Appendix A

Preliminaries

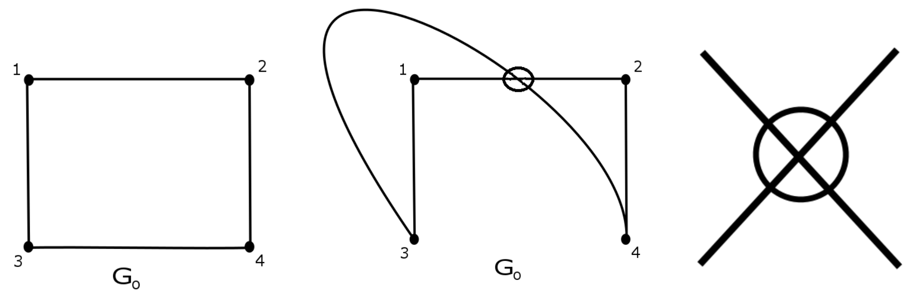
This appendix consists in a brief summary of the basics in Knot Theory and in particular in what is the central topic of the paper, Virtual Knot Theory, in order to gain the necessary background to follow all the exposed.

A.1 Graph Theory

Before entering in the subject and due to the fact that there is a strong relation and common core between knots and graphs, some concepts of these last will be reviewed. *Graph Theory* consists in the study of certain geometrical objects called *graphs*:

Definition A.1.1. We call *abstract graph* any set $G = (V,E)$, where V is a finite set whose elements are called *vertices* and E a multiset whose elements, called *edges*, are sets of two vertices.

These sets are usually represented in $\mathbb{R}^2$, where the vertices are drawn as (different) points and the edges as segments joining its two vertices. We call any of these representations *graphs* and *vertices and egdes of the graph* to the image of the vertices and the edges in the representation. An example of a graph is given in Figure A.1 (a): we have the abstract graph $G_o = (V_o, E_o)$ with $V_o = \{1,2,3,4\}$ and $E_o = \{\{1,2\},\{2,3\},\{3,4\},\{4,1\}\}$ represented in the plane, a graph of G_o.



(a) Planar graph of G_o. (b) Non-planar graph of G_o. (c) Virtual crossing.

Figure A.1: Graphs of G_o and virtual intersection/crossing.

Definition A.1.2. Given an abstract graph $G = (V,E)$, we say $e \in E$ is incident to $v \in V$ if $v \in e$ (if v is one of the endpoints of e in any of its graphs). We denote as $E(v)$ the set of egdes incident to v.

Definition A.1.3. Given a graph $G = (V,E)$ and $n \in \mathbb{N}$, we say G is a *n-valent abstract graph* if $|E(v)| = n \ \forall v \in V$. Any of its graphs are said to be *n-valent graphs*.

On the other hand, in Graph Theory it is very important the concept of planarity:

Definition A.1.4. Given a graph g, we say g is *planar* if the edges do not intersect outside of the vertices.

As we can see, the graph in Figure A.1 (a) is a planar graph, but in Figure A.1 (b) we have a non-planar graph of G_o. The intersection of the edges (outside of the vertices) is represented as in Figure A.1 (c) in order to differenciate it from the vertices. We call this kind of intersections *virtual crossings*.

Definition A.1.5. Given an abstract graph G, we say G is *planar* if there exists a planar graph g of G.

The abstract graph G_o of our example is planar because there exists at least one planar graph of G_o, the given firstly, but not all abstract graphs have a planar graph. We will see that, in the context of Virtual Knot Theory, these concepts serve to clarify and help to understand the complex universe of virtual knots.

A.2 Knot Theory

From the Stone Age to our recent days, knots has been a key element in the development of mankind. Mainly used as tools in devices, constructions and more, its particular structure make them essential for certain specific purposes. Due to this particular geometry, they constitute an object of analysis and study in some disciplines.

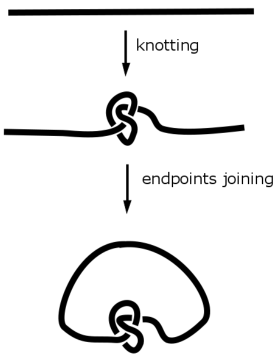


Figure A.2: Description of mathematical knot.

In mathematics, an idealization of what we understand as knot conforms a relatively young branch of topology: routhly speaking, a mathematical knot is the object resultant of knotting a given rope in whatever way and joinning its endpoints, forming a kind of "intertwined circunference" in the 3-dimensional space as represented in Figure A.2. In a more rigurous sense:

Definition A.2.1. Given two topological spaces A and B and an application between them $h : A \longrightarrow B$, we say h is a *homeomorphism* if it is bijective, continuous and h^{-1} is continuous as well.

Definition A.2.2. Given $K : S^1 \longrightarrow \mathbb{R}^3$ map, we say K is a *knot* if it is an embedding (in a topological context), that is, if the map $g : S^1 \longrightarrow K(S^1)$ such that $K = i_{K(S^1)} \circ g$ is a homeomorphism, where $i_{K(S^1)} : K(S^1) \longrightarrow \mathbb{R}^3$ is the inclusion and $K(S^1)$ inherits the topology of R^3.

This topological structure frequently appear describing certain natural geometrical phenomena. Their study is fundamental within Topology and Geometry (where knots appear as selfintersections and boundaries of surfaces as well as in the study of 3-manifolds among other things) and become useful in research in Physics and Biochemistry.

The branch that takes care of understanding them is the so-called *Knot Theory*. Although the first investigations associated to knots took place in the late XVIII Century by the hands of C.F. Gauss or A.

Vandermonde and the (erroneous) atomic model based in knots of Lord Kelvin in the 1860*s* increased the interest in them, they stayed in a secondary position until the beginning of XX Century, in the apogee of topology, with M. Dehn and J.W. Alexander.

Knot Theory has experienced a revolutionary development from then, been faced from many different aproaches from what, among all, we will be particularly interested in the combinatorial one represented by L.H. Kauffman or V. Jones. However, the natural context to define knots is from topology: coming back to the example of the rope, once we have this intertwined circunference, we do not distinguish between the same knot with different sizes, the same knot but being rotated 90 degrees or, in a more general sense, two knots obtained by deforming one to the other with any move that does not cut the knot. In Figure A.3 we have an example of a process in which we get equivalent knots all the time using this type of transformations (notice that we get S^1 (the trivial knot) from other knot that seems more complex).

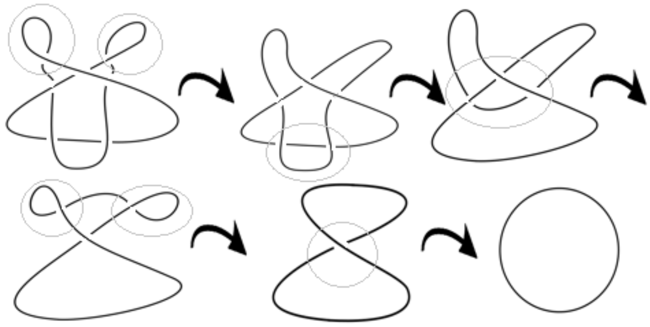


Figure A.3: Equivalent moves.

These moves are captured by a particular map called *isotopy*, a deformation that depends on time and brings one knot to another:

Definition A.2.3. Given two topological spaces X,Y and two embeddings $h_1,h_2 : X \longrightarrow Y$, we call *isotopy from* h_1 *to* h_2 any map

$$\begin{array}{cccc} H: & X\times I & \longrightarrow & Y \\ & (x,t) & \longmapsto & H(x,t) \end{array}$$

such that $H_t(x) = H(x,t)$ is an embedding $\forall t \in I$, $H(x,0) = h_0$ and $H(x,1) = h_1$.

These maps capture the essence of knot equivalence:

Definition A.2.4. Given two topological spaces X,Y and two embeddings $h_1,h_2 : X \longrightarrow Y$, we say h_1 *isotopic to* h_2, $h_1 \sim h_2$, if there exists an isotopy from h_1 to h_2. This is an equivalence relation.

Definition A.2.5. Given two knots K_1 and K_2, we say they are *equivalent*, $K_1 \simeq K_2$, if $K_1 \sim K_2$.

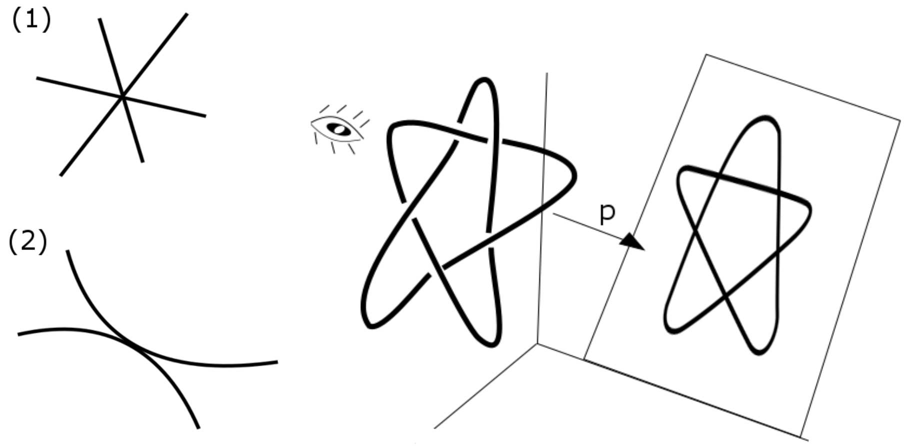
Therefore, what we are actually interested in is these classes of equivalence of knots under isotopy, since englobe knots that are essentially the same for us, knots that can be obtained one from the other by moves that do not cut the knot. All these definitions form the basis of Knot Theory.

However, they just characterize these objects, but in the practice is very hard to find an isotopy between two knots or define knots in terms of embeddings of S^1. Fortunately, we will work with a simpler approach, the universe of diagrams: a diagram is just a representation in the plane of a knot (see Figure A.6 (b)). Before giving a rigurous definition we need to introduce the concept of shadow:

Definition A.2.6. Given a knot K, we call *shadow* of K any closed curve (with or without selfintersections) in the plane that is the resulting of projecting (call p the projection) K into A, where A is some plane in R^3, satisfying:

- There exists no n-points for $n > 2$, where an *n-point* is any point $P \in p(K)$ with n preimages in the projection (see Figure A.4 (a) (1)).
- For every 2-point, the arcs involved intersect transversally (see Figure A.4 (a) (2)).

We call *intersection* of the shadow any 2-point.

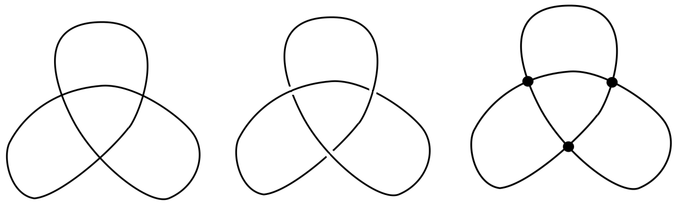


(a) Forbidden situations in conditions 1 and 2 respectively.

(b) Projection of a knot.

Figure A.4: Shadow concepts.

In order to clarify this, we have a projection in Figure A.4 (b), the shadow of the so-called trefoil knot in Figure A.5 (a) and some other shadows in Figure A.6 (a). However, they do not capture all the information: what we want is to represent in the plane a closed curve in $\mathbb{R}^3$. As it is an intertwined curve, it may have intersections when projected in a plane and we have to distinguish between its two preimages. The way to do so is very intuitive, we draw the complete arc that overcrosses and cut in the intersection the one that undercrosses, as we would see it in front of the plane, represented in Figure A.5 (b) and Figure A.6 (b) . Then, intersections are transfromed in what we call *crossings* and every diagram can be seen as a 4-valent graph by substituting crossings by vertices, as in Figure A.5 (c).



(a) Trefoil shadow.

(b) Trefoil diagram.

(c) 4-valent graph of a trefoil diagram.

Figure A.5: Trefoil knot: shadow, diagram and 4-valent graph.

Thus, all the possible diagrams of a given knot represent the knot and from now on, they will be used to develop our study. The next step will be to translate the concept of isotopy to the universe of diagrams.

To do so, we will need to relate the diagrams that come form the same knot.

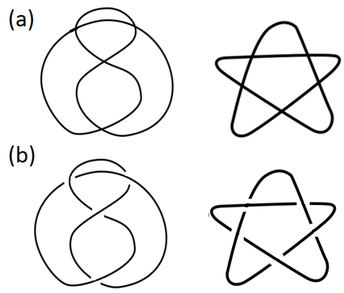


Figure A.6: Examples of shadows and diagrams

Definition A.2.7. Given two diagrams d_1 and d_2, we call *move from d_1 to d_2* any transformation that brings d_1 to d_2.

Equivalence in diagrams will be defined in terms of moves. We will need to be able to determine which moves are allowed and which are not, which moves preserve equivalence in the knots they are representing and which does not. As a necessary condition, it is clear that diagramas that are the same but one is bigger than the other are essentially the same, or that rotations are allowed moves, or in a more general sense, moves that does not erase or generate new crossings, that are just are reordering of the ones we have. These trivial cases are captured by isotopy in the context of graphs: we will see diagrams as graphs (as in Figure A.5 (c)) and define these trivial moves in terms of deformations of these graphs.

Definition A.2.8. Given a planar graph $g_o \subseteq \mathbb{R}^2$, we call *graph embedding (for g_o)* any map $h : g \longrightarrow \mathbb{R}^2$ which is an embedding (in a topological context) and $(h(g) = g_o)$, that is, if the map $f : g \longrightarrow h(g)$ such that $h = i_{h(g))} \circ f$ is a homeomorphism, where $i_{h(g))} : h(g) \longrightarrow \mathbb{R}^3$ is the inclusion and g and $h(g)$ inherits the topology of $\mathbb{R}^2$, and $(h(g) = g_o)$.

Definition A.2.9. Given two planar graphs $g_1, g_2 \subseteq \mathbb{R}^2$, we say g_1 and g_2 are *strongly equivalent*, $g_1 \simeq g_2$, if there exists h_1 a graph embedding for g_1 and h_2 a graph embedding for g_2 that are isotopic $(h_1 \sim h_2)$.

In other words, two graphs are strongly equivalent if one can be brought to the other in a way that its topological structure is preserved. Let´s translate it to diagrams:

Definition A.2.10. We say two diagrams d_1 and d_2 are *strongly equivalent*, $d_1 \simeq_o d_2$, if, seen as graphs, they are strongly equivalent. We call the move from d_1 to d_2 *strong equivalence move*.

In conclusion, given d_1 and d_2 strongly equivalent diagrams, the move that brings d_1 to d_2 is one of these moves that trivially preserves equivalence in its knots, d_1 and d_2 trivially represents two equivalent knots.

However, there are more moves that preserves equivalence in its knots that are not that trivial, as for example, each move involved in Figure A.3. If we pay attention, in each step of the process, the two diagrams involved are equal except from inside the balls drawn. We will define all the moves that preserve equivalence as composition of 3 moves (and its inverses) that leave invariant the diagram except from a local part of it:

Definition A.2.11. We call *Reidemeister move* (i), (ii) or (iii) to any move that leaves invariant the starting diagram except from a local part, where the diagram is transformed as represented in Figure A.7 (a) (i), (ii) or (iii) respectively.

These diagrams form the true basis of diagram equivalence:

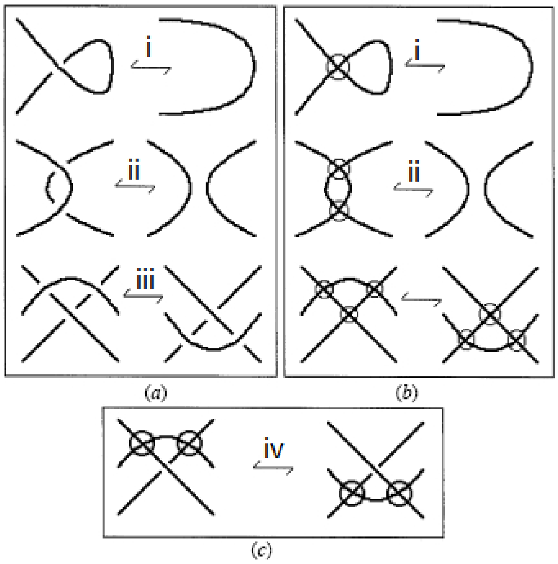


Figure A.7: Generalized Reidemeister moves ([**?**, page 665]).

Definition A.2.12. Given diagrams d_1 and d_2, we say d_1 is *equivalent* to d_2, $d_1 \simeq d_2$, if d_2 can be obtained from d_1 by using a finite number of Reidemeister moves and/or strong equivalence moves. This is an equivalence relation.

To sum up, strong equivalence moves and Reidemeister moves capture in the universe of diagrams the equivalence under isotopy of the knots they represent:

Proposition A.2.1. Given d_1, d_2 diagrams of K_1, K_2 respectively, $K_1 \simeq K_2 \Longleftrightarrow d_1 \simeq d_2$.

Proof. See [**?**]. □

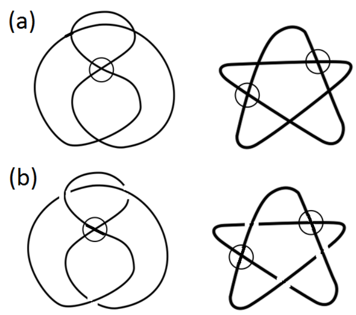
We have defined the concept of knot and equivalence of knots in terms of diagrams in order to have a simpler context to work with knots, but also combinatorial: we have some objects related to each other, related by 3 differents moves. Is not easy to get the way to obtain one from the other, since there exists an infinite number of possiblities to begin with one and continue with others.

This turns the problem of giving a complete classification of knots a hard task: however, partial solutions have been given using more advanced techniques. Following this line, we will continue extending the given before for Virtual Knot Theory, which is the central topic of this paper. So, from now, we will refer to knots as *classical knots* and the concept of knot will be extended.

A.3 Virtual Knot Theory

Discovered in 1996 by L.H. Kauffman, *virtual knots* conform a generalization of the concept of classical knots. In the same way that we have used diagrams to represent classical knots and Reidemeister moves to characterize their equivalence relation, we will identify virtual knots with a new type of diagrams and define a set of moves that characterize its equivalences.

These new diagrams are a generalization of the concept of diagram: being diagrams are a representation of a closed curve of R^3 in the plane that admits objects called *crosings* instead of intersections, we will permit a new type of crossing (Figure A.1 (c)) (called *virtual crossing*), resulting objects as in Figure A.8 (b).



(a) Trefoil shadow.

Figure A.8: Virtual shadows and diagrams.

The idea of virtual crossings is that actually there is no crossing there: although we will define virtual knots in terms of (these new) diagrams, they can be also defined from a topological perspective, as embeddings of S^1 in complex topological spaces. The fact that they do not live in $\mathbb{R}^3$ (as classical knots do) provokes that, when we try to represent them in the plane certain artifacts appear, virtual crossings. In this line, virtual crossings are not real crossings, just a product of the representation in $\mathbb{R}^2$.

Thus, we will redefine and extend the concept of shadow and diagram in order to present virtual knots:

Definition A.3.1. From now, we call *shadow* any 4-valent graph.

This extends the definition given before, and we say a shadow is *planar* if it is a planar 4-valent graph (see Figure A.5 (a)), that is, does not contain virtual intersections, and *non-planar* in other case (see Figure A.8 (a)).

Definition A.3.2. We call *diagram* any shadow with an extra structure in its (non-virtual) intersections as in Figure A.6 (b). In this context, we call *crossing* this resulting intersections and *virtual crossings* to virtual intersections. Moreover, we denote as D the set of all diagrams.

In the same line, we say a diagram is *planar* if it does not contain virtual crossings (see Figure A.5 (b)) and *non-planar* in other case (see Figure A.8 (b)). Notice that these redefinitions effectively extends the concept of shadows and diagrams given for Knot Theory.

The basis of diagrams equivalence will be given as before, in terms of moves: strong equivalence moves and a generalization of Reidemeister moves:

Definition A.3.3. Given a diagram d, we call *planar graphof d*, $p(d)$, to the graph resulting of substituting its crossings and its virtual crossings by vertices.

We redefine the concept of strong equivalence diagrams:

Definition A.3.4. Given two diagrams d_1 and d_2, we say d_1 and d_2 are *strongly equivalent*, $d_1 \simeq_o d_2$, if $p(d_1)$ and $p(d_1)$ are strongly equivalent ($p(d_1) \simeq p(d_2)$).

And generalize Reidemeister moves:

Definition A.3.5. We call *virtual Reidemeister move* (i), (ii), (iii) or (iv) to any move that leaves invariant the starting diagram except from a local part, where the diagram is transformed as represented in Figure A.7 (b) (i), (ii) or (iii) or (c) (iv) respectively. We call *generalized Reidemeister move* to any move that leaves invariant the starting diagram except from a local part, where the diagram is transformed as represented in any of the moves in Figure A.7. From now we call *classical Reidemeister moves (i),(ii) and (iii)* to Reidemeister moves (i), (ii) and (iii).

All these moves form the basis of equivalence in diagrams:

Definition A.3.6. Given diagrams d_1 and d_2, we say d_1 is *equivalent* to d_2, $d_1 \simeq d_2$, if d_2 can be obtained from d_1 by using a finite number of generalized Reidemeister moves and/or strongly equivalence moves. This is an equivalence relation and the classes of equivalence are denoted by $[\cdot]$.

Definition A.3.7. We call *virtual knot* any of these clases.

The concept of virtual knot generalise the concept of classical knot and from now, they will be the object of our study, so, when we say knot we refer to virtual knot. Due to many facts, Virtual Knot Theory represents a promising line of investigation in Knot Theory, which is the main motivation of this paper. After these preliminaries, we will study a specific part of this branch, the capability of certain algebraic elements to represent virtual knots and give us a new interesting and computable perspective to study these particular objects.