On heat properties of AdS black holes in higher dimensions

A. Belhaj,[a,b] **M. Chabab,**[b,1] **H. El Moumni,**[b] **K. Masmar,**[b] **M.B. Sedra**[c,d] **and A. Segui**[e]

[a]*Département de Physique, Faculté Polydisciplinaire, Université Sultan M. Slimane,*
Béni Mellal, Morocco

[b]*High Energy Physics and Astrophysics Laboratory, FSSM, Cadi Ayyad University,*
Marrakesh, Morocco

[c]*Université Mohammed Premier, Ecole Nationale des Sciences Appliquées,*
Al Hoceima, Morocco

[d]*Département de Physique, LHESIR, Faculté des Sciences, Université Ibn Tofail,*
Kénitra, Morocco

[e]*Departamento de Física Teórica, Universidad de Zaragoza,*
E-50009-Zaragoza, Spain

E-mail: belhaj@unizar.es, mchabab@uca.ma, hasan.elmoumni@edu.uca.ma,
karima.masmar@edu.uca.ma, msedra@ictp.it, segui@unizar.es

ABSTRACT: We investigate the heat properties of AdS Black Holes in higher dimensions. We consider the study of the corresponding thermodynamical properties including the heat capacity explored in the determination of the black hole stability. In particular, we compute the heat latent. To overcome the instability problem, the Maxwell construction, in the (T, S)-plane, is elaborated. This method is used to modify the the Hawking-Page phase structure by removing the negative heat capacity regions. Then, we discuss the thermodynamic cycle and the heat engines using the way based on the extraction of the work from a black hole solution.

----

[1]Corresponding author.

Contents

1 Introduction

Recently, many efforts have been devoted to study the thermodynamical properties of black holes, in connection with higher dimensional supergravity models [1, 2]. These properties have been extensively studied via different methods including numerical computation using various codes [3]. In fact, several models based on mathematical methods have been explored to study critical behaviors of black holes having different geometrical configurations in arbitrary dimensions. A particular emphasis has been put on AdS black holes [4–10]. More precisely, a nice interplay between the behaviors of the RN-AdS black hole systems and the Van der Waals fluids has been shown [11–15]. In this context, several landmarks of statistical liquid-gas systems, such as the P-V criticality, the Gibbs free energy, the first order phase transition and the behavior near the critical points have been derived. Also, in arbitrary dimensions of the spacetime, the authors [11, 13] studied the critical behaviors of charged RN-AdS black holes. Extension to other solutions considered as a subject of interest in gravity theory, has also been performed and their corresponding phase transitions and statistical properties have been investigated using different approaches [16–19]. More recently, some authors have worked out the heat properties of AdS charged black holes and their solutions in four dimensions [20, 21].

The aim in this paper is to reconsider the heat properties of AdS black holes in higher spacetime dimensions. More precisely, we will study the corresponding thermodynamical properties including the sign of the heat capacity explored when discussing the stability problem. In particular, we will derive the expression of the latent heat considered as a trivial consequence of the Hawking-Page phase transition. To overcome the instability problem, the Maxwell construction in (T, S) plane is then elaborated to modify the Hawking-Page phase structure [19, 20]. Finally, we will discuss the thermodynamic cycle and the holographic heat engines.

The paper is organized as follows. In section 2, we reconsider the study of thermodynamics of AdS black holes along with the latent heat. Section 3 is devoted to Maxwell's

construction of higher dimensional AdS black holes. In section 4, we discuss the thermodynamical cycle and holographic heat engines. Finally, section 5 contains our conclusions.

2 Thermodynamics and latent heat

This section concerns the study of the latent heat properties of Ads black holes in higher dimensions. This investigation could be supported by the existence of higher dimensional supergravity theories including superstring models, M-theory, and related topics. Here, we consider a non-rotating, neutral, asymptotically anti-de Sitter black holes in high dimensions spacetime $n \geq 4$. The corresponding metric solution reads as

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{n-2}^2 \tag{2.1}$$

where $d\Omega_{n-2}^2$ represents the volume of $(n-2)$-dimensional sphere. This solution is characterized by the function $f(r)$ taking the following general form

$$f(r) = 1 - \frac{2M}{r^{(n-3)}} + \frac{r^2}{\ell^2}. \tag{2.2}$$

It is worth recalling that the parameter M indicates the ADM mass of such a black hole solution while its horizon radius r_+ can be identified to the largest real root of $f(r) = 0$. These two parameters are linked via the relation

$$M = \frac{r_+^{n-3}}{2}\left(\frac{r_+^2}{\ell^2} + 1\right). \tag{2.3}$$

In the non-rotating AdS black holes, M should be interpreted as an enthalpy [4] which can be written as follows

$$H(S, P) = \frac{1}{2}\left(\frac{4S}{\omega}\right)^{\frac{n-3}{n-2}}\left(\frac{16\pi P}{(n-2)(n-1)}\left(\frac{4S}{\omega}\right)^{\frac{2}{n-2}} + 1\right). \tag{2.4}$$

where the Bekenstein-Hawking entropy is given in terms of the horizon

$$S = \frac{\omega}{4}r_+^{n-2}. \tag{2.5}$$

In this equation, the quantity ω reads as

$$\omega = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)}. \tag{2.6}$$

In fact, many other thermodynamical quantities can be also computed using similar technics. Indeed, the pressure can be associated with the cosmological constant Λ and they are related as $P = -\frac{\Lambda}{8\pi} = \frac{(n-2)(n-1)}{16\ell^2\pi}$, whereas the temperature is given in terms of the horizon radius r_+ via the following form

$$T = \frac{f'(r_+)}{4\pi} = -\frac{n(n-5) - 2r_+^2\Lambda + 6}{4\pi r_+(2-n)}. \tag{2.7}$$

By combining thermodynamical relations, the temperature can be re-written as

$$T = \left(\frac{\partial H}{\partial S}\right)_P = \frac{4^{\frac{1}{2-n}}\left(n^2 - 5n + 6\right)\left(\frac{S}{\omega}\right)^{\frac{1}{2-n}} + \pi 2^{\frac{2}{n-2}+4}P\left(\frac{S}{\omega}\right)^{\frac{1}{n-2}}}{4\pi(n-2)}. \tag{2.8}$$

One can see that the temperature presents a minimum associated with the following entropy value

$$S_{T_{\min}} = \pi^{\frac{2-n}{2}}\omega\left(\frac{2^{\frac{2(1-n)}{n-2}}\sqrt{n^2 - 5n + 6}}{\sqrt{P}}\right)^{n-2}. \tag{2.9}$$

This minimum is given by

$$T_{\min} = \frac{2(n-3)\sqrt{P}}{\sqrt{\pi}\sqrt{(n-3)(n-2)}}. \tag{2.10}$$

A similar computation shows that the heat capacity can be expressed as

$$C_p = \left(T\frac{\partial S}{\partial T}\right)_P = \frac{(n-2)S\left(\pi\, 2^{\frac{4}{n-2}+4}P\left(\frac{S}{\omega}\right)^{\frac{2}{n-2}} + n^2 - 5n + 6\right)}{\pi\, 2^{\frac{4}{n-2}+4}P\left(\frac{S}{\omega}\right)^{\frac{2}{n-2}} - n^2 + 5n - 6} \tag{2.11}$$

Note that this quantity is negative for $S < S_{T_{\min}}$. It becomes positive for $S > S_{T_{\min}}$, but diverges at $S = S_{T_{\min}}$.

Besides, recalling that the variation of Gibbs free energy G is

$$dG = -SdT + PdV, \tag{2.12}$$

and knowing that G is the Legendre transform of the enthalpy, one finds

$$G = H - TS = \frac{4^{\frac{n-1}{2-n}}}{\pi}S(T,P)\left(\frac{S(T,P)}{\omega}\right)^{\frac{1}{2-n}}\left(1 - \frac{\pi 4^{\frac{2(1-n)}{2-n}}P\left(\frac{S(T,P)}{\omega}\right)^{\frac{2}{n-2}}}{n^2 - 3n + 2}\right). \tag{2.13}$$

where the entropy function given in terms of T and P reads as

$$S(T,P) = 2^{4-5n}(\pi P)^{2-n}\omega\left[2^{\frac{2n}{n-2}}(n-2)\pi T\right.$$

$$\left. \pm\sqrt{\pi}\sqrt{(n-2)\left(\pi 2^{\frac{4n}{n-2}}(n-2)T^2 - 2^{\frac{8}{n-2}+6}(n-3)P\right)}\right]^{n-2} \tag{2.14}$$

Let

$$\mathcal{B} = \frac{(n^2 - 3n + 2)\omega^{\frac{2}{n-2}}}{\pi\, 4^{\frac{2(1-n)}{2-n}}} \tag{2.15}$$

Then, from eq. (2.13) we see that for $PS^{\frac{2}{n-2}} > \mathcal{B}$ the Gibbs free energy is negative, hence the black hole is a more stable thermodynamical configuration than the Anti-de Sitter one.
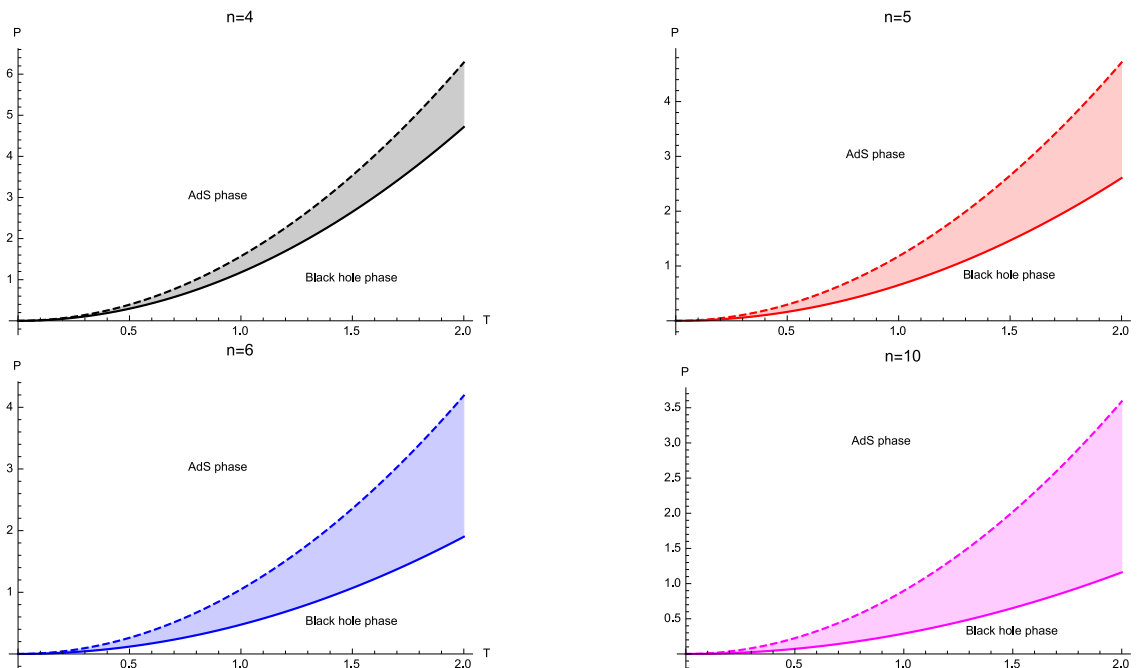
Figure 1. Phase diagram for higher dimensional AdS black holes. The coexistence line of the AdS-Radiation / Black hole phase transition of such a system in (P,T) plane.

For $PS^{\frac{2}{n-2}} < \mathcal{B}$, however, the pure AdS space-time is the more stable which means that the black hole with $S < \left(\frac{\mathcal{B}}{P}\right)^{\frac{n-2}{2}}$ will evaporate. This is associated with the line where the Hawking-Page transition occurs. This line, dubbed coexistence line, is defined by the following equation

$$S = 4^{1-n} \left(n^2 - 3n + 2\right)^{\frac{n}{2}-1} \pi^{1-\frac{n}{2}} \omega P^{1-\frac{n}{2}}. \tag{2.16}$$

where the pressure $P|_{\text{coexistance}}$ can be computed in terms of the temperature for any dimension thanks to equations (2.14) and (2.16),

$$P|_{\text{coexistance}} = \begin{cases} \dfrac{3\pi T^2}{8}, & n = 4 \\ \dfrac{2\left(32 - 85\sqrt[3]{2} + 168\ 2^{2/3}\right)\pi T^2}{1849}, & n = 5 \\ \dfrac{80\pi T^2}{529}, & n = 6 \\ \quad\vdots \\ \dfrac{567\pi T^2}{6241}, & n = 10. \\ \quad\vdots \end{cases} \tag{2.17}$$

This is illustrated in figure 1 which clearly indicates the existence of the two phases.

This figure shows the coexistence curve of the Hawking-Page phase transition, represented by the lower (solid) line. It also indicate that the heat capacity diverges on the upper (dashed) line while the lower branch of the free energy goes to minus infinity on the line given by $P = 0$.
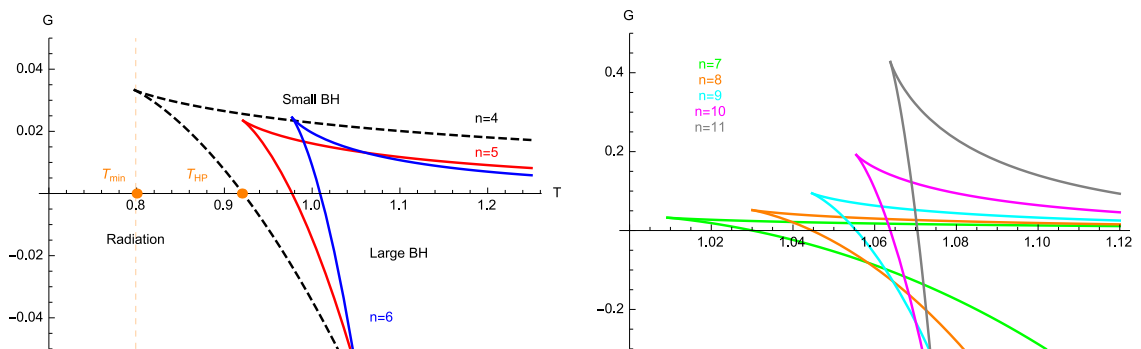
Figure 2. Left: the Gibbs free energy as a function of temperature at fixed $P = 1$ for $4 \leq n \leq 6$. Right: higher dimensional cases associated with $7 \leq n \leq 11$ Schwarzschild-AdS black hole.

Next, we would like to address the stability issue of such solutions. To do so, we plot in figure 2 the behavior of the Gibbs free energy G in terms of the temperature T at fixed pressure.

One can notice a minimum temperature $T_{\min}$ for which no black holes with $T < T_{\min}$ can survive. However, above this temperature, two branches of the black holes appear. In fact, the upper branch describes an unstable small (Schwarzschild-like) black hole involving a negative specific heat. For $(T > T_{\min})$, the black holes, at lower branch, become stable with positive specific heat. In addition since the Hawking-Page temperature T_{HP} is associated with vanishing values of the Gibbs free energy, then the black hole Gibbs free energy becomes negative for $T > T_{HP}$. In fact, at $T = T_{HP}$, a first order Hawking-Page phase transition shows up between the thermal radiations and large black holes as reported in [20, 22, 23].

Moreover, from figure 1 we also note a jump in entropy which becomes more relevant in terms of the dimension of the space time as shown in the following equation

$$\Delta S = 4^{1-n} \left(n^2 - 3n + 2 \right)^{\frac{n}{2}-1} \pi^{1-\frac{n}{2}} \omega P^{1-\frac{n}{2}}. \tag{2.18}$$

Notice that the latent heat of the black hole which nucleates from anti-de Sitter space time can be computed using the following thermodynamical expression

$$L = T\Delta S. \tag{2.19}$$

Indeed, using eq. (2.8), the calculation results in

$$L = \frac{2^{-\frac{n}{n-2}}(n-2)\omega}{\pi} \left(4^{1-n}\pi^{1-\frac{n}{2}} \left(\frac{n^2 - 3n + 2}{P} \right)^{\frac{n-2}{2}} \right)^{\frac{n-3}{n-2}} \tag{2.20}$$

which is equal to the mass on the coexistence curve in the black hole phase. Here, note that for $n = 4$, we reproduce the result reported in [20]. It turns out that the latent heat is nonzero for any finite T and vanishes for very large values of T. In fact, in the case of asymptotically flat space-time with $P = 0$, the latent heat becomes infinite which means that the black hole cannot nucleate in Minkowski space [20].
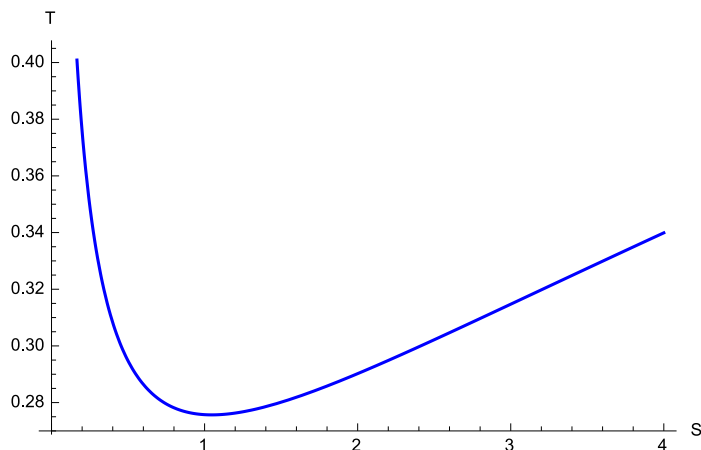
Figure 3. The temperature as function of the entropy in four dimension, with $P = 1$.

It is clear now that the sign of the heat capacity plays an important role in the determination of the stability of the black hole. More precisely, its negative values render the black hole unstable. In the next section we will show how to overcome this problem, by using the Maxwell construction to modify the Hawking-Page phase structure.

3 On Maxwell's construction of high dimensional AdS black holes

In this section, we investigate the corresponding Maxwell's construction. To do so, we first recall that the equal area law was introduced by Maxwell in order to explain the experimental behaviors of real fluids. Usually, this construction is elaborated in the (P, V) plane while keeping constant temperature [16, 18, 19]. However, fixing the pressure in (2.12), such a construction can also be done in the (T, S) plane. For Schwarzschild-AdS black hole, the choice of this plane has been explained in many papers including [16, 17, 22]. The starting point is the temperature as a un function of the entropy given by eq. (2.8). Then, we plot this function in figure 3. It is observed that this function involves minimums at $S_{T_{\min}}$ and $T_{\min}$ given by eq. (2.9) and eq. (2.10) respectively. For any value of the cosmological constant, these quantities produce the following reduced forms

$$t = \frac{1}{2}s^{\frac{1}{n-2}} + \frac{1}{2}\frac{1}{s^{\frac{1}{n-2}}}, \tag{3.1}$$

$$s \equiv \frac{S}{S_{T_{\min}}} \quad \text{and} \quad t \equiv \frac{T}{T_{\min}}. \tag{3.2}$$

It is recalled that the Maxwell area law can be obtained using the fact the Gibbs free energy is the same for coexisting black holes. Exploring Gibbs free energy (2.13)

$$\Delta G_{0,g} = -\int_1^2 S dT = 0, \tag{3.3}$$

$$T^\star \Delta S_{0,g} = \int_0^g T dS, \tag{3.4}$$

n	x_0	x_g	s_0	s_g	$t^\star$
4	0.50000	1.30277	0.25	1.69722	1.03518
5	0.62996	1.19213	0.25	1.69424	1.01548
6	0.70710	1.14071	0.25	1.69319	1.00868
7	0.75785	1.11100	0.25	1.69271	1.00555
8	0.79370	1.09165	0.25	1.69244	1.00385
9	0.82033	1.07805	0.25	1.69229	1.00283
10	0.84089	1.06796	0.25	1.69218	1.00216
11	0.85724	1.06018	0.25	1.69211	1.00171

Table 1. Roots of the polynomial form and the corresponding entropy and temperature.

where $T^\star$ is the temperature of the equal area isotherm. The equal area law, in the reduced variables, gives the entropy of the liquid and gaseous phases. This solves the following equations

$$
\begin{cases}
t = \frac{1}{2s^{\frac{1}{n-2}}} \left(s^{\frac{2}{n-2}} + 1 \right) \\
t^\star = \frac{(n-2)^2}{(n-3)(n-1)} \frac{s^{\frac{n-3}{n-2}} - s_0^{\frac{n-3}{n-2}}}{s_0 - s}.
\end{cases}
\tag{3.5}
$$

Introducing new variable $x \equiv s^{\frac{1}{n-2}}$, we get the following equation

$$
2^{\frac{2}{n-2}+2}(n-3)x^{n+2} - 2^{\frac{2}{n-2}+2}(n-1)x^n - 2^{\frac{2}{n-2}}(n-3)(n-1)x^4
$$
$$
+ \left(n \left(2^{\frac{4}{n-2}}(n-3) + n - 5 \right) + 2^{\frac{n+2}{n-2}} + 6 \right) x^3 - 2^{\frac{2}{n-2}}(n-3)(n-1)x^2 = 0.
\tag{3.6}
$$

The solutions of this equation associated with each dimension are listed in table 1,

We should eliminate the states corresponding to either complex or negatives values since they have no physical meaning. In the (T, S) plane (or equivalently in (t, x)), the system involves similar behaviors associated with the unstable (unphysical) part of the Van der Waals' picture in the (P, V) plane. Roughly, in 4 we show the Maxwell's equal area in the (t, x)-plane for high dimensional Schwarzschild-Ads black hole.

It is observed that a pure radiation phase can survive beyond $T_{\min}$ up to the higher temperature given by the isotherm $T^\star = t^\star T_{\min}$. For $T = T^\star$, the black holes with different entropy values have the same free energy. They are more stable than the pure radiation. For $T > T_{\min}$, there exists a single and stable black hole with a positive heat capacity. When we go back to four dimensions, we recover the results reported in [16] and [17] describing neutral case.

4 Thermodynamic cycle and the heat engines

Having discussed some thermodynamical properties of the Schwarzschild-AdS black hole including the thermodynamical quantities associated with stability and phases transitions, we pave the way to the study of the corresponding work from the heat energy according to
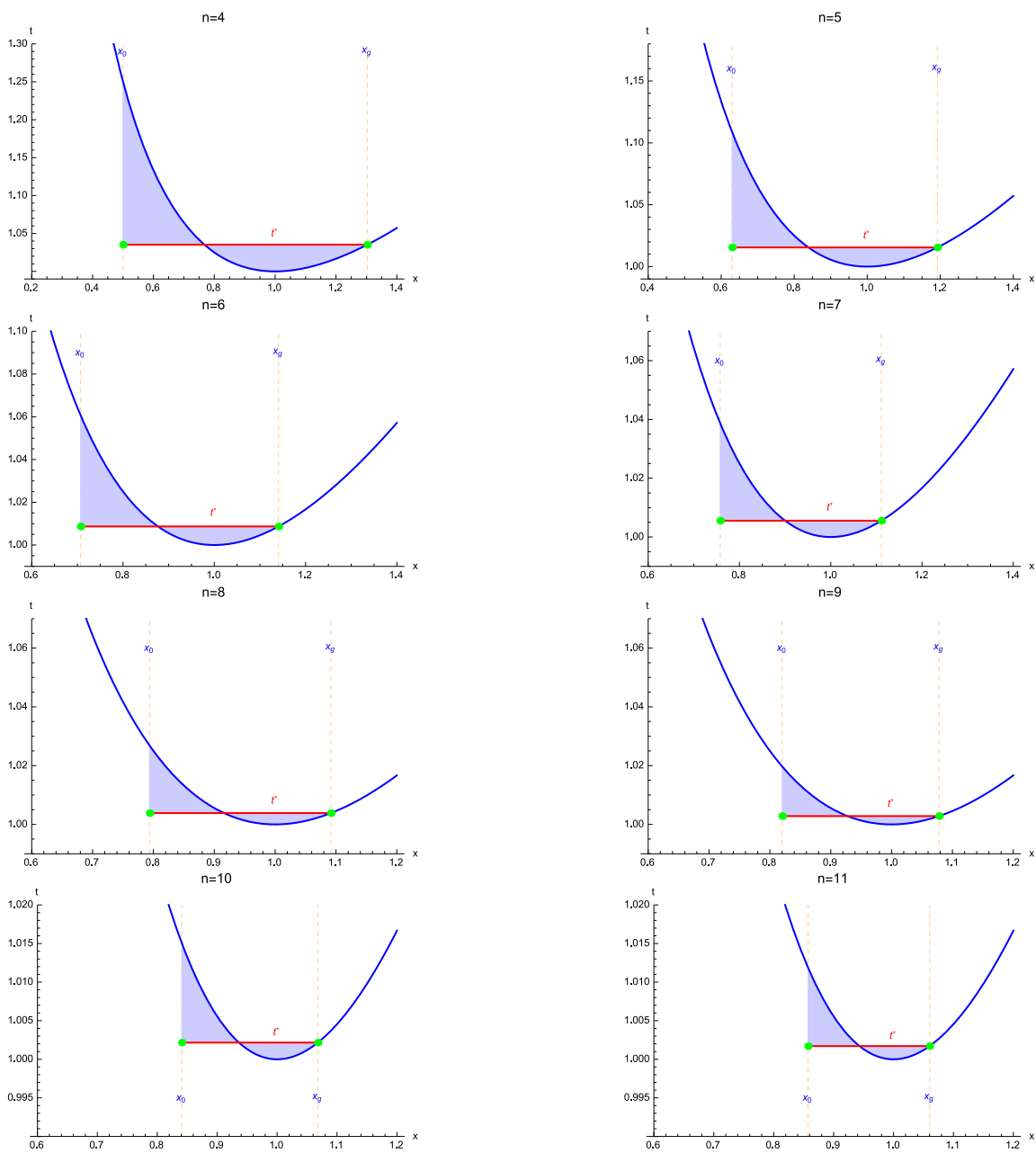
Figure 4. The $t - x$ diagrams for space dimension n between 4 and 11. $t^\star$ and $x_{0,g}$ are given in table 1.
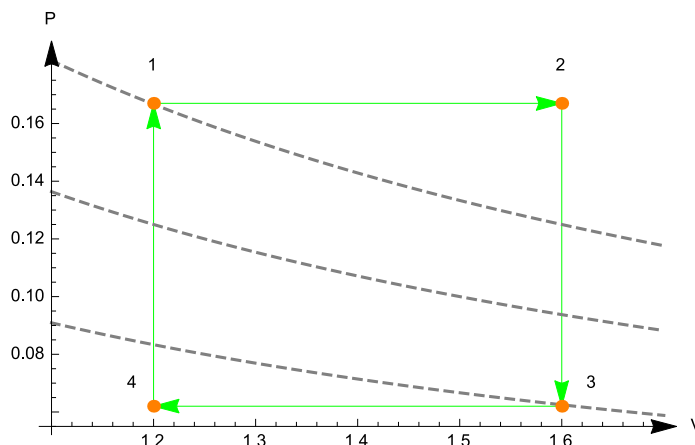
Figure 5. Considered cycle.

a cycle between two sources (cold/hot) with temperature T_C and T_H respectively. Then, we make contact with the Carnot cycle defined as a simple cycle described by two isobars and two isochores as in [21]. This is illustrated in figure 5.

Exploring the equation (2.7) and using the relation between the cosmological constant and the pressure, we can derive the equation of state of the black holes. The calculation gives the following equation

$$P = \frac{1}{4}(n-2)T\left(\frac{(n-1)V}{\omega}\right)^{-\frac{1}{n-1}} - \frac{\left(n^2 - 5n + 6\right)}{16\pi}\left(\frac{(n-1)V}{\omega}\right)^{-\frac{2}{n-1}} \tag{4.1}$$

where the thermodynamical volume V is linked to the horizon radius r_+ via the relation

$$V = \frac{\omega}{n-1}r_+^{n-1}. \tag{4.2}$$

Now it is possible to extract the work of the cycle: expressing the volume in terms of the entropy, which will be used to reduce the number of variables in the final expression for the efficiency, the work takes the following form

$$W = \frac{4^{\frac{1}{n-2}+1}}{n-1}(P_1 - P_4)\left(S_2\left(\frac{S_2}{\omega}\right)^{\frac{1}{n-2}} - S_1\left(\frac{S_1}{\omega}\right)^{\frac{1}{n-2}}\right) \tag{4.3}$$

where the subscripts refer to the quantities evaluated at the corners labeled $(1, 2, 3, 4)$. To derive the efficiency, one has to compute the heat quantity. The upper isobar transformation will produce the net inflow of the heat which will be identified with Q_H. The latter is expressed as follows

$$Q_H = \int_{T_1}^{T_2} C_p(P_1, T)dt. \tag{4.4}$$

The integration do not look nice due to the entropy dependence of C_p giving non-trivial T dependence.

Thus the efficiency is given by

$$\eta = \frac{W}{Q_H}. \tag{4.5}$$

To determine such a quantity, we will use the large value limits of T and P. In this way, the equation (2.14) becomes

$$S = \frac{1}{3} 2^{\frac{-2n^2+3n-4}{n-2}} (n-2)^{n-5} \omega P^{2-n} T^{n-8} \left[3\, 2^{\frac{3n}{n-2}} (n-2)^3 T^6 - \frac{3}{\pi} 2^{\frac{3n}{n-2}} (n-3)(n-2)^3 P T^4 \right.$$

$$+ \frac{3}{\pi^2} 2^{\frac{n}{n-2}+1} (n-3)^2 \left(2^{\frac{4}{n-2}} n^3 - 9\, 2^{\frac{4}{n-2}} n^2 + 3\, 2^{\frac{2n}{n-2}+1} n - 5\, 2^{\frac{2n}{n-2}} \right) P^2 T^2 \qquad (4.6)$$

$$\left. + \frac{4}{\pi^3} \left(-3\, 2^{\frac{2(n+1)}{n-2}} + 3\, 2^{\frac{n+4}{n-2}} (n-3) - 2^{\frac{6}{n-2}} (n-4)(n-3) \right) (n-2)(n-3)^3 P^3 + \cdots \right]$$

In this limit, the heat capacity (2.11) reduces to

$$C_p = 2^{\frac{-2n^2+3n-8}{n-2}} \omega (n-2)^{n-3} \left(4^{\frac{5}{n-2}+2} (n-5)n + 3\, 32^{\frac{n}{n-2}} \right) \frac{T^{n-4}}{\pi P^{n-3}}$$

$$+ 4^{1-n}(n-2)\omega \left(\frac{(n-2)T}{P} \right)^{n-2} + \cdots . \qquad (4.7)$$

which infers the following expression

$$Q_H = \frac{2^{1-2n} \omega \left(\frac{P_1}{n-2} \right)^{2-n}}{\pi^2 (n-1) T_1^5 T_2^5} \left[(n-3)^2(n-1)P_1^2 \left(T_1^5 T_2^n - T_2^5 T_1^n \right) + 2\pi^2(n-2) T_1^4 T_2^4 \left(T_1 T_2^n - T_2 T_1^n \right) \right] + \cdot$$

$$= -\frac{1}{3} 2^{\frac{-4n^2+17n-12}{n-2}} \left(n^2 - 5n + 6 \right)^{n-3} \pi^{3-n} \omega P_1^{4-n} \left(\frac{S_2}{\omega} \right)^{\frac{7}{n-2}}$$

$$\times \left(\pi 2^{\frac{4}{n-2}+4} P_1 \left(\left(\frac{S_1}{S_2} \right)^{\frac{7}{n-2}} \left(\frac{S_1}{\omega} \right)^{\frac{n}{2-n}} - \left(\frac{S_2}{\omega} \right)^{\frac{n}{2-n}} \right) \right.$$

$$\left. + 3(n-2) \left(\frac{S_2}{\omega} \right)^{-\frac{2}{n-2}} \left(\left(\frac{S_1}{S_2} \right)^{\frac{5}{n-2}} \left(\frac{S_1}{\omega} \right)^{\frac{n}{2-n}} - \left(\frac{S_2}{\omega} \right)^{\frac{n}{2-n}} \right) + \cdots \right). \qquad (4.8)$$

Then, the efficiency is finally given by

$$\eta = 3\, 2^{\frac{2n}{n-2} - \frac{20}{n-2} - 19} (n-3)^3 (n-2)^{3-n} \pi^{n-5} P_1^{n-6} (P_1 - P_4) \left(\left(\frac{S_2}{\omega} \right)^{\frac{n-1}{n-2}} - \left(\frac{S_1}{\omega} \right)^{\frac{n-1}{n-2}} \right) \left(\frac{S_2}{\omega} \right)^{-\frac{7}{n-2}}$$

$$\times \left[3(n-2) \left(\frac{S_2}{\omega} \right)^{-\frac{2}{n-2}} \left(\left(\frac{S_1}{S_2} \right)^{\frac{5}{n-2}} \left(4^{\frac{1}{2-n}} (n-3) \left(\frac{S_1}{\omega} \right)^{\frac{1}{2-n}} \right)^n - \left(4^{\frac{1}{2-n}} (n-3) \left(\frac{S_2}{\omega} \right)^{\frac{1}{2-n}} \right)^n \right) \right]$$

$$- \pi 2^{\frac{4}{n-2}+4} P_1 \left(\left(\frac{S_1}{S_2} \right)^{\frac{7}{n-2}} \left(4^{\frac{1}{2-n}} (n-3) \left(\frac{S_1}{\omega} \right)^{\frac{1}{2-n}} \right)^n - \left(4^{\frac{1}{2-n}} (n-3) \left(\frac{S_2}{\omega} \right)^{\frac{1}{2-n}} \right)^n \right) \right] \qquad (4.9)$$

$$\times \left[(n-1) \left(\left(2^{-\frac{2}{n-2}-2} (n-3) \left(\frac{S_2}{\omega} \right)^{\frac{1}{2-n}} \right)^n - \left(\frac{S_1}{S_2} \right)^{\frac{7}{n-2}} \left(2^{-\frac{2}{n-2}-2} (n-3) \left(\frac{S_1}{\omega} \right)^{\frac{1}{2-n}} \right)^n \right)^2 \right]^{-1} + \cdots$$

The efficiency can be calculated at leading order: identifying $T_C = T_4$ and $T_H = T_2$, corresponding to the lowest and highest temperatures of the engine, with $P \sim \frac{1}{4}(n-2)T \left(\frac{(n-1)V}{\omega} \right)^{-\frac{1}{n-1}} + \cdots$, we obtain

$$\eta = 1 - \frac{T_C}{T_H} \left(\frac{V_2}{V_4} \right)^{\frac{1}{n-1}} . \qquad (4.10)$$

In this way, we can compare with the efficiency of the Carnot cycle $\eta_{\text{Carnot}} = 1 - \frac{T_C}{T_H}$. Again, it is worth noting that for $n = 4$, we recover the result reported in [21].

5 Conclusion

In this work, we have investigated the heat properties of AdS black holes in higher dimensions. We have considered the study of corresponding thermodynamical properties along with the sign of the heat capacity explored in the determination of the stability of such black hole. More precisely, we have computed the latent heat as a trivial consequence of the Hawking-Page phase transition. To overcome the instability problem, the Maxwell' construction have been elaborated to modify the Hawking-Page phase structure in the (T, S) plane. We have derived the equal area isotherm for any dimension in the range $4 \leq n \leq 11$. Then, we have analyzed the thermodynamic cycle and the holographic heat engines using the expression of the extracted work and efficiency.

By following [21], it is possible to make contact with Maldacena conjecture known by AdS/CFT holographic correspondence [24, 25]. This could be useful to bring new features in the gauge theories embedded in string theory and related topics.

References

[1] A. Dabholkar and S. Nampuri, *Quantum black holes*, *Lect. Notes Phys.* **851** (2012) 165 [arXiv:1208.4814] [INSPIRE].

[2] S.-J. Rey, *String theory on thin semiconductors: holographic realization of Fermi points and surfaces*, *Prog. Theor. Phys. Suppl.* **177** (2009) 128 [arXiv:0911.5295] [INSPIRE].

[3] H. Witek et al., *Higher dimensional numerical relativity: code comparison*, *Phys. Rev.* **D 90** (2014) 084014 [arXiv:1406.2703] [INSPIRE].

[4] D. Kastor, S. Ray and J. Traschen, *Enthalpy and the mechanics of AdS black holes*, *Class. Quant. Grav.* **26** (2009) 195011 [arXiv:0904.2765] [INSPIRE].

[5] S.W. Hawking and D.N. Page, *Thermodynamics of black holes in anti-de Sitter space*, *Commun. Math. Phys.* **87** (1983) 577 [INSPIRE].

[6] D. Kastor, S. Ray and J. Traschen, *Enthalpy and the mechanics of AdS black holes*, *Class. Quant. Grav.* **26** (2009) 195011 [arXiv:0904.2765] [INSPIRE].

[7] A. Chamblin, R. Emparan, C.V. Johnson and R.C. Myers, *Charged AdS black holes and catastrophic holography*, *Phys. Rev.* **D 60** (1999) 064018 [hep-th/9902170] [INSPIRE].

[8] A. Chamblin, R. Emparan, C.V. Johnson and R.C. Myers, *Holography, thermodynamics and fluctuations of charged AdS black holes*, *Phys. Rev.* **D 60** (1999) 104026 [hep-th/9904197] [INSPIRE].

[9] M. Cvetič, G.W. Gibbons, D. Kubiznak and C.N. Pope, *Black hole enthalpy and an entropy inequality for the thermodynamic volume*, *Phys. Rev.* **D 84** (2011) 024037 [arXiv:1012.2888] [INSPIRE].

[10] B.P. Dolan, D. Kastor, D. Kubiznak, R.B. Mann and J. Traschen, *Thermodynamic volumes and isoperimetric inequalities for de Sitter black holes*, *Phys. Rev.* **D 87** (2013) 104017 [arXiv:1301.5926] [INSPIRE].

[11] D. Kubiznak and R.B. Mann, *P-V criticality of charged AdS black holes*, *JHEP* **07** (2012) 033 [`arXiv:1205.0559`] [ɪɴSPIRE].

[12] S. Chen, X. Liu, C. Liu and J. Jing, *P-V criticality of AdS black hole in $f(R)$ gravity*, *Chin. Phys. Lett.* **30** (2013) 060401 [`arXiv:1301.3234`] [ɪɴSPIRE].

[13] A. Belhaj, M. Chabab, H. El Moumni and M.B. Sedra, *On thermodynamics of AdS black holes in arbitrary dimensions*, *Chin. Phys. Lett.* **29** (2012) 100401 [`arXiv:1210.4617`] [ɪɴSPIRE].

[14] A. Belhaj, M. Chabab, H.E. Moumni, L. Medari and M.B. Sedra, *The thermodynamical behaviors of Kerr-Newman AdS black holes*, *Chin. Phys. Lett.* **30** (2013) 090402 [`arXiv:1307.7421`] [ɪɴSPIRE].

[15] A. Belhaj, M. Chabab, H. El Moumni, K. Masmar and M.B. Sedra, *Critical behaviors of 3D black holes with a scalar hair*, *Int. J. Geom. Meth. Mod. Phys.* **12** (2014) 1550017 [`arXiv:1306.2518`] [ɪɴSPIRE].

[16] E. Spallucci and A. Smailagic, *Maxwell's equal area law for charged anti-de Sitter black holes*, *Phys. Lett.* **B 723** (2013) 436 [`arXiv:1305.3379`] [ɪɴSPIRE].

[17] E. Spallucci and A. Smailagic, *Maxwell's equal area law and the Hawking-Page phase transition*, *J. Grav.* **2013** (2013) 525696 [`arXiv:1310.2186`] [ɪɴSPIRE].

[18] A. Belhaj, M. Chabab, H. El moumni, K. Masmar and M.B. Sedra, *Maxwell's equal-area law for Gauss-Bonnet-anti-de Sitter black holes*, *Eur. Phys. J.* **C 75** (2015) 71 [`arXiv:1412.2162`] [ɪɴSPIRE].

[19] J.-X. Zhao, M.-S. Ma, L.-C. Zhang, H.-H. Zhao and R. Zhao, *The equal area law of asymptotically AdS black holes in extended phase space*, *Astrophys. Space Sci.* **352** (2014) 763 [ɪɴSPIRE].

[20] B.P. Dolan, *Vacuum energy and the latent heat of AdS-Kerr black holes*, *Phys. Rev.* **D 90** (2014) 084002 [`arXiv:1407.4037`] [ɪɴSPIRE].

[21] C.V. Johnson, *Holographic heat engines*, *Class. Quant. Grav.* **31** (2014) 205002 [`arXiv:1404.5982`] [ɪɴSPIRE].

[22] S.W. Hawking and D.N. Page, *Thermodynamics of black holes in anti-de Sitter space*, *Commun. Math. Phys.* **87** (1983) 577 [ɪɴSPIRE].

[23] N. Altamirano, D. Kubiznak, R.B. Mann and Z. Sherkatghanad, *Thermodynamics of rotating black holes and black rings: phase transitions and thermodynamic volume*, *Galaxies* **2** (2014) 89 [`arXiv:1401.2586`] [ɪɴSPIRE].

[24] E. Witten, *Anti-de Sitter space, thermal phase transition and confinement in gauge theories*, *Adv. Theor. Math. Phys.* **2** (1998) 505 [`hep-th/9803131`] [ɪɴSPIRE].

[25] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Int. J. Theor. Phys.* **38** (1999) 1113 [`hep-th/9711200`] [ɪɴSPIRE].