In-evolution operators in Lotka-Volterra coalgebras

Manuel Arenas
Departamento de Matemática, Facultad de Ciencias Naturales, Matemática y Del Medio Ambiente, Universidad Tecnológica Metropolitana, Las Palmeras 3460, 780003, Ñuñoa, Santiago, Región Metropolitana, Chile.
m.arenasc@utem.cl

Alicia labra
Departamento de Matemática, Facultad de Ciencias, Universidad de Chile, Las Palmeras 3425, 780003, Ñuñoa, Santiago, Región Metropolitana, Chile.
alimat@uchile.cl

Irene Paniello
Departamento de Matemáticas, Universidad de Zaragoza, Pedro Cerbuna 12, 50009, Zaragoza, Aragón, España.
ipaniello@unizar.es

We study properties of in-evolution operators in Lotka-Volterra coalgebras. First, we cope with the notion of coalgebra character, and classify characters in special cases as, for instance, those of 2-dimensional Lotka-Volterra coalgebras and of normalized duals of Lotka-Volterra algebras. Then, we study the in-evolution operators defined by nonzero characters of Lotka-Volterra coalgebras, focusing on the case when the in-evolution operators are stochastic maps. For such operators we study the usual equilibrium issues, paying special attention to idempotent in-evolution operators, and the information that their idempotency provides on the structure of Lotka-Volterra coalgebras.

Keywords: Lotka-Volterra coalgebra; Character; In-evolution operator.

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1. Introduction

Lotka-Volterra coalgebras (for short, LV-coalgebras) were introduced in [1], aimed by the notion of Lotka-Volterra algebra considered by Gutierrez and García in [6], together to that of coalgebra with genetic realization introduced by Tian and Li in [22].

Stemming from Itoh's use of nonassociative frameworks to describe ternary interactions in competition systems [9], Lotka-Volterra algebras are nonassociative, but commutative, algebras, over fields of characteristic not 2, whose multiplication with respect to a suitably chosen basis is defined by a skew-symmetric matrix. The algebraic structure of Lotka-Volterra algebras has been studied, for instance, in [6,7,9], and these algebras have been noticed to include some classes of Bernstein algebras [10,18].

Lotka-Volterra algebras are also connected to quadratic stochastic operators [3]. A quadratic stochastic operator is called Volterra when additional defining conditions are added to ensure that, in the underlying genetic population, offspring repeats parents genotype [4]. Finite dimensional Volterra quadratic stochastic operators were studied in [3], whereas their infinite-dimensional counterpart, Volterra quantum quadratic stochastic operators, were considered in [11]. We refer to [11], and references therein, for a more accurate description of Volterra-like quadratic stochastic operators.

As many other algebraic structures aimed by genetic systems, Lotka-Volterra algebras were revisited after Tian seminal work on evolution algebras [21]. A finite-dimensional $\mathbb{K}$-algebra $\mathcal{E}$ is an evolution algebra if it is endowed with a basis $\mathcal{B} = \{e_1, \ldots, e_n\}$, called natural, such that $e_i e_j = 0$ for all $i \neq j$. An evolution algebra $\mathcal{E}$ is then called a Volterra evolution algebra if, moreover, $e_i^2 = \sum_{j=1}^n s_{ij} e_j$, where $\mathbf{S} = (s_{ij})_{i,j=1}^n$ is a skew-symmetric matrix. Volterra evolution algebras were studied in [5,12,17], and later generalized to S-evolution algebras in [13]. The algebraic approach to all these algebraic objects follows the main guidelines settled in [21], and both, an algebraic and a probabilistic approach, can be found in [19].

Simultaneously to the development of the above-mentioned structures, different families of coalgebras with genetic significance had been considered by the third author [14,15], all aimed by Tian and Li work in coalgebras with genetic realization [22], originally designed to algebraically model backward genetic inheritance in Mendelian populations. (A different approach to genetic coalgebras can be found in [12].)

This twofold motivation led the authors to introduce in [1] the notion of Lotka-Volterra coalgebra, and to address their study following the strategies contained in [10,23], but now applied to this new family of noncoassociative coalgebras. As a result, Lotka-Volterra coalgebras arose as finite-dimensional coalgebras admitting a distinguished (natural) basis for which these coalgebras can be endowed with a comultiplication, relative to a distinguished (natural) basis, whose structure constants satisfy four algebraic axioms, labelled LVC-1 to LVC-4 in [1] (see Definition 2.2 below), mirroring different genetic properties. To be more accurate, LVC-1 ensures that the offspring necessarily reproduces the genetic type of one progenitor, LVC-2, that algebraically means cocommutativity, ensures panmixia, that is that paternal or maternal type have no specific weight in offspring type, whereas LVC-3 is a normalizing condition. Finally, in the particular case of real coalgebras, LVC-4 is a nonnegativity condition (usually called being endowed with genetic realization), that together with LCV-3 provides the stochasticity of some related objects. For more details on the biological significance of LV-coalgebras, and their corresponding algebraic properties, we refer the reader to [1].

From the algebraic viewpoint, LV-coalgebras are cocommutative, but not necessarily coassociative [1, Example 4.1]. Besides, these coalgebras do not necessarily admit a counit, though necessary and sufficient conditions for the existence of counits can be easily derived [1, Proposition 4.3].

It is a well-known result that the dual of the underlying vector space of arbitrary coalgebras can be endowed with an algebra structure, whose multiplication is induced by the dual map of the coalgebra comultiplication (the converse, however, only holds in the finite-dimensional case, whereas in the infinite-dimensional case a topological dual needs to be introduced) [20]. This fact led the authors to deal with different issues relative to the duality of Lotka-Volterra structures [1], where it was shown that the dual algebra of a LV-coalgebra is not necessarily a Lotka-Volterra structure [1, Corollary 3.2], and also necessary and sufficient conditions were settled for the dual of a finite-dimensional coalgebra to be a Lotka-Volterra algebra [1, Theorem 3.3]. Based on the behaviour of the (natural) basis elements of LV-coalgebras, a first approach to the LV-coalgebra structure is given in [1, Theorem 6.3], as a direct sum (as vector spaces) of a group subcoalgebra and a coideal. Later in [2], we deepened into the structure of LV-coalgebras achieving to parameterize 2-dimensional real LV-coalgebras having genetic realization. In any case, it became clear in [2] that the problem of classifying LV-coalgebras of a given dimension could be considered as a wild problem, so that new techniques and ideas should be brought in the framework.

This paper is devoted to study in-evolution operators of LV-coalgebras. These operators have been previously considered in [15,16,22]. As for many other algebraic objects arising from population genetics, in the case of LV-coalgebras it becomes of interest to describe the behavior of their evolution operators, with focus on those having a stochastic flavour. Given a LV-coalgebra $(\mathcal{C}, \Delta)$, their in-evolution operators arise defined by their character maps, that is, linear maps $\phi : \mathcal{C} \to \mathbb{K}$, such that $(\phi \otimes \phi)\Delta = \phi$, or equivalently, by the idempotent elements of the dual algebra (with the induced multiplication). For each character ϕ an in-evolution operator $S_\phi : \mathcal{C} \to \mathcal{C}$ arises as $S_\phi = (\phi \otimes id)\Delta$ (or equivalently, by the LV-coalgebra cocommutativity, $S_\phi = (id \otimes \phi)\Delta$). It deserves to be noticed that, as proven un [1], existence of nonzero characters is ensured in any LV-coalgebra as a result of LVC-3. Besides of characterizing in-evolution operators, in this work we also considered different equilibrium issues for these operators, as well as a characterization of those real LV-coalgebras with genetic realization admitting idempotent in-evolution operators.

This work is organized as follows. After this introduction, in section 2 we settle basic definitions and notation for further use. Sections 3 and 4 are devoted to study characters in LV-coalgebras, the general results are given in the third section, whereas the fourth section focuses on characterizing characters in the LV-coalgebras arising after normalizing the coalgebra duals of LV-algebras. The last three sections are devoted to in-evolution operators. In section 5 we recall their definition (as firstly considered in [22]) and settle the main properties, including their matrix realization in terms of the LV-coalgebra structure matrix and the underlying character map. Then, in section 6, we focus on equilibrium issues, using this section also to recall some results on the structure of nonnegative stochastic matrices that are idempotent [8]. Finally, in section 7, we apply the theory on idempotent stochastic matrices to the particular case of the in-evolution operator defined by the weight map (that is, the character map defined as $\phi = \sum_{i=1}^{n} e_i^*$). This allows us to describe the algebraic structure of real LV-coalgebras with genetic realization having idempotent structure matrices.

2. Preliminaries

We will work with algebraic structures defined on finite-dimensional $\mathbb{K}$-vector spaces, where, unless otherwise stated, $\mathbb{K}$ denotes a field of characteristic not 2. We refer to [1] and references therein for basic definitions, notation and results on arbitrary coalgebras.

Definition 2.1. A *coalgebra* $(\mathcal{C}, \Delta)$ is a $\mathbb{K}$-vector space $\mathcal{C}$ with a linear map $\Delta : \mathcal{C} \longrightarrow \mathcal{C} \otimes \mathcal{C}$ called *comultiplication*. If $dim_{\mathbb{K}}(\mathcal{C}) = n$, given a basis $\mathcal{B} = \{e_1, \dots, e_n\}$ of the $\mathbb{K}$-vector space $\mathcal{C}$, we write:

$$\Delta(e_k) = \sum_{i,j=1}^{n} \beta_{ij}^{k} e_i \otimes e_j, \qquad k = 1, \dots, n.$$

Elements $x \in \mathcal{C}$ such that $\Delta(x) = x \otimes x$ are called *group-like elements*. A $\mathbb{K}$-vector subspace $\mathcal{V}$ of a coalgebra $\mathcal{C}$ is a *subcoalgebra* of $\mathcal{C}$ if $\Delta(\mathcal{V}) \subseteq \mathcal{V} \otimes \mathcal{V}$, and a *coideal* if $\Delta(\mathcal{V}) \subseteq \mathcal{V} \otimes \mathcal{C} + \mathcal{C} \otimes \mathcal{V}$.

Definition 2.2. A (finite-dimensional) coalgebra $(\mathcal{C}, \Delta)$ is *Lotka-Volterra* (for short, *LV-coalgebra*) if it admits a basis $\mathcal{B} = \{e_1, \dots, e_n\}$, such that for all $i, j, k \in \{1, \dots, n\}$:

(i) LVC-1: $k \notin \{i, j\} \Longrightarrow \beta_{ij}^{k} = 0$;
(ii) LVC-2: $\beta_{ij}^{k} = \beta_{ji}^{k}$;
(iii) LVC-3: $\sum_{i,j=1}^{n} \beta_{ij}^{k} = 1$.

If, moreover, $\mathbb{K} = \mathbb{R}$, the LV-coalgebra $(\mathcal{C}, \Delta)$ has *genetic realization* if the following condition holds:

(iv) LVC-4: $\beta_{ij}^{k} \geq 0$ for all $i, j, k = 1, \dots, n$.

Bases as in Definition 2.2 are called *natural bases*. All bases considered here are assumed natural bases. We refer the reader to [1,2] for the biological motivation underlying the definition of LV-coalgebras. The following results follow from LVC-1 to LVC-3.

Lemma 2.1. *[2, Lemma 2.1] Let $(\mathcal{C}, \Delta)$ be a LV-coalgebra with natural basis $\mathcal{B} = \{e_1, \dots, e_n\}$. Then:*

(i) $\mathcal{C}$ is cocommutative.
(ii) $\Delta(e_k) = \beta_{kk}^{k} e_k \otimes e_k + \sum_{i=1, i\neq k}^{n} \beta_{ik}^{k} (e_i \otimes e_k + e_k \otimes e_i)$, for all $e_k \in \mathcal{B}$.
(iii) $\beta_{kk}^{k} + 2 \left(\sum_{i=1, i\neq k}^{n} \beta_{ik}^{k} \right) = 1$, for all $k = 1, \dots, n$.

(iv) $\Delta(e_k) \neq 0$, *for all* $e_k \in \mathcal{B}$.

Remark 2.1. Assume $e_k \in \mathcal{B}$ is a (basic) group-like element of a real LV-coalgebra $\mathcal{C}$ satisfying LVC-4, and that there exists $e_i \in \mathcal{B}$, $i \neq k$, such that $\Delta(e_i) = \beta^i_{ii} e_i \otimes e_i + \beta^i_{ki}(e_i \otimes e_k + e_k \otimes e_i)$. Then $\mathcal{D}_k = span_{\mathbb{R}}(e_k)$ and $\mathcal{D}_{k,i} = span_{\mathbb{R}}(e_k, e_i)$ are basic (i.e. spanned by basic elements) subcoalgebras of $\mathcal{C}$. Indeed, both $\mathcal{D}_k$ and $\mathcal{D}_{k,i}$ are LV-coalgebras with natural bases $\{e_k\}$ and $\{e_k, e_i\}$, respectively.

Theorem 2.1. *[2, Theorem 3.2, Corollary 3.3] Let* $\mathcal{C}$ *be a* n*-dimensional* $\mathbb{K}$*-vector space with basis* $\mathcal{B}$*. Then there is a 1:1 correspondence between LV-coalgebras defined on* $\mathcal{C}$ *with natural basis* $\mathcal{B} = \{e_1, \ldots, e_n\}$*, and matrices* $\mathbf{M} \in M_n(\mathbb{K})$ *with column sums equal to 1. More precisely, the LV-coalgebra structure on* $\mathcal{C}$ *is given by the matrix:*

$$\mathbf{M}(\mathcal{C}, \mathcal{B}) = \begin{pmatrix} \sum_{i=1}^n \beta^1_{i1} & \beta^2_{12} & \cdots & \beta^n_{1n} \\ \beta^1_{21} & \sum_{i=1}^n \beta^2_{i2} & \cdots & \beta^n_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \beta^1_{n1} & \beta^2_{n2} & \cdots & \sum_{i=1}^n \beta^n_{in} \end{pmatrix}. \tag{2.1}$$

If, moreover, $\mathbb{K} = \mathbb{R}$*, then the above correspondence relates LV-coalgebras with genetic realization and nonnegative real matrices with column sums equal to one which are (non-necessarily strictly) diagonal dominant by columns.*

We refer to $\mathbf{M}(\mathcal{C}, \mathcal{B})$ to be the (*marginal* or *structure*) *matrix associated with the LV-coalgebra* $\mathcal{C}$ *with respect to the natural basis* $\mathcal{B}$.

The following results, that can be found in [2], will be instrumental along the paper.

Lemma 2.2. *[2, Lemma 3.4] Let* $\mathcal{C}$ *be a LV-coalgebra with natural basis* $\mathcal{B} = \{e_1, \ldots, e_n\}$*. If we denote* $\mathbf{M}(\mathcal{C}, \mathcal{B}) = (m_{ij})^n_{i,j=1}$*, then:*

(i) $\beta^k_{kk} = 0$ *if and only if* $m_{kk} = \frac{1}{2}$.
(ii) e_k *is group-like if and only if* $m_{ik} = \delta_{ik}$*, for* $i = 1, \ldots, n$.
(iii) If $\mathbb{K} = \mathbb{R}$ *and* $\mathcal{C}$ *satisfies LVC-4, then* $m_{kk} \in [\frac{1}{2}, 1]$ *and* $m_{ik} \in [0, \frac{1}{2}]$ *for all* $i \neq k$.

The dual vector space $\mathcal{C}^* = Hom_{\mathbb{K}}(\mathcal{C}, \mathbb{K})$ of any coalgebra $\mathcal{C}$ can be endowed with a multiplication given by the composition map $\mu = \Delta^* \iota : \mathcal{C}^* \otimes \mathcal{C}^* \hookrightarrow (\mathcal{C} \otimes \mathcal{C})^* \to \mathcal{C}^*$, where $\iota : \mathcal{C}^* \otimes \mathcal{C}^* \hookrightarrow (\mathcal{C} \otimes \mathcal{C})^*$. (We note that here, by the finite-dimensional assumption on $\mathcal{C}$, the map ι is indeed the natural isomorphism $\mathcal{C}^* \otimes \mathcal{C}^* \cong (\mathcal{C} \otimes \mathcal{C})^*$.) As a result, for any $\phi, \psi \in \mathcal{C}^*$, $(\phi\psi)(c) = (\phi \otimes \psi)\Delta(c)$, for all $c \in \mathcal{C}$.

Definition 2.3. A *Lotka-Volterra algebra* (*LV-algebra*, for short) is an n-dimensional real algebra admitting a (natural) basis $\mathcal{B} = \{a_1, \ldots, a_n\}$ such that:

$$a_i a_j = \left(\frac{1}{2} + s_{ij}\right) a_i + \left(\frac{1}{2} + s_{ji}\right) a_j, \qquad i, j = 1, \ldots, n,$$

where $\mathbf{S} = (s_{ij})_{i,j=1}^n \in M_n(\mathbb{R})$ is skew-symmetric real matrix.

Definition 2.4. In [9] entries of the above introduced skew-symmetric real matrix $\mathbf{S}$ are assumed to be such that $|s_{ij}| \leq \frac{1}{2}$, for all $i, j = 1, \ldots, n$. We will say that a commutative real algebra $\mathcal{A}$ is a LV-algebra *in Itoh's sense* whenever this condition holds.

Dual algebras of LV-coalgebras do not provide LV-algebras. Necessary and sufficient conditions for the dual algebra $(\mathcal{C}^*, \Delta^*)$ of a coalgebra $(\mathcal{C}, \Delta)$ to be a LV-algebra can be found in [1, Theorem 3.3]. In [1, Example 3.4], a normalized procedure was introduced to obtain new LV-coalgebras stemming from (duals of) algebras with genetic realization. This applies, for instance, to the n-dimensional gametic algebra with simple Mendelian law (see, for instance, [23, p. 135]).

3. Characters in LV-coalgebras

A coalgebra $(\mathcal{C}, \Delta)$ is *baric* if there exists a nonzero linear map $\phi : \mathcal{C} \to \mathbb{K}$ such that $(\phi \otimes \phi)\Delta = \phi$. Then, the map ϕ is called *character* or *weight function.* We will write $(\mathcal{C}, \Delta, \phi)$ to denote that the coalgebra $(\mathcal{C}, \Delta)$ is endowed with a character ϕ. Viewing characters as elements of the dual algebra $\mathcal{C}^* = Hom_{\mathbb{K}}(\mathcal{C}, \mathbb{K})$ of $\mathcal{C}$, one obtains the following characterization.

Proposition 3.1. ***(Character characterization.)*** *[1, Proposition 4.7] Let $\mathcal{C}$ be a LV-coalgebra with natural basis $\mathcal{B} = \{e_1, \ldots, e_n\}$. A linear map $\phi = \sum_{i=1}^n \alpha_i e_i^* \in \mathcal{C}^*$ is a character of $\mathcal{C}$ if and only if*

$$\alpha_k = \alpha_k \left(\beta_{kk}^k \alpha_k + 2 \sum_{i=1, i \neq k}^n \beta_{ik}^k \alpha_i \right), \qquad \textit{for all } k = 1, \ldots, n,$$

where $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$ and $\mathcal{B}^* = \{e_1^*, e_2^*, \ldots, e_n^*\}$ denotes the dual basis of $\mathcal{B}$.

Remark 3.1.

(i) Coalgebra characters are idempotent elements of the dual algebra $\mathcal{C}^*$ with respect to the induced multiplication. Indeed, by definition, any character ϕ of $\mathcal{C}$ satisfies $(\phi \otimes \phi)\Delta = \phi$, which amounts to $\phi^2 = \phi\phi = \phi$ in terms of the induced multiplication in $\mathcal{C}^*$.

(ii) It is a basic linear algebra fact that, with the notation considered in Characterization 3.1 above, $\alpha_i = \phi(e_i)$, for all $i = 1, \ldots, n$.

(iii) Arbitrary coalgebras need not have nontrivial characters [22, Remark 6], but by LVC-3 the element $\phi = \sum_{i=1}^n e_i^* \in \mathcal{C}^*$ is a character for any LV-coalgebra $\mathcal{C}$ [1, Corollary 4.8]. However, characters of LV-coalgebras may not be unique [1, Corollary 4.10].

Lemma 3.1. *Let $\mathcal{C}$ be a LV-coalgebra with natural basis $\mathcal{B} = \{e_1, \ldots, e_n\}$. If $\phi = \sum_{i=1}^n \alpha_i e_i^* \in \mathcal{C}^*$ is a character of $\mathcal{C}$, then $\beta_{kk}^k \alpha_k + 2\sum_{i=1, i \neq k}^n \beta_{ik}^k \alpha_i = 1$ for all $k \in \{1, \ldots, n\}$ such that $\alpha_k \neq 0$.*

Proof. It follows from Proposition 3.1. □

The following result, aimed by [22, Theorem 4.3], generalizes Remark 3.1(iii), holds for arbitrary finite-dimensional coalgebras, and provides an alternative characterization for LVC-3.

Proposition 3.2. *Let $(\mathcal{C}, \Delta)$ be a n-dimensional $\mathbb{K}$-coalgebra with basis $\mathcal{B} = \{e_1, \ldots, e_n\}$. Then, $\phi = \sum_{i=1}^{n} e_i^* \in \mathcal{C}^*$ is a character of $\mathcal{C}$ if and only if $\sum_{i,j=1}^{n} \beta_{ij}^k = 1$, for all $k = 1, \ldots, n$.*

Proof. Let $\phi = \sum_{i=1}^{n} e_i^* \in \mathcal{C}^*$. Since $\phi(e_k) = 1$, for all $k = 1, \ldots, n$, it follows that ϕ is a character of $\mathcal{C}$ if and only if

$$\phi(e_k) = ((\phi \otimes \phi)\Delta)(e_k) = (\phi \otimes \phi)\left(\sum_{i,j=1}^{k} \beta_{ij}^k e_i \otimes e_j\right) =$$

$$= \sum_{i,j=1}^{k} \beta_{ij}^k \phi(e_i)\phi(e_j) = \sum_{i,j=1}^{k} \beta_{ij}^k.$$

That is, ϕ is a character of $\mathcal{C}$ if and only if $\sum_{i,j=1}^{n} \beta_{ij}^k = 1$, for all $k = 1, \ldots, n$. □

Lemma 3.2. *Let $\mathcal{C}$ be a LV-coalgebra with natural basis $\mathcal{B} = \{e_1, \ldots, e_n\}$.*

(i) Let $\phi \in \mathcal{C}^*$ be a character of $\mathcal{C}$. If $g \in \mathcal{C}$ is a group-like element, then $\phi(g) \in \{0, 1\}$.

(ii) Let $e_k \in \mathcal{B}$.

(ii.a) If e_k is group-like, then $\phi = e_k^* \in \mathcal{C}^*$ is a character of $\mathcal{C}$.

(ii.b) If $\phi = e_k^* \in \mathcal{C}^*$ is a character of $\mathcal{C}$, then $\beta_{kk}^k = 1$. If, moreover, $\mathcal{C}$ is a real LV-coalgebra and it satisfies LVC-4, then e_k is a group-like element.

Proof. (i) Assume $\phi \in \mathcal{C}^*$ is a character of $\mathcal{C}$, and let $g \in \mathcal{C}$ be a group-like element. Then, since $(\phi \otimes \phi)\Delta = \phi$, it holds that $\phi(g) = (\phi \otimes \phi)\Delta(g) = (\phi \otimes \phi)(g \otimes g) = \phi(g)^2$. Hence, $\phi(g) \in \{0, 1\}$.

(ii.a) Let now $e_k \in \mathcal{B}$. If e_k is a group-like element. Then $\Delta(e_k) = e_k \otimes e_k$, and we have $\beta_{kk}^k = 1$ and $\beta_{ik}^k = 0$ for all $i \neq k$. It is then straightforward from the Character Characterization (Proposition 3.1) that $\phi = e_k^* \in \mathcal{C}^*$ is a character of $\mathcal{C}$.

(ii.b) Assume next that $e_k \in \mathcal{B}$ is such that $\phi = e_k^* \in \mathcal{C}^*$ is a character of $\mathcal{C}$. Evaluating the Character Characterization (Proposition 3.1) taking $\alpha_i = \delta_{ik}$, $i = 1, \ldots, n$, one obtains $\beta_{kk}^k = 1$. Thus if $\mathbb{K} = \mathbb{R}$ and $\mathcal{C}$ satisfies LVC-4, by Lemma 2.1(iii), $\beta_{ik}^k = 0$, for all $i \neq k$, implying that $\Delta(e_k) = e_k \otimes e_k$ and, therefore, that e_k is group-like. □

Lemma 3.3. *Let $\mathcal{C}$ be a LV-coalgebra with natural basis $\mathcal{B} = \{e_1, \ldots, e_n\}$, and let ϕ be a character of $\mathcal{C}$. For any $e_k, e_i \in \mathcal{B}$, $k \neq i$, such that e_k is group-like and $\Delta(e_i) = \beta_{ii}^i e_i \otimes e_i + \beta_{ik}^i (e_i \otimes e_k + e_k \otimes e_i)$ we have:*

(i) If $\phi(e_k) = 0$, then either $\phi(e_i) = 0$ or $\beta^i_{ii} \neq 0$ (being, in this second case, $\phi(e_i) = 1/\beta^i_{ii}$.)
(ii) If $\phi(e_k) = 1$ and $\beta^i_{ii} \neq 0$, then $\phi(e_i) \in \{0, 1\}$. (If $\beta^i_{ii} = 0$ no conclusive on $\phi(e_i)$ arises.)

Proof. We first note that by Lemma 3.2, it holds that $\phi(e_k) \in \{0, 1\}$. This implies that $\phi(e_i) = (\phi\otimes\phi)\Delta(e_i) = \beta^i_{ii}\phi(e_i)^2+2\beta^i_{ik}\phi(e_i)\phi(e_k)$, where, by LVC-3, $\beta^i_{ii}+2\beta^i_{ik} = 1$.

(i) If $\phi(e_k) = 0$, then $\phi(e_i) = \beta^i_{ii}\phi(e_i)^2$. Thus, either $\phi(e_i) = 0$, or necessarily $\beta^i_{ii} \neq 0$, being then $\phi(e_i) = 1/\beta^i_{ii}$.

(ii) Assume now that $\phi(e_k) = 1$, so that $\phi(e_i) = \beta^i_{ii}\phi(e_i)^2 + 2\beta^i_{ik}\phi(e_i)$. If $\phi(e_i)\beta^i_{ii} \neq 0$, then $1 = \beta^i_{ii}\phi(e_i) + 2\beta^i_{ik}$, implying that $\phi(e_i) = \frac{1-2\beta^i_{ik}}{\beta^i_{ii}} = 1$. □

Example 3.1. Let $\mathcal{C}$ be a 2-dimensional real LV-coalgebra with genetic realization (i.e. satisfying LVC-4) and let $\mathcal{B} = \{e_1, e_2\}$ be a natural basis of $\mathcal{C}$. Following [2, Example 3.5] we denote:

$$\mathbf{M}(\mathcal{C}, \mathcal{B}) = \mathbf{M}(\alpha, \beta) = \begin{pmatrix} 1-\alpha & \beta \\ \alpha & 1-\beta \end{pmatrix}, \qquad 0 \leq \alpha, \beta \leq \frac{1}{2},$$

the marginal (or structure) matrix of $\mathcal{C}$ with respect to the given natural basis $\mathcal{B}$, and recall that (see [2, Theorem 3.2, Corollary 3.3]):

$$\begin{pmatrix} 1-\alpha & \beta \\ \alpha & 1-\beta \end{pmatrix} = \begin{pmatrix} \beta^1_{11}+\beta^1_{21} & \beta^2_{12} \\ \beta^1_{21} & \beta^2_{12}+\beta^2_{22} \end{pmatrix}.$$

Assume $\phi = \phi(e_1)e_1^* + \phi(e_2)e_2^* \in \mathcal{C}^*$ is a nonzero character of $\mathcal{C}$ (see Remark 3.1(ii)). Then, by Characterization 3.1:

$$\begin{cases} \phi(e_1) = \beta^1_{11}\phi(e_1)^2 + 2\beta^1_{21}\phi(e_1)\phi(e_2) \\ \phi(e_2) = \beta^2_{22}\phi(e_2)^2 + 2\beta^2_{12}\phi(e_1)\phi(e_2) \end{cases}$$

whose solutions are comprised in Table 1. Indeed:

(i) If $\alpha = \beta = 0$, that is if $\beta^1_{11} = \beta^2_{22} = 1$ and $\beta^1_{21} = \beta^2_{12} = 0$, then:

$$\begin{cases} \phi(e_1) = \phi(e_1)^2 \\ \phi(e_2) = \phi(e_2)^2 \end{cases}$$

which gives $\phi(e_i) \in \{0, 1\}$, $i = 1, 2$.

(ii) If $\alpha = 0$ and $\beta \neq 0$ (the case $\alpha \neq 0$ and $\beta = 0$ is similar), then $\beta^1_{11} = 1$, $\beta^1_{21} = 0$, $\beta^2_{22} = 1-2\beta$ and $\beta^2_{12} = \beta$, implying that:

$$\begin{cases} \phi(e_1) = \phi(e_1)^2 \\ \phi(e_2) = (1-2\beta)\phi(e_2)^2 + 2\beta\phi(e_1)\phi(e_2) \end{cases}$$

and therefore $\phi(e_1) \in \{0, 1\}$.

(ii.a) If $\phi(e_1) = 0$, then $\phi(e_2) = (1-2\beta)\phi(e_2)^2$ and, since ϕ is assumed to be nonzero, it follows that $\beta \neq \frac{1}{2}$ and $\phi(e_2) = \frac{1}{1-2\beta}$.

(ii.b) If $\phi(e_1) = 1$, then $\phi(e_2) = (1 - 2\beta)\phi(e_2)^2 + 2\beta\phi(e_2)$, and now $\phi(e_2) = 0$, $\phi(e_2) = 1$ if $\beta \neq \frac{1}{2}$, or $\phi(e_2)$ is arbitrary if $\beta = \frac{1}{2}$.

(iii) If $\alpha\beta \neq 0$, we have $\beta_{11}^1 = 1 - 2\alpha$, $\beta_{21}^1 = \alpha$, $\beta_{22}^2 = 1 - 2\beta$ and $\beta_{12}^2 = \beta$. Then:

$$\begin{cases} \phi(e_1) = (1 - 2\alpha)\phi(e_1)^2 + 2\alpha\phi(e_1)\phi(e_2) \\ \phi(e_2) = (1 - 2\beta)\phi(e_2)^2 + 2\beta\phi(e_1)\phi(e_2) \end{cases}$$

(iii.a) If $0 < \alpha, \beta < \frac{1}{2}$, since $\phi(e_1)$ and $\phi(e_2)$ cannot vanish simultaneously, either $\phi(e_1) = 0$ and $\phi(e_2) = \frac{1}{1-2\beta}$, or $\phi(e_1) = \frac{1}{1-2\alpha}$ and $\phi(e_2) = 0$, or $\phi(e_1)\phi(e_2) \neq 0$. But in this last case, the system above becomes:

$$\begin{cases} 1 = (1 - 2\alpha)\phi(e_1) + \quad 2\alpha\phi(e_2) \\ 1 = \quad 2\beta\phi(e_1) \quad + (1 - 2\beta)\phi(e_2) \end{cases}$$

This system has unique solution $\phi(e_1) = \phi(e_2) = 1$ if $\alpha + \beta \neq \frac{1}{2}$, whereas if $\alpha + \beta = \frac{1}{2}$, one obtains $\phi(e_1) = \frac{1-(1-2\beta)\phi(e_2)}{2\beta}$ and $\phi(e_2) \in \mathbb{R}$.

(iii.b) If $0 < \beta < \alpha = \frac{1}{2}$ (analogously if $0 < \beta < \alpha = \frac{1}{2}$), $\phi(e_1) = \phi(e_1)\phi(e_2)$ and $\phi(e_2) = (1 - 2\beta)\phi(e_2)^2 + 2\beta\phi(e_1)\phi(e_2)$. Thus, again using that $\phi(e_1)$ and $\phi(e_2)$ cannot vanish simultaneously, either $\phi(e_1) = 0$ and $\phi(e_2) = \frac{1}{1-2\beta}$, or $\phi(e_1) = \phi(e_2) = 1$.

(iii.c) If $\alpha = \beta = \frac{1}{2}$, we have $\phi(e_1) = \phi(e_1)\phi(e_2) = \phi(e_2)$, and the only nontrivial character ϕ is given by $\phi(e_1) = \phi(e_2) = 1$.

Table 1. Example 3.1 Characters in dimension 2.

Parameters		Characters
$\alpha = \beta = 0$		$\{e_1^*, e_2^*, e_1^* + e_2^*\}$
$\alpha = 0, \beta \neq 0$	$\beta = \frac{1}{2}$	$\{e_1^* + te_2^* \mid t \in \mathbb{R}\}$
	$\beta \neq \frac{1}{2}$	$\{e_1^*, \frac{1}{1-2\beta}e_2^*, e_1^* + e_2^*\}$
$\beta = 0, \alpha \neq 0$	$\alpha = \frac{1}{2}$	$\{te_1^* + e_2^* \mid t \in \mathbb{R}\}$
	$\alpha \neq \frac{1}{2}$	$\{\frac{1}{1-2\alpha}e_1^*, e_2^*, e_1^* + e_2^*\}$
$0 < \alpha, \beta < \frac{1}{2}$	$\alpha + \beta \neq \frac{1}{2}$	$\{\frac{1}{1-2\alpha}e_1^*, \frac{1}{1-2\beta}e_2^*, e_1^* + e_2^*\}$
	$\alpha + \beta = \frac{1}{2}$	$\{se_1^* + te_2^* \mid (s,t) \in \mathbb{R}^2, t\alpha + s\beta = \frac{1}{2}\}$
$0 < \beta < \alpha = \frac{1}{2}$		$\{\frac{1}{1-2\beta}e_2^*, e_1^* + e_2^*\}$
$0 < \alpha < \beta = \frac{1}{2}$		$\{\frac{1}{1-2\alpha}e_1^*, e_1^* + e_2^*\}$
$\alpha = \beta = \frac{1}{2}$		$\{e_1^* + e_2^*\}$

Remark 3.2. The 2-dimensional Example 3.1 clearly corresponds to the solution of a system of quadratic equations. In general, given an arbitrary n-dimensional LV-coalgebra $\mathcal{C}$ its nonzero characters arise as the solutions of a system consisting on n quadratic hypersurfaces (see Characterization 3.1):

$$\beta_{kk}^k X_k^2 + 2\sum_{i=1, i\neq k}^{n} \beta_{ik}^k X_i X_k - X_k = 0, \quad k = 1, \ldots, n.$$

Remark 3.3. For any arbitrary coalgebra (in particular, for any LV-coalgebra) $\mathcal{C}$, the kernel $Ker\,\phi$ of any nonzero character $\phi \in \mathcal{C}^*$ is a codimension one coideal of $\mathcal{C}$, such that $\mathcal{C}\otimes\mathcal{C}$ is not contained in $Ker\,(\phi\otimes\phi) = \Delta(Ker\,\phi) = Ker\,\phi\otimes\mathcal{C}+\mathcal{C}\otimes Ker\,\phi$ [22, Remark 6].

A classical question involving genetic-like algebraic structures endowed with characters is to determine weight one elements. In our current LV-coalgebra setting that amounts to determine elements $x \in \mathcal{C}$ with $\phi(x) = 1$.

Lemma 3.4. *Let $\mathcal{C}$ be a LV-coalgebra with natural basis $\mathcal{B} = \{e_1, \ldots, e_n\}$. For any character ϕ of $\mathcal{C}$ it holds that $\phi = \phi(e_1)e_1^* + \cdots + \phi(e_n)e_n^*$. In particular, $x = \gamma_1 e_1 + \cdots + \gamma_n e_n$ is a weight one element for ϕ if and only if*

$$\sum_{i=1}^{n} \gamma_i \phi(e_i) = \sum_{i,\ e_i \notin Ker\,\phi} \gamma_i \phi(e_i) = 1.$$

Proof. Recall that $\mathcal{C}$ admits at least one character by Remark 3.1(iii). Then, all statements are straightforward linear algebra results. □

Example 3.2. All weight one elements of a 2-dimensional LV-coalgebra with genetic realization can now be computed using Table 1. In general, given a n-dimensional LV-coalgebra baric $\mathcal{C}$, with character ϕ, the weight one elements of $\mathcal{C}$ give rise to a codimension one linear variety of $\mathcal{C}$.

As noticed in Remark 3.1(i), LV-coalgebra characters are exactly the idempotent elements of their dual algebras. We conclude this section with a remark aimed to point out why the results on idempotent elements of LV-algebras given in [6] do not provide a workable tool to study LV-coalgebra characters.

Remark 3.4. As stated in [1, Corollary 3.2], the only LV-coalgebra with genetic realization whose dual algebra is a LV-algebra is the group coalgebra, being the dual algebra of a group coalgebra a (nonzero) trivial evolution algebra. Consider now, for instance, a 2-dimensional real LV-coalgebra $\mathcal{C}$ with genetic realization, natural basis $\mathcal{B} = \{e_1, e_2\}$, and structure matrix $\mathbf{M}(\mathcal{C}, \mathcal{B}) = \mathbf{M}(\alpha, \beta)$, where $0 \leq \alpha, \beta \leq \frac{1}{2}$ (see Lemma 2.2(iii)) Taking into account that $\Delta^*(e_r^* \otimes e_s^*)(e_k) = (e_r^* \otimes e_s^*)\Delta(e_k)$, for all $r, s, k \in \{1, 2\}$ [1], the induced multiplication in $\mathcal{C}^*$, is given by

$$e_1^* \cdot e_1^* = e_1^*, \quad e_2^* \cdot e_2^* = e_2^*, \quad e_1^* \cdot e_2^* = e_2^* \cdot e_1^* = \alpha e_1^* + \beta e_2^*.$$

Hence, $\mathcal{C}^*$ is a LV-algebra if and only if $\alpha = \beta = 0$. If $\alpha\beta \neq 0$, this multiplication can be *normalized* to define a new algebra $Norm(\mathcal{C}^*)$, over the same underlying vector space $\mathcal{C}^*$, but with a new *normalized* multiplication:

$$e_1^* \cdot e_1^* = e_1^*, \quad e_2^* \cdot e_2^* = e_2^*, \quad e_1^* \cdot e_2^* = e_2^* \cdot e_1^* = \frac{\alpha}{\alpha+\beta} e_1^* + \frac{\beta}{\alpha+\beta} e_2^*.$$

Assuming, for instance, $\alpha \geq \beta$, and taking $r = \frac{\alpha-\beta}{2(\alpha+\beta)}$, it is easy to check that $e_1^* \cdot e_2^* = e_2^* \cdot e_1^* = (\frac{1}{2}+r)e_1^* + (\frac{1}{2}-r)e_2^*$, for $r = \frac{\alpha-\beta}{2(\alpha+\beta)}$, which implies that $Norm(\mathcal{C}^*)$ is a LV-algebra. Unfortunately, after this normalization process the cornerstone character of the LV-coalgebra $\mathcal{C}$, that is, $\phi = e_1^* + e_2^*$, is not an idempotent element in $Norm(\mathcal{C}^*)$.

4. Characters of normalized duals of LV-algebras

We devote this section to characterize characters of LV-coalgebras arising by normalizing dual coalgebras of LV-algebras. To achieve this, we first introduce some notation for simplifying forthcoming calculations.

Definition 4.1. Let $\mathcal{C}$ be a LV-coalgebra with natural basis $\mathcal{B} = \{e_1, \ldots, e_n\}$. Given two arbitrary elements $u, v \in \mathcal{C}$ we write $u \bullet v = \frac{1}{2}(u \otimes v + v \otimes u)$, and note that $u \bullet u = u \otimes u$. Then, defining,

$$\gamma_{ik} = \begin{cases} 2\beta_{ik}^k & if\ i \neq k, \\ \beta_{kk}^k & if\ i = k, \end{cases}$$

for $i, k = 1, \ldots, n$, by Definition 2.2, we have:

$$\Delta(e_k) = \gamma_{kk} e_k \bullet e_k + \sum_{i=1, i\neq k}^{n} \gamma_{ik}(e_i \bullet e_k) = \sum_{i=1}^{n} \gamma_{ik}(e_i \bullet e_k) = \left(\sum_{i=1}^{n} \gamma_{ik} e_i\right) \bullet e_k.$$

As a result, letting $\Gamma : \mathcal{C} \to \mathcal{C}$ be the linear map given by $\Gamma(e_k) = \sum_{i=1}^n \gamma_{ik} e_i$, for all $k = 1, \ldots, n$, it holds that $\Delta(e_k) = \Gamma(e_k) \bullet e_k$, for all $k = 1, \ldots, n$.

Remark 4.1. The above definition of the constants $\{\gamma_{ik}\}_{i,k=1}^n$ together to Lemma 2.1(iii) imply that $\sum_{i=1}^n \gamma_{ik} = 1$ for all $k = 1, \ldots, n$. As a result, the operator Γ is stochastic.

Lemma 4.1. *Let $\mathcal{C}$ be a LV-coalgebra with natural basis $\mathcal{B} = \{e_1, \ldots, e_n\}$. Then, $\phi \in \mathcal{C}^*$ is a character of $\mathcal{C}$ if and only if $\phi(e_k) = 0$ or $\phi(\Gamma(e_k)) = 1$ for any $k = 1, \ldots, n$.*

Proof. We first note that for any $\phi \in \mathcal{C}^*$, and any $u, v \in \mathcal{C}$, it holds that:

$$(\phi \otimes \phi)(u \bullet v) = \frac{1}{2}(\phi \otimes \phi)(u \otimes v + v \otimes u) = \frac{1}{2}\left(\phi(u)\phi(v) + \phi(v)\phi(u)\right) = \phi(u)\phi(v).$$

Thus, for any $\phi \in \mathcal{C}^*$ and any $e_k \in \mathcal{B}$:

$$\begin{aligned}(((\phi \otimes \phi)\Delta) - \phi)\,(e_k) &= ((\phi \otimes \phi)\Delta)\,(e_k) - \phi(e_k) = \\ &= (\phi \otimes \phi)(\Gamma(e_k) \bullet e_k) - \phi(e_k) = \\ &= \phi(\Gamma(e_k))\phi(e_k) - \phi(e_k) = \\ &= (\phi(\Gamma(e_k)) - 1)\,\phi(e_k).\end{aligned}$$

Hence, ϕ is a character of $\mathcal{C}$ if and only if $\phi(e_k) = 0$ or $\phi(\Gamma(e_k)) = 1$ for any $k = 1, \ldots, n$. □

Given a finite-dimensional $\mathbb{K}$-coalgebra $(\mathcal{C}, \Delta)$, in [1, Theorem 3.3], the authors settled necessary and sufficient conditions for the dual algebra $\mathcal{C}^*$ of $\mathcal{C}$ to be a LV-algebra. This results into $\mathcal{C}$ satisfying LVC-1 and LVC-2, but not necessarily the normalizing condition LVC-3. However, rescaling the comultiplication structure constants of $\mathcal{C}$, a LV-coalgebra structure can be defined on (the underlying vector space) $\mathcal{C}$. An example illustrating this normalized procedure can be found in [1, Example 3.4].

Theorem 4.1. *Let $\mathcal{A}$ be the n-dimensional real LV-algebra in Itoh's sense defined by a skew-symmetric real matrix $\mathbf{S} = (s_{ij})_{i,j=1}^n$, and let $\mathcal{C}$ the LV-coalgebra arising by normalization of the dual coalgebra $\mathcal{A}^*$ of $\mathcal{A}$. Let $\mathcal{B} = \{e_1, \ldots, e_n\}$ be the natural basis of $\mathcal{C}$, resulting from the normalizing procedure. For any $\phi \in \mathcal{C}^*$, define, for $k = 1, \ldots, n$,*

$$\xi_k^\phi = \phi(S(e_k)) - s_k, \ \textit{where } S(e_k) = \sum_{i=1}^n s_{ik} e_i, \ \textit{and } s_k = \sum_{i=1}^n s_{ik}.$$

Then, $\phi \in \mathcal{C}^*$ is a character of $\mathcal{C}$ if and only if

$$\sum_{i=1}^n \phi(e_i) = n + 2\xi_k^\phi,$$

for every $k \in \{1, \ldots, n\}$ such that $\phi(e_k) \neq 0$.

Remark 4.2. Note that, under the assumptions of Theorem 4.1, $\phi \in \mathcal{C}^*$ is a character of $\mathcal{C}$ if and only if

$$(\phi - e^*) \left(\sum_{i=1}^n \left(\frac{1}{2} - s_{ik} \right) \right) = 0.$$

Proof. Let $\mathcal{A}$ be an n-dimensional LV-algebra with (natural) basis $\{a_1, \ldots, a_n\}$ and multiplication given by $a_i a_j = (\frac{1}{2} + s_{ij})a_i + (\frac{1}{2} + s_{ji})a_j$, where $|s_{ij}| \leq \frac{1}{2}$, for all $1 \leq i, j \leq n$ (see Definition 2.4), and let $\mathcal{C}$ be the LV-coalgebra arising by the normalization of the dual coalgebra $\mathcal{A}^*$ of $\mathcal{A}$, whose natural basis we denote $\mathcal{B} = \{e_1, \ldots, e_n\}$. (We note that the finite-dimensionality of $\mathcal{A}$ ensures us that $\mathcal{A}^*$ can be endowed with a comultiplication [20].)

Let us denote by α_{ij}^k the multiplication constants of $\mathcal{A}$ (that is $a_i a_j = \sum_{k=1}^n \alpha_{ij}^k a_k$), and write:

$$r_k = \sum_{i=1}^n \sum_{j=1}^n \alpha_{ij}^k, \qquad k = 1, \dots, n.$$

By Definition 2.3, $\alpha_{ij}^k = \left(\frac{1}{2} + s_{ij}\right)\delta_{ik} + \left(\frac{1}{2} + s_{ji}\right)\delta_{jk}$, and therefore:

$$\begin{aligned} r_k &= \sum_{i=1}^n \sum_{j=1}^n \alpha_{ij}^k = \sum_{i=1}^n \sum_{j=1}^n \left[\left(\frac{1}{2} + s_{ij}\right)\delta_{ik} + \left(\frac{1}{2} + s_{ji}\right)\delta_{jk}\right] = \\ &= \sum_{i=1}^n \sum_{j=1}^n \left(\frac{\delta_{ik} + \delta_{jk}}{2} + s_{ij}\delta_{ik} + s_{ji}\delta_{jk}\right) = \\ &= n + \sum_{i=1}^n \sum_{j=1}^n (s_{ij}\delta_{ik} + s_{ji}\delta_{jk}) = n + \sum_{i=1}^n \left(\delta_{ik} \sum_{j=1}^n s_{ij} + s_{ki}\right) = \\ &= n + \sum_{i=1}^n \left(-\delta_{ik} \sum_{j=1}^n s_{ji} + s_{ki}\right) = n + \sum_{i=1}^n (-\delta_{ik} s_i - s_{ik}) = \\ &= n + (-s_k - s_k) = n - 2s_k. \end{aligned}$$

Notice now that since $s_{kk} = 0$ for all $k = 1, \dots, n$, and, by Definition 2.4, $|s_{ij}| \leq \frac{1}{2}$ for all $i, j = 1, \dots, n$, it follows that $|s_k| < \frac{n}{2}$. This results in $r_k \neq 0$ for all $k = 1, \dots, n$.

Consider now the comultiplication constants $\{\beta_{ij}^k\}_{i,j,k=1}^n$ of the normalized LV-coalgebra $\mathcal{C}$, and the linear map Γ defined by $\Gamma(e_k) = \sum_{i=1}^n \gamma_{ik} e_i$ and introduced in Definition 4.1. Then:

$$\beta_{ij}^k = \frac{\alpha_{ij}^k}{r_k} = \frac{1}{n - 2s_k}\left[\left(\frac{1}{2} + s_{ij}\right)\delta_{ik} + \left(\frac{1}{2} + s_{ji}\right)\delta_{jk}\right],$$

which results into $\gamma_{kk} = \beta_{kk}^k = \frac{1}{n-2s_k}$, whereas if $i \neq k$, we obtain:

$$\begin{aligned} \gamma_{ik} = 2\beta_{ik}^k &= \frac{2}{n - 2s_k}\alpha_{ik}^k = \\ &= \frac{2}{n - 2s_k}\left[\left(\frac{1}{2} + s_{ik}\right)\delta_{ik} + \left(\frac{1}{2} + s_{ki}\right)\delta_{kk}\right] = \\ &= \frac{2}{n - 2s_k}\left(\frac{1}{2} + s_{ki}\right) = \frac{1 + 2s_{ki}}{n - 2s_k}. \end{aligned}$$

Write now $e = e_1 + \cdots + e_n \in \mathcal{C}$. Then, for all $k = 1, \ldots, n$,

$$\begin{aligned}\Gamma(e_k) = \sum_{i=1}^{n} \gamma_{ik} e_i &= \frac{1}{n - 2s_k} \sum_{i=1}^{n} (e_i + 2s_{ki} e_i) = \\ &= \frac{1}{n - 2s_k} \sum_{i=1}^{n} (e_i - 2s_{ik} e_i) = \\ &= \frac{1}{n - 2s_k} \left(e - 2S(e_k) \right). \end{aligned} \tag{4.1}$$

Applying $\phi \in \mathcal{C}^*$ to (4.1), we obtain the following expression:

$$\phi(\Gamma(e_k)) = \frac{\phi(e) - 2\phi(S(e_k))}{n - 2s_k}. \tag{4.2}$$

We finally have all the machinery needed to prove the desired characterization. Indeed, assume first that $\phi \in \mathcal{C}^*$ is a nonzero character of $\mathcal{C}$. By Lemma 4.1, for any $e_k \in \mathcal{B}$, such that $\phi(e_k) \neq 0$, it holds that $\phi(\Gamma(e_k)) = 1$. Then, by (4.2),

$$\phi(e) = n - 2s_k + 2\phi(S(e_k)) = n + 2\xi_k^{\phi},$$

for any $k \in \{1, \ldots, n\}$ such that $\phi(e_k) \neq 0$.

Conversely, let $\phi \in \mathcal{C}^*$ be a nonzero linear map such that $\phi(e_k) = n + 2\xi_k^{\phi}$, and therefore, such that, $\phi(e) = \sum_{i=1}^{n} \phi(e_i) = n + 2\xi_k^{\phi} = n + 2\left(\phi(S(e_k)) - s_k\right)$ for every $k \in \{1, \ldots, n\}$ with $\phi(e_k) \neq 0$. Then,

$$\phi(\Gamma(e_k)) = \frac{n + 2(\phi(S(e_k)) - s_k) - 2\phi(S(e_k))}{n - 2s_k} = 1,$$

and, finally it follows from Lemma 4.1 that ϕ is a character of $\mathcal{C}$. □

To conclude this section, we consider the particular case of the LV-algebra $\mathcal{A}$ defined by the null skew-symmetric matrix $\mathbf{S} = \mathbf{0}$. The multiplication in $\mathcal{A}$ is then defined by $a_i a_j = \frac{1}{2}(a_i + a_j)$, that is, $\mathcal{A}$ is the gametic algebra for the simple mendelian inheritance. (We refer the reader to [18, Subsection 1.1] and [22, p. 248] for the biological details motivating the following algebraic system.) Now, it is easy to check that $s_k = 0$ and, therefore $\phi(\mathbf{S}(e_k)) = 0$, for all $k = 1, \ldots, n$. Thus, $\xi_k^{\phi} = 0$ for all $\phi \in \mathcal{C}^*$ and all $k = 1, \ldots, n$. The following result follows directly from Theorem 4.1.

Corollary 4.1. *Let $\mathcal{A}$ be the n-dimensional gametic algebra for the simple mendelian inheritance, and let $\mathcal{C}$ be the LV-coalgebra arising from the normalization of the dual coalgebra $\mathcal{A}^*$. Then, a nonzero $\phi \in \mathcal{C}^*$ is a character of $\mathcal{C}$ if and only if $\phi(e) = \sum_{i=1}^{n} \phi(e_i) = n$.*

Remark 4.3. In the simple mendelian inheritance case, since $s_k = 0$ and $\mathbf{S}(e_k) = 0$ holds for all $k = 1, \ldots, n$, identity (4.1) becomes $\Gamma(e_k) = \frac{1}{n} e$ for all $k = 1, \ldots, n$. In particular, this implies that for every nonzero character ϕ of $\mathcal{C}$ it holds that $\phi(\Gamma(e_k)) = 1$ for all $k = 1, \ldots, n$ (even if $\phi(e_k) = 0$).

Remark 4.4. Continuing with the duality issues considered in Remark 3.4, with the same notation consirered there, we notice that for any nonzero $0 < \alpha = \beta \leq \frac{1}{2}$, it turns out that $r = 0$ and, as a result, the normalized algebra $Norm(\mathcal{C}^*)$ of the corresponding LV-coalgebra $\mathcal{C}$ is not only a LV-algebra, but the gametic algebra for the simple mendelian inheritance. However, if we now apply to $Norm(\mathcal{C}^*)$ the procedure described in Theorem 4.1, and consider the dual coalgebra $Norm(\mathcal{C}^*)^*$ of the gametic algebra for the simple mendelian inheritance, the resulting coalgebra is the LV-coalgebra having structure matrix $\mathbf{M}(\frac{1}{4}, \frac{1}{4})$. However, as it can be noticed in Table 1, there exist $0 < \alpha, \alpha' \leq \frac{1}{2}$ such that the LV-coalgebras with structure matrices $\mathbf{M}(\alpha, \alpha)$ and $\mathbf{M}(\alpha', \alpha')$ admit different characters.

5. In-evolution operators

In this section we revisit, in the framework of LV-coalgebras, the notion of (one-sided) in-evolution operator introduced in [22, Section 7] for arbitrary coalgebras with genetic realization. We start noticing that, in the LV-setting, by LVC-2, that is, by the cocommutativity of LV-coalgebras, the notions of left and right in-evolution operators coincide.

Definition 5.1. Let $\mathcal{C}$ be a LV-coalgebra with character ϕ. The *in-evolution operator* of $\mathcal{C}$ defined by ϕ is the linear map $S_\phi : \mathcal{C} \to \mathcal{C}$ given by $S_\phi = (\phi \otimes id)\Delta = (id \otimes \phi)\Delta$ (where $id : \mathcal{C} \to \mathcal{C}$ denotes the identity map).

The following result holds for arbitrary coalgebras endowed with nonzero characters.

Theorem 5.1. *Let $(\mathcal{C}, \Delta, \phi)$ be a baric coalgebra. Then:*

(i) *$\phi(x) = \phi S_\phi(x)$ for all $x \in \mathcal{C}$.*
(ii) *The set $\{x \in \mathcal{C} \mid \phi(x) = 1\}$ of weight one elements of $\mathcal{C}$ (with respect to ϕ) is S_ϕ-invariant.*

Proof. (i) We first prove a general statement.

Claim: Let V be a $\mathbb{K}$-vector space. Then $\varphi(\psi \otimes f) = \psi \otimes (\varphi f)$, for all $f \in End_{\mathbb{K}}(V)$ and $\varphi, \psi \in V^*$ (where we identify $V \cong \mathbb{K} \otimes_{\mathbb{K}} V$ in the obvious way).

To prove this claim it suffices to note that for any $u, v \in V$, it holds that:

$$\begin{aligned}(\varphi(\psi \otimes f))\,(u \otimes v) &= \varphi(\psi(u) \otimes f(v)) = \varphi(\psi(u) f(v)) = \\ &= \psi(u)\varphi(f(v)) = (\psi \otimes (\varphi f))\,(u \otimes v)\end{aligned}$$

Assume now that $(\mathcal{C}, \Delta, \phi)$ is a baric coalgebra with character ϕ, and let $S_\phi = (\phi \otimes id)\Delta$. Then (i) follows from the fact that

$$\phi S_\phi = \phi\left((\phi \otimes id)\Delta\right) = (\phi(\phi \otimes id))\,\Delta = (\phi \otimes (\phi\, id))\,\Delta = (\phi \otimes \phi)\Delta = \phi,$$

and, now, (ii) is straightforward. □

Proposition 5.1. *Let $\mathcal{C}$ be a LV-coalgebra with natural basis $\mathcal{B} = \{e_1, \ldots, e_n\}$. If $\phi = \sum_{i=1}^n \phi(e_i)e_i^*$ is a character of $\mathcal{C}$, then the in-evolution operator S_ϕ defined by ϕ is given by:*

$$S_\phi(e_k) = \left(\sum_{i=1}^n \beta_{ik}^k \phi(e_i)\right) e_k + \sum_{\substack{j=1 \\ j\neq k}}^n \beta_{jk}^k \phi(e_k) e_j, \quad \textit{for all } k = 1, \ldots, n,$$

and extended by linearity to $\mathcal{C}$. In particular, if $\phi = \sum_{i=1}^n e_i^*$, then:

$$S_\phi(e_k) = \left(\sum_{i=1}^n \beta_{ik}^k\right) e_k + \sum_{\substack{j=1 \\ j\neq k}}^n \beta_{jk}^k e_j, \quad \textit{for all } k = 1, \ldots, n.$$

Proof. The statements follow from Characterization 3.1, Remark 3.1, and Theorem 5.1. □

Remark 5.1. For simplicity, we will write S for the in-evolution operator defined by the character $\phi = \sum_{i=1}^n e_i^*$. Otherwise, we will denote by S_ϕ the in-evolution operator defined by a character ϕ.

Example 5.1. Consider the simple Mendelian inheritance for a single gene with two alleles A and a. Let $\mathcal{C}$ be a 2-dimensional real vector space with basis $\mathcal{B} = \{e_1 = A, e_2 = a\}$. Extended by linearity:

$$\Delta(A) = \frac{1}{2} A \otimes A + \frac{1}{4}(A \otimes a + a \otimes A),$$
$$\Delta(a) = \frac{1}{2} a \otimes a + \frac{1}{4}(A \otimes a + a \otimes A),$$

endows $\mathcal{C}$ with a LV-coalgebra structure, satisfying LVC-4, with natural basis $\mathcal{B}$, and structure matrix:

$$\mathbf{M}(\mathcal{C}, \mathcal{B}) = \mathbf{M}\left(\frac{1}{4}, \frac{1}{4}\right) = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}.$$

As it was shown in Example 3.1, $\mathcal{C}$ is baric, being its characters detailed in Table 1 (case $0 < \alpha = \beta = \frac{1}{4} < \frac{1}{2}$ with $\alpha + \beta = \frac{1}{2}$). Consider first $\phi = A^* + a^*$, and let $S = (\phi \otimes id)\Delta = (id \otimes \phi)\Delta$ be the in-evolution operator of $\mathcal{C}$ defined by ϕ. Then, by Proposition 5.1, given an arbitrary element $x = x_1 A + x_2 a \in \mathcal{C}$, it holds that $S(x) = \left(\frac{3x_1+x_2}{4}\right) A + \left(\frac{x_1+3x_2}{4}\right) a$ or, matricially:

$$S\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{3x_1+x_2}{4} \\ \frac{x_1+3x_2}{4} \end{pmatrix}.$$

By Corollary 4.1, and arbitrary (nonzero) linear functional $\phi \in \mathcal{C}^*$ given by $\phi = \phi(A)A^* + \phi(a)a^* \in \mathcal{C}^*$ is a character of $\mathcal{C}$ if and only if $\phi(A) + \phi(a) = 2 = \dim_{\mathbb{R}} \mathcal{C}$. Moreover, it holds that:

$$S_\phi \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{\phi(A)}{2} + \frac{\phi(a)}{4} & \frac{\phi(a)}{4} \\ \frac{\phi(A)}{4} & \frac{\phi(A)}{4} + \frac{\phi(a)}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Remark 5.2. Let $\mathcal{C}$ be a LV-coalgebra with natural basis $\mathcal{B}$ and structure matrix $\mathbf{M}(\mathcal{C},\mathcal{B})$. Given a character ϕ of $\mathcal{C}$ we will denote by $\mathbf{M}_\phi(\mathcal{C},\mathcal{B})$ the matrix associated with the evolution operator S_ϕ in basis $\mathcal{B}$ (or simply $\mathbf{M}_\phi$ if $\mathcal{B}$ is clear).

Theorem 5.2. *Let $\mathcal{C}$ be a LV-coalgebra with natural basis $\mathcal{B}=\{e_1,\ldots,e_n\}$ and structure matrix $\mathbf{M}(\mathcal{C},\mathcal{B})$. If $\phi=\sum_{i=1}^n \phi(e_i)e_i^*$ is a character of $\mathcal{C}$, then:*

$$\mathbf{M}_\phi(\mathcal{C},\mathcal{B}) = \begin{pmatrix} \sum_{i=1}^n \beta_{i1}^1\phi(e_i) & \cdots & \beta_{1k}^k\phi(e_k) & \cdots & \beta_{1n}^n\phi(e_n) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ & \cdots & \sum_{i=1}^n \beta_{ik}^k\phi(e_i) & \cdots & \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \beta_{n1}^1\phi(e_1) & \cdots & \beta_{nk}^k\phi(e_k) & \cdots & \sum_{i=1}^n \beta_{in}^n\phi(e_i) \end{pmatrix}.$$

In particular, if $\phi=\sum_{i=1}^n e_i^*$, then $\mathbf{M}_\phi(\mathcal{C},\mathcal{B})=\mathbf{M}(\mathcal{C},\mathcal{B})$.

Proof. Assume $\phi=\sum_{i=1}^n \phi(e_i)e_i^*$ is a character of $\mathcal{C}$. Then, by Proposition 5.1, the in-evolution operator S_ϕ is given by:

$$S_\phi(e_k) = \left(\sum_{i=1}^n \beta_{ik}^k\phi(e_i)\right)e_k + \sum_{j=1,j\neq k}^n \beta_{jk}^k\phi(e_k)e_j, \quad \text{for all } k=1,\ldots,n,$$

and, extending by linearity, for any $x=\sum_{k=1}^n x_k e_k\in\mathcal{C}$. Equivalently, reordering the above expression, and taking into account LVC-2, for all $k=1,\ldots,n$, we have:

$$S_\phi(e_k) = \beta_{1k}^k\phi(e_k)e_1 + \cdots + \left(\sum_{i=1}^n \beta_{ik}^k\phi(e_i)\right)e_k + \cdots + \beta_{nk}^k\phi(e_k)e_n.$$

Thus, the matrix associated with S_ϕ in basis $\mathcal{B}$ is exactly $\mathbf{M}_\phi(\mathcal{C},\mathcal{B})$. In particular, $S_\phi(\mathbf{x})=\mathbf{M}_\phi(\mathcal{C},\mathcal{B})\mathbf{x}$, for every $\mathbf{x}^T=(x_1,\ldots,x_n)\in\mathbb{K}^n$.

Finally, it is straightforward that if $\phi=\sum_{i=1}^n e_i^*$, then $\mathbf{M}_\phi(\mathcal{C},\mathcal{B})$ is exactly the matrix $\mathbf{M}(\mathcal{C},\mathcal{B})$ given in Theorem 2.1. □

Corollary 5.1. *Let $\mathcal{C}$ be a real LV-coalgebra with natural basis $\mathcal{B}=\{e_1,\ldots,e_n\}$. If $\mathcal{C}$ satisfies LCV-4, then the matrix $\mathbf{M}_\phi(\mathcal{C},\mathcal{B})$ associated with the in-evolution operator S_ϕ defined by the character $\phi=\sum_{i=1}^n e_i^*$ is a (nonnegative) stochastic matrix.*

Proof. The statement follows from Theorem 5.2 and Theorem 2.1, taking into account that the matrix of $S=S_\phi$ is exactly $\mathbf{M}_\phi(\mathcal{C},\mathcal{B})=\mathbf{M}(\mathcal{C},\mathcal{B})$. An alternative proof can also be given. Indeed, let $\phi=\sum_{i=1}^n e_i^*\in\mathcal{C}^*$. Since ϕ is a character of $\mathcal{C}$ and $\phi(e_k)=1$ for all $k=1,\ldots,n$, it follows from Lemma 4.1 that $\phi(\Gamma(e_k))=1$ for all $k = 1, \ldots, n$. Then,

$$\begin{aligned} S(e_k) &= ((\phi \otimes id)\Delta)\,(e_k) = (\phi \otimes id)\,(\Delta(e_k)) = \\ &= (\phi \otimes id)\,(\Gamma(e_k) \bullet e_k) = \frac{1}{2}\,(\phi(\Gamma(e_k))e_k + \phi(e_k)\Gamma(e_k)) = \\ &= \frac{1}{2}\,(e_k + \Gamma(e_k))\,, \end{aligned}$$

which implies that $S = \frac{1}{2}(id + \Gamma)$. Thus, since both id and Γ are stochastic maps, so is the in-evolution operator S. □

Example 5.2. Let $\mathcal{C}$ be a LV-coalgebra with natural basis $\mathcal{B} = \{e_1, \ldots, e_n\}$. Let us assume that e_1 is group-like. Then $\Delta(e_1) = e_1 \otimes e_1$, and therefore $\beta^1_{11} = 1$ and $\beta^1_{i1} = 0$ for all $i = 2, \ldots, n$. Thus:

$$\mathbf{M}(\mathcal{C}, \mathcal{B}) = \begin{pmatrix} 1 & \beta^2_{12} & \cdots & \beta^n_{1n} \\ 0 & \sum_{i=1}^n \beta^2_{i2} & \cdots & \beta^n_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \beta^2_{n2} & \cdots & \sum_{i=1}^n \beta^n_{in} \end{pmatrix}.$$

By Lemma 3.2, $\phi = e_1^*$ is a character of $\mathcal{C}$ and it defines the evolution operator S_ϕ whose associated matrix is given by:

$$\mathbf{M}_\phi(\mathcal{C}, \mathcal{B}) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \beta^2_{12} & 0 & \cdots & 0 \\ 0 & 0 & \beta^3_{13} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \beta^n_{1n} \end{pmatrix}.$$

6. Equilibrium under in-evolution operators

This section focuses on the study of invariance under in-evolution operators. Although all in-evolution operators (i.e., those defined by characters of LV-coalgebras) are considered, we will soon restrict our attention to those defined by the character $\phi = \sum_{i=1}^n e_i^*$.

Proposition 6.1. *Let $\mathcal{C}$ be a LV-coalgebra. All subcoalgebras of $\mathcal{C}$ remain invariant under in-evolution operators.*

Proof. Let $\mathcal{D}$ be a subcoalgebra of a LV-coalgebra $\mathcal{C}$. Then, for any in-evolution operator S_ϕ of $\mathcal{C}$, it holds that $S_\phi(\mathcal{D}) = (\phi \otimes id)\Delta(\mathcal{D}) \subseteq (\phi \otimes id)(\mathcal{D} \otimes \mathcal{D}) \subseteq \phi(\mathcal{D})\mathcal{D} \subseteq \mathbb{K}\mathcal{D} \subseteq \mathcal{D}$. □

Definition 6.1. Let S_ϕ be an in-evolution operator of a LV-coalgebra $\mathcal{C}$. An element $x \in \mathcal{C}$ is an *equilibrium state* for S_ϕ if $S_\phi(x) = x$.

We note that the existence of equilibrium states for arbitrary cocommutative coalgebras with genetic realization (i.e. real coalgebras satisfying LVC-3 and LVC-4) is ensured by Brouwer's fixed point Theorem [22, Theorem 7.1]. However, this result does not describe the equilibrium states.

Example 6.1. Let $\mathcal{C}$ be a LV-coalgebra with natural basis $\mathcal{B} = \{e_1, \ldots, e_n\}$.

(i) By LVC-3, the map $\phi = \sum_{i=1}^{n} e_i^*$ is always a character of $\mathcal{C}$, and any basic group-like element $e_k \in \mathcal{B}$ not contained in $Ker\phi$ results, by Lemma 3.2(i) and Theorem 5.1(ii), in an equilibrium state for $S = (\phi \otimes id)\Delta$.

(ii) Assume $e_k \in \mathcal{B}$ is group-like. By Lemma 3.2(ii), $\phi = e_k^*$ is a character of $\mathcal{C}$ and, therefore (see Example 5.2), $S_\phi(e_i) = \beta_{ki}^i e_i$ for all $i = 1, \ldots, n$. Thus, if $\mathbb{K} = \mathbb{R}$ and LVC-4 holds (so that $0 \leq \beta_{ik}^i \leq \frac{1}{2}$ if $i \neq k$), the equilibrium states of the in-evolution operator S_ϕ are the elements $\mathbb{R}e_k = \{\alpha e_k \mid \alpha \in \mathbb{R}\}$. In particular $\mathcal{C} = \mathbb{R}e_k \dotplus Ker\ \phi$, where $Ker\,\phi = span_{\mathbb{R}}(e_i \mid i \neq k)$ and $\dotplus$ denotes the direct sum as vector spaces.

(iii) Let $\mathcal{C}$ be the LV-coalgebra defined in Example 5.1, and ϕ any of its characters. Then $x = x_1e_1 + x_2e_2 \in \mathcal{C}$ is an equilibrium state for S_ϕ if and only if $x_1 = x_2$.

Proposition 6.2. *Let $\mathcal{C}$ be a LV-coalgebra with natural basis $\mathcal{B} = \{e_1, \ldots, e_n\}$. If ϕ is a character of $\mathcal{C}$, then $S_\phi(x) = x$ if and only if $\mathbf{M}_\phi(\mathcal{C}, \mathcal{B})\mathbf{x} = \mathbf{x}$, that is, if and only if $\mathbf{x}$ is a right eigenvector of $\mathbf{M}_\phi(\mathcal{C}, \mathcal{B})$ (with eigenvalue 1), where $\mathbf{x} = (x_1, \ldots, x_n)$ and $x = \sum_{i=1}^{n} x_i e_i \in \mathcal{C}$.*

Let $\Delta^{(n-1)} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0,\ \sum_{i=1}^{n} x_i = 1\}$ be the $(n-1)$-dimensional simplex. It is clear that arbitrary in-evolution operators do not preserve $\Delta^{(n-1)}$. Consider, for instance, a real n-dimensional LV-coalgebra $\mathcal{C}$ as in Example 5.2. Assuming $\mathcal{C}$ has genetic realization (that is, it satisfies LVC-4), so that all structure constants are nonnegative, and taking S_ϕ the in-evolution operator considered in Example 5.2, it follows that $S_\phi(\Delta^{(n-1)}) \subseteq \Delta^{(n-1)}$ requires that $x_1 + \beta_{12}^2 x_2 + \cdots + \beta_{1n}^n x_n = 1$ for all $(x_1, \ldots, x_n) \in \Delta^{(n-1)}$.

Theorem 6.1. *Let $\mathcal{C}$ be a real LV-coalgebra with genetic realization. For any character ϕ of $\mathcal{C}$, the in-evolution operator S_ϕ preserves the $(n-1)$-dimensional simplex $\Delta^{(n-1)}$ if and only if the following conditions hold for all $\mathbf{x} = (x_1, \ldots, x_n) \in \Delta^{(n-1)}$:*

(i) For all $k = 1, \ldots, n$,

$$\left(\sum_{j=1}^{n} \beta_{jk}^k \phi(e_j) \right) x_k + \phi(e_k) \sum_{\substack{i=1 \\ i \neq k}}^{n} \beta_{ki}^k x_i \geq 0.$$

(ii) The image remains in the simplex:

$$\sum_{k=1}^{n} \left[\left(\sum_{j=1}^{n} \beta_{kj}^k \phi(e_j) \right) x_k + \phi(e_k) \sum_{\substack{i=1 \\ i \neq k}}^{n} \beta_{ki}^k x_i \right] = 1.$$

Proposition 6.3. *Let $\mathcal{C}$ be a real coalgebra with genetic realization and natural basis $\mathcal{B} = \{e_1, \ldots, e_n\}$. Then the in-evolution operator S defined by the character $\phi = \sum_{i=1}^n e_i^*$ satisfies:*

(i) S preserves the $(n-1)$-dimensional simplex $\Delta^{(n-1)}$.
(ii) admits non-trivial equilibrium states, given by the right eigenvectors (of eigenvalue 1) of $\mathbf{M}(\mathcal{C}, \mathcal{B})$.

Proof. The statement (i) is a particular case of Theorem 6.1. Indeed, here the result follows from Theorem 5.2, since for $\phi = \sum_{i=1}^n e_i^*$, it holds that $\mathbf{M}_\phi(\mathcal{C}, \mathcal{B}) = \mathbf{M}(\mathcal{C}, \mathcal{B})$ is a nonnegative column stochastic matrix (see Theorem 2.1). The column stochasticity of $\mathbf{M}(\mathcal{C}, \mathcal{B})$ implies (ii). □

As noticed above, the linearity of the in-evolution operators S_ϕ clearly determines their equilibrium states as eigenvectors of the associated matrix $\mathbf{M}_\phi(\mathcal{C}, \mathcal{B})$. In the particular case of real LV-coalgebras with genetic realization, the existence of equilibrium states for the in-evolution operator defined by the character $\phi = \sum_{i=1}^n e_i^*$ arises as a direct consequence of Perron-Frobenius Theorem.

One of the most classical problems arising in the study of algebraic structures aimed by biological systems, or more precisely from their quadratic evolution operators (also called evolutionary operators), is the Bernstein Problem concerned with describing evolutionary operators such that all states in a generation are of equilibrium in the next generation. Next we will consider an analogous problem in the LV-coalgebra setting. To achieve this we will restrict ourselves to in-evolution operators defined by the character $\phi = \sum_{i=1}^n e_i^*$, so that invariance of the $(n-1)$-dimensional simplex $\Delta^{(n-1)}$ under the in-evolution operators is ensured. Then, as noted in the proof of Corollary 6.3, under this additional assumption, matrices associated with the in-evolution operators are nonnegative column stochastic matrices, also called Markov matrices.

Remark 6.1. Following [10, p. 9], we denote the trajectory defined by an initial state $x^{(0)} \in \Delta^{(n-1)}$, with respect to an in-evolution operator S, as the sequence $x^{(t+1)} = S(x^{(t)})$, for $t = 0, 1, \ldots$. Under the assumption that S is idempotent, it is straightforward that $x^{(2)} = S(x^{(1)}) = S\left(S(x^{(0)})\right) = S^2(x^{(0)}) = S(x^{(0)}) = x^{(1)}$.

Proposition 6.4. *Let $\mathcal{C}$ be a real coalgebra with genetic realization and natural basis $\mathcal{B} = \{e_1, \ldots, e_n\}$. Then, in-evolution operator S, defined by the character $\phi = \sum_{i=1}^n e_i^*$, satisfies $S^2 = S$ if and only if its associated matrix $\mathbf{M}_\phi(\mathcal{C}, \mathcal{B}) = \mathbf{M}(\mathcal{C}, \mathcal{B})$ is an idempotent Markov matrix.*

Immediacy of Proposition 6.4 is clear, but our interest on this result arises from the information that the idempotency of the matrix $\mathbf{M}(\mathcal{C}, \mathcal{B})$ provides on the structure of the LV-coalgebra $\mathcal{C}$. Indeed, as it was already noted in [2], one of the main problems one faces when dealing with LV-coalgebras is how to extract the information encoded in their structure matrices to describe the LV-coalgebra structure. Here we aim to describe particular families of LV-coalgebras by using the properties of their in-evolution operators.

Example 6.2. Let $\mathcal{C}$ be a 2-dimensional real LV-coalgebra $\mathcal{C}$ with natural basis $\mathcal{B} = \{e_1, e_2\}$. We recall (see [2, Example 3.5]) that $\mathcal{C}$ has structure matrix

$$\mathbf{M}(\mathcal{C}, \mathcal{B}) = \mathbf{M}(\alpha, \beta) = \begin{pmatrix} 1-\alpha & \beta \\ \alpha & 1-\beta \end{pmatrix},$$

with $0 \leq \alpha, \beta \leq \frac{1}{2}$. Consider now the in-evolution operator S defined by $\phi = e_1^* + e_2^*$, whose associated matrix is precisely $\mathbf{M}(\mathcal{C}, \mathcal{B}) = \mathbf{M}(\alpha, \beta)$. Since $\mathbf{M}(\alpha, \beta)$ is an idempotent Markov matrix if and only if $\alpha(1-\alpha-\beta) = 0 = \beta(1-\alpha-\beta)$, taking into account that here we are also assuming $0 \leq \alpha, \beta \leq \frac{1}{2}$ (see Lemma 2.2), it follows that the only such LV-coalgebras with idempotent in-evolution operator S are, by [2, Theorem 7.2], those with:

(i) $\alpha = \beta = 0$, and then $\mathcal{C}$ is a 2-dimensional real group coalgebra with structure matrix is $\mathbf{M}(\mathcal{C}, \mathcal{B}) = \mathbf{M}(0,0) = I_2$, and comultiplication given by:

$$\Delta(e_1) = e_1 \otimes e_1,$$
$$\Delta(e_2) = e_2 \otimes e_2.$$

(ii) $\alpha = \beta = \frac{1}{2}$, and then $\mathcal{C}$ is the simple real LV-coalgebra having structure matrix $\mathbf{M}(\mathcal{C}, \mathcal{B}) = \mathbf{M}(\frac{1}{2}, \frac{1}{2}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, and comultiplication:

$$\Delta(e_1) = \frac{1}{2}(e_1 \otimes e_2 + e_2 \otimes e_1),$$
$$\Delta(e_2) = \frac{1}{2}(e_1 \otimes e_2 + e_2 \otimes e_1).$$

We point out that the 2-dimensional real LV-coalgebra $\mathcal{C}$ with structure matrix $\mathbf{M}\left(\frac{1}{2}, \frac{1}{2}\right)$, appearing in Example 6.2 above, is the only 2-dimensional real LV-coalgebra satisfying LVC-4 having singular structure matrix. Matrices as $\mathbf{M}\left(\frac{1}{2}, \frac{1}{2}\right)$ belong to a well-known family of Markov matrices, called equal-input matrices.

Definition 6.2. Let $\mathbf{A} = (a_{ij})_{i,j=1}^n \in M_n(\mathbb{K})$. We will say that the matrix $\mathbf{A}$ is an *equal-input* (columna) matrix if it has equal columns $(a_{1j}, \dots, a_{nj}) = (a_1, \dots, a_n)$, for all $j = 1, \dots, n$. The *parameter sum* of $\mathbf{A}$ is then $a = a_1 + \cdots + a_n$.

A real equal-input matrix is a Markov matrix if it is nonnegative and it has parameter sum equal to 1. Besides, idempotent Markov matrices different from the identity matrix $\mathbf{I}_n$ are not invertible. We recall here the following result characterizing idempotent Markov matrices of a given rank. (For the sake of consistency, the following result appears formulated as in the original reference, that is, for row stochastic matrices. Its application, with the obvious changes, to column stochastic matrices is immediate.)

Theorem 6.2. *[8, Theorem 1.11, Theorem 1.12, Theorem 1.16] Let* $\mathbf{M} = (m_{ij})_{1\leq i,j\leq n} \in M_n(\mathbb{R})$ *be a row stochastic (i.e. row Markov) matrix of rank* $k > 0$. *If* $\mathbf{M}$ *is an idempotent matrix, then there exists a unique partition of* $\{1,\dots,n\}$, *called the basis of* $\mathbf{M}$, *into classes* $\{T_c, C_1, \dots, C_k\}$ *such that the following hold:*

(i) $T_c = \{j \mid$ *the* j*-th column of* $\mathbf{M}$ *is a zero column*$\}$;
(ii) If $j \in T_c$, *then* $\frac{m_{jp}}{m_{pp}} = \frac{m_{jq}}{m_{pq}}$ *for all* $p, q \in C_s$;
(iii) $\mathbf{M}_{|C_s\times C_s}$ *has identical positive rows of sum 1;*
(iv) $\mathbf{M}_{|C_s\times C_t} = 0$ *if* $s \neq t$.

Conversely, any row stochastic matrix with these properties is idempotent.

7. LV-coalgebras with idempotent in-evolution operator

In this section we will exploit Theorem 6.2 to studying real LV-coalgebras with genetic realization having idempotent structure matrices. As a result of Theorem 5.2, this amounts to study LV-coalgebras having an idempotent in-evolution operator defined by the character $\phi = \sum_{i=1}^{n} e_i^*$.

Theorem 7.1. *Let* $\mathcal{C}$ *be a* n*-dimensional real LV-coalgebra with genetic realization and natural basis* $\mathcal{B} = \{e_1,\dots,e_n\}$. *If the structure matrix* $\mathbf{M}(\mathcal{C},\mathcal{B})$ *of* $\mathcal{C}$ *has equal-columns, that is, if* $m_{ij} = m_{ik}$ *for all* $i, j, k = 1\dots, n$, *then:*

(i) $n = 1$ *and* $\mathbf{M}(\mathcal{C},\mathcal{B}) = \mathbf{I}_1$, *or*
(ii) $n = 2$ *and* $\mathbf{M}(\mathcal{C},\mathcal{B}) = \mathbf{M}\left(\frac{1}{2},\frac{1}{2}\right) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

In particular, either $\mathcal{C}$ *is the 1-dimensional real group coalgebra or a 2-dimensional real simple LV-coalgebra.*

Proof. Write $\mathbf{M}(\mathcal{C},\mathcal{B}) = (m_{ij})_{1\leq i,j\leq n}$. Then, by Lemma 2.2 (see [2, Lemma 3.4(iv)]) it holds that $m_{kk} \in [\frac{1}{2}, 1]$, and $m_{ik} \in [0, \frac{1}{2}]$ for all $i, k = 1,\dots,n$, with $i \neq k$. Since, by Theorem 2.1, $\mathbf{M}(\mathcal{C},\mathcal{B})$ is a non-negative matrix with all column sums equal to 1, if, additionally, $\mathbf{M}(\mathcal{C},\mathcal{B})$ has equal columns, then necessarily either $n = 1$ and then $\mathbf{M}(\mathcal{C},\mathcal{B}) = \mathbf{I}_1$, or $n = 2$ and $\mathbf{M}(\mathcal{C},\mathcal{B}) = \mathbf{M}(\frac{1}{2},\frac{1}{2})$. Finally, the last statement is straightforward for $n = 1$, and for $n = 2$, it follows from [2, Theorem 7.2]. □

Theorem 7.2. *Let* $\mathcal{C}$ *be a* n*-dimensional real LV-coalgebra with genetic realization and natural basis* $\mathcal{B} = \{e_1,\dots,e_n\}$. *If the structure matrix* $\mathbf{M}(\mathcal{C},\mathcal{B})$ *of* $\mathcal{C}$ *is an idempotent (column) Markov matrix, then one and only one of the following possibilities hold:*

(i) If $\mathbf{M}(\mathcal{C},\mathcal{B})$ *is regular, then* $\mathbf{M}(\mathcal{C},\mathcal{B}) = \mathbf{I}_n$, *and* $\mathcal{C}$ *is the* n*-dimensional real group coalgebra.*

(ii) If $\mathbf{M}(\mathcal{C},\mathcal{B})$ is singular, then $n = \dim_{\mathbb{R}} \mathcal{C} = m + 2p$, for some $m \geq 0$ and $p > 0$, where $k = m + p < n$ is the rank of $\mathbf{M}(\mathcal{C},\mathcal{B})$, and, up to relabelling, $\mathcal{B} = \{e_1, \ldots, e_m, e_{m+1}, e_{m+2}, \ldots, e_{m+2p-1}, e_{m+2p}\}$, where for each $s = 1, \ldots, p$, $\mathcal{D}_s = span_{\mathbb{R}}(e_{m+2s-1}, e_{m+2s})$ is a simple 2-dimensional LV-subcoalgebra of $\mathcal{C}$ with genetic realization and structure matrix $\mathbf{M}(\mathcal{D}_s, \mathcal{B}_s) = \mathbf{M}_s = \mathbf{M}\left(\frac{1}{2}, \frac{1}{2}\right)$. In particular,

$$\mathbf{M}(\mathcal{C},\mathcal{B}) = \begin{pmatrix} \mathbf{I}_m & & & & & \\ & \frac{1}{2} & \frac{1}{2} & & & \\ & \frac{1}{2} & \frac{1}{2} & & & \\ & & & \ddots & & \\ & & & & \frac{1}{2} & \frac{1}{2} \\ & & & & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

where $\mathbf{I}_m$ is the $m \times m$ identity matrix, and $\mathbf{M}(\mathcal{C},\mathcal{B})$ contains p submatrices of the form $\mathbf{M}\left(\frac{1}{2}, \frac{1}{2}\right)$.

Proof. (i) To obtain this statement it suffices to recall that the only nonsingular $n \times n$ Markov idempotent matrix is $\mathbf{I}_n$, since any non-zero idempotent matrix different from $\mathbf{I}_n$ has minimal polynomial $p(x) = x(x-1)$. Then, by Lemma 2.2(ii), all elements in the natural basis $\mathcal{B}$ of $\mathcal{C}$ are group-like elements, which amounts to $\mathcal{C}$ being the n-dimensional real group coalgebra (see, for instance [1, Theorem 6.3] or [2, Lemma 6.1]).

(ii) Suppose now that $\mathbf{M}(\mathcal{C},\mathcal{B})$ is singular, so that it has rank $k < n$, and let $\{T_r, C_1, \ldots, C_k\}$ be the partition of $\{1, \ldots, n\}$ arising from Theorem 6.2 applied to $\mathbf{M}(\mathcal{C},\mathcal{B})$.

We first note that $T_r = \emptyset$. Indeed, by Lemma 2.2(iii), all diagonal entries of $\mathbf{M}(\mathcal{C},\mathcal{B})$ satisfy $m_{kk} \in [\frac{1}{2}, 1]$ for all $k = 1, \ldots, n$ and, therefore, are nonzero. Hence $\mathbf{M}(\mathcal{C},\mathcal{B})$ has neither zero rows nor zero columns. As a result $\{T_c, C_1, \ldots, C_k\} = \{C_1, \ldots, C_k\}$ and $\mathbf{M}(\mathcal{C},\mathcal{B})$ has a diagonal block structure:

$$\mathbf{M}(\mathcal{C},\mathcal{B}) = \begin{pmatrix} \mathbf{M}_1 & & & & \\ & \mathbf{M}_2 & & & \\ & & \ddots & & \\ & & & \mathbf{M}_{s-1} & \\ & & & & \mathbf{M}_s \end{pmatrix}$$

where, for each $s = 1, \ldots, k$, $\mathbf{M}_s$ is a $n_s \times n_s$ matrix with identical positive columns of sum 1. Therefore $\mathcal{D}_s = span_{\mathbb{R}}(e_i \mid i \in C_s)$ is a real LV-subcoalgebra of $\mathcal{C}$ satisfying LVC-4, having natural basis $\mathcal{B}_s = \{e_i \mid i \in S\}$ and structure matrix $\mathbf{M}_s$. Thus, by Theorem 7.1, either $n_s = 1$ and $\mathbf{M}_s = (1)$ or $n_s = 2$ being then $\mathbf{M}_s = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

Write now $\dim_{\mathbb{R}} \mathcal{C} = n = m + 2p$, where $m = |\{s \mid n_s = 1\}|$ and $p =$

$|\{s \mid n_s = 2\}|$. Then $\mathbf{M}(\mathcal{C},\mathcal{B})$ is the $n \times n$ matrix of rank $k = m + p < n$:

$$\mathbf{M}(\mathcal{C},\mathcal{B}) = \left(\begin{array}{c|ccc} \mathbf{I}_m & & & \\ \hline & \mathbf{M}_1 & & \\ & & \ddots & \\ & & & \mathbf{M}_p \end{array}\right),$$

where $\mathbf{M}_s = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, for $s = 1, \ldots, p$. □

Remark 7.1. Let $(\mathcal{C}, \Delta)$ be a LV-coalgebra with natural basis $\mathcal{B} = \{e_1, \ldots, e_n\}$. Following [2], we denote by $\mathcal{G}(\mathcal{C},\mathcal{B}) = \{e_k \in \mathcal{B} \mid e_k \text{ is group-like}\}$ the set of basic group-like elements of $\mathcal{C}$. By [2, Lemma 6.1], $\mathbb{K}\mathcal{G}(\mathcal{C},\mathcal{B}) = span_{\mathbb{K}}(e_k \mid e_k \in \mathcal{G}(\mathcal{C},\mathcal{B}))$ is a LV-subcoalgebra of $\mathcal{C}$ of dimension $\mid \mathcal{G}(\mathcal{C},\mathcal{B}) \mid$. If, moreover, $\mathbb{K} = \mathbb{R}$, then $\mathbb{R}\mathcal{G}(\mathcal{C},\mathcal{B})$ is a real LV-coalgebra with genetic realization.

Gathering together partial results that have been appearing along this section, we can finally describe the structure of finite dimensional real LV-coalgebras with genetic realization having idempotent structure matrices.

Corollary 7.1. *Let $\mathcal{C}$ be a n-dimensional real LV-coalgebra with genetic realization and natural basis $\mathcal{B} = \{e_1, \ldots, e_n\}$. Then, the structure matrix $\mathbf{M}(\mathcal{C},\mathcal{B})$ of $\mathcal{C}$ is an idempotent (column) Markov matrix if and only if:*

(i) there exist nonnegative integers $m \geq 0$ and $p \geq 0$, such that $\mathbf{M}(\mathcal{C},\mathcal{B})$ has rank $k = m + p > 0$ and, it is as described in Theorem 7.2,

(ii) $\mathcal{C}$ decomposes as the direct sum $\mathcal{C} = \mathcal{C}_0 \oplus \mathcal{D}_1 \oplus \cdots \oplus \mathcal{D}_p$, where:

(ii.a) $\mathcal{C}_0 = \mathbb{R}\mathcal{G}(\mathcal{C},\mathcal{B})$ is a m-dimensional group subcoalgebra of $\mathcal{C}$ spanned by the group-like elements in $\mathcal{B}$,

(ii.a) $\mathcal{D}_s$ is a simple 2-dimensional LV-subcoalgebra with genetic realization and structure matrix $\mathbf{M}(\mathcal{D}_s, \mathcal{B}_s) = \mathbf{M}\left(\frac{1}{2}, \frac{1}{2}\right)$, for $s = 1, \ldots, p$.

Moreover, up to relabelling, $\mathcal{B} = \{e_1, \ldots, e_m, e_{m+1}, e_{m+2}, \ldots, e_{m+2p-1}, e_{m+2p}\}$, $\mathcal{C}_0 = span_{\mathbb{R}}(e_1, \ldots, e_m)$ and, $\mathcal{D}_s = span_{\mathbb{R}}(e_{m+2s-1}, e_{m+2s})$, for all $s = 1, \ldots, p$.

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ORCID

Manuel Arenas - https://orcid.org/0000-0001-9149-0276
Alicia Labra - https://orcid.org/0000-0002-1853-4081
Irene Paniello - https://orcid.org/0000-0002-3240-5067

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