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Geometrical Awareness Enhances Numeracy in Children with Trisomy 21

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ABSTRACT

Background: Studies on cognition in children with Down Syndrome (Trisomy 21) have described poor performance manipulating numbers. Elisabetta Monari Martinez's (2002) research suggest that considering mathematics as a universe of exploration beyond written arithmetic can offer them an opportunity for "human flourishing" (Su, 2020). Geometry offers a suitable starting point. **Objective**: Exploring the use of geometrical activities for introducing children with T21 to integer and rational numbers. **Design**: A series of 7 workshops were designed to convey arithmetic concepts (counting, comparing and measuring) through plane geometry activities. **Setting and Participants**: Seven children aged 9 to 13, who had already completed a 3-year work on geometry, participated in the workshops held at the venue of the Spanish association Sesdown in Zaragoza, in leisure time. **Data collection and analysis**: Raw data consisted of 1) written reflections of lived experience (Van Manen, 1990) by all adults participating in the experiment, following a shared protocol observation guide; 2) photographs; and 3) edited short videos. **Results**: Understanding of counting, cardinality, multiplication, measure and simple fractions was enhanced by previous geometrical conceptions, which came to the forefront and were reinforced. Moreover, activities enhanced speech. Cheerful engagement and increased awareness were also observed. **Conclusions**: The integration of arithmetic and geometry helps children with T21 to enter the mathematical world with understanding and pleasure. Primary school mathematics focuses on written arithmetic, but geometry is hidden in

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many educational aids and models. Explicit geometrical work can help inclusion of all children in mathematics school lessons.
Keywords: Trisomy 21; Down Syndrome; Intellectual disability; Children's Mathematics Education; Geometry.

O conhecimento geométrico aumenta a capacidade numérica em crianças com trissomia 21

RESUMO

Contexto: Estudos sobre cognição em crianças com Síndrome de Down (Trissomia 21) têm descrito baixo desempenho na manipulação de números. A pesquisa de Elisabetta Monari Martinez (2002) sugere que considerar a matemática como um universo de exploração além da aritmética escrita pode oferecer a eles uma oportunidade para o "florescimento humano" (Su, 2020). A geometria oferece um ponto de partida adequado. **Objetivo**: Explorar o uso de atividades geométricas para introduzir crianças com T21 aos números inteiros e racionais. **Design**: Uma série de 7 oficinas foi projetada para transmitir conceitos aritméticos (contagem, comparação e medição) por meio de atividades de geometria plana. **Ambiente e participantes**: Sete crianças com idades compreendidas entre os 9 e os 13 anos, que já tinham concluído um trabalho de geometria de 3 anos, participaram nos workshops realizados na sede da associação espanhola Sesdown em Saragoça, em momentos de lazer. **Coleta e análise de dados**: Os dados brutos consistiram em 1) reflexões escritas da experiência vivida (Van Manen, 1990) por todos os adultos participantes do experimento, seguindo um guia de observação de protocolo compartilhado; 2) fotografias; e 3) vídeos curtos editados. **Resultados**: A compreensão de contagem, cardinalidade, multiplicação, medida e frações simples foi aprimorada por concepções geométricas anteriores, que vieram à tona e foram reforçadas. Além disso, as atividades melhoraram a fala. Envolvimento alegre e maior conscientização também foram observados. **Conclusões**: A integração da aritmética e geometria ajuda as crianças com T21 a entrar no mundo matemático com compreensão e prazer. A matemática da escola primária concentra-se na aritmética escrita, mas a geometria está oculta em muitos subsídios e modelos educacionais. O trabalho geométrico explícito pode ajudar a inclusão de todas as crianças nas aulas de matemática.
Palavras-chave: Trissomia 21; Síndrome de Down; Deficiência intelectual; Educação Matemática Infantil; Geometria.

INTRODUCTION

Trisomy 21 (T21), instead of the more common expression "Down Syndrome", is intended to name a genetic condition that can cause physical challenges and some kind of intellectual disability but does not completely predetermine development. In this development, education plays a central role.

Most of the research about children with T21 learning mathematics has been carried out by psychologists interested in the assessment of their arithmetical achievements (or failures), such as counting performance and cardinality understanding, as well as of the transition to more advanced arithmetic skills (Irwing, 1991; Nye et al, 1995; Porter, 1998; Abdelhammed, 2007; Bruno et al, 2012; see Gil Clemente 2016 for a revision of literature). Difficulties to cope with simple numerical tasks are evident to anyone who has worked in math with children with T21: the learning of numerical facts by heart (addition and multiplication tables) is an overwhelming challenge (Buckley, 2007; Bird & Buckley, 2001; Horstmeier, 2004) and hinders their comprehension of number, at least as it's commonly conceived and taught in preschools and primary schools.

Slow, unsynchronised movement (Zimpel, 2016) as well as short-term memory (Chapman & Hesketh, 2001), limited attention span (Zimpel, 2016) and delay in speech (Faragher & Clarke, 2014) can account for several difficulties that have been described in remembering the oral counting number sequence in their mother tongue(s); matching numbers to objects when counting. Considering their delay in speech (some children began to produce speech around their 3rd birthday, some have motor difficulties regarding the phonetical emission of words), a plausible hypothesis to justify difficulties in the learning of counting numbers lies in the fact already pointed out by Johann Heinrich Pestalozzi (1746-1827) in the early 19th century (Pestalozzi 1894) and deeply analysed by Karen Fuson (1988), that the concept of number is closely linked to language through (oral) counting (the oral sequence of counting numbers). It is precisely in the language areas where children with T21 have one of their biggest problems (Kumin, 1996).

Moreover, subsequent difficulties regard running pen-and-pencil algorithms and understanding of arithmetical problem statements (Bruno et al, 2011); and even using manipulatives (that is, educational materials designed to represent numbers and operations, decompositions and comparisons, Haslam, 2007).

Elisabetta Monari Martinez has shown (Monari Martinez & Benedetti, 2011; Monari Martinez & Pellegrini, 2010) that middle and high school students with T21 with poor numerical skills are able to successfully manage other areas such as algebra (equations geometrically represented) and analytic geometry if a suitable approach tailored to their needs is used. She argues that people with Down syndrome can learn mathematics if they are involved in issues beyond simple calculations, and that they improve their basic skills through the study of advanced subjects. Her work was developed around the crucial question: "Are we sure they cannot learn more? Are we sure about what basic in mathematics means and what the best path to follow is in order to teach to each student?" (Monari Martinez, 2002, p. 19). Rhonda Faragher (2014, p. 179) has faced the dilemma posed by Monari Martinez about the role of mathematics in the upbringing of children with T21 and shares the relevance of approaching mathematics for its own sake because of the pleasure that math (puzzles and curiosity) offers and because of the feeling of fulfilment reached by being able to solve and to understand.

In spite of Monari Martinez's research, the teaching of mathematics to children with T21 is still biased in favour of techniques and exercises on numerals recognition and rote learning procedures regarding money and aspects of daily life. We can better understand this emphasis on arithmetic if we consider two very common assumptions in primary mathematics education: a prevalence of a utilitarian view of mathematics to the detriment of the formative values of the subject and the established belief that mathematics is a hierarchical discipline and that basic (written) arithmetical skills are the foundation on which mathematics is built (Cogolludo-Agustín & Gil Clemente, 2019). Difficulties with arithmetic are not exclusive of people with T21. The way in which it is usually taught in primary schools in many countries oftentimes contributes to this. The large amount of time spent in learning numerals, written arithmetic, and mechanical procedures generates feelings of frustration, anxiety, and even fear on primary-school students about mathematics; there is an excessive focus on algorithms, and not enough on the true meaning of the operations, on memorizing properties and not on understanding them. Moreover, the connection between measure and order and our sensible world usually goes more unnoticed than expected.

Since 2015 a research program has been implemented at the University of Zaragoza (Spain), in collaboration with the association Sesdown (Society for Studies on Down Syndrome) aimed at exploring the early exposure to geometrical activities of children with T21, regarding:

- primordial concepts such as solid figure, point, straight line/segment/distance, rotation/angle, and the plane.
- geometrical operations and relationships: decomposition, comparison, and order.

A second phase of the project, started in 2019, deals with the introduction of numbers (integers and rational numbers) to children who had already developed geometrical awareness. We present here results of a series of 7 workshops that were designed to convey arithmetic concepts (counting, comparing and measuring) through activities involving plane geometry.

We have focused on the educational value of mathematics: we are not worried about *the math performance* of children with T21, but our goal is to explore how exposure to mathematics contributes to their personal growth or human flourishing – using Francis Su's cultural and educational approach (Su, 2020). The design of activities concentrates on exploration, understanding and awareness, on play, pleasure, beauty, and enjoyment, on confidence in struggle and sense of community – rather than on calculation skills for life. Activities are designed so that *precision*, for example, is a challenge, and *error* a source of connection to other people that encourages endurance (Millán Gasca, 2016); skills regarding exact calculations with written numbers regarding money, for example, are not the main focus. It may seem paradoxical, but our underlying pedagogical philosophy is that beauty and engagement enhance learning. Instead of basing the learning of mathematics on arithmetic and focusing on practical examples regarding daily life, our approach is aimed at studying the pedagogical role of early exposure of geometry, to enhance their cognitive strengths to achieve higher levels of understanding and thinking.

THEORETICAL BACKGROUND

The educational role of geometry for human flourishing and intellectual disabilities

We describe here the historical, epistemological, and pedagogical insights that support a change of educational *approach* in mathematics for children with T21, giving up emphasis on performance in number for daily life and promoting exposure to geometry for its own sake.

Werner Jaeger, in his essay on Greek *paideia*, the educational approach to the human formation that he viewed as crucial in Greek culture, quotes the following passage from Plato[1]

[1] Plato, *Laws, Book* V, 747b; English translation by R. G. Bury, see references; Plato's vision of mathematical education in the *Laws* is discussed in Jaeger 1944.

> He must recognize it as a universal rule that the divisions and variations of numbers are applicable to all purposes—both to their own arithmetical variations and to the geometrical variations of surfaces and solids, and also to those of sounds, and of motions, whether in a straight line up and down or circular. The lawgiver must keep all these in view and charge all the citizens to hold fast, so far as they can, to this organized numerical system. For in relation to economics, to politics and to all the arts, no single branch of educational science possesses so great an influence as the study of numbers: its chief advantage is that it wakes up the man who is by nature drowsy and slow of wit, and makes him quick to learn, mindful and sharp-witted, progressing beyond his natural capacity by art divine. (Plato, Laws, Book V, 747b)

Our working hypothesis is that "geometrical and arithmetical variations" offer a way to "wake up" children with intellectual disabilities' awareness, and to foster their wit and thoughtfulness. Two features of the cognitive profile of people with T21 have led us to the hypothesis that geometry is suitable for their initiation to mathematics (Cogolludo-Agustín & Gil Clemente, 2019):

– their relative strength regarding skills in the processing of visually presented information (Bird & Buckley, 2001);

– their interest in abstract symbols as a way to understand several ideas at the same time, thus optimizing possible limited attention span (Zimpel, 2016).

This change of paradigm is also supported from the following historical, epistemological and pedagogical insights.

1) Édouard Séguin's (1812-1880) conceptions on form as the "forceps of intelligence" and his use of educational, solid materials regarding regular figures, decomposition, comparison (bricks and rods) (Gil Clemente & Millán Gasca 2021)

2) identification of primordial geometrical objects and relationships thanks to epistemological ideas on the continuum intuition (Thom, 1971) and on the common root of geometry and arithmetic (Lafforgue, 2010); suggestions from the mathematical foundation of Euclidean geometry (axiomatic assumptions and undefined objects and relationships) combined with recent

> paleoanthropological reflection on embryonic geometric acts (Keller, 1998) and findings in the history of geometry[2] (Giusti, 1999).

A careful historical examination of the role of geometry in the pioneer educator of children with intellectual disabilities ("idiots" in the language of the 19th century) Séguin, shows that 1) he identified the idea of *plane* at the beginning of the difficulties underlying the learning of reading and writing and line drawing[3]; 2) the role of assembling and disassembling of bricks and comparison of size of rods as exercises regarding the connections between perceptual conceptions to develop actual ideas. Let's quote briefly an example of his activities, which were based on the exploitation of bodily *mimesis* that helps to work on the inner conception of the "me" and the "not-me":

> The child being in front of the teacher, a table between them, a few blocks piled near their right hands, the teacher takes one, puts it flat before him on the table, and makes the child do the same. The T. puts his block in various positions relatively to the table and to himself, and shows, not directs, the C. to do the same. The T. puts two blocks in particular relative positions, and the C. does the same each time. What was done with two blocks is done with three, with four, with more, in succession, till the exercise of simple imitation becomes quite intellectual, requiring at least a good deal of attention and power of combination. Later, the T. creates a combination of two or more blocks at once, and the C. must imitate all of it at once; and finally the T. creates a combination of a few blocks, destroys it, and orders the C. to build up the like, whose pattern he now can find only in his mind (Séguin,1866, p. 166).

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[2] Giusti (1999) argues that the origin of basic geometrical conceptions (primitive concepts and Euclidean geometry axioms) lies in human actions and gestures for example in the technical activities of surveying.

[3] Séguin realized that the difficulties on freehand drawing begun from the fact that pupils did not have a proper, abstract notion of plane. The notion of plane is a primordial one and Séguin argued that it was involved in any activity of handwriting, drawing, creating a graphical object; as well as in building or assembling. The idea of plane involves many other conceptions which are connected to each other, such as points, directions, extremities, profiles etc. (Gil Clemente & Millán Gasca, 2021; see also Millán Gasca 2015)

Geometrical exercises (including his form boards) were helpful in the combination of the three targets of his pedagogy: action, intelligence, and will:

> Therefore, the teaching of a geometrical point must not make us forgetful of the line to which this point belongs; the line, of the body it limits; the body, of its accessory properties; the properties, of the possible associations of the subject under consideration with its surroundings: an idea is not an isolated image of one thing, but the representation in a unit of all the facts related to the imaged object. (Séguin, 1866, p. 91)

Séguin's views are consistent with the French mathematician and philosopher René Thom's (1923-2002) outlook on the role of geometrical continuum intuition (shapes, lines, and time) in human consciousness, as a link between the environment and the mind:

> There is hardly any doubt that, from a psychological and, for the writer, ontological point of view, the geometric continuum is the primordial entity. If one has any consciousness at all, it is consciousness of time and space: geometric continuity is in some way inseparably bound to conscious thought (Thom, 1971, p. 698).

Segments, angles, polygons, and solids belong to the continuum realm, independently of the measures of length, amplitude, surface, or volume. Incidence and parallelism of lines, order of points in a line, decomposition are geometrical relationships that can be approached in the framework of our representations and sensations, (the "representative space"), including the visual, the tactile and motion.[4] A starting point from the educational point of view is offered by primordial and primitive objects and relationships borrowed from David Hilbert's axiomatic presentation (Hilbertgue, 1902) of Euclidean geometry, such as *point*, line, *straight line*, extreme, angle, circle, sphere, surface, *plane surface*, betweenness*, to pass through or to lie*, part, congruence; as well as the first three Euclid axioms (to produce a unique straight line between two points was already considered by Séguin in order to draw a quadrilateral or a square).

[4] We use the terminology and ideas of the mathematician and epistemologist Henri Poincaré (1854-1912) in his chapter on "Space and geometry" in the essay *Science and hypothesis* (Poincaré 1905, p. 51-71; see Israel, Millán Gasca 2012, p. 181).

The interconnection between geometrical and arithmetical ideas

Séguin's views and his rods are nowadays applied in the preparation to number concept and the positional decimal numeration system in Maria Montessori (1870-1952) preschools all over the world. In fact, geometry is hidden in many educational aids (Millán Gasca & Spagnoletti Zeuli, 2011); and in models used in mathematics education in preschool or primary school both for whole numbers and rational numbers (such as the use of the number line, which exhibits the order of natural numbers by substituting time in oral sequence of words by the geometrical order of points, and surface decomposition for the concept of fraction, Gunderson et al., 2012; Laisant, 1914), see Figures 1 and 2.

The geometrical procedures used in these 3D and 2D educational aids and models are mostly composition and decomposition, comparison and geometrical addition that correspond to the analogous arithmetical procedures.

Figure 1

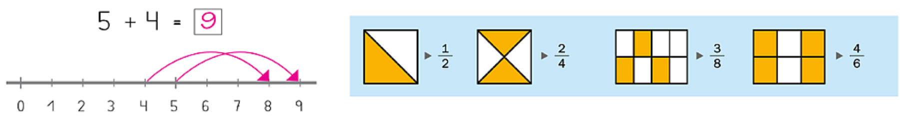
Geometry involved in educational aids for the learning of counting numbers.

Left: compare Seguin's wooden rods intended for exercises on size and (red and blue) Montessori's rods, exploiting the growing length to materialize numbers 1 to 10 in preschool.
Right: The rod or "numbers in colour", developed by George Cuisenaire (1891-1975), and the coloured Montessori pearls exploit growing size to represent natural numbers 1 (white cube and red pearl in one beam) to 9 (blue rod and nine pearls in one blue beam), the numbers that are represented by a single digit in the Indian decimal positional numeration system. The size chosen for the materials, suitable for children's hands, is similar (the rods' edges are actually 1 to 10 cm, thus connecting also to the decimal the metric system). The first number that requires two digits, 10, has a dedicated rod (orange) and a stick with 10 orange beads (linked to the materials

to represent tens, hundreds and thousands). Source (Millán Gasca & Spagnoletti Zeuli, 2015)

Figure 2

Geometrical representation of natural and rational numbers (diagrams).



Left: the number line represents the order of integers and “broken” numbers by means of points indicating positions in a straight line.
Right: fractions are represented with the aid of a surface decomposition and (geometrical) ratios

Notice that this implicit use of geometrical notions depends on individual awareness when there is not a suitable, explicit first approach to geometry in school. In fact, it's fully effective only when the underlying geometric concepts have been thoroughly worked out and properly assimilated by all the pupils in a class. This circumstance hinders the learning of numbers in children with intellectual disabilities; but it can be turned into a resource, thanks to an effective work on geometrical conceptions.

These examples can be seen as elementary instances of the intermediary role of geometry described by Thom:

> […] geometry is a natural and possibly irreplaceable intermediary between ordinary language and mathematical formalism, where each object is reduced to a symbol[…] (Thom, 1971, p. 698).

Lafforgue (2010) places the common roots of arithmetic and geometry in the repetition of identical elements and in the idea of infinite. It's usually emphasyzed that visual representations help to understand complex symbolic relationships (Arcavi, 2003), but geometrical intuition involves vision together with tact and motion: form and numbers are the root of mathematics – historical and epistemologically – and they reinforce each other in education (Clements & Sarama, 2011; Millán Gasca, 2016).

Hands-on activities and mimesis

In order to confirm the suitability of an approach through geometry for its own sake to enhance mathematical understanding of children with T21, a series of educational activities were designed. It is no surprise that children are strongly engaged by hands-on lessons/workshops about mathematics, since this approach has proved to be very effective at any level. An important role is played by mimesis, the ability of the human being to become similar to anything, to imitate in order to resemble another thing or person, and it is very strong in children, a force that pushes to know the world outside of them (Scaramuzzo, 2016, Massenzi & Magrone & Millán Gasca, 2019).

Introducing hands-on sessions helps the learners to get acquainted with the idea that mathematics implies creativity and involves senses. This is obviously all the more important with children, to make their first entry into science enjoyable and genuinely creative and to let them realize that mathematics has to do with beauty, helping them to establish a positive relationship with the study of this subject, which hopefully will stay for the future times. On the other hand, grown-up students often learned mathematics through too many frontal lessons, so the aim is more to change the point of view and to activate them. The hands-on strategy can emerge as an alternative to an exclusively frontal lesson based on the pure speech of the professor supported by the chalkboard, and has various overlapping purposes: let students take part actively (recreating the atmosphere of a manual craftsman's workshop), involve the body, when possible, in geometry (visual representative, tactile and motor space), to increase the role of synthetic geometry by decreasing the excessive use of algebra, which often covers the more intuitive aspects.

Actually the aim is to first engage the class, and the hands-on approach is one possible way; more generally we need a different way of teaching, as Enriques (1921) writes“[…] the teacher would converse with the boys, pretending to be himself a little ignorant, trying to find out with them, suggesting, making attempts towards the way to earn the truth*”*. The hands-on activity helps to identify oneself (mimesis) in the “inner labour through we are able to conquer the truth” (in Enriques’ words). This inner labor is like a moment when we “hold back the breath”, since we perceive that the “yet unknown truth” is getting closer and closer, and we keep on doing the work: this is what Mary Boole (1832-1916) calls rhythmic pulsation law[5] (Everest

[5] Mary Everest Boole designed an activity with needle and thread to approach children to the geometry of the curves: using familiar materials such as cardboard and threads,

Boole, 1904; Magrone & Massenzi & Millán, 2019). So hands-on workshops are chosen as a means of bringing children closer to mathematics, developing a positive and intimate attitude towards the discipline, cultivating dialogue, peer collaboration, interest and the taste of discovery.

We are not merely counterposing the standard frontal lecture to hands-on sessions, it's more about a sequential-procedural-static lecture versus a rhythmic-dynamic-investigating one, "pretending" the evolution of thought, "playing". We point out that if the hands-on approach becomes instructions and procedures, it loses its power, the goal is to overcome the rigid sequences, expecting to persuade students.

Evidence regarding geometry for children with T21

In recent papers we have offered empirical evidence confirming that geometry is indeed a suitable field in the mathematical initiation for young children with T21. The results are based on the design and analysis of activities with a target group starting in 2015. The leisure time workshop educational methodology and the qualitative research methodology will be described in next section. Evidence regards children's naïve geometrical and arithmetical conceptions and the educational effect of being involved in geometrical activities.

Children´s naïve conceptions were explored and observed in children 3-6 years old by means of targeted activities based on single mathematical objects and relationships:[6]

- point as a fixed position,

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they simply observe the forms that grow in front of them, spontaneously and intuitively. The goal is to create a strong and intimate connection, a relationship, between the child and some elements of mathematics, such as numbers and shapes. In this way experience is associated to abstract concepts, preparing the young mind to science in a natural way.

[6] A performance observation guide proposed by Millán Gasca (2016) was used. This exploration of naïve conceptions has been replicated any time a new group of little children (3-6 years old) begins to attend leisure time mathematical workshops organized by the association Sesdown, confirming the outcomes.

- line as a "path" and straight line as the shortest way to go from one point to another. Ability of recognizing the straight line from what we called "curved paths".
- in some cases, plane shapes, probably because their previous exposure to them in toddlers-schools, including sides and vertexes of plane shapes.
- circle as a round.

A special ability was observed to discover similarities among everyday solids, and a remarkable open and willing attitude towards mathematical activities were noticed (Millán Gasca & Gil Clemente & Colella 2017).

In the subsequent learning activities (Cogolludo-Agustín & Gil Clemente, 2019), participants showed a staggering fine geometrical intuition: they can identify the intersection between two straights lines, to understand the relationship of *betweenness* for points leading to the concepts of segment and angle and to count the sides and vertexes of polygons. They are initiated in the comparison of magnitudes, mainly lengths and some surfaces, by superimposing the objects compared. They are initiated in some types of symbolic representation using paper and pencil and in the learning of some geometrical language. Some activities also required counting (number 1 and the sequence of counting numbers being one after another) and included measuring length, area and capacity, applying both geometrical ratio and counting.

OBJECTIVE AND METHODOLOGY

Our evidence regarding the role of geometry for the awakening of the mind of children with T21 suggested the possibility of seizing the power of geometry for developing abstract thinking processes related to arithmetic.

The leisure time workshop educational methodology (LTW Sesdown) that has been developed by our University of Zaragoza/Sesdown research group was applied. Short activities (half an hour), with single mathematical goals are organized in 2-hour leisure time mathematical workshops; mathematical workshops (in Spanish, *talleres*) are offered every two weeks (usually on Saturday morning) in a suitable, pleasant venue, with children of similar ages. Activities were designed according to the cognitive characteristics of children with Trisomy 21 (Faragher, 2010; Zimpel, 2016). Main educational components are: corporeal mimesis; a fictional framework linking activities in single workshops; tight relationship with adults (1-2 children every adult); and the interaction among children (Gil Clemente, 2022). Notice that having these children work on mathematics together with other children without disabilities would have made the observation and conclusions biased, since a slower learning pace can result in a certain inhibition in participation in activities considered difficult (Wishart, 1996). The parallel work of the group of adults itself, analysing activities and sharing reflections, has an impact on the deep feelings and communication emerging in workshops.

The research methodology is based on Van Manen phenomenological approach for researching live experience (Van Manen, 1990, 1995). The workshops are leaded by the principal investigator in our research group and designed by the research group; collaborating instructors are assigned to 1-2 children. Inter-subjective annotations and reflexive reports on the live experience of workshops are elaborated. Instructors are asked to report about each workshop: the writing of live-experience reflections follows a shared protocol observation guide (Appendix 1) regarding single child evolution (including mathematics learning but more generally her/his human flourishing, as shown by speech, play, courage, generosity and sharing, initiative, and so on). To study unique phenomena, Van Manen's methodology includes systematic gathering of raw data based on specifications and questions addressed to all adults involved in the project, step after step. A systematic collection of experiential anecdotes is carried on (including impact at home narrated by parents and siblings); workshops photographical and video documentation is carried on, as well as the production of live-experience short videos. In-depth group discussion among instructors and researchers follows. Analysis of data is carried on through the oral discussion and the reflective narratives.

The present research was based on 7 two-hour leisure time workshops held in the quarters of the Spanish association Sesdown, coordinated by the second author[7]. The time span was May 2019-June 2021[8]. A group of 7 children

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[7] We thank Michelle Stephan, chair of the Topic Study Group 4 (Mathematics for students with special needs), the team members of the Group and the participants in the discussion of the long presentation of this research on July 16, 2021, at the International Congress of Mathematical Education ICME-14 Shanghai.

[8] This project was intended for a time span of 7 months (on a monthly basis) interrupted in March 2020 due to the situation derived from Covid-19 pandemic. Children developed in their homes a specific program *Family math* and we resumed activities in January 2021.

with T21 was involved –ages 9 to 13– who have already completed a three–year geometrical training following the approach above described; 4 adult instructors together with two researchers participated in every workshop, for a total of 6 adults.

Description and qualitative assessment

Four arithmetical conceptions were considered to be introducing on a geometrical basis: in Figure 3 the framework of every workshop as presented to participants is connected to the targeted arithmetical concept and its geometrical basis.

Figure 3

Learning numbers with the help of forms: a learning path through seven workshops.

Arithmetical blocks	*Workshops*	Geometrical basis
Counting to cardinal transition	**Playing with geoboards**	Point, segment, vertexes and sides of polygons
	The beauty of shapes	
Multiplication (natural numbers)	**Building walls**	Rectangle, sides, area
Rational numbers: fractions 1/*n* (inverses of natural numbers)	**Cutting a ribbon**	Geometrical ratio (half, quarter)
	Handing out a cake	
Measure: lengths/distances using natural numbers and fractions	**How tall is my tree?**	Straight line, addition of segments
	How far is my planet?	

We present a qualitative assessment including pictures to show features like how the environment is organized, what the atmosphere was like, how deeply concentrated the children were in the activities, and their corporeal disposition.

The first two workshops (*Playing with geoboards* and *The beauty of shapes*) deal with the counting to cardinal transition. It is possible to access to cardinality by counting but also associating it to a geometrical configuration.

Geometrical configurations for understanding numbers have been used since ancient Greece, Pythagoras talked about polygonal numbers as those that can be arranged in a polygon. We know children with T21 benefit from geometrical configurations to deal with small quantities (dice, dominoes are very easy to understand for them, see Hanrahan & Newman, 1996; Bobis, 2008; Camos, 2009) and that link between a number and a polygon helps them understand its properties.

We can also link the human action of counting with the more abstract task of classifying polygons. Given several types of different polygons, children had to make groups, counting the *corners* (that is how we named the vertices) and make posters identifying each one with a number. This is a clear example of how to use a network of concepts to reinforce mutual understanding. Thanks to their familiarity with polygons and their elements, children used different strategies for their classification such as visual recognition and/or counting of sides and/or vertices.

Geoboards are proper models to work with numbers and polygons. Children design their own –not necessarily regular– polygon placing the rubber bands in the dots. Afterwards they can draw the polygon in a black paperboard and write the number of vertices they have used inside. Having previously worked with the concept of *point* helped children with their counting and the association with particular configurations.

Figure 4

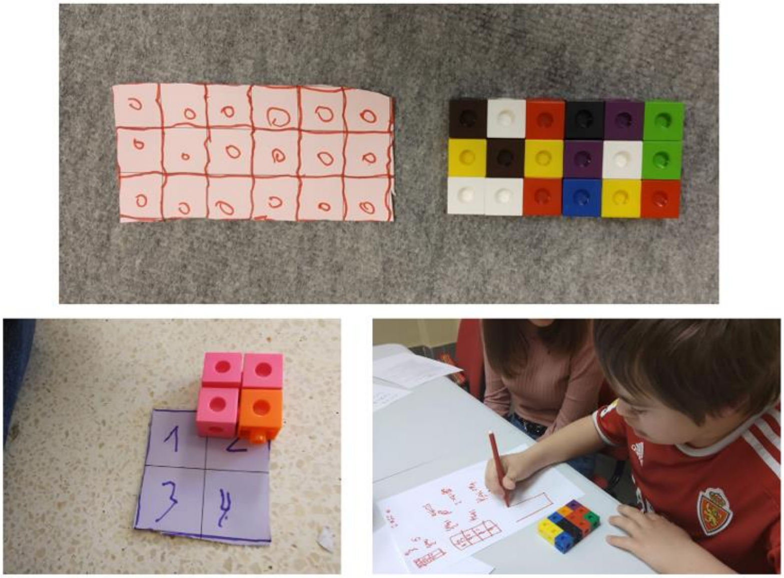
Pictures taken in the first two workshops: (from left to right) the geoplan with rubber bands forming a star polygon; posters showing polygons, corners are marked to highlight them.

Both workshops ended with children verbalizing their experiences with meaningful sentences such as: “my star has six points” or “this is number six’s poster”.

Workshop three (*Building walls*) deals with the arithmetic operation of multiplication. Instead of using the common model of grouping, we propose using the geometrical shape of a rectangle that promotes a better understanding of the multiplication properties. Multiplication is a binary operation: it involves associating two numbers to produce a third one named their product. This operation has two main properties: commutativity and identity element. The usual model to teach multiplication is to use groups: 3 times 4 means “the units that we have in 3 groups of 4 things”. This model entails several problems: for instance, understanding commutativity.

Figure 5

Pictures of the activities of workshop number 3: on top, the drawing of a wall of 6x3 squares and the corresponding wall of polycubes; bottom: again, polycubes and children at work.



Once again, the geometry of a shape such as the rectangle is a better representation. Euclid in his book VII says "And, when two numbers being multiplied one another make some number, the number so produced be called plane, and its sides are the numbers which have multiplied one another". A rectangle exemplifies the association between the factors and the result, producing an image for this: two segments (numbers) produce one rectangle (product). Hence 5 times 2 now means the area of a rectangle of sides 5 and 2. And now commutativity is immediate with else to do than turning the rectangle upside down.

Using a prehistoric setting, we encourage the children to build walls, using *polycubes*, to prevent wild animals from entering their caves. The children start building these walls the way they want and start learning to use the right word for each of the three numbers involved here, for instance, 3 rows, 5 columns, 15 bricks. Starting from this naive game, commutativity emerges in a natural way, and we can introduce children into a systematic representation of the times tables. This activity as it is, also allows the transition to a more symbolic representation: rectangles in a cardboard, where some children can draw the numbers to facilitate counting. Previous work with assembling and decomposing plane shapes allowed children to visualize the whole picture as well as the row, column, and brick decomposition in their heads.

During the activity meaningful comments such as "my wall has 4 rows, 5 columns, and 20 bricks" prove the effectiveness of the approach. The children show more difficulty associating the three numbers without the geometric support.

Workshops *Cutting a ribbon* and *Handing out a cake* deal with the most simple rational or "broken" numbers: 1/2, 1/3, 1/4, the multiplicative inverses of counting numbers. Here we link geometrical and arithmetical ratios, in accordance with Hans Freudenthal (1983) emphasis on fractions as comparers (ratios), rather than "fractures". Recovering the intuitive idea of geometrical ratio, we can better understand such numerical symbol. The arithmetical ratio (for instance, the ratio between 4 and 8) can be linked to the geometrical ratio (half of a shape). We use this network of concepts when designing the activities.

We begin playing with the concept of *half* in their surroundings. After this work, we introduce them slowly in the process of obtaining an exact half using Geometry. We begin with lengths, folding one ribbon into, two or four equal parts, checking that two halves and four quarters have the same length. We continue with surfaces, in our case, a circle (a cookie for children). Through a similar process we fold the circle twice, making sure that the pieces have an equal shape, and then further into four. In these two ways, we arrive to a geometrical representation of the unit fractions, one, one half and one quarter. Finally, we represent them symbolically as unit fractions: a whole, one half and one quarter. These activities lead children with Trisomy 21 to great discoverings: "I have a trick" says a twelve-year old girl: "Two half-circles form the circle I need".

Figure 6

Pictures from workshops 4 and 5. Children working with ribbons and with paper circles, and their fractions.

Workshops *How tall is my tree?* and *How far is my planet?* exploit comparison relationships based upon order among points and addition of magnitudes related to segments to deal with measuring tasks (that is, assigning a number to a geometrical magnitude). Note that this means both physical and mental actions that remain as a historical and epistemological root of mathematics. The process of measuring requires two steps: a geometrical step (to choose a segment or a surface as a unit and to compare any segment or surface with the standard unit by superposition) and then an arithmetical step (counting the number of superimposed units). The human act of measuring magnitude breaks the geometric continuum (see above, "Geometry from a pedagogical and anthropological point of view") into a discrete structure.

We have reproduced these two steps with our group of children measuring lengths and distances. Thanks to the previous work with the primitive concepts of *point* and *line*, children with T21 imagine without any difficulty the straight line that joins two points and are ready to play the *superimposing game*. In the process, they even feel the need and challenge of precision. For instance, when the *galactic rods,* used to measure the distance between a plane and an aircraft, did not fit exactly, one needs to use half a rod, and then meaningful comments such as: “the distance from my aircraft to Pluto is three rods and a half” seemed to show again the network of concepts. Due to the greater difficulty of the task of measuring surfaces we have only measured surfaces whose sides form right angles, using square units.

Figure 7

The activity developed in workshops 6 and 7. Measuring objects and distances, using units of length and of area.

RESULTS

1. Understanding of counting numbers and of some examples of rational numbers was observed:
    a. understanding of cardinal numbers in spite of their difficulties to assign them with accuracy when counting thanks to geometrical configurations
    b. greater awareness when counting (orally) well beyond 10
    c. functional connection among sequence/cardinal/measure meanings of numbers.
    d. understanding of the three numbers involved in any single multiplication (product = factor × factor), for factors lesser than 10, despite their difficulties to calculate single values or remembering them, thanks to the use of rectangles.

   e. understanding of numbers as a result of measurements (lengths and distances) improving precision when compared to saying “greater than”.
   f. understanding of fractions 1/n as comparers (ratio) and their use in several contexts

2. Previous educational work in geometry came to the forefront during the activities. Moreover, we have realized that, even if the target of the activity was numbers, they were also helpful to a better understanding of:
   a. the geometrical ratios regarding segments and rectangles: half, double, quarter, third, sixth as half third…
   b. the length of segments and the area of rectangles (intended as number of fixed “little” sticks or squares to be cover them, rather than as a whole extension).
3. Children engaged cheerfully in the challenging tasks/little problems; showed autonomy while solving them; increase their awareness of the “mathematics world” (for instance they feel the need of precision while measuring distances or they spontaneously understand they must make an effort to place the units with no gaps in order to measure a length)

The live experience of the workshops makes us think that Mathematics connects with something that is deep inside children with T21 as human beings (Su, 2020).

4. Quite unexpectedly, the activities proved to enhance the use of language: children made comments while involved in the tasks and showed the desire to share their discoveries. Our previous work on geometry was also based on the fact that geometry is supported by perception and body experience and not as language-dependent as arithmetic. Nevertheless, we have also found an impact of the activities to reinforce speech because words beyond gestures were considered important for them in order to describe the actions performed or to explain/support the solution.

CONCLUSIONS

The integration of arithmetic and geometry helps children with T21 to enter the mathematical world with understanding and pleasure and to face their initial impairment with speech: they feel the need to communicate their findings and the precision, conciseness and meaningfulness of mathematical language helps them.

Several issues have emerged throughout the work with this group of children with trisomy 21, opening the door to future research:

1) Why children with T21 have a great difficulty to remember simple number facts (addition and multiplication tables)? Is there an educational strategy to cope with this obstacle?
2) Should we invest efforts in teaching number-based algorithms?
3) How can we use our insights for posing and solving problems involving numbers?

Primary school geometry focuses on written arithmetic, but geometry is hidden in many educational aids and models. Explicit geometrical work can help inclusion of all children in mathematics school lessons. Presentations to schoolteachers of the results of this ongoing research on mathematics for children with intellectual disabilities can be found in (Gil Clemente, 2020, 2022); (Bruno et al, 2022) includes a description on some activities on number based on geometry. These resources are intended to support further replication of the mathematics learning activities, as well as to encourage teachers in preschools and primary schools to adopt this approach, for the sake of inclusion and adjusting to every child.

AUTHORS' CONTRIBUTIONS STATEMENTS

EGC and AMG conceived the presented approach. JICA, EGC and AMG developed the theoretical framework. EGC performed the experimental activities and collected the data. PM contributed to the analysis of hands on and mimesis activities. All authors actively participated in the discussion of the results, reviewed and approved the final version of the work.

DATA AVAILABILITY STATEMENT

Data supporting the results of this study will be made available by the corresponding author, [PM], upon reasonable request.

REFERENCES

Abdelhameed, H. (2007). Do children with Down syndrome have difficulty in counting and why? *International Journal of Special Education, 22*(2), 129-139.

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics*, *52*(3), 215-241.

Bird, G. & Buckley S. (2001). *Number skills for individual with Down syndrome: an overview*. The Down Syndrome Educational Trust.

Bobis, J. (2006). Early spatial thinking and the development of number sense. *Australian Primary Mathematics Classroom, 13*(3), 4-9.

Bruno, A., Noda, A., Gonzalez, C., Moreno, L. & Sanabria, H. (2011). Addition and subtraction by students with Down syndrome. *International Journal of Mathematical Education in Science and Technology*, *42*, 13–35.

Bruno, A. & Gil Clemente, E. & Gutiérrez, A. & Jaime, A. & Polo-Blanco, I. (2022). Pensemos en unas matemáticas para todo el alumnado (Let's think about mathematics for all students). In *Orientaciones para la práctica docente ante el cambio curricular: propuestas desde la investigación didáctica* (p. 322-347). SEIEM-Sociedad Española de Investigación en Educación Matemática.

Camos, V. (2009). Numerosity discrimination in children with Down syndrome. *Developmental Neuropsychology, 34*, 435-447.

Chapman, R. Y. & Hesketh, L. (2001). Language, cognition and short-term memory in individuals with Down syndrome. *Down Syndrome Research and Practice 7*(1), 1-7.

Clements, D.H. & Sarama, J. (2011). Early childhood teacher education: the case of geometry. *Journal of Mathematical Teacher Education*. https://doi.org/10.1007/s10857-011-9173-0

Cogolludo Agustín, J.I. & Gil Clemente, E. (2019). The effectiveness of teaching geometry to enhance mathematical understanding in children with Down syndrome. *International Journal of Disability, Development and Education, 66*(2), 1–20.

Enriques, F. (1921). Insegnamento dinamico, *Periodico di Matematiche*, *IV*, 1, 6-16.

Everest Boole, M. (1904). *The Preparation of the Child for Science*, 2s. Clarendon.

Faragher, R. & Clarke, B. (eds) (2014). *Educating learners with Down syndrome.* Routledge.

Faragher, R. (2010). Developing numeracy to enhance quality of life. In Kober, R.: *Enhancing the quality of life of people with intellectual disability: From theory to practice* (p. 401-411). Springer.

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures.* Kluwer.

Fuson, K. (1988). *Children´s counting and concepts of number*. Springer.

Gil Clemente, E. (2016). *Didáctica de las matemáticas para niños con síndrome de Down a partir de una visión integrada de la aritmética y geometría elementales [Teaching mathematics for children with Down syndrome from an integrated view of elementary arithmetic and geometry]*. Unpublished Ph.D. thesis, Universidad de Zaragoza, Zaragoza.

Gil Clemente, E. (2020). *Matemáticas que suman.* Horsori.

Gil Clemente, E. & Millán Gasca, A. (2021). Geometry as "forceps of intelligence": lines, figures and the plane in Édouard Séguin's pedagogical thought. *Bollettino di storia delle scienze matematiche*, *41*(2), 315-339.

Gil Clemente, E. (Ed.) (2022). *Talleres temáticos para la educación matemática de niños con discapacidad intelectual. Guía multimedia*. Servicios de Publicaciones de la Universidad de Zaragoza. https://www2.unavarra.es/gesadj/anfomam/O3_Guia_multimedia_def_21_22_castellano.pdf

Giusti, E. (1999). *Ipotesi sulla natura degli oggetti matematici*. Bollati Boringhieri. (French translation *La naissance des objets mathématiques*, Ellipses, 2000).

Gunderson, E.A. et al (2012). The relation between spatial skill and early number knowledge: the role of the linear number line. *Developmental Psychology, 48*(5), 1229–1241.

Hanrahan, J. Y. & Newman, T. (1996). Teaching addition to children. In Stratford, B. & Gunn, P.: *New approaches to Down syndrome* (p. 300-308). Cassell.

Haslam, L. (2007). Sam´s progress with learning mathematics. *Down Syndrome Research and Practice*, *12*(1), 32-33.

Hilbert, D. (1902). *The foundations of Geometry*. Open Court.

Israel, G. & gui Gasca, A. (2012). *Pensare in matematica.* Zanichelli.

Jaeger, W. (1944) *Paideia: the ideals of Greek culture*. Vol 3: The Conflict of Cultural Ideas in the Age of Plato, Oxford University Press.

Keller, O. (1998). Questions etnographiques et mathématiques de la préhistoire. *Revue de synthèse*, 4th series, *4*, 545-573.

Kumin, L. (1996). Speech and language skills in children with Down syndrome. *Mental retardation and Developmental Disabilities Research Reviews 2,* 109-16

Lafforgue, L. (2010). L´importance du calcul et de la géométrie à l'école primaire. *Journeée « Trans-maître »: instruir aujourd´hui à l'école primaire,* http://www.ihes.fr/~lafforgue/textes/ImportanceCalculGeometrie.pdf

Laisant, C.A. (1914). *Mathematics*. Doubleday.

Magrone, P., Massenzi S. & Millán Gasca, A. (2019). Rhythmical pulsation: art, mimesis and mathematics in primary school following Mary Everest Boole. *Journal of Mathematics and the Arts*, special issue on Education, *13*(1&2), 100-111.

Millán Gasca, A. (2016). *Numeri e forme*. Zanichelli.

Millán Gasca, A. (2015). Mathematics and children's minds: the role of geometry in the European tradition from Pestalozzi to Laisant. *Archives internationales d'histoire des sciences*, *65*(2), 175, 261–277.

Millán Gasca, A., Gil Clemente & E. Colella, I., (2017). Combining historical, foundational, and developmental insights to build children's first steps in mathematics. In Dooley, T. & Gueudet, G.: *Proceedings of the tenth congress of the European society for research in mathematics education* (CERME10, February 1–5, 2017) (pp. 1877–1884). DCU Institute of Education and ERME.

Millán Gasca, A. & Spagnoletti Zeuli, E. (2015). La geometria nei materiali e nelle immagini per apprendere il sistema di numerazione posizionale decimale. Dalla storia alla scuola di oggi, *Periodico di Matematiche* Series IX, *7*(3), 23–40.

Monari Martinez, E. (2002). Learning mathematics at school…and later on. *Down syndrome news and update 2*(1), 19-23.

Monari Martinez, E. & Benedetti, N. (2011). Learning mathematics in mainstream secondary schools: experiences students with Down syndrome. *European Journal of Special Needs Education*, *26*(4), 531-540.

Monari Martinez, E. & Pellegrini, K. (2010). Algebra and problem-solving in Down syndrome: a study with 15 teenagers. *European Journal of Special Needs Education*, *25*(1), 13-29.

Nye, J., Clibbens, J. & Bird, G. (1995). Numerical ability, general ability and language in children with Down syndrome. *Down Syndrome Research and Practice 3*(3), 92-102.

Pestalozzi, H.J. (1894). *How Gertrude teaches her children.* Swan Sonnenschein/C. W. Bardeen. (Original edition, in German, 1801)

Plato (1967-68). *Plato in twelve volumes*, vols. 10-11 translated by R.G. Bury. Harvard University Press/William Heinemann.

Poincaré, H. (1905). *Science and hypothesis*. Scott. (English translation of *La* science *et l'hypothèse* (1902)).

Scaramuzzo, G. (2016) Aristotle's 'homo mimeticus' as an educational paradigm for human coexistence, *Journal of Philosophy of Education 50*(2), 246–260.

Séguin, E. (1866). *Idiocy and its treatment by the physiological method.* W. Wood.

Su, F. (2020). *Mathematics for human flourising.* Yale University Press.

Thom R. (1971). Modern Mathematics: an educational and philosophic error?, *American Scientist*, *59*, 695-699.

Van Manen, M. (1990). *Researching living experience*. State University of New York Press.

Van Manen, M. (1995). On the epistemology of reflective practice, *Teachers and Teaching: theory and practice*, *1*(1), 33-50.

Wishart, J. G. (1996). Avoiding learning styles and cognitive development in young children. In Stratford, B. Y. & Gunn, P.: *New approaches to Down syndrome* (p. 173-205). Cassell.

Zimpel, A. (2016). *Trisomy 21: What we can learn from people with Down syndrome*. Vandenhoeck & Ruprecht.

APPENDIX 1

Guide for Reflexive Observation and Analysis

Workshop How far is my planet?

This guide describes the single steps of the 4 activities included in the workshop session (please refer to the description of activities with the explanation of framework and mathematical contents).

The questions regard the child assigned to you. Please feel free to add comments about any other child interacting with you, indicating her/his name. Please feel free to add comments on the general atmosphere during the session.

0. Entrance, welcome and beginning of activities.
Child's predisposition at the beginning of the session.
Eventually, any child's comment regarding mathematics.
Please jot down any remark, regarding the child or also the overall initial atmosphere.

Activity 1. We measure the distance to different planets.

1.1. We have already measured several times this year. What degree of autonomy does the child have in the process? Does he/she know what he/she has to do? Does he/she place one stick after another? Does he/she reach the planet? Does he/she count the sticks he places? Does he/she write the number on the sheet?

1.2. This is almost the first time we do it. Does he/she realize what the problem is? Can he/she think of a solution? What does he/she do to break the stick in half? Is it easy for him/her to write the measurement?

1.3. How does he/she put the experience into words? Does he/she use the verb to measure? Does he/she use the word half?

Activity 2. We continue the trip and now we see the stars from our ship.

2.1 Does he/she represent something that looks like a star? Does he/she enjoy doing it? Does he/she count the peaks?

2.2. After counting, does he/she state the number of peaks? Does he/she need you to ask him/her?

2.3. In the oral statement of the number of peaks, does he/she say a number word and does she/he add the name peaks?

Activity 3. We prepare some "spatial ropes" to release ourselves from the ship.

3.1 Strategies followed by the child to split the ropes. Do you think that what you worked on in the previous workshop session was helpful for the child?

3.2 What have you done to make the child aware of what "half" means (it is not just two pieces, but two pieces of equal length)

3.3 Does he/she now know how to do it on their own? How have you explained the written, symbolic representation ½, ¼? Do you think he/she has understood? On what basis?

3.4 Now we are relating the geometrical meaning of half (half a segment) with the numerical use of half (3 is half of 6) even if in a context of counting squares… do you notice any difference in understanding? How do children draw pictures? Do they count squares naturally or do you tell them to?

3.5 Describe how the child uses the words half and fourth.

Activity 4. We share the food at Planet Magic

4.1. Strategies followed by the child to split up the cakes. Do you think that what you worked on in the previous session was helpful for the child? What have you done to make the child aware of what half means (it's not just two pieces, but two pieces of equal length)

4.2. Do they know how to do it on their own now? How have you explained the symbolic representation ½, ¼? Do you think he/ she has understood? On what basis?