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Accurate spectral computations and error analysis for r-geometric Min and Max matrices ☆

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ABSTRACT

In this work, we introduce a new two-parameter family of structured matrices, termed r-geometric Min and Max matrices, which generalize both the r-Min/r-Max and geometric Min/Max matrices. We derive explicit bidiagonal factorizations for these matrices using Neville elimination and establish necessary and sufficient conditions under which they are totally positive. Under these conditions, we develop algorithms capable of computing their eigenvalues and singular values to high relative accuracy, as well as closed-form expressions for their determinants. We also apply perturbation theory to analyze the sensitivity of these problems to input data, deriving structured condition numbers that quantify the impact of data perturbations. Numerical experiments confirm the theoretical results and demonstrate the reliability and efficiency of the proposed algorithms across both the general r-geometric case and key special instances.

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1. Introduction

Bidiagonal factorizations constitute a fundamental computational tool in numerical linear algebra, particularly in the context of totally positive matrices—those for which all minors are nonnegative. This property underpins the numerical stability and high accuracy of algorithms based on such factorizations. A seminal contribution in this area is due to Gasca and Peña [1–3], who employed Neville elimination to derive bidiagonal decompositions for totally positive matrices, enabling algorithms that compute eigenvalues and singular values with machine-order precision.

In numerical linear algebra, obtaining accurate approximations of eigenvalues and singular values is a fundamental task. For certain classes of structured matrices, it is possible not only to compute these quantities with small absolute error, but also to high relative accuracy (HRA), meaning that the computed value $\tilde{\lambda}$ of an eigenvalue or singular value λ satisfies $|\tilde{\lambda}-\lambda|/|\lambda| = \mathcal{O}(\varepsilon)$, where ε denotes machine precision. This level of accuracy is particularly important when dealing with eigenvalues or singular values that differ by several orders of magnitude, as absolute error bounds may fail to reflect the true quality of the approximation. Achieving HRA typically requires exploiting special structural properties such as total positivity or bidiagonal factorizations, since classical unstructured algorithms may introduce large relative errors. Foundational results in this direction were established in [4], and further developed in the context of totally positive matrices by Koev [5,6]. The design of HRA-preserving algorithms remains a central goal when computing spectra of structured matrix families.

Structured matrices frequently arise in matrix theory and applications, valued for their algebraic regularity and the resulting simplification in analysis and computation. Within this class, a prominent role is played by the Min and Max matrices—defined elementwise by

$$(M)_{i,j} := \min(i,j), \quad (\mathbb{M})_{i,j} := \max(i,j), \quad 1 \le i,j \le n,$$

which yield matrices with distinctive “Γ”-shaped or reversed-“L” patterns:

$$M = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{bmatrix}, \quad \mathbb{M} = \begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 2 & 3 & \cdots & n \\ 3 & 3 & 3 & \cdots & n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n & n & n & \cdots & n \end{bmatrix}. \tag{1}$$

These matrices, first implicitly considered in the early 20th-century work of Pólya and Szegő [7], have attracted continued interest due to their simplicity and rich structure. The spectral properties of the Min matrix were initially investigated by Trench [8], with later refinements and extensions in works such as [9,10]. Generalizations—such as the generalized Min and Max matrices introduced in [11]—exhibit further interesting algebraic features, including explicit expressions for determinants, inverses, and criteria for positive definiteness [12].

Beyond structural analysis, Min and Max matrices have been linked to various mathematical domains. For example, connections to Fibonacci sequences have been explored in [13], while applications in trigonometric inequalities were discussed in [14]. These matrices also appear naturally in the context of meet and join matrices on partially ordered sets [15,16], with recent advancements documented in [17,18].

Several families of generalized Min/Max matrices have emerged in the literature. Notable among them are the r-Min and r-Max matrices [19], which were inspired by the theory of r-circulant matrices [20–22], and allow closed-form expressions for a variety of matrix characteristics, including norms, inverses, and determinants. Other significant extensions include the geometric Min and Max matrices, introduced in [23] and further developed in [24], featuring results on LU factorizations, characteristic polynomials, and determinant identities.

Despite this rich body of work, the numerical analysis of these matrices—particularly regarding the accurate computation of eigenvalues and singular values—remains relatively underdeveloped. Notably, in the case of the sequence-dependent Min and Max matrices introduced in [11], total positivity under specific conditions has been established, and accurate spectral algorithms have been proposed in [18].

In this work, we introduce and analyze a novel two-parameter family of structured matrices, termed r-geometric Min/Max matrices. Both classes simultaneously generalize the r-Min/r-Max matrices studied in [19] and the geometric Min/Max matrices introduced in [23,24]. We derive explicit bidiagonal factorizations for these matrices using Neville elimination, which provides a foundation for several theoretical and computational results. In particular, we establish necessary and sufficient conditions under which these matrices are totally positive. Leveraging the bidiagonal structure, we also obtain closed-form expressions for their determinants. When total positivity is guaranteed, we develop numerically stable algorithms for computing eigenvalues and singular values to HRA. Furthermore, we perform a detailed analysis of roundoff error propagation and derive structured condition numbers relevant to determinant evaluation and spectral computations.

The structure of this paper is as follows. Section 2 presents the necessary preliminaries and notation. Section 3 provides the bidiagonal factorizations and total positivity conditions for the r-geometric Min and Max matrices, respectively. In Section 4, we develop perturbation bounds, while Section 5 focuses on the analysis of the determinants of the analyzed matrices. Finally, Section 6 reports numerical experiments that confirm the accuracy and stability of the proposed methods across several matrix instances, including several r-geometric Min/Max variants.

2. Preliminaries

A matrix is said to be totally positive (TP) if all its minors are nonnegative. Matrices with total positivity properties appear in a wide range of mathematical and applied contexts, with notable applications discussed in [25–27]. According to [3, Theorem 4.2, p. 116], any nonsingular TP matrix $A \in \mathbb{R}^{n\times n}$ admits the following bidiagonal decomposition:

$$A = F_{n-1} \cdots F_1 D G_1 \cdots G_{n-1}, \tag{2}$$

where $F_i, G_i \in \mathbb{R}^{n\times n}$ for $i = 1, \ldots, n-1$, are TP lower and upper triangular bidiagonal matrices, respectively. More specifically,

$$\begin{aligned} F_i &= \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & m_{i+1,1} & 1 & & \\ & & & \ddots & \ddots & \\ & & & & m_{n,n-i} & 1 \end{pmatrix}, \\ G_i^T &= \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & \widetilde{m}_{i+1,1} & 1 & & \\ & & & \ddots & \ddots & \\ & & & & \widetilde{m}_{n,n-i} & 1 \end{pmatrix}, \end{aligned} \tag{3}$$

and $D \in \mathbb{R}^{n\times n}$ is a diagonal matrix with positive diagonal entries $p_{i,i}$. Moreover, if the following conditions hold

$$m_{i,j} = 0 \Rightarrow m_{h,j} = 0 \ \forall\, h > i, \qquad \widetilde{m}_{i,j} = 0 \Rightarrow \widetilde{m}_{h,j} = 0 \ \forall\, h > i,$$

then the decomposition (2) is unique. This bidiagonal decomposition can be encoded into a matrix $BD(A) \in \mathbb{R}^{n\times n}$, which adopting the notation of [28] is defined by

$$BD(A)_{i,j} := \begin{cases} m_{i,j}, & i > j, \\ p_{i,i}, & i = j, \\ \widetilde{m}_{j,i}, & i < j. \end{cases}$$

A standard way of obtaining the entries of the above decomposition consists in performing the Neville elimination process of the matrix A. This procedure is an alternative to Gauss elimination, in which suitable multiples of preceding rows are added at each step to eliminate all entries below the main diagonal. When this process is completed, the so-called pivots $p_{i,i}$ and multipliers $m_{i,j}$, $\widetilde{m}_{i,j}$ of the Neville elimination are obtained [1–3] (see, e.g., Section 2 of [2] for a detailed description of Neville elimination).

The Non-Inaccurate Cancellation (NIC) condition is a critical criterion in numerical linear algebra that ensures HRA in computations. An algorithm satisfies the NIC condition if all subtractions occur between numbers of opposite signs or involve exact initial data. This property prevents significant digit loss due to subtractive cancellation, a common source of numerical instability. When the NIC condition is met, algorithms achieve HRA, making them highly reliable even for ill-conditioned problems. This condition is particularly pertinent when computing eigenvalues, singular values, or solving linear systems involving TP matrices, where maintaining numerical stability is paramount.

If the computation of the Neville pivots and multipliers satisfy the NIC condition, i.e., the matrix $BD(A)$ can be provided to HRA, then the MATLAB functions `TNEigenValues`, `TNSingularValues`, `TNInverseExpand`, and `TNSolve` from the library `TNTools` [29] can use $BD(A)$ as input to accurately compute: eigenvalues and singular values of A, its inverse A^{-1} (via the algorithm in [30]), and the solution to systems $Ax = b$ where b has alternating signs.

Finally, we recall a standard model in floating-point arithmetic used throughout this work. For a real number $x \in \mathbb{R}$, its computed value is denoted by $\mathrm{fl}(x)$ or $\tilde{x}$. We adopt the rounding model from [31, Section 2.2]:

$$\mathrm{fl}(x \,\mathrm{op}\, y) = (x \,\mathrm{op}\, y)(1+\delta), \quad |\delta| \le \varepsilon, \tag{4}$$

where ε is the unit roundoff and $\mathrm{op} \in \{+, -, \times, /\}$ is any basic arithmetic operation.

3. Bidiagonal factorization of r-geometric Min and Max matrices

Let us start this section by introducing a new generalization of Min matrices, termed r-geometric Min matrices. For a given $n \in \mathbb{N}$, parameters $r, g > 0$, and the n-tuple $\mathbf{x} = (x_1, \dots, x_n)$, we define the associated r-geometric Min matrix $M_{r,g,\mathbf{x}} = (M_{i,j})_{1\le i,j\le n}$ as follows:

$$M_{i,j} := \begin{cases} rg^{i-j+1} x_{\min(i,j)}, & i > j, \\ x_{\min(i,j)}, & i \le j, \end{cases} \tag{5}$$

resulting in the matrix with the explicit structure

$$M_{r,g,\mathbf{x}} = \begin{bmatrix} x_1 & x_1 & x_1 & \cdots & x_1 \\ rgx_1 & x_2 & x_2 & \cdots & x_2 \\ rg^2x_1 & rgx_2 & x_3 & \cdots & x_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ rg^{n-1}x_1 & rg^{n-2}x_2 & rg^{n-3}x_3 & \cdots & x_n \end{bmatrix}. \tag{6}$$

Let us observe that for $r = 1$, the corresponding r-geometric Min matrix coincides with the geometric Min matrix introduced in [24].

The following theorem provides the bidiagonal factorization for the class of r-geometric Min matrices.

Theorem 1. *For $r, g > 0$ and the tuple $\mathbf{x} = (x_1, \dots, x_n)$, the r-geometric Min matrix $M_{r,g,\mathbf{x}}$ in (6) admits a bidiagonal factorization (2), such that:*

$$m_{2,1} = rg, \quad m_{i,1} = g, \quad i = 3, \dots, n; \tag{7}$$

$$m_{j+1,j} = \frac{g(r-1)x_j}{x_j - rgx_{j-1}}, \quad m_{i,j} = 0, \quad i = j+2, \dots, n, \quad j = 2, \dots, n-1; \tag{8}$$

$$\widetilde{m}_{i,1} = 1, \quad i = 2, \dots, n; \quad \widetilde{m}_{i,j} = 0, \quad j = 2, \dots, n-1; \tag{9}$$

and

$$p_{1,1} = x_1, \quad p_{i,i} = x_i - rgx_{i-1}, \quad i = 2, \dots, n. \tag{10}$$

Proof. The proof proceeds by induction, analyzing the steps of the Neville elimination procedure applied to $M_{r,g,\mathbf{x}}$. Let $M^{(k)} = (M_{i,j}^{(k)})_{1 \le i,j \le n}$, $k = 2, \dots, n+1$, denote the matrices obtained after $k-1$ steps of the Neville elimination for $M^{(1)} := M_{r,g,\mathbf{x}}$. Now, let us see that for $k = 2, \dots, n$,

$$\begin{aligned} &M_{1,j}^{(k)} = x_1, \quad j = 1, \dots, n; \\ &M_{i,j}^{(k)} = 0, \quad j = 1, \dots, i-1; \quad M_{i,j}^{(k)} = x_i - rgx_{i-1}, \quad j = i, \dots, n, \ i = 2, \dots, k; \end{aligned} \tag{11}$$

and, for $i = k+1, \dots, n$,

$$\begin{aligned} &M_{i,j}^{(k)} = 0, \quad j = 1, \dots, i-2, \quad M_{i,i-1}^{(k)} = g(r-1)x_{i-1}, \\ &M_{i,j}^{(k)} = x_i - gx_{i-1}, \quad j = i, \dots, n. \end{aligned} \tag{12}$$

From the pattern of the entries of $M_{r,g,\mathbf{x}}$, we deduce the Neville elimination multipliers satisfy $m_{2,1} = rg$ and $m_{i,1} = g$, for $i = 3, \dots, n$, ensuring (7). Then,

$$M_r^{(2)} = \begin{bmatrix} x_1 & x_1 & x_1 & x_1 & \cdots & x_1 \\ 0 & x_2 - rgx_1 & x_2 - rgx_1 & x_2 - rgx_1 & \cdots & x_2 - rgx_1 \\ 0 & g(r-1)x_2 & x_3 - gx_2 & x_3 - gx_2 & \cdots & x_3 - gx_2 \\ 0 & 0 & g(r-1)x_3 & x_4 - gx_3 & \cdots & x_4 - gx_3 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & g(r-1)x_{n-1} & x_n - gx_{n-1} \end{bmatrix},$$

which satisfies the conditions in (11) for $i = 1, 2$, and (12) for $i > 2$.

Assuming that, for some $k \in \{2, \dots, n\}$, the matrix $M^{(k)}$ satisfies (11) for $i = 1, \dots, k$, and (12) for $i > k$, the unique nonzero entry below $M_{k,k}^{(k)} = x_k - rgx_{k-1}$ is $M_{k+1,k}^{(k)} = g(r-1)x_k$. Thus,

$$\frac{M_{k+1,k}^{(k)}}{M_{k,k}^{(k)}} = \frac{g(r-1)x_k}{x_k - rgx_{k-1}},$$

and, for $j = k+1, \dots, n$, we also have

$$M_{k+1,j}^{(k+1)} = M_{k+1,j}^{(k)} - \frac{M_{k+1,k}^{(k)}}{M_{k,k}^{(k)}} M_{k,j}^{(k)} = x_{k+1} - gx_k - \frac{g(r-1)x_k}{x_k - rgx_{k-1}}(x_k - rgx_{k-1}) = x_{k+1} - rgx_k,$$

ensuring that $M^{(k+1)}$ satisfies (11) for $i = 2, \ldots, k+1$, and (12) for $i \geq k+2$.

From the derived expressions for the elements of $M^{(k)}$, we obtain the Neville elimination multipliers,

$$m_{j+1,j} = \frac{M_{j+1,j}^{(j)}}{M_{j,j}^{(j)}} = \frac{g(r-1)x_j}{x_j - rgx_{j-1}}, \quad m_{i,j} = 0, \; i = j+2, \ldots, n, \quad j = 2, \ldots, n-1,$$

as stated in (8). Furthermore, the diagonal pivots are:

$$p_{1,1} = x_1, \quad p_{i,i} = M_{i,i}^{(i)} = x_i - rgx_{i-1}, \quad i = 2, \ldots, n,$$

as in (10).

To derive the multipliers $\widetilde{m}_{i,j}$, we consider the matrix U^T where

$$U = (M_{i,j}^{(n)})_{1 \leq i,j \leq n} = \begin{bmatrix} x_1 & x_1 & x_1 & x_1 & \cdots & x_1 \\ 0 & x_2 - rgx_1 & x_2 - rgx_1 & x_2 - rgx_1 & \cdots & x_2 - rgx_1 \\ 0 & 0 & x_3 - rgx_2 & x_3 - rgx_2 & \cdots & x_3 - rgx_2 \\ 0 & 0 & 0 & x_4 - rgx_3 & \cdots & x_4 - rgx_3 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & x_n - rgx_{n-1} \end{bmatrix}.$$

Noting that

$$\frac{M_{i,j-1}^{(n)}}{M_{i,j}^{(n)}} = 1, \quad j = i+1, \ldots, n; \quad i = 1, \ldots, n-1,$$

we deduce that $\widetilde{m}_{i,1} = 1$, $i = 2, \ldots, n$, and the matrix obtained after the first step of the Neville elimination of U^T is a diagonal matrix with the diagonal pivots $p_{i,i}$ along the diagonal. This also confirms (9). □

The bidiagonal decomposition of the r-geometric Min matrix $M_{r,g,\mathbf{x}}$ can be represented as:

$$BD(M_{r,g,\mathbf{x}}) = \begin{bmatrix} x_1 & 1 & 1 & 1 & 1 & \cdots & 1 \\ rg & x_2 - rgx_1 & 0 & 0 & 0 & \cdots & 0 \\ g & \frac{g(r-1)x_2}{x_2 - rgx_1} & x_3 - rgx_2 & 0 & 0 & \cdots & 0 \\ g & 0 & \frac{g(r-1)x_3}{x_3 - rgx_2} & x_4 - rgx_3 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ g & 0 & \cdots & 0 & \frac{g(r-1)x_{n-2}}{x_{n-2} - rgx_{n-3}} & x_{n-1} - rgx_{n-2} & 0 \\ g & 0 & \cdots & 0 & 0 & \frac{g(r-1)x_{n-1}}{x_{n-1} - rgx_{n-2}} & x_n - rgx_{n-1} \end{bmatrix}. \tag{13}$$

Analogously, for $r, g > 0$ and the tuple $\mathbf{x} = (x_1, \dots, x_n)$, we define the r-geometric Max matrix $\mathbb{M}_{r,g,\mathbf{x}} = (\widetilde{M}_{i,j})_{1\le i,j\le n}$ as follows,

$$\widetilde{M}_{i,j} := \begin{cases} rg^{i-j+1}x_{\max(i,j)}, & i > j, \\ x_{\max(i,j)}, & i \le j, \end{cases} \tag{14}$$

resulting in the matrix with the explicit structure:

$$\mathbb{M}_{r,g,\mathbf{x}} = \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ rgx_2 & x_2 & x_3 & \cdots & x_n \\ rg^2x_3 & rgx_2 & x_3 & \cdots & x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ rg^{n-1}x_n & rg^{n-2}x_2 & rg^{n-3}x_3 & \cdots & x_n \end{bmatrix}. \tag{15}$$

The following result describes the Neville elimination of r-geometric Max matrices and provides their bidiagonal factorization (2).

Theorem 2. *For $r, g > 0$ and the tuple $\mathbf{x} = (x_1, \dots, x_n)$, the r-geometric Max matrix $\mathbb{M}_{r,g,\mathbf{x}}$ in (15) admits a bidiagonal factorization (2), such that*

$$m_{2,1} = rg\frac{x_2}{x_1}, \; m_{i,1} = g\frac{x_i}{x_{i-1}}, \; i = 3, \dots, n; \tag{16}$$

$$m_{j+1,j} = g(r-1)\frac{x_{j+1}x_{j-1}}{x_j(x_{j-1} - rgx_j)}, \quad m_{i,j} = 0, \quad i = j+2, \dots, n, \; j = 2, \dots, n-1; \tag{17}$$

$$\widetilde{m}_{i,1} = \frac{x_i}{x_{i-1}}, \; i = 2, \dots, n; \quad \widetilde{m}_{i,j} = 0, \; j = 2, \dots, n-1, \tag{18}$$

and

$$p_{1,1} = x_1, \quad p_{i,i} = \frac{x_i}{x_{i-1}}(x_{i-1} - rgx_i), \quad i = 2, \dots, n. \tag{19}$$

Proof. The bidiagonal decomposition (2) of $\mathbb{M}_{r,g,\mathbf{x}}$ can be derived using a similar inductive approach to that used for the r-geometric Min matrices in Theorem 1. By induction on $k = 2, \dots, n$, we can establish that the matrices $M^{(k)} := (M^{(k)}_{i,j})_{1\le i,j\le n}$, obtained after $k-1$ steps of the Neville elimination procedure for $\mathbb{M}_{r,g,\mathbf{x}}$, satisfy the following conditions:

$$\begin{aligned} &M^{(k)}_{1,j} = x_j, \quad j = 1, \dots, n; \\ &M^{(k)}_{i,j} = 0, \quad j = 1, \dots, i-1; \quad M^{(k)}_{i,j} = \frac{x_j}{x_{i-1}}(x_{i-1} - rgx_i), \quad j = i, \dots, n, \quad i = 2, \dots, k; \end{aligned} \tag{20}$$

and, for $i = k+1, \dots, n$,

$$\begin{aligned} &M_{i,j}^{(k)} = 0, \quad j = 1, \dots, i-2; \\ &M_{i,i-1}^{(k)} = g(r-1)x_i; \quad M_{i,j}^{(k)} = \frac{x_j}{x_{i-1}}(x_{i-1} - gx_i), \quad j = i, \dots, n. \end{aligned} \tag{21}$$

In the first step, the multipliers satisfy $m_{2,1} = rgx_2/x_1$ and $m_{i,1} = gx_i/x_{i-1}$ for $i = 3, \dots, n$. Consequently, (16) holds. The matrix $M^{(2)} = (M_{i,j}^{(2)})_{1 \le i,j \le n}$ is given by:

$$M^{(2)} = \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_{n-1} & x_n \\ 0 & \tfrac{x_2}{x_1}(x_1 - rgx_2) & \tfrac{x_3}{x_1}(x_1 - rgx_2) & \cdots & \tfrac{x_{n-1}}{x_1}(x_1 - rgx_2) & \tfrac{x_n}{x_1}(x_1 - rgx_2) \\ 0 & g(r-1)x_3 & \tfrac{x_3}{x_2}(x_2 - gx_3) & \cdots & \tfrac{x_{n-1}}{x_2}(x_2 - gx_3) & \tfrac{x_n}{x_2}(x_2 - gx_3) \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & g(r-1)x_n & \tfrac{x_n}{x_{n-1}}(x_{n-1} - gx_n) \end{bmatrix},$$

which satisfies the conditions in (20) for $i = 1, 2$ and (21) for $i > 2$.

Assume that for $k \in \{2, \dots, n\}$, $M^{(k)}$ satisfies (20) for $i = 1, \dots, k$ and (21) for $i > k$. The unique nonzero entry below $M_{k,k}^{(k)} = \frac{x_k}{x_{k-1}}(x_{k-1} - rgx_k)$ is $M_{k+1,k}^{(k)} = g(r-1)x_{k+1}$. Thus,

$$M_{k+1,j}^{(k+1)} = M_{k+1,j}^{(k)} - \frac{M_{k+1,k}^{(k)}}{M_{k,k}^{(k)}} M_{k,j}^{(k)} = \frac{x_j}{x_k}(x_k - gx_{k+1}) - \frac{g(r-1)x_{k+1}x_j}{x_k} = \frac{x_j}{x_k}(x_k - rgx_{k+1}),$$

for $j = k+1, \dots, n$. This shows that $M^{(k+1)}$ also satisfies conditions (20) for $i = 1, \dots, k+1$ and (21) for $i = k+2, \dots, n$.

From the derived expressions for the entries of matrix $M^{(k)}$, we can deduce the expression (17) for the multipliers of the Neville elimination as follows:

$$m_{j+1,j} = \frac{M_{j+1,j}^{(j)}}{M_{j,j}^{(j)}} = \frac{g(r-1)x_{j+1}x_{j-1}}{x_j(x_{j-1} - rgx_j)}, \quad m_{i,j} = 0, \quad i = j+2, \dots, n, \quad j = 2, \dots, n-1.$$

Moreover, the diagonal pivots are $p_{1,1} = x_1$ and

$$p_{i,i} = M_{i,i}^{(i)} = \frac{x_i}{x_{i-1}}(x_{i-1} - rgx_i), \quad i = 2, \dots, n,$$

as stated in (19).

Finally, since the entries of the upper triangular matrix $U = M^{(n)} = (M_{i,j}^{(n)})_{1 \le i,j \le n}$ obtained from the Neville elimination satisfy:

$$\frac{M_{i,j-1}^{(n)}}{M_{i,j}^{(n)}} = \frac{x_{j-1}}{x_j}, \quad j = i+1, \dots, n,$$

we deduce that the multipliers of the Neville elimination of U^T are

$$\widetilde{m}_{i,1} = \frac{x_{i-1}}{x_i}, \quad i = 2, \ldots, n,$$

and the matrix obtained after the first step of the Neville elimination of U^T is a diagonal matrix with the diagonal pivots $p_{i,i}$ along the diagonal, implying (18). □

The bidiagonal decomposition (2) of $\mathbb{M}_{r,g,\mathbf{x}}$ can be represented by

$$BD(\mathbb{M}_{r,g,\mathbf{x}}) =$$
$$\begin{bmatrix} x_1 & \frac{x_2}{x_1} & \frac{x_3}{x_2} & \frac{x_4}{x_3} & \cdots & \frac{x_n}{x_{n-1}} \\ \frac{rgx_2}{x_1} & \frac{x_2(x_1-rgx_2)}{x_1} & 0 & 0 & \cdots & 0 \\ \frac{gx_3}{x_2} & \frac{g(r-1)x_3x_1}{x_2(x_1-rgx_2)} & \frac{x_3(x_2-rgx_3)}{x_2} & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \frac{gx_{n-1}}{x_{n-2}} & & & \frac{g(r-1)x_{n-1}x_{n-3}}{x_{n-2}(x_{n-3}-rgx_{n-2})} & \frac{x_{n-1}(x_{n-2}-rgx_{n-1})}{x_{n-2}} & 0 \\ \frac{gx_n}{x_{n-1}} & & & 0 & \frac{g(r-1)x_nx_{n-2}}{x_{n-1}(x_{n-2}-rgx_{n-1})} & \frac{x_n(x_{n-1}-rgx_n)}{x_{n-1}} \end{bmatrix}. \tag{22}$$

The total positivity of r-geometric Min and Max matrices can be characterized studying the signs of their multipliers and pivots, given by Theorems 1 and 2.

Theorem 3. *Let $M_{r,g,\mathbf{x}}, \mathbb{M}_{r,g,\mathbf{x}} \in \mathbb{R}^{n\times n}$ be the r-geometric Min and Max matrices in (6) and (15) for $r, g > 0$ and the n-tuple $\mathbf{x} = (x_1, \ldots, x_n)$. Then, these matrices are TP if and only if $r \geq 1$, $g > 0$, and $(x_1, \ldots, x_n)$ is a sequence of positive values such that*

(i) $x_i \geq rgx_{i-1}$, $i = 2, \ldots, n$, for $M_{r,g,\mathbf{x}}$,
(ii) $rgx_{i+1} < x_i$, $i = 2, \ldots, n-2$, for $\mathbb{M}_{r,g,\mathbf{x}}$.

In this case, the bidiagonal decomposition (2) satisfies the NIC condition and, therefore, can be computed to HRA.

The aforementioned bidiagonal factorizations of the r-geometric Min and Max matrices enable the solution of various linear algebra problems with remarkable relative accuracy, as will be demonstrated in the subsequent sections.

Finally, it is worth noting that the converse, defined as the matrix obtained by reflecting all its entries, that is

$$A^{\#} := (a_{n+1-i,n+1-j})_{1\leq i,j\leq n},$$

plays an important role in the study of structured matrices. The following result shows that the class of nonsingular TP matrices is preserved under this operation.

Proposition 1. *A nonsingular matrix $A \in \mathbb{R}^{n\times n}$ is TP if and only if its converse $A^{\#}$ is TP.*

Proof. The converse can be written as $A^\# = RAR$, where R is the matrix that reverses the rows (or columns) of the identity matrix. Since $R^2 = I_n$, we obtain:

$$A^\# = RAR = F_{n-1}^\# \cdots F_1^\# D G_1^\# \cdots G_{n-1}^\#,$$

where $F_i^\# := RF_iR$, and similarly for $G_i^\#$. Clearly, when considering bidiagonal matrices of the form (3), total positivity is preserved under such similarity transformations and the result follows, taking into account that the product of TP matrices is also TP. The converse direction holds because $(A^\#)^\# = R^2AR^2 = A$. □

Min and Max classical matrices (1) exhibit a reciprocal relationship. Note that Min and Max matrices are each other converses with respect to reverse tuples. Leveraging this duality, it becomes feasible to solve the same algebraic problems using an alternative factorization. This property extends to their r-geometric counterparts. We formalize this observation in the following result.

Proposition 2. *Let* $\mathbf{x} = (x_1, \ldots, x_n)$ *be an* n*-tuple, and let* $\mathbf{x}^\# = (x_n, \ldots, x_1)$ *denote its reverse. Then,*

$$(M_{r,g,\mathbf{x}}^\#)^T = \mathbb{M}_{r,g,\mathbf{x}^\#}, \quad (\mathbb{M}_{r,g,\mathbf{x}}^\#)^T = M_{r,g,\mathbf{x}^\#},$$

where $M_{r,g,\mathbf{x}}$ *and* $\mathbb{M}_{r,g,\mathbf{x}}$ *denote the* r*-geometric Min and Max matrices defined in* (6) *and* (15)*, respectively, for the values* r*,* g *and the tuple* $\mathbf{x}$*.*

Proof. Denoting $M_{r,g,\mathbf{x}} = (M_{i,j})_{1\le i,j\le n}$, $M_{r,g,\mathbf{x}}^\# = (M_{i,j}^\#)_{1\le i,j\le n}$, setting $y_i := x_{n+1-i}$ for $i = 1, \ldots, n$, and taking into account formulae (5) and (14) for the entries of the r-geometric Min and Max matrices, we can write

$$\begin{aligned} M_{i,j}^\# &= \begin{cases} x_{\min(n+1-i,n+1-j)}, & i \ge j, \\ rg^{j-i+1} x_{\min(n-i+1,n-j+1)}, & i < j, \end{cases} \\ &= \begin{cases} y_{\max(i,j)}, & i \ge j, \\ rg^{j-i+1} y_{\max(i,j)}, & i < j, \end{cases} \end{aligned}$$

for $i, j = 1, \ldots, n$, deriving that $M_{r,g,\mathbf{x}}^\# = (\mathbb{M}_{r,g,\mathbf{x}^\#})^T$. With a similar reasoning, $\mathbb{M}_{r,g,\mathbf{x}}^\# = (M_{r,g,\mathbf{x}^\#})^T$ follows. □

Taking into account Proposition 2, the eigenvalues, singular values and inverse of $(\mathbb{M}_{r,g,\mathbf{x}}^\#)^T$ can be computed through the bidiagonal factorization of the equivalent Min matrix $M_{r,g,\mathbf{x}^\#}$, and then the solutions of the original Max matrix $\mathbb{M}_{r,g,\mathbf{x}}$ are recovered. A similar invariance property under the converse operation holds for Min matrices, which allows analogous computational techniques to be applied in both cases.

4. Perturbation theory

Under the specific conditions given by Theorem 3, computing the bidiagonal decomposition of r-geometric Min or Max matrices requires only subtractions of initial inputs, along with multiplications and divisions. As a result, this decomposition can be performed to HRA for solving linear algebra problems with the MATLAB functions provided in the software library `TNTools` (see [29]).

However, in practical applications, input values are rarely known with perfect precision, and the reliability of the bidiagonal decomposition depends on its sensitivity to perturbations. Therefore, understanding how small variations in the input tuples affect the decomposition is essential.

In this section, we show that the sensitivity of the bidiagonal factorization in (2) to small perturbations in the input tuple depends on their relative separations rather than their absolute distances. This behavior is similar to the sensitivity of determinants of certain structured matrices. For example, the smallest relative gap,

$$\operatorname{relgap} := \min_{i \neq j} \frac{|x_i - x_j|}{|x_i| + |x_j|}, \tag{23}$$

provides a lower bound on the sensitivity of the determinant of the Vandermonde matrix associated with the nodes $(x_1, \ldots, x_n)$ (see [32]).

To derive a suitable structured condition number for the bidiagonal factorization of r-geometric Min matrices, we define the following quantity, inspired by similar analyses in [32,28,33,34]:

$$\operatorname{relgap}_{r,g,\mathbf{x}} := \min_{2 \leq i \leq n} \frac{|x_i - rgx_{i-1}|}{|x_i| + rg|x_{i-1}|}. \tag{24}$$

Theorem 4. *For $r, g > 0$, let $M_{r,g,\mathbf{x}}$ and $M_{r,g,\tilde{\mathbf{x}}}$ be the r-geometric Min matrices described in (5) corresponding to the tuples $\mathbf{x} = (x_1, \ldots, x_n)$ and $\tilde{\mathbf{x}} = (\tilde{x}_1, \ldots, \tilde{x}_n)$, respectively, such that $\tilde{x}_i = x_i(1 + \delta_i)$ with $|\delta_i| \leq \delta < \varepsilon$ for $i = 1, \ldots, n$. Then, the matrices $BD(M_{r,g,\mathbf{x}}) = (M_{i,j})_{1 \leq i,j \leq n}$ and $BD(M_{r,g,\tilde{\mathbf{x}}}) = (\tilde{M}_{i,j})_{1 \leq i,j \leq n}$ in (13) satisfy*

$$\left| \frac{M_{i,j} - \tilde{M}_{i,j}}{M_{i,j}} \right| \leq \mathcal{O}(\kappa_M \delta), \quad \kappa_M := 1 + \frac{1}{\mathit{relgap}_{r,g,\mathbf{x}}}, \tag{25}$$

for every $i, j = 1, \ldots, n$.

Proof. From (24), we can derive

$$\tilde{M}_{1,1} = M_{1,1}(1 + \delta_1), \quad \tilde{M}_{i,i} = \tilde{x}_i - rg\tilde{x}_{i-1} = (x_i - rgx_{i-1})(1 + \tilde{\delta}_i) = M_{i,i}(1 + \tilde{\delta}_i), \tag{26}$$

where $|\tilde{\delta}_i| \leq \delta/\mu$, for $i = 2, \ldots, n$. On the other hand, accumulating relative errors in the style of Higham (see Chapter 3 of [31]), we have

$$\tilde{M}_{2,1} = M_{2,1}, \quad \tilde{M}_{i,i-1} = M_{i,i-1}(1+\bar{\delta}), \quad i = 2, \ldots, n-1,$$

with

$$|\bar{\delta}| \leq \frac{(1+1/\mu)\delta}{1-(1+1/\mu)\delta}, \tag{27}$$

for any $\mu > 0$ such that $\mu \leq \mathrm{relgap}_{r,g,\mathbf{x}}$ as defined in (24). Taking into account (26) and (27), the result follows. $\square$

Now, defining

$$\mathrm{relgap}'_{r,g,\mathbf{x}} := \min_{1 \leq i \leq n-1} \frac{|x_i - rgx_{i+1}|}{|x_i| + rg\,|x_{i+1}|}, \tag{28}$$

and using a reasoning similar to Theorem 4, the following structured condition number for the computation of the bidiagonal decomposition of r-geometric Max matrices and their determinant can be derived.

Theorem 5. *For $r, g > 0$, let $\mathbb{M}_{r,g,\mathbf{x}}$ and $\mathbb{M}_{r,g,\tilde{\mathbf{x}}}$ be the r-geometric Max matrices described in (14) corresponding to the tuples $\mathbf{x} = (x_1, \ldots, x_n)$ and $\tilde{\mathbf{x}} = (\tilde{x}_1, \ldots, \tilde{x}_n)$, respectively, such that $\tilde{x}_i = x_i(1+\delta_i)$ with $|\delta_i| \leq \delta < \varepsilon$ for $i = 1, \ldots, n$. Then, the matrices $BD(\mathbb{M}_{r,g,\mathbf{x}}) = (M_{i,j})_{1\leq i,j\leq n}$ and $BD(\mathbb{M}_{r,g,\tilde{\mathbf{x}}}) = (\tilde{M}_{i,j})_{1\leq i,j\leq n}$ in (22) satisfy*

$$\left| \frac{M_{i,j} - \tilde{M}_{i,j}}{M_{i,j}} \right| \leq \mathcal{O}(\kappa_{\mathbb{M}}\delta), \quad \kappa_{\mathbb{M}} := 3 + \frac{1}{\mathrm{relgap}'_{r,g,\mathbf{x}}}, \tag{29}$$

for $i, j = 1, \ldots, n$.

These results clearly show that small relative perturbations in the input tuple lead to small relative variations in the bidiagonal factorization of the r-geometric Min and Max matrices, provided the relative gaps defined in (24) and (28), respectively, are sufficiently large. As a direct consequence, their spectra can be accurately computed, since by Corollary 7.3 of [6] small component-wise relative perturbations in $BD(M)$ result in only small relative perturbations in the eigenvalues λ_i and singular values μ_i of the TP matrix M. Moreover, the structured condition number of each eigenvalue or singular value with respect to perturbations in $BD(M)$ is bounded above by $2n^2$. This is gathered in the following result.

Corollary 1. *For $r, g > 0$, consider the tuples $\mathbf{x} = (x_1, \ldots, x_n)$ and $\tilde{\mathbf{x}} = (\tilde{x}_1, \ldots, \tilde{x}_n)$, such that $\tilde{x}_i = x_i(1+\delta_i)$ with $|\delta_i| \leq \delta < \varepsilon$ for $i = 1, \ldots, n$, and let $M_{r,g,\mathbf{x}}$ and $M_{r,g,\tilde{\mathbf{x}}}$ (respectively, $\mathbb{M}_{r,g,\mathbf{x}}$ and $\mathbb{M}_{r,g,\tilde{\mathbf{x}}}$) be the associated r-geometric Min (respectively, Max) matrices. Then, the relative perturbations in the eigenvalues λ_i and singular values μ_i are bounded as*

$$|\tilde{\lambda}_i - \lambda_i| \leq \mathcal{O}(2n^2\kappa\delta)\,\lambda_i, \quad |\tilde{\mu}_i - \mu_i| \leq \mathcal{O}(2n^2\kappa\delta)\,\mu_i,$$

where κ corresponds to the structured condition number associated with perturbations in the input tuple $\mathbf{x}$*, specifically,* $\kappa = \kappa_M$ *as defined in* (25) *(respectively,* $\kappa = \kappa_{\mathbb{M}}$ *as defined in* (29)*).*

5. Explicit formulae for the determinant of *r*-geometric Min and Max matrices

Let us recall that, if a bidiagonal factorization (2) is known for given a matrix A, its determinant can be easily computed as the product of the entries $p_{i,i}$, $i = 1, \dots, n$—the diagonal pivots of the Neville elimination of A. As a consequence, Theorems 1 and 2 allow to derive a closed form expression for the determinant of r-geometric Min and Max matrices, respectively, which generalize those given in the literature for particular cases of these matrices [19,23,24]:

$$\det M_{r,g,\mathbf{x}} = x_1 \prod_{i=2}^{n} (x_i - rgx_{i-1}), \quad \det \mathbb{M}_{r,g,\mathbf{x}} = x_n \prod_{i=1}^{n-1} (x_i - rgx_{i+1}). \tag{30}$$

In the following result, based on formula (30), we derive a structured condition number for computing the determinant of r-geometric Min and Max matrices.

Theorem 6. *Let D and $\tilde{D}$ be the expression in* (30) *for the determinant of* $M_{r,g,\mathbf{x}}$*,* $M_{r,g,\tilde{\mathbf{x}}}$ *(respectively,* $\mathbb{M}_{r,g,\mathbf{x}}$ *and* $\mathbb{M}_{r,g,\tilde{\mathbf{x}}}$*) the r-geometric Min matrices (respectively, Max) for* $r, g \neq 0$ *and the tuples* $\mathbf{x} = (x_1, \dots, x_n)$ *and* $\tilde{\mathbf{x}} = (\tilde{x}_1, \dots, \tilde{x}_n)$ *such that* $\tilde{x}_i = x_i(1 + \delta_i)$ *with* $|\delta_i| \leq \delta < \varepsilon$ *for* $i = 1, \dots, n$*. Then, we have*

$$\left| \frac{D - \hat{D}}{D} \right| \leq \frac{(1 + (n-1)/\mu)\delta}{1 - (1 + (n-1)/\mu)\delta}, \tag{31}$$

for any μ such that $\mu \leq relgap_{r,g,\mathbf{x}}$ *as in* (24) *(respectively,* $\mu \leq relgap'_{r,g,\mathbf{x}}$ *as in* (28)*).*

Proof. Using (26), and accumulating the perturbations in the style of Higham (see Chapter 3 of [31]), we can write

$$\hat{D} = D \cdot (1 + \delta_1) \prod_{i=2}^{n} (1 + \tilde{\delta}_i) = D(1 + \bar{\delta}), \quad |\bar{\delta}| \leq \frac{(1 + (n-1)/\mu)\,\delta}{1 - (1 + (n-1)/\mu)\,\delta}.$$

Proceeding analogously for the r-geometric Max case, the result follows. □

Having closed form expressions for the determinants of the studied matrices, it is possible to provide a posteriori bounds for the errors due to the finite precision of floating-point arithmetics at the same time as the computed determinant without increasing significantly their computational cost (see [31] for a detailed explanation—whose notation we follow hereafter—or, e.g., [35] for a practical application of the method in a similar context).

Algorithm 1 Evaluation of det $M_{r,g,\mathbf{x}}$.

```
Require: $n, r, g, (x_1, \dots, x_n)$
  $d_1 \leftarrow x_1$
  for $i = 2 : n$ do
      $d_i \leftarrow d_{i-1} * (x_i - r * g * x_{i-1})$
  end for
det $M_{r,g,\mathbf{x}} \leftarrow d_n$
```

In the following, we will bound the error in the i-th step of Algorithm 1, which is defined as

$$D_i := \tilde{d}_i - d_i, \quad i = 2, \dots, n, \tag{32}$$

where $\tilde{d}_i$ denotes the computed value, which is obtained after one subtraction ($-$) and three multiplications ($*$).

Following the model (4), $\tilde{d}_i$ can be expressed in terms of $\tilde{d}_{i-1}$ as

$$\tilde{d}_i = \tilde{d}_{i-1}(x_i - rgx_{i-1}(1+\delta_i)(1+\bar{\delta}_i))(1+\lambda_i)(1+\varepsilon_i)^{-1},$$

where $|\delta_i|$, $|\bar{\delta}_i|$, $|\lambda_i|$ and $|\varepsilon_i|$ are values bounded by the unit roundoff or machine precision ε.

Now, taking into account (32) and performing some manipulations, we derive

$$D_i + \varepsilon_i \tilde{d}_i = D_{i-1}(x_i - rgx_{i-1}) - \tilde{d}_{i-1} rgx_{i-1}(\delta_i + \bar{\delta}_i + \delta_i\bar{\delta}_i) + \tilde{d}_{i-1}(x_i - rgx_{i-1}(1+\delta_i)(1+\bar{\delta}_i))\lambda_i.$$

Consequently, $|D_i|$ can be bounded as follows

$$|D_i| \le |D_{i-1}|\,|x_i - rgx_{i-1}| + \varepsilon\left(|\tilde{d}_i| + 2rg\,|\tilde{d}_{i-1}|\,|x_{i-1}| + |\tilde{d}_{i-1}||x_i - rgx_{i-1}|\right) + \mathcal{O}(\varepsilon^2).$$

Moreover, since d_1 does not require arithmetic operations, $D_1 = 0$ and so,

$$|D_i| \le \varepsilon\,\pi_i + \mathcal{O}(\varepsilon^2), \tag{33}$$

with the majorizing sequence π_i, which is defined as

$$\pi_1 := 0, \quad \pi_i := (\pi_{i-1} + |\tilde{d}_{i-1}|)\,|x_i - rgx_{i-1}| + |\tilde{d}_i| + 2rg\,|\tilde{d}_{i-1}|\,|x_{i-1}|, \quad i = 2, \dots, n. \tag{34}$$

To obtain an algorithm with the error bound, we can slightly reduce the number of operations required to compute the majorizing sequence by defining the sequence

$$M_i := (\pi_i + |\tilde{d}_i|)/2, \quad i = 1, \dots, n,$$

that satisfies

$$M_1 := |\tilde{d}_1|/2, \quad M_i = M_{i-1}|x_i - rgx_{i-1}| + |\tilde{d}_i| + rg\,|\tilde{d}_{i-1}|\,|x_{i-1}|, \quad i = 2, \ldots, n.$$

From (33), we derive the following error bound

$$|D_n| = |\tilde{d}_n - \det M_{r,g,\mathbf{x}}| \leq \mu\,\varepsilon + \mathcal{O}(\varepsilon^2), \quad \mu := 2M_n - |\tilde{d}_n|.$$

The value for $\mu\,\varepsilon$ can be used as a running error bound and can be computed with Algorithm 2. Furthermore, it is possible to obtain a relative error bound just as $\mu\,\varepsilon/|\hat{d}_n|$, if the condition $|\hat{d}_n| > \mu\,\varepsilon$ is fulfilled (see Theorem 3.1 in [36]).

Algorithm 2 Evaluation of $\det M_{r,g,\mathbf{x}}$ with running error bound.

```
Require: $n, (x_1, \ldots, x_n)$
  $d_1 \leftarrow x_1$
  $M_1 \leftarrow |\tilde{d}_1|/2$
  for $i = 2 : n$ do
      $d_i \leftarrow d_{i-1} * (x_i - r * g * x_{i-1})$
      $M_i \leftarrow M_{i-1} * |x_i - r * g * x_{i-1}| + |d_i| + r * g * |d_{i-1}| * |x_{i-1}|$
  end for
$\det M_{r,g,\mathbf{x}} \leftarrow d_n$
$\mu \leftarrow 2 * M_n - |d_n|$
```

With a similar reasoning we can derive an analogous algorithm with running error bound for the r-geometric Max matrix. The value for $\mu\,\varepsilon$ is again the running error bound in the computation of $\det \mathbb{M}_{r,g,\mathbf{x}}$, which is provided by Algorithm 3.

Algorithm 3 Evaluation of $\det \mathbb{M}_{r,g,\mathbf{x}}$ with running error bound.

```
Require: $n, (x_1, \ldots, x_n)$
  $d_1 \leftarrow x_n$
  $M_1 \leftarrow |\tilde{d}_1|/2$
  for $i = 2 : n$ do
      $d_i \leftarrow d_{i-1} * (x_{i-1} - r * g * x_i)$
      $M_i \leftarrow M_{i-1} * |x_{i-1} - r * g * x_i| + |d_i| + r * g * |d_{i-1} * x_i|$
  end for
$\det \mathbb{M}_{r,g,\mathbf{x}} \leftarrow d_n$
$\mu \leftarrow 2 * M_n - |d_n|$
```

6. Numerical experiments

In the following, we consider specific instances of r-geometric Min and Max matrices to illustrate the behavior of the algorithms introduced in the previous sections. We provide numerical evidence and compare the results with those obtained using standard general-purpose routines.

For each matrix under analysis, its bidiagonal decomposition—given by Theorems 1 and 2—is computed to HRA and used as input to the `TNTool` package routines [29]. These routines solve various linear algebra problems to HRA, including the computation

Table 1
Relative errors for the smallest eigenvalue and singular value of $\mathbb{M}_{r,g,\mathbf{x}}$ with $r = 4$, $g = 1/3$, and $x_i = F_{n-i+2}$, the first n distinct Fibonacci numbers in reverse order.

n	κ_2	$n^2\kappa_{\mathbb{M}}$	`eig`	`TNEigenValues`	`svd`	`TNSingularValues`	Δ
10	$2.6e+07$	$4.0e+02$	$1.7e-10$	$1.1e-15$	$2.9e-11$	$2.5e-15$	$1.8e-13$
20	$9.0e+14$	$1.6e+03$	$9.9e-03$	$1.4e-15$	$5.3e-06$	$7.3e-15$	$7.1e-13$
30	$3.2e+22$	$3.6e+03$	$1.8e+01$	$1.4e-15$	$3.7e+00$	$1.1e-14$	$1.6e-12$
40	$1.1e+30$	$6.4e+03$	$1.6e+01$	$1.8e-15$	$5.3e+06$	$7.1e-15$	$2.8e-12$
50	$3.9e+37$	$1.0e+04$	$3.2e+01$	$1.8e-15$	$2.9e+11$	$1.4e-14$	$4.4e-12$

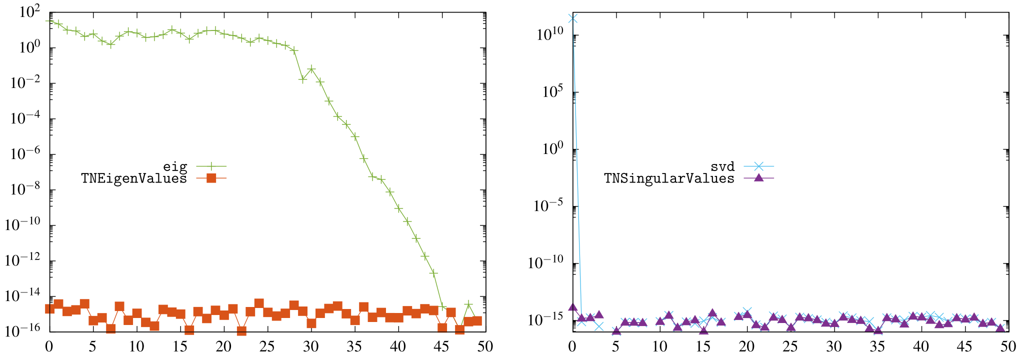

Fig. 1. Relative errors in the computation of the eigenvalues λ_i (left) and singular values σ_i (right) of $\mathbb{M}_{r,g,\mathbf{x}}$, with $n = 50$, $r = 4$, $g = 1/3$, and $x_i = F_{n-i+2}$, for $i = 1, \ldots, n$. Two methods are compared: the standard MATLAB command `eig` and those obtained by the function `TNEigenValues` taking as input the bidiagonal decomposition (22).

of eigenvalues via `TNEigenValues` and singular values via `TNSingularValues`, both with computational cost $\mathcal{O}(n^3)$. Additionally, the determinant of each matrix is computed using (30), requiring only $\mathcal{O}(n)$ operations.

For comparison, standard MATLAB routines—`eig`, `svd`, and `det`—are used to compute the eigenvalues, singular values, and determinant, respectively. Relative errors are then computed by comparing both approximations with reference values obtained using Wolfram Mathematica with 200-digit precision, assumed to be exact. Specifically, for a computed value $\hat{x}$ and its exact counterpart x, the relative error is defined as $|\hat{x} - x|/|x|$.

In the case of determinant computations, Algorithms 2 and 3 are also applied to obtain an upper bound on the relative running error. For the eigenvalues and the singular values, an a priori error bound Δ is given by Corollary 1, considering the bidiagonal decomposition to be known to HRA.

To give a measure of the ill-conditioning of the analyzed matrices, we compute the 2-norm condition number κ_2 for each case. Additionally, structured condition numbers κ derived from the perturbation analysis in Section 4 are provided: $n^2\kappa$ for both eigenvalue and singular value computations, and $n\kappa$ for the determinant.

Computations with r-geometric Max matrices with Fibonacci sequences In this experiment we study the r-geometric Max matrix $\mathbb{M}_{r,g,\mathbf{x}}$, as defined in (15), with $r = 4$,

Table 2
Relative errors for the determinant of $\mathbb{M}_{r,g,\mathbf{x}}$ with $r = 4$, $g = 1/3$, and $x_i = F_{n-i+2}$, the first n distinct Fibonacci numbers in reverse order.

n	κ_2	$n\kappa_{\mathbb{M}}$	`det`	Formula (30)	Running error
10	$2.6e+07$	$4.0e+01$	$3.4e-12$	$2.4e-15$	$1.3e-14$
20	$9.0e+14$	$8.0e+01$	$7.2e-07$	$6.0e-15$	$2.8e-14$
30	$3.2e+22$	$1.2e+02$	$1.8e+00$	$9.1e-15$	$4.3e-14$
40	$1.1e+30$	$1.6e+02$	$8.4e+03$	$1.1e-14$	$5.8e-14$
50	$3.9e+37$	$2.0e+02$	$1.3e+11$	$1.4e-14$	$7.3e-14$

Table 3
Relative errors for the smallest eigenvalue and singular value of $M_{r,g,\mathbf{x}}$ with $r = 2$, $g = 1$ and the sequence $x_i = Q_i$, $i = 1, \ldots, n$, being the first distinct n Pell–Lucas numbers.

n	κ_2	$n^2\kappa_M$	`eig`	`TNEigenValues`	`svd`	`TNSingularValues`	Δ
5	$4.8e+03$	$8.8e+01$	$6.1e-14$	$1.3e-16$	$1.3e-13$	$5.3e-16$	$3.9e-14$
10	$3.3e+07$	$3.5e+02$	$3.6e-10$	$1.6e-16$	$6.7e-10$	$1.9e-15$	$1.6e-13$
25	$9.9e+18$	$2.2e+03$	$6.5e+00$	$7.9e-16$	$1.3e+02$	$2.1e-15$	$9.8e-13$
50	$1.4e+38$	$8.8e+03$	$8.5e+00$	$1.3e-16$	$1.5e+19$	$5.9e-15$	$3.9e-12$
100	$2.6e+76$	$3.5e+04$	$7.2e+01$	$1.6e-15$	$1.9e+35$	$7.9e-15$	$1.6e-11$
200	$9.3e+152$	$1.4e+05$	$8.8e+29$	$1.1e-15$	$1.4e+67$	$1.3e-14$	$6.2e-11$

$g = 1/3$, and x_i a sequence of Fibonacci numbers in decreasing order. Specifically, recalling that this well-known recurrent sequence is defined as

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+1} = F_n + F_{n-1}, \quad n \geq 1,$$

the sequence that generates $\mathbb{M}_{r,g,\mathbf{x}}$ is chosen as $x_i = F_{n-i+2}$, $i = 1, \ldots, n$. It can be checked that for this choice of parameters $\mathbb{M}_{r,g,\mathbf{x}}$ is TP, since the entries of its bidiagonal decomposition (22) are positive. The relative errors in the determination of the smallest eigenvalue and the smallest singular value of the matrix are gathered in Table 1. Additionally, to provide a more exhaustive picture of the spectra, Fig. 1 depicts the relative errors of all eigenvalues and singular values for the particular case of $n = 50$. As can be seen, the standard approach given by MATLAB methods `eig` and `svd` cannot provide accurate answers as the size of the matrix grows, while the results of the bidiagonal approach achieve HRA in all cases. Notably, the only singular value which cannot be accurately determined by the standard method is the smallest one, although for other sequences (e.g., the next experiment) this is not the case. An equivalent scenario is obtained for the determinant computation, as shown in Table 2, where additionally the running error analysis provides sharp upper relative error bounds.

Computations with r-Min matrices with Pell–Lucas sequences In this second experiment the r-Min matrix, introduced in [19], is analyzed as a relevant particular case of the r-geometric Min matrix (6), obtained just by setting $g = 1$. For the sequence x_i the first n distinct Pell–Lucas numbers are chosen; let us recall that these numbers are defined as

$$Q_0 = 2, \quad Q_1 = 2, \quad Q_{n+1} = 2Q_n + Q_{n-1}, \quad n \geq 1.$$

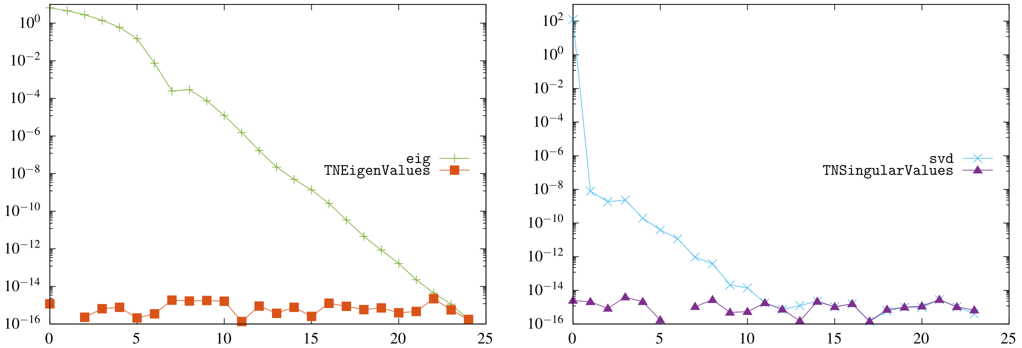

Fig. 2. Relative errors in the computation of the eigenvalues λ_i (left) and singular values σ_i (right) of $M_{r,g,\mathbf{x}}$, with $n = 25$, $r = 2$, $g = 1$, and $x_i = Q_i$, for $i = 1, \ldots, n$. Two methods are compared: the standard MATLAB command `eig` and those obtained by the function `TNEigenValues` taking as input the bidiagonal decomposition (13).

Table 4
Relative errors in the computation of the determinant of $M_{r,g,\mathbf{x}}$ with $r = 5/2$, $g = 1$ and the sequence $x_i = Q_i$, $i = 1, \ldots, n$, being the first distinct n Pell–Lucas numbers.

n	κ_2	$n\kappa_M$	`det`	Formula (30)	Running error
10	$2.3e+14$	$7.0e+02$	$1.0e-15$	0	$1.1e-13$
20	$4.1e+30$	$1.4e+03$	$1.4e-15$	$1.8e-16$	$2.4e-13$
30	$7.4e+46$	$2.1e+03$	$1.3e+13$	$3.3e-16$	$3.7e-13$
40	$1.3e+63$	$2.8e+03$	$1.3e+13$	$2.8e-16$	$5.1e-13$

The smallest eigenvalue and the smallest singular value are determined for $r = 2$; it can be checked that considered r-Min matrix is TP. The relative errors of all eigenvalues and singular values for the case $n = 25$ can be seen in Fig. 2. As depicted in Table 3, for every value of n the approximations based on the bidiagonal decomposition are computed to HRA, whereas the general purpose methods fail to provide accurate results. Moreover, in this case, the standard method has difficulties beyond the smallest singular value, in comparison with the first experiment. With respect to the determinant computation, the value of $r = 5/2$ is chosen, making the associated r-Min matrix no longer TP. This is to stress that formula (30) provides the determinant to HRA in all cases, since it only involves subtractions of initial data. This is confirmed by the relative errors presented in Table 4, which also show the inconsistent results provided by the standard MATLAB subroutine `det`.

Computations with geometric Max matrices with a high structured condition number Lastly, another particular case of relevance is studied, namely that of geometric Max matrices, introduced in [24], which in this work correspond to the r-geometric Max matrix as defined in (15), for $r = 1$. It should be noted that, in the previous experiments, the structured condition number takes only moderate values, even for high values of κ_2. To test the discussed methods under more demanding conditions, we chose the sequence to be in geometric progression with a ratio very close to the value of g. Specifically, we

Table 5
Relative errors for the smallest eigenvalue and singular value of $\mathbb{M}_{r,g,\mathbf{x}}$ with $r = 1$, $g = 2$, and the sequence $x_i = (g+\delta)^{n-i}$, $i = 1\dots, n$.

n	κ_2	$n^2\kappa_{\mathbb{M}}$	`eig`	`TNEigenValues`	`svd`	`TNSingularValues`
10	$6.9e+13$	$1.0e+13$	$5.3e-06$	$1.1e-16$	$8.3e-08$	$2.4e-16$
20	$9.9e+16$	$4.0e+13$	$6.2e-06$	$4.5e-16$	$8.3e-08$	$1.1e-15$
30	$1.2e+20$	$9.0e+13$	$1.0e-05$	$1.1e-16$	$8.3e-08$	$1.1e-15$
40	$1.5e+23$	$1.6e+14$	$9.5e-06$	$1.1e-16$	$4.6e-06$	$5.9e-16$
50	$1.7e+26$	$2.5e+14$	$6.3e-06$	$1.1e-16$	$3.8e-06$	$1.4e-15$
60	$1.9e+29$	$3.6e+14$	$3.6e-03$	$1.1e-16$	$5.3e-05$	$1.2e-15$

Table 6
Relative errors for the determinant of $\mathbb{M}_{r,g,\mathbf{x}}$ with $r = 1$, $g = 2$, and the sequence $x_i = (g+\delta)^{n-i}$, $i = 1\dots, n$.

n	κ_2	$n\kappa_{\mathbb{M}}$	`det`	Formula (30)	Running error
10	$6.9e+13$	$1.0e+12$	$7.4e-07$	$6.5e-16$	$2.0e-05$
20	$9.9e+16$	$2.0e+12$	$1.6e-06$	$1.4e-15$	$4.2e-05$
30	$1.2e+20$	$3.0e+12$	$2.4e-06$	$2.6e-15$	$6.4e-05$
40	$1.5e+23$	$4.0e+12$	$3.2e-06$	$4.7e-15$	$8.7e-05$
50	$1.7e+26$	$5.0e+12$	$4.0e-06$	$6.5e-15$	$1.1e-04$
60	$1.9e+29$	$6.0e+12$	$1.4e-05$	$8.7e-15$	$1.3e-04$

take $g = 2$ and $x_i = (g+\delta)^{(n-i)}$, $i = 1, \dots, n$ and $\delta = 10^{-10}$. Relative errors in the determination of the smallest eigenvalue and the smallest singular value are collected in Table 5, showing how the bidiagonal approach methods are capable of retaining HRA in all cases, whereas the standard MATLAB routines cannot offer the same level of accuracy, although the situation is not as severe as in previous experiments—this is to be attributed to a favorable structure of the matrix, regardless of the node distribution. Note that, due to the high values of the structured condition numbers, the a priori error bound Δ looses sharpness and it has not been included. In 6 results concerning the determinant are shown, exhibiting a similar pattern. Note that, in this last case, the running error bound loose sharpness due to the high value of the structured condition number $\kappa_{\mathbb{M}}$.

CRediT authorship contribution statement

The authors contributed equally to this work.

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Declaration of competing interest

The authors have no competing interests to declare that are relevant to the content of this article.

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Data availability

The source code used to run the numerical experiments is available upon request.

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