$$

for $x, y, z \in \mathcal{A}$.

Moreover, $\mathcal{S}$ *is a Jordan algebra with multiplication*

$$a \cdot b = \frac{1}{2}\big(a(sb) + b(sa)\big),$$

for $a, b \in \mathcal{S}$*;* $\mathcal{A}$ *is a special Jordan* $\mathcal{S}$*-module with*

$$a \bullet x = a(sx),$$

for $a \in \mathcal{S}$ *and* $x \in \mathcal{A}$*; and* $\mathcal{A}$ *is an* $\mathcal{S}$*-ternary algebra with the triple product above and with*

$$\langle x \mid y \rangle = y\bar{x} - x\bar{y},$$

for $x, y \in \mathcal{A}$.

Moreover, if t *is the element in* $\mathcal{A}$ *with* $L_s(t) = 1$*, then* t *is in* $\mathcal{S}$ *and the Lie algebra* $S(\mathcal{A}, \mathcal{A}) \leq \operatorname{End}_{\mathbb{F}}(\mathcal{A})^{(-)}$ *in* (2.19) *is given by*

$$S(\mathcal{A}, \mathcal{A}) = \{T \in \mathfrak{instrl}(\mathcal{A}, -) \mid T^{\delta}(t) = 0\}.$$

Proof Since L_s is invertible, if $t \in \mathcal{A}$ is the element such that $st = 1$ then, by skew-alternativity, we have $s(ts) = (st)s = s = s1$, which gives $L_s(ts - 1) = 0$ and $ts = 1$. Taking conjugates we get $-s\bar{t} = 1$, so $L_s(t + \bar{t}) = 0$ and $t \in \mathcal{S}$. Moreover, by skew-alternativity, we have the Moufang identity (proof as in [27, §2.1]) $s(t(sx)) = (sts)x = sx$ for any $x \in \mathcal{A}$. Thus $L_s L_t L_s = L_s$ and the invertibility of L_s gives $L_s L_t = L_t L_s = \mathrm{id}$.

Now, the elements $E = (0, t)$ and $F = (0, s)^{\sim}$ in $\mathcal{K}(\mathcal{A}, -)$ (Proposition 3.4) satisfy the hypotheses of Proposition 3.7, which imply that $\mathcal{K}(\mathcal{A}, -)_1 = \{(x, 0) \mid x \in \mathcal{A}\}$ is a J-ternary algebra with the triple product

$$
\begin{aligned}
((x,0),(y,0),(z,0)) &= \frac{1}{2}[[(x,0),[F,(y,0)]],(z,0)] = \frac{1}{2}[[(x,0),(sy,0)^{\sim}],(z,0)] \\
&= \frac{1}{2}[V_{x,sy},(z,0)] = \frac{1}{2}(V_{x,sy}(z),0),
\end{aligned}
$$

for $x, y, z \in \mathcal{A}$, and hence we obtain that $\mathcal{A}$ is a J-ternary algebra with the given triple product after scaling by 2. (Alternatively, the version of $\mathcal{K}(\mathcal{A}, -)$ in Remark 3.5 can be used to avoid the appearance of $\frac{1}{2}$ in the formula above.)

Let us consider now the operators in (2.3) for this J-ternary algebra (i.e., special $(1, 1)$ Freudenthal-Kantor triple system), where $(x, y, z) = V_{x,sy}(z)$. To distinguish it from the K-operators $K_{a,b}$ in (3.5) we will denote them by K^s remarking its dependence on the fixed element $s \in \mathcal{S}$. For $x, y, z \in \mathcal{A}$, we get

$$
\begin{aligned}
K^s(x,y)(z) &= (x,z,y) - (y,z,x) = V_{x,sz}(y) - V_{y,sz}(x) = K_{x,y}(sz) \\
&= L_{x\overline{y}-y\overline{x}}(sz) = L_{x\overline{y}-y\overline{x}}L_s(z),
\end{aligned}
$$

so that $K^s(\mathcal{A}, \mathcal{A}) = L_{\mathcal{S}}L_s$. As $L_tL_s = \mathrm{id}$, the Jordan algebra $\mathcal{J} = \mathbb{F}\mathrm{id} + K^s(\mathcal{A}, \mathcal{A})$ is just $\mathcal{J} = K^s(\mathcal{A}, \mathcal{A}) = L_{\mathcal{S}}L_s$.

For any $a \in \mathcal{S}$, by skew-alternativity we have, as before, the Moufang identity $L_aL_sL_a = L_{a(sa)}$, and hence, for any $a, b \in \mathcal{S}$, $L_aL_sL_b + L_bL_sL_a = L_{a(sb)+b(sa)}$, so that the Jordan product in $\mathcal{J} = L_{\mathcal{S}}L_s$ becomes

$$
\frac{1}{2}\big((L_aL_s)(L_bL_s) + (L_bL_s)(L_aL_s)\big) = \frac{1}{2}L_{a(sb)+b(sa)}L_s,
$$

for $a, b \in \mathcal{S}$. Transferring this Jordan product to $\mathcal{S}$ through the linear bijection $\mathcal{S} \to \mathcal{J}$, $a \mapsto L_aL_s$, shows that $\mathcal{S}$ is a Jordan algebra with the product

$$
a \cdot b = \frac{1}{2}\big(a(sb) + b(sa)\big),
$$

and that $\mathcal{A}$ is a special module for $\mathcal{S}$ with $a \bullet x = L_aL_s(x) = a(sx)$, for $a \in \mathcal{S}$ and $x \in \mathcal{A}$.

Theorem 2.8 shows that $\mathcal{A}$ is an $\mathcal{S}$-ternary algebra with the given triple product and with

$$
\langle x \mid y\rangle = y\overline{x} - x\overline{y}
$$

for $x, y \in \mathcal{A}$, because $-K^s(x, y) = -L_{x\overline{y}-y\overline{x}}L_s = L_{y\overline{x}-x\overline{y}}L_s$, which corresponds, through the linear bijection above, to the element $y\overline{x} - x\overline{y} \in \mathcal{S}$.

Finally, from the definition of the Lie algebra $S(\mathcal{A}, \mathcal{A})$ we get

$$
\begin{aligned}
S(\mathcal{A},\mathcal{A}) &= \mathrm{span}\left\{V_{x,sy} + V_{y,sx} \mid x, y \in \mathcal{A}\right\} \\
&= \mathrm{span}\left\{[(x,0),[(0,s)^{\sim},(y,0)]] + [(y,0),[(0,s)^{\sim},(x,0)]] \mid x, y \in \mathcal{A}\right\} \\
&= \mathrm{Cent}_{\mathcal{K}(\mathcal{A},-)}(\mathfrak{sl}_2) \quad \left(\mathfrak{sl}_2 = \mathrm{span}\left\{(0,t), \mathrm{id}, (0,s)^{\sim}\right\}\right) \\
&= \{T \in \mathfrak{instrl}(\mathcal{A},-) \mid [(0,t),T] = 0\} \\
&= \{T \in \mathfrak{instrl}(\mathcal{A},-) \mid T^{\delta}(t) = 0\},
\end{aligned}
$$

where we have used items (v) and (ii) of Lemma 3.6. (Note that $\mathcal{K}(\mathcal{A}, -)_0 = \mathfrak{instrl}(\mathcal{A}, -) = [\mathcal{K}(\mathcal{A}, -)_1, \mathcal{K}(\mathcal{A}, -)_{-1}]$ and $\mathcal{K}(\mathcal{A}, -)_2 = [\mathcal{K}(\mathcal{A}, -)_1, \mathcal{K}(\mathcal{A}, -)_1]$.) □

Example 3.9 Let $(\mathcal{A} = \mathcal{E} \oplus \mathcal{W}, -)$ be a structurable algebra of a hermitian form (see [6, Example 6.5]), that is,

- $(\mathcal{E}, -)$ is an associative algebra with involution, with multiplication denoted by juxtaposition,
- $\mathcal{W}$ is a left $\mathcal{E}$-module with action denoted by $e \circ x$ for $e \in \mathcal{E}$ and $x \in \mathcal{W}$,
- $\mathcal{W}$ is endowed with a hermitian form $\mathrm{h} : \mathcal{W} \times \mathcal{W} \to \mathcal{E}$, so that $\mathrm{h}(y, x) = \overline{\mathrm{h}(x, y)}$ and $\mathrm{h}(e \circ x, y) = e\mathrm{h}(x, y)$, for any $x, y \in \mathcal{W}$ and $e \in \mathcal{E}$,
- the involution on $\mathcal{A}$ is defined by $\overline{e + x} = \bar{e} + x$, for $e \in \mathcal{E}$ and $x \in \mathcal{W}$, that is, the involution on $\mathcal{E}$ is extended to an order 2 linear automorphism of $\mathcal{A}$ by imposing that $\mathcal{W}$ consists of symmetric elements.
- The multiplication in $\mathcal{A}$ is defined as follows:

$$(e_1 + x_1)(e_2 + x_2) = \big(e_1 e_2 + \mathrm{h}(x_2, x_1)\big) + \big(\overline{e_1} \circ x_2 + e_2 \circ x_1\big)$$

  for $e_1, e_2 \in \mathcal{E}$ and $x_1, x_2 \in \mathcal{W}$.

Take an element $s \in \mathcal{S} = \{x \in \mathcal{A} \mid \bar{x} = -x\} = \{a \in \mathcal{E} \mid \bar{a} = -a\}$, with L_s invertible. Theorem 3.8 shows that $\mathcal{S}$ is a Jordan algebra with $a \cdot b = \frac{1}{2}(asb + bsa)$ for $a, b \in \mathcal{S}$ ($\mathcal{E}$ is associative, so there is no need of parentheses), and that $\mathcal{A}$ is an $\mathcal{S}$-ternary algebra with

$$a \bullet x = a(sx), \quad \langle x \mid y \rangle = y\bar{x} - x\bar{y}, \quad (x, y, z) = V_{x,sy}(z),$$

for $a \in \mathcal{S}$, $x, y, z \in \mathcal{A}$.

Let us show that this ternary triple product is prototypical (Example 2.9).

To begin with, consider the *isotope* $\mathcal{E}^{(s)}$, which is the associative algebra defined on $\mathcal{E}$ with new multiplication $a * b := asb$ for any $a, b \in \mathcal{E}$, and with involution $\tau(a) = -\bar{a}$ for any $a \in \mathcal{E}$. Note that τ is indeed an involution because $\bar{s} = -s$. The unit element of $\mathcal{E}^{(s)}$ is $t = s^{-1}$, and the Jordan algebra of symmetric elements is $H(\mathcal{E}^{(s)}, \tau) = \{a \in \mathcal{E} \mid \tau(a) = a\} = \mathcal{S}$, with the product $\frac{1}{2}(a * b + b * a) = \frac{1}{2}(asb + bsa) = a \cdot b$ for $a, b \in \mathcal{S}$.

Moreover, $\mathcal{W}$ is a left $\mathcal{E}^{(s)}$-module with

$$a * x := (as) \circ x = a \circ (s \circ x)$$

for any $a \in \mathcal{E}$ and $x \in \mathcal{W}$. Hence, for any $a \in \mathcal{S}$, $x \in \mathcal{W}$, and $e \in \mathcal{E}$, $a \bullet x = a(sx) = \bar{a} \circ (\bar{s} \circ x) = a \circ (s \circ x) = a * x$, while $a \bullet e = ase = a * e$. We conclude that $\mathcal{A}$ is a left $\mathcal{E}^{(s)}$-module with action given by $a \bullet x$ for $a \in \mathcal{E}$ and $x \in \mathcal{A}$.

For $x, y \in \mathcal{W}$ and $a, b \in \mathcal{E}$, we have

$$\begin{aligned}
\langle x \mid y \rangle &= y\bar{x} - x\bar{y} = yx - xy = \mathrm{h}(x, y) - \mathrm{h}(y, x),\\
\langle a \mid x \rangle &= x\bar{a} - a\bar{x} = \bar{a} \circ x - \bar{a} \circ x = 0,\\
\langle a \mid b \rangle &= b\bar{a} - a\bar{b}.
\end{aligned}$$

Define a bilinear form

$$\tilde{\mathrm{h}} : \mathcal{A} \times \mathcal{A} \to \mathcal{E}^{(s)}$$

by imposing $\tilde{\mathrm{h}}(\mathcal{E}, \mathcal{W}) = \tilde{\mathrm{h}}(\mathcal{W}, \mathcal{E}) = 0$, $\tilde{\mathrm{h}}(x, y) = \mathrm{h}(x, y)$ for $x, y \in \mathcal{W}$, and $\tilde{\mathrm{h}}(a, b) = -a\bar{b}$ for $a, b \in \mathcal{E}$. By its own definition we get

$$\langle x \mid y \rangle = \tilde{\mathrm{h}}(x, y) - \tilde{\mathrm{h}}(y, x),$$

for any $x, y \in \mathcal{A}$. For $x, y \in \mathcal{W}$, $e, f \in \mathcal{E}$, and $a \in \mathcal{S}$, we have:

$$\begin{aligned}
\tilde{\mathrm{h}}(y, x) &= \mathrm{h}(y, x) = \overline{\mathrm{h}(x, y)} = -\tau\big(\tilde{\mathrm{h}}(x, y)\big),\\
\tilde{\mathrm{h}}(f, e) &= -f\bar{e} = -\overline{e\bar{f}} = \tau\big(e\bar{f}\big) = -\tau\big(\tilde{\mathrm{h}}(e, f)\big),\\
\tilde{\mathrm{h}}(a \bullet x, y) &= \mathrm{h}((as) \circ x, y) = (as)\mathrm{h}(x, y) = a \bullet \mathrm{h}(x, y) = a \bullet \tilde{\mathrm{h}}(x, y),\\
\tilde{\mathrm{h}}(a \bullet e, f) &= -ase\bar{f} = a \bullet \tilde{\mathrm{h}}(e, f).
\end{aligned}$$

This shows that $\tilde{\mathrm{h}}$ is a skew-hermitian form. We must finally check that $(x, y, z) = \tilde{\mathrm{h}}(x, y) \bullet z + \tilde{\mathrm{h}}(z, x) \bullet y + \tilde{\mathrm{h}}(z, y) \bullet x$ for any $x, y, z \in \mathcal{A}$. We split this into several cases.

For $x, y, z \in \mathcal{W}$ we get

$$\begin{aligned}
(x, y, z) &= V_{x,sy}(z) = \big(x(\overline{sy})\big)z + \big(z(\overline{sy})\big)x - \big(z\bar{x}\big)(sy)\\
&= -\big(x(ys)\big)z - \big(z(ys)\big)x - (zx)(sy)\\
&= -\mathrm{h}(s \circ y, x)z - \mathrm{h}(s \circ y, z)x - \mathrm{h}(x, z)(sy)\\
&= -\mathrm{h}(x, s \circ y) \circ z - \mathrm{h}(z, s \circ y) \circ x + \mathrm{h}(z, x) \circ (s \circ y) \quad (\text{as } ex = \bar{e} \circ x)\\
&= \big(\mathrm{h}(x, y)s\big) \circ z + \big(\mathrm{h}(z, y)s\big) \circ x + \mathrm{h}(z, x) \circ (s \circ y)\\
&= \tilde{\mathrm{h}}(x, y) \bullet z + \tilde{\mathrm{h}}(z, y) \bullet x + \tilde{\mathrm{h}}(z, x) \bullet y,
\end{aligned}$$

as desired.

For $e, f, g \in \mathcal{E}$ we get

$$\begin{aligned}
(e, f, g) &= V_{e,sf}(g) = \big(e(\overline{sf})\big)g + \big(g(\overline{sf})\big)e - (g\bar{e})(sf)\\
&= -e\bar{f}sg - g\bar{f}se - g\bar{e}sf\\
&= \tilde{\mathrm{h}}(e, f) \bullet g + \tilde{\mathrm{h}}(g, f) \bullet e + \tilde{\mathrm{h}}(g, e) \bullet f.
\end{aligned}$$

Now, for $x, y \in \mathcal{W}$ and $e \in \mathcal{E}$,

$$\begin{aligned}
(x, e, y) &= V_{x,se}(y) = \big(x(\overline{se})\big)y + \big(y(\overline{se})\big)x - (y\bar{x})(se)\\
&= -\big((\bar{e}s) \circ x\big)y - \big((\bar{e}s) \circ y\big)x - (yx)(se)\\
&= -\mathrm{h}(y, (\bar{e}s) \circ x) - \mathrm{h}(x, (\bar{e}s) \circ y) - \mathrm{h}(x, y)se\\
&= -\mathrm{h}(y, x)\bar{s}e - \mathrm{h}(x, y)\bar{s}e - \mathrm{h}(x, y)se\\
&= \tilde{\mathrm{h}}(y, x) \bullet e,
\end{aligned}$$

as required, because $\tilde{\mathrm{h}}(\mathcal{W}, \mathcal{E}) = 0$. We also have

$$(e, x, y) = (x, e, y) + \langle e \mid x\rangle \bullet y = (x, e, y) = \tilde{\mathrm{h}}(y, x) \bullet e$$

using (2.2d), and

$$(x, y, e) = (e, y, x) - \langle x \mid e\rangle \bullet y = (e, y, x) = (y, e, x) = \tilde{\mathrm{h}}(x, y) \bullet e$$

using (2.2c), (2.2d), and $\langle \mathcal{W} \mid \mathcal{E}\rangle = 0$.

Finally, for $e, f \in \mathcal{E}$ and $x \in \mathcal{W}$, we get

$$\begin{aligned}
(x, e, f) &= V_{x,se}(f) = \big(x(\overline{se})\big)f + \big(f(\overline{se})\big)x - (f\bar{x})(se)\\
&= -\big((\bar{e}s) \circ x\big)f - (f\bar{e}s)x - (se) \circ (fx)\\
&= -f \circ \big((\bar{e}s) \circ x\big) + (se\bar{f}) \circ x - (se\bar{f}) \circ x\\
&= -(f\bar{e}) \bullet x = \tilde{\mathrm{h}}(f, e) \bullet x
\end{aligned}$$

as desired, and also, using (2.2d), (2.2c), and $\langle \mathcal{W} \mid \mathcal{E} \rangle = 0$ we have

$$(e, x, f) = (x, e, f) + \langle e \mid x \rangle \bullet f = (x, e, f) = \tilde{\mathrm{h}}(f, e) \bullet x$$

and

$$(e, f, x) = (x, f, e) - \langle e \mid x \rangle \bullet f = (x, f, e) = \tilde{\mathrm{h}}(e, f) \bullet x.$$

We conclude that indeed, $\mathcal{A}$ is an $\mathcal{S}$-ternary algebra of prototypical type.

Also note that the structurable algebras of hermitian forms include the associative algebras with involution ($\mathcal{W} = 0$).

4 From algebras to superalgebras via tensor categories

This section is devoted to reviewing the process to get Lie superalgebras from Lie algebras in the symmetric tensor categories $\mathsf{Rep}\,\boldsymbol{\alpha}_p$ over fields of prime characteristic p. Details may be found in [12, 17, 25] and the references there in.

Given a symmetric tensor category $\mathfrak{C}$, an *operadic Lie algebra* is an object $\mathfrak{g}$ in $\mathfrak{C}$, with a morphism $\beta : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, such that the following relations hold:

$$\begin{aligned} &\beta \circ (\mathrm{id}_{\mathfrak{g}\otimes\mathfrak{g}} + c_{\mathfrak{g},\mathfrak{g}}) = 0 \quad \text{(anticommutativity)},\\ &\beta \circ (\beta \otimes \mathrm{id}_{\mathfrak{g}}) \circ \Big(\mathrm{id}_{\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}} + (\mathrm{id}_{\mathfrak{g}} \otimes c_{\mathfrak{g},\mathfrak{g}}) \circ (c_{\mathfrak{g},\mathfrak{g}} \otimes \mathrm{id}_{\mathfrak{g}})\\ &\qquad\qquad + (c_{\mathfrak{g},\mathfrak{g}} \otimes \mathrm{id}_{\mathfrak{g}}) \circ (\mathrm{id}_{\mathfrak{g}} \otimes c_{\mathfrak{g},\mathfrak{g}})\Big) = 0 \quad \text{(Jacobi identity)}, \end{aligned}$$

where $c_{X,Y}$ is the braiding $X \otimes Y \to Y \otimes X$ in $\mathfrak{C}$.

For instance, an operadic Lie algebra in the category of vector superspaces sVec over our ground field $\mathbb{F}$, where the braiding is given by $c(x \otimes y) = (-1)^{|x||y|} y \otimes x$ for homogeneous elements x and y, where $|x|$ is the parity: 0 or 1, of x, is a vector superspace $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, with a degree preserving bilinear map $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, $x \otimes y \to [x, y]$, satisfying the conditions

$$\begin{aligned} &[x, y] = -(-1)^{|x||y|}[y, x],\\ &[[x, y], z] + (-1)^{|x|(|y|+|z|)}[[y, z], x] + (-1)^{|z|(|x|+|y|)}[[z, x], y] = 0, \end{aligned}$$

for homogeneous elements $x, y, z \in \mathfrak{g}$.

Remark 4.1 There is a subtle point here, and that is the reason to use the adjective 'operadic'. In characteristic 3, an operadic Lie algebra in sVec is also called a *weak Lie superalgebra*. A Lie superalgebra is a weak Lie superalgebra satisfying the extra condition $[[x, x], x] = 0$ for any odd element x. (Note that this does not follow from the Jacobi identity in characteristic 3, being a stronger condition than the Jacobi identity for odd elements.)

Given a weak Lie superalgebra $\mathfrak{g}$, the subspace

$$\mathfrak{i} = \mathrm{span}\left\{[[x, x], x] \mid x \in \mathfrak{g}_{\bar{1}}\right\}$$

is an ideal of $\mathfrak{g}$, contained in $\mathfrak{g}_{\bar{1}}$, that satisfies $[\mathfrak{g}, \mathfrak{i}] = 0$, due to Jacobi identity. The quotient $\mathfrak{g}/\mathfrak{i}$ is a Lie superalgebra (with the same even part). In particular, if $\mathfrak{g}_{\bar{1}}$ is a sum of irreducible nontrivial modules for $\mathfrak{g}_{\bar{0}}$, then $\mathfrak{g}$ is a Lie superalgebra.

In the same vein, in characteristic 2, an operadic Lie algebra in sVec is a Lie superalgebra if it satisfies the extra condition $[x, x] = 0$ for any even element x.

If the characteristic is > 3, an operadic Lie algebra in sVec is just a Lie superalgebra.

Assume that the characteristic of the ground field $\mathbb{F}$ is $p > 0$. The affine group scheme $\boldsymbol{\alpha}_p$, looked at as a functor from the category of unital, commutative, associative algebras over $\mathbb{F}$ to the category of groups, is the functor that assigns to any R the additive group $\boldsymbol{\alpha}_p(R) = \{r \in R \mid r^p = 0\}$, and that acts naturally on morphisms. Its representing object is the Hopf algebra $\mathbb{F}[X]/(X^p)$. A representation of $\boldsymbol{\alpha}_p$ is nothing else but a vector space endowed with a nilpotent endomorphism δ satisfying $\delta^p = 0$. A Lie algebra in $\mathsf{Rep}\,\boldsymbol{\alpha}_p$ is just a Lie algebra $\mathcal{L}$ over $\mathbb{F}$ endowed with a nilpotent derivation $\delta \in \mathrm{Der}(\mathcal{L})$ with $\delta^p = 0$.

There are, up to isomorphism, p indecomposable objects in $\mathsf{Rep}\,\boldsymbol{\alpha}_p$: $L_1, \dots, L_p$, where L_i denotes the indecomposable object of dimension i, $1 \le i \le p$.

A morphism $f : X \to Y$ in $\mathsf{Rep}\,\boldsymbol{\alpha}_p$ is said to be *negligible* if $\mathrm{tr}(f \circ g) = 0$ for any morphism $g : Y \to X$. The symmetric tensor category $\mathsf{Rep}\,\boldsymbol{\alpha}_p$ is not semisimple. Its semisimplification, which is the category with the same objects but with morphisms the classes of morphisms in $\mathsf{Rep}\,\boldsymbol{\alpha}_p$ modulo the subspace of negligible ones, is the Verlinde category Ver_p. In Ver_p, the indecomposable objects $L_1, \dots, L_{p-1}$ become irreducible, while L_p is a zero object.

Assuming $p > 2$, the full tensor subcategory of Ver_p generated by L_1 and L_{p-1} is equivalent to the symmetric tensor category sVec of vector superspaces (over our ground field). Therefore, given a Lie algebra $\mathcal{L}$ over $\mathbb{F}$ endowed with a nilpotent derivation $\delta \in \mathrm{Der}(\mathcal{L})$ such that $\delta^p = 0$, that is, given a Lie algebra $\mathcal{L}$ in $\mathsf{Rep}\,\boldsymbol{\alpha}_p$, we obtain, by considering the Lie bracket modulo negligible homomorphisms, an operadic Lie algebra in Ver_p. Moreover, this operadic Lie algebra in Ver_p contains a subalgebra, which lies in the subcategory above, and hence it is an operadic Lie algebra in sVec. Note that if $p = 3$, the tensor categories Ver_3 and sVec are equivalent, because L_3 is a zero object in Ver_3.

Let us make these comments more precise.

Theorem 4.2 *Let $\mathcal{L}$ be a finite-dimensional Lie algebra over a field $\mathbb{F}$ of characteristic $p > 2$, endowed with a nilpotent derivation $\delta \in \mathrm{Der}(\mathcal{L})$ such that $\delta^p = 0$. Decompose $\mathcal{L}$ in a sum of indecomposable modules for the action of δ, so that*

$$\mathcal{L} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_p,$$

where $\mathcal{L}_i$ is a sum of indecomposable modules of dimension i for the action of δ, $1 \le i \le p$. Write $\mathcal{L}_{\bar 0} = \mathcal{L}_1$ and choose a subspace $\mathcal{L}_{\bar 1}$ of $\mathcal{L}_{p-1}$ complementing $\delta(\mathcal{L}_{p-1})$: $\mathcal{L}_{p-1} = \mathcal{L}_{\bar 1} \oplus \delta(\mathcal{L}_{p-1})$.

Define a bracket on $\mathcal{L}^{\mathrm{ss}} := \mathcal{L}_{\bar 0} \oplus \mathcal{L}_{\bar 1}$ as follows:

$$\begin{aligned}
[x_{\bar 0}, y_{\bar 0}] &= \mathrm{proj}_{\mathcal{L}_{\bar 0}}([x_{\bar 0}, y_{\bar 0}])\\
[x_{\bar 0}, y_{\bar 1}] &= \mathrm{proj}_{\mathcal{L}_{\bar 1}}([x_{\bar 0}, y_{\bar 1}])\\
[x_{\bar 1}, y_{\bar 0}] &= \mathrm{proj}_{\mathcal{L}_{\bar 1}}([x_{\bar 1}, y_{\bar 0}])\\
[x_{\bar 1}, y_{\bar 1}] &= \mathrm{proj}_{\mathcal{L}_{\bar 0}}\big([x_{\bar 1}, \delta^{p-2}(y_{\bar 1})]\big)
\end{aligned} \tag{4.1}$$

for all $x_{\bar 0}, y_{\bar 0} \in \mathcal{L}_{\bar 0}$ and $x_{\bar 1}, y_{\bar 1} \in \mathcal{L}_{\bar 1}$, where the projections are taken with respect to the decomposition

$$\mathcal{L} = \mathcal{L}_{\bar 0} \oplus \mathcal{L}_2 \oplus \cdots \oplus \mathcal{L}_{p-2} \oplus \mathcal{L}_{\bar 1} \oplus \delta(\mathcal{L}_{p-1}) \oplus \mathcal{L}_p. \tag{4.2}$$

Then $\mathcal{L}^{\mathrm{ss}}$, with this bracket, is an operadic Lie algebra in sVec, which is isomorphic to a subalgebra of the operadic Lie algebra in Ver_p obtained from the Lie algebra $\mathcal{L}$ in $\mathsf{Rep}\,\boldsymbol{\alpha}_p$.

Proof This is proved in [17, Recipe 2.8, Corollary 2.9, Remark 2.11]. □

We are interested in the following consequence of Theorem 4.2 for J-ternary algebras.

Corollary 4.3 *Let $\mathcal{T}$ be a J-ternary algebra over a field $\mathbb{F}$ of characteristic 3. Then the $\mathbb{Z}/2\mathbb{Z}$-graded vector space $\mathcal{L}^{ss}(\mathcal{T})$ with*

$$\mathcal{L}^{ss}(\mathcal{T})_{\bar{0}} = S(\mathcal{T}, \mathcal{T}), \quad \mathcal{L}^{ss}(\mathcal{T})_{\bar{1}} = \mathcal{T},$$

with $S(\mathcal{T}, \mathcal{T})$ in (2.19), is a weak Lie superalgebra with the bracket given by the usual bracket in $S(\mathcal{T}, \mathcal{T}) \leq \mathrm{End}_{\mathbb{F}}(\mathcal{T})$, by $[d, x] = d(x)$ for $d \in S(\mathcal{T}, \mathcal{T})$ and $x \in \mathcal{T}$, and by

$$[x, y] = S(x, y)$$

for $x, y \in \mathcal{T} = \mathcal{L}^{ss}(\mathcal{T})_{\bar{1}}$.

Proof Fix a symplectic basis $\{p, q\}$ in the two-dimensional vector space V in Theorem 2.14, that is, $(p \mid q) = 1$; take the endomorphism $F \in \mathfrak{sl}(V)$ with coordinate matrix $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ in this basis (i.e., $p \to q \to 0$) and consider the Lie algebra $\mathcal{L}(\mathcal{T})$ in (2.18). Then $\delta = \mathrm{ad}_{F \otimes \mathrm{id}}$ is a nilpotent derivation of $\mathcal{L}(\mathcal{T})$ with $\delta^3 = 0$. The summand $\mathfrak{sl}(V) \otimes \mathcal{J}$ in (2.18) is a sum of indecomposable modules of dimension 3 for the action of δ, the summand $V \otimes \mathcal{T}$ is a sum of indecomposable modules of dimension 2, while $S(\mathcal{T}, \mathcal{T})$ is annihilated by δ, and hence it is a sum of trivial (indecomposable) modules of dimension 1. Besides, $V \otimes \mathcal{T} = (p \otimes \mathcal{T}) \oplus \delta(V \otimes \mathcal{T})$, so that we get a decomposition as in (4.2), with

$$\mathcal{L}_{\bar{0}} = S(\mathcal{T}, \mathcal{T}), \quad \mathcal{L}_{\bar{1}} = p \otimes \mathcal{T}.$$

Note that $S(\mathcal{T}, \mathcal{T})$ is a subalgebra of $\mathcal{L}(\mathcal{T})$, and for $d \in S(\mathcal{T}, \mathcal{T})$ and $x \in \mathcal{T}$, $[d, p \otimes x] = p \otimes d(x)$, by (2.20), which also gives

$$\mathrm{proj}_{\mathcal{L}_{\bar{0}}}\big([p \otimes x, \delta(p \otimes y)]\big) = \mathrm{proj}_{S(\mathcal{T},\mathcal{T})}[p \otimes x, q \otimes y] = S(x, y),$$

for any $x, y \in \mathcal{T}$. The result then follows at once from Theorem 4.2, by identifying $p \otimes \mathcal{T}$ with $\mathcal{T}$ by means of $p \otimes x \leftrightarrow x$ for $x \in \mathcal{T}$. □

Remark 4.4 Corollary 4.3 can be proved without using the semisimplification process, but it has been this process the one that has allowed to realize that it was possible to define $\mathcal{L}^{ss}(\mathcal{T})$ and to check that this is a weak Lie superalgebra.

On the other hand, $\mathcal{L}^{ss}(\mathcal{T})$ may fail to be a bona fide Lie superalgebra. For example, let $\mathcal{T}$ be a two-dimensional vector space over a field $\mathbb{F}$ of characteristic 3 with basis $\{x, y\}$ and triple product determined by $xxx = y$ and $uvw = 0$ if at least one of u, v, or w equals y. With this triple product, $\mathcal{T}$ becomes trivially a J-ternary algebra. Besides, $S(x, x)$ takes x to $2y = -y$ and hence we have $[[x, x], x] = S(x, x)x = -y \neq 0$ in $\mathcal{L}^{ss}(\mathcal{T})$. Therefore, the weak Lie superalgebra $\mathcal{L}^{ss}(\mathcal{T})$ is not a Lie superalgebra.

5 From *J*-ternary algebras to Lie superalgebras in characteristic 3

Corollary 4.3 shows us how to get Lie superalgebras from J-ternary algebras over fields of characteristic 3. This section will give examples of this situation.

It must be remarked that there is no known classification of the simple finite-dimensional J-ternary algebras over fields of characteristic 3. The known classification [22], [23], needs characteristic $\neq 2, 3$. However, all the simple algebras that appear in these classifications are either of prototypical type or they are obtained from a structurable algebra with a skew-symmetric element with invertible left multiplication as in Theorem 3.8. Again, there is no known classification of the simple finite-dimensional structurable algebras over fields of characteristic 3, but the known simple algebras in other characteristics make sense in characteristic 3 (see [6, §6]).

5.1 Lie superalgebras from prototypical *J*-ternary algebras

Let us start with the Lie superalgebras obtained from simple J-ternary algebras fitting in the prototypical example (Example 2.9).

Theorem 5.1 *Let $(\mathcal{A}, -)$ be a finite-dimensional simple algebra with involution (i.e., there is no proper ideal closed under the involution!) over an algebraically closed field $\mathbb{F}$ of characteristic 3. Let $\mathcal{T}$ be a left $\mathcal{A}$-module and let $\mathrm{h} : \mathcal{T} \times \mathcal{T} \to \mathcal{A}$ be a nondegenerate skew-hermitian form. Then $\mathcal{T}$ is a J-ternary algebra with triple product*

$$(x, y, z) = \mathrm{h}(x, y)z + \mathrm{h}(z, x)y + \mathrm{h}(z, y)x$$

for $x, y, z \in \mathcal{T}$. Let $\mathfrak{L}^{\mathrm{ss}}(\mathcal{T})$ be the associated weak Lie superalgebra in Corollary 4.3. Then $\mathfrak{L}^{\mathrm{ss}}(\mathcal{T})$ is a Lie superalgebra and the following conditions hold:

- *If the involution $-$ on $\mathcal{A}$ is of the first kind, then there are $n, m \in \mathbb{Z}_{>0}$, with m even, such that $\mathfrak{L}^{\mathrm{ss}}(\mathcal{T})$ is isomorphic to the orthosymplectic Lie superalgebra $\mathfrak{osp}(n|m)$.*
- *If the involution $-$ on $\mathcal{A}$ is of the second kind, then there are $n, m \in \mathbb{Z}_{>0}$ such that $\mathfrak{L}^{\mathrm{ss}}(\mathcal{T})$ is isomorphic to the projective special linear Lie superalgebra $\mathfrak{psl}(n|m)$.*

Proof The proof will be done in several steps.

Assume first that the involution is of the first kind and orthogonal. Then the algebra $\mathcal{A}$ is simple and there exists a finite-dimensional vector space X over $\mathbb{F}$, endowed with a nondegenerate symmetric bilinear form $\mathrm{b}_X : X \times X \to \mathbb{F}$, such that $(\mathcal{A}, -)$ is isomorphic to $(\mathrm{End}_{\mathbb{F}}(X), -)$, with the involution given by the adjunction relative to b_X:

$$\mathrm{b}_X\big(f(u), v\big) = b_X(u, \overline{f}(v)\big)$$

for any $u, v \in X$ and $f \in \mathrm{End}_{\mathbb{F}}(X)$. We may then assume that $\mathcal{A} = \mathrm{End}_{\mathbb{F}}(X)$ with this involution.

Any finite-dimensional simple left module is, up to isomorphism, of the form $\mathcal{T} = X \otimes Y$ for a finite-dimensional vector space Y, where the action of $\mathcal{A}$ is on X. Let $\mathrm{h} : \mathcal{T} \times \mathcal{T} \to \mathcal{A}$ be a nondegenerate skew-hermitian form. The left $\mathcal{A}$ module $\mathcal{T}$ is also a right module with $ta := \overline{a}t$ for any $a \in \mathcal{A}$ and $t \in \mathcal{T}$. Then h may be seen as a homomorphism of $\mathcal{A}$-bimodules $\mathrm{h} : \mathcal{T} \otimes \mathcal{T} \to \mathcal{A}$. Up to scalars, the unique homomorphism of $\mathcal{A}$-bimodules $X \otimes X \to \mathcal{A} = \mathrm{End}_{\mathbb{F}}(X)$ is given by $x_1 \otimes x_2 \mapsto x_1 \mathrm{b}_X(x_2, \cdot)$, for $x_1, x_2 \in X$. As a consequence, there exists a bilinear form $\mathrm{b}_Y : Y \times Y \to \mathbb{F}$ such that

$$\mathrm{h}(x_1 \otimes y_1, x_2 \otimes y_2) = \mathrm{b}_Y(y_1, y_2) x_1 \mathrm{b}_X(x_2, \cdot),$$

for any $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. The fact that h is nondegenerate and skew-symmetric implies that b_Y is a nondegenerate skew-symmetric bilinear form. Let us compute the action of the operators

$$S(u, v) = L(u, v) + L(v, u) = (u, v, \cdot) + (v, u, \cdot) \in \mathrm{End}_{\mathbb{F}}(\mathcal{T}),$$

for $u_1 = x_1 \otimes y_1$ and $u_2 = x_2 \otimes y_2$ as above:

$$\begin{aligned}
S(x_1 \otimes y_1, x_2 \otimes y_2)(x \otimes y)&\\
&= \big(\mathrm{h}(x_1 \otimes y_1, x_2 \otimes y_2) + \mathrm{h}(x_2 \otimes y_2, x_1 \otimes y_1)\big)(x \otimes y)\\
&\quad + 2\big(\mathrm{h}(x \otimes y, x_1 \otimes y_1)(x_2 \otimes y_2) + \mathrm{h}(x \otimes y, x_2 \otimes y_2)(x_1 \otimes y_1)\big)\\
&= \big(\mathrm{h}(x_1 \otimes y_1, x_2 \otimes y_2) + \mathrm{h}(x_2 \otimes y_2, x_1 \otimes y_1)\big)(x \otimes y)\\
&\quad - \big(\mathrm{h}(x \otimes y, x_1 \otimes y_1)(x_2 \otimes y_2) + \mathrm{h}(x \otimes y, x_2 \otimes y_2)(x_1 \otimes y_1)\big)\\
&= x_1\mathrm{b}_X(x_2, x) \otimes \mathrm{b}_Y(y_1, y_2)y + x_2\mathrm{b}_X(x_1, x) \otimes \mathrm{b}_Y(y_2, y_1)y\\
&\quad - x\mathrm{b}_X(x_1, x_2) \otimes \mathrm{b}_Y(y, y_1)y_2 - x\mathrm{b}_X(x_2, x_1) \otimes \mathrm{b}_Y(y, y_2)y_1\\
&= \big(x_1\mathrm{b}_X(x_2, x) - x_2\mathrm{b}_X(x_1, x)\big) \otimes \mathrm{b}_Y(y_1, y_2)y\\
&\quad + \mathrm{b}_X(x_1, x_2)x \otimes \big(\mathrm{b}_Y(y_1, y)y_2 + \mathrm{b}_Y(y_2, y)y_1\big)
\end{aligned}$$

for $x \in X$ and $y \in Y$. Hence, we have

$$S(x_1 \otimes y_1, x_2 \otimes y_2) = \sigma_{x_1,x_2} \otimes \mathrm{b}_Y(y_1, y_2)\mathrm{id}_Y + \mathrm{b}_X(x_1, x_2)\mathrm{id}_X \otimes \gamma_{y_1,y_2} \tag{5.1}$$

where

$$\sigma_{x_1,x_2} = x_1\mathrm{b}_X(x_2, \cdot) - x_2\mathrm{b}_X(x_1, \cdot) \quad \text{and} \quad \gamma_{y_1,y_2} = y_1\mathrm{b}_Y(y_2, \cdot) + y_2\mathrm{b}_Y(y_1, \cdot).$$

(Note that the operators σ_{x_1,x_2} span the orthogonal Lie algebra of (X, b_X), and the operators γ_{y_1,y_2} span the symplectic Lie algebra of (Y, b_Y).)

Consider the vector superspace with even part X, odd part Y, and endowed with the supersymmetric even bilinear form $\mathbf{b}$ such that

$$\mathbf{b}|_X = \mathrm{b}_X \quad \text{and} \quad \mathbf{b}|_Y = -\mathrm{b}_Y.$$

The corresponding orthosymplectic Lie superalgebra is given by

$$\mathfrak{osp}(X|Y, \mathbf{b})_i = \{\varphi \in \mathfrak{gl}(X|Y)_i \mid \mathbf{b}(\varphi(u), v) + (-1)^{i|u|}\mathbf{b}(u, \varphi(v)) = 0\ \forall u, v \in X \cup Y\}$$

for $i = \bar{0}, \bar{1}$, where $|u|$ is the parity of the homogeneous element u. This is spanned by the operators

$$\Lambda_{u,v} = u\mathbf{b}(v, \cdot) - (-1)^{|u||v|}v\mathbf{b}(u, \cdot),$$

for $u, v \in X \cup Y$. The odd part of $\mathfrak{osp}(X|Y, \mathbf{b})$ is $\Lambda_{X,Y}$ which can be identified with $\mathcal{T} = X \otimes Y$ by means of $\Lambda_{x,y} \leftrightarrow x \otimes y$. An easy computation gives

$$\begin{aligned}
[\Lambda_{x_1,y_1}, \Lambda_{x_2,y_2}] &= (x_1\mathbf{b}(y_1, \cdot) - y_1\mathbf{b}(x_1, \cdot)) \circ (x_2\mathbf{b}(y_2, \cdot) - y_2\mathbf{b}(x_2, \cdot))\\
&\quad + (x_2\mathbf{b}(y_2, \cdot) - y_2\mathbf{b}(x_2, \cdot)) \circ (x_1\mathbf{b}(y_1, \cdot) - y_1\mathbf{b}(x_1, \cdot))\\
&= -x_1\mathbf{b}(y_1, y_2)\mathbf{b}(x_2, \cdot) - y_1\mathbf{b}(x_1, x_2)\mathbf{b}(y_2, \cdot)\\
&\quad - x_2\mathbf{b}(y_2, y_1)\mathbf{b}(x_1, \cdot) - y_2\mathbf{b}(x_2, x_1)\mathbf{b}(y_1, \cdot)\\
&= \mathrm{b}_Y(y_1, y_2)\sigma_{x_1,x_2} + \mathrm{b}_X(x_1, x_2)\gamma_{y_1,y_2}
\end{aligned}$$

for $x_1, x_2 \in X$, $y_1, y_2 \in Y$. Comparing with (5.1), it turns out that, after identifying $\mathcal{T}$ with $\mathfrak{osp}(X|Y, \mathbf{b})_{\bar{1}}$ as above, we have

$$S(x_1 \otimes y_1, x_2 \otimes y_2) = \mathrm{ad}([\Lambda_{x_1,y_1}, \Lambda_{x_2,y_2}])|_{\mathcal{T}}$$

and this shows that $S(\mathcal{T}, \mathcal{T})$ coincides with $\mathrm{ad}(\mathfrak{osp}(X|Y, \mathbf{b})_{\bar{0}})|_{\mathcal{T}}$ and, since the action of $\mathfrak{osp}(X|Y, \mathbf{b})_{\bar{0}}$ on $\mathfrak{osp}(X|Y, \mathbf{b})_{\bar{1}}$ is faithful, that $\mathcal{L}^{ss}(\mathcal{T}) = S(\mathcal{T}, \mathcal{T}) \oplus \mathcal{T}$ is isomorphic to $\mathfrak{osp}(X|Y, \mathbf{b})$.

The situation for a symplectic involution is completely analogous.

Let us consider finally the situation of a second kind involution. In this case, up to isomorphism, we may assume $\mathcal{A} = \mathcal{B} \oplus \mathcal{B}^{\mathrm{op}}$, where $\mathcal{B}^{\mathrm{op}}$ is the opposite algebra, and that the involution is given by $\overline{b_1 + b_2^{\mathrm{op}}} = b_2 + b_1^{\mathrm{op}}$, where we write b^{op} for the element $b \in \mathcal{B}$ when considered as an element of $\mathcal{B}^{\mathrm{op}}$. A left $\mathcal{A}$-module $\mathcal{T}$ is then, up to isomorphism, the direct sum of a left $\mathcal{B}$-module and a left $\mathcal{B}^{\mathrm{op}}$-module (i.e., a right $\mathcal{B}$-module): $\mathcal{T} = \mathcal{M} \oplus \mathcal{N}$. Denote by $\rho\colon \mathcal{A} \to \mathrm{End}_{\mathbb{F}}(\mathcal{T})$ the associated representation. Write $e_1 = (1, 0)$ and $e_2 = (0, 1^{\mathrm{op}})$.

If $\mathrm{h} : \mathcal{T} \times \mathcal{T} \to \mathcal{A}$ is a skew-hermitian form, then we get $\mathrm{h}(\mathcal{M}, \mathcal{M}) = \mathrm{h}(e_1\mathcal{T}, e_1\mathcal{T}) = e_1\mathrm{h}(\mathcal{T}, \mathcal{T})\overline{e_1} \subseteq e_1\mathcal{A}e_2 = 0$, and also $\mathrm{h}(\mathcal{N}, \mathcal{N}) = 0$. Moreover, we have $\mathrm{h}(\mathcal{M}, \mathcal{N}) \subseteq e_1\mathcal{A}e_1 = \mathcal{B}$ and $\mathrm{h}(\mathcal{N}, \mathcal{M}) \subseteq \overline{\mathcal{B}} = \mathcal{B}^{\mathrm{op}}$. In other words, there is a homomorphism of $\mathcal{B}$-bimodules $\mu\colon \mathcal{M} \otimes \mathcal{N} \to \mathcal{B}$ such that $\mathrm{h}(m, n) = \mu(m \otimes n) \in \mathcal{B}$ and $\mathrm{h}(n, m) = -\overline{\mathrm{h}(m, n)} = -\overline{\mu(m \otimes n)} = -\mu(m \otimes n)^{\mathrm{op}} \in \mathcal{B}^{\mathrm{op}}$.

For $x, y, z \in \mathcal{T}$, the operator $S(x, y) \in \mathrm{End}_{\mathbb{F}}(\mathcal{T})$ works as follows:

$$
\begin{aligned}
S(x, y)(z) &= (x, y, z) + (y, x, z)\\
&= \big(\mathrm{h}(x, y) + \mathrm{h}(y, x)\big)z + 2\big(\mathrm{h}(z, x)y + \mathrm{h}(z, y)x\big)\\
&= \big(\mathrm{h}(x, y) - \overline{\mathrm{h}(x, y)}\big)z - \big(\mathrm{h}(z, x)y + \mathrm{h}(z, y)x\big).
\end{aligned}
$$

For $x, y \in \mathcal{M}$, $\mathrm{h}(x, y) = 0$ and $\mathrm{h}(z, x)y, \mathrm{h}(z, y)x \in \mathcal{B}^{\mathrm{op}}\mathcal{M} = 0$, and the same happens for $x, y \in \mathcal{N}$. Now, for $x \in \mathcal{M}$ and $y \in \mathcal{N}$, $\mathrm{h}(x, y) = \mu(x \otimes y) \in \mathcal{B}$. In other words, we have, for $x \in \mathcal{M}$ and $y \in \mathcal{N}$:

$$
\begin{aligned}
S(x, y) &= \rho\big(\mathrm{h}(x, y) - \mathrm{h}(x, y)^{\mathrm{op}}\big) - \mathrm{h}(\cdot, x)y - \mathrm{h}(\cdot, y)x :\\
z &\mapsto \begin{cases} \mu(x \otimes y)z - \mu(z \otimes y)x, & \text{if } z \in \mathcal{M},\\ -z\mu(x \otimes y) + y\mu(x \otimes z), & \text{if } z \in \mathcal{N}. \end{cases}
\end{aligned}
\tag{5.2}
$$

Now, $\mathcal{B}$ is a finite-dimensional simple algebra, so we may assume $\mathcal{B} = \mathrm{End}_{\mathbb{F}}(X)$ for a finite-dimensional vector space X. The finite-dimensional left $\mathcal{B}$-module $\mathcal{M}$ is, up to isomorphism, of the form $X \otimes Z$ for a vector space Z, where the action on $\mathcal{B}$ is on X. In the same vein, the right $\mathcal{B}$-module $\mathcal{N}$ is, up to isomorphism, of the form $Y \otimes X^*$, where the action of $\mathcal{B}$ is on the dual X^*, that is, $\varphi b\colon x \mapsto \varphi(bx)$ for $\varphi \in X^*$, $b \in \mathcal{B}$, and $x \in X$. As a bimodule for $\mathcal{B}$, $X \otimes X^*$ is isomorphic to $\mathcal{B} = \mathrm{End}_{\mathbb{F}}(X)$, with $x \otimes \varphi$ identified with the linear map $z \mapsto x\varphi(z)$, and the homomorphism of $\mathcal{B}$-bimodules $\mu : \mathcal{M} \otimes \mathcal{N} \to \mathcal{B}$ is of the form

$$
(x \otimes z) \otimes (y \otimes \varphi) \mapsto \eta(z, y)x \otimes \varphi,
$$

for $x \in X$, $\varphi \in X^*$, $y \in Y$ and $z \in Z$, for a bilinear form $\eta : Z \times Y \to \mathbb{F}$.

The skew-hermitian form h is nondegenerate if and only if so is the bilinear form η, and then η can be used to identify Z with Y^*.

Summarizing, for involutions of the second kind, we may assume that $\mathcal{A} = \mathcal{B} \oplus \mathcal{B}^{\mathrm{op}}$, the involution being the exchange involution, with $\mathcal{B} = \mathrm{End}_{\mathbb{F}}(X)$ for a finite-dimensional vector space X, while $\mathcal{T} = (X \otimes Y^*) \oplus (Y \otimes X^*)$ for another finite-dimensional vector space Y. Moreover, the skew-hermitian form h satisfies that $X \otimes Y^*$ and $Y \otimes X^*$ are isotropic for h, while we have

$$
\mathrm{h}(x \otimes \omega, y \otimes \varphi) = \mu\big((x \otimes \omega) \otimes (y \otimes \varphi)\big) = \omega(y)x \otimes \varphi,
$$

for $x \in X$, $\varphi \in X^*$, $y \in Y$, and $\omega \in Y^*$.

Equation (5.2) takes now the following form:

$$S(x\otimes\omega, y\otimes\varphi)\colon \begin{cases} x'\otimes\omega'\mapsto \omega(y)\varphi(x')x\otimes\omega' - \omega'(y)\varphi(x)x'\otimes\omega,\\ y'\otimes\varphi'\mapsto -\omega(y)\varphi'(x)y'\otimes\varphi + \omega(y')\varphi(x)y\otimes\varphi',\end{cases} \tag{5.3}$$

for $x, x'\in X$, $\varphi,\varphi'\in X^*$, $y, y'\in Y$, and $\omega,\omega'\in Y^*$.

Consider the vector superspace with even part X and odd part Y, and the corresponding general linear Lie superalgebra $\mathfrak{gl}(X|Y)=\mathrm{End}_{\mathbb{F}}(X\oplus Y)$, with its natural $\mathbb{Z}/2\mathbb{Z}$-grading and bracket. Use the natural identifications

$$\mathfrak{gl}(X|Y)_{\bar 0} = (X\otimes X^*)\oplus(Y\otimes Y^*),$$
$$\mathfrak{gl}(X|Y)_{\bar 1} = (X\otimes Y^*)\oplus(Y\otimes X^*)\ (=\mathcal{T}).$$

Note that $\mathfrak{gl}(X|Y)$ is 3-graded with

$$\mathfrak{gl}(X|Y)_{-1}=X\otimes Y^*,\quad \mathfrak{gl}(X|Y)_0=(X\otimes X^*)\oplus(Y\otimes Y^*),\quad \mathfrak{gl}(X|Y)_1=Y\otimes X^*.$$

For any $x\in X$, $\varphi\in X^*$, $y\in Y$, and $\omega\in Y^*$, the bracket in $\mathfrak{gl}(X|Y)$ of the odd elements $x\otimes\omega$ and $y\otimes\varphi$ is

$$[x\otimes\omega, y\otimes\varphi] = (x\otimes\omega)\circ(y\otimes\varphi)+(y\otimes\varphi)\circ(x\otimes\omega)=\omega(y)x\otimes\varphi+\varphi(x)y\otimes\omega.$$

These elements span the even part of the special linear Lie superalgebra $\mathfrak{sl}(X|Y)$.

The action of $[x\otimes\omega, y\otimes\varphi]$ on the elements of $\mathfrak{gl}(X|Y)_{\bar 1}=\mathcal{T}$ works as follows:

$$\begin{aligned}[][[x\otimes\omega, y\otimes\varphi], x'\otimes\omega'] &= [\omega(y)x\otimes\varphi+\varphi(x)y\otimes\omega, x'\otimes\omega']\\ &= \omega(y)\varphi(x')x\otimes\omega' - \varphi(x)\omega'(y)x'\otimes\omega,\\ [[x\otimes\omega, y\otimes\varphi], y'\otimes\varphi'] &= [\omega(y)x\otimes\varphi+\varphi(x)y\otimes\omega, y'\otimes\varphi']\\ &= \varphi(x)\omega(y')y\otimes\varphi' - \omega(y)\varphi'(x)y'\otimes\varphi,\end{aligned}$$

for $x, x'\in X$, $\varphi,\varphi'\in X^*$, $y, y'\in Y$, and $\omega,\omega'\in Y^*$, so we conclude that

$$S(x\otimes\omega, y\otimes\varphi)=\mathrm{ad}([x\otimes\omega, y\otimes\varphi])|_{\mathcal{T}},$$

and this shows that $S(\mathcal{T},\mathcal{T})$ equals $\mathrm{ad}(\mathfrak{sl}(X|Y)_{\bar 0})|_{\mathcal{T}}$, which is isomorphic to the even part of the projective special linear Lie superalgebra $\mathfrak{psl}(X|Y)$, and that $\mathcal{L}^{ss}(\mathcal{T})=S(\mathcal{T},\mathcal{T})\oplus\mathcal{T}$ is isomorphic to the projective special linear Lie superalgebra $\mathfrak{psl}(X|Y)$, as required. □

5.2 Lie superalgebras from simple structurable algebras

Now we will turn our attention to the J-ternary algebras obtained from finite-dimensional simple structurable algebras with a suitable regular skew-symmetric element (Theorem 3.8). Example 3.9 shows that, if our structurable algebra is an associative algebra with involution or the structurable algebra of a hermitian form, the corresponding J-ternary algebra is prototypical. If the structurable algebra is a Jordan algebra (with identity involution), then there are no nonzero skew-symmetric elements.

On the other hand, if $(\mathcal{A},-)$ is a simple finite-dimensional structurable algebra with one-dimensional space of skew-symmetric elements, then $\mathcal{S}(\mathcal{A},-)=\mathbb{F}s$ for a skew-symmetric element s with invertible L_s and with $s^2\in\mathbb{F}1$. Denote by $\mathcal{T}$ the associated J-ternary algebra in Theorem 3.8. This last theorem and (2.2c) show that the operators $K(x,y)\colon z\mapsto (x,z,y)-(y,z,x)$ are scalar multiples of the identity. Hence, the Jordan algebra $\mathcal{J}=\mathbb{F}\mathrm{id}+K(\mathcal{T},\mathcal{T})$

in Theorem 2.14 is just the one-dimensional unital Jordan algebra. The outcome is that $\mathcal{T}$ is then a so called *symplectic triple system*, and the associated Lie superalgebra $\mathcal{L}^{ss}(\mathcal{T})$ has been computed in [14, Theorem 3.2]. The Lie superalgebras that appear are the Brown superalgebra $\mathfrak{br}(2;3)$ of superdimension $(10|8)$, i.e., the even part has dimension 10 and the odd part has dimension 8 (see [16]), and the Lie superalgebras $\mathfrak{g}(r,6)$, $r=1,2,4,8$, in the "supermagic square" in [10] (see [11, Corollary 5.9]). The notation above for these superalgebras follows [9].

We are left with two types of simple structurable algebras, assuming the ground field is algebraically closed: the tensor product of two unital composition algebras, with their natural involutions, such that one of the factors is a Cayley algebra (dimension 8), as otherwise the algebra is associative and hence the associated J-ternary algebra is prototypical, or the 35-dimensional Smirnov algebra [29].

For the Smirnov algebra, this may be defined as the subalgebra $T(\mathcal{C})$ of the tensor product $\mathcal{C}\otimes\mathcal{C}$, where $\mathcal{C}$ is the Cayley algebra over $\mathbb{F}$, obtained as the kernel of the map $S^2(\mathcal{C})\to\mathbb{F}$ that takes $x\otimes x$ to $\mathrm{n}(x)$ (its norm), where $S^2(\mathcal{C})$ denotes the subspace of symmetric tensors (see [7]). The skew-symmetric elements of $T(\mathcal{C})$ are of the form $s\otimes 1+1\otimes s$ for $s\in\mathcal{S}(\mathcal{C},-)$ (skew-symmetric in $\mathcal{C}$). Due to the skew-alternativity, for any $r\in\mathcal{S}(\mathcal{C},-)$, the next product does not need parentheses, and an easy computation gives:

$$(s\otimes 1+1\otimes s)(r\otimes 1+1\otimes r)(s\otimes 1+1\otimes s)=-2\mathrm{n}(r,s)(s\otimes 1+1\otimes s),$$

where n denotes the norm of the Cayley algebra $\mathcal{C}$. If $L_{s\otimes 1+1\otimes s}$ were invertible, then we would have

$$0=(r\otimes 1+1\otimes r)(s\otimes 1+1\otimes s)=rs\otimes 1+r\otimes s+s\otimes r+1\otimes rs$$

for any $r\in\mathcal{S}=\mathcal{S}(\mathcal{C},-)$ such that $\mathrm{n}(r,s)=0$. Since $\mathcal{C}\otimes\mathcal{C}=(\mathcal{C}\otimes 1+1\otimes\mathcal{C})\oplus(\mathcal{S}\otimes\mathcal{S})$, this would give $r\otimes s+s\otimes r=0$ for any $r\in\mathcal{S}=\mathcal{S}(\mathcal{C},-)$ such that $\mathrm{n}(r,s)=0$, which is not the case. Therefore, there are no suitable skew-symmetric elements in $T(\mathcal{C})$.

This leaves us with the tensor products of two unital composition algebras, and this case will be dealt with in the next section.

6 A magic square of Lie superalgebras

Unless otherwise stated, *the ground field* $\mathbb{F}$ *will be assumed to be algebraically closed of characteristic* 3 *throughout this section.*

Let $\mathcal{C}_1$ be the Cayley algebra over $\mathbb{F}$, that is, the (unique) eight-dimensional unital composition algebra, and let $\mathcal{C}_2$ be another unital composition algebra. Consider the structurable algebra $\mathcal{A}=\mathcal{C}_1\otimes\mathcal{C}_2$, where the involution is the tensor product of the canonical involutions in $\mathcal{C}_1$ and $\mathcal{C}_2$: $\overline{a\otimes b}=\bar{a}\otimes\bar{b}$, for any $a\in\mathcal{C}_1$ and $b\in\mathcal{C}_2$.

Denote by n_i the norm in $\mathcal{C}_i$, and by $\mathcal{S}_i$ the subspace of trace zero elements (i.e., $\bar{s}=-s$) in $\mathcal{C}_i$, for $i=1,2$. The skew-symmetric part of $\mathcal{A}$ is $\mathcal{S}=\mathcal{S}_1\otimes 1+1\otimes\mathcal{S}_2$, which we will identify with $\mathcal{S}_1\oplus\mathcal{S}_2$.

As in [4] consider the *Albert form* $Q:\mathcal{S}\to\mathbb{F}$, which is the nondegenerate quadratic form given by

$$Q(s_1+s_2):=\mathrm{n}_1(s_1)-\mathrm{n}_2(s_2), \tag{6.1}$$

for $s_1\in\mathcal{S}_1$ and $s_2\in\mathcal{S}_2$. Consider also the linear map $\sharp$ given by

$$(s_1+s_2)^\sharp:=s_1-s_2.$$

This is an isometry of Q.

The following result is [4, Proposition 3.3]. The proof there works in characteristic $\neq 2$.

Proposition 6.1 *Let $(\mathcal{A}, -)$ be the structurable algebra obtained as the tensor product of the Cayley algebra $\mathcal{C}_1$ and a unital composition algebra $\mathcal{C}_2$, and consider its Albert form $Q : \mathcal{S} \to \mathbb{F}$ in (6.1). Then the following conditions hold for any $a, b, c \in \mathcal{S}$, where $L_x : \mathcal{A} \to \mathcal{A}$ denotes, as usual, the left multiplication by an element $x \in \mathcal{A}$, and $Q(a, b) = Q(a+b)-Q(a)-Q(b)$ is the polar form of Q:*

$$L_a L_{a^\sharp} = L_{a^\sharp} L_a = -Q(a)\mathrm{id}, \tag{6.2}$$

$$ab^\sharp a = Q(a)b - Q(a, b)a, \tag{6.3}$$

$$L_a L_{b^\sharp} L_a = Q(a)L_b - Q(a, b)L_a, \tag{6.4}$$

$$L_a L_{b^\sharp} L_a L_{c^\sharp} = Q(a)L_b L_{c^\sharp} - Q(a, b)L_a L_{c^\sharp}, \tag{6.5}$$

$$(ab^\sharp)^2 + Q(a, b)ab^\sharp + Q(a)Q(b)1 = 0. \tag{6.6}$$

6.1 The inner structure Lie algebra

Our first goal is the description of the inner structure Lie algebra of these structurable algebras.

We need the following preliminary result.

Lemma 6.2 *Let $(\mathcal{A}, -)$ be the structurable algebra obtained as the tensor product of the Cayley algebra $\mathcal{C}_1$ and a unital composition algebra $\mathcal{C}_2$. Then the subspace $L_\mathcal{S} L_\mathcal{S} = \mathrm{span}\,\{L_a L_b \mid a, b \in \mathcal{S}\}$ is an ideal of $\mathfrak{instrl}(\mathcal{A}, -)$ that splits as the direct sum of its central ideal $\mathbb{F}\mathrm{id}$ and the ideal*

$$M_{\mathcal{S},\mathcal{S}} = \mathrm{span}\left\{M_{a,b} := L_a L_{b^\sharp} - L_b L_{a^\sharp} \mid a, b \in \mathcal{S}\right\}.$$

Moreover, the representation $\delta\colon \mathfrak{instrl}(\mathcal{A}, -) \to \mathrm{End}_\mathbb{F}(\mathcal{S})$ involved in the definition of $\mathcal{K}(\mathcal{A}, -)$ in (3.10) takes id to $2\mathrm{id}_\mathcal{S}$ and takes $M_{\mathcal{S},\mathcal{S}}$ isomorphically onto the orthogonal Lie algebra $\mathfrak{so}(\mathcal{S}, Q)$.

Proof The fact that $L_\mathcal{S} L_\mathcal{S}$ is an ideal of $\mathfrak{instrl}(\mathcal{A}, -)$ follows from $\mathfrak{instrl}(\mathcal{A}, -)$ being equal to $\mathcal{K}(\mathcal{A}, -)_0$, and $L_\mathcal{S} L_\mathcal{S}$ to $[\mathcal{K}(\mathcal{A}, -)_2, \mathcal{K}(\mathcal{A}, -)_{-2}]$ in Proposition 3.4. The arguments in [4, Proposition 4.3], which are valid in characteristic $\neq 2$, give $L_\mathcal{S} L_\mathcal{S} = \mathbb{F}\mathrm{id} \oplus M_{\mathcal{S},\mathcal{S}}$ and

$$M^\delta_{a,b}(c) = 2\bigl(Q(a, c)b - Q(b, c)a\bigr) := 2\sigma^Q_{a,b}(c) \tag{6.7}$$

for any $a, b, c \in \mathcal{S}$. By skew-symmetry, the dimension of $M_{\mathcal{S},\mathcal{S}}$ is at most $\binom{\dim \mathcal{S}}{2} = \dim \mathfrak{so}(\mathcal{S}, Q)$, and the orthogonal Lie algebra $\mathfrak{so}(\mathcal{S}, Q)$ is spanned by the operators $\sigma^Q_{a,b}$, so we obtain that δ takes $M_{\mathcal{S},\mathcal{S}}$ isomorphically to $\mathfrak{so}(\mathcal{S}, Q)$. The result follows. □

Theorem 6.3 *Let $(\mathcal{A}, -)$ be the structurable algebra obtained as the tensor product of the Cayley algebra $\mathcal{C}_1$ and a unital composition algebra $\mathcal{C}_2$.*

- *If the dimension of $\mathcal{C}_2$ is 1, 2 or 8, then we have the equality $\mathfrak{instrl}(\mathcal{A}, -) = L_\mathcal{S} L_\mathcal{S}$.*
- *In the remaining case: $\dim \mathcal{C}_2 = 4$, the Lie algebra $\mathfrak{instrl}(\mathcal{A}, -)$ is the direct sum of its ideals $R_{\mathcal{S}_2} = \mathrm{span}\,\{R_a \mid a \in \mathcal{S}_2\}$ and $L_\mathcal{S} L_\mathcal{S}$. (As usual, R_a denotes the right multiplication by a, and $\mathcal{S}_2$ is identified with $1 \otimes \mathcal{S}_2$.)*

Remark 6.4 Over fields of characteristic $\neq 2, 3$ the above result fails for $\dim \mathcal{C}_2 = 2$, where $\mathfrak{instrl}(\mathcal{A}, -) = R_{\mathcal{S}_2} \oplus L_{\mathcal{S}} L_{\mathcal{S}}$ ([4, Theorem 4.4]).

The proof of Theorem 6.3 will be given in several steps and in a quite indirect way. In the process, some results that have their own interest will be proved.

Let us recall first the definition in [6] of the Steinberg unitary algebra $\mathfrak{stu}_3(\mathcal{A}, -, \gamma)$ attached to a structurable algebra $(\mathcal{A}, -)$ and a triple $(\gamma_1, \gamma_2, \gamma_3) \in (\mathbb{F}^\times)^3$. This is the Lie algebra with generators $u_{ij}[x]$, for $1 \leq i \neq j \leq 3$ and $x \in \mathcal{A}$, subject to the following relations for any $x, y \in \mathcal{A}$ and indices $i \neq j$:

- $u_{ij}[x]$ is linear in x,
- $u_{ij}[x] = -\gamma_i \gamma_j^{-1} u_{ji}[\overline{x}]$,
- $[u_{12}[x], u_{23}[y]] = u_{13}[xy]$, and cyclically on 1, 2, 3.

Let us write explicitly the last condition, taking into account the previous one:

$$
\begin{aligned}
[u_{12}[x], u_{23}[y]] &= -\gamma_1 \gamma_3^{-1} u_{31}[\overline{xy}], \\
[u_{23}[x], u_{31}[y]] &= -\gamma_2 \gamma_1^{-1} u_{12}[\overline{xy}], \\
[u_{31}[x], u_{12}[y]] &= -\gamma_3 \gamma_2^{-1} u_{23}[\overline{xy}].
\end{aligned}
\tag{6.8}
$$

It follows that $\mathfrak{stu}_3(\mathcal{A}, -, \gamma)$ is $(\mathbb{Z}/2\mathbb{Z})^2$-graded:

$$
\mathfrak{stu}_3(\mathcal{A}, -, \gamma) = \mathfrak{g} \oplus u_{12}[\mathcal{A}] \oplus u_{23}[\mathcal{A}] \oplus u_{31}[\mathcal{A}],
$$

with $\mathfrak{stu}_3(\mathcal{A}, -, \gamma)_{(\bar{0},\bar{0})} = \mathfrak{g} = [u_{12}[\mathcal{A}], u_{21}[\mathcal{A}]] + [u_{23}[\mathcal{A}], u_{32}[\mathcal{A}]] + [u_{31}[\mathcal{A}], u_{13}[\mathcal{A}]]$, $\mathfrak{stu}_3(\mathcal{A}, -, \gamma)_{(\bar{1},\bar{0})} = u_{12}[\mathcal{A}] (= u_{21}[\mathcal{A}])$, $\mathfrak{stu}_3(\mathcal{A}, -, \gamma)_{(\bar{0},\bar{1})} = u_{23}[\mathcal{A}] (= u_{32}[\mathcal{A}])$, and $\mathfrak{stu}_3(\mathcal{A}, -, \gamma)_{(\bar{1},\bar{1})} = u_{31}[\mathcal{A}] (= u_{13}[\mathcal{A}])$.

On the other hand, the Lie algebra $\mathcal{K}(\mathcal{A}, -)$ in Proposition 3.4 is endowed with two natural order 2 commuting automorphisms (see [18]):

$$
\begin{aligned}
\epsilon &: (x, a)^\sim + T + (y, b) \mapsto (y, b)^\sim + T^\varepsilon + (x, a), \\
\tau &: (x, a)^\sim + T + (y, b) \mapsto (-x, a)^\sim + T + (-y, b).
\end{aligned}
$$

These two automorphisms induce a grading by $(\mathbb{Z}/2\mathbb{Z})^2$:

$$
\begin{aligned}
\mathcal{K}(\mathcal{A}, -)_{(\bar{0},\bar{0})} &= \{T \in \mathfrak{instrl}(\mathcal{A}, -) \mid T^\varepsilon = T\} \oplus \{(0, a)^\sim + (0, a) \mid a \in \mathcal{S}\}, \\
\mathcal{K}(\mathcal{A}, -)_{(\bar{1},\bar{0})} &= \{T \in \mathfrak{instrl}(\mathcal{A}, -) \mid T^\varepsilon = -T\} \oplus \{(0, a)^\sim - (0, a) \mid a \in \mathcal{S}\}, \\
\mathcal{K}(\mathcal{A}, -)_{(\bar{0},\bar{1})} &= \{(x, 0)^\sim + (x, 0) \mid x \in \mathcal{A}\}, \\
\mathcal{K}(\mathcal{A}, -)_{(\bar{1},\bar{1})} &= \{(x, 0)^\sim - (x, 0) \mid x \in \mathcal{A}\}.
\end{aligned}
$$

The subspace $\{T \in \mathfrak{instrl}(\mathcal{A}, -) \mid T^\varepsilon = -T\}$ equals $\operatorname{span}\left\{V_{x,y} - V^\varepsilon_{x,y} \mid x, y \in \mathcal{A}\right\} = \operatorname{span}\left\{V_{x,y} + V_{y,x} \mid x, y \in \mathcal{A}\right\}$. But $V_{x,y} + V_{y,x} = L_{x\bar{y}+y\bar{x}}$ for any $x, y \in \mathcal{A}$, which gives:

$$
\{T \in \mathfrak{instrl}(\mathcal{A}, -) \mid T^\varepsilon = -T\} = L_{\mathcal{H}},
$$

where $\mathcal{H} = \mathcal{H}(\mathcal{A}, -) = \{x \in \mathcal{A} \mid \bar{x} = x\}$ is the subspace of symmetric elements in $\mathcal{A}$. Therefore, we have $\mathcal{K}(\mathcal{A}, -)_{(\bar{1},\bar{0})} = L_{\mathcal{H}} \oplus \{(0, a)^\sim - (0, a) \mid a \in \mathcal{S}\}$.

As in [18, §4] write, for $x \in \mathcal{A}$,

$$\begin{cases} \epsilon_1(x) = (x,0) + (x,0)^{\sim} \in \mathcal{K}(\mathcal{A},-)_{(\bar{0},\bar{1})}, \\ \epsilon_2(x) = (\bar{x},0) - (\bar{x},0)^{\sim} \in \mathcal{K}(\mathcal{A},-)_{(\bar{1},\bar{1})} \\ \epsilon_0(x) = \dfrac{1}{2}\Big(L_{x+\bar{x}} + \big((0,x-\bar{x}) - (0,x-\bar{x})^{\sim}\big)\Big) \in \mathcal{K}(\mathcal{A},-)_{(\bar{1},\bar{0})}. \end{cases}$$

The bracket in (3.10) of these elements in $\mathcal{K}(\mathcal{A},-)$ behaves as follows, for any $x, y \in \mathcal{A}$ (see [18, (4.2)], noting that all the computations there work also in characteristic 3):

$$\begin{aligned} [\epsilon_0(x), \epsilon_1(y)] &= \epsilon_2(\overline{xy}), \\ [\epsilon_1(x), \epsilon_2(y)] &= -2\epsilon_0(\overline{xy}), \\ [\epsilon_2(x), \epsilon_0(y)] &= -\epsilon_1(\overline{xy}). \end{aligned}$$

(We refrain to use $-2 = 1$ here, because these formulas are valid in any characteristic $\neq 2$.)

With $\tilde{\epsilon}_0(x) = \epsilon_0(x)$, $\tilde{\epsilon}_1(x) = -\frac{1}{2}\epsilon_1(x)$, and $\tilde{\epsilon}_2(x) = \epsilon_2(x)$, these last equations become

$$\begin{aligned} [\tilde{\epsilon}_0(x), \tilde{\epsilon}_1(y)] &= -\frac{1}{2}\tilde{\epsilon}_2(\overline{xy}), \\ [\tilde{\epsilon}_1(x), \tilde{\epsilon}_2(y)] &= \tilde{\epsilon}_0(\overline{xy}), \\ [\tilde{\epsilon}_2(x), \tilde{\epsilon}_0(y)] &= 2\tilde{\epsilon}_1(\overline{xy}). \end{aligned} \tag{6.9}$$

Note that with $\gamma = (1,-1,2)$, $-\gamma_1\gamma_3^{-1} = -\frac{1}{2}$, $-\gamma_2\gamma_1^{-1} = 1$, and $-\gamma_3\gamma_2^{-1} = 2$, and hence there is a unique Lie algebra homomorphism as follows:

$$\begin{aligned} \Phi : \mathfrak{stu}_3(\mathcal{A},-,\gamma) &\longrightarrow \mathcal{K}(\mathcal{A},-) \\ u_{12}[x] &\mapsto \tilde{\epsilon}_0(x) \\ u_{23}[x] &\mapsto \tilde{\epsilon}_1(x) \\ u_{31}[x] &\mapsto \tilde{\epsilon}_2(x) \end{aligned}$$

for any $x \in \mathcal{A}$. Note that Φ is surjective because $\tilde{\epsilon}_1(\mathcal{A}) + \tilde{\epsilon}_2(\mathcal{A}) = \mathcal{K}(\mathcal{A},-)_1 \oplus \mathcal{K}(\mathcal{A},-)_{-1}$, and this subspace generates $\mathcal{K}(\mathcal{A},-)$. Moreover, Φ is homogeneous of trivial degree for the $(\mathbb{Z}/2\mathbb{Z})^2$-gradings on $\mathfrak{stu}_3(\mathcal{A},-,\gamma)$ and $\mathcal{K}(\mathcal{A},-)$, being bijective on $u_{12}[\mathcal{A}]$, $u_{23}[\mathcal{A}]$, and $u_{31}[\mathcal{A}]$. As a consequence, the ideal $\ker\Phi$ is contained in $\mathfrak{stu}_3(\mathcal{A},-,\gamma)_{(\bar{0},\bar{0})} = \mathfrak{g}$ and, therefore, $[\ker\Phi, u_{12}[\mathcal{A}]]$ is contained in both $[\mathfrak{g}, u_{12}[\mathcal{A}]] \subseteq u_{12}[\mathcal{A}]$ and in $\ker\Phi$, so it is trivial. It follows that $\ker\Phi$ annihilates $u_{12}[\mathcal{A}]$ and similarly $u_{23}[\mathcal{A}]$ and $u_{31}[\mathcal{A}]$, which generate $\mathfrak{stu}_3(\mathcal{A},-,\gamma)$. Thus $\ker\Phi$ is contained in the center $Z\big(\mathfrak{stu}_3(\mathcal{A},-,\gamma)\big)$. On the other hand, the image under Φ of this center is contained in the center of $\mathcal{K}(\mathcal{A},-)$, which is trivial.

We summarize our findings in the next result.

Proposition 6.5 *Let $(\mathcal{A},-)$ be a structurable algebra, then its Kantor algebra $\mathcal{K}(\mathcal{A},-)$ is isomorphic, as a $(\mathbb{Z}/2\mathbb{Z})^2$-graded Lie algebra, to the quotient of the Lie algebra $\mathfrak{stu}_3(\mathcal{A},-,\gamma)$ by its center, with $\gamma = (1,-1,2)$.*

Assume now that $\mathcal{A} = \mathcal{C}_1 \otimes \mathcal{C}_2$ for a Cayley algebra $\mathcal{C}_1$ and a unital composition algebra $\mathcal{C}_2$. Consider the so called *para-Hurwitz* algebras $\overline{\mathcal{C}}_1^{\text{op}}$ and $\overline{\mathcal{C}}_2^{\text{op}}$ attached to the opposite algebras $\mathcal{C}_1^{\text{op}}$ and $\mathcal{C}_2^{\text{op}}$. That is, the multiplication in $\overline{\mathcal{C}}_1^{\text{op}}$ is given by $x_1 \bullet y_1 = \overline{y_1}\,\overline{x_1} = \overline{x_1 y_1}$ for $x_1, y_1 \in \mathcal{C}_1$, and similarly for $\overline{\mathcal{C}}_2^{\text{op}}$.

These para-Hurwitz algebras are examples of *symmetric composition algebras* and, as shown in [13, Remark 3.2], for each $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in (\mathbb{F}^\times)^3$, there is a Lie algebra $\mathfrak{g}_\alpha(\overline{\mathcal{C}}_1^{\mathrm{op}}, \overline{\mathcal{C}}_2^{\mathrm{op}})$ defined on the vector space

$$\big(\mathfrak{tri}(\overline{\mathcal{C}}_1^{\mathrm{op}}) \oplus \mathfrak{tri}(\overline{\mathcal{C}}_2^{\mathrm{op}})\big) \oplus \Big(\oplus_{i=0}^{2} \iota_i(\overline{\mathcal{C}_1}^{\mathrm{op}} \otimes \overline{\mathcal{C}}_2^{\mathrm{op}})\Big),$$

where $\iota_i(\mathcal{C}_1 \otimes \mathcal{C}_2)$ is a copy of $\mathcal{C}_1 \otimes \mathcal{C}_2 = \mathcal{A}$, $\mathfrak{tri}(\overline{\mathcal{C}}_1^{\mathrm{op}})$ is the triality Lie algebra of $\overline{\mathcal{C}}_1^{\mathrm{op}}$, whose elements are the triples $(d_0, d_1, d_2) \in \mathfrak{so}(\mathcal{C}_1, \mathrm{n}_1)$ such that

$$d_0(x_1 \bullet y_1) = d_1(x_1) \bullet y_1 + x_1 \bullet d_2(y_1)$$

for any $x_1, y_1 \in \mathcal{C}_1$, and similarly for $\mathfrak{tri}(\overline{\mathcal{C}}_2^{\mathrm{op}})$. The bracket in $\mathfrak{g}_\alpha(\overline{\mathcal{C}}_1^{\mathrm{op}}, \overline{\mathcal{C}}_2^{\mathrm{op}})$ satisfies

$$[\iota_0(x_1 \otimes x_2), \iota_1(y_1 \otimes y_2)] = \alpha_2\iota_2\big((x_1 \bullet y_1) \otimes (x_2 \bullet y_2)\big) = \alpha_2\iota_2\big(\overline{x_1 y_1} \otimes \overline{x_2 y_2}\big),$$

and cyclically on 0, 1, 2. Then, with $\alpha = (1, 2, -\frac{1}{2})$, this last equation works exactly as (6.9), which shows that there is a Lie algebra homomorphism determined as follows:

$$\begin{aligned} \Psi : \mathfrak{stu}_3(\mathcal{A}, -, \gamma) &\longrightarrow \mathfrak{g}_\alpha(\overline{\mathcal{C}}_1^{\mathrm{op}}, \overline{\mathcal{C}}_2^{\mathrm{op}}) \\ u_{12}[x \otimes y] &\mapsto \iota_0(x \otimes y) \\ u_{23}[x \otimes y] &\mapsto \iota_1(x \otimes y) \\ u_{31}[x \otimes y] &\mapsto \iota_2(x \otimes y) \end{aligned}$$

for any $x \in \mathcal{C}_1$ and $y \in \mathcal{C}_2$. The image of Ψ is the subalgebra of $\mathfrak{g}_\alpha(\overline{\mathcal{C}}_1^{\mathrm{op}}, \overline{\mathcal{C}}_2^{\mathrm{op}})$ generated by $\sum_{i=0}^{2} \iota_i(\mathcal{C}_1^{\mathrm{op}} \otimes \mathcal{C}_2^{\mathrm{op}})$, which is the derived subalgebra $\mathfrak{g}_\alpha(\overline{\mathcal{C}}_1^{\mathrm{op}}, \overline{\mathcal{C}}_2^{\mathrm{op}})^{(1)}$.

The same arguments as above give our next result.

Proposition 6.6 *Let $(\mathcal{A}, -)$ be the structurable algebra obtained as the tensor product of the Cayley algebra $\mathcal{C}_1$ and a unital composition algebra $\mathcal{C}_2$. Then the Lie algebra $\mathfrak{g}_\alpha(\overline{\mathcal{C}}_1^{\mathrm{op}}, \overline{\mathcal{C}}_2^{\mathrm{op}})^{(1)}$, with $\alpha = (1, 2, -\frac{1}{2})$, is isomorphic, as a $(\mathbb{Z}/2\mathbb{Z})^2$-graded Lie algebra, to the quotient of $\mathfrak{stu}_3(\mathcal{A}, -, \gamma)$ by its center, with $\gamma = (1, -1, 2)$.*

Corollary 6.7 *Let $(\mathcal{A}, -)$ be the structurable algebra obtained as the tensor product of the Cayley algebra $\mathcal{C}_1$ and a unital composition algebra $\mathcal{C}_2$. Then its Kantor algebra $\mathcal{K}(\mathcal{A}, -)$ is isomorphic, as a $(\mathbb{Z}/2\mathbb{Z})^2$-graded Lie algebra, to the Lie algebra $\mathfrak{g}_\alpha(\overline{\mathcal{C}}_1^{\mathrm{op}}, \overline{\mathcal{C}}_2^{\mathrm{op}})^{(1)}$, with $\alpha = (1, 2, -\frac{1}{2})$.*

***Proof of Theorem* 6.3** To begin with, given a structurable algebra $(\mathcal{A}, -)$, the dimension of the associated Lie algebra $\mathcal{K}(\mathcal{A}, -)$ is $\dim \mathfrak{instrl}(\mathcal{A}, -) + 2\dim \mathcal{A} + 2\dim \mathcal{S}$. Hence, for $\mathcal{A} = \mathcal{C}_1 \otimes \mathcal{C}_2$ for a Cayley algebra $\mathcal{C}_1$ and a unital composition algebra $\mathcal{C}_2$ of dimension n ($n = 1, 2, 4$ or 8), we get

$$\begin{aligned} &\dim \mathcal{K}(\mathcal{A}, -) = \dim \mathfrak{instrl}(\mathcal{A}, -) + 2 \times 8 \times n + 2 \times (7 + n - 1) \\ &= \dim \mathfrak{instrl}(\mathcal{A}, -) + 18n + 12. \end{aligned}$$

On the other hand, the dimension of $\mathfrak{g}_\alpha(\overline{\mathcal{C}}_1^{\mathrm{op}}, \overline{\mathcal{C}}_2^{\mathrm{op}})^{(1)}$ is 52, 77, 133, or 248, depending on n being 1, 2, 4 or 8 (see [13, §3]). This gives that the dimension of $\mathfrak{instrl}(\mathcal{A}, -)$ is 22, 29, 49, or 92, respectively, while the dimension of $L_{\mathcal{S}} L_{\mathcal{S}}$ is, according to Lemma 6.2, 22, 29, 46, or 92.

We conclude that if the dimension of $\mathcal{C}_2$ is not 4, these dimensions coincide, and hence the ideal $L_{\mathcal{S}} L_{\mathcal{S}}$ is the whole $\mathfrak{instrl}(\mathcal{A}, -)$. However, if the dimension of $\mathcal{C}_2$ is 4, then $\mathcal{C}_2$ is isomorphic to the algebra of 2×2-matrices over $\mathbb{F}$ (recall that we are assuming in this section that the field $\mathbb{F}$ is algebraically closed). In this case, for any $0 \neq a \in \mathcal{S}_2$, there are elements $b, c \in \mathcal{S}_2$ such that $a = bc$ (exercise for the reader!) and, because of the associativity of $\mathcal{C}_2$, $L_a = L_b L_c$ lies in $L_\mathcal{S} L_\mathcal{S} \subseteq \mathfrak{instrl}(\mathcal{A}, -)$. But $V_{a,1} = L_a + 2R_a$, so we conclude that the right multiplication R_a also belongs to $\mathfrak{instrl}(\mathcal{A}, -)$. Besides, $R_a^\delta = R_a + R_{\overline{R_a(1)}} = R_a - R_a = 0$ (see Proposition 3.4). Hence, $R_{\mathcal{S}_2}$ lies in $\ker \delta$, while δ is one-to-one on $L_\mathcal{S} L_\mathcal{S}$ (Lemma 6.2). Therefore, $R_{\mathcal{S}_2} \cap L_\mathcal{S} L_\mathcal{S} = 0$ and, by dimension count, $\mathfrak{instrl}(\mathcal{A}, -) = R_{\mathcal{S}_2} \oplus L_\mathcal{S} L_\mathcal{S}$. □

6.2 The Lie algebra $S(\mathcal{A}, \mathcal{A})$

Let $(\mathcal{A}, -)$ be the structurable algebra obtained as the tensor product of the Cayley algebra $\mathcal{C}_1$ and a unital composition algebra $\mathcal{C}_2$. Fix an element $s \in \mathcal{S}$ with L_s invertible, and take the element $t \in \mathcal{S}$ with $L_s L_t = L_t L_s = \mathrm{id}$. Equation (6.2) shows that $Q(s) \neq 0$ and $t = -\frac{1}{Q(s)} s^\sharp$, where Q is the Albert form.

Recall from Theorem 3.8 that $\mathcal{A}$ becomes a J-ternary algebra with triple product $(x, y, z) = V_{x,sy}(z)$ for $x, y, z \in \mathcal{A}$, and that the Lie algebra $S(\mathcal{A}, \mathcal{A})$ consists of those elements $T \in \mathfrak{instrl}(\mathcal{A}, -)$ such that $T^\delta(t) = 0$. Because of (3.10), this can be expressed in terms of the associated Lie algebra $\mathcal{K}(\mathcal{A}, -)$ in Proposition 3.4 as follows:

$$S(\mathcal{A}, \mathcal{A}) = \{T \in \mathfrak{instrl}(\mathcal{A}, -) \mid [T, (0, t)] = 0\}.$$

As in the proof of Theorem 3.8, consider the elements $E = (0, t)$, $F = (0, s)^\sim$ and $H = \mathrm{id}$ in $\mathcal{K}(\mathcal{A}, -)$. Denote by $\mathfrak{sl}_2$ the subalgebra of $\mathcal{K}(\mathcal{A}, -)$ spanned by E, F, and H. Hence, an element $T \in \mathfrak{instrl}(\mathcal{A}, -)$ lies in $S(\mathcal{A}, \mathcal{A})$ if and only if $[E, T] = 0$. By Lemma 3.6 this is equivalent to centralizing the subalgebra $\mathfrak{sl}_2$:

$$S(\mathcal{A}, \mathcal{A}) = \mathrm{Cent}_{\mathcal{K}(\mathcal{A}, -)}(\mathfrak{sl}_2).$$

Consider the subspace $\mathcal{S}' := (\mathbb{F}t)^\perp$ (orthogonal relative to the Albert form Q in Proposition 6.1). Lemma 6.2 shows that the representation $\delta\colon \mathfrak{instrl}(\mathcal{A}, -) \to \mathrm{End}_\mathbb{F}(\mathcal{S})$ induces an isomorphism of Lie algebras:

$$\begin{aligned} \delta'\colon M_{\mathcal{S}',\mathcal{S}'} &\longrightarrow \mathfrak{so}(\mathcal{S}', Q) \\ T &\mapsto T^\delta|_{\mathcal{S}'}, \end{aligned}$$

where, as usual, $M_{\mathcal{S}',\mathcal{S}'} = \mathrm{span}\left\{M_{a,b} = L_a L_{b^\sharp} - L_b L_{a^\sharp} \mid a, b \in \mathcal{S}'\right\}$.

A straightforward consequence of Theorem 6.3 is the following result:

Corollary 6.8 *Let $(\mathcal{A}, -)$ be the structurable algebra obtained as the tensor product of the Cayley algebra $\mathcal{C}_1$ and a unital composition algebra $\mathcal{C}_2$. Fix elements $s, t \in \mathcal{S}$ such that $L_s L_t = L_t L_s = \mathrm{id}$, and consider the subspace $\mathcal{S}'$ of $\mathcal{S}$ orthogonal to t relative to the Albert form Q.*

- *If the dimension of $\mathcal{C}_2$ is 1, 2, or 8, then the Lie subalgebra $S(\mathcal{A}, \mathcal{A})$ in Theorem 3.8 equals $M_{\mathcal{S}',\mathcal{S}'}$, which is isomorphic to the orthogonal Lie algebra $\mathfrak{so}(\mathcal{S}', Q)$.*
- *If the dimension of $\mathcal{C}_2$ is 4, then the Lie subalgebra $S(\mathcal{A}, \mathcal{A})$ in Theorem 3.8 equals $R_{\mathcal{S}_2} \oplus M_{\mathcal{S}',\mathcal{S}'}$, which is the direct sum of the three-dimensional simple Lie algebra $R_{\mathcal{S}_2}$, isomorphic to $\mathfrak{sl}_2$, and the subalgebra $M_{\mathcal{S}',\mathcal{S}'}$, which is isomorphic to the orthogonal Lie algebra $\mathfrak{so}(\mathcal{S}', Q)$.*

6.3 Structure of $\mathcal{A}$ as a module for $S(\mathcal{A},\mathcal{A})$

In order to determine the Lie superalgebra $\mathcal{L}^{ss}(\mathcal{A}) = S(\mathcal{A},\mathcal{A}) \oplus \mathcal{A}$ attached to the J-ternary algebra associated to a structurable algebra $(\mathcal{A}, -)$, obtained as the tensor product of a Cayley algebra $\mathcal{C}_1$ and a unital composition algebra $\mathcal{C}_2$, and a fixed element $s \in \mathcal{S}$ with L_s invertible, we need to know the structure of $\mathcal{A}$ as a module for $S(\mathcal{A},\mathcal{A})$.

We will follow now some arguments in [4]. Consider the linear map

$$\begin{aligned} \Phi\colon \mathcal{S}' &\longrightarrow \mathrm{End}_{\mathbb{F}}(\mathcal{A}) \\ a &\mapsto L_a L_s, \end{aligned}$$

which satisfies

$$\Phi(a)^2 = L_a L_s L_a L_s = Q(a) L_{s^\sharp} L_s = -Q(a)Q(s)\mathrm{id},$$

for any $a \in \mathcal{S}'$, because of (6.4) and (6.2). Hence, Φ extends to an algebra homomorphism, still denoted by Φ:

$$\Phi\colon \mathfrak{Cl}(\mathcal{S}', \tilde{Q}) \longrightarrow \mathrm{End}_{\mathbb{F}}(\mathcal{A}), \tag{6.10}$$

where $\tilde{Q}$ is the quadratic form defined on $\mathcal{S}'$ by $\tilde{Q}(a) = -Q(a)Q(s)$ and $\mathfrak{Cl}(\mathcal{S}', \tilde{Q})$ denotes the associated Clifford algebra. Denote by $u \cdot v$ the multiplication in $\mathfrak{Cl}(\mathcal{S}', \tilde{Q})$. The orthogonal Lie algebra $\mathfrak{so}(\mathcal{S}', Q) = \mathfrak{so}(\mathcal{S}', \tilde{Q})$ lives inside $\mathfrak{Cl}(\mathcal{S}', \tilde{Q})$ as the subspace span $\{[a,b]^{\cdot} := a\cdot b - b\cdot a \mid a, b \in \mathcal{S}'\}$.

Actually, for any $a, b, c \in \mathcal{S}'$, $a\cdot b\cdot a = a\cdot b\cdot a + a\cdot a\cdot b - \tilde{Q}(a)b = a\cdot(a\cdot b + b\cdot a) - \tilde{Q}(a)b = \tilde{Q}(a,b)a - \tilde{Q}(a)b$, so that, by linearization, we have $a\cdot b\cdot c + c\cdot b\cdot a = \tilde{Q}(a,b)c + \tilde{Q}(c,b)a - \tilde{Q}(a,c)b$. As a consequence,

$$\begin{aligned} [a\cdot b - b\cdot a, c]^{\cdot} &= (a\cdot b\cdot c + c\cdot b\cdot a) - (b\cdot a\cdot c + c\cdot a\cdot b) \\ &= -2\big(\tilde{Q}(a,c)b - \tilde{Q}(b,c)a\big) = -2\sigma^{\tilde{Q}}_{a,b}(c). \end{aligned}$$

In other words, the linear map $\iota : \mathfrak{so}(\mathcal{S}', \tilde{Q}) \to \mathfrak{Cl}(\mathcal{S}', \tilde{Q})$ such that

$$\iota(\sigma^{\tilde{Q}}_{a,b}) = -\frac{1}{2}(a\cdot b - b\cdot a),$$

for $a, b \in \mathcal{S}'$, is a one-to-one homomorphism of Lie algebras.

Lemma 6.9 *The diagram*

$$\begin{array}{ccc} \mathfrak{so}(\mathcal{S}', \tilde{Q}) & \xleftarrow[\delta']{\simeq} & M_{\mathcal{S}',\mathcal{S}'} \\ \iota\big\downarrow & & \big\downarrow \\ \mathfrak{Cl}(\mathcal{S}', \tilde{Q}) & \xrightarrow{\Phi} & \mathrm{End}_{\mathbb{F}}(\mathcal{A}) \end{array}$$

commutes.

Proof For $a, b \in \mathcal{S}'$ we have

$$\Phi(a\cdot b - b\cdot a) = L_a L_s L_b L_s - L_b L_s L_a L_s = Q(s)(L_a L_{b^\sharp} - L_b L_{a^\sharp}) = Q(s) M_{a,b},$$

where we have used (6.4) and that $Q(s, b^\sharp) = Q(s^\sharp, b) = -Q(s)Q(t,b) = 0 = Q(s, a^\sharp)$, and

$$\begin{aligned} \delta'(M_{a,b}) &= 2\sigma^{Q}_{a,b} = 2\big(Q(a,.)b - Q(b,.)a\big) \quad \text{by (6.7)} \\ &= -\frac{2}{Q(s)}\big(\tilde{Q}(a,.)b - \tilde{Q}(b,.)a\big) = -\frac{2}{Q(s)}\sigma^{\tilde{Q}}_{a,b}. \end{aligned}$$

Then we compute

$$\Phi \circ \iota \circ \delta'(M_{a,b}) = -\frac{2}{Q(s)}\Phi \circ \iota(\sigma^{\tilde{Q}}_{a,b}) = \frac{1}{Q(s)}\Phi(a \cdot b - b \cdot a) = M_{a,b},$$

as required. □

We will consider the different possibilities, according to the dimension of $\mathcal{C}_2$:

$\dim \mathcal{C}_\mathbf{2} = \mathbf{1}$: In this case we get the following dimensions:

$$\dim \mathcal{S} = 7, \quad \dim \mathcal{S}' = 6, \quad \dim \mathrm{End}_{\mathbb{F}}(\mathcal{A}) = 8^2 = 2^6 = \dim \mathfrak{Cl}(\mathcal{S}', \tilde{Q}),$$

so that the homomorphism Φ in (6.10) is an isomorphism, because $\mathfrak{Cl}(\mathcal{S}', \tilde{Q})$ is a simple (associative) algebra. Then $\mathcal{A}$ is the sum of the two half-spin modules for the orthogonal Lie algebra $M_{\mathcal{S}',\mathcal{S}'} \simeq \mathfrak{so}(\mathcal{S}', \tilde{Q}) \simeq \mathfrak{so}_6 \simeq \mathfrak{sl}_4$. These half-spin modules, thought as modules for $\mathfrak{sl}_4$, are the natural four-dimensional module for $\mathfrak{sl}_4$ and its dual.

Therefore, Corollary 6.8 shows that the even part of $\mathcal{L}^{\mathrm{ss}}(\mathcal{A})$ is, up to isomorphism, $\mathfrak{sl}_4 \simeq \mathfrak{sl}(W)$ for a four-dimensional vector space W, while the odd part is isomorphic, as a module for $\mathfrak{sl}(W)$ to $W \oplus W^*$. But, as is well-known (and valid in characteristic 3), one has $\mathrm{Hom}_{\mathfrak{sl}(W)}(W \otimes W, \mathfrak{sl}(W)) = 0 = \mathrm{Hom}_{\mathfrak{sl}(W)}(W^* \otimes W^*, \mathfrak{sl}(W))$ and $\mathrm{Hom}_{\mathfrak{sl}(W)}(W \otimes W^*, \mathfrak{sl}(W))$ is one-dimensional. Hence, the Lie bracket in $\mathcal{L}^{\mathrm{ss}}(\mathcal{A})_{\bar{1}} \times \mathcal{L}^{\mathrm{ss}}(\mathcal{A})_{\bar{1}} \to \mathcal{L}^{\mathrm{ss}}(\mathcal{A})_{\bar{0}}$ is uniquely determined, and hence $\mathcal{L}^{\mathrm{ss}}(\mathcal{A})$ is isomorphic to the projective special linear Lie superalgebra $\mathfrak{psl}(4|1)$.

$\dim \mathcal{C}_\mathbf{2} = \mathbf{2}$: Here we get the following dimensions:

$$\dim \mathcal{S} = 8, \quad \dim \mathcal{S}' = 7, \quad \dim \mathfrak{Cl}(\mathcal{S}', \tilde{Q}) = 2^7, \quad \dim \mathfrak{Cl}_{\bar{0}}(\mathcal{S}', \tilde{Q}) = 2^6.$$

The even Clifford algebra is simple of dimension 2^6, so its unique irreducible module has dimension 8, and hence the representation of $S(\mathcal{A}, \mathcal{A}) = M_{\mathcal{S}',\mathcal{S}'}$ on $\mathcal{A}$, given by the composition in Lemma 6.9, is the direct sum of two copies of the spin module for $M_{\mathcal{S}',\mathcal{S}'} \simeq \mathfrak{so}_7$. But if W is the spin module for $\mathfrak{so}_7$, then $\mathrm{Hom}_{\mathfrak{so}_7}(W \otimes W, \mathfrak{so}_7)$ is one-dimensional and spanned by a skew-symmetric map (see, e.g., [15, Proposition 2.12]). Therefore, up to scalars, there is a unique $\mathfrak{so}_7$-invariant symmetric bilinear map $\mathcal{L}^{\mathrm{ss}}(\mathcal{A})_{\bar{1}} \times \mathcal{L}^{\mathrm{ss}}(\mathcal{A})_{\bar{1}} \to \mathcal{L}^{\mathrm{ss}}(\mathcal{A})_{\bar{0}}$, and $\mathcal{L}^{\mathrm{ss}}(\mathcal{A})$ is necessarily isomorphic to the Lie superalgebra $\mathfrak{g}(3, 3) = \mathfrak{g}(S_{1,2}, S_{1,2})$ in [10, Proposition 5.19].

$\dim \mathcal{C}_\mathbf{2} = \mathbf{4}$: In this case we get the following dimensions:

$$\dim \mathcal{S} = 10, \quad \dim \mathcal{S}' = 9, \quad \dim \mathfrak{Cl}(\mathcal{S}', \tilde{Q}) = 2^9, \quad \dim \mathfrak{Cl}_{\bar{0}}(\mathcal{S}', \tilde{Q}) = 2^8.$$

Corollary 6.8 shows that $S(\mathcal{A}, \mathcal{A})$ is the direct sum of the simple three-dimensional Lie algebra $R_{\mathcal{S}_2}$ and the ideal $M_{\mathcal{S}',\mathcal{S}'} \simeq \mathfrak{so}_9$. The action of the quaternion algebra $\mathcal{C}_2 \simeq 1 \otimes \mathcal{C}_2$ by multiplication on the right on $\mathcal{A}$ commutes with the action of $M_{\mathcal{S}',\mathcal{S}'}$, because of the associativity of $\mathcal{C}_2$. By dimension count, and using that the even Clifford algebra $\mathfrak{Cl}_{\bar{0}}(\mathcal{S}', \tilde{Q})$ is simple, Φ induces an isomorphism, denoted by the same letter:

$$\Phi : \mathfrak{Cl}_{\bar{0}}(\mathcal{S}', \tilde{Q}) \to \mathrm{End}_{\mathcal{C}_2}(\mathcal{C}_1 \otimes \mathcal{C}_2) \simeq \mathrm{End}_{\mathbb{F}}(\mathcal{C}_1) \otimes \mathcal{C}_2.$$

As a module for $M_{\mathcal{S}',\mathcal{S}'} \simeq \mathfrak{so}_9$, $\mathcal{A}$ is a sum of two copies of the spin module W (the irreducible module for the simple algebra $\mathfrak{Cl}_{\bar{0}}(\mathcal{S}', \tilde{Q})$), and thus $R_{\mathcal{S}_2}$ lies in $\mathrm{End}_{M_{\mathcal{S}',\mathcal{S}'}}(\mathcal{A}) \simeq \mathrm{End}_{\mathfrak{so}_9}(W \oplus W) \simeq \mathrm{Mat}_2(\mathbb{F})$. As a consequence, $\mathcal{L}^{\mathrm{ss}}(\mathcal{A})_{\bar{0}}$ is isomorphic to $\mathfrak{sl}_2 \oplus \mathfrak{so}_9$, while

$\mathcal{L}^{ss}(\mathcal{A})_{\bar{1}}$ is, as a module for the even part, isomorphic to the tensor product of the two-dimensional natural module V for $\mathfrak{sl}_2$ and the spin module W for $\mathfrak{so}_9$. The uniqueness, up to scalars, of the elements in the vector spaces $\mathrm{Hom}_{\mathfrak{sl}_2}(V \otimes V, \mathbb{F})$, $\mathrm{Hom}_{\mathfrak{sl}_2}(V \otimes V, \mathfrak{sl}_2)$, $\mathrm{Hom}_{\mathfrak{so}_9}(W \otimes W, \mathbb{F})$, and $\mathrm{Hom}_{\mathfrak{so}_9}(W \otimes W, \mathfrak{so}_9)$ (see [15, Proposition 2.12]) forces that the bracket $\mathcal{L}^{ss}(\mathcal{A})_{\bar{1}} \times \mathcal{L}^{ss}(\mathcal{A})_{\bar{1}} \to \mathcal{L}^{ss}(\mathcal{A})_{\bar{0}}$ is uniquely determined up to scalars (because of the Jacobi identity), and hence $\mathcal{L}^{ss}(\mathcal{A})$ turns out to be isomorphic to the exceptional Lie superalgebra $\mathfrak{el}(5; 3)$ ([16, Theorem 3.3]).

$\dim \mathcal{C}_2 = \mathbf{8}$: Here we have the following dimensions:

$$\dim \mathcal{S} = 14, \quad \dim \mathcal{S}' = 13, \quad \dim \mathfrak{Cl}_{\bar{0}}(\mathcal{S}', \tilde{Q}) = 2^{12} = (2^6)^2 = \dim \mathrm{End}_{\mathbb{F}}(\mathcal{A}),$$

and hence the homomorphism Φ in (6.10) restricts to an isomorphism from the even Clifford algebra $\mathfrak{Cl}_{\bar{0}}(\mathcal{S}', \tilde{Q})$, which is simple, onto $\mathrm{End}_{\mathbb{F}}(\mathcal{A})$. It turns out that $\mathcal{A}$ is the spin module for $S(\mathcal{A}, \mathcal{A}) = M_{\mathcal{S}',\mathcal{S}'} \simeq \mathfrak{so}_{13}$. As the space $\mathrm{Hom}_{\mathfrak{so}_{13}}(W \otimes W, \mathfrak{so}_{13})$ is one-dimensional (see [15, Proposition 2.12 and Theorem 3.1]), where W is the spin module for $\mathfrak{so}_{13}$, we conclude that $\mathcal{L}^{ss}(\mathcal{A})$ is isomorphic to the Lie superalgebra $\mathfrak{g}(6, 6) = \mathfrak{g}(S_{4,2}, S_{4,2})$ in [10, Proposition 5.10].

We summarize our findings in the next theorem.

Theorem 6.10 *Let* $(\mathcal{A}, -)$ *be the structurable algebra obtained as the tensor product of the Cayley algebra* $\mathcal{C}_1$ *and a unital composition algebra* $\mathcal{C}_2$ *over an algebraically closed field of characteristic* 3. *Let* s *be a skew-symmetric element such that* L_s *is invertible, and let* $\mathcal{L}^{ss}(\mathcal{A})$ *be the Lie superalgebra attached to the* J*-ternary algebra defined on* $\mathcal{A}$ *in Theorem 3.8. Then the Lie superalgebra* $\mathcal{L}^{ss}(\mathcal{A})$ *is given, up to isomorphism, by the following table:*

$\dim \mathcal{C}_2$	1	2	4	8
$\mathcal{L}^{ss}(\mathcal{A})$	$\mathfrak{psl}(4\|1)$	$\mathfrak{g}(3, 3)$	$\mathfrak{el}(5; 3)$	$\mathfrak{g}(6, 6)$

6.4 A magic square of Lie superalgebras

If our structurable algebra $(\mathcal{A}, -)$ is the tensor product of two associative unital composition algebras, then $(\mathcal{A}, -)$ is associative with involution and hence it fits in Example 3.9 with $\mathcal{W} = 0$. According to this Example, the isotope $\mathcal{B} := \mathcal{A}^{(s)}$ is endowed with the involution $\tau : x \mapsto -\bar{x}$, $\mathcal{A}$ is a left $\mathcal{B}$-module by means of $b \bullet x := bsx$, for all $b, x \in \mathcal{A}$, and it is endowed with a nondegenerate skew-hermitian form $\tilde{\mathrm{h}} : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ given by $\tilde{\mathrm{h}}(x, y) = -x\bar{y}$. The triple product on $\mathcal{A}$ is given by

$$(x, y, z) = V_{x,sy}(z) = \tilde{\mathrm{h}}(x, y)sz + \tilde{\mathrm{h}}(z, y)sx + \tilde{\mathrm{h}}(z, x)sy,$$

for $x, y, z \in \mathcal{A}$.

In case $\dim \mathcal{C}_1 = \dim \mathcal{C}_2 = 4$, $\mathcal{A}$ (and hence $\mathcal{B}$) is a simple algebra, so that it is isomorphic to $\mathrm{Mat}_4(\mathbb{F})$, the involution $-$ on $\mathcal{A}$ is orthogonal, and hence the involution τ on $\mathcal{B}$ is symplectic, and Theorem 5.1 implies that $\mathcal{L}^{ss}(\mathcal{A})$ is isomorphic to $\mathfrak{osp}(4|4)$. The case of $\dim \mathcal{C}_1 = 1$, $\dim \mathcal{C}_2 = 4$ is similar, but $\mathcal{B}$ is isomorphic to $\mathrm{Mat}_2(\mathbb{F})$ in this case, the involution τ is orthogonal, and $\mathcal{L}^{ss}(\mathcal{A})$ is isomorphic to $\mathfrak{osp}(2|2)$.

For $\dim \mathcal{C}_1 = 2$ and $\dim \mathcal{C}_2 = 4$, $(\mathcal{A}, -)$ is a simple algebra with involution, but it is not simple as an algebra, the involution being of the second kind, and Theorem 5.1 gives that

$\mathcal{L}^{ss}(\mathcal{A})$ is isomorphic to $\mathfrak{psl}(2|2)$. The same happens for $\dim \mathcal{C}_1 = 2$ and $\dim \mathcal{C}_2 = 1$, in which case $\mathcal{L}^{ss}(\mathcal{A})$ is isomorphic to $\mathfrak{psl}(1|1)$.

The case of $\dim \mathcal{C}_1 = \dim \mathcal{C}_2 = 1$ does not give a J-ternary algebra, as $\mathcal{S} = 0$. Finally, if $\dim \mathcal{C}_1 = \dim \mathcal{C}_2 = 2$, $(\mathcal{A}, -)$ is the direct sum of two two-dimensional ideals, both with involution of the second kind, so it follows that $\mathcal{L}^{ss}(\mathcal{A})$ is isomorphic to $\mathfrak{psl}(1|1) \oplus \mathfrak{psl}(1|1)$.

We finish the paper by collecting all this information in the following *magic square* of Lie superalgebras:

$\dim \mathcal{C}_2$ \ $\dim \mathcal{C}_1$	1	2	4	8
1		$\mathfrak{psl}(1\|1)$	$\mathfrak{osp}(2\|2)$	$\mathfrak{psl}(4\|1)$
2	$\mathfrak{psl}(1\|1)$	$\mathfrak{psl}(1\|1) \oplus \mathfrak{psl}(1\|1)$	$\mathfrak{psl}(2\|2)$	$\mathfrak{g}(3,3)$
4	$\mathfrak{osp}(2\|2)$	$\mathfrak{psl}(2\|2)$	$\mathfrak{osp}(4\|4)$	$\mathfrak{el}(5;3)$
8	$\mathfrak{psl}(4\|1)$	$\mathfrak{g}(3,3)$	$\mathfrak{el}(5;3)$	$\mathfrak{g}(6,6)$

that contains three of the exceptional finite-dimensional Lie superalgebras specific of characteristic 3.

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