VOLUME INEQUALITIES FOR THE i-TH-CONVOLUTION BODIES

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Abstract. We obtain a new extension of Rogers-Sephard inequality providing an upper bound for the volume of the sum of two convex bodies K and L. We also give lower bounds for the volume of the k-th limiting convolution body of two convex bodies K and L. Special attention is paid to the $(n-1)$-th limiting convolution body, for which a sharp inequality, which is equality only when $K=-L$ is a simplex, is given. Since the n-th limiting convolution body of K and $-K$ is the polar projection body of K, these inequalities can be viewed as an extension of Zhang's inequality.

1. Introduction and notation

Given $K\in\mathcal{K}_0^n$ an n-dimensional convex body (*i.e.* convex, compact subset of $\mathbb{R}^n$ with non-empty interior) and $\theta\in S^{n-1}$ a vector in the unit Euclidean sphere, we denote by $P_{\theta^\perp}(K)$ the projection of K onto the hyperplane orthogonal to θ. An important object in the study of hyperplane projections of a convex body is its polar projection body, since it gathers the information about the volume of all of its hyperplane projections. Namely, the polar projection body of K, which is denoted by $\Pi^*(K)$, is the centrally symmetric convex body which is the unit ball of the norm

$$\|x\|_{\Pi^*(K)}=|x||P_{x^\perp}(K)|,$$

where by $|\cdot|$ we denote, when no confusion is possible, indistincly the usual Lebesgue measure of a set and the Euclidean norm of a vector.

For any $T\in GL(n)$ we have that $\Pi^*(TK)=|\det T|^{-1}T\Pi^*(K)$ and then the quantity $|K|^{n-1}|\Pi^*(K)|$ is affine invariant. Perhaps the most important inequalities involving the polar projection body are Petty's projection [P] and Zhang's inequality [Z]. On one hand, Petty's projection inequality states that the afforementioned affine invariant quantity is maximized when K is an ellipsoid. Thus, denoting by B_2^n the n-dimensional Euclidean ball,

$$|K|^{n-1}|\Pi^*(K)|\leq|B_2^n|^{n-1}|\Pi^*(B_2^n)|=\left(\frac{|B_2^n|}{|B_2^{n-1}|}\right)^n. \tag{1.1}$$

On the other hand, Zhang proved a reverse form of (1.1), showing that this quantity is minimized when K is a simplex. Thus, denoting by Δ^n the n-dimensional regular simplex,

$$|K|^{n-1}|\Pi^*(K)|\geq|\Delta_n|^{n-1}|\Pi^*(\Delta_n)|=\frac{1}{n^n}\left(\begin{array}{c}2n\\ n\end{array}\right). \tag{1.2}$$

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For any $K \in \mathcal{K}_0^n$, Steiner's formula says that the volume of $K + tB_2^n$ (where the sum is the Minkowski addition of two sets) can be expressed as a polynomial in t

$$|K + tB_2^n| = \sum_{k=0}^{n} \binom{n}{k} W_k(K) t^k.$$

The coefficients $W_k(K)$ are called the quermaßintegrals of K and, by Kubota's formula, they can be expressed

$$W_{n-k}(K) = \frac{|B_2^n|}{|B_2^k|} \int_{G_{n,k}} |P_E(K)| d\nu_{n,k}(E),$$

where $G_{n,k}$ denotes the Grassmannian manifold of the linear k-dimensional subspaces of $\mathbb{R}^n$, $d\nu_{n,k}$ is the unique Haar probability measure, invariant under orthogonal maps, on $G_{n,k}$ and P_E denotes the orthogonal projection onto the subspace E. Notice that $W_0(K) = |K|$, $nW_1(K) = |\partial K|$ (the surface area of K) and $W_{n-1}(K) = |B_2^n| w(K)$, (the mean width of K). We refer the reader to [SCH] for these and many other well-known facts in the Brunn-Minkowski theory.

In the same way as the volume of the $(n-1)$-dimensional projections of K define a norm in $\mathbb{R}^n$, the quermaßintegrals of the $(n-1)$-dimensional projections also define a norm, whose unit ball is the i-th polar projection body. Namely, if $1 \leq i \leq n-1$, $\Pi_i^*(K)$ is the unit ball of the norm given by

$$\|x\|_{\Pi_i^*(K)} = |x| W_{n-i-1}(P_{x^\perp}(K)) = \frac{1}{2} \int_{S^{n-1}} |\langle u, x \rangle| dS_i(K, u),$$

where $dS_i(K,u)$ denotes the i-th surface area measure of K. Notice that the $(n-1)$-th polar projection body is exactly the polar projection body defined before, $\Pi^*(K) = \Pi_{n-1}^*(K)$. However, when $i \neq n-1$, it is no longer true that $|K|^i |\Pi_i^*(K)|$ is an affine invariant.

In [L1], [L2] and [L3], the author studied the class of mixed projection bodies and gave sharp inequalities for them and their polars. Since the i-th polar projection bodies belong to this class, the following inequality which extends (1.1) was obtained:

$$|K|^i |\Pi_i^*(K)| \leq |B_2^n|^i |\Pi_i^*(B_2^n)| = \frac{|B_2^n|^{i+1}}{|B_2^{n-1}|^n}, \tag{1.3}$$

with equality if and only if $K = B_2^n$.

This inequality was strengthened in [L3]. When $i = n-1$, Zhang's inequality gives a lower bound for the quantity $|K|^i |\Pi_i^*(K)|$. From the results in [L3], one can easily deduce (see Section 3) the following lower bound for any i

$$|K|^i |\Pi_i^*(K)| \geq \frac{1}{n^n} \binom{2n}{n} \frac{|K|^{i+1}}{W_{n-i-1}(K)^n}. \tag{1.4}$$

However, there are no equality cases in this inequality unless $i = n-1$.

In [AJV], the authors studied the behavior of the θ-convolution body of two convex bodies

$$K +_\theta L = \{x \in K + L : |K \cap (x - L)| \geq \theta M(K, L)\},$$

where $M(K,L) = \max_{z \in \mathbb{R}^n} |K \cap (z - L)|$. In particular, since

$$\lim_{\theta \to 1^-} \frac{K +_\theta (-K)}{1 - \theta^{\frac{1}{n}}} = n|K| \Pi^*(K)$$

(see [S]), a new proof of Zhang's inequality (1.2) was obtained and this inequality was extended to the limiting convolution body of two different convex bodies:

$$\left|\lim_{\theta\to 1^-}\frac{K+_\theta L}{1-\theta^{\frac{1}{n}}}\right| \geq \binom{2n}{n}\frac{|K||L|}{M(K,L)}.$$

The results in this paper also characterized the equality cases in Rogers-Sephard inequality [RS]:

$$M(K,L)|K+L| \leq \binom{2n}{n}|K||L|. \tag{1.5}$$

In [TS], the author considered a different class of convolution bodies of two convex bodies (k-th θ-convolution bodies) and studied their limiting behavior when θ tends to 1. Changing slightly the definition in [TS], the k-th θ-convolution body of K and L is:

$$K+_{k,\theta} L := \{x\in K+L \,:\, W_{n-k}(K\cap(x-L)) \geq \theta M_{n-k}(K,L)\},$$

where $M_{n-k}(K,L) = \max\limits_{x\in K+L} W_{n-k}(K\cap(x-L))$. Notice that $K+_{n,\theta} L = K+_\theta L$.

In this paper we are going to follow the lines of [AJV] and study some properties of this class of convolution bodies, all this in order to prove some volume inequalities for the limiting convolution body and $K+L$ that can be viewed as an extension of Zhang's inequality and Rogers-Sephard inequality for the volume of the difference body.

We give an upper bound for the volume of the sum of K and L and a lower bound for the volume of the limiting k-th convolution body of K and L

$$C_k(K,L) := \lim_{\theta\to 1^-}\frac{K+_{k,\theta} L}{1-\theta^{\frac{1}{k}}}.$$

Special attention is paid to the case $k=n-1$, for which the inequalities we obtain are sharp and improve inequality (1.4):

Theorem 1.1. *Let $K,L\in\mathcal{K}_0^n$. Then*

$$|C_{n-1}(K,L)| \geq \binom{2n}{n}\frac{|K|W_1(L)+|L|W_1(K)}{2M_1(K,L)} \geq |K+L|$$

with equality in each one of the inequalities if and only if $K=-L$ is a simplex.

The left-hand side inequality improves inequality (1.4), when $L=-K$ and $k=i+1=n-1$ since, as we will see in Section 3, for any $1\leq k\leq n$ and any $K\subseteq\mathbb{R}^n$

$$C_k(K,-K) \subseteq nW_{n-k}(K)\Pi^*_{k-1}(K). \tag{1.6}$$

The right hand-side inequality gives an upper bound for the volume of the sum of two convex bodies K and L of a different nature than Rogers-Shephard inequality. Excluding the case when $L=-K$ is a simplex, for which we know Rogers-Shephard inequality is sharp, the upper bound in Theorem 1.1 seems to give a better bound for the volume $|K+L|$ than (1.5). Indeed, it is easy to see the latter for K and $L=-\lambda K$ with $\lambda>1$.

In [R], the author gave an upper bound for the volume of the sections of the difference body. Namely, he proved that for any $E\in G_{n,k}$

$$|(K-K)\cap E| \leq C^k\varphi(n,k)^k \max_{x\in\mathbb{R}^n}|K\cap(x+E)|, \tag{1.7}$$

where
$$\varphi(n,k)=\min\left\{\frac{n}{k},\sqrt{k}\right\}.$$

This estimate was used in [R2] to give an upper bound of $M(K)M^*(K)$ for any convex body K and consequently gave an upper bound for the Banach-Mazur distance between any two convex bodies (non-necessarily symmetric). In order to prove the $\frac{n}{k}$ upper bound the author proved some estimates than can be seen as volume inequalities for the k-th, θ convolution bodies of K and $-K$. We will provide some volume estimates for the sections of the sum of two convex bodies that, as a particular case, will recover Rudelson's $\frac{n}{k}$ upper bound providing a simpler proof of it.

The paper is organized as follows: In Section 2 we define the class of convolution bodies we will use and study some of their general properties. Since inequality (1.4) is not explicitly written in [L3], we show how it is deduced from the results there in Section 3. We also prove (1.6) to show that Theorem 1.1 is really an improvement of equation (1.4) when $k=i+1=n-1$. In Section 4 we give a lower bound for the volume of $C_k(K,L)$ which in particular gives the proof of Theorem 1.1. Finally in Section 5 we provide bounds for the volume of sections of the limiting convolution body $C_n(K,L)$ and the body $K+L$.

We denote by $\mathrm{span}\{x_1,\dots,x_m\}$ the smallest linear subspace that contains the vectors $x_1,\dots,x_m$. The 1-dimensional linear subspace generated by a vector x will be denoted by $\langle x\rangle$. The interior of a set A will be denoted by $\mathrm{int}(A)$. If A is contained in an affine subspace, $\mathrm{int}(A)$ refers to the relative interior of A in such subspace.

2. The h,θ-convolution bodies.

Definition 2.1. *Let $h:\mathcal{K}_0^n\to\mathbb{R}$ satisfying*

(*i*) *If $K\subseteq L$ then $h(K)\le h(L)$, for any $K,L\in\mathcal{K}_0^n$.*
(*ii*) *$h(a+K)=h(K)$, for any $a\in\mathbb{R}^n$ and $K\in\mathcal{K}_0^n$.*
(*iii*) *$h(\lambda K)=\lambda^k h(K)$ for any $0\le\lambda\le1$, $K\in\mathcal{K}_0^n$ and some integer k,*
(*iv*) *h satisfies a Brunn-Minkowski type inequality*
$$h((1-\lambda)K+\lambda L)^{\frac1k}\ge\lambda h(K)^{\frac1k}+\lambda h(L)^{\frac1k}.$$

We define the h,θ-convolution of K and L by
$$K+_{h,\theta}L:=\{x\in K+L\ :\ h(K\cap(x-L))\ge\theta M_h(K,L))\},$$
where $M_h(K,L)=\max\limits_{z\in K+L}h(K\cap(z-L))$. For all of our results, we can assume without loss of generality that $M_h(K,L)=K\cap(-L)$.

Remark. The quermaßintegrals $W_{n-k}(K)$ satisfy these hypotheses. In that case we have denoted $K+_{W_{n-k},\theta}L=K+_{k,\theta}L$.

The following proposition gives an inclusion relation between the h,θ-convolution bodies.

Proposition 2.1. *Let $K,L\in\mathcal{K}_0^n$. Then for every $\theta_1,\theta_2,\lambda_1,\lambda_2\in[0,1]$ such that $\lambda_1+\lambda_2\le1$ we have*
$$\lambda_1(K+_{h,\theta_1}L)+\lambda_2(K+_{h,\theta_2}L)\subseteq K+_{h,\theta}L,$$
where $1-\theta^{\frac1k}=\lambda_1(1-\theta_1^{\frac1k})+\lambda_2(1-\theta_2^{\frac1k})$.

Proof. Let $x_1 \in K +_{h,\theta_1} L$ and $x_2 \in K +_{h,\theta_2} L$. From the general inclusion

$$K \cap (\lambda_0 A_0 + \lambda_1 A_1 + \lambda_2 A_2) \supset \lambda_0 K \cap A_0 + \lambda_1 K \cap A_1 + \lambda_2 K \cap A_2$$

where K is convex and $\lambda_0 + \lambda_1 + \lambda_2 = 1$, and using the convexity of K and L, we have

$$K \cap (\lambda_1 x_1 + \lambda_2 x_2 - L) \supseteq (1 - \lambda_1 - \lambda_2)(K \cap (-L)) + \lambda_1 [K \cap (x_1 - L)] + \lambda_2 [K \cap (x_2 - L)].$$

By the properties of h and the fact that $x_i \in K +_{h,\theta_i} L$ we have

$$h(K \cap (\lambda_1 x_1 + \lambda_2 x_2 - L)) \geq [1 - \lambda_1 (1 - \theta_1^{\frac{1}{k}}) - \lambda_2 (1 - \theta_2^{\frac{1}{k}})]^k M(K, L),$$

which proves that $\lambda_1 x_1 + \lambda_2 x_2 \in K +_{h,\theta} L$ for $\theta = [1 - \lambda_1 (1 - \theta_1^{\frac{1}{k}}) - \lambda_2 (1 - \theta_2^{\frac{1}{k}})]^k$. □

Taking $\theta_1 = \theta_2$ and $\lambda_2 = 1 - \lambda_1$ we have

Corollary 2.1. *Let $K, L \in \mathcal{K}_0^n$ and $\theta \in [0, 1]$. Then $K +_{h,\theta} L$ is convex.*

Corollary 2.2. *Let $K, L \in \mathcal{K}_0^n$. Then, for every $0 \leq \theta_0 \leq \theta < 1$ we have*

$$\frac{K +_{h,\theta_0} L}{1 - \theta_0^{\frac{1}{k}}} \subseteq \frac{K +_{h,\theta} L}{1 - \theta^{\frac{1}{k}}}.$$

Proof. Taking $\theta_1 = \theta_2 = \theta_0$ in the above proposition, for any $\lambda_1, \lambda_2 \in [0, 1]$ such that $\lambda_1 + \lambda_2 \leq 1$

$$(\lambda_1 + \lambda_2)(K +_{h,\theta_0} L) = \lambda_1 (K +_{h,\theta_0} L) + \lambda_2 (K +_{h,\theta_0} L) \subseteq K +_{h,\theta} L,$$

with $1 - \theta^{\frac{1}{k}} = (\lambda_1 + \lambda_2)(1 - \theta_0^{\frac{1}{k}})$. Since $\lambda_1 + \lambda_2 = \dfrac{1 - \theta^{\frac{1}{k}}}{1 - \theta_0^{\frac{1}{k}}}$,

$$\frac{1 - \theta^{\frac{1}{k}}}{1 - \theta_0^{\frac{1}{k}}} (K +_{h,\theta_0} L) \subseteq K +_{h,\theta} L$$

whenever $\lambda_1 + \lambda_2 \leq 1$, which means $0 \leq \theta_0 \leq \theta \leq 1$.

□

The next proposition shows that if the equality cases in (*iv*) of Definition 2.1 occur K and L must be homothetic. Thus, it is a necessary condition for $K = -L$ to be a simplex in order to attain equality in all inequalities in Corollary 2.2. This is the case if $h(K) = W_{n-k}(K)$ $(k > n - 1)$.

Lemma 2.1. *Let h be like in Definition 2.1, such that equality in (iv) occurs if and only if K and L are homothetic. Assume that for every $0 \leq \theta_0 \leq \theta < 1$ we have*

$$\frac{K +_{h,\theta_0} L}{1 - \theta_0^{\frac{1}{k}}} = \frac{K +_{h,\theta} L}{1 - \theta^{\frac{1}{k}}}.$$

Then $K = -L$ is a simplex.

Proof. In particular, we have that for any $0 \leq \theta < 1$

$$K +_{h,\theta} L = (1 - \theta^{\frac{1}{k}})(K + L)$$

and

$$K +_{h,1} L = \{0\}.$$

Thus, for any $x \in K + L$, $x \in \partial(K +_{h,\theta} L)$ for some θ and

$$x = \theta^{\frac{1}{k}} 0 + (1 - \theta^{\frac{1}{k}}) y,$$

with $y \in K+L$. Since $x \in \partial(K +_{h,\theta} L)$ we have $h(K \cap (x-L)) = \theta M_h(K,L)$ and so, we have equality in

$$h^{\frac{1}{k}}(K \cap (x-L)) \geq h^{\frac{1}{k}}(\theta^{\frac{1}{k}}(K \cap (-L)) + ((1-\theta^{\frac{1}{k}})(K \cap (y-L))) \geq \theta^{\frac{1}{k}} M(K,L)^{\frac{1}{k}}.$$

Thus, $K \cap (x-L)$, $K \cap (-L)$ and $K \cap (y-L)$ are homothetic. By Soltan's characterization of a simplex ([S]), $K=-L$ is a simplex if and only if for every $x \in K+L$ $K \cap x - L$ is homothetic to $K \cap (-L)$. Thus, K and $-L$ are homothetic simplices. Since $K +_{h,1} L = \{0\}$, $K = -L$. □

The following proposition gives an upper inclusion for the h,θ-convolution bodies.

Proposition 2.2. *Let $K, L \in \mathcal{K}_0^n$ and h like in Definition 2.1 such that for any $v \in S^{n-1}$ $h(K \cap (tv-L))$ is differentiable in an interval $[0,\epsilon)$. Then, for any $\theta \in [0,1)$*

$$\frac{K +_{h,\theta} L}{1-\theta^{\frac{1}{k}}} \subseteq L_h(K,L),$$

where

$$L_h(K,L) := \left\{ x \in \mathbb{R}^n \,:\, -|x| \frac{d^+}{dt}\, h\left(K \cap \left(t\frac{x}{|x|} - L\right)\right)\Big|_{t=0} \leq k M_h(K,L) \right\}.$$

Proof. The concavity of the function $x \to h(K \cap (x-L))^{\frac{1}{k}}$ implies

$$\begin{aligned} h(K \cap (\lambda x - L)) &\geq \left((1-\lambda) M_h(K,L)^{\frac{1}{k}} + \lambda h(K \cap (x-L))^{\frac{1}{k}}\right)^k \\ &= M_h(K,L)\left[1 + \lambda\left(\frac{h(K\cap(x-L))^{\frac{1}{k}}}{M_h(K,L)^{\frac{1}{k}}} - 1\right)\right]^k \\ &\geq M_h(K,L)\left[1 + \lambda k\left(\frac{h(K\cap(x-L))^{\frac{1}{k}}}{M_h(K,L)^{\frac{1}{k}}} - 1\right)\right] \end{aligned}$$

for $\lambda \in [0,1]$ and $x \in K+L$. On the other hand,

$$\begin{aligned} h(K \cap (\lambda x - L)) &= M_h(K,L) + \int_0^{\lambda|x|} \frac{d^+}{dt} h\left(K \cap \left(t\frac{x}{|x|} - L\right)\right) dt \\ &\leq M_h(K,L) + \lambda |x| \max_{t \in [0,\lambda|x|]} \frac{d^+}{dt} h\left(K \cap \left(t\frac{x}{|x|} - L\right)\right) \end{aligned}$$

again using the concavity of $x \to h(K \cap (x-L))^{\frac{1}{k}}$. Comparing these two inequalities, and letting $\lambda \to 0^+$, we obtain

$$k M_h(K,L)\left(\frac{h(K\cap(x-L))^{\frac{1}{k}}}{M_h(K,L))^{\frac{1}{k}}} - 1\right) \leq |x| \frac{d^+}{dt} h\left(K \cap \left(t\frac{x}{|x|} - L\right)\right)\Big|_{t=0}.$$

Since the lateral derivative is non positive, we get the desired inclusion. □

The following lemmas show that, when $K=-L$ is a simplex, all the inclusions above are identities. The first lemma shows that when $K=-L$ is a simplex, then the h,θ-convolution of a linear image of the body is the linear image of the h,θ convolution.

Lemma 2.2. *Let K be a simplex. Then, for any $T \in GL(n)$*

$$TK +_{h,\theta} (-TK) = T(K +_{h,\theta} (-K)).$$

Proof. By Soltan's result [S], K is a simplex if and only if for every $x \in K - K$ $K \cap x + K$ is homothetic to K. Thus, if K is a simplex, for every $x \in K - K$

$$K \cap (x + K) = a(x) + \lambda(x)K.$$

Consequently

$$\begin{aligned} K +_{h,\theta} (-K) &= \{x \in K - K \,:\, h(\lambda(x)K) \geq \theta h(K)\} \\ &= \{x \in K - K \,:\, \lambda(x)^k \geq \theta\}. \end{aligned}$$

For any $T \in GL(n)$ we have

$$\begin{aligned} TK +_{h,\theta} (-TK) &= \{x \in TK - TK \,:\, h(TK \cap (x + TK)) \geq \theta h(TK)\} \\ &= \{x \in T(K - K) \,:\, h(T(K \cap (T^{-1}x + K)) \geq \theta h(TK)\} \\ &= \{x \in T(K - K) \,:\, h(T\lambda(T^{-1}x)K) \geq \theta h(TK)\} \\ &= \{x \in T(K - K) \,:\, \lambda(T^{-1}x)^k \geq \theta\} \\ &= T(K +_{h,\theta} (-K)). \end{aligned}$$

□

Lemma 2.3. *Let $K \subseteq \mathbb{R}^n$ be a simplex. Then, for any $\theta \in [0, 1]$*

$$K +_{h,\theta} (-K) = (1 - \theta^{\frac{1}{k}})(K - K).$$

Proof. The $\supseteq$ part of the identity is a consequence of Corollary 2.2. By the previous lemma we can assume, without loss of generality, that $K = \text{conv}\{0, e_1, \dots, e_n\}$. Then, as it was shown in [AJV],

$$K \cap (x + K) = a(x) + \lambda(x)K,$$

with

$$\lambda(x) = \frac{1}{2}\left(2 - \left|\sum_{i=1}^{n} x_i\right| - \sum_{i=1}^{n} |x_i|\right).$$

Consequently,

$$\begin{aligned} K +_{h,\theta} (-K) &= \left\{x \in K - K \,:\, \left|\sum_{i=1}^{n} x_i\right| + \sum_{i=1}^{n} |x_i| \leq 2(1 - \theta^{\frac{1}{k}})\right\} \\ &= \left(1 - \theta^{\frac{1}{k}}\right)\left\{x \in K - K \,:\, \left|\sum_{i=1}^{n} x_i\right| + \sum_{i=1}^{n} |x_i| \leq 2\right\} \\ &= \left(1 - \theta^{\frac{1}{k}}\right)(K +_{h,0} (-K)) \\ &= \left(1 - \theta^{\frac{1}{k}}\right)(K - K). \end{aligned}$$

□

Lemma 2.4. *Let $K \subseteq \mathbb{R}^n$ be a simplex. Then, the set $L_h(K, -K)$ defined in Proposition 2.2 is*

$$L_h(K, -K) = K - K.$$

Proof. We can assume, without loss of generality, that $K = \text{conv}\{0, e_1, \dots e_n\}$. Then for any $v \in S^{n-1}$

$$h(K \cap (tv + K)) = h(\lambda(tv)K) = \lambda^k(tv)h(K).$$

with

$$\lambda(tv) = 1 - \frac{|t|}{2}\left(\left|\sum_{i=1}^{n} v_i\right| + \sum_{i=1}^{n} |v_i|\right).$$

Consequently

$$\begin{aligned}\frac{d}{dt^+}h(K\cap(tv+K))\Big|_{t=0} &= -kh(K)\lambda^{k-1}(tv)\frac{1}{2}\left(\left|\sum_{i=1}^n v_i\right|+\sum_{i=1}^n|v_i|\right)\Bigg|_{t=0}\\ &= -kh(K)\frac{1}{2}\left(\left|\sum_{i=1}^n v_i\right|+\sum_{i=1}^n|v_i|\right).\end{aligned}$$

Thus

$$L_h(K,-K)=\left\{x\in\mathbb{R}^n\,:\,\left|\sum_{i=1}^n x_i\right|+\sum_{i=1}^n|x_i|\leq 2\right\}=K-K.$$

□

3. Lower bound for the volume of the $i-th$ polar projection body

In this section we are going to show how inequality (1.4) is deduced from the results in [L3], and the relation between this inequality and the inequality in Theorem 1.1. In [L3], the author studied the volume of mixed bodies. A particular case of these bodies is the body $[K]_i$ defined by

$$dS_{n-1}([K]_i,\theta)=dS_{n-i-1}(K,\theta).$$

The following estimate for their volume was given:

$$|[K]_i|^{n-1}\leq\frac{W_i(K)^n}{|K|},$$

with equality if and only if $[K]_i$ and K are homothetic. This reduces to the fact that K is an $(n-i-1)$ tangential body of B_2^n *i.e.*, a body such that every support hyperplane of K that is not a support hyperplane of B_2^n contains only $(n-i-2)$ singular points of K.

On the other hand, from the definition of $[K]_i$

$$\Pi^*([K]_{n-i-1})=\Pi^*_i(K).$$

Thus, using Zhang's inequality we obtain

$$|K|^i|\Pi^*_i(K)|\geq\frac{|K|^i}{|[K]_{n-i-1}|^{n-1}}\frac{1}{n^n}\binom{2n}{n}\geq\frac{1}{n^n}\binom{2n}{n}\frac{|K|^{i+1}}{W_{n-i-1}(K)^n}.$$

There is equality in the above inequalities if and only if K is an i-tangential body of a ball and $[K]_{n-i-1}$, which has to be homothetic to K, is a simplex. Since the simplex is a p-tangential body of B_2^n only for $p=n-1$ there is no equality unless $i=n-1$.

Let $L_k(K)=L_{W_{n-k}}(K,-K)$. The following result shows that the inequality given in Theorem 1.1 improves inequality (1.4):

Proposition 3.1. *Let $K\in\mathcal{K}_0^n$. Then*

$$C_k(K,-K)\subseteq L_k(K)\subseteq nW_{n-k}(K)\Pi^*_{k-1}(K).$$

Proof. The first inclusion has been shown in Section 2. For the second one, let $v\in S^{n-1}$. Then

$$\begin{aligned}&\frac{d^+}{dt}W_{n-k}\left(K\cap(tv+K)\right)|_{t=0}=\\ &=\frac{|B_2^n|}{|B_2^k|}\lim_{t\to 0^+}\int_{G_{n,k}}\frac{|P_E(K\cap(tv+K))|-|P_E(K)|}{t}d\nu_{n,k}(E)\end{aligned}$$

$$
\begin{aligned}
&= \frac{|B_2^n|}{|B_2^k|}\int_{G_{n,k}} \lim_{t\to 0^+}\frac{|P_E(K\cap(tv+K))|-|P_E(K)|}{t}d\nu_{n,k}(E)\\
&\leq \frac{|B_2^n|}{|B_2^k|}\int_{G_{n,k}} \lim_{t\to 0^+}\frac{|P_E(K)\cap(tP_Ev+P_E(K))|-|P_E(K)|}{t}d\nu_{n,k}(E)\\
&= -\frac{|B_2^n|}{|B_2^k|}\int_{G_{n,k}} |P_Ev||P_{(P_Ev)^\perp\cap E}(K)|d\nu_{n,k}(E).
\end{aligned}
$$

For any k-dimensional subspace E, if $u_1,\dots,u_{n-k}$ is an orthonormal basis of $E^\perp$, we have that

$$
\begin{aligned}
|P_Ev| &= \sqrt{1-\sum_{i=1}^{n-k}\langle v,u_i\rangle^2}\\
&= \sqrt{1-\sum_{i=1}^{n-k}|P_{\text{span}\{u_1,\dots,u_{i-1}\}^\perp}v|^2\langle\frac{P_{\text{span}\{u_1,\dots,u_{i-1}\}^\perp}v}{|P_{\text{span}\{u_1,\dots,u_{i-1}\}^\perp}v|},u_i\rangle^2}
\end{aligned}
$$

and

$$
(P_Ev)^\perp\cap E = \text{span}\{v,u_1,\dots,u_{n-k}\}^\perp = \text{span}\{v,\xi_1,\dots,\xi_{n-k}\}^\perp,
$$

where $\xi_1=P_{v^\perp}u_1$ and $\xi_i=P_{\text{span}\{v,\xi_1,\dots,\xi_{i-1}\}^\perp}u_i$ $(i>1)$.

By uniqueness of the Haar probability measure on $G_{n,k}$, the above integral equals

$$
-\frac{|B_2^n|}{|B_2^k|}\int\int\dots\int g_v(u_1,\dots,u_{n-k})d\sigma(u_{n-k})\dots d\sigma(u_1),
$$

where u_1 runs over S^{n-1}, u_i runs over $S^{n-1}\cap\text{span}\{u_1,\dots,u_{i-1}\}^\perp$ $(i>1)$ and

$$
\begin{aligned}
g_v(u_1,\dots,u_{n-k}) &= \sqrt{1-\sum_{i=1}^{n}|P_{\text{span}\{u_1,\dots,u_{i-1}\}^\perp}v|^2\langle\frac{P_{\text{span}\{u_1,\dots,u_{i-1}\}^\perp}v}{|P_{\text{span}\{u_1,\dots,u_{i-1}\}^\perp}v|},u_i\rangle^2}\times\\
&\times\quad |P_{\text{span}\{\xi_1,\dots,\xi_{n-k}\}^\perp}P_{v^\perp}(K)|.
\end{aligned}
$$

Now, using the slice integration formula on each one of the spheres, in the direction $\frac{P_{\text{span}\{u_1,\dots,u_{i_1}\}^\perp}v}{|P_{\text{span}\{u_1,\dots,u_{i_1}\}^\perp}v|}$, we obtain that the previous integral equals

$$
\begin{aligned}
&-\quad \frac{k}{n}\int_{-1}^1\dots\int_{-1}^1(1-x_1^2)^{\frac{n-2}{2}}(1-x_2^2)^{\frac{n-3}{2}}\dots(1-x_{n-k}^2)^{\frac{k-1}{2}}dx_{n-k}\dots dx_1\times\\
&\times\quad \int\int\dots\int|P_{\text{span}\{\xi_1,\dots,\xi_{n-k}\}^\perp}P_v^\perp(K)|d\sigma(\xi_{n-k})\dots d\sigma(\xi_1),
\end{aligned}
$$

where ξ_1 runs over $S^{n-1}\cap v^\perp$ and ξ_i runs over $S^{n-1}\cap\text{span}\{v,\xi_1,\dots,\xi_{i-1}\}^\perp$. By uniqueness of the Haar measure in $G_{v^\perp,k-1}$ equals

$$
\begin{aligned}
&-\quad \frac{k}{n}\int_{-1}^1\dots\int_{-1}^1(1-x_1^2)^{\frac{n-2}{2}}(1-x_2^2)^{\frac{n-3}{2}}\dots(1-x_{n-k}^2)^{\frac{k-1}{2}}dx_{n-k}\dots dx_1\times\\
&\times\quad \int_{G_{v^\perp,k-1}}|P_EP_{v^\perp}(K)|d\nu_{n-1,k-1}\\
&=\quad -\frac{k|B_2^{k-1}|}{n|B_2^{n-1}|}\int_{-1}^1\dots\int_{-1}^1(1-x_1^2)^{\frac{n-2}{2}}(1-x_2^2)^{\frac{n-3}{2}}\dots(1-x_{n-k}^2)^{\frac{k-1}{2}}dx_{n-k}\dots dx_1\times\\
&\times\quad W_{n-k}(P_{v^\perp}(K))
\end{aligned}
$$

$$= \quad -\frac{k|B_2^{k-1}|}{n|B_2^{n-1}|}\frac{(\sqrt{\pi})^{n-k}\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}W_{n-k}(P_{v^\perp}(K)) = -\frac{k}{n}\|v\|_{\Pi^*_{k-1}(K)}.$$

Consequently

$$L_k(K) \subseteq nW_{n-k}(K)\Pi^*_{k-1}(K).$$

□

4. Rogers-Sephard inequality and Zhang's inequality for $C_{n-1}(K)$

In this section we prove Theorem 1.1. It is a consequence of Theorem 4.2 and the following:

Theorem 4.1. *Let $K \in \mathcal{K}^n_0$, h a function like in Definition 2.1 and*

$$C_h(K,L) := \lim_{\theta\to 1^-}\frac{K +_{h,\theta} L}{1-\theta^{\frac{1}{k}}}$$

Then

$$|C_h(K,L)| \geq \binom{n+k}{n}\int_{\mathbb{R}^n}\frac{h(K\cap(x-L))}{M_h(K,L)}dx \geq |K+L|,$$

with equality when $K=-L$ is a simplex. If h is like in Lemma 2.1, then there is equality if and only if $K=-L$ is a simplex.

Proof. By Proposition 2.2, for any $\theta \in [0,1)$

$$|C_h(K,L)|(1-\theta^{\frac{1}{k}})^n \geq |K +_{h,\theta} L| \geq |K+L|(1-\theta^{\frac{1}{k}})^n.$$

Thus

$$|C_h(K,L)|\int_0^1(1-\theta^{\frac{1}{k}})^n d\theta \geq \int_0^1 |K +_{h,\theta} L| d\theta \geq |K+L|\int_0^1(1-\theta^{\frac{1}{k}})^n d\theta.$$

Since

$$\begin{aligned}\int_0^1 |K +_{h,\theta} L| d\theta &= \int_0^1\int_{\mathbb{R}^n}\chi_{h(K\cap(y-L))\geq\theta M_h(K,L)}(x)dxd\theta\\ &= \int_{\mathbb{R}^n}\frac{h(K\cap(x-L))}{M_h(K,L)}dx\end{aligned}$$

we obtain the result. By the Lemmas in the previous Section, all the inequalities are equalities when $K=-L$ is a simplex and if h is like in Lemma 2.1, then there is equality if and only if $K=-L$ is a simplex. □

Taking $h(K) = W_{n-k}(K)$, we obtain the following Theorem, which in particular gives Theorem 1.1, since the inequality we obtain computing the integral $\int_{\mathbb{R}^n}\frac{h(K\cap(x-L))}{M_h(K,L)}dx$ is an equality when $h(K)=W_1(K)$:

Theorem 4.2. *Let $K \in \mathcal{K}^n_0$. Then, for any $1 \leq k \leq n$*

$$|C_k(K,L)| \geq \binom{n+k}{n}\frac{|K|W_{n-k}(L)+|L|W_{n-k}(K)}{W_{n-k}(K\cap(-L))}.$$

If $L=-K$ we can slightly improve this to

$$|C_k(K,-K)| \geq \binom{2n}{n}\binom{2n}{n-k}^{-1}\left(2\binom{n}{k}+2^{n-k}-2\right)|K|.$$

When $k=n-1$ these inequalities are sharp and we have equality if and only if $K=-L$ is a simplex.

Proof. If we take $h(K) = W_{n-k}(K)$ we have, by Crofton's intersection formula (see [SCH], page 235) that

$$W_{n-k}(K) = C_{n,k}\mu_{n,n-k}\{E \in \mathbb{A}_{n,n-k} \,:\, K \cap E \neq \emptyset\},$$

where $C_{n,k}$ is a constant depending only on n and k and $d\mu_{n,n-k}$ is the Haar measure on the set of affine $(n-k)$-dimensional subspaces of $\mathbb{R}^n$, $\mathbb{A}_{n,n-k}$. Thus

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{h(K\cap(x-L))}{M_h(K,L)}dx &= \frac{\int_{\mathbb{R}^n}\int_{\mathbb{A}_{n,n-k}} \chi_{\{K\cap(x-L)\cap E\neq\emptyset\}}(E)d\mu_{n,n-k}(E)dx}{\mu_{n,n-k}\{E \in \mathbb{A}_{n,n-k} \,:\, K\cap(-L)\cap E \neq \emptyset\}}\\ &= \frac{\int_{\{E\in\mathbb{A}_{n,n-k}\,:\,K\cap E\neq\emptyset\}} |(K\cap E)+L| d\mu_{n,n-k}(E)}{\mu_{n,n-k}\{E \in \mathbb{A}_{n,n-k} \,:\, K\cap(-L)\cap E \neq \emptyset\}}\end{aligned}$$

For every $E \in \mathbb{A}_{n,n-k}$, calling E_0 the linear subspace parallel to E,

$$|(K\cap E)+L| = \int_{P_{E_0^\perp}L} |(K\cap E)+(L\cap(y+E_0))|dy.$$

Thus, since for any subspace $E_0 \in G_{n,k}$, $\binom{n}{k}\max_{x\in E_0^\perp}|K\cap(x+E_o)||P_{E_0^\perp}(K)| \leq |K|$ (see [Pi], Lemma 8.8 for a proof in the symmetric case, which also works in the non-symmetric case).

$$\begin{aligned}&\int_{\{E\in\mathbb{A}_{n,n-k}\,:\,K\cap E\neq\emptyset\}} |(K\cap E)+L|d\mu_{n,n-k}(E)\\ =\ &\int_{G_{n,n-k}}\int_{P_{E_0^\perp}(K)}\int_{P_{E_0^\perp}(L)} |(K\cap(z+E_0))+(L\cap(y+E_0))|dydzd\nu_{n,n-k}(E_0)\\ \geq\ &\int_{G_{n,n-k}}\int_{P_{E_0^\perp}(K)}\int_{P_{E_0^\perp}(L)} \left(|(K\cap(z+E_0))|^{\frac{1}{n-k}}+|(L\cap(y+E_0))|^{\frac{1}{n-k}}\right)^{n-k}\times\\ \times\ & dydzd\nu_{n,n-k}(E_0)\\ \geq\ & |K|\int_{G_{n,n-k}}|P_{E_0^\perp}(L)|d\nu_{n,n-k}+|L|\int_{G_{n,n-k}}|P_{E_0^\perp}(K)|d\nu_{n,n-k},\end{aligned}$$

where the first inequality follows from the $(n-k)$-dimensional version of Brunn-Minkowski inequality and the second one follows from the fact that $(a+b)^{n-k} \geq a^{n-k}+b^{n-k}$ for any $a,b \geq 0$.

Since

$$\begin{aligned}\mu_{n,n-k}\{E\in\mathbb{A}_{n,1} \,:\, K\cap(-L)\cap E\neq\emptyset\} &= \int_{G_{n,n-k}} |P_{E_0}^\perp(K\cap(-L))|d\nu_{n,n-n}(E_0)\\ &= \frac{|B_2^k|}{|B_2^n|}W_{n-k}(K\cap(-L))\end{aligned}$$

we have

$$\int_{\mathbb{R}^n}\frac{W_{n-k}(K\cap(x-L))}{W_{n-k}(K\cap(-L))}dx \geq \frac{|K|W_{n-k}(L)+|L|W_{n-k}(K)}{W_{n-k}(K\cap(-L))}$$

Thus

$$|C_k(K,L)| \quad\geq\quad \binom{n+k}{n}\frac{|K|W_{n-k}(L)+|L|W_{n-k}(K)}{W_{n-k}(K\cap L)}.$$

Notice that if $k=n-1$ the above inequalities become equalities. If $L=-K$, we have

$$\begin{aligned}
&\int_{\{E\in\mathbb{A}_{n,n-k}\,:\,K\cap E\neq\emptyset\}}|(K\cap E)-K|d\mu_{n,n-k}(E)\\
=\;&\int_{G_{n,n-k}}\int_{P_{E_0^\perp}(K)}\int_{P_{E_0^\perp}(-K)}|(K\cap(z+E_0))+((-K)\cap(y+E_0))|\times\\
\times\;&dydzd\nu_{n,n-k}(E_0)\\
\geq\;&\int_{G_{n,n-k}}\int_{P_{E_0^\perp}(K)}\int_{P_{E_0^\perp}(-K)}\left(|K\cap(z+E_0)|^{\frac{1}{n-k}}+|(-K)\cap(y+E_0)|^{\frac{1}{n-k}}\right)^{n-k}\\
\times\;&dydzd\nu_{n,n-k}(E_0)\\
\geq\;&\int_{G_{n,n-k}}\int_{P_{E_0^\perp}(K)}\int_{P_{E_0^\perp}(-K)}\sum_{i=0}^{n-k}\binom{n-k}{i}|K\cap(z+E_0)|^{\frac{i}{n-k}}\times\\
\times\;&|(-K)\cap(y+E_0)|^{\frac{n-k-i}{n-k}}dydzd\nu_{n,n-k}(E_0)\\
=\;&2|K|\int_{G_{n,n-k}}|P_{E_0^\perp}(K)|d\nu_{n,n-k}\\
+\;&\sum_{i=1}^{n-k-1}\binom{n-k}{i}\int_{G_{n,n-k}}\int_{P_{E_0^\perp}(K)}\int_{P_{E_0^\perp}(-K)}\frac{|K\cap(z+E_0)|}{|K\cap(z+E_0)|^{\frac{n-k-i}{n-k}}}\times\\
\times\;&\frac{|(-K)\cap(y+E_0)|}{|(-K)\cap(y+E_0)|^{\frac{i}{k}}}dydzd\nu_{n,n-k}\\
\geq\;&2|K|\int_{G_{n,n-k}}|P_{E_0^\perp}(K)|d\nu_{n,n-k}\\
+\;&\sum_{i=1}^{n-k-1}\binom{n-k}{i}\int_{G_{n,n-k}}\int_{P_{E_0^\perp}(K)}\int_{P_{E_0^\perp}(-K)}|K\cap(z+E_0)|\times\\
\times\;&\frac{|(-K)\cap(y+E_0)|}{\max_{x\in P_E(K)}|K\cap(x+E_0)|}dydzd\nu_{n,n-k}\\
\geq\;&2|K|\int_{G_{n,n-k}}|P_{E_0^\perp}(K)|d\nu_{n,n-k}\\
+\;&(2^{n-k}-2)\int_{G_{n,n-k}}\frac{|K|^2}{\max_{x\in P_E(K)}|K\cap(x+E_0)|}d\nu_{n,n-k}\\
\geq\;&2|K|\int_{G_{n,n-k}}|P_{E_0^\perp}(K)|d\nu_{n,n-k}\\
+\;&(2^{n-k}-2)\binom{n}{k}^{-1}|K|\int_{G_{n,n-k}}|P_{E_0^\perp(K)}|d\nu_{n,n-k}\\
=\;&\left(2\binom{n}{k}+2^{n-k}-2\right)\binom{n}{k}^{-1}|K|\frac{|B_2^k|}{|B_2^n|}W_{n-k}(K).
\end{aligned}$$

and then

$$\int_{\mathbb{R}^n}\frac{W_{n-k}(K\cap(x+K))}{W_{n-k}(K)}dx\geq\left(2\binom{n}{k}+2^{n-k}-2\right)\binom{n}{k}^{-1}|K|.$$

Thus

$$|C_k(K,-K)|\quad\geq\quad\binom{n+k}{n}\binom{n}{k}^{-1}\left(2\binom{n}{k}+2^{n-k}-2\right)|K|$$

$$= \binom{2n}{n}\binom{2n}{n-k}^{-1}\left(2\binom{n}{k}+2^{n-k}-2\right)|K|.$$

□

5. Sections of the difference body and the polar projection body

In the following proposition we use the inclusion relation we obtained for the h,θ-convolution bodies (for h being the volume of the projection onto a subspace) to give an estimate for the volume of the sections of the Minkowski sum of two convex bodies. In particular, taking h the volume (which is the volume the projection onto $\mathbb{R}^n$) we can give a simpler proof of the upper bound in (1.7) involving the $\frac{n}{k}$ term.

Proposition 5.1. *Let $E\in G_{n,k}$ be a linear subspace and let $F\in G_{n,l}$ be a linear subspace such that $E\subseteq F$. Then, for any K,L convex bodies we have*

$$|(K+L)\cap E|\leq\binom{l+k}{k}\int_{F\cap E^\perp}\frac{|P_F(K)\cap(x+E)||P_F(-L)\cap(x+E)|}{\max_{z\in\mathbb{R}^n}|P_F(K\cap(z-L)|}dx$$

In particular, if $L=-K$ we obtain the following estimate for the volume of the sections of the difference body

$$|(K-K)\cap E| \quad\leq\quad \binom{l+k}{k}\inf_{F\in G_{n,l},E\subseteq F}\max_{x\in F}|P_F(K)\cap(x+E)|$$

Proof. Let $h(K)$=$P_F(K)$. By Corollary 2.2, we have that

$$(1-\theta^{\frac{1}{l}})^k((K+L)\cap E)\subseteq(K+_{h,\theta}L)\cap E.$$

Thus, taking volumes and integrating in $[0,1]$ we obtain

$$\binom{k+l}{k}^{-1}|(K+L)\cap E|\leq\int_0^1|(K+_{h,\theta}L)\cap E|d\theta.$$

Now, since $E\subseteq F$,

$$\begin{aligned}
\int_0^1|(K+_{h,\theta}L)\cap E|d\theta &= \int_E\frac{|P_F(K\cap(x-L))|}{M_h(K,L)}dx\\
&\leq \int_E\frac{|P_F(K)\cap(x-P_F(L)))|}{M_h(K,L)}dx\\
&= \frac{1}{M_h(K,L)}\int_E\int_F\chi_{P_F(K)}(y)\chi_{x-P_F(L)}(y)dydx\\
&= \frac{1}{M_h(K,L)}\int_F\int_E\chi_{P_F(K)}(y)\chi_{y+P_F(L)}(x)dxdy\\
&= \frac{1}{M_h(K,L)}\int_F\chi_{P_F(K)}(y)|(y+P_F(L))\cap E|dy\\
&= \int_{F\cap E^\perp}\frac{|P_F(K)\cap(z+E)||(-P_F(L))\cap(z+E)|}{M_h(K,L)}dz
\end{aligned}$$

In particular, if $L=-K$

$$\begin{aligned}
|(K-K)\cap E| &\leq \binom{l+k}{k}\inf_{F\in G_{n,l},E\subseteq F}\int_{F\cap E^\perp}\frac{|P_F(K)\cap(x+E)|^2}{|P_F(K)|}dx\\
&\leq \binom{l+k}{k}\inf_{F\in G_{n,l},E\subseteq F}\max_{x\in F}|P_F(K)\cap(x+E)|
\end{aligned}$$

□

Remark. If we take $L=-K$, $F=\mathbb{R}^n$, we obtain

$$\begin{aligned}|(K-K)\cap E| &\leq \binom{n+k}{k}\max_{x\in\mathbb{R}^n}|P_F(K)\cap(x+E)|\\ &\leq e^k\left(1+\frac{n}{k}\right)^k\max_{x\in\mathbb{R}^n}|K\cap(x+E)|\end{aligned}$$

and recover one of the two upper bounds proved in (1.7) for the volume of the sections of the difference body.

In the same way we can give a lower bound for the volume of the sections of the polar projection body of a convex body:

Proposition 5.2. *Let $E\in G_{n,k}$ be a linear subspace. Then, for any K,L convex bodies we have*

$$|C_n(K,L)\cap E|\geq\binom{n+k}{n}\int_{E^\perp}\frac{|K\cap(x+E)||(-L)\cap(x+E)|}{M_0(K,L)}dx.$$

When $L=-K$

$$n^k|K|^k|\Pi^*(K)\cap E|\geq\binom{n+k}{n}\frac{|K|}{|P_{E^\perp}(K)|}.$$

Proof. By Corollary 2.2, we have that

$$(1-\theta^{\frac{1}{n}})C_n(K,L)\cap E\supseteq(K+_{n,\theta}L)\cap E.$$

Taking volumes and integrating in $[0,1]$ we have

$$\binom{n+k}{n}^{-1}|C_n(K,L)\cap E|\geq\int_0^1|(K+_{n,\theta}L)\cap E|d\theta.$$

Now,

$$\begin{aligned}\int_0^1|(K+_{n,\theta}L)\cap E|d\theta &= \int_E\int_0^1\chi_{\{x\in\mathbb{R}^n\,:\,|K\cap(x-L)|\geq\theta M_0(K,L)\}}(z)d\theta dz\\ &= \int_E\frac{|K\cap(z-L)|}{M_0(K,L)}dz=\frac{\int_E\int_{\mathbb{R}^n}\chi_K(y)\chi_{z-L}(y)dydz}{M_0(K,L)}\\ &= \frac{\int_E\int_{\mathbb{R}^n}\chi_K(y)\chi_{y+L}(z)dydz}{M_0(K,L)}\\ &= \frac{\int_{\mathbb{R}^n}\chi_K(y)|(y+L)\cap E|dy}{M_0(K,L)}\\ &= \frac{\int_{\mathbb{R}^n}\chi_K(y)|(-L)\cap(y+E)|dy}{M_0(K,L)}\\ &= \int_{E^\perp}\frac{|K\cap(x+E)||(-L)\cap(x+E)|dx}{M_0(K,L)}.\end{aligned}$$

In particular, if $L=-K$, this integral equals

$$\begin{aligned}\frac{1}{|K|}\int_{E^\perp}|K\cap(x+E)|^2dx &= \frac{|P_{E^\perp}(K)|}{|K|}\frac{1}{|P_{E^\perp}(K)|}\int_{E^\perp}|K\cap(x+E)|^2dx\\ &\geq \frac{|P_{E^\perp}(K)|}{|K|}\left(\frac{1}{|P_{E^\perp}(K)|}\int_{E^\perp}|K\cap(x+E)|dx\right)^2\\ &= \frac{|K|}{|P_{E^\perp}(K)|}.\end{aligned}$$

□

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