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A class of nonselfadjoint spectral differential operators of interest in physics

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ABSTRACT

It is shown that the nonselfadjoint (and non-normal) linear ordinary differential operators of a certain class are spectral operators of scalar type in the sense of Dunford and Bade. Operators of this kind appear in physical problems such as the scattering of spin waves by magnetic solitons.

1. Introduction

Besides its intrinsic mathematical interest, spectral analysis of operators is an important tool in physical and engineering problems. Many theoretical results are best interpreted and confronted with experiments through some kind of spectral analysis. From this point of view, theoretical problems involving spectral operators [4,5] are considerably simplified. Although there are known classes of non-normal spectral differential operators (for instance some ordinary differential operators with purely discrete spectrum [15,10,17], some ordinary differential operators with periodic coefficients [13], or some classes of second order elliptic differential operators defined in $L^2(\mathbb{R}^n)$, with $n \geq 3$ [14,6]), it is in general difficult to proof that concrete operators are spectral. The root of the difficulties lies in the possible appearance of spectral singularities, which are absent in the case of normal operators but are typical otherwise [14].

In this work it is shown that a class of non-normal ordinary differential operators, not considered before, are spectral operators of scalar type in the sense of Dunford [4,5] (actually, in the sense of Bade, since the operators are unbounded [1]). These operators appear, for instance, in the theory of scattering of spin waves by one-dimensional solitons [11], and are defined through a differential expression of the form $Lu = D_2 D_1 u$, where u is a function on $\mathbb{R}$ and

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$$D_i = -D^2 + q_i + h_i, \quad (i = 1, 2). \tag{1}$$

In the above expression D represents the derivative, $Du = u'$, $h_2 > h_1 > 0$ are constants, and q_1 and q_2 are real functions defined on $\mathbb{R}$. We require, for $i = 1, 2$,

1. $q_i \in C^1(\mathbb{R})$, $q_i' \in AC(\mathbb{R})$, $q_i'' \in L^2(\mathbb{R})$,
2. $\int_{\mathbb{R}} (1 + |x|^3)|q_i^{(k)}(x)|dx < \infty$, $0 \leq k \leq 2$,

where $AC(\mathbb{R})$ is the set of complex functions on $\mathbb{R}$ which are absolutely continuous on any compact interval, and the primes and the superscript (k) on a function denote, respectively, the derivatives and the k-th derivative of the function.

To avoid symbol proliferation, we use the same symbols, D_1, D_2, L, L^* (the last is defined below), for the differential expressions and for the operators defined by the differential expressions on appropriate domains. Since in this work each differential expression is associated with a unique operator, there is no ambiguity in this respect. It should be clear from the context whether the symbol represents the operator or the differential expression applied to some suitable function.

With the above assumptions, D_1 and D_2 are selfadjoint operators in $L^2(\mathbb{R})$, with domain $H^2(\mathbb{R})$, and with spectra bounded from below. For $i = 1, 2$, the essential spectrum of D_i is $[h_i, \infty)$, and the point spectrum is contained in $(-\infty, h_i]$. We require further that D_2 be positive definite and have a bounded inverse. That is, denoting the scalar product and the norm in $L^2(\mathbb{R})$ by $(\cdot, \cdot)$ and $\|\cdot\|$, respectively, there is $c_2 > 0$ such that $(D_2 u, u) \geq c_2 \|u\|^2$ for any $u \in \mathrm{dom}(D_2)$.

It may be that $\ker(D_1) \neq \{0\}$ (this happens generically in applications to spin wave dynamics). The orthogonal projection onto $\ker(D_1)$, denoted by P_0, is a finite rank operator of rank at most two. Let I_{d} be the identity operator and define $P_R = I_{\mathrm{d}} - P_0$. The restriction of D_1 to the subspace $\mathrm{dom}(D_1) \cap \mathrm{ran}(P_R)$, denoted by D_{1R}, is injective and has a bounded inverse, $D_{1R}^{-1} : \mathrm{ran}(D_1) \to \mathrm{ran}(P_R)$, and the operator $D_1 + P_0$ is injective and has a bounded inverse given by $(D_1 + P_0)^{-1} = D_{1R}^{-1} P_R + P_0$.

From D_1 and D_2 we build the linear operator $L : \mathrm{dom}(L) \to L^2(\mathbb{R})$ as

$$\begin{gathered} \mathrm{dom}(L) = \{u \in \mathrm{dom}(D_1) \mid D_1 u \in \mathrm{dom}(D_2)\}, \\ Lu = D_2 D_1 u, \quad u \in \mathrm{dom}(L). \end{gathered} \tag{2}$$

The domain of L is dense in $L^2(\mathbb{R})$ since it contains $\mathcal{S}(\mathbb{R})$, the Schwartz space of rapidly decreasing functions. By elementary means it is proved that L^*, the adjoint of L, is given by

$$\begin{gathered} \mathrm{dom}(L^*) = \{u \in \mathrm{dom}(D_2) \mid D_2 u \in \mathrm{dom}(D_1)\}, \\ L^* u = D_1 D_2 u, \quad u \in \mathrm{dom}(L^*). \end{gathered} \tag{3}$$

Again $\mathcal{S}(\mathbb{R}) \subset \mathrm{dom}(L^*)$. By similar means it is proved that $L^{**} = L$, and therefore, L and L^* are closed operators. They are not selfadjoint unless $q_1 = q_2$, since, on $\mathrm{dom}(L) \cap \mathrm{dom}(L^*)$,

$$L^* - L = 2(q_1' - q_2')D + q_1'' - q_2''. \tag{4}$$

We notice that L^* is a relatively bounded perturbation of L, with zero relative bound.

Let us remark two facts that will be used in the proof of some results. First, if $f \in \mathcal{S}(\mathbb{R})$ then Lf and $L^* f$ are in $L^1(\mathbb{R})$. Second, $\ker(L) = \ker(D_1)$ and if $f \in \ker(D_1)$ then $f \in \mathrm{dom}(L^*)$, because in this case $D_2 f = Bf$, where B is the operator of multiplication by $h_2 - h_1 + q_2 - q_1$, and then $Bf \in H^2(\mathbb{R})$. Therefore, $\mathrm{ran}(P_0) \subset \mathrm{dom}(L^*)$.

Notation. We denote by primes the derivatives with respect to the *first variable* of any function. The k-th derivative with respect to the *first variable* of a function is denoted by the superscript (k). For a complex number z the symbols $z^{1/2} = \sqrt{z}$ represent the principal branch of the square root of z. *Almost all*, abbreviated *a.a.*, and almost everywhere (*a.e.*) are to be understood with respect to the Lebesgue measure.

By $\rho(A)$, $\sigma(A)$, $\sigma_p(A)$, $\sigma_r(A)$, and $\sigma_c(A)$, we denote, respectively, the resolvent set, the spectrum, the point spectrum, the residual spectrum, and the continuous spectrum of the operator A. If A is selfadjoint, its essential spectrum is denoted by $\sigma_e(A)$. Otherwise, the five subsets of the spectrum defined in chapter 9 of Edmunds and Evans [7], all of which are occasionally called essential spectrum, are denoted by $\sigma_{ek}(A)$, $1 \leq k \leq 5$. For $z \in \rho(A)$ the resolvent of A is defined by $R_A(z) = (A - zI_\mathrm{d})^{-1}$.

We use the symbol I_d for any identity operator. It should be clear from the context in which space acts this identity operator. To lighten the notation we introduce the symbols $\mathcal{H} = L^2(\mathbb{R})$ and $\mathcal{S} = \mathcal{S}(\mathbb{R})$. As already said, we denote the scalar product and the norm in $\mathcal{H}$ by $(\cdot, \cdot)$ and $\|\cdot\|$, respectively.

2. Summary of main results

The main results of this work are easily summarized: there is a spectral resolution of the identity for L (Theorem 21), and L is a spectral operator of scalar type in the sense of Dunford and Bade (Theorem 22). The same statements hold for L^*. Theorems 21 and 22 follow from the three key results described below.

1. The spectral properties of L and L^* are related to those of the selfadjoint operator $\Lambda = D_2^{1/2} D_1 D_2^{1/2}$. This fact allows to show that the spectra of L and L^* lie on the real axis and to obtain a homomorphism B_L between the Boolean algebra of bounded Borel sets of $\mathbb{R}$ and a Boolean algebra of projections contained in $\mathcal{L}(\mathcal{H})$, the elements of which reduce L (the adjoint projections reduce L^*). The projections $B_L(b)$ are uniformly bounded when restricted to the Borel sets b contained of any bounded subset of $\mathbb{R}$, what implies that L and L^* do not have spectral singularities. These ideas are developed in sections 3 and 4. From these results it remains unclear whether the homomorphism is bounded and whether it can be extended to a homomorphism between the Boolean algebra of Borel set of $\mathbb{R}$ and a Boolean algebra of projections contained in $\mathcal{L}(\mathcal{H})$.

2. There is a spectral expansion associated to L (an expansion in terms of the bounded solutions of $Lu = \lambda u$, or "generalized eigenfunctions", see section 6), which can be obtained after a thorough analysis of the resolvent of L, exploiting the fact that L is an ordinary differential operator (part of this analytic work was made long ago by Kemp [9] for a more general class of ordinary differential operators which include L and L^*, but his results have to be sharpened for this work). The analysis of the Green function is carried out in section 5, and the spectral expansion (Theorem 9) and its consequences (Corollary 10) are proved in section 6.

3. The spectral expansions provide linear maps between $\mathcal{H}$ and a certain space of square integrable functions on which the "transformed L" acts multiplicatively. The fact that these maps are continuous (Theorem 11), and their restriction certain subspaces of $\mathcal{H}$ are injective and have a continuous inverse (Theorem 18) allow to show that B_L is bounded and can be extended to a bounded homomorphism E_L between the Boolean algebra of Borel sets of $\mathbb{R}$ and a Boolean algebra of projections contained in $\mathcal{L}(\mathcal{H})$. Theorems 21 and 22 follow from these results.

3. Spectrum of L and L^*

The spectral properties of L and L^* are related to the properties of the selfadjoint operator Λ, defined by

$$\begin{aligned} \mathrm{dom}(\Lambda) &= \mathrm{ran}\big({D_2}^{-1/2}(D_1 + P_0)^{-1}{D_2}^{-1/2}\big), \\ \Lambda f &= D_2^{1/2} D_1 D_2^{1/2} f, \quad f \in \mathrm{dom}(\Lambda), \end{aligned} \tag{5}$$

where $D_2^{1/2}$ is the positive square root of D_2. It is clear that $\mathrm{dom}(\Lambda)$ is dense in $\mathcal{H}$. That Λ is selfadjoint is proved by elementary means.

The operator $D_2^{1/2}$ is relatively bounded with respect to Λ, since for any $f \in \mathrm{dom}(\Lambda)$,

$$D_2^{1/2} f = (D_1 + P_0)^{-1} D_2^{-1/2} \Lambda f + P_0 D_2^{1/2} f, \tag{6}$$

and $P_0 D_2^{1/2}$ is a bounded finite rank operator. It can be seen similarly that D_1 and D_2 are both relatively bounded with respect to both L and L^*.

Theorem 1. *$\rho(L) = \rho(L^*) = \rho(\Lambda)$, and if $z \in \rho(L)$ then*

$$R_L(z) = D_2^{1/2} R_\Lambda(z) D_2^{-1/2}, \quad R_\Lambda(z) = D_2^{1/2} R_{L^*}(z) D_2^{-1/2}. \tag{7}$$

Proof. We prove first that $\rho(L) = \rho(L^*)$. If $z \in \rho(L^*)$ then the operator $L - zI_\mathrm{d} = D_2(L^* - zI_\mathrm{d}){D_2}^{-1}$ is injective and has a dense range, and its inverse, $D_2 R_{L^*}(z){D_2}^{-1}$, is bounded, since D_2 is relatively bounded with respect to L^*. Hence, $z \in \rho(L)$, what implies $\rho(L^*) \subseteq \rho(L)$.

Take now $z \in \rho(L) \setminus \{0\}$. Consider the equation $(L^* - zI_\mathrm{d})g = f$, with $f \in \mathcal{H}$. Acting with P_0 we get $P_0 g = -z^{-1} P_0 f$. Using $g = P_R g + P_0 g$ and acting with P_R we have

$$(L^* - zI_\mathrm{d}) P_R g = P_R f + \frac{1}{z} L^* P_0 f. \tag{8}$$

The above equation is a relation between elements of $\mathrm{ran}(P_R)$, and then we can apply D_{1R}^{-1} to both sides, obtaining

$$(L - zI_\mathrm{d}) D_{1R}^{-1} P_R g = D_{1R}^{-1} \Big(P_R f + \frac{1}{z} L^* P_0 f \Big). \tag{9}$$

Hence we have

$$P_R g = D_1 R_L(z) D_{1R}^{-1} \Big(P_R f + \frac{1}{z} L^* P_0 f \Big). \tag{10}$$

Then, equation $(L^* - zI_\mathrm{d})g = f$ has a unique solution $g \in \mathcal{H}$ for each $f \in \mathcal{H}$, which is clearly bounded by $\|g\| \le c\|f\|$, where c does not depend on f. Therefore, $z \in \rho(L^*)$ and thus $\rho(L) \setminus \{0\} \subseteq \rho(L^*)$. If $0 \in \rho(L)$ then $\ker(D_1) = \{0\}$ and D_1 is an injective operator with a bounded inverse. Thus it is clear that L^* is injective and has a bounded inverse and therefore $0 \in \rho(L^*)$. All together we have proved $\rho(L) \subseteq \rho(L^*)$. Thus, $\rho(L) = \rho(L^*)$.

Now we prove that $\rho(\Lambda) \subseteq \rho(L)$ and $\rho(L^*) \subseteq \rho(\Lambda)$. For $z \in \rho(\Lambda)$ it is clear that $L - zI_\mathrm{d} = D_2^{1/2}(\Lambda - zI_\mathrm{d}) D_2^{-1/2}$ is an injective operator with a dense range. Its inverse is bounded since $D_2^{1/2}$ is relatively bounded with respect to Λ. Hence, $z \in \rho(L)$ and $R_L(z)$ is given by the first of equations (7). Similarly, for $z \in \rho(L^*)$ the operator $\Lambda - zI_\mathrm{d} = D_2^{1/2}(L^* - zI_\mathrm{d}) D_2^{-1/2}$ is injective and has a dense range. Its inverse is bounded since $D_2^{1/2}$ is relatively bounded with respect to L^*, what implies that $z \in \rho(\Lambda)$ and that $R_\Lambda(z)$ is given by the second of equations (7). □

We notice that equations (7) can be cast to the following useful forms:

$$R_L(z) = (D_1 + P_0)^{-1} D_2^{-1/2} \Big(I_\mathrm{d} + z R_\Lambda(z) \Big) D_2^{-1/2} + P_0 D_2^{1/2} R_\Lambda(z) D_2^{-1/2}, \tag{11}$$

$$R_\Lambda(z) = D_2^{-1/2} (D_1 + P_0)^{-1} \Big(I_\mathrm{d} + z R_{L^*}(z) \Big) D_2^{-1/2} + P_0 D_2^{1/2} R_{L^*}(z) D_2^{-1/2}. \tag{12}$$

Theorem 1 implies that $\sigma(L) = \sigma(L^*) = \sigma(\Lambda) \subset \mathbb{R}$. In the next theorem we analyze the components of the spectrum.

Notation. It is convenient to introduce the notation $h_\mathrm{a} = (h_1 + h_2)/2$, $h_\mathrm{p} = h_1 h_2$, $h_m = -(h_2 - h_1)^2/4$, and the open interval $I = (h_\mathrm{p}, \infty)$. We denote by $\sigma_p'(L)$ the set of accumulation points of $\sigma_p(L)$.

Theorem 2. *The spectra of L, L^*, and Λ are equal, component by component: $\sigma_p(L) = \sigma_p(L^*) = \sigma_p(\Lambda)$, $\sigma_r(L) = \sigma_r(L^*) = \sigma_r(\Lambda) = \emptyset$, $\sigma_c(L) = \sigma_c(L^*) = \sigma_c(\Lambda) = \overline{I}$, and $\sigma_{ek}(L) = \sigma_{ek}(L^*) = \sigma_e(\Lambda) = \overline{I} \cup \sigma_p'(L)$, for $1 \leq k \leq 5$. The point spectrum is a countable bounded nowhere dense set which has at most two accumulation points: $\sigma_p'(L) \subseteq \{h_\mathrm{m}, h_\mathrm{p}\}$.*

Proof. To prove that the point spectra of the three operators coincide we notice that if f is an eigenfunction of Λ corresponding to the eigenvalue λ, then $D_2^{1/2} f$ and $D_2^{-1/2} f$ are eigenfunctions of L and L^*, respectively, corresponding to the eigenvalue λ; if f is an eigenfunction of L corresponding to the eigenvalue λ, then $D_2^{-1/2} f$ and $D_2{}^{-1} f$ are eigenfunctions of Λ and L^*, respectively, corresponding to the eigenvalue λ; and if f is an eigenfunction of L^* corresponding to the eigenvalue λ, then $D_2^{1/2} f$ and $D_2 f$ are eigenfunctions of Λ and L, respectively, corresponding to the eigenvalue λ. Since Λ is selfadjoint and $\mathcal{H}$ separable, $\sigma_p(\Lambda)$ is countable. That $\sigma_p(\Lambda)$ is a nowhere dense set follows from the fact that each eigenvalue of L is a zero of the function W on $\mathbb{C}$ defined in section 5, which is analytic in the upper half-plane of $\mathbb{C}$ and continuous on the real axis. The zeros of W form a bounded subset of the real axis which may have accumulation points only in $\{h_\mathrm{m}, h_\mathrm{p}\}$ (see section 5).

It is now obvious that the residual spectra of the three operators are empty and that their continuous spectra coincide. Since the spectrum of L lies on the real axis, the five sets $\sigma_{ek}(L)$, $1 \leq k \leq 5$, are equal, and are obtained by removing from $\sigma(L)$ the isolated eigenvalues, which have finite multiplicity (see point (i) of theorem 1.6 of chapter 9 of Edmunds and Evans [7], which clearly holds not only for selfadjoint operators in Hilbert spaces but for any operator in a Banach space whose spectrum contains only accumulation points of its resolvent set). Evidently, these statements hold also for L^*. Hence, the ten sets $\sigma_{ek}(L)$, $\sigma_{ek}(L^*)$, $1 \leq k \leq 5$, are all equal and coincide with $\sigma_e(\Lambda) = \overline{I} \cup \sigma_p'(L)$. □

4. Spectral projections for bounded Borel sets

Equation (7) for the resolvent of L suggests a way to define a family of spectral projections for L, as follows. Let $\mathcal{B}_b$ the algebra of bounded Borel sets of $\mathbb{R}$, $\mathcal{B}$ the Borel σ-algebra of $\mathbb{R}$, and $\mathcal{L}(\mathcal{H})$ the Banach algebra of bounded linear operators in $\mathcal{H}$. Let $E_\Lambda : \mathcal{B} \to \mathcal{L}(\mathcal{H})$ be the resolution of the identity for Λ. Since $\Lambda E_\Lambda(b)$ is a bounded operator if $b \in \mathcal{B}_b$, we can define a map $B_L : \mathcal{B}_b \to \mathcal{L}(\mathcal{H})$ by

$$B_L(b) = D_2^{1/2} E_\Lambda(b) D_2^{-1/2} = D_2^{1/2} R_\Lambda(z)(\Lambda - z I_\mathrm{d}) E_\Lambda(b) D_2^{-1/2} \tag{13}$$

for $b \in \mathcal{B}_b$ and $z \in \rho(L)$. The second equality of the above expression shows that $B_L(b)$ is a bounded projection for each bounded Borel set b. It is clear that B_L inherits from E_Λ all the properties of spectral projections, except the uniform boundedness. That is, B_L satisfies

$$B_L(b \cap d) = B_L(b) B_L(d), \tag{14}$$

$$B_L(b \cup d) = B_L(b) + B_L(d) - B_L(b \cap d), \tag{15}$$

for $b, d \in \mathcal{B}_b$. Therefore, the image of B_L is a Boolean algebra of projections and B_L is a homomorphism from the boolean algebra $\mathcal{B}_b$ onto its image. Moreover, if $\{b_i, i \in \mathbb{N}\}$ is a family of pairwise disjoint bounded Borel sets whose union is a bounded set, then

$$B_L\big(\cup_{i=1}^{\infty} b_i\big) f = \sum_{i=1}^{\infty} B_L(b_i) f, \quad f \in \mathcal{H}. \tag{16}$$

This relation implies the continuity of the projections from above and below, that is, if $\{b_i,\ i \in \mathbb{N}\}$ is a descending sequence of bounded Borel sets, then

$$B_L\big(\cap_i b_i\big)f = \lim_{i\to\infty} B_L(b_i)f, \quad f \in \mathcal{H}, \tag{17}$$

and if $\{b_i, i \in \mathbb{N}\}$ is an ascending sequence of bounded Borel sets whose union is a bounded set, then

$$B_L\big(\cup_i b_i\big)f = \lim_{i\to\infty} B_L(b_i)f, \quad f \in \mathcal{H}. \tag{18}$$

An analogue of the Stone formula is also inherited from E_L: if $a, b \in \mathbb{R}$, with $a < b$, then

$$\operatorname*{s-lim}_{\epsilon\to 0^+} \frac{1}{2\pi\mathrm{i}} \int_a^b \Big(R_L(\lambda+\mathrm{i}\epsilon) - R_L(\lambda-\mathrm{i}\epsilon)\Big)\, d\lambda = \frac{1}{2}\Big(B_L\big([a,b]\big) + B_L\big((a,b)\big)\Big). \tag{19}$$

It is straightforward to see that the projections reduce L, that is, for any $b \in \mathcal{B}_b$,

$$B_L(b)R_L(z)f = R_L(z)B_L(b)f, \quad f \in \mathcal{H}, \tag{20}$$

and if $f \in \operatorname{dom}(L)$ then $B_L(b)f \in \operatorname{dom}(L)$ and

$$B_L(b)Lf = LB_L(b)f. \tag{21}$$

We also have the following useful formula, analogous to equation (11):

$$B_L(b) = (D_1 + P_0)^{-1} D_2^{-1/2} \Lambda E_\Lambda(b) D_2^{-1/2} + P_0 D_2^{1/2} E_\Lambda(b) D_2^{-1/2}. \tag{22}$$

The above equation implies that $\operatorname{ran}\big(B_L(b)\big) \subset \operatorname{dom}(L)$, since $\operatorname{ran}\big(\Lambda E_\Lambda(b)\big) \subset \operatorname{dom}(\Lambda) \subset \operatorname{dom}(D_2^{1/2})$ and $\operatorname{ran}(P_0) \subset \operatorname{dom}(L)$.

We notice the following property, which will be used later in the proof of some results: if b is a non-empty bounded subset of $\mathbb{R}$, then the family of projections associated to the Borel subsets of b is uniformly bounded, in the sense that there is a constant c, which may depend on b, such that $\|B_L(e)\| \leq c$ if e is a Borel set contained in b. This follows from equation (22), since $\|\Lambda E_\Lambda(e)\| \leq \sup\{|z|, z \in b\}$ for any Borel set $e \subseteq b$.

It is not apparent that $\mathcal{B}_L$ is a bounded homomorphism, in the sense that there is $c > 0$ such that $\|B_L(b)\| \leq c$ for all $b \in \mathcal{B}_b$. In what follows we shall show that B_L is actually bounded and that it can be extended to a bounded homomorphism of $\mathcal{B}$ into a Boolean algebra of projections contained in $\mathcal{L}(\mathcal{H})$. To do this we take advantage of the fact that L and L^* are ordinary differential operators and their resolvents are integral operators whose kernel, the Green function, can be studied by standard analytic means.

5. The Green function of L

The Green function of L is constructed from the solutions of the differential equations $Lu = zu$, with $z \in \rho(L)$. Then, the starting point is the analysis of the solutions of these equations. For $z \in \mathbb{C}$, the fourth order differential equation $Lu = zu$ is equivalent to the system of four first order equations

$$y' = (A + B)y, \tag{23}$$

where $y : \mathbb{R} \to \mathbb{C}^4$ is a vector function, A is a 4×4 constant matrix, and $B : \mathbb{R} \to M_4(\mathbb{C})$ is a matrix-valued function. The only non-zero matrix elements of A and B are

$$\begin{aligned} A_{12} = A_{23} = A_{34} = 1, \quad A_{41} = z - h_{\mathrm{p}}, \quad A_{43} = -2h_{\mathrm{a}}, \\ B_{41} = q_2'' - (h_1 + q_1)(h_2 + q_2), \quad B_{42} = 2q_2', \quad B_{43} = h_1 + h_2 + q_1 + q_2. \end{aligned} \tag{24}$$

The characteristic polynomial of A is $p_{\mathrm{c}}(-\mathrm{i}\mu) - z$, where

$$p_{\mathrm{c}}(\mu) = \mu^4 + 2h_{\mathrm{a}}\,\mu^2 + h_{\mathrm{p}}, \tag{25}$$

and therefore the eigenvalues of A are $\mathrm{i}\mu_k$, $1 \leq k \leq 4$, where μ_k are the roots of the polynomial $p_{\mathrm{c}}(\mu) - z$. The four roots are simple if $z \notin \{h_{\mathrm{m}}, h_{\mathrm{p}}\}$. An eigenvector of A corresponding to the eigenvalue $\mathrm{i}\mu_k$ is the column vector p_k that has $(\mathrm{i}\mu_k)^{j-1}$ in the j-th row ($1 \leq j \leq 4$). Let $\Pi(z)$ be the 4×4 matrix whose k-th column is p_k, so that $\Pi_{jk}(z) = (\mathrm{i}\mu_k)^{j-1}$. If all the eigenvalues of A are different, then $\Pi(z)$ is regular. Let us also introduce the 4×4 diagonal matrix $D(x, z)$ whose matrix elements are given by $D_{jk}(x, z) = \exp(\mathrm{i}\mu_k x)\delta_{jk}$ for $1 \leq j, k \leq 4$. Bear in mind that the roots μ_k are functions of z.

Let e_i, $1 \leq i \leq n$, be the vectors of the canonical basis of $\mathbb{C}^n$. We use the norms for vectors of $v \in \mathbb{C}^n$ and for matrices $M \in M_n(\mathbb{C})$ given by $|v| = \sum_i |e_i \cdot v|$ and $|M| = \sum_{ij} |M_{ij}|$. Let us denote by $\mathbb{C}^+$ and $\mathbb{C}^-$, respectively, the open upper and lower half-planes of $\mathbb{C}$. The following theorem provides us with two fundamental matrices of the system (23) for the case $z \in \mathbb{C} \setminus \mathbb{R}$.

Theorem 3. *There are two matrix-valued functions Φ_+ and Φ_-, with domain $\mathbb{R} \times (\mathbb{C} \setminus \mathbb{R})$ and taking values on $M_4(\mathbb{C})$, written as*

$$\begin{aligned} \Phi_+(x, z) = \Pi(z)\big(I_{\mathrm{d}} + \Theta_+(x, z)\big)D(x, z), \\ \Phi_-(x, z) = \Pi(z)\big(I_{\mathrm{d}} + \Theta_-(x, z)\big)D(x, z), \end{aligned} \tag{26}$$

for $x \in \mathbb{R}$ and $z \in \mathbb{C} \setminus \mathbb{R}$, which have the following properties:

1. *For fixed z, $\Phi_+(\cdot, z)$ and $\Phi_-(\cdot, z)$ are fundamental matrices of the system (23).*
2. *For fixed x, $\Phi_+(x, \cdot)$ and $\Phi_-(x, \cdot)$ are analytic.*
3. *For fixed z, $|\Theta_+(x, z)|$ and $|\Theta_-(-x, z)|$ are $o(1)$ as $x \to \infty$.*
4. *For fixed $x \geq 0$, $|\Theta_+(x, z)|$ and $|\Theta_-(-x, z)|$ are $O(|z|^{-1/4})$ for $|z| \to \infty$.*
5. *The restrictions of Φ_+ and Φ_- to $\mathbb{R} \times \mathbb{C}^\sigma$, with $\sigma = \pm$, can be extended to continuous functions, $\Phi_+^{(\sigma)}$ and $\Phi_-^{(\sigma)}$, on $\mathbb{R} \times \overline{\mathbb{C}^\sigma}$.*
6. *For fixed $\lambda \in \mathbb{R}$, the functions $\Phi_+^{(\sigma)}(\cdot, \lambda)$ and $\Phi_-^{(\sigma)}(\cdot, \lambda)$ are solution matrices of the system (23).*
7. *For fixed $\lambda \in \mathbb{R}$, $|\Theta_+^{(\sigma)}(x, \lambda)|$ and $|\Theta_-^{(\sigma)}(-x, \lambda)|$ are $o(1)$ as $x \to \infty$.*
8. *For fixed $x \geq 0$ and if $\lambda \in \mathbb{R}$, $|\Theta_+^{(\sigma)}(x, \lambda)|$ and $|\Theta_-^{(\sigma)}(-x, \lambda)|$ are $O(\lambda^{-1/4})$ as $\lambda \to \infty$.*

Proof. The proof of this theorem is not given here. It only uses standard arguments, similar to those of Theorem 23, and it is a modification of the proof of theorem 8.1 of chapter 6 of Coddington and Levinson [3]. □

Of special interest are the solutions of $Lu = \lambda u$ for $\lambda \in [h_{\mathrm{p}}, \infty)$. In this case $p_{\mathrm{c}}(\mu) - \lambda$ has four roots, two real and two purely imaginary, which are denoted by $\mu_1 = \mathrm{i}\theta$, $\mu_2 = \nu$, $\mu_3 = -\nu$, $\mu_4 = -\mathrm{i}\theta$, where $\theta > 0$ and $\nu \geq 0$ are the functions of $\lambda \in [h_{\mathrm{p}}, \infty)$ given by

$$\theta(\lambda) = \left(\sqrt{\lambda + h_{\mathrm{a}}^2 - h_{\mathrm{p}}} + h_{\mathrm{a}}\right)^{1/2}, \quad \nu(\lambda) = \left(\sqrt{\lambda + h_{\mathrm{a}}^2 - h_{\mathrm{p}}} - h_{\mathrm{a}}\right)^{1/2}. \tag{27}$$

Notice that θ and ν are nonnegative increasing functions, with $\nu/\theta \leq 1$ and $\theta \geq \theta_0 = \sqrt{2h_{\mathrm{a}}} > 0$.

We need fundamental sets of solutions of $Lu = \lambda u$ that satisfy the estimates (29) of Theorem 4. To this end, we separate the interval I into two subintervals, $(h_\mathrm{p}, \lambda_s]$ and (λ_s, ∞), where $\lambda_s > h_\mathrm{p}$ is sufficiently large.

Theorem 4. *Let $\lambda_s > h_\mathrm{p}$ sufficiently large. For $1 \leq k \leq 4$ there are functions $\phi_k : \mathbb{R} \times \bar{I} \to \mathbb{C}$ and $\chi_k : \mathbb{R} \times \bar{I} \to \mathbb{C}$ such that:*

1. *For fixed $\lambda \in I$, $\{\phi_k(\cdot, \lambda), 1 \leq k \leq 4\}$ and $\{\chi_k(\cdot, \lambda), 1 \leq k \leq 4\}$ are fundamental sets of solutions of $Lu = \lambda u$.*
2. *For fixed $x \in \mathbb{R}$, $\phi_k(x, \cdot)$ and $\chi_k(x, \cdot)$ are analytic in $I \setminus \{\lambda_s\}$.*
3. *The restrictions to $\mathbb{R} \times [h_\mathrm{p}, \lambda_s]$ and to $\mathbb{R} \times (\lambda_s, \infty)$ of each of these functions are continuous functions.*
4. *For each $x \in \mathbb{R}$, the limits of $\phi_k(x, \lambda)$ and $\chi_k(x, \lambda)$ as $\lambda \to \lambda_s^+$ exist and provide fundamental sets of solutions (as in point 1) of $Lu = \lambda_s u$.*
5. *For $x \in \mathbb{R}$ and $\lambda \in \bar{I}$ the functions have the form*

$$\begin{aligned} \phi_k(x, \lambda) &= \exp(\mathrm{i}\mu_k x)\big(1 + r_k(x, \lambda)\big), \\ \chi_k(x, \lambda) &= \exp(\mathrm{i}\mu_k x)\big(1 + s_k(x, \lambda)\big), \end{aligned} \tag{28}$$

where r_k and s_k satisfy the bounds

$$\begin{aligned} |r_k(x, \lambda)| &\leq c\, \lambda^{-1/4}(1 + x)^{-1}, \quad x \geq 0, \\ |s_k(x, \lambda)| &\leq c\, \lambda^{-1/4}(1 - x)^{-1}, \quad x \leq 0, \end{aligned} \tag{29}$$

where $c > 0$ is a constant.

Proof. The proof of this theorem relies on standard arguments similar to those of theorem 8.1 of chapter 3 of Coddington and Levinson [3]. Since this theorem is a key point for this work, we provide the proof in Appendix A. □

In the remaining of the paper the functions ϕ_k and χ_k, $1 \leq k \leq 4$, have the properties of Theorem 4. There is a function $C : \bar{I} \to GL(4, \mathbb{C})$, bounded, analytic in $I \setminus \{\lambda_s\}$, such that

$$\chi_k(\cdot, \lambda) = \sum_{j=1}^{4} c_{jk}(\lambda)\, \phi_j(\cdot, \lambda), \quad 1 \leq k \leq 4, \quad \lambda \in \bar{I}, \tag{30}$$

where $c_{jk}(\lambda)$ are the matrix elements $C(\lambda)$. For $\lambda \to \infty$ we have the estimate $C(\lambda) = I_\mathrm{d} + O(1/\lambda^{1/4})$, which follows from the estimates of Theorem 4.

Since the differential expression L^*u has the same structure as Lu, there is an analogue to Theorem 4 concerning the solutions of $L^*u = \lambda u$. That is, there are functions ϕ_k^* and χ_k^*, $1 \leq k \leq 4$, with domain $\mathbb{R} \times \bar{I}$, such that, for each $\lambda \in \bar{I}$, $\{\phi_k^*(\cdot, \lambda), 1 \leq k \leq 4\}$ and $\{\chi_k^*(\cdot, \lambda), 1 \leq k \leq 4\}$ are fundamental sets of solutions of $L^*u = \lambda u$. The functions ϕ_k^* and χ_k^* have the properties of ϕ_k and χ_k, respectively, listed in points 2-5 of Theorem 4. The functions corresponding to r_k and s_k of Theorem 4, and to c_{jk} of equation (30), are represented by r_k^*, s_k^*, and c_{jk}^*, respectively.

Lemma 5. *If $\lambda \in I$ is not an eigenvalue of L, the linear space of bounded solutions of $Lu = \lambda u$ has dimension two, and the linear space of bounded solutions of $L^*u = \lambda u$ has also dimension two.*

Proof. In this proof the indices l and k take the values $1 \leq l \leq 3$ and $2 \leq k \leq 4$. Any bounded solution of $Lu = \lambda u$ is of the form $\phi = \sum_l \alpha_l \phi_l(\cdot, \lambda) = \sum_k \beta_k \chi_k(\cdot, \lambda)$, where α_l and β_k are complex numbers, which are not necessarily independent, since relation (30) implies $\alpha_l = \sum_k c_{lk}\beta_k$ and $\sum_k c_{4k}\beta_k = 0$. Unless $c_{42} = c_{43} = c_{44} = 0$, the last constraint on the values of β_k determines a two dimensional linear subspace of the linear space of all triples $(\beta_2, \beta_3, \beta_4)$, and thus the linear space of bounded solutions of $Lu = \lambda u$ has dimension two. In the case $c_{42} = c_{43} = c_{44} = 0$ the values of β_k are unconstrained and the linear space of bounded solutions of $Lu = \lambda u$ has dimension three. But then $\chi_4 = c_{41}\phi_1$, with $c_{41} \neq 0$, and then χ_4 is a nontrivial square integrable solution of $Lu = \lambda u$, what implies that λ is an eigenvalue of L, a possibility excluded by the hypothesis. The proof of the statement for the solutions of $L^*u = \lambda u$ is similar. $\square$

The Green matrix of the system (23) is built as follows. For $z \in \mathbb{C} \setminus \mathbb{R}$ we order the roots of $p_c(\mu) - z$ so that $\operatorname{Im}\mu_1 \geq \operatorname{Im}\mu_2 > 0 > \operatorname{Im}\mu_3 \geq \operatorname{Im}\mu_4$. This ordering determines the ordering of the columns of the fundamental matrices $\Phi_+(\cdot, z)$ and $\Phi_-(\cdot, z)$. For $x \in \mathbb{R}$ and $z \in \mathbb{C} \setminus \mathbb{R}$ we define the matrix

$$\Phi(x, z) = \Phi_+(x, z)P_+ + \Phi_-(x, z)P_-, \tag{31}$$

where $P_+ = \operatorname{diag}(1, 1, 0, 0)$ and $P_- = I_d - P_+$. The matrix $\Phi(x, z)$ is regular for all $x \in \mathbb{R}$, and its determinant, $W(z)$, is independent of x. For $x, \tau \in \mathbb{R}$ and $z \in \mathbb{C} \setminus \mathbb{R}$ the Green matrix of the system (23) is

$$K(x, \tau, z) = \begin{cases} \Phi_+(x, z)P_+\Phi^{-1}(\tau, z), & x \geq \tau, \\ -\Phi_-(x, z)P_-\Phi^{-1}(\tau, z), & x < \tau, \end{cases} \tag{32}$$

and satisfies $|K(x, \tau, z)| \leq k \exp(-\delta|x-\tau|)$, for $x, \tau \in \mathbb{R}$, where $k > 0$ and $\delta > 0$ depend on z but not on x or τ [9]. The Green function of L is obtained from the Green matrix of the system (23) as $G(x, \tau, z) = K_{14}(x, \tau, z)$.

The properties of Φ_+ and Φ_- (Theorem 3) imply that the function W, defined in $\mathbb{C} \setminus \mathbb{R}$, is analytic and can be continuously extended to the real axis both from the upper and lower open half-planes of $\mathbb{C}$. These continuous extensions define the functions W_+ and W_- on $\mathbb{R}$ by

$$W_\pm(\lambda) = \lim_{\epsilon \to 0^+} W(\lambda \pm \mathrm{i}\epsilon), \quad \lambda \in \mathbb{R}. \tag{33}$$

The properties of Φ_+ and Φ_- also guarantee that for $\lambda \in \mathbb{R}$ the limits

$$G_+(x, \tau, \lambda) = \lim_{\epsilon \to 0^+} G(x, \tau, \lambda + \mathrm{i}\epsilon), \quad G_-(x, \tau, \lambda) = \lim_{\epsilon \to 0^+} G(x, \tau, \lambda - \mathrm{i}\epsilon), \tag{34}$$

exist if $W_+(\lambda) \neq 0$ and $W_-(\lambda) \neq 0$, respectively.

Let N_+ and N_- be the set of zeros of W_+ and W_-, respectively, and $N = N_+ \cup N_-$. Since $W(z) = 1 + o(1)$ as $|z| \to \infty$, N_+, N_-, and N are bounded closed nowhere dense subsets of $\mathbb{R}$, and are contained in the spectrum of L. For $n \in \mathbb{N}$, let us define the sets

$$M_n = \big\{\lambda \in I \,\big|\, |W_+(\lambda)| > 1/(2n), |W_-(\lambda)| > 1/(2n)\big\}, \tag{35}$$

and $N_n = \sigma(L) \setminus M_n$, so that $\sigma(L) = M_n \cup N_n$. Notice that M_n is open and unbounded and N_n is closed and bounded, and that $\overline{M_n} \subseteq M_{n+1}$ and $N_{n+1} \subseteq N_n$. Clearly $\cap_n N_n = N$. We define $M = \cup_n M_n$, and then we have $M \cap N = \emptyset$ and $M \cup N = \sigma(L)$. We take λ_s (Theorem 4) large enough so that $[\lambda_s, \infty) \subseteq M_n$ for all $n \in \mathbb{N}$.

The expression $G_+(x, \tau, \lambda) - G_-(x, \tau, \lambda)$, for $x, \tau \in \mathbb{R}$ and $\lambda \in M$, defines a continuous function on $\mathbb{R} \times \mathbb{R} \times M$, bounded on each subset $\mathbb{R} \times \mathbb{R} \times M_n$, with $n \in \mathbb{N}$. For fixed τ and λ the function $G_+(\cdot, \tau, \lambda) - G_-(\cdot, \tau, \lambda)$ is a bounded solution of $Lu = \lambda u$, and for fixed x and λ the function $\overline{G_+(x, \cdot, \lambda)} - \overline{G_-(x, \cdot, \lambda)}$ is a bounded solution of $L^*u = \lambda u$. Therefore, there are continuous functions $\varphi_j : \mathbb{R} \times M \to \mathbb{C}$ and $\varphi_j^* : \mathbb{R} \times M \to \mathbb{C}$, $(j = 1, 2)$, bounded on each subset $\mathbb{R} \times M_n$ $(n \in \mathbb{N})$, such that, for each fixed $\lambda \in M$, $\varphi_1(\cdot, \lambda)$ and $\varphi_2(\cdot, \lambda)$

are two linearly independent bounded solutions of $Lu = \lambda u$, and $\varphi_1^*(\cdot,\lambda)$ and $\varphi_2^*(\cdot,\lambda)$ are two bounded solutions of $L^*u = \lambda u$, and

$$G_+(x,\tau,\lambda) - G_-(x,\tau,\lambda) = 2\pi\mathrm{i}\sum_{j=1}^{2}\varphi_j(x,\lambda)\,\overline{\varphi_j^*(\tau,\lambda)}. \tag{36}$$

The asymptotic analysis shows [9] that for fixed x and τ and $\lambda \to \infty$

$$G_+(x,\tau,\lambda) - G_-(x,\tau,\lambda) = \frac{2\mathrm{i}\cos\big(\nu(\lambda)(x-\tau)\big)}{p_{\mathrm{c}}'\big(\nu(\lambda)\big)}\Big(1+o(1)\Big), \tag{37}$$

where $\nu(\lambda)$ is given by the second of equations (27). Then, we can choose the functions φ_j so that for $n \in \mathbb{N}$

$$|\varphi_j(x,\lambda)| \le \frac{c_n}{\lambda^{3/8}}, \quad |\varphi_j^*(x,\lambda)| \le \frac{c_n}{\lambda^{3/8}}, \quad x \in \mathbb{R}, \quad \lambda \in M_n, \tag{38}$$

where the sequence c_n may be unbounded.

Lemma 6. *For $1 \le j \le 2$, $1 \le l \le 3$, and $2 \le k \le 4$, there are functions α_{jl}, β_{jk}, α_{jl}^*, β_{jk}^*, with domain M, continuous on $M \setminus \{\lambda_s\}$, and bounded on each subset $\overline{M_n}$, with $n \in \mathbb{N}$, such that for each $x \in \mathbb{R}$ and $\lambda \in M$,*

$$\varphi_j(x,\lambda) = \sum_{l=1}^{3}\alpha_{jl}(\lambda)\phi_l(x,\lambda) = \sum_{k=2}^{4}\beta_{jk}(\lambda)\chi_k(x,\lambda), \tag{39}$$

$$\varphi_j^*(x,\lambda) = \sum_{l=1}^{3}\alpha_{jl}^*(\lambda)\phi_l^*(x,\lambda) = \sum_{k=2}^{4}\beta_{jk}^*(\lambda)\chi_k^*(x,\lambda). \tag{40}$$

Moreover, there is $c > 0$ such that, for each $\lambda \ge \lambda_s$,

$$\lambda^{p_l}|\alpha_{jl}(\lambda)| \le c, \;\; \lambda^{p_k}|\beta_{jk}(\lambda)| \le c, \;\; \lambda^{p_l}|\alpha_{jl}^*(\lambda)| \le c, \;\; \lambda^{p_k}|\beta_{jk}^*(\lambda)| \le c, \tag{41}$$

where $p_i = 5/8$ for $i = 1,4$ and $p_i = 3/8$ for $i = 2,3$.

Proof. In this proof the indices take the values $1 \le j \le 2$, $1 \le l \le 3$, and $2 \le k \le 4$. For $\lambda \in M$, $\varphi_j(\cdot,\lambda)$ is a unique linear combination of $\phi_m(\cdot,\lambda)$, with $1 \le m \le 4$. The boundedness of $\varphi_j(\cdot,\lambda)$ for each $\lambda \in M$ implies that the coefficient of $\phi_4(\cdot,\lambda)$ has to be zero. This linear combination defines the functions α_{jl}. The functions φ_j, ϕ_l are continuous and bounded on each subset $\overline{\mathbb{R}^+} \times (M_n \setminus \{\lambda_s\})$. By Theorem 4 and Lemma 25 of Appendix B the functions α_{jl} are continuous on $M_n \setminus \{\lambda_s\}$, for each $n \in \mathbb{N}$. Since $\overline{M_n} \subset M_{n+1}$, the restriction of φ_j to $\mathbb{R} \times \overline{M_n}$ is continuous and bounded. By Theorem 4, the restrictions of ϕ_l to $\overline{\mathbb{R}^+} \times \big(\overline{M_n} \cap [h_{\mathrm{p}}, h_s]\big)$ are continuous and bounded, and the restrictions of ϕ_l to $\overline{\mathbb{R}^+} \times \big(\overline{M_n} \cap (h_s,\infty)\big)$ can be extended to continuous bounded functions on $\overline{\mathbb{R}^+} \times \big(\overline{M_n} \cap [h_s,\infty)\big)$. Then, from Lemma 25 we infer that the functions α_{jl} are bounded on each bounded subset of $\overline{M_n}$. To see that they are bounded on $\overline{M_n}$, take $\lambda \in \overline{M_n}$ large enough and consider the sequences of positive numbers $x_m = 2\pi m/\nu(\lambda)$, $y_m = x_m + \pi/\big(2\nu(\lambda)\big)$, $m \in \mathbb{N}$. From the boundedness of the sequences $\varphi_j(x_m,\lambda)$ and $\varphi_j(y_m,\lambda)$ it is easily obtained that the restrictions of α_{j2} and α_{j3} to $\overline{M_n}$ are bounded functions. Then, from the boundedness of $\varphi_j(0,\lambda)$ and the estimates (29), we find that α_{j1} is also bounded on $\overline{M_n}$. The proofs of the continuity and boundedness of β_{jk}, α_{jl}^*, and β_{jk}^* are similar.

The estimates (38) imply that $\alpha_{j2}(\lambda)$, $\alpha_{j3}(\lambda)$, $\beta_{j2}(\lambda)$, and $\beta_{j3}(\lambda)$ are $O\big(1/\lambda^{3/8}\big)$ for $\lambda \to \infty$. From the relations $\alpha_{jl}(\lambda) = \sum_k c_{lk}(\lambda)\beta_{jk}(\lambda)$ and $\sum_k c_{4k}(\lambda)\beta_{jk}(\lambda) = 0$, and the asymptotics $c_{im}(\lambda) = \delta_{im} + O(1/\lambda^{1/4})$, $1 \le i,m \le 4$, we find that $\alpha_{j1}(\lambda)$ and $\beta_{j4}(\lambda)$ are both $O(1/\lambda^{5/8})$. The estimates (41) follow taking into account the boundedness of α_{jl} and β_{jk} on each set $\overline{M_n}$. The estimates for α^*_{jl} and β^*_{jk} are proved in a similar way. □

6. Spectral expansion

For $z \in \rho(L)$ the resolvent $R_L(z)$ is an integral operator whose kernel is the Green function, $G(x,\tau,z)$. We obtain an integral formula for the Green function in the following standard way. Let λ_0 be a real number such that (λ_0,∞) contains the spectrum of L and the point h_m, and for each $\epsilon > 0$ let C_ϵ be the curve on the complex plane formed by the points which are at a distance ϵ from (λ_0,∞), oriented in clockwise sense. The curve separates the complex plane into two disjoint subsets, one of which contains the spectrum of L. For fixed x and τ, the function $G(x,\tau,\cdot)$ is analytic in the subset that does not contain the spectrum of L. From the behavior of $G(x,\tau,\xi)$ for large $|\xi|$ in the subset that does not contain the spectrum of L and the Cauchy theorem we obtain, for $z \notin (\lambda_0,\infty)$ and ϵ sufficiently small,

$$G(x,\tau,z) = \frac{1}{2\pi\mathrm{i}} \int\limits_{C_\epsilon} \frac{G(x,\tau,\xi)}{\xi - z}\, d\xi. \tag{42}$$

We divide C_ϵ into three pieces, as follows. Take $\eta > 0$. The first piece, S_ϵ, is the subset of points $\xi \in C_\epsilon$ such that $\mathrm{Re}\,\xi < \lambda_0$. The second piece consists of the points $\xi \in C_\epsilon$ with $\lambda_0 \le \mathrm{Re}\,\xi \le h_\mathrm{p} + \eta$, and the third piece is formed by the points of C_ϵ with real part larger than $h_\mathrm{p} + \eta$. Then, we have

$$\begin{aligned} G(x,\tau,z) = \frac{1}{2\pi\mathrm{i}} \int\limits_{S_\epsilon} \frac{G(x,\tau,\xi)}{\xi - z} d\xi + \frac{1}{2\pi\mathrm{i}} \int\limits_{\lambda_0}^{h_\mathrm{p}+\eta} F_\epsilon(x,\tau,z,\lambda) d\lambda \\ + \frac{1}{2\pi\mathrm{i}} \int\limits_{h_\mathrm{p}+\eta}^{\infty} F_\epsilon(x,\tau,z,\lambda) d\lambda, \end{aligned} \tag{43}$$

where

$$F_\epsilon(x,\tau,z,\lambda) = \frac{G(x,\tau,\lambda+\mathrm{i}\epsilon)}{\lambda + \mathrm{i}\epsilon - z} - \frac{G(x,\tau,\lambda-\mathrm{i}\epsilon)}{\lambda - \mathrm{i}\epsilon - z}. \tag{44}$$

The next lemma is used in the proof of part of Theorem 9.

Lemma 7. *For each $a, b \in \mathbb{R}$, with $a < b$, and $z \in \rho(L) \setminus [a,b]$, we have*

$$\begin{aligned} \underset{\epsilon\to 0^+}{\text{s-lim}}\, \frac{1}{2\pi\mathrm{i}} \int\limits_a^b \left(\frac{R_L(\lambda+\mathrm{i}\epsilon)}{\lambda+\mathrm{i}\epsilon - z} - \frac{R_L(\lambda-\mathrm{i}\epsilon)}{\lambda-\mathrm{i}\epsilon - z} \right) d\lambda \\ = \frac{1}{2} R_L(z) \Big(B_L\big((a,b)\big) + B_L\big([a,b]\big) \Big). \end{aligned} \tag{45}$$

Proof. Using equation (11) we can rewrite the left-hand side of equation (45) as the strong limit as $\epsilon \to 0^+$ of

$$\begin{aligned} (D_1 + P_0)^{-1} D_2^{-1/2} \big(A_\epsilon(z) + z B_\epsilon(z) \big) D_2^{-1/2} + P_0 D_2^{1/2} B_\epsilon(z) D_2^{-1/2} + \\ h_\epsilon(z) (D_1 + P_0)^{-1} D_2^{-1}, \end{aligned} \tag{46}$$

where

$$A_\epsilon(z) = \frac{1}{2\pi \mathrm{i}} \int_a^b \Big(R_\Lambda(\lambda + \mathrm{i}\epsilon) - R_\Lambda(\lambda - \mathrm{i}\epsilon) \Big) d\lambda, \tag{47}$$

$$B_\epsilon(z) = \frac{1}{2\pi \mathrm{i}} \int_a^b \Big(\frac{R_\Lambda(\lambda + \mathrm{i}\epsilon)}{\lambda + \mathrm{i}\epsilon - z} - \frac{R_\Lambda(\lambda - \mathrm{i}\epsilon)}{\lambda - \mathrm{i}\epsilon - z} \Big) d\lambda, \tag{48}$$

$$h_\epsilon(z) = \frac{1}{2\pi \mathrm{i}} \int_a^b \Big(\frac{1}{\lambda + \mathrm{i}\epsilon - z} - \frac{1}{\lambda - \mathrm{i}\epsilon - z} \Big) d\lambda. \tag{49}$$

It is clear that $h_\epsilon(z)$ vanishes as $\epsilon \to 0^+$ since $z \notin [a, b]$. The strong limit of $A_\epsilon(z)$ is given by the analogue of Stone's formula (19). To obtain an expression for $B_\epsilon(z)$ we take advantage of the operational calculus for the selfadjoint operator Λ. On $[a, b] \times \mathbb{R}$ we define the bounded Borel function f_ϵ by

$$f_\epsilon(\lambda, \xi) = \frac{1}{2\pi \mathrm{i}} \left(\frac{1}{(\lambda + \mathrm{i}\epsilon - z)(\xi - \lambda - \mathrm{i}\epsilon)} - \frac{1}{(\lambda - \mathrm{i}\epsilon - z)(\xi - \lambda + \mathrm{i}\epsilon)} \right), \tag{50}$$

for $\lambda \in [a, b]$ and $\xi \in \mathbb{R}$. From f_ϵ we define the function g_ϵ on $\mathbb{R}$ by

$$g_\epsilon(\xi) = \int_a^b f_\epsilon(\lambda, \xi)\, d\lambda, \quad \xi \in \mathbb{R}. \tag{51}$$

Then, by lemma 4.1 of Teschl [16], $B_\epsilon(z) = g_\epsilon(\Lambda)$. The integral defining $g_\epsilon(\xi)$ can be performed by elementary means, and it is easy to see that g_ϵ is bounded uniformly in ϵ if $z \neq a, b$ and $\epsilon > 0$ is sufficiently small. For $z \notin [a, b]$ we obtain

$$g(\xi) = \lim_{\epsilon \to 0^+} g_\epsilon(\xi) = \frac{1}{2} \Big(\chi_{(a,b)}(\xi) + \chi_{[a,b]}(\xi) \Big) \frac{1}{\xi - z}, \quad \xi \in \mathbb{R}, \tag{52}$$

where χ_G is the characteristic function of the set G. The operational calculus for Λ (lemma 4.2 and theorem 3.1 of Teschl [16]) gives

$$\operatorname*{s-lim}_{\epsilon \to 0^+} B_\epsilon(z) = \operatorname*{s-lim}_{\epsilon \to 0^+} g_\epsilon(\Lambda) = g(\Lambda) = R_\Lambda(z) \frac{1}{2} \Big(E_\Lambda\big((a, b)\big) + E_\Lambda\big([a, b]\big) \Big). \tag{53}$$

Then, the left-hand side of equation (46) is equal to

$$\begin{gathered} \frac{1}{2}(D_1 + P_0)^{-1} D_2^{-1/2} \big(I_{\mathrm{d}} + z R_\Lambda(z)\big) \Big(E_\Lambda\big((a, b)\big) + E_\Lambda\big([a, b]\big) \Big) D_2^{-1/2} + \\ + \frac{1}{2} P_0 D_2^{1/2} R_\Lambda(z) \Big(E_\Lambda\big((a, b)\big) + E_\Lambda\big([a, b]\big) \Big) D_2^{-1/2}, \end{gathered} \tag{54}$$

which, using $I_{\mathrm{d}} + z R_\Lambda(z) = \Lambda R_\Lambda(z)$, the definition of Λ, and $(D_1 + P_0)^{-1} D_1 f = P_R f$ for $f \in \mathrm{dom}(D_1)$, can be cast to the form

$$D_2^{1/2} R_\Lambda(z) \frac{1}{2} \Big(E_\Lambda\big((a, b)\big) + E_\Lambda\big([a, b]\big) \Big) D_2^{-1/2}. \tag{55}$$

The statement of the lemma is obtained from the above expression by inserting $D_2^{-1/2} D_2^{1/2}$ between $R_\Lambda(z)$ and the spectral projections, what is allowed since the range of $E_\Lambda([a, b])$ is contained in the domain of $D_2^{1/2}$. □

For $f \in \mathcal{S}$ we define on M the functions $T_j f$, $S_j f$, $j = 1, 2$, by

$$T_j f(\lambda) = \int_{-\infty}^{\infty} f(\tau)\, \overline{\varphi_j^*(\tau,\lambda)}\, d\tau, \quad S_j f(\lambda) = \int_{-\infty}^{\infty} f(\tau)\, \overline{\varphi_j(\tau,\lambda)}\, d\tau, \tag{56}$$

for $\lambda \in M$. For each $n \in \mathbb{N}$ the restrictions of the functions φ_j and φ_j^* to $\mathbb{R} \times M_n$ are continuous and bounded, and therefore the restrictions of $T_j f$ and $S_j f$ ($j = 1, 2$) to M_n are continuous, bounded, integrable and square integrable functions, since, for $\lambda \in M_n$,

$$\lambda \big| T_j f(\lambda) \big| = \left| \int_{-\infty}^{\infty} Lf(x) \overline{\varphi_j^*(x,\lambda)}\, dx \right| \leq \frac{c_n}{\lambda^{3/8}} \int_{\mathbb{R}} |Lf(x)|\, dx. \tag{57}$$

To derive the above inequality we use of integration by parts, which is allowed since $f \in \mathcal{S}$, the estimates (38), and the fact that $Lf \in L^1(\mathbb{R})$ if $f \in \mathcal{S}$. The analogous properties for $S_j f$ are shown similarly.

Remark 8. The bounds of the restrictions of φ_j and φ_j^* to $\mathbb{R} \times M_n$ need not be uniform in n, and thus $T_j f$ and $S_j f$ may be unbounded on M (but they are continuous). Therefore, they do not belong to $L^2(M)$ necessarily.

Theorem 9. *Let $n \in \mathbb{N}$. For $z \in \rho(L) \setminus (\lambda_0, \infty)$, for each $f \in \mathcal{S}$, and for $x \in \mathbb{R}$,*

$$R_L(z) f(x) = R_L(z) B_L(N_n) f(x) + \int_{M_n} \sum_{j=1}^{2} \frac{\varphi_j(x,\lambda) T_j f(\lambda)}{\lambda - z}\, d\lambda, \tag{58}$$

$$R_{L^*}(z) f(x) = R_{L^*}(z) B_L^*(N_n) f(x) + \int_{M_n} \sum_{j=1}^{2} \frac{\varphi_j^*(x,\lambda) S_j f(\lambda)}{\lambda - z}\, d\lambda. \tag{59}$$

Proof. We prove relation (58). The proof of relation (59) is similar. Take $f \in \mathcal{S}$, multiply equation both sides of (43) by $f(\tau)$ and integrate over $\tau \in \mathbb{R}$. Hence we get $R_L(z) f(x) = A_\epsilon f(x) + B_{\eta\epsilon} f(x) + C_{\eta\epsilon} f(x)$, where each term of the right-hand side of this equation is the contribution to $R_L(z) f(x)$ of the corresponding term of the right-hand side of equation (43). It is clear that $A_\epsilon f$ vanishes as $\epsilon \to 0^+$, and that

$$B_{\eta\epsilon} f = \frac{1}{2\pi \mathrm{i}} \int_{\lambda_0}^{h_\mathrm{p}+\eta} \left(\frac{R_L(\lambda + \mathrm{i}\epsilon)}{\lambda + \mathrm{i}\epsilon - z} - \frac{R_L(\lambda - \mathrm{i}\epsilon)}{\lambda - \mathrm{i}\epsilon - z} \right) d\lambda\, f. \tag{60}$$

Applying Lemma 7, we get that the limit of $B_{\eta\epsilon} f$ as $\epsilon \to 0^+$ and $\eta \to 0^+$ is $R_L(z) B_L\big([\lambda_0, h_\mathrm{p}]\big) f$.

To obtain $C_{\eta\epsilon} f$ we define $M_{n\eta} = M_n \cap I_\eta$ and $N_{n\eta} = N_n \cap I_\eta$, where $I_\eta = (h_\mathrm{p} + \eta, \infty)$. Notice that $M_{n\eta} \subseteq M_{n\eta'}$ and $N_{n\eta} \subseteq N_{n\eta'}$ if $\eta > \eta'$, and $\cup_{\eta > 0} M_{n\eta} = M_n$ and $\cup_{\eta>0} N_{n\eta} = N_n \cap I$. The integral over I_η that defines $C_{\eta\epsilon} f$ is equal to the sum of an integral over $N_{n\eta}$ and an integral over $M_{n\eta}$, so that we have $C_{\eta\epsilon} f = D_{\eta\epsilon} f + E_{\eta\epsilon} f$, where $D_{\eta\epsilon} f$ and $E_{\eta\epsilon} f$ are the contribution of the integrals over $N_{n\eta}$ and $M_{n\eta}$, respectively.

Let us study first $D_{\eta\epsilon} f$, which is given by an expression like the right-hand side of equation (60), with the integral over $N_{n\eta}$ instead of over $(\lambda_0, h_\mathrm{p} + \eta)$. Let us introduce the bounded interval $I_{s\eta} = (h_\mathrm{p} + \eta, \lambda_s)$. It is clear that $N_{n\eta} = N_n \cap I_{s\eta}$, and therefore the integral over $N_{n\eta}$ is equal to the integral over $I_{s\eta}$ minus the integral over $I_{s\eta} \cap N_n^c$. Since N_n^c is an open set, it is equal to the union of an at most countable family of disjoint open intervals, I_k, with k in some subset K of $\mathbb{N}$. Hence, $I_{s\eta} \cap N_n^c = \cup_k (I_{s\eta} \cap I_k)$. Thus $D_{\eta\epsilon} f$

is expressed as an integral over $I_{s\eta}$ minus the sum of integrals over $I_{s\eta} \cap I_k$. Applying Lemma 7 to each of these integrals we get that the limit as $\epsilon \to 0^+$ of $D_{\eta\epsilon} f$ is

$$\frac{1}{2} R_L(z) \Big(B_L(I_{s\eta}) + B_L(\overline{I_{s\eta}}) - \sum_{k \in K} \Big(B_L\big(I_{s\eta} \cap I_k\big) + B_L\big(\overline{I_{s\eta} \cap I_k}\big) \Big) \Big) f. \tag{61}$$

It is clear that $\sigma_p(L) \cap I$ is contained in the interior of N_n, and therefore $\sigma_p(L) \cap \overline{I_k} = \emptyset$ for all $k \in K$. This implies $B_L\big(\overline{I_{s\eta} \cap I_k}\big) = B_L(\overline{I_{s\eta}} \cap I_k)$, and then from the expression (61) we get

$$\begin{aligned} \lim_{\epsilon \to 0^+} D_{\eta\epsilon} f &= \frac{1}{2} R_L(z) \Big(B_L(I_{s\eta} \cap N_n) + B_L(\overline{I_{s\eta}} \cap N_n) \Big) f \\ &= \frac{1}{2} R_L(z) \Big(B_L(I_{s\eta}) + B_L(\overline{I_{s\eta}}) \Big) B_L(N_n) f. \end{aligned} \tag{62}$$

The limit $\eta \to 0^+$ of the above expression is equal to $R_L(z) B_L(I \cap N_n) f$. Since the spectral projections $B_L(b)$ are supported on the spectrum of L, and since $\sigma(L) \cap [\lambda_0, h_{\mathrm{p}}] \subset N_n \cap [\lambda_0, h_{\mathrm{p}}]$, we obtain that the limit $\epsilon \to 0^+$ and $\eta \to 0^+$ of $B_{\eta\epsilon} f + D_{\eta\epsilon} f$ is equal to $R_L(z) B_L(N_n)$, which is the first term of the right-hand side of equation (58).

It remains to analyze the contribution of $E_{\eta\epsilon} f$, which is given by

$$E_{\eta\epsilon} f(x) = \frac{1}{2\pi \mathrm{i}} \int\limits_{\mathbb{R}} \int\limits_{M_{n\eta}} F_\epsilon(x, \tau, z, \lambda) f(\tau) \, d\lambda \, d\tau. \tag{63}$$

By Fubini's theorem, the iterated integrals of the right-hand side are equal to an integral over $M_{n\eta} \times \mathbb{R}$, the integrand of which, $F_\epsilon(x, \tau, z, \lambda) f(\tau)$, is bounded by a function $g(\lambda, z)|f(\tau)|$ independent of ϵ and integrable on $M_{n\eta} \times \mathbb{R}$. Hence, the limit $\epsilon \to 0^+$ can be interchanged with the integral and we obtain the integral over $M_{n\eta} \times \mathbb{R}$ of the function

$$F(\lambda, \tau) = \lim_{\epsilon \to 0^+} F_\epsilon(x, \tau, z, \lambda) f(\tau) = \frac{\sum_{j=1}^{2} \varphi_j(x, \lambda) \overline{\varphi_j^*(\tau, \lambda)}}{\lambda - z} f(\tau), \tag{64}$$

for $(\lambda, \tau) \in M_{n\eta} \times \mathbb{R}$. In the second equality of the above expression we used equation (36). Now we can use Fubini's theorem again to perform first the integral in the variable τ. Taking afterwards the limit $\eta \to 0^+$ we obtain the second term of the right-hand side of equation (58). □

Corollary 10. *For each $n \in \mathbb{N}$ and for each $f \in \mathcal{S}$ we have two spectral expansions:*

$$f(x) = B_L(N_n) f(x) + \int\limits_{M_n} \sum_{j=1}^{2} \varphi_j(x, \lambda) T_j f(\lambda) \, d\lambda, \tag{65}$$

$$f(x) = B_L^*(N_n) f(x) + \int\limits_{M_n} \sum_{j=1}^{2} \varphi_j^*(x, \lambda) S_j f(\lambda) \, d\lambda. \tag{66}$$

Moreover, the following analogues of the Parseval equality hold for $f, g \in \mathcal{S}$:

$$(f, g) = \big(B_L(N_n) f, g \big) + \int\limits_{M_n} \sum_{j=1}^{2} T_j f(\lambda) \overline{S_j g(\lambda)} \, d\lambda, \tag{67}$$

$$(f,g) = \big(B_L^*(N_n)f,g\big) + \int\limits_{M_n} \sum_{j=1}^{2} S_j f(\lambda)\overline{T_j g(\lambda)}\,d\lambda. \tag{68}$$

Proof. Apply $L - zI_\mathrm{d}$ to equation (58) and $L^* - zI_\mathrm{d}$ to equation (59), and notice that we can differentiate four times under the integral sign since, for $j = 1,2$, $T_j f, S_j f \in L^1(M_n)$, and φ_j, φ_j^*, and their derivatives with respect to their first variable up to third order, are bounded functions on $\mathbb{R} \times M_n$. Equalities (67) and (68) are obtained, respectively, by multiplying equations (65) and (66) by $\overline{g(x)}$ and integrating in x over $\mathbb{R}$, interchanging the order of integration in the second terms, which is clearly allowed. □

7. The bounded linear maps U_n and V_n

Let us introduce the Hilbert space $\mathcal{J} = L^2(M) \oplus L^2(M)$, and the projections $P_j : \mathcal{J} \to L^2(M)$, $j = 1,2$, onto each of the two components of $\mathcal{J}$. The scalar product of $f,g \in \mathcal{J}$ is given by

$$(f,g)_{\mathcal{J}} = \int\limits_{M} \sum_{j=1}^{2} P_j f(\lambda)\,\overline{P_j g(\lambda)}\,d\lambda, \tag{69}$$

and the norm of $f \in \mathcal{J}$ is defined by $\|f\|_{\mathcal{J}}^2 = (f,f)_{\mathcal{J}}$.

Let $\chi_b : M \to \mathbb{R}$ be the characteristic function of the set $b \subseteq M$. For any Borel set $b \subseteq \mathbb{R}$ we define the orthogonal projection $\hat{\chi}(b)$ in $\mathcal{J}$ by $P_j\hat{\chi}(b)f = \chi_{M\cap b}P_j f$, for $j = 1,2$, and $f \in \mathcal{J}$. For $n \in \mathbb{N}$ and $b \in \mathcal{B}$ we also define the orthogonal projection $\hat{\chi}_n(b) = \hat{\chi}(M_n)\hat{\chi}(b)$ in $\mathcal{J}$. Notice that $\hat{\chi}_n(b) \le \hat{\chi}_m(b)$ if $n \le m$ and s-$\lim_n \hat{\chi}_n(b) = \hat{\chi}(b)$. It is convenient to introduce the notation $\mathcal{J}_n = \operatorname{ran}\big(\hat{\chi}_n(\mathbb{R})\big)$. Finally, we define the selfadjoint linear operator Q in $\mathcal{J}$ given by

$$\begin{gathered}\operatorname{dom}(Q) = \left\{ f \in \mathcal{J},\ \int\limits_{M} \lambda^2 \sum_{j=1}^{2} \big|P_j f(\lambda)\big|^2 d\lambda < \infty \right\}, \\ P_j Q f(\lambda) = \lambda P_j f(\lambda), \quad j = 1,2, \quad \lambda \in M, \quad f \in \operatorname{dom}(Q).\end{gathered} \tag{70}$$

Evidently $\sigma(Q) = M$ and $\hat{\chi}$ is the resolution of the identity for Q. For $n \in \mathbb{N}$ we denote by Q_n the restriction of Q to $\mathcal{J}_n \cap \operatorname{dom}(Q)$. Clearly, $\sigma(Q_n) = M_n$.

For $f \in \mathcal{S}$ and $n \in \mathbb{N}$ the functions on M given by $\chi_{M_n}T_j f$ and $\chi_{M_n}S_j f$ ($j = 1,2$) belong to $L^2(M)$. Therefore, each of these functions define linear maps from $\mathcal{S}$ to $L^2(M)$. We are going to show that these maps are bounded and, therefore, can be continuously extended from $\mathcal{S}$ to $\mathcal{H}$. This fact allows us to define the bounded linear maps U_n and V_n from $\mathcal{H}$ to $\mathcal{J}$ of Theorem 11 below, which is one of the key points of this work.

Theorem 11. *For each $n \in \mathbb{N}$ there are two bounded linear maps, $U_n : \mathcal{H} \to \mathcal{J}$ and $V_n : \mathcal{H} \to \mathcal{J}$, such that, for any $f \in \mathcal{S}$*

$$P_j U_n f = \chi_{M_n} T_j f, \quad P_j V_n f = \chi_{M_n} S_j f, \quad j = 1,2. \tag{71}$$

Proof. We proof the statement for V_n. The proof of the statement for U_n is similar. Along this proof we use the indices j, k, and l and m, which take the values $1 \le j \le 2$, $2 \le k \le 4$, $1 \le l \le 3$, and $2 \le m \le 3$. We denote by $\|\cdot\|_{L_n^2}$ the norm in $L^2(M_n)$.

Let $f \in \mathcal{S}$. From the definition of $S_j f$ and Lemma 6, equation (39), we have

$$S_j f = \sum_{k=2}^{4} \beta_{jk}\big(g_k + u_k\big) + \sum_{l=1}^{3} \alpha_{jl}\big(h_l + v_l\big), \tag{72}$$

where, for $\lambda \in M$,

$$g_k(\lambda) = \int_{-\infty}^{0} f(x)\,\overline{\exp(\mathrm{i}\mu_k x)}dx, \quad u_k(\lambda) = \int_{-\infty}^{0} f(x)\,\overline{\exp(\mathrm{i}\mu_k x)\, s_k(x,\lambda)}\,dx, \tag{73}$$

and,

$$h_l(\lambda) = \int_{0}^{\infty} f(x)\,\overline{\exp(\mathrm{i}\mu_l x)}dx, \quad v_l(\lambda) = \int_{0}^{\infty} f(x)\,\overline{\exp(\mathrm{i}\mu_l x)\, r_l(x,\lambda)}\,dx. \tag{74}$$

Recall $\mu_1 = -\mu_4 = \mathrm{i}\theta$ and $\mu_2 = -\mu_3 = \nu$, where θ and ν are functions (of λ in the previous expressions) on $[h_{\mathrm{p}}, \infty)$ given by equation (27). For fixed $\lambda \in M$ the functions $\exp(\mathrm{i}\mu_l x) r_l(x,\lambda)$ are square integrable in x over $(0,\infty)$, and the functions $\exp(\mathrm{i}\mu_k x) s_k(x,\lambda)$ are square integrable in x over $(-\infty, 0)$. The Schwarz inequality gives

$$|u_k(\lambda)| \leq c\lambda^{-1/4}\|f\|, \quad |v_l(\lambda)| \leq c\lambda^{-1/4}\|f\|, \quad \lambda \in M_n, \tag{75}$$

where we used the bounds on r_k and s_k of Theorem 4, possibly with a larger value of c. Taking into account the bounds on β_{jk} and α_{jl} of Lemma 6, the restrictions of $\beta_{jk} u_k$ and $\alpha_{jl} v_l$ to M_n are square integrable functions and satisfy the estimates

$$\|\beta^*_{jk} u_k\|_{L^2_n} \leq c_n\|f\|, \quad \|\alpha^*_{jl} v_l\|_{L^2_n} \leq c_n\|f\|. \tag{76}$$

The Schwarz inequality implies also that both $|g_4|$ and $|h_1|$ are bounded by $(2\theta)^{-1/2}\|f\|$. The bounds on β_{j4} and α_{j1} imply that the restrictions of $\beta_{j4} g_4$ and $\alpha_{j1} h_1$ to M_n are square integrable functions and satisfy the estimates

$$\|\beta_{j4} g_4\|_{L^2_n} \leq c_n\|f\|, \quad \|\alpha_{j1} h_1\|_{L^2_n} \leq c_n\|f\|. \tag{77}$$

To analyze g_m and h_m (recall $2 \leq m \leq 3$) we split $f = f_+ + f_-$, where $f_+ = f(x)$ if $x \geq 0$ and $f_+(x) = 0$ if $x < 0$. Then $g_m(\lambda) = \tilde{f_-}\big(\mu_m(\lambda)\big)$ and $h_m(\lambda) = \tilde{f_+}\big(\mu_m(\lambda)\big)$, where $\tilde{f_+}$ and $\tilde{f_-}$ are the Fourier transforms of f_+ and f_-, respectively. For the restriction of $\alpha_{jm} h_m$ to M_n we have,

$$\|\alpha_{jm} h_m\|^2_{L^2_n} = \int_{M_n} \big|\alpha_{jm}(\lambda)\big|^2 \big|\tilde{f_+}\big(\mu_m(\lambda)\big)\big|^2 d\lambda. \tag{78}$$

Performing the change of variable $\lambda = p_{\mathrm{c}}(\xi)$ we obtain

$$\|\alpha_{jm} h_m\|^2_{L^2_n} = \int_{\widehat{M}_n} \big|\alpha_{jm}\big(p_{\mathrm{c}}(\xi)\big)\big|^2 \big|\tilde{f_+}\big((-1)^m\xi\big)\big|^2 p'_{\mathrm{c}}(\xi) d\xi, \tag{79}$$

where $\widehat{M}_n$ is the image of M_n under the inverse of p_{c}. The bounds on α_{jm} (Lemma 6) imply that $p'_{\mathrm{c}}(\xi)|\alpha_{jm}\big(p_{\mathrm{c}}(\xi)\big)|^2$ is bounded on $\widehat{M}_n$, so that

$$\|\alpha_{jm} h_m\|^2_{L^2_n} \leq c_n \int_{\widehat{M}_n} \big|\tilde{f_+}\big((-1)^m\xi\big)\big|^2 d\xi \leq c_n\|f_+\|^2 \leq c_n\|f\|^2, \tag{80}$$

where we used the genuine Parseval equality. Thus, we get the estimate $\|\alpha_{jm}h_m\|_{L^2_n} \leq c_n\|f\|$. In a similar way we obtain $\|\beta_{jm}g_m\|_{L^2_n} \leq c_n\|f\|$. Putting all the pieces together, we find that the restriction of S_jf to M_n satisfies $\|S_jf\|_{L^2_n} \leq c_n\|f\|$.

Thus, we can define the bounded linear transformation $V_n : \mathcal{H} \to \mathcal{J}$ by setting $P_jV_nf = \chi_{M_n}S_jf$ for $f \in \mathcal{S}$, and the results proved above provide the bound $\|V_nf\|_{\mathcal{J}} \leq c_n\|f\|$. The image of $\mathcal{H} \setminus \mathcal{S}$ under V_n is obtained by continuity. □

The analogues of Parseval's equality, equations (67) and (68), are extended by continuity to any two functions $f, g \in \mathcal{H}$, and can be written in the concise ways

$$(f, g) = \big(B_L(N_n)f, g\big) + \big(U_nf, V_ng\big)_{\mathcal{J}} = \big(B_L^*(N_n)f, g\big) + \big(V_nf, U_ng\big)_{\mathcal{J}}. \tag{81}$$

The next goal is to show that $\operatorname{ran}(U_n)$ and $\operatorname{ran}(V_n)$ are dense in $\mathcal{J}_n$. This is an immediate consequence, which we state in Corollary 16, of the technical results collected in Lemmas 12, 13, 14, and 15.

Lemma 12. *Let $n \in \mathbb{N}$ and $f_1, f_2 \in L^1(M_n) \cap L^2(M_n)$, and define the functions g and h on $\mathbb{R}$ by*

$$g(x) = \int\limits_{M_n} \sum_{j=1}^{2} \varphi_j(x, \lambda)f_j(\lambda)\, d\lambda, \quad h(x) = \int\limits_{M_n} \sum_{j=1}^{2} \varphi_j^*(x, \lambda)f_j(\lambda)\, d\lambda, \tag{82}$$

for $x \in \mathbb{R}$. Then $g, h \in \mathcal{H}$.

Proof. Since φ_j and φ_j^* are continuous and bounded on $\mathbb{R} \times M_n$ and f_j integrable over M_n, the functions g and h are well defined (and continuous) on $\mathbb{R}$. We prove $g \in \mathcal{H}$. That $h \in \mathcal{H}$ is proved similarly.

Let us split $g = g_+ + g_-$, where $g_+(x) = g(x)$ if $x \geq 0$ and $g_+(x) = 0$ if $x < 0$. We shall proof $g_+, g_- \in \mathcal{H}$. Using equation (39) for φ_j we write $g_+ = u_1 + u_2 + u_3$, where, for $1 \leq k \leq 3$,

$$u_k(x) = \int\limits_{M_n} \exp(\mathrm{i}\mu_kx)\Big(1 + r_k(x, \lambda)\Big) \sum_{j=1}^{2} \alpha_{jk}(\lambda)f_j(\lambda)\, d\lambda, \quad x \geq 0, \tag{83}$$

and $u_k(x) = 0$ if $x < 0$. By Lemma 6 the functions α_{jk} are bounded on M_n. Since $\theta(\lambda) \geq \theta_0 > 0$ for all $\lambda \in M$, it is clear that $|u_1(x)|$ is bounded by a constant times $\exp(-\theta_0|x|)$ and hence $u_1 \in \mathcal{H}$. To analyze u_2 we notice that it has two contributions, one which involves the function r_2 and another one that does not. From the bound on r_2 (Theorem 4) it is clear that the contribution which involves r_2 is bounded by a constant times $1/(1 + |x|)$ and thus it belongs to $\mathcal{H}$. Making the change of variable $\lambda = p_{\mathrm{c}}(\xi)$ in the integral, the contribution to u_2 of the term that does not involve r_2 is given by

$$\int\limits_{\widehat{M}_n} \exp(\mathrm{i}\xi x) \sum_{j=1}^{2} \alpha_{j2}\big(p_{\mathrm{c}}(\xi)\big) f_j\big(p_{\mathrm{c}}(\xi)\big) p_{\mathrm{c}}'(\xi)\, d\xi, \quad x \geq 0, \tag{84}$$

where $\widehat{M}_n$ is the image of M_n by the inverse of p_{c}. We prove below that the above integral is the Fourier transform of a square integrable function on $\mathbb{R}$, and thus it belongs to $\mathcal{H}$, what implies $u_2 \in \mathcal{H}$. To see that (84) is the Fourier transform of a square integrable function on $\mathbb{R}$, notice that, for $j = 1, 2$,

$$\begin{aligned}\int\limits_{\widehat{M}_n} \big|\alpha_{j2}\big(p_{\mathrm{c}}(\xi)\big) f_j\big(p_{\mathrm{c}}(\xi)\big)\big|^2 p_{\mathrm{c}}'^2(\xi)\, d\xi &= \int\limits_{M_n} p_{\mathrm{c}}'\big(\nu(\lambda)\big) \big|\alpha_{j2}(\lambda)\big|^2 \big|f_j(\lambda)\big|^2\, d\lambda \\ &\le c \int\limits_{M_n} \big|f_j(\lambda)\big|^2\, d\lambda < \infty,\end{aligned} \tag{85}$$

where we used that the bounds (41) imply that $p_{\mathrm{c}}'(\nu)|\alpha_{jk}|^2$ is bounded on M_n. The proof that $u_3 \in \mathcal{H}$ is similar. Thus, $g_+ \in \mathcal{H}$. In the same way we can proof that $g_- \in \mathcal{H}$, and therefore we obtain $g \in \mathcal{H}$. □

Lemma 13. *Let n, f_1, f_2, g and h as in Lemma 12, and G a bounded open subset of $\mathbb{R}$. Then:*

$$B_L(N_n)g = 0, \quad B_L^*(N_n)h = 0, \tag{86}$$

$$B_L(G)g(x) = \int\limits_{M_n \cap G} \sum_{j=1}^{2} \varphi_j(x,\lambda) f_j(\lambda)\, d\lambda, \quad x \in \mathbb{R}, \tag{87}$$

$$B_L^*(G)h(x) = \int\limits_{M_n \cap G} \sum_{j=1}^{2} \varphi_j^*(x,\lambda) f_j(\lambda)\, d\lambda, \quad x \in \mathbb{R}. \tag{88}$$

Proof. We prove the statements for g. The statements for h are proved similarly. We start with the proof of equation (87), which relies on the analogue of the Stone formula, equation (19). For $z \in \rho(L)$ it is clear that $R_L(z)g$ is given by substituting $f_j(\lambda)$ by $f_j(\lambda)/(\lambda - z)$ in the expression that defines g. Hence, for $\xi \in \mathbb{R}$ and $\epsilon > 0$ we have

$$R_L(\xi + \mathrm{i}\epsilon)g(x) - R_L(\xi - \mathrm{i}\epsilon)g(x) = \int\limits_{M_n} \frac{2\mathrm{i}\epsilon}{(\lambda - \xi)^2 + \epsilon^2} \sum_{j=1}^{2} \varphi_j(x,\lambda) f_j(\lambda)\, d\lambda. \tag{89}$$

Let (a,b) be an open bounded interval. Inserting the above equation into equation (19), interchanging the order of integration in the iterated integral (Fubini's theorem is clearly applicable), and manipulating the resultant expression in a standard way we get

$$\frac{1}{2}\Big(B_L\big((a,b)\big) + B_L\big([a,b]\big)\Big)g(x) = \int\limits_{M_n \cap (a,b)} \sum_{j=1}^{2} \varphi_j(x,\lambda) f_j(\lambda)\, d\lambda\, d\xi. \tag{90}$$

Relation (90) shows that if λ_e is an eigenvalue of L then $B_L\big(\{\lambda_e\}\big)g = 0$, since λ_e is contained in an open interval $(\alpha,\beta) \subset N_n$, and then

$$B_L\big(\{\lambda_e\}\big)g = B_L\big(\{\lambda_e\}\big)\frac{1}{2}\Big(B_L\big((\alpha,\beta)\big) + B_L\big([\alpha,\beta]\big)\Big)g = 0, \tag{91}$$

because $(\alpha,\beta) \cap M_n = \emptyset$. Hence, the left-hand side of equation (90) can be replaced by $B_L\big((a,b)\big)g(x)$ and then equation (87) holds if G is a bounded open interval. Since any bounded open set is the union of an at most countable family of disjoint bounded open intervals, equation (87) holds for any open bounded set G.

Now, since $N_n \subset (\lambda_0, \lambda_s)$ we have $B_L(N_n)g = B_L\big((\lambda_0,\lambda_s)\big)g - B_L\big((\lambda_0,\lambda_s) \cap N_n^c\big)g$. Applying equation (87) to the right-hand side of this equation we obtain $B_L(N_n)g = 0$, which is the first of equations (86). □

If $f \in \mathcal{S}$ and G is a bounded open subset of M_n, then the spectral resolutions (65) and (66), the definition of U_n and V_n, and Lemma 13 imply

$$B_L(G)f(x) = \int\limits_G \sum_{j=1}^{2} \varphi_j(x,\lambda) P_j U_n f(\lambda)\, d\lambda, \tag{92}$$

$$B_L^*(G)f(x) = \int\limits_G \sum_{j=1}^{2} \varphi_j^*(x,\lambda) P_j V_n f(\lambda)\, d\lambda. \tag{93}$$

Lemma 14. *Let $n \in \mathbb{N}$ and $f_1, f_2 \in C(M_n) \cap L^1(M_n) \cap L^2(M_n)$, and define g and h as in Lemma 12. Then $U_n g = V_n h = f$, where, for $j = 1, 2$, $P_j f(\lambda) = f_j(\lambda)$ if $\lambda \in M_n$ and $P_j f(\lambda) = 0$ if $\lambda \in M \setminus M_n$.*

Proof. Consider a sequence $g_m \in \mathcal{S}$, $m \in \mathbb{N}$, that converges to g in $\mathcal{H}$. Then $U_n g = \lim_m U_n g_m$. Using the spectral resolution (65) for g_m and taking into account that $B_L(N_n)g = 0$ (Lemma 12) we get $\lim_m v_m = 0$, where

$$v_m(x) = \int\limits_{M_n} \sum_{j=1}^{2} \varphi_j(x,\lambda)\Big(f_j(\lambda) - P_j U_n g_m(\lambda)\Big)\, d\lambda, \quad x \in \mathbb{R}. \tag{94}$$

Take a bounded interval $(a,b) \subset M_n$. The continuity of $B_L\big((a,b)\big)$ and Lemma 12 imply

$$\lim_{m\to\infty} \int\limits_a^b \sum_{j=1}^{2} \varphi_j(x,\lambda)\Big(f_j(\lambda) - P_j U_n g_m(\lambda)\Big)\, d\lambda = 0, \quad a.a.\ x \in \mathbb{R}. \tag{95}$$

Since for each fixed x the functions $\varphi_j(x,\cdot)$ are square integrable on (a,b), the convergence of $U_n g_m$ to $U_n g$ in $L^2(M_n)$ gives

$$\int\limits_a^b \sum_{j=1}^{2} \varphi_j(x,\lambda)\Big(f_j(\lambda) - P_j U_n g(\lambda)\Big)\, d\lambda = 0, \quad a.a.\ x \in \mathbb{R}. \tag{96}$$

Notice that $P_j U_n g$ is integrable on (a,b) since it is square integrable. Therefore, the left-hand side of the above equation defines a continuous function of x that has to vanish for all $x \in \mathbb{R}$ since it vanishes for *a.a.* $x \in \mathbb{R}$. Since equation (96) holds for every bounded interval contained in the open set M_n, the integrand has to vanish *a.e.* in M_n for each fixed $x \in \mathbb{R}$:

$$\sum_{j=1}^{2} \varphi_j(x,\lambda)\Big(f_j(\lambda) - P_j U_n g(\lambda)\Big) = 0, \quad x \in \mathbb{R}, \quad a.a.\ \lambda \in M_n. \tag{97}$$

The linear independence of $\varphi_1(\cdot,\lambda)$ and $\varphi_2(\cdot,\lambda)$ requires $P_j U_n g(\lambda) = f_j(\lambda)$ for *a.a.* $\lambda \in M_n$, that is, $U_n g = f$.

The proof of $U_n g = f$ is basically identical, but it requires that $\varphi_1^*(\cdot,\lambda)$ and $\varphi_2^*(\cdot,\lambda)$ be linearly independent for *a.a.* $\lambda \in M_n$. This is the thesis of Lemma 15, the proof of which uses the part of this lemma that has been proved. □

Lemma 15. *The functions $\varphi_1^*(\cdot,\lambda)$ and $\varphi_2^*(\cdot,\lambda)$ are linearly independent for almost all $\lambda \in M$.*

Proof. Along this proof the index j takes the values 1 and 2. We proceed by contradiction. If the thesis is false, there are functions c_j on M, which vanish simultaneously on at most a set of zero Lebesgue measure, and such that $\sum_j c_j(\lambda)\varphi_j^*(\cdot,\lambda) = 0$ for *a.a.* $\lambda \in M$. Therefore, for any $n \in \mathbb{N}$ and any $h \in \mathcal{S}$ we have $\sum_j c_j(\lambda) P_j U_n h(\lambda) = 0$ for *a.a.* $\lambda \in M$. Take functions f_j as in Lemma 14, so that there is a sequence of functions $U_n g_m$, $m \in \mathbb{N}$, where $g_m \in \mathcal{S}$, such that $P_j U_n g_m(\lambda) \to f_j(\lambda)$ for *a.a.* $\lambda \in M_N$. We have

$\sum_j c_j(\lambda) P_j U_n g_m(\lambda) = 0$, and therefore $\sum_j c_j(\lambda) f_j(\lambda) = 0$, for *a.a.* $\lambda \in M_n$. This property has to hold for all continuous functions f_1 and f_2 on M_n that are in $L^1(M_n) \cap L^2(M_n)$, and this is only possible if c_1 and c_2 vanish *a.e.* in M_n. Since n is arbitrary, c_1 and c_2 do vanish *a.e.* in M, what contradicts our initial assumption. Hence, $\varphi_1^*(\cdot, \lambda)$ and $\varphi_2^*(\cdot, \lambda)$ have to be linearly independent for *a.a.* $\lambda \in M$. □

The following corollary is an immediate consequence of Lemma 14, since $C(M_n) \cap L^1(M_n) \cap L^2(M_n)$ is dense in $L^2(M_n)$.

Corollary 16. $\operatorname{ran}(U_n)$ *and* $\operatorname{ran}(V_n)$ *are dense in* $\mathcal{J}_n$.

Next theorem collects some important properties of U_n and V_n that will be used later.

Theorem 17. *For all* $n \in \mathbb{N}$ *the linear maps* U_n *and* V_n *satisfy:*

1. $\ker(U_n) = \operatorname{ran}\big(B_L(N_n)\big)$ *and* $\ker(V_n) = \operatorname{ran}\big(B_L^*(N_n)\big)$.
2. $U_n R_L(z) = R_{Q_n}(z) U_n$ *and* $V_n R_{L^*}(z) = R_{Q_n}(z) V_n$ *for all* $z \in \rho(L)$.
3. $U_n L f = Q_n U_n f$ *if* $f \in \operatorname{dom}(L)$, *and* $V_n L^* f = Q_n V_n f$ *if* $f \in \operatorname{dom}(L^*)$.

Proof. Point 1 follows almost immediately from the analogue of the Parseval equality (81) and the density of $\operatorname{ran}(V_n)$ in $\mathcal{J}_n$. To prove the first relation of point 2 we notice it holds for $f \in \mathcal{S}$ due to equation (58) of Lemma 9, equation (20), point 1 of this lemma, and Lemma 14. Then, the relation is extended to $\mathcal{H}$ by continuity. The second relation of point 2 follows in a similar way. Point 3 is obtained easily from point 2. □

The projections $B_L(N_n)$ and $B_L^*(N_n)$ induce decompositions of $\mathcal{H}$ as a direct sum of two closed subspaces:

$$\mathcal{H} = \operatorname{ran}\big(B_L(N_n)\big) \oplus \mathcal{U}_n = \operatorname{ran}\big(B_L^*(N_n)\big) \oplus \mathcal{V}_n, \tag{98}$$

where we introduce the notation $\mathcal{U}_n = \ker\big(B_L(N_n)\big)$ and $\mathcal{V}_n = \ker\big(B_L^*(N_n)\big)$. Let $\mathbb{U}_n$ be the restriction of U_n to $\mathcal{U}_n$ and $\mathbb{V}_n$ the restriction of V_n to $\mathcal{V}_n$.

Theorem 18. *For each* $n \in \mathbb{N}$, $\mathbb{U}_n$ *is a continuous linear bijection from* $\mathcal{U}_n$ *onto* $\mathcal{J}_n$ *which has a continuous inverse, and* $\mathbb{V}_n$ *is a continuous linear bijection from* $\mathcal{V}_n$ *onto* $\mathcal{J}_n$ *which has a continuous inverse.*

Proof. Clearly, $\mathbb{U}_n$ is injective and bounded, and $\operatorname{ran}(\mathbb{U}_n) = \operatorname{ran}(U_n)$. Let us show that the inverse of $\mathbb{U}_n$ is bounded. If $f \in \operatorname{ran}(U_n)$, then there is a unique $g \in \mathcal{U}_n$ such that $U_n g = f$, and $g = \mathbb{U}_n^{-1} f$. Since $g \in \mathcal{U}_n$ implies $B_L(N_n) g = 0$, the Parseval equality (81) gives $\|g\|^2 = (V_n g, U_n g)_{\mathcal{J}}$, and since V_n is bounded we get $\|g\| \le \|V_n\| \|U_n g\|_{\mathcal{J}}$, that is, $\|\mathbb{U}_n^{-1} f\| \le \|V_n\| \|f\|_{\mathcal{J}}$. Since $\mathbb{U}_n$ has a bounded inverse, its range of is closed, and then $\operatorname{ran}(\mathbb{U}_n) = \mathcal{J}_n$, because, by Corollary 16, $\operatorname{ran}(\mathbb{U}_n)$ is dense in $\mathcal{J}_n$. This proves that $\mathbb{U}_n$ is a bounded bijection of $\mathcal{U}_n$ onto $\mathcal{J}_n$ which has a bounded inverse. The claim for $\mathbb{V}_n$ is proved similarly. □

An important byproduct of Theorem 18 is emphasized in the following corollary.

Corollary 19. $\operatorname{ran}(U_n) = \operatorname{ran}(V_n) = \mathcal{J}_n$ *for all* $n \in \mathbb{N}$.

The extension to $\mathcal{H}$ of the spectral expansions (65) and (66) can be written concisely as

$$I_{\mathrm{d}} = B_L(N_n) + \mathbb{U}_n^{-1} U_n = B_L^*(N_n) + \mathbb{V}_n^{-1} V_n. \tag{99}$$

8. Spectral resolution of the identity for L

The results of the previous sections allow us to show that there are spectral resolutions of the identity for L and L^*, and that both L and L^* are spectral operators of scalar type. We start noticing that for any $n \in \mathbb{N}$ and $b \in \mathcal{B}$ the linear operators $\mathbb{U}_n^{-1}\hat{\chi}_n(b)U_n$ and $\mathbb{V}_n^{-1}\hat{\chi}_n(b)V_n$ are bounded projections on $\mathcal{H}$. From Lemma 13 we obtain

$$\mathbb{U}_n^{-1}\hat{\chi}_n(b)U_n f(x) = \int\limits_{M_n \cap b} \sum_{j=1}^{2} \varphi_j(x,\lambda) P_j U_n f(\lambda)\, d\lambda, \tag{100}$$

$$\mathbb{V}_n^{-1}\hat{\chi}_n(b)V_n f(x) = \int\limits_{M_n \cap b} \sum_{j=1}^{2} \varphi_j^*(x,\lambda) P_j V_n f(\lambda)\, d\lambda, \tag{101}$$

for any $f \in \mathcal{S}$ and $x \in \mathbb{R}$.

Lemma 20. *If $n \in \mathbb{N}$, $b_1 \in \mathcal{B}_b$, and $b_2 \in \mathcal{B}$, then*

$$B_L(b_1)\mathbb{U}_n^{-1}\hat{\chi}_n(b_2)U_n = \mathbb{U}_n^{-1}\hat{\chi}_n(b_1 \cap b_2)U_n, \tag{102}$$

$$B_L^*(b_1)\mathbb{V}_n^{-1}\hat{\chi}_n(b_2)V_n = \mathbb{V}_n^{-1}\hat{\chi}_n(b_1 \cap b_2)V_n. \tag{103}$$

Proof. We prove only equality (102). The other equality is proved similarly. If b_1 is open, equation (100) and Lemma 13 show that (102) holds in $\mathcal{S}$ and then, by continuity, in $\mathcal{H}$. Consider now a general bounded Borel set b_1 and take an open bounded set G_1 such that $b_1 \subset G_1$. All projections $B_L(b)$, where b is a Borel set contained in G_1, are uniformly bounded: there is $c > 0$ such that $\|B_L(b)\| < c$ (section 4). Let μ_ℓ be the Lebesgue measure on $\mathbb{R}$. There is an open set G_2 such that $b_1 \subseteq G_2 \subset G_1$ and $\mu_\ell(G_2 \backslash b_1)$ is as small as desired. Let $f \in \mathcal{H}$ and $g = \mathbb{U}_n^{-1}\hat{\chi}_n(b_2)U_n f$. Since G_2 is open and bounded we have $B_L(G_2)g = \mathbb{U}_n^{-1}\hat{\chi}_n(G_2 \cap b_2)U_n f$, and thus

$$B_L(b_1)g - \mathbb{U}_n^{-1}\hat{\chi}_n(b_1 \cap b_2)U_n f = \mathbb{U}_n^{-1}\hat{\chi}_n\big((G_2 \setminus b_1) \cap b_2\big)U_n f - B_L(G_2 \setminus b_1)g. \tag{104}$$

There is also an open set G_3 such that $G_2 \setminus b_1 \subseteq G_3$ and $\mu_\ell(G_3) - \mu_\ell(G_2 \setminus b_1)$ is as small as desired. We have

$$\begin{aligned} \|B_L(G_2 \setminus b_1)g\| &= \|B_L(G_2 \setminus b_1)B_L(G_3)g\| \le c\|B_L(G_3)g\| \\ &\le c\|\mathbb{U}_n^{-1}\|\|\hat{\chi}_n(G_3 \cap b_2)U_n f\|_{J_n}, \end{aligned} \tag{105}$$

where we used equality (102) for G_3, which is open and bounded. Take any $\epsilon > 0$. By a proper choice of G_2 and G_3 we can have

$$\|\hat{\chi}_n\big((G_2 \setminus b_1) \cap b_2\big)U_n f\|_{J_n} \le \frac{\epsilon}{2\|\mathbb{U}_n^{-1}\|}, \tag{106}$$

$$\|\hat{\chi}_n(G_3 \cap b_2)U_n f\|_{J_n} \le \frac{\epsilon}{2c\|\mathbb{U}_n^{-1}\|}. \tag{107}$$

Then, from equation (104) we obtain $\|B_L(b_1)g - \mathbb{U}_n^{-1}\hat{\chi}(b_1 \cap b_2)U_n f\| \le \epsilon$ for any $\epsilon > 0$, what proves equality (102). $\square$

We extend B_L to a bounded algebra homomorphism, E_L, from $\mathcal{B}$ onto a subset of $\mathcal{L}(\mathcal{H})$, by defining the map $E_L : \mathcal{B} \to \mathcal{L}(\mathcal{H})$ by

$$E_L(b) = B_L(N_n \cap b) + \mathbb{U}_n^{-1}\hat{\chi}_n(b)U_n, \quad b \in \mathcal{B}, \quad n \in \mathbb{N}. \tag{108}$$

The right-hand side of the above equation is independent of n, what can be easily proved using the following equalities, which hold for $n \geq m$: first, from the definitions it is obvious that $U_m = \hat{\chi}_n(M_m)U_n$ and $V_m = \hat{\chi}_n(M_m)V_n$; second, $\mathbb{U}_n^{-1}\hat{\chi}_n(M_m)\hat{\chi}_n(b)U_n = \mathbb{U}_m^{-1}\hat{\chi}_m(b)U_m$ (this is also rather obvious but it can be proved using the analogue of the Parseval equality); and third, $N_n \cup (M_n \setminus M_m) = N_m$. From these facts, and using Lemma 20, we have

$$\begin{aligned} B_L(N_n \cap b) + \mathbb{U}_n^{-1}\hat{\chi}_n(b)U_n &= B_L(N_n \cap b) + \mathbb{U}_n^{-1}\hat{\chi}_n(M_1)\hat{\chi}_n(b)U_n \\ +\mathbb{U}_n^{-1}\hat{\chi}_n(M_n \setminus M_1)\hat{\chi}_n(b)U_n &= B_L(N_n \cap b) + \mathbb{U}_1^{-1}\hat{\chi}_1(b)U_1 \\ +B_L\big((M_n \setminus M_1) \cap b\big)\mathbb{U}_n^{-1}U_n &= B_L(N_1 \cap b) + \mathbb{U}_1^{-1}\hat{\chi}_1(b)U_1. \end{aligned} \tag{109}$$

The strong limit of $\mathbb{U}_n^{-1}\hat{\chi}_n(b)U_n$ as $n \to \infty$ exits,

$$\operatorname*{s-lim}_{n\to\infty} \mathbb{U}_n^{-1}\hat{\chi}_n(b)U_n = B_L\big((M \setminus M_1) \cap b\big) + \mathbb{U}_1^{-1}\hat{\chi}_1(b)U_1, \tag{110}$$

and is a bounded projection, with a bound independent of b. Then, to fully exploit the spectral expansions, equations (65) and (100), it may be interesting to express E_L as

$$E_L(b) = B_L(N \cap b) + \operatorname*{s-lim}_{n\to\infty} \mathbb{U}_n^{-1}\hat{\chi}_n(b)U_n, \quad b \in \mathcal{B}. \tag{111}$$

We are ready to discuss the conclusions of this work, collected in the next two theorems.

Theorem 21. *The map E_L extends B_L from $\mathcal{B}_b$ to $\mathcal{B}$, and is a spectral resolution of the identity for L. That is:*

1. *$E_L(b) = B_L(b)$ for each bounded Borel set b.*
2. *There is $c > 0$ such that $\|E_L(b)\| \leq c$ for all $b \in \mathcal{B}$.*
3. *$E_L(\emptyset) = 0$ and $E_L(\mathbb{R}) = E_L\big(\sigma(L)\big) = I_{\mathrm{d}}$.*
4. *$E_L(b_1 \cap b_2) = E_L(b_1)E_L(b_2)$ for $b_1, b_2 \in \mathcal{B}$.*
5. *$E_L(b_1 \cup b_2) = E_L(b_1) + E_L(b_2) - E_L(b_1)E_L(b_2)$ for $b_1, b_2 \in \mathcal{B}$.*
6. *If $\{b_i, i \in \mathbb{N}\}$ is a sequence of pairwise disjoint Borel sets and $f \in \mathcal{H}$, then $E_L\big(\cup_i b_i\big)f = \sum_{i=1}^{\infty} E_L(b_i)f$.*
7. *If $\{b_i, i \in \mathbb{N}\}$ is an ascending sequence of Borel sets and $f \in \mathcal{H}$, then $E_L\big(\cup_i b_i\big)f = \lim_i E_L(b_i)f$. If the sequence is descending, then we have $E_L\big(\cap_i b_i\big)f = \lim_i E_L(b_i)f$.*
8. *If $b \in \mathcal{B}$ and $z \in \rho(L)$ then $E_L(b)R_L(z) = R_L(z)E_L(b)$.*
9. *If $b \in \mathcal{B}$ is bounded, then $\operatorname{ran}\big(E_L(b)\big) \subset \operatorname{dom}(L)$.*
10. *If $b \in \mathcal{B}$ and $f \in \operatorname{dom}(L)$ then $E_L(b)f \in \operatorname{dom}(L)$ and*

$$E_L(b)Lf = LE_L(b)f. \tag{112}$$

11. *If $b \in \mathcal{B}$ and L_b is the restriction of L to $\operatorname{ran}\big(E_L(b)\big)$, then $\sigma(L_b) \subseteq \overline{b}$.*

Proof. Point 1 follows easily from Lemma 20 (with $b_1 = b$ and $b_2 = \mathbb{R}$) and equality (99), since for any $b \in \mathcal{B}_b$ we have

$$E_L(b) = B_L(b)\Big(B_L(N_n) + \mathbb{U}_n^{-1}U_n\Big) = B_L(b). \tag{113}$$

Point 2 follows from the fact that the projections $B_L(b)$ corresponding to Borel sets b contained in the bounded set N_1 are uniformly bounded, and that $\|\hat{\chi}_1(b)\| \leq 1$ for all $b \in \mathcal{B}$. Point 3 follows from $B_L(\emptyset) = 0$

and $\hat{\chi}(\emptyset) = 0$, from $\sigma(L) = M_n \cup N_n$, and from relation (99). Points 4, 5, and 6 follow from the corresponding properties of the projections B_L and $\hat{\chi}_n$, and from $B_L(N_n \cap b_1)\mathbb{U}_n^{-1}\hat{\chi}_n(b_2)U_n = 0$ for any $b_1, b_2 \in \mathcal{B}$. Point 7 follows from point 6 and simple standard arguments. Point 8 follows from the corresponding property for B_L, equation (20), and point 2 of Theorem 17. Point 9 follows from equality (22), since $E_L(b) = B_L(b)$ for $b \in \mathcal{B}_b$. Point 10 follows easily from point 9 of this theorem.

To prove point 11, we determine $\rho(L_b)$ by considering $(L_b - zI_\mathrm{d})f = g$ for $g \in \operatorname{ran}\big(E_L(b)\big)$ and $z \in \mathbb{C}$. Applying $B_L(N_n \cap b)$ and $\hat{\chi}_n(b)U_n$ to both sides of this equation, and using point 3 of Theorem 17, we get

$$E_\Lambda(N_n \cap b)(\Lambda - zI_\mathrm{d})D_2^{-1/2}f = E_\Lambda(N_n \cap b)D_2^{-1/2}g, \tag{114}$$

$$\big(Q_n - zI_\mathrm{d}\big)\hat{\chi}_n(b)U_nf = \hat{\chi}_n(b)U_ng. \tag{115}$$

We will have $z \in \rho(L_b)$ if and only if the above equations have a unique uniformly bounded solution for each $g \in \operatorname{ran}\big(E_L(b)\big)$. If $h \in \operatorname{ran}\big(E_\Lambda(N_n \cap b)\big)$ then $h \in \operatorname{dom}(\Lambda) \subset \operatorname{dom}(D_2^{1/2})$ and $h = D_2^{-1/2}E_L(N_n \cap b)D_2^{1/2}h$. Therefore, equation (114) has a unique uniformly bounded solution if and only if $z \in \rho(\Lambda_{N_n\cap b})$, where $\Lambda_{N_n\cap b}$ is the restriction of Λ to $\operatorname{ran}\big(E_\Lambda(N_n \cap b)\big)$. Since the image of $\operatorname{ran}\big(E_L(b)\big)$ under $\hat{\chi}_n(b)U_n$ is equal to $\operatorname{ran}\big(\hat{\chi}_n(b)\big)$, equation (115) will have a unique uniformly bounded solution if and only if $z \in \rho(Q_{M_n\cap b})$, where $Q_{M_n\cap b}$ is the restriction of Q_n to $\operatorname{ran}\big(\hat{\chi}(M_n \cap b)\big)$. Therefore, $\rho(L_b) = \rho(\Lambda_{N_n\cap b}) \cap \rho(Q_{M_n\cap b})$, and then $\sigma(L_b) = \sigma(\Lambda_{N_n\cap b}) \cup \sigma(Q_{M_n\cap b}) \subseteq (\overline{N_n \cap b}) \cup (\overline{M_n \cap b}) \subseteq \overline{b}$, where we used the spectral properties of the selfadjoint operators Λ and Q. □

The integral with respect to a projection valued measure such as E_L or E_Λ, which appears in the next theorem, is defined and studied, for instance, in Dunford and Schwartz, chapter XI [6].

Theorem 22. *If $\{b_n,\, n \in \mathbb{N}\}$ is an ascending sequence of bounded Borel sets such that $\sigma(L) \subseteq \cup_n b_n$, then, for $f \in \operatorname{dom}(L)$,*

$$Lf = \lim_{n\to\infty} \int_{b_n} \lambda\, E_L(d\lambda)\, f. \tag{116}$$

Proof. We notice that $E_L(b_n) = B_L(b_n)$ since b_n is bounded for all $n \in \mathbb{N}$. For any $b \subseteq b_n$ we can write equation (22) as

$$B_L(b) = (D_1 + P_0)^{-1}D_2^{-1/2}\Lambda E_\Lambda(b_n)E_\Lambda(b)D_2^{-1/2} + P_0D_2^{1/2}E_\Lambda(b)D_2^{-1/2}. \tag{117}$$

Then, since $\Lambda E_\Lambda(b_n)$ is a bounded operator, we have

$$\begin{aligned}\int_{b_n} \lambda\, E_L(d\lambda) &= (D_1 + P_0)^{-1}D_2^{-1/2}\Lambda E_\Lambda(b_n) \int_{b_n} \lambda\, E_\Lambda(d\lambda)\, D_2^{-1/2} \\ &+ P_0D_2^{1/2} \int_{b_n} \lambda\, E_\Lambda(d\lambda)\, D_2^{-1/2} = LB_L(b_n) = LE_L(b_n),\end{aligned} \tag{118}$$

where in the last equality we used $\int_{b_n} \lambda E_\Lambda(d\lambda) = \Lambda E_\Lambda(b_n)$. For $f \in \operatorname{dom}(L)$ we have $LE_L(b_n)f = E_L(b_n)Lf$ (point 10 of Theorem 21). Using equation (118) and taking the limit $n \to \infty$ (using points 7 and 3 of Theorem 21) we obtain Equation (116). □

Theorem 22 shows that L is an unbounded spectral operator of scalar type, in the sense of Bade [1]. Furthermore, Theorems 21 and 22 ensure that it is possible to establish an operational calculus for L as it is done for selfadjoint operators. It is clear that L^* is also an unbounded spectral operator of scalar type, and its spectral resolution of the identity is $E_{L^*}(b) = E_L(b)^*$ for $b \in \mathcal{B}$.

9. Final remarks

If, instead of $\int_{\mathbb{R}}(1+|x|^3)|q_i^{(k)}(x)|dx < \infty$, we require that the functions q_i satisfy $\int_{\mathbb{R}} \exp(\gamma|x|)|q_i^{(k)}(x)|dx < \infty$ for some $\gamma > 0$, $i = 1, 2$, and $0 \leq k \leq 2$, as it happens in the case of the spin wave dynamics in presence of a magnetic soliton, it can be shown that the fundamental matrices of Theorem 3 are such that, for fixed x, $\Phi_+(x, \cdot)$ and $\Phi_-(x, \cdot)$ can both be extended analytically from the upper and lower half-planes to open sets that contain $\mathbb{R} \setminus \{h_\mathrm{m}, h_\mathrm{p}\}$. This implies that the zeros of W_+ and W_- are isolated and can accumulate only at h_m and h_p. Therefore, in this case $N = \overline{\sigma_p(L)} \cup \{h_\mathrm{p}\}$ and $M = (h_\mathrm{p}, \infty) \setminus \sigma_p(L)$, and we have, for any $f \in \mathcal{H}$,

$$B_L(N)f = \sum_n \sum_{\alpha=1}^{d_n} (f, \xi^*_{n\alpha})\xi_{n\alpha}, \tag{119}$$

where n runs over the eigenvalues of L and $d_n \leq 2$ is the dimension of the fundamental subspace corresponding to the n-th eigenvalue, λ_n. For each n, the functions $\xi_{n\alpha}$ span the fundamental subspace of L corresponding to λ_n, and the functions $\xi^*_{n\alpha}$ span the fundamental subspace of L^* corresponding to λ_n, and satisfy the condition $(\xi_{n\alpha}, \xi^*_{m\beta}) = \delta_{nm}\delta_{\alpha\beta}$. If $\psi_{n\alpha}$ constitute an orthogonal basis the fundamental subspace of Λ corresponding to λ_n, we can take $\xi_{n\alpha} = D_2^{1/2}\psi_{n\alpha}$ and $\xi^*_{n\alpha} = D_2^{-1/2}\psi_{n\alpha}$.

The main obstacle to proof that an operator is spectral is found in proving that it has no spectral singularity (that is, that the spectral projections are uniformly bounded). For instance, Huige proved [8] that ordinary differential operators of a certain class, which include L and L^*, have, under certain hypothesis, spectral projections associated to the Borel sets whose closure exclude a finite set of exceptional points. However, these projections need not be uniformly bounded. Moreover, among the hypothesis there is one which implies the assumption that the only spectral singularities belong to the exceptional set.

Mackey proved (theorem 55 of [12], see also [18]) that any bounded spectral measure, such as E_L, is similar to a selfadjoint spectral measure, in the sense that there is a continuous linear bijection A of $\mathcal{H}$, hence with continuous inverse, such that $A^{-1}E_L(b)A$ is selfadjoint for each $b \in \mathcal{B}$. This means that any spectral operator of scalar type is similar to a self-adjoint operator. In our case, there is a selfadjoint operator H such that $A^{-1}LA = H$. A similar statement is true for L^*. Evidently, A is not equal to $D_2^{1/2}$ and thus H is not equal to Λ. Nonselfadjoint operators that are similar to selfadjoint operators are sometimes called quasi-selfadjoint operators in the literature [2]. Actually, quasi-selfadjoint operators are defined as operators which are similar to their adjoints, but it is easy to show (proposition 5.5.2 of [2]) that quasi-selfadjointness is equivalent to being similar to a selfadjoint operator. Thus, using this terminology, the conclusion of this work may be quickly summarized by saying that L and L^* are quasi-selfadjoint operators.

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Appendix A. Proof of Theorem 4

To prove Theorem 4 we analyze the solutions of $Lu = \lambda u$ for $\lambda \in [h_\mathrm{p}, \lambda_s]$ and for $\lambda \in [\lambda_s, \infty)$ separately. Recall that $\lambda_s > h_\mathrm{p}$ is a sufficiently large positive number. This fourth order differential equation is equivalent to the first order linear system (23). Recall also that for $\lambda \in [h_\mathrm{p}, \infty)$ the roots of $p_\mathrm{c}(\mu) - \lambda$ are $\mu_1 = \mathrm{i}\theta$, $\mu_2 = \nu$, $\mu_3 = -\nu$, $\mu_4 = -\mathrm{i}\theta$, with θ and ν functions of λ given by equations (27). We denote by e_j $(1 \leq j \leq 4)$ the vectors of the canonical basis of $\mathbb{C}^4$. For $1 \leq k \leq 4$ the eigenvalues of the matrix A entering equation (23) are $\mathrm{i}\mu_k$, and an eigenvector corresponding to $\mathrm{i}\mu_k$ is $p_k = \sum_j (\mathrm{i}\mu_k)^{j-1}e_j$. The 4×4 matrix whose k-th column is p_k is denoted by Π. If all the eigenvalues of A are different, then Π is the transpose of a non-singular Vandermonde matrix and we have

$$\left(\Pi^{-1}\right)_{k4} = \frac{\mathrm{i}}{p_{\mathrm{c}}'(\mu_k)} = \frac{\mathrm{i}}{\mu_k(\mu_k^2 + h_{\mathrm{a}})}. \tag{A.1}$$

Let $D(x)$ be the 4×4 diagonal matrix with matrix elements $D(x)_{jk} = \exp(\mathrm{i}\mu_j x)\delta_{jk}$. Notice that $\Pi D(x)$ is a fundamental matrix of the system $y' = Ay$ if $\lambda > h_{\mathrm{p}}$. The λ dependence of μ_k, p_k, Π, and $D(x)$ is not always explicitly shown to avoid clumsy expressions.

Theorem 4 is a corollary of the following two theorems about the solutions of the linear system (23).

Theorem 23. *If $\lambda_s > h_{\mathrm{p}}$ is large enough, there are functions y_k^+ and y_k^-, $1 \leq k \leq 4$, with domain $\mathbb{R} \times [\lambda_s, \infty)$ and image in $\mathbb{C}^4$, which are written in the form*

$$\begin{aligned} y_k^+(x,\lambda) &= \exp(\mathrm{i}\mu_k x)\Pi\big(e_k + v_k^+(x,\lambda)\big), \\ y_k^-(x,\lambda) &= \exp(\mathrm{i}\mu_k x)\Pi\big(e_k + v_k^-(x,\lambda)\big), \end{aligned} \tag{A.2}$$

for $(x,\lambda) \in \mathbb{R} \times [\lambda_s, \infty)$, such that

1. *$\{y_k^+(\cdot,\lambda),\ 1 \leq k \leq 4\}$ and $\{y_k^-(\cdot,\lambda),\ 1 \leq k \leq 4\}$ are two fundamental sets of solutions of the system (23) for each $\lambda \in [\lambda_s, \infty)$.*
2. *The functions v_k^+ and v_k^- are continuous on $\mathbb{R} \times [\lambda_s, \infty)$, and satisfy*

$$|v_k^+(x,\lambda)| \leq c\,\nu(\lambda)^{-1}\big(1 + x^3\big)^{-1}, \quad x \geq 0, \quad \lambda \in [\lambda_s, \infty), \tag{A.3}$$

$$|v_k^-(x,\lambda)| \leq c\,\nu(\lambda)^{-1}\big(1 - x^3\big)^{-1}, \quad x \leq 0, \quad \lambda \in [\lambda_s, \infty), \tag{A.4}$$

   where c is a constant.
3. *For each fixed $x \in \mathbb{R}$, $v_k^+(x,\cdot)$ and $v_k^-(x,\cdot)$ are analytic in $[\lambda_s, \infty)$.*

Proof. For $1 \leq k \leq 4$ we obtain y_k^+ as follows. For $1 \leq j \leq 4$ define $s_{1j} = 1$ if $\operatorname{Im}\mu_j - \operatorname{Im}\mu_k > 0$ and $s_{1j} = 0$ otherwise, and $s_{2j} = 1 - s_{1j}$. For $l = 1, 2$, let $D_l(x)$ be the 4×4 diagonal matrix whose matrix elements are $D_l(x)_{ij} = s_{lj}\exp(\mathrm{i}\mu_j x)\delta_{ij}$. Notice that $\Pi D_l(x)$ are solution matrices of the system (23), and $D(x) = D_1(x) + D_2(x)$. For $(x,\lambda) \in [0,\infty) \times [\lambda_s, \infty)$ consider the integral equation

$$\begin{aligned} y(x,\lambda) = \exp(\mathrm{i}\mu_k x)p_k &+ \int_0^x \Pi D_1(x)D(-\tau)\Pi^{-1}B(\tau)y(\tau,\lambda)d\tau \\ &- \int_x^\infty \Pi D_2(x)D(-\tau)\Pi^{-1}B(\tau)y(\tau,\lambda)d\tau. \end{aligned} \tag{A.5}$$

If this equation has a solution, then it is a solution of the system (23). The function $\tilde{y}(x,\lambda) = \Pi^{-1}(\lambda)y(x,\lambda)$, for $(x,\lambda) \in [0,\infty) \times [\lambda_s, \infty)$, satisfies the equation

$$\begin{aligned} \tilde{y}(x,\lambda) = \exp(\mathrm{i}\mu_k x)e_k &+ \int_0^x D_1(x)D(-\tau)\tilde{B}(\tau)\tilde{y}(\tau,\lambda)d\tau \\ &- \int_x^\infty D_2(x)D(-\tau)\tilde{B}(\tau)\tilde{y}(\tau,\lambda)d\tau, \end{aligned} \tag{A.6}$$

where $\tilde{B}(\tau) = \Pi B(\tau)\Pi^{-1}$. We need a bound on the norm of $\tilde{B}$. From equations (24) and (A.1) we get for $1 \le i, j \le 4$:

$$\tilde{B}_{ij}(\tau) = \frac{\mathrm{i}}{p_{\mathrm{c}}'(\mu_i)} \sum_{l=1}^{3} \big(\mathrm{i}\mu_j\big)^{l-1} B_{4l}(\tau). \tag{A.7}$$

Since $|\mu_j|^{l-1}/p_{\mathrm{c}}'(\mu_i) \le \theta^{l-1}/\big(2\nu(\theta^2+\nu^2)\big) \le \theta^{l-3}/(2\nu)$ and $\theta \ge \theta_0 > 0$, we obtain $|\tilde{B}(\tau)| \le (c_0/\nu)|B(\tau)|$, where c_0 is a constant. Let us try successive approximations to the solution of equation (A.6): $\tilde{y}^{(0)}(x,\lambda) = 0$ and for $j \in \mathbb{N}$

$$\begin{aligned}\tilde{y}^{(j)}(x,\lambda) = \exp(\mathrm{i}\mu_k x)e_k &+ \int_0^x D_1(x)D(-\tau)\tilde{B}(\tau)\,\tilde{y}^{(j-1)}(\tau,\lambda)d\tau \\ &- \int_x^\infty D_2(x)D(-\tau)\tilde{B}(\tau)\,\tilde{y}^{(j-1)}(\tau,\lambda)d\tau.\end{aligned} \tag{A.8}$$

We have $\tilde{y}^{(1)}(x,\lambda) = \exp(\mathrm{i}\mu_k x)e_k$ and $|\tilde{y}^{(1)}(x,\lambda)| = \exp(-\mathrm{Im}\,\mu_k x)$. The functions $\tilde{y}^{(j)}$ are continuous on $[0,\infty)\times[\lambda_s,\infty)$, and, for fixed x, $\tilde{y}^{(j)}(x,\cdot)$ is analytic in $[\lambda_s,\infty)$. Now we show that if $x \ge 0$ and $\lambda \ge \lambda_s$, with λ_s large enough, then

$$\big|\tilde{y}^{(j+1)}(x,\lambda) - \tilde{y}^{(j)}(x,\lambda)\big| \le (1/2)^j \exp\big(-\mathrm{Im}\,\mu_k x\big) \tag{A.9}$$

for $j \in \mathbb{N}$. We proceed by induction. For $j = 0$ the inequality holds. Assuming that it holds for some $j \in \mathbb{N}\cup\{0\}$, equation (A.8) imply

$$\big|\tilde{y}^{(j+2)}(x,\lambda) - \tilde{y}^{(j+1)}(x,\lambda)\big| \le (1/2)^j \exp(-\mathrm{Im}\,\mu_k x)\frac{c_0}{\nu(\lambda)}\int_0^\infty |B(\tau)|d\tau. \tag{A.10}$$

If λ_s is large enough so that $c_0\,\nu(\lambda_s)^{-1}\int_0^\infty |B(\tau)|d\tau \le 1/2$, then inequality (A.9) is proved (recall that ν is an increasing unbounded function of λ). Hence, the sequence $\tilde{y}^{(j)}$ converges uniformly on each compact set $[0,x_1]\times[\lambda_s,\lambda_1]$, with $x_1 \in \mathbb{R}^+$ and $\lambda_1 > \lambda_s$, to a function $\tilde{y}_k^+$ which, consequently, is continuous in $[0,\infty)\times[\lambda_s,\infty)$. Moreover, for fixed $x \in [0,\infty)$, the function $\tilde{y}_k^+(x,\cdot)$ is analytic in a neighborhood of $[\lambda_s,\infty)$. The bound $|\tilde{y}_k^+(x,\lambda)| \le 2\exp(-\mathrm{Im}\,\mu_k x)$ is easily obtained by summing the inequality

$$|\tilde{y}^{(j+1)}(x,\lambda)| - |\tilde{y}^{(j)}(x,\lambda)| \le (1/2)^j \exp(-\mathrm{Im}\,\mu_k x), \tag{A.11}$$

from $j = 0$ to l and taking the limit $l \to \infty$. From $\tilde{y}_k^+(x,\lambda)$ we obtain $y_k^+(x,\lambda) = \Pi\tilde{y}_k^+(x,\lambda)$, and $v_k^+(x,\lambda) = \exp(-\mathrm{i}\mu_k x)\tilde{y}_k^+(x,\lambda)$, so that

$$\begin{aligned}v_k^+(x,\lambda) = \exp(-\mathrm{i}\mu_k x)\Bigg(&\int_0^x D_1(x)D(-\tau)\tilde{B}(\tau)\tilde{y}_k^+(\tau,\lambda)d\tau \\ &- \int_x^\infty D_2(x)D(-\tau)\tilde{B}(\tau)\tilde{y}_k^+(\tau,\lambda)d\tau\Bigg).\end{aligned} \tag{A.12}$$

From this equation we get $|v_k^+(x,\lambda)| \le (2c_0/\nu)\big(I_1(x) + I_2(x)\big)$, where

$$I_1(x) = \int_0^x \sum_{j=1}^4 s_{1j} \exp\Big(- (\mathrm{Im}\mu_j - \mathrm{Im}\mu_k)(x-\tau)\Big)|B(\tau)|d\tau, \tag{A.13}$$

$$I_2(x) = \int_x^\infty \sum_{j=1}^4 s_{2j} \exp\Big(- (\mathrm{Im}\mu_j - \mathrm{Im}\mu_k)(x-\tau)\Big)|B(\tau)|d\tau. \tag{A.14}$$

We have

$$I_2(x) \leq 4\int_x^\infty |B(\tau)|d\tau \leq 4\int_x^\infty \frac{1+\tau^3}{1+x^3}|B(\tau)|d\tau \leq \frac{c_2}{1+x^3}, \tag{A.15}$$

where c_2 is a constant, independent of λ. We also have

$$\begin{gathered} I_1(x) \leq 4\int_0^x \exp\big(- \theta_0(x-\tau)\big)|B(\tau)|d\tau \leq \\ 4\left(\int_0^{x/2} \exp\big(- \theta_0(x-\tau)\big)|B(\tau)|d\tau + \int_{x/2}^x |B(\tau)|d\tau\right) \leq \\ 4\left(\exp(-\theta_0 x/2)\int_0^{x/2} |B(\tau)|d\tau + \int_{x/2}^x \frac{1+8\tau^3}{1+x^3}|B(\tau)|d\tau\right) \leq \frac{c_1}{1+x^3}, \end{gathered} \tag{A.16}$$

where c_1 is another constant. In this way we get the bound on $|v_k^+(x,\lambda)|$ of equation (A.3). The function y_k^+ extended to $\mathbb{R}\times[\lambda_s,\infty)$ with the required properties is obtained by solving the system (23) with the initial condition at $x=0$ given by $y(0) = \Pi(\lambda)\big(e_k + v_k^+(0,\lambda)\big)$.

The existence of the functions y_k^- with the required properties is proved similarly. □

The matrix $\Pi(\lambda)$ is singular at $\lambda = h_\mathrm{p}$ because in this case $\nu = 0$ and $p_2 = p_3$. A fundamental matrix of the system $y' = Ay$ with an inverse bounded in $[h_\mathrm{p}, \lambda_s]$ has to be built in a different way. Let $\Psi(x)$ the 4×4 matrix whose first, second, and fourth columns are p_1, p_2, and p_4, respectively, and whose third column is

$$\frac{1}{\nu}\big(p_3 - \exp(-\mathrm{i}2\nu x)p_2\big) \;\text{ if }\; \lambda \in (h_\mathrm{p}, \lambda_s], \quad xe_1 + e_2 \;\text{ if }\; \lambda = h_\mathrm{p}. \tag{A.17}$$

Again the λ dependence of $\Psi(x)$ is not explicitly shown to avoid unwieldy expressions. Clearly, $\Psi(x)D(x)$ is a fundamental matrix of $y' = Ay$. We notice

$$\det\Psi(x) = \frac{1}{\nu}\det\Pi = 2\theta(\theta^2+\nu^2) \geq 2\theta_0^3 > 0, \quad \lambda \geq h_\mathrm{p}. \tag{A.18}$$

From this it is easy to see that there is c_0 is independent of x and λ such that $|\Psi(x)| \leq c_0(1+|x|)$ and $|\Psi^{-1}(x)| \leq c_0(1+|x|)$ hold for $(x,\lambda) \in \mathbb{R}\times[h_\mathrm{p},\lambda_s]$.

Theorem 24. *For any $\lambda_s > h_\mathrm{p}$ there are functions z_k^+ and z_k^-, $1 \leq k \leq 4$, with domain $\mathbb{R}\times[h_\mathrm{p},\lambda_s]$ and image in $\mathbb{C}^4$, which are written in the form*

$$\begin{aligned} z_k^+(x,\lambda) &= \exp(\mathrm{i}\mu_k x)\big(p_k + w_k^+(x,\lambda)\big), \\ z_k^-(x,\lambda) &= \exp(\mathrm{i}\mu_k x)\big(p_k + w_k^-(x,\lambda)\big), \end{aligned} \tag{A.19}$$

such that

1. $\left\{z_k^+(\cdot,\lambda),\, 1 \le k \le 4\right\}$ *and* $\left\{z_k^-(\cdot,\lambda),\, 1 \le k \le 4\right\}$ *are two fundamental sets of solutions of the system (23) for each* $\lambda \in (h_\mathrm{p}, \lambda_s]$.
2. *The functions* w_k^+ *and* w_k^- *are continuous on* $\mathbb{R} \times [h_\mathrm{p}, \lambda_s]$, *and satisfy*

$$|w_k^+(x,\lambda)| \le c\,(1+x)^{-1}, \quad x \ge 0, \quad \lambda \in [h_\mathrm{p}, \lambda_s], \tag{A.20}$$

$$|w_k^-(x,\lambda)| \le c\,(1-x)^{-1}, \quad x \le 0, \quad \lambda \in [h_\mathrm{p}, \lambda_s], \tag{A.21}$$

where c *is a constant (independent of* x *and* λ, *it may depend on* λ_s*).*

3. *For fixed* $x \in \mathbb{R}$, $w_k^+(x,\cdot)$ *and* $w_k^-(x,\cdot)$ *are analytic in* $(h_\mathrm{p}, \lambda_s]$.

Proof. For $1 \le k \le 4$ we obtain z_k^+ as follows. For $l = 1, 2$ and $1 \le j \le 4$ define s_{lj} and $D_l(x)$ as in the proof of Theorem 24. Notice that $D(x) = D_1(x) + D_2(x)$ and that $\Psi(x)D_l(x)$, $l = 1, 2$, are solution matrices of the system (23). Let $x_0 \ge 0$ be a constant to be determined, and for $(x,\lambda) \in [x_0,\infty) \times [h_\mathrm{p}, \lambda_s]$ consider the integral equation

$$\begin{aligned} y(x,\lambda) = \exp(\mathrm{i}\mu_k x)p_k + \int\limits_{x_0}^{x} \Psi(x)D_1(x)D(-\tau)\Psi^{-1}(\tau)B(\tau)y(\tau,\lambda)d\tau \\ - \int\limits_{x}^{\infty} \Psi(x)D_2(x)D(-\tau)\Psi^{-1}(\tau)B(\tau)y(\tau,\lambda)d\tau. \end{aligned} \tag{A.22}$$

If this equation has a solution, then it is a solution of the system (23). Let us consider the successive approximations $y^{(0)}(x,\lambda) = 0$ and, for $j \in \mathbb{N}$,

$$\begin{aligned} y^{(j)}(x,\lambda) = \exp(\mathrm{i}\mu_k x)p_k + \int\limits_{x_0}^{x} \Psi(x)D_1(x)D(-\tau)\Psi^{-1}(\tau)B(\tau)\,y^{(j-1)}(\tau,\lambda)d\tau \\ - \int\limits_{x}^{\infty} \Psi(x)D_2(x)D(-\tau)\Psi^{-1}(\tau)B(\tau)\,y^{(j-1)}(\tau,\lambda)d\tau. \end{aligned} \tag{A.23}$$

We have $y^{(1)}(x,\lambda) = \exp(\mathrm{i}\mu_k x)p_k$ and $|y^{(1)}(x,\lambda)| = \exp(-\mathrm{Im}\,\mu_k x)|p_k|$. The functions $y^{(j)}$ are continuous on $[x_0,\infty) \times [h_\mathrm{p}, \lambda_s]$ and, for fixed $x \ge x_0$, $y^{(j)}(x,\cdot)$ are analytic in $(h_\mathrm{p}, \lambda_s]$. Define $c_1 = \max\{|p_k|,\, 1 \le k \le 4,\, h_\mathrm{p} \le \lambda \le \lambda_s\}$. We now show that if x_0 is large enough then

$$\left|y^{(j+1)}(x,\lambda) - y^{(j)}(x,\lambda)\right| \le c_1(1/2)^j \exp(-\mathrm{Im}\,\mu_k x) \tag{A.24}$$

for $x \ge x_0$, $\lambda \in [h_\mathrm{p}, \lambda_s]$, and $j \in \mathbb{N} \cup \{0\}$. We proceed by induction. For $j = 0$ the inequality holds. Suppose that it holds for some $j \in \mathbb{N} \cup \{0\}$. This assumption, equation (A.23), and the bounds on $\Psi(x)$ and $\Psi^{-1}(x)$ imply

$$\left|y^{(j+2)}(x,\lambda) - y^{(j+1)}(x,\lambda)\right| \le c_2(1+x)(1/2)^j \exp(-\mathrm{Im}\,\mu_k x)J(x), \tag{A.25}$$

where c_2 is a constant, $J(x) = I_1(x) + I_2(x)$, and

$$I_1(x) = \int_{x_0}^{x} \sum_{j=1}^{4} s_{1j} \exp\Big(-(\mathrm{Im}\mu_j - \mathrm{Im}\mu_k)(x-\tau)\Big)(1+\tau)|B(\tau)|d\tau, \tag{A.26}$$

$$I_2(x) = \int_{x}^{\infty} \sum_{j=1}^{4} s_{2j} \exp\Big(-(\mathrm{Im}\mu_j - \mathrm{Im}\mu_k)(x-\tau)\Big)(1+\tau)|B(\tau)|d\tau. \tag{A.27}$$

As in the proof of Theorem 23, inequalities (A.13) and (A.14), we get that $I_1(x)$ and $I_2(x)$ are bounded by $c_3/(1+x)^2$ if $x \geq x_0$. Then, choosing x_0 so that $2c_3c_2/(1+x_0) \leq 1/2$ we get

$$\big|y^{(j+2)}(x,\lambda) - y^{(j+1)}(x,\lambda)\big| \leq c_1(1/2)^{j+1}\exp(-\mathrm{Im}\,\mu_k x), \tag{A.28}$$

for all $x \geq x_0$ and $\lambda \in [h_{\mathrm{p}}, \lambda_s]$, and inequality (A.24) is proved. Hence, the sequence $y^{(j)}$ converges uniformly on each set $[x_0, x_1] \times [h_{\mathrm{p}}, \lambda_s]$, with $x_1 > x_0$, to a function z_k^+ which, consequently, is continuous on $[x_0, \infty) \times [h_{\mathrm{p}}, \lambda_s]$. Moreover, for fixed $x \in [x_0, \infty)$, the function $z_k^+(x, \cdot)$ is analytic on $(h_{\mathrm{p}}, \lambda_s]$. The bound $|z_k^+(x,\lambda)| \leq 2c_1\exp(-\mathrm{Im}\,\mu_k x)$, for $x \geq x_0$, is easily obtained by summing the inequality

$$|y^{(j+1)}(x,\lambda)| - |y^{(j)}(x,\lambda)| \leq c_1(1/2)^j \exp(-\mathrm{Im}\,\mu_k x), \tag{A.29}$$

from $j = 0$ to l and taking the limit $l \to \infty$. From $z_k^+(x,\lambda)$ we obtain $w_k^+(x,\lambda) = \exp(-\mathrm{i}\mu_k x) z_k^+(x,\lambda) - p_k$, so that

$$\begin{aligned} w_k^+(x,\lambda) = \exp(-\mathrm{i}\mu_k x)\Bigg(&\int_{x_0}^{x} \Psi(x)D_1(x)D(-\tau)\Psi^{-1}(\tau)B(\tau)z_k^+(\tau,\lambda)d\tau \\ &- \int_{x}^{\infty} \Psi(x)D_2(x)D(-\tau)\Psi^{-1}(\tau)B(\tau)z_k^+(\tau,\lambda)d\tau\Bigg). \end{aligned} \tag{A.30}$$

From this equation and the bounds on the different factors of the integrands, we get $|w_k^+(x,\lambda)| \leq 2c_1c_0^2(1+x)\exp(-\mathrm{Im}\,\mu_k x)\big(I_1(x)+I_2(x)\big)$, for $x \geq x_0$, where $I_1(x)$ and $I_2(x)$ are given by equations (A.26) and (A.27), respectively. From the bounds on $I_1(x)$ and $I_2(x)$ obtained above we get the estimate $|w_k^+(x,\lambda)| \leq c/(1+x)$ for $(x,\lambda) \in [x_0,\infty) \times [h_{\mathrm{p}}, \lambda_s]$. The function z_k^+ extended to $\mathbb{R} \times [h_{\mathrm{p}}, \lambda_s]$ with the required properties is obtained by solving the system (23) with the initial condition at $x = x_0$ given by $y(x_0) = p_k + w_k^+(x_0, \lambda)$. The functions z_k^+ thus obtained are continuous $\mathbb{R} \times [h_{\mathrm{p}}, \lambda_s]$ and hence bounded on $[0, x_0] \times [h_{\mathrm{p}}, \lambda_s]$. Then, the bound on w_k^+ can be extended to $[0,\infty) \times [h_{\mathrm{p}}, \lambda_s]$, by increasing c if necessary.

The existence of the functions z_k^- with the required properties is proved similarly. □

For $1 \leq k \leq 4$ the functions ϕ_k and χ_k of Theorem 4 are obtained from the functions y_k^+, y_k^-, z_k^+, and z_k^- of Theorems 23 and 24 as follows. For $x \in \mathbb{R}$:

$$\begin{aligned} \phi_k(x,\lambda) &= \begin{cases} e_1 \cdot z_k^+(x,\lambda), & \lambda \in [h_{\mathrm{p}}, \lambda_s], \\ e_1 \cdot y_k^+(x,\lambda), & \lambda \in (\lambda_s, \infty), \end{cases} \\ \chi_k(x,\lambda) &= \begin{cases} e_1 \cdot z_k^-(x,\lambda), & \lambda \in [h_{\mathrm{p}}, \lambda_s], \\ e_1 \cdot y_k^-(x,\lambda), & \lambda \in (\lambda_s, \infty). \end{cases} \end{aligned} \tag{A.31}$$

Appendix B. Proof of an ancillary result

The next lemma is used in the proof of Lemma 6.

Lemma 25. *Let n be a natural number, I and M subsets of $\mathbb{R}$, with I open, and $f_j : I \times M \to \mathbb{C}$, $1 \le j \le n$, continuous bounded functions. Suppose that for each $\lambda \in M$ the functions $f_j(\cdot, \lambda)$ are linearly independent, consider functions $c_j : M \to \mathbb{C}$, and define $f : I \times M \to \mathbb{C}$ by*

$$f(x,\lambda) = \sum_{j=1}^{n} c_j(\lambda) f_j(x,\lambda), \quad (x,\lambda) \in I \times M. \tag{B.1}$$

If f is continuous and bounded, then the functions c_j are continuous.

Proof. The functions c_j are bounded on any compact subset $J \subseteq M$. Otherwise, there is a sequence $\lambda_m \in J$ such that $\lim_m a_m = \infty$, where $a_m = \max\{|c_j(\lambda_m)|, 1 \le j \le n\}$. Since J is compact, the sequence can be taken convergent, so that $\lambda_m \to \lambda_0 \in J$. Obviously, we have $|c_j(\lambda_m)|/a_m \le 1$ and $\max\{|c_j(\lambda_m)|/a_m, 1 \le j \le n\} = 1$, and therefore the sequence λ_m can be taken so that $|c_j(\lambda_m)|/a_m$ is convergent for $1 \le j \le n$. Let α_j denote the corresponding limit. We have

$$\sum_{j=1}^{n} \frac{|c_j(\lambda_m)|}{a_m} f_j(x,\lambda_m) = \frac{1}{a_m} f(x,\lambda_m). \tag{B.2}$$

Taking the limit $m \to \infty$ in the above equality we obtain $\sum_j \alpha_j f_j(x,\lambda_0) = 0$, and the linear independence of the functions $f_j(\cdot,\lambda_0)$ implies $\alpha_j = 0$ for $1 \le j \le n$. However, this is not possible since $\max\{|c_j(\lambda_m)|/a_m, 1 \le j \le n\} = 1$ for all $m \in \mathbb{N}$. Therefore, the functions c_j are bounded on each compact set J contained in M.

Now take a sequence $\lambda_m \in M$ which converges to $\lambda \in M$. The set $\{(c_1(\lambda_m), \ldots, c_n(\lambda_m)), m \in \mathbb{N}\}$ is bounded in $\mathbb{C}^n$. Let $(a_1, \ldots, a_n) \in \mathbb{C}^n$ be one of its accumulation points. There is a subsequence λ'_m such that $c_j(\lambda'_m)$ converges to a_j for $1 \le j \le n$. Taking the limit $m \to \infty$ in the expression

$$\begin{aligned} \sum_{j=1}^{n} \big(c_j(\lambda) - c_j(\lambda'_m)\big) f_j(x,\lambda) &= f(x,\lambda) - f(x,\lambda'_m) \\ &+ \sum_{j=1}^{n} c_j(\lambda'_m)\big(f_j(x,\lambda) - f_j(x,\lambda'_m)\big) \end{aligned} \tag{B.3}$$

we get $\sum_{j=1}^{n} \big(c_j(\lambda) - a_j\big) f_j(x,\lambda) = 0$ for all $x \in \mathbb{R}$. The linear independence of the functions $f_j(\cdot,\lambda)$ implies that $a_j = c_j(\lambda)$. Therefore, the accumulation point of the sequence $c_j(\lambda_m)$ is unique and the same for all sequences λ_m that converge to λ, and is given by $c_j(\lambda)$. Hence the functions c_j are continuous on M. □

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