



# Stora's fine notion of divergent amplitudes

Joseph C. Várilly<sup>a,\*</sup>, José M. Gracia-Bondía<sup>b,c</sup>

<sup>a</sup> *Escuela de Matemática, Universidad de Costa Rica, San José 11501, Costa Rica*

<sup>b</sup> *Departamento de Física Teórica, Universidad de Zaragoza, Zaragoza 50009, Spain*

<sup>c</sup> *Departamento de Física, Universidad de Costa Rica, San Pedro 11501, Costa Rica*

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Raymond Stora, 1930–2015: *in memoriam*

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## Abstract

Stora and coworkers refined the notion of divergent quantum amplitude, somewhat upsetting the standard power-counting recipe. This unexpectedly clears the way to new prototypes for free and interacting field theories of bosons of any mass and spin.

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## 1. Exordium

One of us (JMG-B) learned of a flaw in the standard notion of “superficially divergent amplitude” from the lips of Raymond Stora, quickly becoming aware of *some* of the vistas opened by his alternative notion during intense conversations at CERN in the winter of 2013. In fairness, the notion should be attributed as well to Nikolay M. Nikolov and Ivan Todorov, with whom Raymond was working at the time on paper [1], wherein the matter is expounded in convincing detail. We shall refer to the new notion of convergent Feynman amplitude as the NST renormalization prescription.

We begin by a review of causal Riesz distributions as introduced in [2]. This prelude smooths the way for the new notion of divergent graph, valid for physical quantum fields (what “physical”

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\* Corresponding author.

E-mail address: [joseph.varilly@ucr.ac.cr](mailto:joseph.varilly@ucr.ac.cr) (J.C. Várilly).

means will be declared in due course). This helps to open the door to the brave new world of string-local fields. Finally, in Section 6 we show that, although homogeneity of the amplitudes is lost, the concept in [1] makes perfect sense for massive theories.

## 2. Causal Riesz distributions and massless field amplitudes

Let us invoke in somewhat simplified form a meromorphic family of distributions on Minkowski space  $\mathbb{M}^4$  studied in [2]:

$$G(x; \alpha) := \frac{e^{-i\pi\alpha} \Gamma(-\alpha)}{4^{\alpha+2} \pi^2 \Gamma(\alpha+2)} (t^2 - r^2 - i0)^\alpha \equiv \frac{e^{-i\pi\alpha} \Gamma(-\alpha)}{4^{\alpha+2} \pi^2 \Gamma(\alpha+2)} (x^2 - i0)^\alpha. \tag{1}$$

The distribution  $(x^2 - i0)^\alpha$  is well defined for  $-2 < \Re\alpha < 0$ ; it can be extended analytically to non-integer  $\Re\alpha < -2$  by repeated applications of  $\square$ ; so  $(x^2 - i0)^\alpha$  can be regarded as meromorphic in  $\alpha$  with (simple) poles at  $-2 - n$  for  $n = 0, 1, 2, \dots$ . These are canceled in (1) by the poles of  $\Gamma(\alpha + 2)$ . The extension prescription of analytic renormalization, obtained by discarding the pole part in the Laurent expansion of  $(x^2 - i0)^{\alpha+\varepsilon}$ , is therefore straightforward whenever  $\Re\alpha > -2$ , i.e., there is a homogeneous extension. The relation

$$\square G(x; \alpha) = G(x; \alpha - 1)$$

holds, just as for the ordinary Riesz distributions. This is clear from

$$\square(x^2 - i0)^\alpha = 4\alpha(\alpha + 1)(x^2 - i0)^{\alpha-1},$$

valid on the chosen domain, and then analytically extended. Note that  $iG(x; -1) = D_0^F(x)$ , the Feynman propagator for massless scalars; so  $G(x; -l)$  for integer  $l \geq 2$  is proportional to  $\square^{l-2} \delta(x)$ . This is confirmed by a direct calculation of the residues at  $\alpha = -2, -3, \dots$ .

The first aim of this paper is to investigate a generalization of all this for massless particles of higher (integer) helicity. The quantum Maxwell field can be built from the helicity  $\pm 1$  massless unirreps of the Poincaré group, under the form:

$$F^{\mu\nu}(x) := i \sum_r \int d\mu(p) \left[ e^{i(p x)} (p^\mu e_r^\nu(p) - p^\nu e_r^\mu(p)) a_r^\dagger(p) - e^{-i(p x)} (p^\mu e_r^\nu(p)^* - p^\nu e_r^\mu(p)^*) a_r(p) \right], \tag{2}$$

for appropriate creation operators  $a_r^\dagger(p)$  and polarization vectors  $e_r^\nu(p)$ . With  $g_{\mu\nu}$  denoting the Minkowski metric with  $(+---)$  signature, routine computation establishes for the vacuum expectation value of the two-point time-ordered product [3]:

$$\begin{aligned} \langle\langle \text{T} F_{\mu\nu}(x) F_{\rho\sigma}(x') \rangle\rangle &:= \langle 0 | \text{T} F_{\mu\nu}(x) F_{\rho\sigma}(x') | 0 \rangle \\ &= (g_{\mu\rho} \partial_\nu \partial_\sigma - g_{\nu\rho} \partial_\mu \partial_\sigma - g_{\mu\sigma} \partial_\nu \partial_\rho + g_{\nu\sigma} \partial_\mu \partial_\rho) D_0^F(x - x') \\ &=: f_{\mu\nu, \rho\sigma}(\partial) D_0^F(x - x') \end{aligned} \tag{3}$$

valid outside the diagonal  $x = x'$ . On the face of it, this expression seems logarithmically divergent, since it homogeneously scales like  $x^{-4}$ ; the field itself scales like  $x^{-2}$ .

For brevity, let us write  $x^2 \equiv x^2 - i0$  hereinafter. In the Epstein–Glaser program [4], to renormalize a distribution like  $\langle\langle \text{T} F_{\mu\nu}(x) F_{\rho\sigma}(x') \rangle\rangle$  in position space is to find a suitable extension to the diagonal. “Suitable” means keeping the scaling behavior of the original distribution as

much as possible. It also means satisfying physically motivated and mathematically convenient requirements, in particular Lorentz covariance and other symmetries.

Using translation invariance, extension of a distribution  $f(x - x')$  to the diagonal is equivalent to extending  $f(x)$ , defined for  $x \neq 0$ , to the origin in Minkowski space. Then the distribution  $x^{2\alpha} \equiv (x^2)^\alpha$  extends homogeneously for  $\alpha > -2$ ; and for integer  $\alpha \leq -2$ , its extensions can be determined by the complex-analytic methods in [1] or the real-variable methods in [5], adopted in [6]. Thus for instance the extensions of  $x^{-4}$  are given by:

$$R_4[x^{-4}] = -\frac{1}{4} \square \left( x^{-2} \log \frac{x^2}{\ell^2} \right) - i\pi^2 \delta(x),$$

with a length scale  $\ell$ . This is log-homogeneous of bidegree  $(-4, 1)$  in the terminology of [6]. (The Euclidean version is  $R_4[x^{-4}] = -\frac{1}{4} \Delta(x^{-2} \log(x^2/\ell^2)) + \pi^2 \delta(x)$ ; the two cases differ only in the coefficient of  $\delta(x)$ , arising from the fundamental solutions of the Laplacian,  $\Delta(x^{-2}) = -4\pi^2 \delta(x)$  in  $\mathbb{R}^4$ ; and of the d'Alembertian,  $\square(x^{-2}) = 4i\pi^2 \delta(x)$  in  $\mathbb{M}^4$ .)

For two-point functions which are polynomials in  $x^{-2}$ , these procedures go a long way. For the sunset graph in massless  $\phi_4^4$ , demanding Lorentz invariance, one can show [6, Eq. (2.19)] that

$$R_4[x^{-6}] = -\frac{1}{32} \square^2 \left( x^{-2} \log \frac{x^2}{\ell^2} \right) - \frac{5i\pi^2}{16} \square \delta(x),$$

whose second term incidentally differs from the one in [1, Eq. (5.29)] due to the precise usage of the multiplicativity property of [5].

One concludes that while unrenormalized two-point amplitudes are *homogeneous* functions for  $x \neq x'$ , they admit *log-homogeneous* extensions to the diagonal. The second index in the bidegree indicates the power of the logarithm, counting the number of successive extensions for distributions presenting subdivergences, in general: the sunset graph is quadratically divergent, but still primitive in this dispensation. The matter was treated in detail for many graphs of the massless  $\phi_4^4$  theory in [6], albeit in the Euclidean signature; happily, only minor modifications are needed for the Minkowskian version. There has been a crop of relatively recent papers dealing with this kind of problem [1,6,7], reaching similar conclusions.

Things appear to be more complicated when the unrenormalized amplitude has an angular dependence, as in our present case (3). Since  $\partial_\mu \partial_\rho (x^{-2}) = -2(g_{\mu\rho} x^2 - 4x_\mu x_\rho) x^{-6}$ , we compute (for  $x \neq 0$ ):

$$\begin{aligned} & (g_{\mu\rho} \partial_\nu \partial_\sigma - g_{\nu\rho} \partial_\mu \partial_\sigma - g_{\mu\sigma} \partial_\nu \partial_\rho + g_{\nu\sigma} \partial_\mu \partial_\rho) [x^{-2}] \\ &= -4 \left( (g_{\mu\rho} g_{\nu\sigma} - g_{\nu\rho} g_{\mu\sigma}) x^2 - 2(g_{\mu\rho} x_\nu x_\sigma - g_{\nu\rho} x_\mu x_\sigma - g_{\mu\sigma} x_\nu x_\rho + g_{\nu\sigma} x_\mu x_\rho) \right) x^{-6} \\ &=: h_{\mu\nu, \rho\sigma}(x) x^{-6}, \end{aligned} \quad (4)$$

where each  $h_{\mu\nu, \rho\sigma}(x)$  is a homogeneous quadratic polynomial.

In fact, each of these polynomials is *harmonic* in the Minkowskian sense. To see that, it is enough to apply  $\square(x^2) = 8$  and  $\square(x_\mu x_\nu) = 2g_{\mu\nu}$  to the quadratic polynomial in (4), to get

$$\square h_{\mu\nu, \rho\sigma}(x) = -4(8 - 8)(g_{\mu\rho} g_{\nu\sigma} - g_{\nu\rho} g_{\mu\sigma}) = 0.$$

Actually, these  $h_{\mu\nu, \rho\sigma}$  form a *basis* for the vector space of quadratic harmonic polynomials on  $\mathbb{M}^4$ . Due to (skew)symmetry under the exchanges  $\mu \leftrightarrow \nu$  and  $\rho \leftrightarrow \sigma$ , and symmetry under  $(\mu, \nu) \leftrightarrow (\rho, \sigma)$  and  $(\mu, \nu) \leftrightarrow (\sigma, \rho)$ , there are 9 linearly independent  $h_{\mu\nu, \rho\sigma}$ ; whereas the harmonic homogeneous polynomials of degree  $k$  on  $\mathbb{M}^4$  (or on  $\mathbb{R}^4$ , for that matter) form a space of dimension  $(k + 1)^2$  [8, Sect. 9.3].

### 3. The NST renormalization prescription

The task then becomes to extend to the origin functions of the form  $x^{2\alpha} H_k(x)$ , where  $H_k$  is a homogeneous polynomial of degree  $k$  that is also (Minkowskian) harmonic. There are two reasons to hope that the “radial” extensions of [1,6] may prove equal to the task. The first is the off-origin calculation:

$$\begin{aligned} \square(x^{2\alpha} H_k(x)) &= \square(x^{2\alpha}) H_k(x) + 2\partial^\mu (x^{2\alpha}) \partial_\mu (H_k(x)) + x^{2\alpha} \square(H_k(x)) \\ &= 4\alpha(\alpha + 1)x^{2\alpha-2} H_k(x) + 4\alpha x^{2\alpha-2} x^\mu \partial_\mu (H_k(x)) \\ &= 4\alpha(\alpha + k + 1)x^{2\alpha-2} H_k(x), \end{aligned} \tag{5}$$

where we have used harmonicity:  $\square H_k = 0$ , and homogeneity:  $x^\mu \partial_\mu H_k = k H_k$ . These relations show that the family of  $x^{2\alpha} H_k(x)$  also act like the causal Riesz distributions (1); a suitable normalization is

$$G(x; \alpha, k) := \frac{e^{-i\pi\alpha} \Gamma(-\alpha)}{4^{\alpha+2} \pi^2 \Gamma(\alpha + k + 2)} x^{2\alpha} H_k(x);$$

and from (5) we get at once:

$$\square G(x; \alpha, k) = G(x; \alpha - 1, k). \tag{6}$$

The extension prescription of analytic renormalization now tells us that there is a *homogeneous* extension whenever  $\alpha > -k - 2$ . In particular, the case of interest (4) has  $\alpha = -3$  and  $k = 2$ . Since  $-3 > -4$ , the naïve power-counting recipe is overridden: the time-ordered product (3) does extend homogeneously to the origin, the result being none other than:

$$\langle\langle T F_{\mu\nu}(x) F_{\rho\sigma}(x') \rangle\rangle = \frac{i}{4\pi^2} f_{\mu\nu,\rho\sigma}(\partial) \frac{1}{(x - x')^2 - i0},$$

as many a physicist, taking a cue from the commutation relations [9, Aufgabe 7.5], would have written at the outset. In other words, the apparent singularity was removable; according to the lore of renormalization of massless amplitudes, truly renormalization has not taken place.

The general criterion [1, Corl. 5.4] is: a two-point unrenormalized Feynman amplitude in Minkowski space of the form  $h_k(x)/(x^2 \pm i0)^s$  for  $x \neq 0$  has an homogeneous extension if and only if its “degree of harmonicity”  $k$  is greater than the “degree of divergence”  $2s - k - 4$ .

Furthermore, in this case the homogeneous extension is unique if we impose Lorentz covariance. This needs to be properly understood. Once a homogeneous extension of  $x^{2\alpha} H_k(x)$  has been found, any other such extension can differ from it only by a distribution  $P(\partial) \delta(x)$  supported at the origin, where  $P(x)$  is a homogeneous polynomial of degree  $2(-\alpha) - k - 4$ , the superficial degree of divergence. In our example, this degree is 0, so  $P(x)$  would be a constant. However,  $H_k(x)$  is not constant: indeed, it transforms under a representation of the Lorentz group on the space of harmonic homogeneous polynomials of degree  $k$ , and  $P(x)$  must transform likewise. The upshot is that  $P(x)$  must be at least divisible by such a harmonic homogeneous polynomial, so that  $\text{deg } P \geq k$ . Thus, again the condition  $k > 2(-\alpha) - k - 4$  [in the example:  $2 > 0$ ] is enough to ensure that the Lorentz-covariant extension of  $x^{2\alpha} H_k(x)$  is unique. In fine: the off-diagonal function (3) extends to a Lorentz-covariant time-ordered product, without ambiguity. Equivalently, one can argue in the spirit of the *on-shell extension* of amplitudes by Bahns and Wrochna [10]: the decisive fact is that the differential equation (6) is extended to the origin, too.

#### 4. The prescription for higher helicities ...

Similarly to the above, there is a free quantum field  $R_{\alpha\beta\rho\tau}(x)$ , the linearized Riemann tensor, corresponding to helicity-2 particles and transforming as a rank 4 tensor, with the symmetry properties:

$$R_{\alpha\beta\kappa\tau}(x) = -R_{\beta\alpha\kappa\tau}(x) = -R_{\alpha\beta\tau\kappa}(x) = R_{\kappa\tau\alpha\beta}(x).$$

One analogously finds for this:

$$\begin{aligned} \langle\langle T R_{\alpha\beta\kappa\tau}(x) R_{\rho\sigma\lambda\gamma}(x') \rangle\rangle &= \sum \pm G_{\beta\tau,\sigma\gamma} \partial_\alpha \partial_\kappa \partial_\rho \partial_\lambda D_0^F(x-x') + 15 \text{ similar terms} \\ &=: \frac{16\pi^8}{3} h_{\alpha\beta\kappa\tau,\rho\sigma\lambda\gamma}(x) D_0^F(x-x')^5; \end{aligned} \quad (7)$$

where  $G_{\beta\tau,\sigma\gamma} := \frac{1}{8}(g_{\beta\sigma}g_{\tau\gamma} + g_{\beta\gamma}g_{\tau\sigma} - g_{\beta\tau}g_{\sigma\gamma})$  and the “similar terms” are obtained by permuting the indices under exchange of  $(\alpha, \beta, \rho, \sigma)$  with  $(\kappa, \tau, \lambda, \gamma)$ ,  $(\tau, \kappa, \lambda, \gamma)$ ,  $(\kappa, \tau, \gamma, \lambda)$  and  $(\tau, \kappa, \gamma, \lambda)$  respectively; the signs are those that respect the aforementioned symmetries.<sup>1</sup> Therefore,  $h_{\alpha\beta\kappa\tau,\rho\sigma\lambda\gamma}(x) = \sum \pm G_{\beta\tau,\sigma\gamma} q_{\alpha\kappa\rho\lambda}(x)$  is likewise a sum of 16 quartic harmonic polynomials, coming from  $\partial_\alpha \partial_\kappa \partial_\rho \partial_\lambda [x^{-2}] =: q_{\alpha\kappa\rho\lambda}(x) x^{-10}$  by direct calculation, such as:

$$\begin{aligned} q_{\alpha\kappa\rho\lambda}(x) &:= 48x_\alpha x_\kappa x_\rho x_\lambda + (g_{\alpha\kappa}g_{\rho\lambda} + g_{\alpha\lambda}g_{\kappa\rho} + g_{\alpha\rho}g_{\kappa\lambda})x^4 \\ &\quad - 6(g_{\alpha\kappa}x_\rho x_\lambda + g_{\alpha\rho}x_\kappa x_\lambda + g_{\alpha\lambda}x_\kappa x_\rho + g_{\kappa\rho}x_\alpha x_\lambda + g_{\kappa\lambda}x_\alpha x_\rho + g_{\rho\lambda}x_\alpha x_\kappa)x^2. \end{aligned}$$

The harmonic property  $\square q_{\alpha\kappa\rho\lambda}(x) = 0$  is easily checked directly, using:

$$\begin{aligned} \square(x^4) &= 24x^2, & \square(x_\rho x_\lambda x^2) &= 2g_{\rho\lambda}x^2 + 16x_\rho x_\lambda, \\ \square(x_\alpha x_\kappa x_\rho x_\lambda) &= 2g_{\alpha\kappa}x_\rho x_\lambda + 5 \text{ similar terms.} \end{aligned}$$

Just as before, these  $h_{\alpha\beta\kappa\tau,\rho\sigma\lambda\gamma}$  constitute a basis of the 25-dimensional space of quartic homogeneous harmonic polynomials on  $\mathbb{M}^4$ . Indeed, taking into account the 20 independent components of  $R_{\alpha\beta\rho\tau}(x)$  and the four mentioned symmetries of the cross-indexes, the number of independent  $h_{\bullet}$ -polynomials in this case is  $(20)^2/2^4 = 25$ .

Now, on the face of it there is a quadratic divergence here – the field scales like  $x^{-3}$ . However, since  $4 > 10 - 4 - 4$ , by the same token as above, the finer NST criterion shows that the vacuum expectation value of the time-ordered 2-point function for the  $R$ -tensor field is a *convergent* amplitude.

How to generalize to higher integer helicities should be clear now: among the free point-local fields for helicity  $h$  there are two tensor fields with apparently optimal ultraviolet behavior in relative terms, namely, they scale like  $x^{-h-1}$ : the *field strength*  $F_{\mu_1\nu_1,\dots,\mu_h\nu_h}$  of rank  $2h$ , symmetric under exchange of any of the pairs  $(\mu_i, \nu_i) \leftrightarrow (\mu_j, \nu_j)$  and skewsymmetric under exchange inside the pairs; and its potential  $A_{\mu_1,\dots,\mu_h}$  of rank  $h$ , which is totally symmetric [27,12]. The quantum fields associated to the representation  $(h, 0) \oplus (0, h)$  are “physical” in that their classical counterparts are measurable.

“Apparently” we say, because in fact  $\langle\langle T F_{\mu_1\nu_1,\dots,\mu_h\nu_h} F_{\alpha_1\beta_1,\dots,\alpha_h\beta_h} \rangle\rangle$  is a convergent amplitude, as we have seen for  $h = 1, 2$ . Whereas the 2-point function for the potentials carries a problematic existence, due to gauge freedom (or slavery) and the impossibility, starting with the photon, for  $A_{\mu_1,\dots,\mu_h}$  to live on Hilbert space.

<sup>1</sup> The expression for  $G_{\beta\tau,\sigma\gamma}$  appears in the graviton propagator, see for instance [11, Eq. 1.77].

### 5. ...and its consequence: a gauge-free world?

By abandoning point-localization, it is feasible to construct  $A$ -fields for any boson particle that share in the good ultraviolet properties of the field strengths. This fact has been known for over ten years now [13,14], and has the potentiality to drastically change the game of perturbative quantum field theory.

The field strengths remain pointlike. To keep notations simple, here we just exhibit a (lightlike) string-local potential field for the photon:

$$A^\mu(x, l) := \int_0^\infty dt F^{\mu\nu}(x + tl) l_\nu,$$

with  $l = (l^0, \mathbf{l})$  a null vector. The definition depends only on the ray of  $l$ , which is a point of the celestial sphere  $\mathbb{S}^2$ , or the light front uniquely associated to it.

A comment is in order here. Previous formulations of string-local fields were based on modular localization theory, which naturally suggests the use of spacelike strings [15]. However, in interacting models this leads to almost intractable complications at third order of perturbation theory. For purely massive models, there is a huge advantage in employing null strings, since then the field is actually a well-behaved function on the  $l$ -variable, not just a distribution like in the spacelike case. In models containing massless particles, use of null strings generates a *sui generis* ultraviolet-infrared problem, which needs to be and can be dealt with by appropriate recipes. Note that all null directions are on the same footing: each one carries its own cyclic subspace, and these are shuffled around by the Lorentz transformations – see right below.

The operator-valued distribution  $A$  “lives” on the same Fock space as  $F$ , and its main properties are the following:

- ★ Transversality:  $(l A(x, l)) = 0$ .
- ★ Pointlike differential:  $\partial^\mu A^\nu(x, l) - \partial^\nu A^\mu(x, l) = F^{\mu\nu}(x)$ .
- ★ Covariance: let  $U$  denote the lifting (or “second quantization”) of Wigner’s unirrep of the Poincaré group on the one-particle states. Then

$$U(a, \Lambda) A^\mu(x, l) U^\dagger(a, \Lambda) = A^\nu(\Lambda x + a, \Lambda l) \Lambda_\nu^\mu = (\Lambda^{-1})^\mu_\nu A^\nu(\Lambda x + a, \Lambda l).$$

- ★ Locality:  $[A_\mu(x, l), A_\nu(x', l')] = 0$  when the strings  $x + tl$  and  $x' + t'l'$  are causally disjoint.

The very concept of gauge disappears, since this potential *vector*, with all the good properties, is uniquely defined. The formalism appears more exotic than the usual one, in that a new variable is invoked. “The choice of what kind of field describes an observed particle is really a matter of choice: try what type of field describes best the observed data” [16]. It is however more mundane, in that it allows us to remain in physical Hilbert spaces: the ghosts can depart, since there is need for them no longer.

Of course, the string “ought not to be seen”, and the program becomes to demonstrate whether, and how, this simple criterion is enough to determine interaction vertices and govern perturbative renormalization of string-local models of so-called (Abelian and non-Abelian) gauge interactions [17] from the Lie algebra structure, down to every relevant detail [18,19]. This includes models with massive intermediate vector bosons – see the following section.

The above construction works in a parallel way for *all* the other integer-helicity cases, like linear gravity, which now are gauge-free, and seen to possess the same ultraviolet properties as

scalar particles.<sup>2</sup> What we realize is that the construction of string-local fields [13,14] rests on the bedrock of a never-ambiguous time-ordered product of the field strengths.

## 6. Massive field amplitudes

With a suitable change of the polarization vielbeins  $e_r^\nu$ , the very formula (2) describes a skewsymmetric quantum field for massive spin 1 particles [3]. In the massive case, Eq. (3) holds as well. A small miracle is involved here, since

$$F_{\mu\nu}(x) = \partial_\mu B_\nu(x) - \partial_\nu B_\mu(x),$$

where  $B$  denotes the Proca field, and for it, outside the diagonal  $x = x'$ :

$$\langle\langle T B_\mu(x) B_\nu(x') \rangle\rangle = i(g_{\mu\nu} + \partial_\mu \partial_\nu / m^2) D^F(x - x'),$$

with just  $D^F$  denoting the massive scalar Feynman propagator. Thus one would expect fourth-order derivatives (a quadratic divergence) in  $\langle\langle T F F' \rangle\rangle$ . But they all cancel, so the 2-point time-ordered function off the diagonal  $x = x'$  looks exactly like the one in (3):

$$\langle\langle T F_{\mu\nu}(x) F_{\rho\sigma}(x') \rangle\rangle = (g_{\mu\rho} \partial_\nu \partial_\sigma - g_{\nu\rho} \partial_\mu \partial_\sigma - g_{\mu\sigma} \partial_\nu \partial_\rho + g_{\nu\sigma} \partial_\mu \partial_\rho) D^F(x - x'), \quad (8)$$

but with the massive propagator replacing the massless one.

That still looks logarithmically divergent. However, since the ultraviolet properties in both cases are the same, most physicists would conclude without hesitation that the formula makes sense and extends  $\langle\langle T F_{\mu\nu}(x) F_{\rho\sigma}(x') \rangle\rangle$  to the diagonal. We cite Todorov in this context: “Introducing ... masses in the analysis of small distance behavior seems to be just adding technical details to the general picture” [21].

The conclusion is correct, and can be substantiated in at least two rather different ways.

- ★ We recall the expansion of  $D^F$  in the vicinity of  $m = 0$ :

$$D^F(x) = D_0^F(x) + m^2 [f_1(m^2 x^2) \log(-m^2(x^2 - i0)) + f_2(m^2 x^2)], \quad (9)$$

where  $f_1, f_2$  are analytic. In [22, Sect. 6], it is shown that the basic postulate of Epstein–Glaser renormalization, to wit, that the renormalized amplitudes scale like the unrenormalized ones, up to logarithmic corrections, can be strengthened, in that these corrections – albeit necessarily introducing a new mass scale – do not change the dependence on  $m$  in (9); so (8) extends to the diagonal without further ado.

- ★ A method in the spirit of the present paper is as follows [23].<sup>3</sup>

We can modify  $G(x; \alpha)$  in (1) by extracting the finite part of  $\Gamma(-\alpha)x^{2\alpha}$  for  $\alpha = 0, 1, 2, \dots$ . This is equivalent to renormalizing the convolution powers of the massless Feynman propagator; these are all primitives, which means that only the first power of the logarithm appears in:

<sup>2</sup> It appears tempting to redo some of the graviton-scattering calculations in [20], performed in the framework of unimodular gravity, using the  $A(x, l)$ -field companion of the linearized Riemann tensor.

<sup>3</sup> This is actually the same paper as [2], but in the published version the pertinent section was withdrawn, because the referee could not make head or tail of it.

$$\begin{aligned}
 F(x; \alpha) &:= G(x; \alpha) \quad \text{for } \alpha \neq 0, 1, \dots; \\
 F(x; n) &:= \frac{e^{-i\pi n} x^{2n}}{4^{n+2} \pi^2 n!(n+1)!} \left( \log \frac{m^2 x^2}{4} - \psi(n+2) - \psi(n+1) - i\pi \right), \\
 &\quad \text{for } n = 0, 1, \dots; \text{ where } \psi \text{ is the digamma function.}
 \end{aligned}$$

Note the choice  $m = 1/l$  here.

Now  $\square F(x; \alpha) = F(x; \alpha - 1)$  holds without restriction [24], so in fact we may write

$$F(x; \alpha) = -i \square^{-1-\alpha} D_0^F(x),$$

for all  $\alpha \in \mathbb{C}$ , and we have a perfect generalization of Riesz theory. Moreover, the series  $\sum_{n=-1}^{\infty} m^{2n+2} F(x, n)$  solves the massive Klein–Gordon equation with the convolution unit as source [25,26]:

$$\sum_{n=-1}^{\infty} m^{2n+2} F(x, n) = -im \frac{K_1(m\sqrt{-x^2})}{4\pi^2 \sqrt{-x^2}} = D^F(x).$$

So let us define, for  $H_k$  homogeneous harmonic of order  $k$ :

$$\begin{aligned}
 F(x; \alpha, k) &= G(x; \alpha, k) \quad \text{for } \alpha \neq 0, 1, \dots; \\
 F(x; n, k) &:= H_k(x) \frac{e^{-i\pi n} x^{2n}}{4^{n+2} \pi^2 n!(n+k+1)!} \left( \log \frac{m^2 x^2}{4} - \psi(n+2) - \psi(n+1) - i\pi \right).
 \end{aligned}$$

Finally, it is clear that the formula

$$\langle\langle T F_{\mu\nu}(x) F_{\rho\sigma}(x') \rangle\rangle = f_{\mu\nu, \rho\sigma}(\partial) D^F(x - x'),$$

valid for  $x \neq x'$ , extends to the diagonal without further renormalization being necessary.

What about higher spins? Following [27], we compute the expected value of the time-ordered product of the linearized Riemann tensor for massive gravitons, with a result identical to (7), except that instead of  $G_{\beta\tau, \sigma\gamma}$  as in Sect. 4, one finds  $\frac{1}{8}(g_{\beta\sigma} g_{\tau\gamma} + g_{\beta\gamma} g_{\tau\sigma} - \frac{2}{3} g_{\beta\tau} g_{\sigma\gamma})$ .<sup>4</sup> This difference between the massive and the massless cases is immaterial for harmonicity since, as we remarked earlier, the polynomials  $q_{\alpha\kappa\rho\lambda}$  are already harmonic. Therefore  $\langle\langle T R_{\alpha\beta\kappa\tau}(x) R_{\rho\sigma\lambda\gamma}(x') \rangle\rangle$  extends to the diagonal, without further ado.

We conjecture that our conclusions extend to all the massive  $F_{\mu_1\nu_1, \dots, \mu_h\nu_h}$ -fields.

## 7. Conclusion

Two small miracles do not a big miracle make. Nevertheless, it is surprising and gratifying that, against appearances, for massive or massless particles of respectively integer spin or helicity  $j$ , the quantum fields associated to the representation  $(j, 0) \oplus (0, j)$  enjoy the same optimal UV properties. These are inherited by the string-local true tensor fields  $A_{\mu_1, \dots, \mu_h}(x, l)$  constructed from them.<sup>5</sup>

<sup>4</sup> A similar expression with the  $\frac{2}{3}$  coefficient appears in the massive graviton propagator given in [28, Sect. 1.5].

<sup>5</sup> As we were readying this paper for publication, we were made aware of the article [29]. It also seeks to transfer results from massless to massive models, in a direction different from ours.



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