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Accurate computations with Riordan arrays associated with Schröder matrices

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Abstract
Large and little Schröder matrices as well as with their inverses are triangular matrices arising in Combinatorics. Through their total positivity and bidiagonal decomposition it is shown that their singular values, inverses and some associated linear systems can be solved with high relative accuracy. Numerical experiments confirm the theoretical results.

Keywords Schröder matrices · Riordan arrays · Total positivity · High relative accuracy

1 Introduction

Finding classes of matrices for which many algebraic computations can be performed with high relative accuracy (HRA) has been a very active research field in the last years. As an important source of such matrices, we can point out subclasses of nonsingular totally positive matrices. In fact, let us recall that, given a nonsingular totally positive matrix, if we know its bidiagonal decomposition with high relative accuracy,

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then we can apply algorithms in [18] to calculate with high relative accuracy all its eigenvalues, singular values or its inverse. Totally positive matrices are matrices with nonnegative minors (cf. [1, 15, 26]). They arise in many fields, including Combinatorics.

Riordan matrices arise in Combinatorics in the framework of Riordan arrays, which in turn play an important role when dealing with recurrence relations, walk problems and combinatorial identities, among others fields (cf. [28]). For example, they naturally arise in the inversion of combinatorial sums of the form $a_n = \sum_{k=0}^{n} d_{n,k} b_k$, a recurring problem in enumerative combinatorics. As shown in [4], such sums can be interpreted as multiplications by Riordan matrices, and their left-inversion of these combinatorial sums reduces to the inversion of the associated Riordan array. We observe that Riordan matrices are also related to recursive matrices in the umbral calculus (cf. [22]). Large and little Schröder numbers arise in the problem of counting certain types of lattice paths, and both lead to corresponding Riordan arrays and triangular matrices. Inverses of large and little Schröder matrices have also an interpretation in Combinatorics (cf. Sect. 3). In this paper, we show that many linear algebra computations can be performed with high relative accuracy when dealing with large and little Schröder matrices as well as with their inverses. Schröder–type Riordan arrays and their associated triangular matrices also arise in the study of lattice paths such as Łukasiewicz paths and related polynomial families, where Riordan matrices encode concrete combinatorial transformations [30]. From a numerical perspective, bidiagonal factorizations with total positivity provide a natural and reliable framework for handling combinatorial matrices [17, 19, 20].

We now present the organization of this paper. Section 2 presents basic concepts and results on total positivity, bidiagonal decompositions and high relative accuracy. Section 3 introduces Riordan arrays and it presents some related matrices that will be considered in the paper. Section 4 focuses on recursive matrices, providing in Sect. 4.1 the bidiagonal decomposition of a family of recursive matrices that is totally positive. Section 4 shows the total positivity and gives the bidiagonal decomposition to HRA of large and little Schröder matrices as well as with their inverses. This will guarantee the high relative accurate computation of all singular values and inverses of these matrices, as well as the accurate computation of some associated linear systems. Observe that, since these matrices are triangular, the eigenvalues are the diagonal entries. Section 5 includes some numerical examples confirming the theoretical results.

2 Total positivity, Neville elimination and bidiagonal decompositions

A matrix is totally positive (TP) if all its minors are nonnegative. TP matrices are also called totally nonnegative matrices. This class of matrices has applications in many fields: see the surveys [1, 10], the classical book [15] and the recent books [11, 26].

A remarkable property of TP matrices is that their product is also a TP matrix (cf. Theorem 3.1 of [1]). This is a consequence of the Cauchy-Binet formula for determinants (cf. [1] or [26]).

Let us introduce a matrix notation for submatrices. Given positive integer numbers k, n, with $1 \leq k \leq n$, the set $Q_{k,n}$ denotes the set of all increasing sequences of k positive integers less than or equal to n. Let A be a $n \times n$ matrix. For $k \leq n$, $m \leq n$, and for any $\alpha \in Q_{k,n}$ and $\beta \in Q_{m,n}$, $A[\alpha|\beta]$ denotes the $k \times m$ submatrix of A containing the rows numbered by α and the columns numbered by β. Finally, if $\alpha = \beta$, the corresponding principal submatrix will be denoted by $A[\alpha] := A[\alpha|\alpha]$.

Neville elimination (NE) is an alternative procedure to Gaussian elimination that produces zeros in a column of a matrix by adding to each row an appropriate multiple of the previous one. This elimination procedure is very useful when dealing with some classes of matrices such as TP matrices or sign-regular matrices. For nonsingular TP matrices, it is always possible to perform NE without row exchanges (for more details on NE, see [12, 13]). Given a nonsingular matrix $A = (a_{ij})_{0\leq i,j\leq n}$, the Neville elimination procedure without row exchanges consists of n steps and leads to the following sequence of matrices:

$$A =: A^{(0)} \to A^{(1)} \to A^{(2)} \to \cdots \to A^{(n)} = U, \tag{1}$$

where U is an upper triangular matrix.

The entries of $A^{(k+1)} = (a_{ij}^{(k+1)})_{0\leq i,j\leq n}$ can be obtained from $A^{(k)}$ using the formula

$$a_{ij}^{(k+1)} = \begin{cases} a_{ij}^{(k)} - \frac{a_{ik}^{(k)}}{a_{i-1,k}^{(k)}} a_{i-1,j}^{(k)}, & \text{if } k+1 \leq i \leq n, k \leq j \leq n \text{ and } a_{i-1,k}^{(k)} \neq 0, \\ a_{ij}^{(k)}, & \text{otherwise}, \end{cases}$$

for $k = 0, \ldots, n-1$.

The (i, j) pivot of the NE of A is given by $p_{ij} = a_{ij}^{(j)}$, for $0 \leq j \leq i \leq n$. If $i = j$ we say that p_{ii} is a diagonal pivot. The (i, j) multiplier of the NE of A with $1 \leq j < i \leq n$, is defined as

$$m_{ij} = \begin{cases} \frac{a_{ij}^{(j)}}{a_{i-1,j}^{(j)}} = \frac{p_{ij}}{p_{i-1,j}}, & \text{if } a_{i-1,j}^{(j)} \neq 0, \\ 0, & \text{if } a_{i-1,j}^{(j)} = 0. \end{cases}$$

The multipliers satisfy that $m_{ij} = 0 \Rightarrow m_{hj} = 0$, for all $h > i$. The (i,j) multiplier of the Neville elimination of A^T is denoted by $\widetilde{m}_{ij}$.

The following theorem is a consequence of Theorem 4.2 of [13] and characterizes nonsingular TP matrices by their bidiagonal decompositions.

Theorem 1 *A nonsingular matrix $A = (a_{ij})_{0\le i,j\le n}$ is TP if and only if it admits the following decomposition:*

$$A = F_{n-1}F_{n-2}\cdots F_0 D G_0 \cdots G_{n-2}G_{n-1}, \tag{2}$$

where D is the diagonal matrix diag$(p_{00}, p_{11}, \ldots, p_{nn})$ with positive diagonal entries and F_i, $G_i \in \mathbb{R}^{(n+1)\times(n+1)}$ are the nonnegative bidiagonal matrices given by

$$F_i = \begin{pmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ & \ddots & \ddots & & & & \\ & & 0 & 1 & & & \\ & & & m_{i+1,0} & 1 & & \\ & & & & \ddots & \ddots & \\ & & & & & m_{n,n-i-1} & 1 \end{pmatrix}, \quad G_i = \begin{pmatrix} 1 & 0 & & & & & \\ & 1 & \ddots & & & & \\ & & \ddots & 0 & & & \\ & & & 1 & \widetilde{m}_{i+1,0} & & \\ & & & & 1 & \ddots & \\ & & & & & \ddots & \widetilde{m}_{n,n-i-1} \\ & & & & & & 1 \end{pmatrix}, \tag{3}$$

for all $i \in \{0, 1, \ldots, n-1\}$. If, in addition, the entries m_{ij} and $\widetilde{m}_{ij}$ satisfy

$$\begin{array}{l} m_{ij} = 0 \Rightarrow m_{hj} = 0 \quad \forall h > i, \\ \widetilde{m}_{ij} = 0 \Rightarrow \widetilde{m}_{hj} = 0 \quad \forall h > i, \end{array} \tag{4}$$

then the decomposition is unique.

In the bidiagonal decomposition given by (2)–(4), the entries m_{ij} and p_{ii} are the multipliers and diagonal pivots, respectively, corresponding to the NE of *A*, and the entries $\widetilde{m}_{ij}$ are the multipliers of the NE of A^T (see pp. 116 and 120 of [13]).

Bidiagonal decompositions can be used to factorize more classes of matrices (cf. [2]). If we consider the factorization given by (2) and (3) without any further requirement than the nonsingularity of *D*, the matrix notation $\mathcal{BD}(A)$ can be used to represent a bidiagonal decomposition (2) of a nonsingular matrix,

$$(\mathcal{BD}(A))_{ij} = \begin{cases} m_{ij}, & \text{if } i > j, \\ \widetilde{m}_{ji}, & \text{if } i < j, \\ p_{ii}, & \text{if } i = j, \end{cases} \tag{5}$$

for $0 \le i, j \le n$.

Some results of [21] help us to show that a bidiagonal decomposition of the inverse of a TP matrix *A* is directly derived from a bidiagonal decomposition of *A*. For the particular case of lower triangular matrices with all of its diagonal entries equal to one, we have the following result, whose proof is closely related to Theorem 5 of [5].

Proposition 2 *Let $A = (a_{ij})_{0\le i,j\le n}$ be a lower triangular and TP matrix with all of its diagonal entries equal to one, so it has a bidiagonal decomposition given by*

$$(\mathcal{BD}(A))_{ij} = \begin{cases} 1, & if\, i = j, \\ m_{ij}, & if\, i > j, \\ 0, & otherwise. \end{cases}$$

Then, $JA^{-1}J$, with $J = diag(1, -1, 1, -1, ..., (-1)^{n-1})$, is a lower triangular and TP matrix with all of its diagonal entries equal to one too, having a bidiagonal decomposition given by

$$(\mathcal{BD}(JA^{-1}J))_{ij} = \begin{cases} 1, & if i = j, \\ m_{i,i-j-1}, & if i > j, \\ 0, & otherwise. \end{cases}$$

Let us finish this section by recalling that a sufficient condition to assure that an algorithm can be performed with high relative accuracy (HRA) is the non-inaccurate cancellation condition, denoted as NIC condition, which is satisfied if the algorithm does not include subtractions (except of initial data), that is, if it only includes products, divisions, sums (of numbers of the same sign) and subtractions of initial data (cf. [7–9]). In particular, a subtraction-free algorithm satisfies the NIC condition and it can be performed with HRA.

The nontrivial entries of the matrices in $\mathcal{BD}(A)$ (see (5)) have been considered natural parameters associated with A in many recent references [16, 17, 20]. In many cases, we know them with high relative accuracy. If we assume it for a nonsingular totally positive matrix A, then algorithms with high relative accuracy can be applied (see [17, 21]) to compute the singular values of A, the eigenvalues, the inverse or solving certain linear systems $Ax = b$ (those where b has chessboard pattern of signs).

For many subclasses of nonsingular TP matrices it has been possible to obtain the bidiagonal decomposition $\mathcal{BD}(A)$ of their matrices A with HRA, so that the mentioned linear algebra computations can also be solved with HRA (cf. [5, 6, 19, 20]).

3 Riordan arrays

Riordan arrays were introduced in 1991 in [27] and play an important role in combinatorics, where they are related to recurrence relations, walk problems and combinatorial identities, among others fields (cf. [28]).

The concept of generating function is important for the definition of a Riordan array. Before defining it, note that although the following definitions talk about commutative rings, in general it will suffice for us that the commutative ring be $\mathbb{R}$.

Definition 1 Let R be a commutative ring, and let $(a_n)_{n\geq 0}$ be a sequence with coefficients in R. The generating function (GF) for $(a_n)_{n\geq 0}$ is the formal power series

$$a(t) = \sum_{n=0}^{\infty} a_n t^n = a_0 + a_1 t + a_2 t^2 + \cdots ,$$

where t is an auxiliary variable.

Let us introduce a formal definition of a Riordan array and explain its two characterizations (by two formal power series and by two sequences).

Definition 2 Let R be a commutative ring, and let $f(t)=\sum_{n=0}^{\infty} f_n t^n$ and $g(t)=\sum_{n=1}^{\infty} g_n t^n$ be formal power series with coefficients in *R*. The Riordan array associated to the pair (*f*, *g*) is the infinite lower triangular matrix $\mathcal{R}(f,g)=(r_{nk})_{n,k\geq 0}$ defined by

$$r_{nk}=[t^n]f(t)g(t)^k, \quad n\geq k,$$

where $[t^n]$ denotes the operator for extracting the *n*-th coefficient of a generating function.
That is, $\mathcal{R}(f,g)=(r_{nk})_{n,k\geq 0}$ is a Riordan array if the GF for the sequence $(r_{nk})_{n\geq 0}$ of numbers in the *k*-th column of $\mathcal{R}(f,g)$ is $f(t)g(t)^k$, for all $k\geq 0$. From now on, we will use the notation $\mathcal{R}:=\mathcal{R}(f,g)=(f(t),g(t))$. Note that *g* has zero constant term.

Remark 1 By Section 1 of [27], the inverse of a Riordan array $\mathcal{R}(f,g)$ is given by

$$(f(t),g(t))^{-1}=(1/f(\bar{g}(t)),\bar{g}(t)), \tag{6}$$

where $\bar{g}$ is the composition inverse of *g* (so it satisfies $\bar{g}(g(t))=g(\bar{g}(t))=t$). Furthermore, the Riordan array of the product $(f(t),g(t))\cdot(h(t),l(t))$ is given by

$$(f(t),g(t))\cdot(h(t),l(t))=(f(t)h(g(t)),l(g(t))). \tag{7}$$

The characterization of a Riordan array $\mathcal{R}=(r_{nk})_{n,k\geq 0}$ by two sequences $(a_n)_{n\geq 0}$ and $(z_n)_{n\geq 0}$ and an element of the commutative ring $r\in R$ is described by the following recurrence (see [22])

$$r_{0,0}=r, \qquad r_{n+1,0}=\sum_{j\geq 0} z_j r_{n,j}, \quad r_{n+1,k+1}=\sum_{j\geq 0} a_j r_{n,k+j}, \tag{8}$$

for $n,k\geq 0$. Then, $(a_n)_{n\geq 0}$ and $(z_n)_{n\geq 0}$ are called the A- and Z-sequences of $\mathcal{R}$ respectively.

It is very common (and in all the examples we study it will be like this) that $r=1$, which induces that Riordan arrays can be characterized only by two sequences $A=(a_n)_{n\geq 0}$ and $Z=(z_n)_{n\geq 0}$.

The previous characterization of Riordan arrays motivates the following result.

Proposition 3 *Let $\mathcal{R}=(r_{ij})_{i,j\geq 0}$ be a Riordan array, $\mathcal{R}_n=(r_{ij})_{0\leq i,j\leq n}$ denotes the corresponding truncated matrix. Given r and the two sequences $A=(a_n)_{n\geq 0}$ and $Z=(z_n)_{n\geq 0}$ in (8) that characterize the Riordan array, we have that*

$$\mathcal{R}_n = \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{R}_{n-1} \end{pmatrix} L_n, \quad where \quad L_n = \begin{pmatrix} r & & & & & \\ z_0 & a_0 & & & & \\ z_1 & a_1 & a_0 & & & \\ z_2 & a_2 & a_1 & a_0 & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ z_n & a_n & a_{n-1} & a_{n-2} & \cdots & a_0 \end{pmatrix}, \tag{9}$$

that is, the entries of $L_n = (\ell_{ij})_{0 \le i,j \le n}$ *are* $\ell_{ij} = \begin{cases} r, & if i = j = 0, \\ z_{i-1}, & if i > j = 0, \\ a_{i-j}, & if i \ge j \ne 0, \\ 0, & otherwise. \end{cases}$

Proof See Theorems 2.1 and 2.2. of [22]. □

Several important combinatorial sequences, fundamental in areas such as algebra, graph theory, and theoretical computer science [29], are introduced here, along with a combinatorial application, as they are closely related to the Riordan arrays studied in this paper.

The n-th Catalan number, denoted by C_n, is given by

$$C_n := \frac{1}{n+1} \binom{2n}{n}, \quad n \ge 0,$$

and counts, among other things, the number of monotonic lattice paths along the edges of a grid from (0, 0) to (*n*, *n*) that do not rise above the main diagonal. A monotonic lattice path is one on a grid that only moves rightward or upward.

The Narayana numbers, *N*(*n*, *k*), are defined as

$$N(n,k) := \frac{1}{n} \binom{n}{k} \binom{n}{k-1}, \quad n \ge k \ge 1,$$

and count the number of lattice paths from (0, 0) to (2*n*, 0), where permitted steps are (1, 1) and $(1, -1)$, not straying below the x-axis, with *k* peaks.

The n-th large Schröder number, denoted S_n, is expressed as

$$S_n := \sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} \binom{n+k}{k} = \sum_{k=0}^{n} \binom{n+k}{2k} C_k, \quad n \ge 0,$$

and counts certain types of lattice paths, including those that start from the point (0, 0) to (*n*, *n*) with steps (1, 0), (0, 1) and (1, 1) and that never rise above the line $y = x$.

Finally, the n-th little Schröder number, s_n, is defined by

$$s_n := \sum_{k=1}^{n} N(n,k)2^{k-1}, \quad n \geq 1,$$

and arises in various counting problems, such as the counting of number of ways to insert parentheses into an expression of $n+1$ terms with two or more items within a parentheses. It is well known that $S_n = 2s_n$ for every $n \geq 1$ (cf. [29]).

In the following examples, we explore the relationships between the Schröder numbers and their corresponding Riordan arrays, which provide a way to study their generating functions and combinatorial properties.

Example 1 The Riordan array

$$\mathcal{P} := \left(\frac{1-x-\sqrt{1-6x+x^2}}{2x}, \frac{1-x-\sqrt{1-6x+x^2}}{2}\right) = \begin{pmatrix} 1 & & & & & \\ 2 & 1 & & & & \\ 6 & 4 & 1 & & & \\ 22 & 16 & 6 & 1 & & \\ 90 & 68 & 30 & 8 & 1 & \\ \vdots & & & & & \ddots \end{pmatrix} \quad (10)$$

is characterized by the sequences $A = (1, 2, 2, \ldots)$ and $Z = (2, 2, 2, \ldots)$.

Since its first column is formed by the large Schröder numbers, in [3] it is called large Schröder triangle or matrix. In addition, row sums are little Schröder numbers.

The entries p_{ij} of the Riordan array $\mathcal{P} = (p_{ij})_{i,j\geq 0}$ are given by

$$p_{ij} = \frac{j+1}{i+1} \sum_{k=0}^{i-j} \binom{i+1}{j+k+1} \binom{i+k}{k}$$

(this is the sequence A080247 in [24]). In combinatorial terms, it counts the lattice paths running from (0, 0) to $(n, n-k)$ using steps $E = (1, 0)$, $N = (0, 1)$ and $D = (1, 1)$ which stay weakly below the main diagonal. The resulting matrix forms a lower triangular Riordan array that encodes these constrained paths, known as Schröder paths. This Schröder matrix was widely studied (see [25, 31, 32]).

Suppose $\mathcal{P}_n = (p_{ij})_{0\leq i,j\leq n}$ is the truncated large Schröder triangle. Following Proposition 3, $\mathcal{P}_n$ satisfies

$$\mathcal{P}_n = \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{P}_{n-1} \end{pmatrix} L_n^{(\mathcal{P})}, \quad (11)$$

where

$$L_n^{(\mathcal{P})} := \begin{pmatrix} 1 & & & & \\ 2 & 1 & & & \\ 2 & 2 & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ 2 & 2 & \ldots & 2 & 1 \end{pmatrix}, \quad (12)$$

that is, the entries of $L_n^{(\mathcal{P})} = (\ell_{ij})_{0\le i,j\le n}$ are $\ell_{ij} = \begin{cases} 1, & \text{i} \\ 2, & \text{if } i > j, \\ 0, & \text{otherwise.} \end{cases}$

Example 2 The Riordan array

$$\mathcal{Q} := \left(\frac{1+x-\sqrt{1-6x+x^2}}{4x}, \frac{1-x-\sqrt{1-6x+x^2}}{2}\right) = \begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 3 & 3 & 1 & & & \\ 11 & 11 & 5 & 1 & & \\ 45 & 45 & 23 & 7 & 1 & \\ \vdots & & & & & \ddots \end{pmatrix} \tag{13}$$

is characterized by the sequences $A = Z = (1, 2, 2, 2, \ldots)$.

Since its first column is formed by the little Schröder numbers, in [3] it is called little Schröder triangle or matrix.

The entries q_{ij} of the Riordan array $\mathcal{Q} = (q_{ij})_{i,j\ge 0}$ are given by

$$q_{ij} = \frac{1}{i+1}\sum_{k=j}^{i}(-1)^{k-j}(k+1)\sum_{\ell=0}^{i-j}\binom{i+1}{k+\ell+1}\binom{i+\ell}{\ell}$$

(see [31]). This is the sequence A186826 in [24]. In combinatorial terms, it counts the lattice paths running from (0, 0) to $(n, n-k)$ with steps $E = (1, 0)$, $N = (0, 1)$ and $D = (1, 1)$, which stay weakly below the main diagonal and avoid $D = (1, 1)$ steps on the line $y = x$. The resulting matrix forms a lower triangular Riordan array that encodes these constrained paths, known as small Schröder paths. This matrix has been extensively studied in the literature (see [14, 31, 32]).

Suppose $\mathcal{Q}_n = (q_{ij})_{0\le i,j\le n}$ is the truncated little Schröder triangle. Following Proposition 3, $\mathcal{Q}_n$ satisfies

$$\mathcal{Q}_n = \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{Q}_{n-1} \end{pmatrix} L_n^{(\mathcal{Q})}, \tag{14}$$

where

$$L_n^{(\mathcal{Q})} := \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 2 & 2 & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ 2 & 2 & \ldots & 2 & 1 \end{pmatrix}, \tag{15}$$

that is, the entries of $L_n^{(\mathcal{Q})} = (\ell_{ij})_{0\le i,j\le n}$ are $n_{ij} = \begin{cases} 1, & \text{if } i = j \text{ or } (i,j) = (1,0), \\ 2, & \text{if } i > j \text{ and } (i,j) \ne (1,0), \\ 0, & \text{otherwise.} \end{cases}$

The following examples are motivated by Proposition 2.

Example 3 Using the property of the inverse matrix of a Riordan array (see Remark 1), we have that the inverse of the large Schröder matrix is $\mathcal{P}^{-1}=\left(\frac{1-x-\sqrt{1-6x+x^2}}{2x},\frac{1-x-\sqrt{1-6x+x^2}}{2}\right)^{-1}=\left(\frac{1-x}{1+x},\frac{x(1-x)}{1+x}\right)$. Taking into account that $J=diag(1,-1,1,-1,...)$ is the Riordan array $J=(1,-t)$, by using the rule of the multiplication of Riordan arrays (see Remark 1), we have that

$$\hat{\mathcal{P}}:=J\mathcal{P}^{-1}J=\left(\frac{1+x}{1-x},\frac{x(1+x)}{1-x}\right)=\begin{pmatrix}1&&&&&\\2&1&&&&\\2&4&1&&&\\2&8&6&1&&\\2&12&18&8&1&\\\vdots&&&&&\ddots\end{pmatrix}. \tag{16}$$

This matrix is also characterized by the sequences $A=(1,1+S_0,-S_1,S_2,-S_3,\dots)$ and $Z=(1+S_0,-S_1,S_2,-S_3,\dots)$, where S_n are the large Schröder numbers. The entries $\hat{p}_{ij}$ of the Riordan array $\hat{\mathcal{P}}=(\hat{p}_{ij})_{i,j\geq 0}$ are given by

$$\hat{p}_{ij}=\sum_{k=0}^{i-j}\binom{j+k}{k}\binom{j+1}{i-j-k}$$

(this is the sequence A113413 in [24]).

Example 4 Using again the property of the inverse matrix of a Riordan array and the rule of the multiplication (see Remark 1), we have that the inverse of the little Schröder matrix is $\mathcal{Q}^{-1}=\left(\frac{1+x-\sqrt{1-6x+x^2}}{4x},\frac{1-x-\sqrt{1-6x+x^2}}{2}\right)^{-1}=\left(1-x,\frac{x(1-x)}{1+x}\right)$, and also

$$\hat{\mathcal{Q}}:=J\mathcal{Q}^{-1}J=\left(1+x,\frac{x(1+x)}{1-x}\right)=\begin{pmatrix}1&&&&&\\1&1&&&&\\0&3&1&&&\\0&4&5&1&&\\0&4&12&7&1&\\\vdots&&&&&\ddots\end{pmatrix}. \tag{17}$$

This matrix is also characterized by the sequences $A=(1,1+S_0,-S_1,S_2,-S_3,\dots)$ and $Z=(1,-s_0,s_1,-s_2,s_3,\dots)$, where S_n and s_n are the large and little Schröder numbers, respectively. The entries $\hat{q}_{ij}$ of the Riordan array $\hat{\mathcal{Q}}=(\hat{q}_{ij})_{i,j\geq 0}$ are given by

$$\hat{q}_{ij}=\sum_{k=0}^{j+1}\binom{j+1}{k}\binom{i-k-1}{i-j-k}$$

(this is the non-signed version of the sequence A186827 in [24]).

4 Recursive matrices

Let $A = (a_{ij})_{i,j\geq 0}$ be an infinite lower triangular matrix. We say that A is recursive if it satisfies a recurrence relation of the following form

$$a_{0,0} = \ell_{00} a_{i+1,0} = \sum_{k\geq 0} \ell_{k+1,0} a_{ik} \quad a_{i+1,j+1} = \sum_{k\geq 0} \ell_{k+j+1,j+1} a_{i,j+k}.$$

This linear recursion can also be expressed as a product of matrices, given the truncated matrix $A_n = (a_{ij})_{0\leq i,j\leq n}$, such that

$$A_n = \begin{pmatrix} 1 & 0 \\ 0 & A_{n-1} \end{pmatrix} L_n,$$

where $L_n = (\ell_{ij})_{0\leq i,j\leq n}$ is also a lower triangular matrix. From here, we have

$$\begin{aligned} A_n &= \begin{pmatrix} 1 & 0 \\ 0 & A_{n-1} \end{pmatrix} L_n = \begin{pmatrix} I_2 & 0 \\ 0 & A_{n-2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & L_{n-1} \end{pmatrix} L_n = \cdots \\ &= \begin{pmatrix} I_{n-1} & 0 \\ 0 & A_1 \end{pmatrix} \begin{pmatrix} I_{n-2} & 0 \\ 0 & L_2 \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 \\ 0 & L_{n-1} \end{pmatrix} L_n \\ &= \begin{pmatrix} I_{n-1} & 0 \\ 0 & L_1 \end{pmatrix} \begin{pmatrix} I_{n-2} & 0 \\ 0 & L_2 \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 \\ 0 & L_{n-1} \end{pmatrix} L_n. \end{aligned} \tag{18}$$

This expression is interesting because it has a very useful application in the following result. As a consequence, a further analysis of $\mathcal{BD}(L_n)$ is also interesting.

Lemma 4 Let A_n and L_n be the lower triangular matrices connected by the relation given in (18). If the bidiagonal decomposition of L_n is known, then the bidiagonal decomposition of A_n can be computed with a subtraction-free algorithm, and so with HRA.

Proof In Section 5.2 of [17], Koev developed a subtraction-free algorithm where given the bidiagonal decomposition A and B of two matrices F and G, respectively, $A = \mathcal{BD}(F)$ and $B = \mathcal{BD}(G)$, then the bidiagonal decomposition of the product of those matrices, $\mathcal{BD}(FG)$, is provided with HRA. Then by (18), it is evident that knowing the bidiagonal decomposition of the matrices L_k, for $k = 1, 2 \ldots, n$, is sufficient to obtain the bidiagonal decomposition of A_n. □

4.1 A family of recursive matrices

Among all matrices L_n as in (18), we focus on those with the following form

$$H_n := \begin{pmatrix} r_{-1} & & & & & & \\ s_0 & r_0 & & & & & \\ t_1 & s_1 & r_1 & & & & \\ t_1 & t_2 & s_2 & r_2 & & & \\ t_1 & t_2 & t_3 & s_3 & r_3 & & \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \\ t_1 & t_2 & t_3 & \cdots & t_{n-1} & s_{n-1} & r_{n-1} \end{pmatrix}, \quad r_i, s_i, t_i > 0, \tag{19}$$

i.e., $H_n = (h_{ij})_{0\le i,j\le n}$ such that $h_{ij} = \begin{cases} r_{i-1}, & \text{if } i = j, \\ s_{i-1}, & \text{if } i = j+1, \\ t_{j+1}, & \text{if } i \ge j+2, \\ 0, & \text{otherwise.} \end{cases}$

The following result obtains the bidiagonal decomposition of H_n and gives conditions for its total positivity.

Theorem 5 *Suppose H_n is the $(n+1)\times(n+1)$ matrix defined in (19). Then the bidiagonal decomposition of H_n is given by*

$$(\mathcal{BD}(H_n))_{ij} = \begin{cases} r_{i-1}, & if i = j, \\ \beta_{i-1}/r_{i-2}, & if i = j+1, \\ (t_{i-1} - s_{i-2})/\beta_{i-2}, & if i = j+2 \ne 2, \\ t_1/s_0, & if (i,j) = (2,0), \\ 1, & if i \ge 3 and j = 0, \\ 0, & otherwise, \end{cases} \tag{20}$$

for $0 \le i, j \le n$, where

$$\beta_i = \begin{cases} s_0, & if i = 0, \\ s_i - r_{i-1} - \frac{r_{i-1}(t_i - s_{i-1})}{\beta_{i-1}}, & if i = 1,2,3\ldots \end{cases} \tag{21}$$

Besides, H_n is nonsingular TP if $t_i \ge s_{i-1}$, for $i \ge 1$, and if one of the following conditions is satisfied

$$\begin{array}{rll} (i) & s_1 - \frac{t_1 r_0}{s_0} \ge r_1, & s_k \ge r_{k-1} + r_k + t_k - s_{k-1}, for k \ge 2. \\ (ii) & s_1 - \frac{t_1 r_0}{s_0} \ge t_2 - s_1, & s_k \ge t_{k+1}/2 + r_{k-1}, for k \ge 2. \\ (iii) & s_1 - \frac{t_1 r_0}{s_0} \ge 1, & s_k \ge r_{k-1}(t_k - s_{k-1} + 1) + 1, for k \ge 2. \\ (iv) & s_1 - \frac{t_1 r_0}{s_0} \ge r_1(t_2 - s_1), & s_k \ge r_{k-1} + r_k(t_{k+1} - s_k) + 1, for k \ge 2. \end{array}$$

Proof First of all, H_n is a lower triangular matrix with all of its diagonal entries positives, so it is also nonsingular. Then its bidiagonal decomposition (2) is formed only by the lower triangular matrices F_i, and the diagonal matrix D.

Let $H^{(1)} := (h_{ij}^{(1)})_{0\le i,j\le n}$ be the matrix obtained in the first step of the NE of H_n (see (1)). In fact, we obtain $h_{ij}^{(1)} = 0$ for $i < j$, $h_{ii}^{(1)} = r_{i-1}$ for $0 \le i \le n$, and $h_{i0}^{(1)} = 0$ for $i > 0$ when performing the NE of H_n. Hence, the multipliers of the first step of the NE, for $1 \le i \le n$, are

$$m_{i0} = \frac{h_{i0}}{h_{i-1,0}} = \begin{cases} s_0/r_{-1}, & \text{if } i = 1, \\ t_1/s_0, & \text{if } i = 2, \\ 1, & \text{if } i \geq 3. \end{cases} \tag{22}$$

So we get, for $1 \leq j < i \leq n$,

$$h_{ij}^{(1)} = h_{ij} - m_{i0}\ h_{i-1,j} = \begin{cases} h_{21} - (t_1/s_0)h_{11} = s_1 - \dfrac{t_1 r_0}{s_0}, & \text{if } (i,j) = (2,1), \\ h_{ij} - h_{i-1,j} = s_{i-1} - r_{i-2}, & \text{if } i = j+1 \neq 2, \\ h_{ij} - h_{i-1,j} = t_{i-1} - s_{i-2}, & \text{if } i = j+2, \\ h_{ij} - h_{i-1,j} = 0, & \text{if } i > j+2. \end{cases}$$

Therefore, $H^{(1)}$ is a lower triangular matrix with non-zero entries only in the three principal lower diagonals. We can express the entries of $H^{(1)}$, for $0 \leq i, j \leq n$, as

$$h_{ij}^{(1)} = \begin{cases} \widetilde{r}_{i-1}, & \text{if } i = j, \\ \widetilde{s}_{i-1}, & \text{if } i = j+1, \\ \widetilde{t}_{i-1}, & \text{if } i = j+2, \\ 0, & \text{otherwise}, \end{cases}$$

where

$$\widetilde{r}_i = r_i, \quad \widetilde{s}_i = \begin{cases} 0, & \text{if } i = 0, \\ s_1 - \frac{t_1 r_0}{s_0}, & \text{if } i = 1, \\ s_i - r_{i-1}, & \text{if } i \geq 2, \end{cases} \quad \widetilde{t}_i = \begin{cases} 0, & \text{if } i = 1, \\ t_i - s_{i-1}, & \text{if } i \geq 2. \end{cases} \tag{23}$$

The matrix $H^{(1)}[2, \ldots, n+1]$ has the structure of the matrix L_h of Theorem 3.2 of [23]. Then, using this theorem and (22), the bidiagonal decomposition of $H^{(1)}$ is given by

$$\left(\mathcal{BD}(H^{(1)})\right)_{ij} = \begin{cases} \widetilde{r}_{i-1}, & \text{if } i = j, \\ \widetilde{\beta}_{i-1}/\widetilde{r}_{i-2}, & \text{if } i = j+1 \neq 1, \\ \widetilde{t}_{i-1}/\widetilde{\beta}_{i-2}, & \text{if } i = j+2 \neq 2, \\ 0, & \text{otherwise}, \end{cases} \tag{24}$$

for $0 \leq i, j \leq n$, where

$$\widetilde{\beta}_i = \begin{cases} \widetilde{s}_1, & \text{if } i = 1, \\ \widetilde{s}_i - \frac{\widetilde{r}_{i-1}\widetilde{t}_i}{\widetilde{\beta}_{i-1}}, & \text{if } i = 2, 3, 4, \ldots \end{cases} \tag{25}$$

In the bidiagonal decomposition of H_n, the first column is determined by (22), and the remaining columns are obtained from (24). Taken together, these two expressions specify the following bidiagonal decomposition

$$
(\mathcal{BD}(H_n))_{ij} = \begin{cases} r_{-1}, & \text{if } i=j=0, \\ s_0/r_{-1}, & \text{if } i=1, j=0, \\ t_1/s_0, & \text{if } i=2, j=0, \\ 1, & \text{if } i \geq 3, j=0, \\ \widetilde{r}_{i-1}, & \text{if } i=j \neq 0, \\ \widetilde{\beta}_{i-1}/\widetilde{r}_{i-2}, & \text{if } i=j+1 \neq 1, \\ \widetilde{t}_{i-1}/\widetilde{\beta}_{i-2}, & \text{if } i=j+2 \neq 2, \\ 0, & \text{otherwise,} \end{cases} \tag{26}
$$

for $0 \leq i, j \leq n$, where $\widetilde{\beta}_i$ in (25) can be expanded as

$$
\widetilde{\beta}_i = \begin{cases} \widetilde{s}_1 = s_1 - \frac{t_1 r_0}{s_0} = s_1 - r_0 - \frac{r_0(t_1 - s_0)}{s_0}, & \text{if } i=1, \\ \widetilde{s}_i - \frac{\widetilde{r}_{i-1}\widetilde{t}_i}{\widetilde{\beta}_{i-1}} = s_i - r_{i-1} - \frac{r_{i-1}(t_i - s_{i-1})}{\widetilde{\beta}_{i-1}}, & \text{if } i = 2,3,4,\dots, \end{cases}
$$

and thus the definition can be extended as

$$
\beta_i = \begin{cases} s_0, & \text{if } i=0, \\ \widetilde{\beta}_i = s_i - r_{i-1} - \frac{r_{i-1}(t_i - s_{i-1})}{\beta_{i-1}}, & \text{if } i=1,2,3,\dots, \end{cases}
$$

which is the expression (21).

Finally, it can be observed from (23) and (26) that when $i = j$, we have $(\mathcal{BD}(H_n))_{ii} = r_{i-1}$, for $0 \leq i \leq n$. Moreover, since $s_0 = \beta_0$, it follows that $(\mathcal{BD}(H_n))_{i,i-1} = \beta_{i-1}/r_{i-2}$, for $1 \leq i \leq n$. Combining these two relations with (26) yields (20), as required.

From Lemma 3.1 of [23], the total positivity of H_n is obtained by substituting the entries $(\widetilde{r}_i)_{i\geq 0}$, $(\widetilde{s}_i)_{i\geq 1}$ and $(\widetilde{t}_i)_{i\geq 2}$ of the matrix $H^{(1)}$ by the expressions of the lemma, and verifying that the entries that arise in the first step of NE of H_n are positive. These entries are

$$
s_1 - \frac{t_1 r_0}{s_0}, \qquad s_i - r_{i-1},\ 2 \leq i \leq n-1 \qquad t_i - s_{i-1},\ 1 \leq i \leq n-1. \tag{27}
$$

From the conditions of the lemma, one can directly deduce the first two conditions of (27). Therefore, the only additional requirement is $t_i - s_{i-1} > 0$, for $1 \leq i \leq n-1$. In fact, since these entries occupy the first sub-diagonal, $i = j + 2$, they may also be zero, making $H^{(1)}$ bidiagonal, so we consider the condition $t_i - s_{i-1} \geq 0$. □

From Lemma 4 and Theorem 5, the Algorithm 1 is proposed to obtain the bidiagonal decomposition of a matrix A_n whose recurrence is associated with H_n, defined in (19).

Algorithm 1 Computing a bidiagonal decomposition with recursive matrices

Require: $\{r_i\}_{i\geq -1}, \{s_i\}_{i\geq 0}, \{t_i\}_{i\geq 1}, n$
Ensure: $B = \mathcal{BD}(A_n)$
$B_n = zeros(n+1, n+1)$
$B_n(0,0) = r(-1)$
$\beta(0) = s(0)$
for $i = 1 : n$ **do**
  $B_n(i,0) = 1$
  $\beta(i) = s(i) - r(i-1) - r(i-1)\cdot(t(i) - s(i-1))/\beta(i-1)$
  $B_n(i,i) = r(i-1)$
  $B_n(i,i-1) = \beta(i-1)/r(i-2)$
end for
$B_n(2,0) = t(1)/s(0)$
for $i = 3 : n$ **do**
  $B_n(i,i-2) = (t(i-1) - s(i-2))/\beta(i-2)$
end for
for $i = 1 : n-1$ **do**
  $B_i := B_n(0:i, 0:i)$
end for
for $i = 1 : n-1$ **do**
  $B_i := \mathrm{blkdiag}(1, B_i)$
  $B_{i+1} := \mathrm{TNProduct}(B_i, B_{i+1})$
end for
$B = B_n$

The family of recursive matrices generated by (19) contains, as a particular case, several subfamilies, including matrices that form Riordan arrays. Hence, we define the Riordan array $\mathcal{R}$ by the parameter sequences

$$A = (a_0, a_1, a_2, a_2, \dots), \qquad Z = (z_0, z_1, z_1, \dots).$$

The *n*-th principal submatrix of $\mathcal{R}$ is $\mathcal{R}_n$, and, motivated by Proposition 3, it can be expressed as

$$\mathcal{R}_n = \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{R}_{n-1} \end{pmatrix} L_n^{(\mathcal{R})}, \quad \text{where} \quad L_n^{(\mathcal{R})} := \begin{pmatrix} r & & & & & \\ z_0 & a_0 & & & & \\ z_1 & a_1 & a_0 & & & \\ z_1 & a_2 & a_1 & a_0 & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ z_1 & a_2 & \cdots & a_2 & a_1 & a_0 \end{pmatrix}. \tag{28}$$

This Riordan array is generated by $L_n^{(\mathcal{R})}$, which is a special type of matrix H_n where

$$r_i = \begin{cases} r & \text{if } i = -1 \\ a_0 & \text{if } i = 0, 1, 2, \dots \end{cases} \qquad s_i = \begin{cases} z_0 & \text{if } i = 0 \\ a_1 & \text{if } i = 1, 2, 3, \dots \end{cases}$$
$$t_i = \begin{cases} z_1 & \text{if } i = 1 \\ a_2 & \text{if } i = 2, 3, 4, \dots \end{cases} \tag{29}$$

Therefore, the bidiagonal decomposition of $L_n^{(\mathcal{R})}$ (and also that of $\mathcal{R}_n$) is known.

The following corollary follows directly from Theorem 5, for the particular case of (29).

Corollary 6 *Suppose $L_n^{(\mathcal{R})}$ is the $(n+1) \times (n+1)$ matrix defined in (28). Then the bidiagonal decomposition of $L_n^{(\mathcal{R})}$ is given by*

$$\left(\mathcal{BD}(L_n^{(\mathcal{R})})\right)_{ij} = \begin{cases} r, & if i = j = 0, \\ z_0/r, & if i = j+1 = 1, \\ z_1/z_0, & if i = j+2 = 2, \\ 1, & if i \geq 3 and j = 0, \\ a_0, & if i = j \neq 0, \\ \beta_{i-1}/a_0, & if i = j+1 \neq 1, \\ (a_2 - a_1)/\beta_{i-2}, & if i = j+2 \neq 2, \\ 0, & otherwise, \end{cases} \tag{30}$$

for $0 \leq i, j \leq n$, where

$$\beta_i = \begin{cases} z_0, & if i = 0, \\ a_1 - a_0 - \frac{a_0(z_1 - z_0)}{\beta_{i-1}}, & if i = 1, \\ a_1 - a_0 - \frac{a_0(a_2 - a_1)}{\beta_{i-1}}, & if i = 2, 3, 4 \dots \end{cases} \tag{31}$$

Besides $L_n^{(\mathcal{R})}$ is nonsingular TP if $a_1 \leq a_2$, $z_1 \leq z_2$, and if one of the following conditions is satisfied

$$\begin{array}{rlll} (i) & a_1 - \frac{z_1 a_0}{z_0} \geq a_0, & a_1 \geq a_0 + a_2/2. \\ (ii) & a_1 - \frac{z_1 a_0}{z_0} \geq a_2 - a_1, & a_1 \geq a_0 + a_2/2. \\ (iii) & a_1 - \frac{z_1 a_0}{z_0} \geq 1, & a_1 \geq a_0(a_2 - a_1 + 1) + 1. \\ (iv) & a_1 - \frac{z_1 a_0}{z_0} \geq a_0(a_2 - a_1), & a_1 \geq a_0(a_2 - a_1 + 1) + 1. \end{array}$$

As can be observed, the computation of $\mathcal{BD}(H_n)$ and $\mathcal{BD}(L_n^{(\mathcal{R})})$ involves divisions by β_i. In the general case, the expression for β_i includes subtractions. However, for some important Riordan arrays the corresponding β_i can be known with HRA. Therefore, the bidiagonal decomposition of H_n can also be obtained with HRA. Moreover, by Lemma 4, the same holds for any matrix A_n defined as in (18).

We can now apply previous results to some particular important examples. Recovering the Riordan array examples presented in Sect. 3, we proceed to study their bidiagonal decompositions.

In the case of the large Schröder triangle $\mathcal{P}$, defined in (10) and characterized by the sequences $A = (1, 2, 2, \dots)$ and $Z = (2, 2, 2, \dots)$, we seek the bidiagonal decomposition of $L_n^{(\mathcal{P})}$, as defined in (12). The following result then follows directly from Corollary 6, for $r = a_0 = 1$ and $a_1 = a_2 = z_0 = z_1 = 2$.

Corollary 7 *Let $L_n^{(\mathcal{P})}$ be the $(n+1) \times (n+1)$ matrix defined in (12). Then its bidiagonal decomposition is given by*

$$\left(\mathcal{BD}(L_n^{(\mathcal{P})})\right)_{ij} = \begin{cases} 1, & if(i=j)or(i=j+1\neq 1)or(i\neq 1, j=0) \\ 2, & if(i,j)=(1,0) \\ 0, & otherwise \end{cases} \tag{32}$$

for $0 \leq i, j \leq n+1$. Hence, $L_n^{(\mathcal{P})}$ is nonsingular TP, and (32) is exactly known.

As a consequence, by Lemma 4, we also know the bidiagonal decomposition of the truncated large Schröder triangle $\mathcal{P}_n$ with HRA.

Corollary 8 *$\mathcal{BD}(\mathcal{P}_n)$ can be computed with a subtraction-free algorithm (and so, with HRA), where $\mathcal{P}_n$ is the matrix defined in Example 1. Therefore, we can compute with HRA all singular values, the inverse and the solution of linear systems $\mathcal{P}_n x = b$, where b has chessboard pattern of signs.*

From Corollaries 7 and 8, we propose an algorithm to obtain the bidiagonal decomposition of the large Schröder triangle. This algorithm is based on the **TNProduct** function defined by Koev in [17], where finding $\mathcal{BD}(\mathcal{P}_n)$ with HRA is guaranteed. Hence, the pseudocode providing $\mathcal{BD}(\mathcal{P}_n)$ with HRA can be seen in Algorithm 2.

Algorithm 2 Computing a bidiagonal decomposition of $\mathcal{P}_n$

Require: n
Ensure: $B = \mathcal{BD}(\mathcal{P}_n)$
$B_n = eye(n+1)$
for $i = 1 : n$ **do**
  $B_n(i, 0) = 1$
  $B_n(i, i-1) = 1$
end for
$B_n(1, 0) = 2$
for $i = 1 : n-1$ **do**
  $B_i := B_n(0 : i, 0 : i)$
end for
for $i = 1 : n-1$ **do**
  $B_i := \text{blkdiag}(1, B_i)$
  $B_{i+1} := \text{TNProduct}(B_i, B_{i+1})$
end for
$B = B_n$

In the case of the little Schröder triangle $\mathcal{Q}$, defined in (13) and characterized by the sequences $A = Z = (1, 2, 2, \ldots)$, we seek the bidiagonal decomposition of

$L_n^{(\mathcal{Q})}$, as defined in (15). The following result then follows directly from Corollary 6, for $r = a_0 = z_0 = 1$ and $a_1 = a_2 = z_1 = 2$.

Corollary 9 *Suppose $L_n^{(\mathcal{Q})}$ is the $(n+1) \times (n+1)$ matrix defined in (15). Then its bidiagonal decomposition is given by*

$$\left(\mathcal{BD}(L_n^{(\mathcal{Q})})\right)_{ij} = \begin{cases} 1, & if (i = j) or (i = j+1 \neq 2) or (i \neq 2, j = 0) \\ 2, & if (i,j) = (2,0) \\ 0, & otherwise \end{cases} \tag{33}$$

for $0 \leq i, j \leq n+1$. Hence, $L_n^{(\mathcal{Q})}$ is nonsingular TP, and (33) is exactly known.

As a consequence, by Lemma 4, we also know the bidiagonal decomposition of the truncated little Schröder triangle $\mathcal{Q}_n$ with HRA.

Corollary 10 *$\mathcal{BD}(\mathcal{Q}_n)$ can be computed with a subtraction-free algorithm (and so, with HRA), where $\mathcal{Q}_n$ is the matrix defined in Example 2. Therefore, we can compute with HRA all singular values, the inverse and the solution of linear systems $\mathcal{Q}_n x = b$, where b has chessboard pattern of signs.*

Analogously to the large Schröder case, from Corollaries 9 and 10 we develop an algorithm to obtain the bidiagonal decomposition of the little Schröder triangle. The procedure also relies on Koev's **TNProduct** function in [17], ensuring $\mathcal{BD}(\mathcal{Q}_n)$ with HRA, as summarized in Algorithm 3.

Algorithm 3 Computing a bidiagonal decomposition of $\mathcal{Q}_n$

Require: n
Ensure: $B = \mathcal{BD}(\mathcal{Q}_n)$
$B_n = eye(n+1)$
for $i = 1 : n$ **do**
  $B_n(i, 0) = 1$
  $B_n(i, i-1) = 1$
end for
$B_n(2, 0) = 2$
$B_n(2, 1) = 0$
for $i = 1 : n-1$ **do**
  $B_i := B_n(0 : i, 0 : i)$
end for
for $i = 1 : n-1$ **do**
  $B_i := \mathrm{blkdiag}(1, B_i)$
  $B_{i+1} := \mathrm{TNProduct}(B_i, B_{i+1})$
end for
$B = B_n$

On the other hand, taking as reference Proposition 2, we can also write a pseudocode to know the bidiagonal decomposition of matrices $\hat{\mathcal{P}}_n = J\mathcal{P}_n^{-1}J$ and $\hat{\mathcal{Q}}_n = J\mathcal{Q}_n^{-1}J$, defined in Examples 3 and 4, given the bidiagonal decomposition of $\mathcal{P}_n$ and $\mathcal{Q}_n$, respectively. It can be seen in Algorithm 4, and it is performed with HRA.

Algorithm 4 Computing $\mathcal{BD}(JA^{-1}J)$ from $\mathcal{BD}(A)$ of a lower triangular TP matrix A with all of its diagonal entries equal to one.

Require: $M = \mathcal{BD}(A)$
Ensure: $B = \mathcal{BD}(JA^{-1}J)$
$n = size(M, 1)$
for $i = 1 : n$ **do**
  for $j = 0 : i - 1$ **do**
    $B(i, j) = M(i, i - j - 1)$
  end for
end for

5 Numerical experiments

Known the bidiagonal factorization, $B = \mathcal{BD}(A)$, of a TP matrix A, in [17] algorithms to compute the eigenvalues, the singular values of A and the solution of linear systems of equations $Ax = b$ were introduced. Moreover, if $\mathcal{BD}(A)$ is known to HRA, those linear algebra problems are solved to HRA with these algorithms (for the case of the linear system of equations when b has a chessboard sign pattern). Under the same hypotheses, in [21] an algorithm to compute the inverse of A, A^{-1}, with HRA was presented. The four algorithms have been implemented in Matlab/Octave functions. These functions are contained in the TNTool software library, which can be downloaded in [18]. In particular, the names of the functions are **TNEigenValues(B)** for the eigenvalues, **TNSingularValues(B)** for the singular values, **TNSolve(B,b)** for the solution of the linear system of equations $Ax = b$ and **TNInverseExpand(B)** for the inverse A^{-1}.

In this section, we will provide numerical examples illustrating the high relative accuracy when solving the previous algebra problems for Large and Little Schröder triangles by using the bidiagonal decompositions of these matrices computed to HRA with the algorithms developed in this paper. Since both matrices are triangular its eigenvalues are the diagonal entries of them. So, in the numerical experiments we will only compute the singular values, the inverse and the solution of linear systems.

5.1 Large Schröder triangle

Algorithm 2 provides $\mathcal{BD}(\mathcal{P}_n)$ with HRA. We have implemented this algorithm in a Matlab function with the name **TNBDLargeSchroder**. To illustrate the accuracy of using this function together with the functions of TNTool, matrices $\mathcal{P}_{30}$ and $\mathcal{P}_{80}$ have been considered.

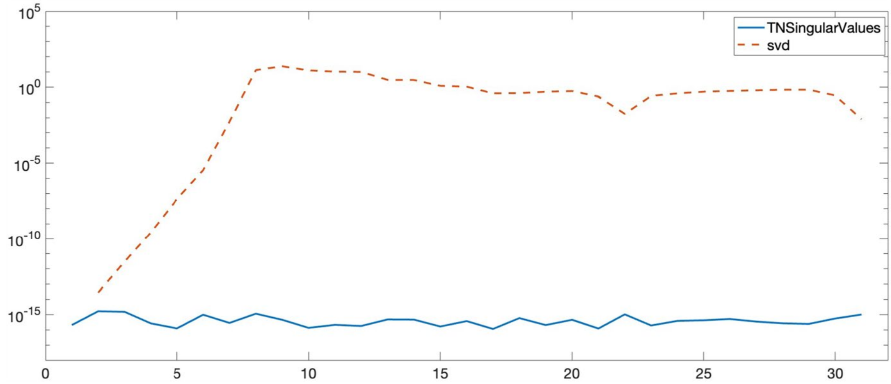

Fig. 1 Relative errors when computing the singular values σ_i, $i = 1, \ldots, 31$ of $\mathcal{P}_{30}$

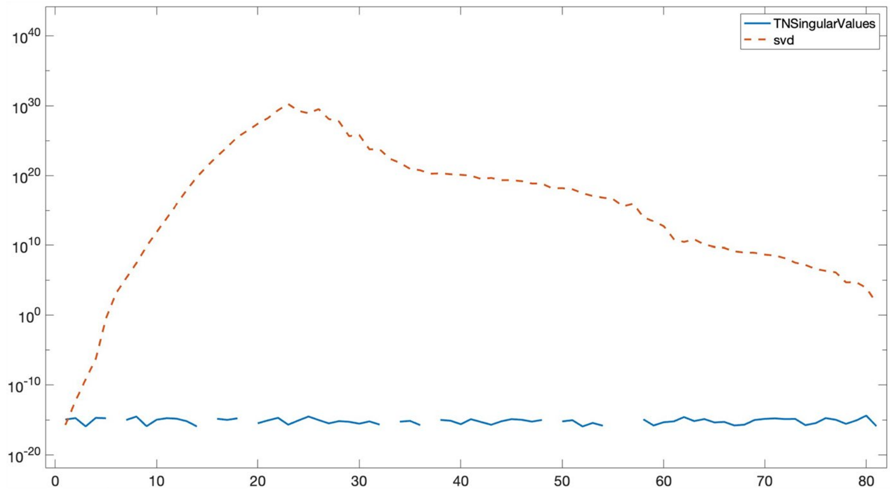

Fig. 2 Relative errors when computing the singular values σ_i, $i = 1, \ldots, 81$ of $\mathcal{P}_{80}$

First, the singular values $\sigma_1 > \cdots > \sigma_{31} > 0$ of $\mathcal{P}_{30}$ have been computed with Matlab by both methods **svd** and **TNSingularValues**. In order to calculate the corresponding relative errors, we have also obtained the singular values with Mathematica using extended 100-digit precision. Figure 1 shows these relative errors. The maximum relative error is $1,68346e-15$ for **TNSingularValues** and $2,34687e+01$ for **svd**.

The same process has been carried out for the singular values of $\mathcal{P}_{80}$. The corresponding relative errors are shown in Fig. 2. The maximum relative error is $4,19989e-15$ for **TNSingularValues** and $1,75529e+30$ for **svd**. As we can observe, our new approach is much more accurate than the standard Matlab method for the computation of the singular values. In addition, when the size of the matrix increases our methods keep robust in contrast to the usual method.

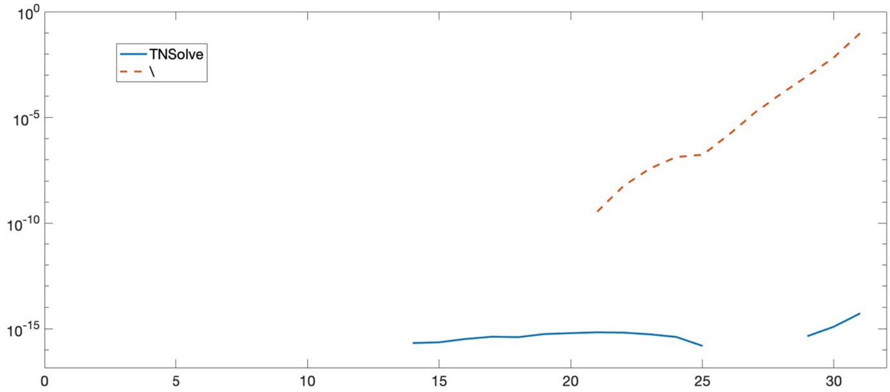

Fig. 3 Componentwise relative errors when solving $\mathcal{P}_{30}x = b$

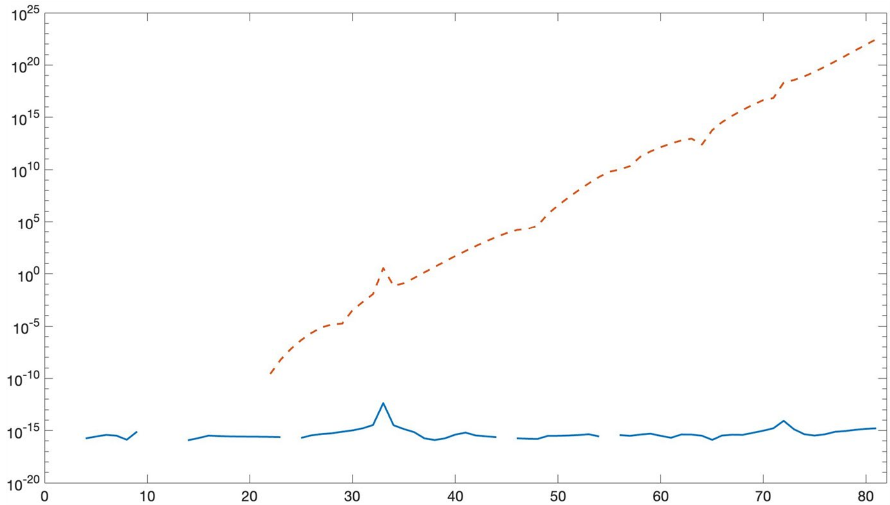

Fig. 4 Componentwise relative errors when solving $\mathcal{P}_{80}x = b$

Let us now consider the system of linear equations $\mathcal{P}_{30}x = b$, where b has a chessboard sign pattern. The absolute values of the components of b have been randomly chosen from the set of integers not greater than 1000. Then, an approximation to the solution of the system has been obtained with Matlab with $\backslash$ and **TNSolve**. In order to compare the accuracy of both methods we have computed the corresponding relative errors solving in exact arithmetic the linear system with Mathematica. These componentwise relative errors are shown in Fig. 3.

The same process has been performed for solving a system of linear equations $\mathcal{P}_{80}x = b$, where b also has a chessboard sign pattern. Figure 4 shows the corresponding componentwise relative errors. The results illustrate that our approach for solving linear systems is also much better and robust than the standard Matlab algorithm.

Table 1 Relative errors when computing $\mathcal{P}_{30}^{-1}$

	TNInverseExpand	**inv**
Average rel. error	1,22337E-16	9,50206E+04
Maximum rel. error	7,40677E-16	3,00983E+07

Table 2 Relative errors when computing $\mathcal{P}_{80}^{-1}$

	TNInverseExpand	**inv**
Average rel. error	3,03146E-16	5,46595E+46
Maximum rel. error	1,96362E-15	9,24624E+49

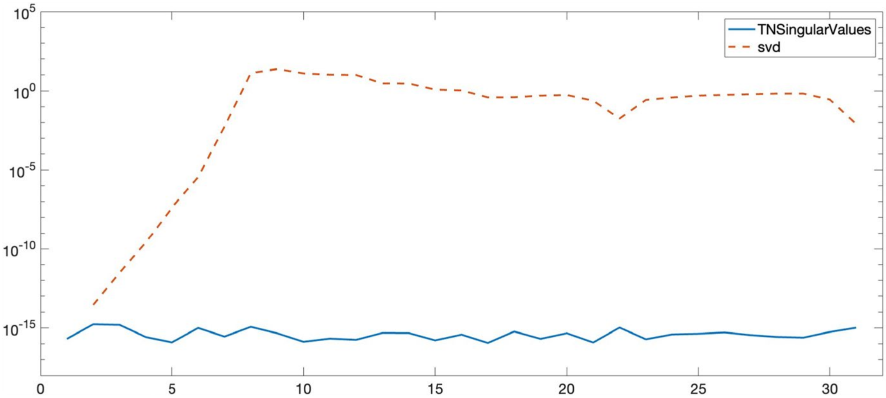

Fig. 5 Relative errors when computing the singular values σ_i, $i = 1, \ldots, 31$ of $\mathcal{Q}_{30}$

The inverses $\mathcal{P}_{30}^{-1}$ and $\mathcal{P}_{80}^{-1}$ have been computed with Matlab by using both methods **inv** and **TNInverseExpand**. Then, the componentwise relative errors have been calulated using the exact inverse obtained with Mathematica. In this case, the average and maximum relative errors are shown in Tables 1 and 2. Again, our approach outperforms that of the standard Matlab function also for the calculation of the inverses.

5.2 Little Schröder triangle

Finally, the same numerical tests performed in the previous section have been performed for the Little Schröder triangles $\mathcal{Q}_{30}$ and $\mathcal{Q}_{80}$.

Figures 5 and 6 show the relative errors for the calculation of the singular values of $\mathcal{Q}_{30}$ and $\mathcal{Q}_{80}$, respectively. For the singular values of $\mathcal{Q}_{30}$, the maximum relative error is $1,80103e-15$ for **TNSingularValues** and $2,37151e+01$ for **svd**. And for $\mathcal{Q}_{80}$, the maximum relative error is $5,93639e-15$ for **TNSingularValues** and $1,16345e+30$ for **svd**. Like in the case of the Large Schröder triangle the approach using the bidiagonal decomposition provides much more accurate and robust results than the given by the standard Matlab function.

The componentwise relative errors for the solution of linear systems $\mathcal{Q}_{30}x = b$ and $\mathcal{Q}_{80}x = b$ are shown in Figs. 7 and 8. The solutions provided by **TNSolve** are more accurate and robust than those provided by the command \.

Finally, Tables 3 and 4 show the average and maximum relative errors when computing the inverses $\mathcal{Q}_{30}^{-1}$ and $\mathcal{Q}_{80}^{-1}$ by both **inv** and **TNInverseExpand**. The same conclusions about accuracy that in the case of the inverses $\mathcal{P}_{30}^{-1}$ and $\mathcal{P}_{80}^{-1}$ can be derived.

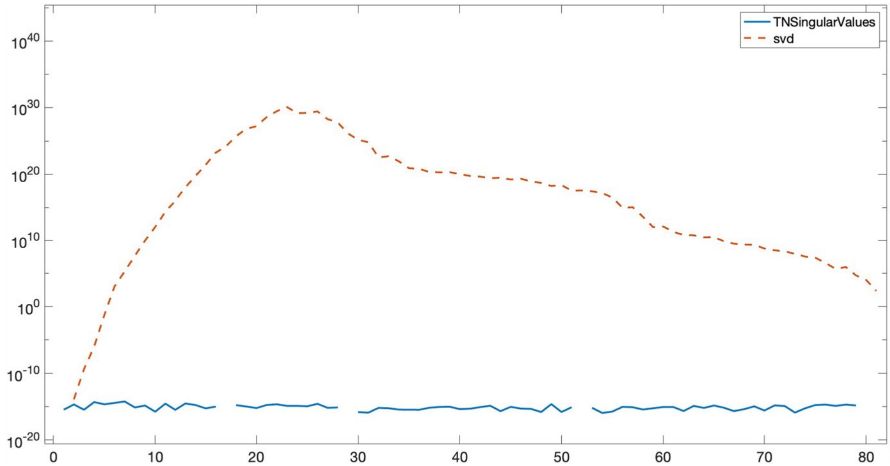

Fig. 6 Relative errors when computing the singular values σ_i, $i = 1, \ldots, 81$ of $\mathcal{Q}_{80}$

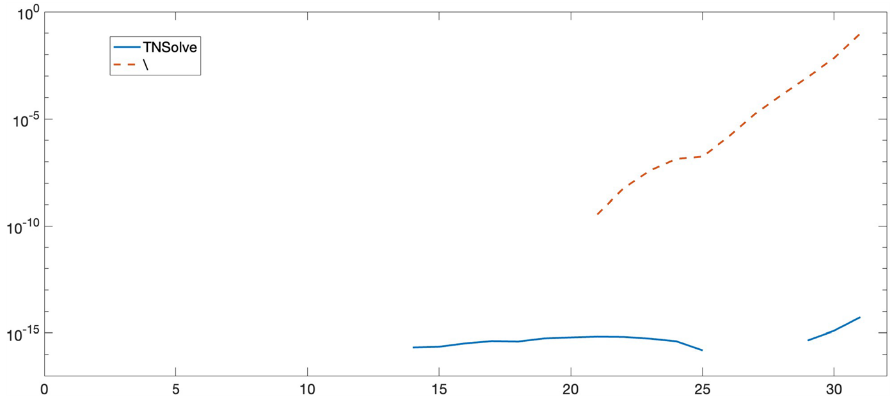

Fig. 7 Componentwise relative errors when solving $\mathcal{Q}_{30}x = b$

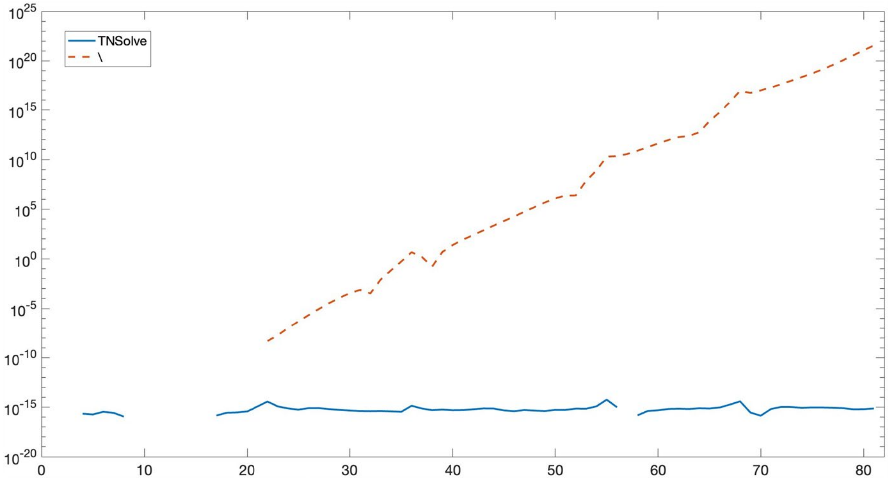

Fig. 8 Componentwise relative errors when solving $\mathcal{Q}_{80}x = b$

Table 3 Relative errors when computing $\mathcal{Q}_{30}^{-1}$

	TNInverseExpand	inv
Average rel. error	1,09837E-16	7,67890+03
Maximum rel. error	5,12256E-16	1,86574E+06

Table 4 Relative errors when computing $\mathcal{Q}_{80}^{-1}$

	TNInverseExpand	inv
Average rel. error	2.65887E-16	Inf
Maximum rel. error	1.64034E-15	Inf

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Declarations

Conflict of interest The authors have no conflict of interest to declare that are relevant to the content of this article.

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