A Test of Independence Between Nonhomogeneous Poisson Processes and its Application to Heat Wave Modeling

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Abstract The main contribution of this work is a test to check the independence between nonhomogeneous Poisson processes. It is developed for the three-process case, but it can be generalized to any number of processes. Moreover, it can be applied not only to Poisson processes but to any point process which can be simulated. It is a Monte Carlo test if the parameters describing the processes are known, and a parametric bootstrap test in the most usual case of unknown parameters. A simulation study shows that the size of the test is close to the nominal one. The power is also analyzed, considering two approaches for generating dependent nonhomogeneous processes and different levels of dependence, with satisfactory results in all the cases.

This test is a valuable tool in the validation analysis of a common Poisson shock process. For the bivariate case, this process can be decomposed into three indepen-

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dent Poisson processes, and the independence of the three marginal processes has to be checked for validating the model. As an application, the modeling of the occurrence process of extreme heat events in daily maximum and minimum temperatures is described.

1 Introduction

This work aims to develop a test to check the independence between nonhomogeneous point processes, in particular, nonhomogeneous Poisson processes (NH-PPs). As far as we know, there does not exist a test of independence between one-dimensional NHPPs. Tests of independence between point processes have been studied in the framework of marked spatial point processes, see Schlather et al. (2004), Eckel et al. (2007), Guan and Afshartous (2007) and Comas et al. (2010). There also exist quite a few functions aiming to quantify the dependence between spatial point processes, such as the ones suggested by Diggle (1983) and van Lieshout and Baddeley (1999), and some of them are also used in time point processes (Doss 1989). Some of the previous independence tests can be adapted to time point processes, but little work has been done with nonhomogeneous (NH) processes. Cuzick and Edwards (1990) suggested a test to detect clustering, and Allard et al. (2001) a test of local independence based on approximate isotropy for NH spatial point processes. Baddeley et al. (2000) and Marcon and Puech (2009) developed a generalization of the Ripley's K function to an inhomogeneous spatial setting.

The need of developing this test stemmed from a real problem, the modelling of extreme heat events. A bivariate model is needed in order to model the distribution of the occurrence of extremes in maximum and minimum daily temperatures, accounting for the dependence between the two signals. A reasonable model is a bivariate nonhomogenoeus common Poisson shock process, which can be described by three independent NHPPs. The proposed test is necessary to check the independence between the three marginal processes.

The interest of characterizing extreme heat events in a bivariate framework is clear. First, because heat waves are known to produce serious impacts on regional economies, ecosystems and human health. Secondly, because Palecki et al. (2001), Grintzevitch (2006) and other studies about the effect of extreme heat on mortality rates found that both maximum and minimum temperatures have an impact on human health. As a consequence, an increasing number of heat wave definitions include information on both temperatures, see for example Plummer et al. (1999), or the definition of Excessive Heat Watch by the U.S. National Weather Service.

Section 2 describes the proposed test to check independence. In Section 3, a simulation study of the size and the power of this test is carried out. Section 4 shows an application, the modeling process of the occurrence of the extreme heat events in Zaragoza (Spain). The most relevant conclusions are summarized in Section 5.

2 Methodology

2.1 A test to check the independence of several nonhomogeneous Poisson processes

One might think that a way to simplify the problem of testing the independence of NH processes is to transform them into homogeneous ones as a previous step. However, it is noteworthy that with conditional intensities $\lambda^*(t)$, this is not a feasible approach since the dependence structure disappears under simultaneous random time transformations of the type $t' = \int_0^t \lambda^*(u)du$ (Daley and Vere-Jones 2003, prop. 7.4.VI).

Hence, we propose a test to check the independence between one-dimensional NH point processes, conditionally on their corresponding intensities. Let us denote (N_x, N_y, N_z) a vector of three point processes with intensities $\lambda_x(t)$, $\lambda_y(t)$ and $\lambda_z(t)$, where n_x, n_y and n_z points have been observed. If dependence exists, it is reasonable to assume that it will manifest itself in events occurring close to each other, so we tried to generalize the crossed nearest neighbor distance ideas to the NH case. These distances only consider the dependence between the occurrence of points, but dependence can also manifest in the non occurrence. To account for this aspect, each point is linked to the time interval until its previous neighbor, which will be called the *time interval* of the point. We define the *close point relation* $\rightsquigarrow$ between two points in two different processes as follows: a point in a process is close to a point in another process, if the time interval of the first overlaps the time interval of the second. For example, in the left plot in Figure 1, $tx_i \rightsquigarrow ty_j$ and in the right one, $ty_j \rightsquigarrow tz_k$ and $ty_j \rightsquigarrow tz_{k+1}$.

Then, given the vector process (N_x, N_y, N_z), the *set of close points* of tx_i in N_x, the big point in Figure 1 (left), is defined as,

$$S_{x_i} = \left\{ (ty_j, tz_k) \in N_y \times N_z : tx_i \rightsquigarrow ty_j, ty_j \rightsquigarrow tz_k \right\}.$$

In the example in Figure 1, $S_{x_i} = \{(ty_j, tz_k), (ty_j, tz_{k+1})\}$.

To check if there is a longer dependence in the process, the close point relation could be broadened by including the previous and/or the following points to the overlapping intervals: ty_{j-1} and ty_{j+1} in Figure 1 (left) and tz_{k-1} and tz_{k+2} in Figure 1 (right). If no relevant difference is observed, it will be concluded that the actual dependence has been properly taken into account. Otherwise, it should be checked if more points must be added to capture the dependence structure.

If only the points with overlapping time intervals are considered, the close point relation is symmetric, but if more points are added, it is not. For example, in Figure 1 (left) $tx_i \rightsquigarrow ty_{j-1}$ but ty_{j-1} is not close to tx_i. In that case, and if there is not an intrinsic order between the processes, all the permutations resulting from interchanging the order of the processes should be considered to build the sets of close points.

The underlying idea of the test is to compare, for each point of N_x, the behavior of its set of close points in the observed vector process (N_x, N_y, N_z), and in simulated processes with the same characteristics (same N_x process and second and third processes with the same intensities) and mutually independent components. If the observed behavior is significantly different, the independence will be rejected. The multivariate character of the points and the nonhomogeneity make it difficult to obtain a statistic with a known probability distribution. However, the distribution can be
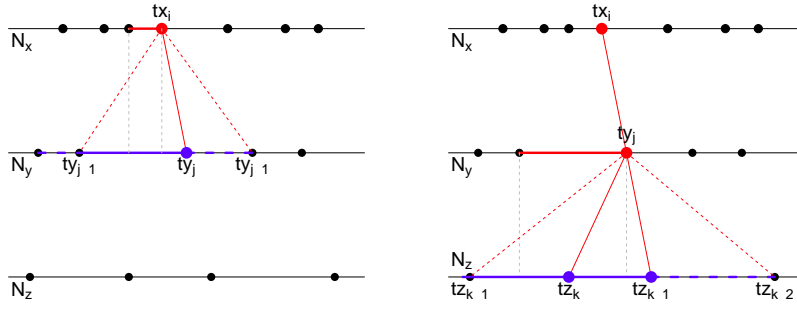
Fig. 1 Definition of the set of close points of a point tx_i. The dashed lines to ty_{j-1} and ty_{j+1} in the left plot and to tz_{k-1} and tz_{k+2} in the right one, mark the previous and the following points of the overlapping intervals.

obtained using a Monte Carlo approach if the intensities $\lambda_y(t)$ and $\lambda_z(t)$ are known, or a parametric bootstrap if they have to be estimated.

The steps of the test are the following:

Step 1. For each point tx_i in N_x, its set of close points in the observed vector process (N_x, N_y, N_z), S_{x_i}, is built and $\bar{d}x_i$, the mean of the distances $|ty_j - tx_i| + |tz_k - tx_i|$ of the pairs in that set, is calculated as a summary.

Step 2.1. A vector process (N_x, N_y^*, N_z^*) is generated, where N_y^* and N_z^* are independent processes with intensities $\lambda_y(t)$ and $\lambda_z(t)$, also independent of the observed process N_x.

Step 2.2. For each point tx_i in N_x, its set of close points in the simulated vector process (N_x, N_y^*, N_z^*), $S_{x_i}^*$, is built and $\bar{d}x_i^*$, the mean of the distances $|ty_j^* - tx_i| + |tz_k^* - tx_i|$, calculated.

Step 2 is repeated n_S times so that, for each tx_i, a set of one observed and n_S simulated distances is obtained. Under the independence hypothesis, this should be

a sample of i.i.d. observations and px_i, the sample percentile of $\bar{d}x_i$, an observation from a uniform distribution.

The final result of this process is a series of n_x (the number of points in N_x) percentiles px_i. These percentiles should be $U(0,1)$ under the null hypothesis, but it is not guaranteed that they are uncorrelated. The serial correlation comes from the possible presence of some equal pairs (ty_j, tz_k) in the sets of close points of nearby points tx_i and tx_l.

Kolmogorov-Smirnov (KS) is a well-tried statistic for testing whether a sample stems from a given distribution, for example the uniform one. Although the distribution of this statistic in a correlated sample is not known, its empirical distribution can be obtained from the previously simulated trajectories. To that aim, the uniform KS statistic for the percentile sample obtained from each simulated vector of independent processes (D_{n_x} herein) is calculated, so that a size n_S sample of D_{n_x} under the null hypothesis is obtained. The p-value of the test is calculated from that sample. As in the standard KS test, the null hypothesis is rejected if $d > d_{1-\alpha}$, where d is the observed value of D_{n_x} and $d_{1-\alpha}$ is the $1 - \alpha$ percentile of the statistic distribution under the null hypothesis. The simulated NHPP trajectories are used to calculate both the percentiles and the distribution of D_{n_x}, so that the computing time cost is not noticeably increased.

2.2 Validity of the test

When the parameters (λ_y and λ_z in this case) are known, this type of tests are Monte Carlo tests and they are exact (Dufour 2006). However, in most real problems the in-

tensities are unknown. In that case, the p-value is obtained by a parametric bootstrap, where the unknown parameters λ_y and λ_z of the bootstrap data generating process (DGP) are substituted by their estimators.

Most of the results for parametric bootstrap tests are developed for pivotal or asymptotically pivotal statistics, whose distribution does not depend on the unknown parameters, see Beran (1988), Davidson and MacKinnon (1999) and Davidson and MacKinnon (2006b). In our test, the structure of the statistic makes it difficult to prove if it is pivotal, or to make it pivotal by a scale transformation. On the other hand, the nuisance parameters λ_y and λ_z do not appear in the null hypothesis H_0: processes are independent, unlike most parametric bootstrap tests (Beran 1988) (Andrews 1997). That makes the study of the test properties simpler.

First, in order to be valid, the bootstrap distribution of the statistic must mimic its real distribution under the null, even though the data were generated under the alternative hypothesis (Andrews 1997). If λ_y and λ_z are estimated using the likelihood function of the marginal processes, with no restrictions, they are estimated under the null, and the previous condition is guaranteed.

Concerning the bootstrap distribution, Davidson and MacKinnon (2006a) remark that 'it is desirable to get as good an estimate as possible of the true DGP for the bootstrap DGP'. This requires a good estimation method which provides consistent estimators, such as maximum likelihood, and a thorough validation analysis of the models used to estimate the parameters. Given n_x, the distribution $G_{n_x}(d, F(\hat{\lambda}_y, \hat{\lambda}_z))$ of D_{n_x}, considered as a functional of the distribution of the DGP $F(\hat{\lambda}_y, \hat{\lambda}_z)$, is contin-

uous. Then, according to Horowitz (2001), if MLE $\hat{\lambda}_y$ and $\hat{\lambda}_z$ are used, it is expected that $G_{n_x}(d, F(\hat{\lambda}_y, \hat{\lambda}_z))$ will be close to $G_{n_x}(d, F(\lambda_y, \lambda_z))$.

To show empirically the closeness of $G_{n_x}(d, F(\hat{\lambda}_y, \hat{\lambda}_z))$ and $G_{n_x}(d, F(\lambda_y, \lambda_z))$ distributions, the following simulation study is done. We generate three independent NHPP N_x, N_y and N_z with given intensities $\lambda_x(t)$, $\lambda_y(t)$ and $\lambda_z(t)$, which play the role of observed processes. First, applying the test with these intensities, a sample of size n_S of D_{n_x} is obtained and the distribution $G_{n_x}(d, F(\lambda_y, \lambda_z))$ calculated from that sample. Then, using N_y and N_z and adequate covariates, the estimators $\hat{\lambda}_y$ and $\hat{\lambda}_z$ are calculated, and applying the test with them, the distribution $G_{n_x}(d, F(\hat{\lambda}_y, \hat{\lambda}_z))$ is obtained. A KS test is used to check if the distribution of both samples is the same. Since the simulation study aims to check the behavior of the bootstrap test assuming that good estimators are used, we only consider the cases where the fitted model satisfies some basic validation analysis and the real intensities are covered by the 95% confidence intervals estimated from the model. The validation analysis consists of checking the distribution and the uncorrelation of the uniform residuals of the NHPP, see section 4.5 for more details.

The simulation study is performed with the intensities,

$$\lambda_x(t) = \exp\left[-56 - 48\cos(2\pi t) - 23\sin(2\pi t) - 0.02t + 0.001t^2\right]$$

$$\lambda_y(t) = \exp\left[-8 - 2\cos(2\pi t) + 3\sin(2\pi t) + 0.03t + 0.0002t^2\right]$$

$$\lambda_z(t) = \exp\left[-10.5 - 6\cos(2\pi t) - 3\sin(2\pi t) + 0.01t + 0.0003t^2\right]$$

which include the most usual features of a real evolution in time: a seasonal term (one harmonic) and a polynomial trend (quadratic in this case). The qqplot comparing the

Table 1 Summary of the p-values from the KS test comparing the empirical distributions $G_{n_x}(d, F(\hat{\lambda}_y, \hat{\lambda}_z))$ and $G_{n_x}(d, F(\lambda_y, \lambda_z))$

KS p-value>	0.05	0.25	0.5	0.75	0.95
Proportion	0.950	0.714	0.504	0.302	0.056

samples from the distributions $G_{n_x}(d, F(\hat{\lambda}_y, \hat{\lambda}_z))$ and $G_{n_x}(d, F(\lambda_y, \lambda_z))$ is shown in Figure 2 and the corresponding KS p-value is 0.26. This analysis is repeated 500 times. The resulting p-values of the KS test comparing both distributions are summarized in Table 1. In most cases high values are obtained, and the equality of the distributions is rejected, at a $\alpha = 0.05$ significance level, in approximately 5% of the cases. Hence, there is no evidence that the distribution of the test statistic changes when the real parameters are substituted by their ML estimators.

The study about G_∞, the asymptotic distribution of the statistic is not relevant in this test. In each case, the bootstrap distribution of the statistic is calculated for the given sample size n_x with a low time computing cost and, consequently, the approximation provided by G_∞ is not needed. As far as we know, there does not exist any exact or asymptotic test to check the independence of NHPPs, and comparative analyses of the performance and the order of approximation of the proposed test cannot be carried out.

3 Size and power of the test

This section summarizes the main results from an analysis of the size and the power of the test. All the experiments are based on 1000 trials, and in each trial the p-value is calculated with $n_S = 1000$. Given the equality of the distributions $G_{n_x}(d, F(\hat{\lambda}_y, \hat{\lambda}_z))$
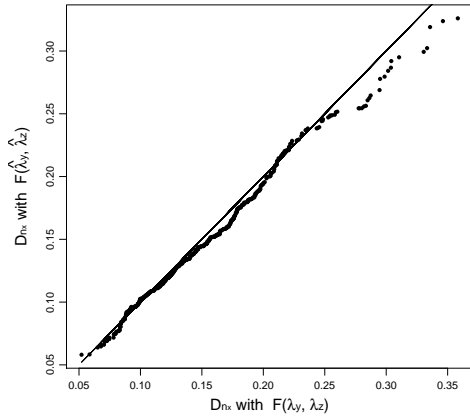
Fig. 2 QQplot comparing the distributions $G_{n_x}(d,F(\hat{\lambda}_y,\hat{\lambda}_z))$ and $G_{n_x}(d,F(\lambda_y,\lambda_z))$

and $G_{n_x}(d,F(\lambda_y,\lambda_z))$ and in order to simplify the analysis, the properties of the test are studied in its Monte Carlo version.

3.1 Size of the test

Table 2 summarizes, for nominal levels $\alpha = 0.05$ and 0.10, the rate of rejection under the null hypothesis, i.e. given three independent processes with different intensities and observation period T. In the first case, the intensities are the fitted ones in the application described in section 4, $\hat{\lambda}_{(1)}(t)$, $\hat{\lambda}_{(2)}(t)$ and $\hat{\lambda}_{(12)}(t)$ and $T = 8415$. The second and third cases are HPPs with intensities equal to the mean of these fitted intensities and $T = 8415$ and $T = 4000$, respectively. In the last case, three NHPPs with a linearly increasing intensity function $f(t)$ and $T = 8000$ are considered. The estimated sizes are close to the nominal levels for both the homogeneous and the NH

processes. The size of the test does not seem to be influenced neither by the length T,

nor by the shape of the intensity function.

3.2 Power of the test

The alternative hypothesis (three dependent processes) is very general, and it is not

possible to systematically study it. Two approaches are considered here, the first one

generates dependent Poisson processes using two queues in tandem. The second gen-

erates dependent point processes, not Poisson, using marks based on a Markov chain.

In both cases, the power of the test is studied according to the degree of the induced

dependence.

Queue approach. Burke's theorem states that, in a stable stationary system, the out-

put process from any $M/M/m$ queue is a PP with the same rate λ as the input process

(Kleinrock 1975). In a tandem of $M/M/1$ queues with exponential time service of

rates μ_1 and μ_2, the input and the output processes of the first queue and the output

of the second queue are three dependent PPs. This approach, considering a constant

input intensity $\lambda = 0.01$, is used to generate dependent HPPs.

Table 2 Size of the test for different DGPs and nominal levels.

T	$\lambda_x(t)$	$\lambda_y(t)$	$\lambda_z(t)$	$\alpha = 0.10$	$\alpha = 0.05$
8415	$\hat{\lambda}_{(12)}(t)$	$\hat{\lambda}_{(2)}(t)$	$\hat{\lambda}_{(1)}(t)$	0.101	0.046
8415	$\overline{\hat{\lambda}}_{(12)}$	$\overline{\hat{\lambda}}_{(2)}$	$\overline{\hat{\lambda}}_{(1)}$	0.095	0.052
4000	$\overline{\hat{\lambda}}_{(12)}$	$\overline{\hat{\lambda}}_{(2)}$	$\overline{\hat{\lambda}}_{(1)}$	0.100	0.052
8000	$f(t)$	$f(t)$	$f(t)$	0.115	0.049

There is a relationship between the utilization factors of the two queues, ρ_1 and ρ_2 ($\rho_i = \lambda/\mu_i$), and the degree of dependence between the input/output processes. This relation is not simple since utilization factors near zero lead to high correlations, whereas a high variability in the correlation is observed with higher factors (lower than 1).

To study the power according to the degree of dependence, the empirical Pearson correlation coefficient between the inter-arrival times of each pair of processes is categorized into four levels, very low (L), low (l), high (h) and very high (H), depending on which interval the coefficients lie in: [0, 0.25), [0.25, 0.5), [0.50, 0.75) and [0.75, 1]. The different combinations of correlation levels between the first and the second process and between the second and the third are considered. The power of the test at levels $\alpha = 0.10$ and $\alpha = 0.05$, using processes with $n_x = 100$ points, are shown in Table 3. The critical value of the Pearson correlation coefficient at a significance level $\alpha = 0.01$ for a sample of size 100 is 0.25, so that correlations in level L can be considered null. The generated trajectories which have not the desired correlation values are discarded. The values of μ_1 and μ_2 are selected in order to get discarding rates as low as possible, column *Disc.* in the table.

The power for $\alpha = 0.10$ is higher than 0.70 when very high or high correlations are observed. The power decreases when two of the three processes are independent or almost independent (levels Hl, HL and hl). However, the test performance perceptibly improves when the sample size increases to $n_x = 200$, see the right columns of the table. Even in those cases where the estimated power is not very high, the p-value

Table 3 Power of the test for HPPs with different correlation levels, $M/M/1$ queue approach

Level	ρ_1	ρ_2	Disc.	$n_x = 100$			$n_x = 200$		
				$\alpha = 0.10$	$\alpha = 0.05$	$pv_{0.5}$	$\alpha = 0.10$	$\alpha = 0.05$	$pv_{0.5}$
HH	0.4	0.4	18%	0.990	0.981	0.000	-	-	-
Hh	0.4	0.6	40%	0.900	0.824	0.007	-	-	-
Hl	0.4	0.8	57%	0.600	0.442	0.070	0.895	0.812	0.007
HL	0.4	0.9	58%	0.531	0.381	0.089	0.836	0.720	0.014
hh	0.6	0.6	54%	0.701	0.544	0.042	0.951	0.898	0.003
hl	0.6	0.8	64%	0.359	0.234	0.180	0.622	0.491	0.053
hL	0.6	0.9	77%	0.220	0.122	0.306	0.427	0.306	0.143
ll	0.8	0.8	76%	0.163	0.082	0.460	0.292	0.182	0.236
lL	0.8	0.9	70%	0.114	0.061	0.450	0.172	0.094	0.366

distribution is clearly concentrated on small values, as the median of the p-values $pv_{0.5}$ shows.

Generation of NHPPs with the queue approach. The queue approach can also be used to generate dependent NHPPs applying the result by Keilson and Servi (1994): the output process from a $M(t)/G/1$ queue is also a PP with a NH intensity equal to the convolution $\lambda_{out}(t) = \lambda_{inp}(t) * f_\mu(t)$, where $\lambda_{inp}(t)$ is the input intensity and $f_\mu(x)$ the density function of the service time, in this case an exponential distribution of rate μ.

In the NH case the correlation between the processes cannot be easily characterized, and the power cannot be studied according to the correlation level. It can be expected that, as in the homogeneous case, high utilization factors will often generate

processes which are not under the alternative hypothesis. By this reason, the power analysis for NH intensities is restricted to the case $\rho_1 = \rho_2 = 0.4$ where, in the homogeneous case, 99% of the trajectories had a high or very high correlation. We defer to the Markov chain approach, a more thorough study of the power according to the dependence in the NH case.

In this experiment, the test performance is analyzed in relation to the shape of the intensity, including the homogeneous case as a reference. In all the cases the mean intensity is 0.011 and $T = 8415$, so that the number of points in each trajectory is $n_x \approx 100$. The first NH input intensity is the sum of the fitted rates in the data application, $\lambda_o(t) = \hat{\lambda}_{(1)}(t) + \hat{\lambda}_{(2)}(t) + \hat{\lambda}_{(12)}(t)$, which shows a complex evolution, see Figure 4. $\lambda_{o_R}(t)$ and $\lambda_{o_A}(t)$ are two transformations of $\lambda_o(t)$, which reduce and amplify, respectively, its variability:

$$\lambda_{o_R}(t) = (\lambda_o(t) - \lambda_c)/2 + \lambda_c$$

$$\lambda_{o_A}(t) = 0.01\lambda_o(t)(\lambda_o(t) < \lambda_c) + 1.14\lambda_o(t)(\lambda_o(t) > \lambda_c)$$

being $\lambda_c = \overline{\lambda_o(t)} = 0.011$, and 0.01 and 1.14 the most extreme factors that keep positive the intensity function. A process with input intensity λ_{ps}, defined by a linearly increasing function of time, and the homogeneous intensity λ_c are also considered. In all the cases $\bar{\rho}_1 = \bar{\rho}_2 = 0.4$ and their minimum and maximum utilization values are shown in Table 4.

The homogeneous case shows the highest rejection rates but, even for the most complex intensities, the results are satisfactory. The power improves with the sample size, as the results for a period of length $2T$ with intensity $\lambda_{o,2T}(t)$, obtained by concatenating twice $\lambda_o(t)$, show.

Table 4 Rejection rates of the test for NHPPs, $M(t)/M/1$ queue approach, T=8415

Int.	$\max(\rho_1)$	$\max(\rho_2)$	$\min(\rho_1)$	$\min(\rho_2)$	$\alpha = 0.10$	$\alpha = 0.05$	$pv_{0.5}$
λ_c	0.4	0.4	0.4	0.4	0.974	0.950	0.001
$\lambda_o(t)$	19.3	4.3	6e-6	9e-3	0.881	0.791	0.009
$\lambda_{o_A}(t)$	22.1	4.9	6e-8	7e-4	0.823	0.703	0.018
$\lambda_{o_R}(t)$	9.9	2.4	0.2	0.2	0.957	0.918	0.002
$\lambda_{ps}(t)$	0.8	0.8	1e-4	1e-2	0.848	0.740	0.013
$\lambda_{o,2T}(t)$	19.3	4.3	6e-6	9e-3	0.996	0.987	0.000

Markov chain approach. In this study, dependent homogeneous and NH point pro-
cesses are generated by marking a PP using a three-state Markov chain. The processes
defined by the marks are not Poisson, since the generated marks are dependent ob-
servations (Isham 1980). However, as previously mentioned, the test can be applied
to any point process which can be simulated. A transition matrix $P = (p_{ij})$ with three
equal rows leads to three independent point processes, and the more similar the rows
of P, the less dependent the resulting processes. Two different dependence measures
are considered in the power analysis. The first one is the spectral gap *SG*, which as-
sesses the convergence speed of P to a matrix P_I with all the rows equal to (π_1, π_2, π_3),
the stationary distribution of P. *SG* takes values in [0,1].

The results for four doubly stochastic Markovian matrices (M1 to M4) with an
increasing convergence speed are shown in the upper rows of Table 5. M1 is a matrix
which does not converge, $SG = 0$, and defines a completely deterministic marking
processes. The aforementioned intensity $\lambda_o(t)$ and $T = 8415$ ($n_x \approx 100$) are used to
generate the underlying PP. The study is also run using the intensities λ_c and $\lambda_{o,2T}(t)$

(with period length $2T$). The power at significance levels $\alpha = 0.05$ and $\alpha = 0.10$ is summarized in the table, and the effect of SG on the performance of the test is clear. The power is satisfactory for $SG < 0.6$. For matrices closer to the independence, such as M4, the results are poorer but they improve when the sample size increases. Slightly better results are observed in the homogeneous case.

For matrices with the same convergence speed, we consider a second dependence measure related to the time until leaving the current state i, which has a Geometric$(1 - p_{ii})$ distribution. We define $D_I = \sum_{i=1}^{3} p_{ii}/3$, which varies from 0 to 1, being 1/3 the value for the limit matrix P_I. If the test behaves properly, the further D_I is from 1/3, the better the performance of the test should be. The lower part of Table 5 shows the doubly stochastic matrices $M3b$ and $M3c$ obtained by permuting the probabilities in $M3$, which have the same SG as $M3$ but different D_I. D_I values far from 1/3 lead to a better power and, since 1/3 is not centered on the D_I sample space, the increase is not symmetric around that value. The results for the other three permuted matrices of $M3$ and for permuted matrices of M2 and M4, not shown here, confirm that the test performance for matrices with the same SG and D_I is equivalent.

4 Data application

In order to jointly model the distribution of the occurrence of extreme heat events in maximum and minimum temperatures, taking into account the dependence between the two signals, a bivariate model is needed. The seasonality and a possible time evolution of the phenomenon require a nonstationary process. We propose a nonhomogeneous common Poisson shock process (NHCPSP) which, for the bivariate case,

Table 5 Power analysis: dependent processes generated by the Markov chain approach, $T = 8415$

Markov matrix	SG	D_I	α	$\lambda_o(t)$	λ_c	$\lambda_{o,2T}(t)$
$M1 = \begin{pmatrix} 0\ 1\ 0 \\ 0\ 0\ 1 \\ 1\ 0\ 0 \end{pmatrix}$	0.00		0.05	1	1	-
			0.10	1	1	
$M2 = \begin{pmatrix} 0.7\ 0.2\ 0.1 \\ 0.1\ 0.7\ 0.2 \\ 0.2\ 0.1\ 0.7 \end{pmatrix}$	0.44		0.05	0.987	0.997	-
			0.10	0.994	1	
$M3 = \begin{pmatrix} 0.6\ 0.3\ 0.1 \\ 0.1\ 0.6\ 0.3 \\ 0.3\ 0.1\ 0.6 \end{pmatrix}$	0.56	0.6	0.05	0.751	0.857	0.937
			0.10	0.828	0.919	0.969
$M4 = \begin{pmatrix} 0.5\ 0.2\ 0.3 \\ 0.3\ 0.5\ 0.2 \\ 0.2\ 0.3\ 0.5 \end{pmatrix}$	0.74		0.05	0.418	0.560	0.634
			0.10	0.526	0.649	0.744
$M3b = \begin{pmatrix} 0.3\ 0.6\ 0.1 \\ 0.1\ 0.3\ 0.6 \\ 0.6\ 0.1\ 0.3 \end{pmatrix}$	0.56	0.3	0.05	0.462	0.475	-
			0.10	0.623	0.609	
$M3c = \begin{pmatrix} 0.1\ 0.6\ 0.3 \\ 0.3\ 0.1\ 0.6 \\ 0.6\ 0.3\ 0.1 \end{pmatrix}$	0.56	0.1	0.05	0.837	0.860	-
			0.10	0.918	0.918	

can be decomposed into three independent NHPPs. Since a model assumption is the independence of the three marginal NHPPs, the proposed test is applied in its validation analysis.

4.1 Common Poisson shock process

A common Poisson shock process (CPSP), denoted by N, is a multivariate point process which assumes that there is an underlying PP of shocks N_0, that can yield d different types of events. The counting process of the jth type of events, for $j = 1, \ldots, d$, is the Poisson component process N_j (Lindskog and McNeil 2003).

For the bivariate case, $d = 2$, events can be divided into 3 types, $(1,0), (0,1), (1,1)$, depending on the components where the occurrence has been observed. Given two component processes N_1 and N_2, see Figure 3, the bivariate process can be decomposed into three independent processes: $N_{(1)}$ which includes the events occurring only in the N_1 process, $N_{(2)}$ which includes the events occurring only in N_2, and $N_{(12)}$ which contains the events occurring simultaneously in N_1 and N_2. $N_{(1)}, N_{(2)}$ and $N_{(12)}$ are the marginal processes with intensities $\lambda_{(1)}, \lambda_{(2)}$ and $\lambda_{(12)}$. The distribution of a CPSP is completely specified by its marginal processes. This decomposition into independent PPs allows the model to be readily applied for data modeling and easily generalizable to the NH case, by permitting the marginal intensities to be a function of time-varying covariates. These covariates also model the dependence induced by the systematic part of the intensities and, consequently, the independence between the processes becomes conditional independence, given the covariates.

4.2 Data

The daily maximum and minimum temperature series of Zaragoza (Spain) from May to September (MJJAS) for the period [1951, 2004], Tx and Tn herein, are used for
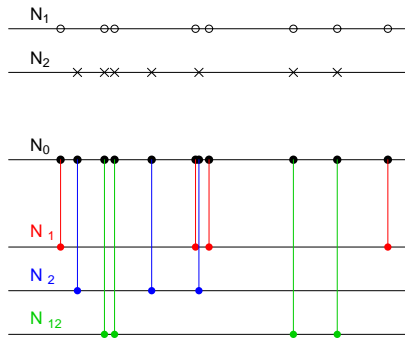
Fig. 3 Decomposition of the CPSP into three independent marginal Poisson process

illustration. Data are provided by AEMET, the Spanish meteorological office. A thorough exploratory analysis shows a time evolution not seasonally homogeneous, and a time-varying dependence between the occurrence of both type of extremes. A measure summarizing the extremal dependence between two variables is the coefficient $\chi = \lim_{u \to 1} \chi(u)$, with $\chi(u) = P(U > u | V > u)$ and (U, V) the transformed uniform marginals (Coles et al. 1999). For Tx and Tn, $\hat{\chi}(0.95) = 0.43$ and $\hat{\chi} = 0.2$ while, under independence, they should be 0.05 and 0, respectively. Under these conditions, two different but dependent models for describing the occurrence of the Tx and Tn extremes are required, and the suggested NHCPSP provides an adequate flexible framework.

4.3 Extreme event definition

An extreme heat event (EHE) only in Tx is a run of consecutive days where Tx exceeds its extreme threshold U_x but Tn does not exceed its extreme threshold U_n. An EHE only in Tn is defined analogously. A simultaneous EHE is a run where Tx and Tn exceed U_x and U_n, respectively. In Zaragoza, these thresholds are $U_x = 37.0^o C$

and $U_n = 21.2^oC$, the 95% percentiles in the reference period (months from June to August in 1971-2000). Tools for selecting adequate threshold can be found in Coles (2001) and a specific justification for the use of the 95% percentile in temperature series in Abaurrea et al. (2007).

In the EHE only in Tx or Tn, the occurrence point is located at the maximum temperature time and, in the simultaneous events, at the time where the sum of both temperatures is maximum. In the Zaragoza temperature series, there are 120 EHEs only in Tx, 92 only in Tn and 58 simultaneous EHEs.

4.4 Estimation of the model

The estimation of a NHCPSP reduces to the estimation of the three marginal NHPPs, whose intensities are modeled as parametric functions of time dependent covariates. An important application of this model is to obtain projections of the occurrence of EHEs in climate change scenarios. For that reason, we need covariates whose future 21st century values should be available to feed the model. Since General Circulation Model projections for temperature are reliable, but only at an aggregated time scale (Abaurrea et al. 2007) the following covariates were considered:

- **Temperature terms**. Two possible influential factors on EHEs are the current temperature level (short term temperature) and the decadal temperature trend (long term temperature). The short term signal is defined as the moving average in 15 or 31 day intervals around day t, and denoted Tx_{m15}, Tn_{m15}, Tx_{m31} and Tn_{m31}. The long term covariates, TTx and TTn, are the lowess with a 30%

window of the daily temperature series. Polynomial terms are also considered to model possible non linear effects.

- **Seasonal terms**. The part of annual harmonics corresponding to the five summer months, defined by the pairs of covariates $\cos(2k\pi t)$ and $\sin(2k\pi t)$ with $t = 121/365,..., 273/365$ (the time index from 1st of May to 30th of September) and $k = 1, 2,...$

- **Interaction terms**. Harmonic and temperature interaction terms allow us to model changing effects of temperature over the summer.

Parameters are estimated by conditional (on the covariates) maximum likelihood; the conditional loglikelihood of a NHPP can be found in Coles (2001). A logarithmic link is used to guarantee positive intensities. To keep the notation simple, the conditional intensity $\lambda(t|\mathbf{X(t)}) = \exp(\mathbf{x'(t)}\beta)$ is denoted by $\lambda(t)$, being $\mathbf{X(t)} = (\mathbf{x(1)}, \ldots, \mathbf{x(t)})'$ the covariate matrix and $\mathbf{x(t)}$ the vector of covariates at time t. It is worth to recall that efficient estimation based only on the conditional model (ignoring the information of the marginal process), as we do, can only be carried out if weak exogeneity holds (Ericsson and Irons 1994, chap. 1). The weak exogeneity of the covariates in our model has been checked using the definition given by (Engle et al. 1983, p. 282).

Covariate selection is based on likelihood ratio tests with a $\alpha = 0.05$ significance level. The final intensities fitted to the three marginal processes are,

$$\log(\lambda_{(1)}(t)) = -68.9 - 47.6\cos(2\pi t) - 22.6\sin(2\pi t) + 0.08Tx_{m15} + 0.21Tn_{m15} +$$

$$0.25Tn_{m15}\cos(2\pi t) + 0.12Tn_{m15}\sin(2\pi t) + 0.01TTx$$

$$\log(\lambda_{(2)}(t)) = -13.3 - 0.35\cos(2\pi t) + 0.26\sin(2\pi t) + 0.11Tn_{m15} - 0.02Tx_{m31}$$

$$\log(\lambda_{(12)}(t)) = -32.1 - 6.6\cos(2\pi t) - 3.1\sin(2\pi t) + 0.01 Tn_{m15} + 0.04 Tx_{m15} +$$

$$0.07 TTn$$

4.5 Validation analysis

The assumptions to be checked are the NHPP behavior of the three marginal processes and their mutual independence.

Checking the Poisson character. The usual approach to check a univariate NHPP($\lambda(t)$) is to test whether the process after the time transformation $t^* = \int_0^t \lambda(u)du$ is a unit rate HPP. Then, the validation analysis is standard and it is based on the study of the uncorrelation and the exponentiality of the inter-event distances $d_i^* = t_i^* - t_{i-1}^*$, or the uniform character of $\exp(-d_i^*)$. A more detailed example of the application of these validation techniques can be found in Abaurrea et al. (2007). The point process raw residuals (Baddeley et al. 2005) are also checked.

The validation results, briefly summarized here, do not show any evidence against the fitted model: the p-values of the uniform KS test are 0.56 for $N_{(1)}$, 0.16 for $N_{(2)}$ and 0.55 for $N_{(12)}$, and all the Kendall correlation p-values are greater than 0.7. The empirical and the fitted rates $\int_{t_1}^{t_2} \hat{\lambda}(u)du/(t_2 - t_1)$, both calculated in moving 5-month intervals, are plotted in Figure 4. For $N_{(1)}$, its lowess with a 20% window is also shown (top right plot). The observed evolution is satisfactorily reproduced by the fitted intensity, and a clear increase of the occurrence rate from the late 90's is observed.

Concerning the other processes, an increase of the occurrence rate, higher in $N_{(2)}$, is observed from the 80's, and it becomes steeper from the 90's onwards.
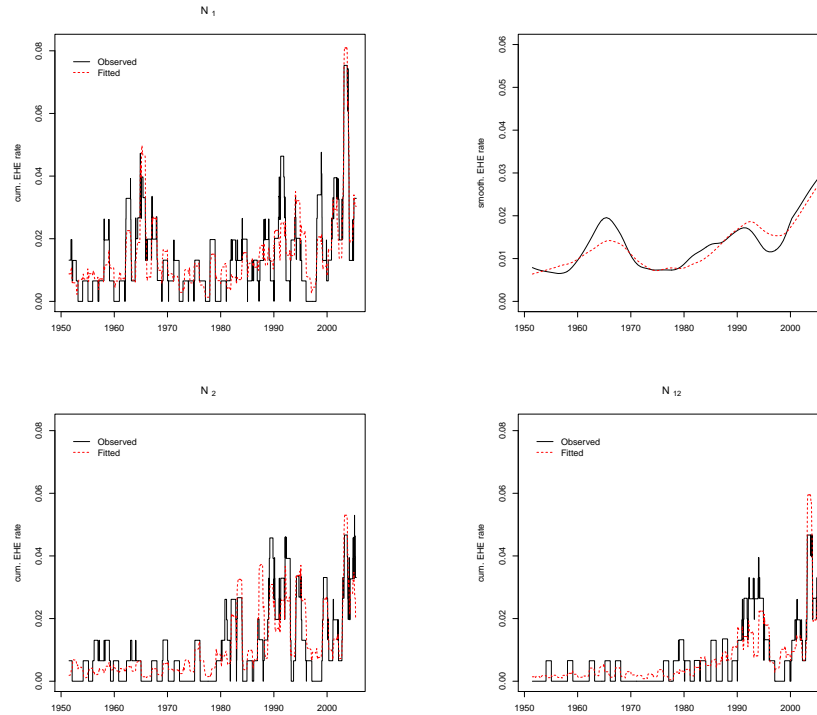
Fig. 4 Empirical and fitted intensities of the marginal processes of the NHCPSP for the extreme heat events in Zaragoza. The top right plot is the 20% lowess of the signals of $N_{(1)}$

Checking the independence of the marginal processes. To check the independence between $N_{(1)}, N_{(2)}$ and $N_{(12)}$, the test in Section 2.1 is applied with $n_S = 1000$ simulations. Since the real intensities of the PP are unknown, a parametric bootstrap p-value is calculated using the MLE $\hat{\lambda}_y(t)$ and $\hat{\lambda}_z(t)$, estimated under the null hypothesis. The marginal PP have been satisfactorily validated, so that it can be assumed that the in-

tensities are well estimated. The p-value of the independence test is 0.423 when close

points are defined as points with overlapping intervals, and 0.569 when the previous

and the following points are included in the definition. In both cases, the null hypoth-

esis cannot be rejected at any usual significance level, and the three processes can be

considered independent.

5 Conclusions

A test to check the independence between NH point processes is developed. When

the parameters describing the point processes are known, it is a Monte Carlo test but,

in most real problems, the parameters are unknown and it is a parametric bootstrap

test. The test is developed for the three-process case, but it can be generalized to any

number of processes. It is noteworthy that it can be used to check the independence

of any point process which can be simulated, since the type of process (Poisson for

example) is only used to generate independent trajectories.

The use of a computational algorithm which reuses the generated processes in

two steps of the test (the calculation of the distance percentiles and the distribution of

the test statistic) allows us to keep low the computation time.

A simulation study shows that the size of the test reaches its nominal level and

that its power is satisfactory and improves with the sample size. Two approaches for

generating dependent NH processes are considered: NHPP based on two queues in

tandem and NH point processes resulting from marking a PP using a Markov chain.

Different measures for quantifying the dependence between the NH processes are

suggested.

This test is a necessary tool in the validation analysis of a common Poisson shock process for checking the independence of the marginal Poisson processes. It could also be used in different applications to check the independence between point processes characterizing the time evolution of different variables. This type of problems appear in biomedical sciences, economy and specially in environmental sciences, for example to represent the occurrence of earthquakes in several regions.

Concerning the temperature application, it is shown that the CPSP is a valuable model to characterize the occurrence of extreme events in two dependent variables. It is successfully applied to model the occurrence of EHEs in daily maximum and minimum temperatures.

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Bio-note

Dr. Ana C. Cebrián is senior lecturer at the Department of Statistics of the University of Zaragoza. Her background is on Mathematics and she completed her doctoral dissertation on extreme value theory with application on drought process modelling. Since 1993, she has been teaching at the Faculty of Sciences, first as assistant and associate professor and then as senior lecturer (2003). After obtaining the Prix FSR 2000 from the Catholic University of Leuven, she occupied a 1-year post-doctoral position at the Institute of Statistics of that university. At present, she is working with the Stochastic Processes research group and more specifically on the development of statistic and probability models for climate data.

Research

The general topic of my research is statistical modelling applied to climate data. More specifically, it focuses on extreme value analysis; that is the development of statistical and probabilistic models to handle extreme data. I apply these models to the analysis of drought, periods with extremely low rainfall, or heat waves, periods with extremely high temperatures. The objective is to design models capable of describing not only what has happened in the past but also to use information to make long term forecasts of the characteristics of the extreme events in the future. Currently, I am working with bivariate models based on copulas.