Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2015, Article ID 915358, 11 pages
http://dx.doi.org/10.1155/2015/915358

Research Article

Quantitative Estimates for Positive Linear Operators in terms of the Usual Second Modulus

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Received 16 February 2015; Accepted 24 April 2015

Academic Editor: Milan Pokorny

We give accurate estimates of the constants $C_n(\mathscr{A}(I),x)$ appearing in direct inequalities of the form $|L_nf(x)-f(x)| \le C_n(\mathscr{A}(I),x)\omega_2(f;\sigma(x)/\sqrt{n})$, $f\in\mathscr{A}(I)$, $x\in I$, and $n=1,2,\ldots$, where L_n is a positive linear operator reproducing linear functions and acting on real functions f defined on the interval I, $\mathscr{A}(I)$ is a certain subset of such functions, $\omega_2(f;\cdot)$ is the usual second modulus of f, and $\sigma(x)$ is an appropriate weight function. We show that the size of the constants $C_n(\mathscr{A}(I),x)$ mainly depends on the degree of smoothness of the functions in the set $\mathscr{A}(I)$ and on the distance from the point x to the boundary of I. We give a closed form expression for the best constant when $\mathscr{A}(I)$ is a certain set of continuous piecewise linear functions. As illustrative examples, the Szàsz-Mirakyan operators and the Bernstein polynomials are discussed.

1. Introduction

Let I be a closed real interval with nonempty interior set $\mathring{I}$. The usual second modulus of smoothness of a function $f: I\to\mathbb{R}$ is defined as

$$\omega_2(f;\tau) = \sup\left\{\left|\Delta_h^2 f(x)\right| : 0\le h\le\tau, [x-h,x+h]\subseteq I\right\},\quad \tau\ge 0, \tag{1}$$

where

$$\Delta_h^2 f(x) = f(x+h) - 2f(x) + f(x-h). \tag{2}$$

Denote by $\mathscr{M}(I)$ the set of measurable functions $f: I\to\mathbb{R}$ such that $\omega_2(f;\tau)<\infty$, $\tau\ge 0$. Many sequences $(L_n, n=1,2,\ldots)$ of positive linear operators acting on $\mathscr{M}(I)$ allow for a probabilistic representation of the form (cf. [1])

$$L_nf(x) = Ef(Y_n(x)),\quad f\in\mathscr{M}(I),\ x\in I,\ n=1,2,\ldots, \tag{3}$$

where E stands for mathematical expectation and $Y_n(x)$ is an I-valued random variable whose mean and standard deviation are given, respectively, by

$$EY_n(x) = x,\quad \sqrt{E(Y_n(x)-x)^2} = \frac{\sigma(x)}{\sqrt{n}},\quad x\in I,\ n=1,2,\ldots, \tag{4}$$

for some nonnegative function $\sigma: I\to\mathbb{R}$. The condition $EY_n(x)=x$ is equivalent to say that L_n reproduces linear functions.

It is well known (see, for instance, [2–6] and the references therein) that such operators satisfy pointwise inequalities of the form

$$\left|L_nf(x)-f(x)\right| \le C_n(x)\,\omega_2\left(f;\frac{\sigma(x)}{\sqrt{n}}\right),\quad x\in I,\ n=1,2,\ldots, \tag{5}$$

which measure the rate of convergence from $L_nf(x)$ to $f(x)$ according to the degree of smoothness of f. In (5), $C_n(x)$ is a positive constant only depending upon n and x. It is also interesting to consider in (5) the uniform constant

$$C = \sup \{C_n(x) : x \in I, n = 1, 2, \dots\}. \tag{6}$$

Several authors have obtained estimates of this uniform constant. For instance, Adell and Sangüesa [7] gave $C = 1.385$ for the Weierstrass operator. Păltănea [5, Corollary 4.1.2, pp. 93-94] obtained $C = 11/8$ for the Bernstein polynomials, and Gonska and Păltănea [8] showed that $C \simeq 3/2$ for a certain class of Bernstein-Durrmeyer operators. More generally, in Păltănea's book [5, Corollary 2.2.1, p. 31] it is shown that $C \simeq 3/2$ for a large class of positive linear operators reproducing linear functions.

The aim of this paper is to give a general method to provide accurate estimates of the constants $C_n(\mathscr{A}(I), x)$ satisfying the inequalities

$$\begin{gathered}\left|L_n f(x) - f(x)\right| \le C_n(\mathscr{A}(I), x)\, \omega_2\left(f; \frac{\sigma(x)}{\sqrt{n}}\right), \\ f \in \mathscr{A}(I), \ x \in I, \ n = 1, 2, \dots,\end{gathered} \tag{7}$$

where $\mathscr{A}(I)$ is a certain subset of $\mathscr{M}(I)$. Such a problem is meaningful, because in specific examples the estimates of the constants in (6) and (7) may be quite different, mainly depending on two facts: the degree of smoothness of the functions in the set $\mathscr{A}(I)$ and the distance from the point x to the boundary of I. In this way, we complete the general results shown by Păltănea [5].

The method is based on the approximation of any function $f \in \mathscr{M}(I)$ by a quasi interpolating piecewise linear function having an appropriate set of nodes. In doing this, special attention must be paid to the nodes near the endpoints of I, if any. The main results are Theorems 6 and 7 stated in Section 3. In particular, Theorem 6 provides inequalities of form (7), where the upper bound consists of various terms involving $\omega_2(f;\cdot)$ evaluated at different lengths. Theorem 7 gives a closed form expression for the best constant in (7) when $\mathscr{A}(I)$ is a certain set of continuous piecewise linear functions.

As illustrative examples, we consider the Szàsz-Mirakyan operator (Section 4) and the Bernstein polynomials (Section 5). Although the kind of estimates is similar in both examples, the results take on a simpler form in the first case, because the interval of definition $I = [0, \infty)$ has only one endpoint. In any case, both examples show that the size of the constants in front of $\omega_2(f;\cdot)$ heavily depends on the set of functions $\mathscr{A}(I)$ under consideration and on the distance from point x to boundary of I.

We believe that the methods proposed in this paper could be applied to a wide class of positive linear operators, such as Baskakov operators, Stancu operators, and their q-analogues, among others (see [9, 10] and the references therein). To obtain accurate estimates of the constants involved in each case, we essentially need to compute second moments (see Theorem 8 in Section 3) and tail probabilities of the underlying random variables defining the operators under consideration (see Lemmas 9 and 11 in Sections 4 and 5, resp.).

2. Continuous Piecewise Linear Functions

Throughout this paper, I is a closed real interval of positive length and $\mathring{I}$ is the interior set of I. If $I = [a, b]$, we denote by $\mathscr{N}$ a finite ordered set of nodes $a = x_{-(m+1)} < x_{-m} < \cdots < x_{-1} < x_0 < x_1 < \cdots < x_k < x_{k+1} = b$, for some $m, k = 0, 1, \dots$. If I is an infinite interval, $\mathscr{N}$ could also be infinite. In such a case, the finite endpoint of I, if any, is always in $\mathscr{N}$. We denote by $\mathscr{L}(I)$ the set of continuous piecewise linear functions $g : I \to \mathbb{R}$ whose set of nodes is $\mathscr{N}$. Unless otherwise specified, we assume from now on that $I = [a, b]$. Given a sequence $(c_i, i \in \mathbb{Z})$, we denote by $\delta c_i = c_{i+1} - c_i$, $i \in \mathbb{Z}$. We set $y_+ = \max(0, y)$, $y_- = \max(0, -y)$ and denote by 1_A the indicator function of the set A.

Lemma 1. *For any $g \in \mathscr{L}(I)$, one has the representations*

$$\begin{aligned} g(y) - g(x_0) &= c_1 (y - x_0)_+ - c_0 (y - x_0)_- \\ &\quad + \sum_{i=1}^{k} \delta c_i (y - x_i)_+ + \sum_{i=-m}^{-1} \delta c_i (y - x_i)_- \\ &= \frac{c_0 + c_1}{2}(y - x_0) + \frac{\delta c_0}{2}\left|y - x_0\right| \\ &\quad + \sum_{i=1}^{k} \delta c_i (y - x_i)_+ \\ &\quad + \sum_{i=-m}^{-1} \delta c_i (y - x_i)_-, \quad y \in I, \end{aligned} \tag{8}$$

where

$$c_i = \frac{g(x_i) - g(x_{i-1})}{x_i - x_{i-1}}, \quad i = -m, \dots, k+1. \tag{9}$$

Proof. The first equality in (8) follows from the fact that the two functions involved have the same Radon-Nikodym derivative in (x_{i-1}, x_i), $i = -m, \dots, k+1$, given by the constant c_i defined in (9). The second equality in (8) follows from the first one and the equalities

$$\begin{gathered} y_+ = \frac{1}{2}(|y| + y), \\ y_- = \frac{1}{2}(|y| - y), \\ y \in \mathbb{R}. \end{gathered} \tag{10}$$

The proof is complete. □

The following auxiliary result is taken from [5, Lemma 2.5.7] (see also [11]). We give a simple proof of it for the sake of completeness.

Lemma 2. *Let $f : I \to \mathbb{R}$ be a function such that $f(c) = f(d) = 0$, for some $c, d \in I$ with $c \le d$. Then,*

$$M := \sup_{c \le y \le d} \left|f(y)\right| \le \omega_2\left(f; \frac{d - c}{2}\right). \tag{11}$$

Proof. Assume that $y \in (c, (c+d)/2]$, the case $y \in [(c+d)/2, d)$ being similar. Set $\tilde{y} = y + (y-c) = 2y - c \in [c, d]$. Then,

$$|f(y)| = \frac{1}{2}|f(c) - 2f(y) + f(\tilde{y}) - f(\tilde{y})| \leq \frac{1}{2}\omega_2\left(f; \frac{d-c}{2}\right) + \frac{M}{2}. \quad (12)$$

The proof is complete. □

For any $0 < \varepsilon \leq (b-a)/3$, denote by $\mathscr{L}_\varepsilon(I)$ the set of functions in $\mathscr{L}(I)$ whose set of nodes $\mathscr{N}_\varepsilon = \{x_i : i = -(m+1), \ldots, k+1\}$ satisfies

$$\varepsilon \leq \min\{x_i - x_{i-1} : i = -m, \ldots, k+1\}. \quad (13)$$

Lemma 3. *Let $g \in \mathscr{L}_\varepsilon(I)$, for some $0 < \varepsilon \leq (b-a)/3$. Then,*

$$\omega_2(g; \tau) = \tau \max_{-m \leq i \leq k} |\delta c_i|, \quad 0 \leq \tau \leq \varepsilon. \quad (14)$$

Proof. Let $0 \leq h \leq \tau \leq \varepsilon$ and $i = -m, \ldots, k$. Denote by $s_i(y) = (y - x_i)_+$. We claim that

$$\Delta_h^2 s_i(y) = (h - |y - x_i|)_+ =: q_i(y), \quad y \in [a+h, b-h]. \quad (15)$$

Formula (15) is obvious if $|y - x_i| \geq h$; suppose that $|y - x_i| < h$. If $y \in (x_i - h, x_i] \cap [a+h, b-h]$, then

$$\Delta_h^2 s_i(y) = (y + h - x_i)_+ = (h - |y - x_i|)_+, \quad (16)$$

whereas if $y \in [x_i, x_{i+h}) \cap [a+h, b-h]$, then

$$\Delta_h^2 s_i(y) = -2(y - x_i) + y + h - x_i = (h - |y - x_i|)_+, \quad (17)$$

thus showing claim (15). By virtue of (10), formula (15) is also true if we replace s_i by any one of the functions $\bar{s}_i(y) = (y - x_i)_-$ or $\bar{\bar{s}}_i(y) = |y - x_i|/2$, $y \in I$. We therefore have from (8) and (15)

$$\Delta_h^2 g(y) = \sum_{i=-m}^{k} \delta c_i q_i(y), \quad y \in [a+h, b-h]. \quad (18)$$

Since $q_i(y) = 0$ for $y \in [a, x_{i-1}] \cup [x_{i+1}, b]$, as follows from (13), we have from (18)

$$\sup_{a \leq y \leq x_{-m}} |\Delta_h^2 g(y)| = \sup_{a \leq y \leq x_{-m}} |\delta c_{-m} q_{-m}(y)| = |\delta c_{-m}| q_{-m}(x_{-m}) = |\delta c_{-m}| h. \quad (19)$$

Similarly,

$$\sup_{x_k \leq y \leq b} |\Delta_h^2 g(y)| = |\delta c_k| h, \quad (20)$$

and, for $i = -m, \ldots, k-1$

$$\sup_{x_i \leq y \leq x_{i+1}} |\Delta_h^2 g(y)| = \sup_{x_i \leq y \leq x_{i+1}} |\delta c_i q_i(y) + \delta c_{i+1} q_{i+1}(y)| \leq \max(|\delta c_i|, |\delta c_{i+1}|)(q_i(y) + q_{i+1}(y)) = h \max(|\delta c_i|, |\delta c_{i+1}|), \quad (21)$$

thus showing that

$$\omega_2(g; \tau) \leq \tau \max_{-m \leq i \leq k} |\delta c_i|. \quad (22)$$

By assumption (13), $x_i \in [a+h, b-h]$. We thus have from (18)

$$\Delta_h^2 g(x_i) = \delta c_i q_i(x_i) = h \delta c_i. \quad (23)$$

This shows the converse inequality to (22) and completes the proof. □

Remark 4. If assumption (13) is dropped, Lemma 3 is no longer true. To see this, consider the function $s(y) = (y - x)_+$, $y \in I$, where $a < x < (b+a)/2$. Then,

$$\omega_2(s, \tau) = \min(\tau, x - a), \quad 0 \leq \tau \leq \frac{(b-a)}{2}. \quad (24)$$

Actually, let $0 \leq h \leq \tau$. If $x - a \leq h$, we have from (15)

$$\sup_{a+h \leq y \leq b-h} \Delta_h^2 s(y) = \sup_{a+h \leq y \leq b-h} (h - (y - x))_+ = x - a, \quad (25)$$

whereas if $h \leq x - a$, we have

$$\sup_{a+h \leq y \leq b-h} \Delta_h^2 s(y) = \sup_{a+h \leq y \leq b-h} (h - |y - x|)_+ = h = \Delta_h^2 s(x), \quad (26)$$

thus showing (24).

We close this section with the following auxiliary result concerning the symmetric functions

$$\varphi(y) = \sum_{i=1}^{\infty} (|y| - i)_+, \quad \psi(y) = \frac{1}{2}|y| + \varphi(y), \quad y \in \mathbb{R}. \quad (27)$$

For any $y \in \mathbb{R}$, let $\lfloor y \rfloor$ and $\lceil y \rceil$ be the floor and the ceiling of y, respectively; that is,

$$\lfloor y \rfloor = \sup\{k \in \mathbb{Z} : k \leq y\}, \quad \lceil y \rceil = \inf\{k \in \mathbb{Z} : k \geq y\}. \quad (28)$$

Lemma 5. *Let φ and ψ be as in (27). Then,*

$$\varphi(y) \le \frac{1}{2}y^2 1_{(1,\infty)}(|y|), \quad \psi(y) \le \frac{1}{2}y^2 + \frac{1}{8}, \quad y \in \mathbb{R}. \tag{29}$$

Proof. Let $y \ge 0$. Then,

$$\varphi(y) = \sum_{i=1}^{\lfloor y \rfloor}(y-i) = \frac{\lfloor y \rfloor}{2}(2y-(1+\lfloor y \rfloor)) \le \frac{\lfloor y \rfloor}{2} y 1_{(1,\infty)}(y) \le \frac{y^2}{2}1_{(1,\infty)}(y). \tag{30}$$

Thanks to (30), the second inequality in Lemma 5 is equivalent to

$$\eta(y) := \lfloor y \rfloor (2y-(1+\lfloor y \rfloor)) \le \left(y-\frac{1}{2}\right)^2 =: \nu(y), \quad y \ge 0. \tag{31}$$

It is easily checked that

$$\eta\left(m+\frac{1}{2}\right) = \nu\left(m+\frac{1}{2}\right), \quad \eta'\left(m+\frac{1}{2}\right) = \nu'\left(m+\frac{1}{2}\right), \quad m = 0,1,\ldots. \tag{32}$$

These equalities imply (31), since ν is convex and η is linear in each interval $[m, m+1]$, $m = 0,1,\ldots$. The proof is complete. □

3. Main Results

Denote by $\mathscr{C}(I)$ the set of convex functions in $\mathscr{M}(I)$. Given $0 < \varepsilon \le (b-a)/3$ and $x \in \mathring{I}$, we consider the set

$$\mathscr{A}_{\varepsilon,x} = \{x_i = x + i\varepsilon, i = -m,\ldots,k\}, \quad m = \left\lceil \frac{x-a}{\varepsilon} \right\rceil - 1, \; k = \left\lceil \frac{b-x}{\varepsilon} \right\rceil - 1. \tag{33}$$

If $I = [a,\infty)$, the preceding set should be defined as

$$\mathscr{A}_{\varepsilon,x} = \{x_i = x + i\varepsilon, i = -m,\ldots,0,1,2,\ldots\}, \quad m = \left\lceil \frac{x-a}{\varepsilon} \right\rceil - 1, \tag{34}$$

and analogously if $I = (-\infty, b]$ or $I = \mathbb{R}$. Observe that $x_{-m} \in (a, a+\varepsilon]$ and $x_k \in [b-\varepsilon, b)$. We define the function

$$g_{\varepsilon,x}(y) = \frac{r_0}{2}\frac{|y-x|}{\varepsilon} + \sum_{i=1}^{k} r_i \frac{(y-x_i)_+}{\varepsilon} + \sum_{i=-m}^{-1} r_i \frac{(y-x_i)_-}{\varepsilon}, \quad y \in I, \tag{35}$$

where

$$r_i = 1_{[a+\varepsilon, b-\varepsilon]}(x_i), \quad i = -m,\ldots,k. \tag{36}$$

Note that $g_{\varepsilon,x} \in \mathscr{L}_\varepsilon(I) \cap \mathscr{M}(I)$ and its set of nodes is

$$\mathscr{N}_{\varepsilon,x} = (\mathscr{A}_{\varepsilon,x} \cap [a+\varepsilon, b-\varepsilon]) \cup \{a,b\}. \tag{37}$$

If $x \in (a, a+\varepsilon)$, then $x = x_{-m}$ and therefore $x \notin \mathscr{N}_{\varepsilon,x}$. The same is true if $x \in (b-\varepsilon, b)$. Since $\varepsilon \le (b-a)/3$, we see that $\mathscr{A}_{\varepsilon,x} \cap [a+\varepsilon, b-\varepsilon] \ne \emptyset$ and therefore $\mathscr{N}_{\varepsilon,x}$ has at least three nodes. From (35), (36), and Lemma 3, we have

$$\omega_2(g_{\varepsilon,x};\tau) = \frac{\tau}{\varepsilon}, \quad 0 \le \tau \le \varepsilon. \tag{38}$$

Finally, let Y be a random variable taking values in I such that

$$EY = x, \quad E(Y-x)^2 < \infty. \tag{39}$$

Since $EY = x$, we have from (10)

$$E(Y-x)_+ = E(Y-x)_- = \frac{1}{2}E|Y-x|. \tag{40}$$

With these notations, we enunciate our first main result.

Theorem 6. *Let $0 < \varepsilon \le (b-a)/3$ and $x \in \mathring{I}$. Then one has the following.*

(a) *If $f \in \mathscr{M}(I)$, then*

$$|Ef(Y) - f(x)| \le \omega_2(f;\varepsilon)Eg_{\varepsilon,x}(Y) + \omega_2\left(f;\frac{\varepsilon}{2}\right) + \omega_2(f;x_{-m}-a)P(Y < x_{-m}) + \omega_2(f;b-x_k)P(Y > x_k). \tag{41}$$

(b) *If $f \in \mathscr{C}(I)$, then*

$$|Ef(Y) - f(x)| \le \omega_2(f;\varepsilon)Eg_{\varepsilon,x}(Y) + \left(\omega_2\left(f;\frac{\varepsilon}{2}\right) + \omega_2(f;x_{-m}-a)\right)P(Y < x_{-m}) + \left(\omega_2\left(f;\frac{\varepsilon}{2}\right) + \omega_2(f;b-x_k)\right)P(Y > x_k). \tag{42}$$

Proof. Fix $0 < \varepsilon \le (b-a)/3$ and $x \in \mathring{I}$. Let $\tilde{f} \in \mathscr{L}_\varepsilon(I)$ be the function having representation (8), whose set of nodes is $\mathscr{N}_{\varepsilon,x}$, as defined in (37), and satisfying the following properties:

(a) $\tilde{f}(x_i) = f(x_i)$, $x_i \in \mathscr{A}_{\varepsilon,x}$, $i = -m,\ldots,k$;

(b) if $x_{-m} = a+\varepsilon$, then $\tilde{f}$ is linear in $[a, a+\varepsilon]$ and $\tilde{f}(a) = f(a)$; if $x_{-m} \in (a, a+\varepsilon)$, then $\tilde{f}$ is linear in $[a, x_{-m+1}]$ (in such a case, it could happen that $\tilde{f}(a) \ne f(a)$);

(c) if $x_k = b - \varepsilon$, then $\tilde{f}$ is linear in $[b-\varepsilon, b]$ and $\tilde{f}(b) = f(b)$; if $x_k \in (b-\varepsilon, b)$, then $\tilde{f}$ is linear in $[x_{k-1}, b]$ (in such a case, it could happen that $\tilde{f}(b) \neq f(b)$).

These properties, together with (8) and (39), allow us to write

$$E\tilde{f}(Y) - f(x) = r_0 \frac{\delta c_0}{2} E|Y-x| + \sum_{i=1}^{k} r_i \delta c_i E(Y - x_i)_+ + \sum_{i=-m}^{-1} r_i \delta c_i E(Y-x_i)_-, \tag{43}$$

where r_i is defined in (36) and

$$\delta c_i = \frac{f(x_i+\varepsilon) - f(x_i)}{\varepsilon} - \frac{f(x_i) - f(x_i - \varepsilon)}{\varepsilon}, \quad x_i \in \mathscr{A}_{\varepsilon,x} \cap [a+\varepsilon, b-\varepsilon]. \tag{44}$$

We therefore have from (35) and (43)

$$\left|E\tilde{f}(Y) - f(x)\right| \leq \omega_2(f;\varepsilon) Eg_{\varepsilon,x}(Y). \tag{45}$$

On the other hand, applying Lemma 2 to the function $f^\star := f - \tilde{f}$, we have

$$\sup_{x_i \leq y \leq x_{i+1}} |f^\star(y)| \leq \omega_2\left(f; \frac{\varepsilon}{2}\right), \quad i = -m, \ldots, k-1. \tag{46}$$

If $y \in [a, x_{-m})$, we set $y^\star = x_{-m} + (x_{-m} - y) \in [x_{-m}, x_{-m+1}]$ and obtain thanks to (46)

$$|f^\star(y)| = |f^\star(y) - 2f^\star(x_{-m}) + f^\star(y^\star) - f^\star(y^\star)| \leq \omega_2(f; x_{-m} - a) + \omega_2\left(f; \frac{\varepsilon}{2}\right). \tag{47}$$

In the same way,

$$\sup_{x_k < y \leq b} |f^\star(y)| \leq \omega_2(f; b - x_k) + \omega_2\left(f; \frac{\varepsilon}{2}\right). \tag{48}$$

Thus, we have from (46)–(48)

$$\left|Ef(Y) - E\tilde{f}(Y)\right| \leq \omega_2\left(f; \frac{\varepsilon}{2}\right) + \omega_2(f; x_{-m} - a) P(Y < x_{-m}) + \omega_2(f; b - x_k) P(Y > x_k). \tag{49}$$

This, together with (45), shows part (a).

Suppose that $f \in \mathscr{C}(I)$. By substracting an affine function, if necessary, we can assume without loss of generality that $f(y) \geq f(x) = 0$, $y \in I$. The convexity of f implies that

$$f(y) \leq \tilde{f}(y) + \left(f(y) - \tilde{f}(y)\right)\left(1_{[a,x_{-m})}(y) + 1_{(x_k,b]}(y)\right), \quad y \in I. \tag{50}$$

We therefore have from (45), (47), and (48)

$$\begin{aligned} Ef(Y) &\leq E\tilde{f}(Y) + Ef^\star(Y)\left(1_{[a,x_{-m})}(Y) + 1_{(x_k,b]}(Y)\right) \\ &\leq \omega_2(f;\varepsilon) Eg_{\varepsilon,x}(Y) \\ &\quad + \left(\omega_2\left(f;\frac{\varepsilon}{2}\right) + \omega_2(f; x_{-m} - a)\right) P(Y < x_{-m}) \\ &\quad + \left(\omega_2\left(f;\frac{\varepsilon}{2}\right) + \omega_2(f; b - x_k)\right) P(Y > x_k). \end{aligned} \tag{51}$$

The proof is complete. □

Let $0 < \varepsilon \leq (b-a)/3$ and $x \in [a+\varepsilon, b-\varepsilon]$. Denote by $\mathscr{L}_{\varepsilon,x}(I)$ the set of functions $g \in \mathscr{L}_\varepsilon(I)$ having a node at x and not being linear in I (in other words, $\omega_2(g;\tau) > 0$, $\tau > 0$). It turns out that the function $g_{\varepsilon,x}$ defined in (35) is a maximal function in $\mathscr{L}_{\varepsilon,x}(I)$, as shown in the following result.

Theorem 7. *Let $0 < \varepsilon \leq (b-a)/3$ and $x \in [a+\varepsilon, b-\varepsilon]$. Then,*

$$\sup_{g \in \mathscr{L}_{\varepsilon,x}(I)} \frac{|Eg(Y) - g(x)|}{\omega_2(g;\varepsilon)} = Eg_{\varepsilon,x}(Y). \tag{52}$$

Proof. Let $g \in \mathscr{L}_{\varepsilon,x}(I)$ with representation (8) and set of nodes $a = \overline{x}_{-(\overline{m}+1)} < \overline{x}_{-\overline{m}} < \cdots < \overline{x}_{-1} < \overline{x}_0 = x < \overline{x}_1 < \cdots < \overline{x}_{\overline{k}} < \overline{x}_{\overline{k}+1} = b$, for some $\overline{m}, \overline{k} = 0, 1, \ldots$. From (39) and Lemma 3, we have

$$\begin{aligned} |Eg(Y) - g(x)| &= \left|\frac{\delta c_0}{2} E|Y-x| + \sum_{i=1}^{\overline{k}} \delta c_i E(Y - \overline{x}_i)_+ + \sum_{i=-\overline{m}}^{-1} \delta c_i E(Y - \overline{x}_i)_-\right| \leq \frac{\omega_2(g;\varepsilon)}{\varepsilon}\Bigg(E|Y-x| \\ &\quad + \sum_{i=1}^{\overline{k}} E(Y - \overline{x}_i)_+ + \sum_{i=-\overline{m}}^{-1} E(Y - \overline{x}_i)_-\Bigg). \end{aligned} \tag{53}$$

Let $g_{\varepsilon,x}$ be as in (35) with set of nodes $\mathscr{N}_{\varepsilon,x}$ as defined in (37). By assumption, $\overline{x}_i - \overline{x}_{i-1} \geq \varepsilon$, $i = -\overline{m}, \ldots, \overline{k}+1$. Therefore,

$$\begin{aligned} \overline{x}_i &\geq x_i, \quad i = 1, \ldots, \overline{k}, \\ \overline{x}_i &\leq x_i, \quad i = -\overline{m}, \ldots, -1. \end{aligned} \tag{54}$$

This implies, by virtue of (35) and (53), that

$$|Eg(Y) - g(x)| \leq \omega_2(g;\varepsilon) Eg_{\varepsilon,x}(Y). \tag{55}$$

This, in conjunction with (38), completes the proof. □

In order to apply Theorems 6 and 7 to concrete examples, we need to estimate the expectation $Eg_{\varepsilon,x}(Y)$ and the tail probabilities of the random variable Y under consideration. With regard to the first question, we give the following.

Theorem 8. *Let* $0 < \varepsilon \leq (b-a)/3$. *Then one has the following.*

(a) *If* $x \in [a+\varepsilon, b-\varepsilon]$, *then*

$$Eg_{\varepsilon,x}(Y) \leq \frac{1}{2}E\left(\frac{Y-x}{\varepsilon}\right)^2 + \frac{1}{8}. \tag{56}$$

(b) *If* $x \in \mathring{I} \setminus [a+\varepsilon, b-\varepsilon]$, *then*

$$Eg_{\varepsilon,x}(Y) \leq \frac{1}{2}E\left(\frac{Y-x}{\varepsilon}\right)^2 1_{(1,\infty)}\left(\left|\frac{Y-x}{\varepsilon}\right|\right). \tag{57}$$

Proof. Suppose that $x \in [a+\varepsilon, b-\varepsilon]$. Using definitions (33)–(36) and Lemma 5, we have

$$\begin{aligned} g_{\varepsilon,x}(y) &= \frac{1}{2}\frac{|y-x|}{\varepsilon} + \sum_{i=1}^{k} r_i\left(\frac{y-x}{\varepsilon} - i\right)_+ \\ &\quad + \sum_{i=-m}^{-1} r_i\left(\frac{y-x}{\varepsilon} - i\right)_- \leq \psi\left(\frac{y-x}{\varepsilon}\right) \\ &\leq \frac{1}{2}\left(\frac{y-x}{\varepsilon}\right)^2 + \frac{1}{8}, \quad y \in I. \end{aligned} \tag{58}$$

Part (a) follows by replacing y by Y in the preceding inequality and then taking expectations. Part (b) follows in a similar manner, by noting that if $x \in \mathring{I} \setminus [a+\varepsilon, b-\varepsilon]$, then

$$\begin{aligned} g_{\varepsilon,x}(y) &\leq \varphi\left(\frac{y-x}{\varepsilon}\right) \\ &\leq \frac{1}{2}\left(\frac{y-x}{\varepsilon}\right)^2 1_{(1,\infty)}\left(\left|\frac{y-x}{\varepsilon}\right|\right), \quad y \in I, \end{aligned} \tag{59}$$

as follows from Lemma 5. This completes the proof. □

Theorem 8 gives an upper bound for $Eg_{\varepsilon,x}(Y)$ in terms of the variance of the random variable Y, which is easy to compute in many usual examples. Such an upper bound also suggests the choice

$$\varepsilon = \varepsilon(x) = E^{1/2}(Y-x)^2, \quad x \in \mathring{I}. \tag{60}$$

4. Example 1: The Szàsz Operator

Let $(N_\lambda, \lambda \geq 0)$ be the standard Poisson process, that is, a stochastic process starting at the origin, having independent stationary increments such that

$$P(N_\lambda = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, 2, \ldots, \ \lambda \geq 0. \tag{61}$$

Let $n = 1, 2, \ldots$ and $x \geq 0$. Thanks to (61), the classical Szàsz-Mirakyan operator L_n can be written as

$$L_n f(x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\frac{(nx)^k e^{-nx}}{k!} = Ef\left(\frac{N_{nx}}{n}\right), \tag{62}$$

where $f \in \mathscr{M}([0,\infty))$. It is well known that

$$\begin{aligned} E\left(\frac{N_{nx}}{n}\right) &= x, \\ E\left(\frac{N_{nx}}{n} - x\right)^2 &= \frac{x}{n}. \end{aligned} \tag{63}$$

Concerning the tail probabilities of the standard Poisson process, we give the following lemma.

Lemma 9. *Let* $(N_\lambda, \lambda \geq 0)$ *be as in (61). Then,*

(a) *one has*

$$\begin{aligned} &\sup_{1\leq\lambda\leq4} P\left(N_\lambda < \lambda - \sqrt{\lambda}\right) \leq e^{-1}, \\ &\sup_{4<\lambda\leq9} P\left(N_\lambda < \lambda - 2\sqrt{\lambda}\right) \leq e^{-1}; \end{aligned} \tag{64}$$

(b) *for any* $\lambda > 1$, *one has*

$$\begin{aligned} P\left(N_\lambda < \sqrt{\lambda}\right) &\leq e^{-\tau(\lambda)}, \\ \tau(\lambda) &= \lambda - \sqrt{\lambda}\left(1 + \frac{1}{2}\log\lambda\right), \end{aligned} \tag{65}$$

$\tau(\lambda)$ *being strictly increasing in* $(1,\infty)$.

Proof. (a) Suppose that $1 \leq \lambda \leq 4$. Denote by $\lambda_0 = (3+\sqrt{5})/2$ the solution to the equation $\lambda - \sqrt{\lambda} = 1$. If $1 \leq \lambda \leq \lambda_0$, we obviously have from (61)

$$P\left(N_\lambda < \lambda - \sqrt{\lambda}\right) = P(N_\lambda = 0) = e^{-\lambda} \leq e^{-1}, \tag{66}$$

whereas if $\lambda_0 \leq \lambda \leq 4$, we have

$$\begin{aligned} P\left(N_\lambda < \lambda - \sqrt{\lambda}\right) &= P(N_\lambda \leq 1) = e^{-\lambda}(1+\lambda) \\ &\leq e^{-\lambda_0}(1+\lambda_0) = 0.263\cdots < e^{-1}. \end{aligned} \tag{67}$$

Suppose that $4 < \lambda \leq 9$. We have from (63) and Markov's inequality

$$\begin{aligned} P\left(N_\lambda < \lambda - 2\sqrt{\lambda}\right) &\leq P\left(|N_\lambda - \lambda| > 2\sqrt{\lambda}\right) \\ &\leq \frac{1}{4\lambda}E(N_\lambda - \lambda)^2 = \frac{1}{4} < e^{-1}. \end{aligned} \tag{68}$$

(b) Let $u > 0$. Again by Markov's inequality, we have

$$\begin{aligned} P\left(N_\lambda < \sqrt{\lambda}\right) &= P\left(e^{-uN_\lambda} > e^{-u\sqrt{\lambda}}\right) \leq e^{u\sqrt{\lambda}}Ee^{-uN_\lambda} \\ &= e^{\lambda e^{-u} - \lambda + u\sqrt{\lambda}}. \end{aligned} \tag{69}$$

It suffices to choose $u = \log\sqrt{\lambda}$ in the preceding inequality. The proof is complete. □

Theorem 10. *Let* L_n *be as in (62),* $n = 1, 2, \ldots$, *and let* $f \in \mathscr{M}([0,\infty))$. *Then,*

(a) *if* $0 < x < 1/n$, *then*

$$\left|L_n f(x) - f(x)\right| \le \frac{1}{2}\left(1 - nxe^{-nx}\right)\omega_2\left(f; \sqrt{\frac{x}{n}}\right) + \omega_2\left(f; \frac{1}{2}\sqrt{\frac{x}{n}}\right) + e^{-nx}\omega_2(f;x); \quad (70)$$

(b) *if* $1/n \le x \le 9/n$, *then*

$$\left|L_n f(x) - f(x)\right| \le \left(\frac{5}{8} + \frac{1}{e}\right)\omega_2\left(f; \sqrt{\frac{x}{n}}\right) + \omega_2\left(f; \frac{1}{2}\sqrt{\frac{x}{n}}\right); \quad (71)$$

(c) *if* $9/n < x$, *then*

$$\left|L_n f(x) - f(x)\right| \le \left(\frac{5}{8} + e^{-\tau(nx)}\right)\omega_2\left(f; \sqrt{\frac{x}{n}}\right) + \omega_2\left(f; \frac{1}{2}\sqrt{\frac{x}{n}}\right); \quad (72)$$

where $\tau(\cdot)$ *is defined in (65).*

Proof. For any $n = 1, 2, \ldots$ and $x > 0$, denote by $\varepsilon = \sqrt{x/n}$ and $\lambda = nx$. In view of (62), we will apply Theorems 6 and 8 with $Y = N_{nx}/n$.

(a) If $x < 1/n$, then $x \in (0, \varepsilon)$ and $x_{-m} = x$, as follows from (33). Thus, we have from Theorem 8(b) and (63)

$$Eg_{\varepsilon,x}\left(\frac{N_{nx}}{n}\right) \le \frac{1}{2}E\left(\frac{N_\lambda - \lambda}{\sqrt{\lambda}}\right)^2 1_{(1,\infty)}\left(\left|\frac{N_\lambda - \lambda}{\sqrt{\lambda}}\right|\right) \le \frac{1}{2}E\left(\frac{N_\lambda - \lambda}{\sqrt{\lambda}}\right)^2 1_{(0,\infty)}(N_\lambda) = \frac{1}{2}\left(1 - \lambda e^{-\lambda}\right), \quad (73)$$

as well as

$$P\left(\frac{N_{nx}}{n} < x_{-m}\right) = P(N_\lambda < \lambda) = P(N_\lambda = 0) = e^{-\lambda}. \quad (74)$$

Therefore, the conclusion follows from Theorem 6(a).

(b) If $1/n \le x \le 9/n$, we see that $x \in [\varepsilon, \infty)$. By Theorem 8(a) and (63), we have

$$Eg_{\varepsilon,x}\left(\frac{N_{nx}}{n}\right) \le \frac{1}{2}E\left(\frac{N_\lambda - \lambda}{\sqrt{\lambda}}\right)^2 + \frac{1}{8} = \frac{5}{8}. \quad (75)$$

If $x = 1/n$, then $x_{-m} = x$, as follows from (33). Thus,

$$P\left(\frac{N_{nx}}{n} < x_{-m}\right) = P(N_1 = 0) = e^{-1}. \quad (76)$$

If $1/n < x \le 4/n$, then $x_{-m} = x - \varepsilon$, again by (33). We therefore have from Lemma 9(a)

$$P\left(\frac{N_{nx}}{n} < x_{-m}\right) = P\left(N_\lambda < \lambda - \sqrt{\lambda}\right) < e^{-1}. \quad (77)$$

Similarly, if $4/n < x \le 9/n$, then $x_{-m} = x - 2\varepsilon$. Again by Lemma 9(a), we have

$$P\left(\frac{N_{nx}}{n} < x_{-m}\right) = P\left(N_\lambda < \lambda - 2\sqrt{\lambda}\right) < e^{-1}. \quad (78)$$

In any of the previous cases, we always have $x_{-m} \le \varepsilon$ and therefore

$$\omega_2(f; x_{-m}) \le \omega_2(f; \varepsilon). \quad (79)$$

In view of the preceding discussion, part (b) follows from Theorem 6(a).

(c) If $x > 9/n$, we have as in (75)

$$Eg_{\varepsilon,x}\left(\frac{N_{nx}}{n}\right) \le \frac{5}{8}. \quad (80)$$

As in part (b), $x_{-m} \le \varepsilon$ and inequality (79) holds. Also, we have from Lemma 9(b)

$$P\left(\frac{N_{nx}}{n} < x_{-m}\right) \le P\left(\frac{N_{nx}}{n} < \varepsilon\right) = P\left(N_\lambda < \sqrt{\lambda}\right) \le e^{-\tau(\lambda)}. \quad (81)$$

By Theorem 6(a), this shows part (c) and completes the proof. □

Theorem 10 could also be stated for functions $f \in \mathscr{C}([0,\infty))$ using Theorem 6(b) instead of Theorem 6(a). In such a case, we obtain better estimates. For instance, if $x > 9/n$, we get

$$\left|L_n f(x) - f(x)\right| \le \left(\frac{5}{8} + e^{-\tau(nx)}\right)\omega_2\left(f; \sqrt{\frac{x}{n}}\right) + e^{-\tau(nx)}\omega_2\left(f; \frac{1}{2}\sqrt{\frac{x}{n}}\right). \quad (82)$$

Observe that, for fixed $x > 0$, the constant $e^{-\tau(nx)}$ exponentially decreases to zero, as $n \to \infty$, as follows from (65).

5. Example 2: Bernstein Polynomials

Let $n = 1, 2, \ldots$ and let $(U_k)_{k\ge1}$ be a sequence of independent identically distributed random variables having the uniform distribution on $[0,1]$. We consider the (uniform) empirical process $(S_n(x)/n, 0 \le x \le 1)$ defined as

$$S_n(x) = \sum_{k=1}^{n} 1_{[0,x]}(U_k), \quad 0 \le x \le 1. \quad (83)$$

Observe that the random variable $S_n(x)$ has the binomial law with parameters n and x; that is,

$$P(S_n(x) = k) = \binom{n}{k} x^k (1-x)^{n-k}, \quad k = 0, 1, \ldots, n, \; 0 \le x \le 1. \quad (84)$$

Also observe that the paths of the empirical process are nondecreasing, since we have from (83)

$$S_n(x) \le S_n(y), \quad 0 \le x \le y \le 1. \tag{85}$$

It is well known that

$$E\left(\frac{S_n(x)}{n}\right) = x, \qquad E\left(\frac{S_n(x)}{n} - x\right)^2 = \frac{x(1-x)}{n}, \qquad 0 \le x \le 1. \tag{86}$$

For any function $f : [0,1] \to \mathbb{R}$, the Bernstein polynomials of f can be written as

$$B_n f(x) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k} = Ef\left(\frac{S_n(x)}{n}\right), \quad 0 \le x \le 1. \tag{87}$$

In view of (86), we define

$$\varepsilon_n(x) = \sqrt{\frac{x(1-x)}{n}}, \quad n = 3, 4, \ldots, \quad 0 < x < 1. \tag{88}$$

The following auxiliary result will be needed.

Lemma 11. *Let $n = 3, 4, \ldots$ and $0 < x < 1$. Let $\tau(\cdot)$ and $\varepsilon_n(x)$ be as in (65) and (88), respectively. Then,*

$$P\left(\frac{S_n(x)}{n} < \varepsilon_n(x)\right) \le e^{-\tau(nx)}, \tag{89}$$

$$P\left(\frac{S_n(x)}{n} > 1 - \varepsilon_n(x)\right) \le q_n(x) := \min\left(\frac{x}{\left(\sqrt{n(1-x)} - \sqrt{x}\right)^2}, e^{-\tau(n(1-x))}\right). \tag{90}$$

Proof. Let $u > 0$. By Markov's inequality and (84), we have

$$\begin{aligned} P\left(\frac{S_n(x)}{n} < \varepsilon_n(x)\right) &= P\left(e^{-uS_n(x)} > e^{-un\varepsilon_n(x)}\right) \\ &\le e^{un\varepsilon_n(x)} E e^{-uS_n(x)} \\ &= e^{u\sqrt{nx(1-x)}} \left(1 - x\left(1 - e^{-u}\right)\right)^n \\ &\le e^{u\sqrt{nx} - nx(1-e^{-u})}, \end{aligned} \tag{91}$$

where we have used the inequality

$$(1 + y) \le e^y, \quad y \in \mathbb{R}. \tag{92}$$

Inequality (89) follows by choosing $u = \log\sqrt{nx}$ in (91). On the other hand, the random variables $S_n(x)$ and $n - S_n(1-x)$ have the same law, as follows from (84). We therefore have from (88) and (89)

$$P\left(\frac{S_n(x)}{n} > 1 - \varepsilon_n(x)\right) = P\left(\frac{S_n(1-x)}{n} < \varepsilon_n(1-x)\right) \le e^{-\tau(n(1-x))}. \tag{93}$$

Again by Markov's inequality, (86), and (88), we get

$$\begin{aligned} P\left(\frac{S_n(x)}{n} - x > 1 - x - \varepsilon_n(x)\right) &\le \frac{1}{\left(1 - x - \varepsilon_n(x)\right)^2} E\left(\frac{S_n(x)}{n} - x\right)^2 \\ &= \frac{x}{\left(\sqrt{n(1-x)} - \sqrt{x}\right)^2}. \end{aligned} \tag{94}$$

This, together with (93), shows (90) and completes the proof. □

Denote by

$$r_n = \max\left(e^{-n/n+1}, \frac{7n-3}{2n-3} e^{-5n/2(n+1)}\right), \quad n = 3, 4, \ldots. \tag{95}$$

Numerical computations show that $r_n = e^{-n/n+1}$, for $n \ge 11$.

Theorem 12. *Let $n = 3, 4, \ldots$ and $x \in (0, 1/2]$. Let $\tau(\cdot)$, $q_n(x)$, and r_n be as in (65), (90), and (95), respectively. For any $f \in \mathscr{M}([0,1])$, one has the following.*

(a) *If $x < 1/(n+1)$, then*

$$\begin{aligned} \left|B_n f(x) - f(x)\right| &\le \left(\frac{1}{2}\left(1 - nx(1-x)^{n-1}\right) + q_n(x)\right) \omega_2\left(f; \varepsilon_n(x)\right) \\ &\quad + \omega_2\left(f; \frac{\varepsilon_n(x)}{2}\right) + (1-x)^n \omega_2(f; x). \end{aligned} \tag{96}$$

(b) *If $1/(n+1) \le x \le 9/(n+9)$, then*

$$\left|B_n f(x) - f(x)\right| \le \left(\frac{5}{8} + r_n + q_n(x)\right) \omega_2\left(f; \varepsilon_n(x)\right) + \omega_2\left(f; \frac{\varepsilon_n(x)}{2}\right). \tag{97}$$

(c) *If $9/(n+9) < x$, then*

$$\left|B_n f(x) - f(x)\right| \le \left(\frac{5}{8} + q_n(x) + e^{-\tau(nx)}\right) \omega_2\left(f; \varepsilon_n(x)\right) + \omega_2\left(f; \frac{\varepsilon_n(x)}{2}\right). \tag{98}$$

Proof. In view of (87), we will apply Theorems 6 and 8 with $Y = S_n(x)/n$ and $\varepsilon = \varepsilon_n(x)$, as defined in (88). In the first place, we have from (90)

$$\begin{aligned}\omega_2(f; 1-x_k) P\left(\frac{S_n(x)}{n} > x_k\right) &\le \omega_2(f; \varepsilon_n(x)) P\left(\frac{S_n(x)}{n} > 1-\varepsilon_n(x)\right) \\ &\le q_n(x)\,\omega_2(f; \varepsilon_n(x)).\end{aligned} \tag{99}$$

(a) If $x < 1/(n+1)$, then $x \in (0, \varepsilon_n(x))$ and $x_{-m} = x$, as follows from (33). Thus, we have from Theorem 8(b) and (86)

$$\begin{aligned} Eg_{\varepsilon,x}\left(\frac{S_n(x)}{n}\right) &\le \frac{1}{2}E\left(\frac{S_n(x)-nx}{\sqrt{nx(1-x)}}\right)^2 1_{(1,\infty)}\left(\left|\frac{S_n(x)-nx}{\sqrt{nx(1-x)}}\right|\right) \\ &\le \frac{1}{2}E\left(\frac{S_n(x)-nx}{\sqrt{nx(1-x)}}\right)^2 1_{(0,\infty)}(S_n(x)) \\ &= \frac{1}{2}\left(1-nx(1-x)^{n-1}\right),\end{aligned} \tag{100}$$

and also

$$\begin{aligned}\omega_2(f; x_{-m}) P\left(\frac{S_n(x)}{n} < x_{-m}\right) &= \omega_2(f;x) P(S_n(x)=0) = (1-x)^n \omega_2(f;x).\end{aligned} \tag{101}$$

This, together with (99) and (100), shows part (a).

(b) If $1/(n+1) \le x \le 9/(n+9)$, see that $x \in [\varepsilon_n(x), 1/2]$. By Theorem 8(a) and (86), we have

$$Eg_{\varepsilon,x}\left(\frac{S_n(x)}{n}\right) \le \frac{1}{2}E\left(\frac{S_n(x)-nx}{\sqrt{nx(1-x)}}\right)^2 + \frac{1}{8} = \frac{5}{8}. \tag{102}$$

We distinguish the following subcases.

Case 1. One has $x = 1/(n+1)$. In this case, $x_{-m} = x$, as follows from (33). We thus have from (92)

$$\begin{aligned} P\left(\frac{S_n(x)}{n} < x_{-m}\right) &= P\left(S_n\left(\frac{1}{n+1}\right) = 0\right) \\ &= \left(1-\frac{1}{n+1}\right)^n \le e^{-n/(n+1)} \le r_n.\end{aligned} \tag{103}$$

Case 2. One has $1/(n+1) < x \le 4/(n+4)$. Then $x_{-m} = x - \varepsilon_n(x)$, again by (33), and therefore

$$P\left(\frac{S_n(x)}{n} < x_{-m}\right) = P(S_n(x) < nx - n\varepsilon_n(x)). \tag{104}$$

Let $\overline{x}_0$ be the solution in $(1/(n+1), 1/2]$ to the equation $nx - n\varepsilon_n(x) = 1$; that is,

$$\overline{x}_0 = \frac{3+\sqrt{5-4/n}}{2(n+1)}. \tag{105}$$

If $1/(n+1) < x < \overline{x}_0$, we have from (104)

$$\begin{aligned} P\left(\frac{S_n(x)}{n} < x_{-m}\right) &= P(S_n(x)=0) \le \left(1-\frac{1}{n+1}\right)^n \\ &\le e^{-n/(n+1)} \le r_n.\end{aligned} \tag{106}$$

If $n=3$, then $\overline{x}_0 > 1/2$. If $n = 4, 5, \ldots$, then $\overline{x}_0 \le 1/2$ (actually, $\overline{x}_0 = 1/2$, for $n=4$). Thus, we can assume without loss of generality that $n \ge 4$. In such a case,

$$\overline{x}_0 \ge \frac{5}{2(n+1)} =: x_0. \tag{107}$$

If $\overline{x}_0 \le x \le 4/(n+4)$, we have from (85), (92), and (107)

$$\begin{aligned} P\left(\frac{S_n(x)}{n} < x_{-m}\right) &= P(S_n(x) \le 1) \\ &\le P(S_n(x_0) \le 1) \\ &= (1-x_0)^n + nx_0(1-x_0)^{n-1} \\ &= \left(1-\frac{5}{2(n+1)}\right)^n \frac{7n-3}{2n-3} \\ &\le \frac{7n-3}{2n-3}e^{-5n/(2(n+1))} \le r_n.\end{aligned} \tag{108}$$

Case 3. One has $4/(n+4) < x \le 9/(n+9)$. Again by (33), we see that $x_{-m} = x - 2\varepsilon_n(x)$. Therefore, we have from Markov's inequality and (86)

$$\begin{aligned} P\left(\frac{S_n(x)}{n} < x_{-m}\right) &= P(S_n(x) - nx < -2n\varepsilon_n(x)) \\ &\le P(|S_n(x)-nx| > 2n\varepsilon_n(x)) \\ &\le \frac{1}{4n^2\varepsilon_n^2(x)}E(S_n(x)-nx)^2 = \frac{1}{4} \\ &\le e^{-n/(n+1)} \le r_n.\end{aligned} \tag{109}$$

The preceding discussion shows that

$$\omega_2(f; x_{-m}) P\left(\frac{S_n(x)}{n} < x_{-m}\right) \le r_n \omega_2(f; \varepsilon_n(x)). \tag{110}$$

This, in conjunction with (99) and (102), shows part (b).

(c) If $9/(n+9) < x$, we have as in part (b)

$$Eg_{\varepsilon,x}\left(\frac{S_n(x)}{n}\right) \le \frac{5}{8}. \tag{111}$$

By (89), we have

$$\begin{aligned}\omega_2(f; x_{-m}) P\left(\frac{S_n(x)}{n} < x_{-m}\right) &\le \omega_2(f; \varepsilon_n(x)) P\left(\frac{S_n(x)}{n} < \varepsilon_n(x)\right) \\ &\le e^{-\tau(nx)}\omega_2(f; \varepsilon_n(x)).\end{aligned} \tag{112}$$

Therefore, part (c) follows from (99). The proof is complete. □

Remark 13. For $x \in [1/2, 1)$, Theorem 12 remains true if we replace x by $1-x$ in the right-hand sides of the corresponding inequalities. This is due to the fact that

$$B_n f(x) - f(x) = B_n g(1-x) - g(1-x), \quad g(y) = f(1-y), \quad y \in [0,1]. \tag{113}$$

Theorem 12 can also be stated for convex functions, the estimates being better. For instance, if $f \in \mathscr{C}([0,1])$ and $x \in (9/(n+9), 1/2]$, then

$$\begin{aligned} &\left|B_n f(x) - f(x)\right| \\ &\quad \le \left(\frac{5}{8} + q_n(x) + e^{-\tau(nx)}\right)\omega_2\left(f; \varepsilon_n(x)\right) \\ &\quad + \left(q_n(x) + e^{-\tau(nx)}\right)\omega_2\left(f; \frac{\varepsilon_n(x)}{2}\right). \end{aligned} \tag{114}$$

The proof of (114) follows along the lines of that in Theorem 12(c), using Theorem 6(b) instead of Theorem 6(a).

Finally, to illustrate the size of the constants in Theorem 12, consider the Lipschitz class $L_\alpha([0,1])$ consisting of those functions in $\mathscr{M}([0,1])$ such that $\omega_2(f;\delta) \le \delta^\alpha$, $\delta \ge 0$, $\alpha \in (0,2]$.

Corollary 14. *Let $x \in (0,1)$ and $\alpha \in (0,2]$. If $f \in L_\alpha([0,1])$, then*

$$\overline{\lim_{n\to\infty}} \frac{\left|B_n f(x) - f(x)\right|}{\left(\varepsilon_n(x)\right)^\alpha} \le \frac{5}{8} + \frac{1}{2^\alpha}. \tag{115}$$

If, in addition, $f \in \mathscr{C}([0,1]) \cap L_\alpha([0,1])$, then

$$\overline{\lim_{n\to\infty}} \frac{\left|B_n f(x) - f(x)\right|}{\left(\varepsilon_n(x)\right)^\alpha} \le \frac{5}{8}. \tag{116}$$

Proof. Taking into account Remark 13, inequalities (115) and (116) readily follow from Theorems 12(c) and (114), respectively. □

Observe that the asymptotic constant in (115) is less than or equal to 1 if

$$\alpha \ge \frac{\log 8 - \log 3}{\log 2} = 1.415\cdots. \tag{117}$$

Also, suppose that $g \in \mathscr{L}_{\varepsilon_n(x),x}([0,1])$, where $n = 3, 4, \ldots$, $\varepsilon_n(x)$ is defined in (88), and $x \in [\varepsilon_n(x), 1-\varepsilon_n(x)]$ or, equivalently, $1/(n+1) \le x \le n/(n+1)$. Then, it follows from Theorems 7 and 8(a) that

$$\left|B_n g(x) - g(x)\right| \le \frac{5}{8}\omega_2\left(g; \varepsilon_n(x)\right). \tag{118}$$

To close the paper, let us mention some known results concerning the Bernstein polynomials. Gonska and Zhou [12] showed that there exists a constant $0 < c < 1$ such that, for any $1/2 \le a < 1$, there exists $N(a)$ such that

$$\sup_{1-a\le k/n<a} \left|B_n f\left(\frac{k}{n}\right) - f\left(\frac{k}{n}\right)\right| \le c\,\omega_2\left(f; \frac{1}{\sqrt{n}}\right), \quad n \ge N(a). \tag{119}$$

On the other hand, Kacsó [13] showed that if $f \in \mathscr{C}([0,1])$, then

$$\begin{aligned} &B_n f\left(\frac{k}{\lfloor\sqrt{n}\rfloor + 1}\right) - f\left(\frac{k}{\lfloor\sqrt{n}\rfloor + 1}\right) \\ &\quad \le \frac{5}{8}\omega_2\left(f; \frac{1}{\sqrt{n}}\right). \end{aligned} \tag{120}$$

Finally, Păltănea [5, Corollary 4.2.1, pp. 93-94] has obtained the uniform estimate for $x \in (0,1)$

$$\left|B_n f(x) - f(x)\right| \le \frac{11}{8}\omega_2\left(f; \varepsilon_n(x)\right), \quad f \in \mathscr{M}([0,1]), \; n = 1, 2, \ldots. \tag{121}$$

Theorem 12(c) and inequality (115) complete in certain sense inequality (119), with $\omega_2(f; 1/\sqrt{n})$ replaced by $\omega_2(f;\varepsilon_n(x))$. Similarly, formulas (114), (116), and (118) add new information to inequality (120). Finally, Theorem 12 and Corollary 14 complete the uniform estimate in (121).

Conflict of Interests

The authors declare that there is no conflict of interests.

Acknowledgments

The authors are partially supported by Research Project MTM2011-23998 and by FEDER funds.

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