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DOCTORAL THESIS

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I am indebted to Dr. Mikhail Kochetov, supervisor of my research visit of three months to the Memorial University of Newfoundland (St.John's, Canada). During that time, together with my thesis supervisor, Prof. Kochetov and myself wrote a paper entitled *A* $\mathbb{Z}_4^3$*-grading on a* 56*-dimensional simple exceptional structurable algebra and related fine gradings on the simple Lie algebras of type* E. This paper has been published in Comment. Math. Univ. Carolin. **55** (2014), no. 3, 285–313. Most of its results are included in the last chapter of this Thesis.

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Contents

Introduction

The main goal of this thesis is to shed some light on the problem of classification of fine gradings on some simple Lie algebras. (Note that the classification of equivalence classes of fine gradings is already known for simple Lie algebras over an algebraically closed field of characteristic zero, but the problem is still open for the case of positive characteristic.) The objects that we consider in this work are the exceptional simple Jordan pairs and triple systems, and a 56-dimensional exceptional simple structurable algebra referred to as the Brown algebra. Gradings on these structures can be used to construct gradings on the simple exceptional Lie algebras of types E_6, E_7 and E_8, via the Kantor construction.

In this work, by grading we usually mean group grading. The base field is assumed to be algebraically closed of characteristic different from 2.

Simple Lie algebras over the field of complex numbers $\mathbb{C}$ were classified by W. Killing and E. Cartan in 1894, and Cartan decompositions (i.e., gradings by the associated root system) were used in the proof. After much work on some particular gradings, a systematic study of gradings was initiated by Patera and Zassenhaus in the context of Lie algebras (see [PZ89]), as a generalization of Cartan decompositions. Since then, gradings have been studied by many authors, and are an important tool for a better understanding of algebras and other structures.

It is well-known that, if the base field is algebraically closed of characteristic 0, fine gradings by abelian groups are in bijective correspondence with maximal abelian diagonalizable subgroups of the automorphism group and, in that correspondence, equivalence classes of gradings correspond to conjugation classes of subgroups. Note that the same correspondence holds for any characteristic if we consider the automorphism group-scheme instead of the automorphism group. Therefore, the problem of classification of fine gradings by abelian groups on an algebra (and other algebraic structures) can be reformulated as an equivalent problem in group theory.

This work is organized as follows:

In Chapter 1 we recall the basic definitions of the algebraic structures that we study in this work. We also recall some well-known results of classifications of fine gradings on Cayley and Albert algebras, which constitute an extremely important tool to construct gradings in other structures, as shown in further sections. This chapter has no original results of the author.

The basic definitions of gradings (by abelian groups) on Jordan systems are given in Chapter 2, and hereinafter some general results are proven. We recall the definition of the exceptional simple Jordan pairs and triple systems, i.e., the bi-Cayley and Albert pairs and triple systems. Some results related to the orbits and automorphism groups of the bi-Cayley pair and triple system are given too.

In Chapters 3 and 4 we obtain classifications of the equivalence classes of fine gradings (by abelian groups) on the exceptional simple Jordan pairs and triple systems (over an algebraically closed field of characteristic not 2). The associated Weyl group is determined for each grading. We also study the induced fine gradings on $\mathfrak{e}_6$ and $\mathfrak{e}_7$ via the TKK-construction.

Finally, in Chapter 5 we give a construction of a $\mathbb{Z}_4^3$-grading on the Brown algebra. We then compute the Weyl group of this grading and study how this grading can be used to construct some very special fine gradings on $\mathfrak{e}_6$, $\mathfrak{e}_7$ and $\mathfrak{e}_8$.

Introducción

El objetivo principal de esta tesis es arrojar algo de luz sobre el problema de clasificación de las graduaciones finas en algunas álgebras de Lie simples. (Recalquemos que ya se conoce la clasificación de las clases de equivalencia de graduaciones finas para álgebras de Lie simples sobre un cuerpo algebraicamente cerrado de característica cero, pero el problema sigue abierto en el caso de característica positiva). Los objetos de estudio en este trabajo son los pares y sistemas triples de Jordan simples excepcionales, y un álgebra estructurable simple excepcional de dimensión 56 conocida como álgebra de Brown. Las graduaciones en estas estructuras pueden ser usadas para construir graduaciones en las álgebras de Lie simples excepcionales de los tipos E_6, E_7 y E_8, mediante la construcción de Kantor.

En este trabajo, por graduación normalmente nos referimos a graduaciones de grupo. Asumiremos que el cuerpo base es algebraicamente cerrado de característica diferente de 2.

Las álgebras de Lie simples sobre el cuerpo $\mathbb{C}$ de los números complejos fueron clasificadas por W. Killing y E. Cartan en 1894, y en la demostración se utilizaron descomposiciones de Cartan (es decir, graduaciones por el sistema de raíces asociado). El estudio sistemático de graduaciones fue iniciado por Patera y Zassenhaus en el contexto de álgebras de Lie ([PZ89]), como una generalización de las descomposiciones de Cartan. Desde entonces, muchos autores han estudiado las graduaciones, y éstas son una herramienta importante para entender mejor las álgebras y otras estructuras.

Es bien conocido que, en el caso de que el cuerpo base sea algebraicamente cerrado de característica cero, las graduaciones finas de grupo abeliano se corresponden biyectivamente con los subgrupos abelianos diagonalizables maximales del grupo de automorfismos, y en esa correspondencia, las clases de equivalencia de graduaciones se corresponden con clases de conjugación de subgrupos. Notemos que se tiene la misma correspondencia para cualquier característica si consideramos el esquema-grupo de automorfismos en lugar del grupo de automorfismos. Por tanto, el problema de clasificación de graduaciones finas de grupo abeliano en un álgebra (y en otras estructuras) puede ser reformulado como un problema equivalente en teoría de grupos.

Este trabajo está organizado de la siguiente manera:

En el Capítulo 1 recordamos las definiciones básicas sobre las estructuras algebraicas que estudiamos en este trabajo. También recordamos algunos resultados bien conocidos sobre clasificaciones de graduaciones finas en álgebras de Cayley y de Albert, que son una herramienta extremadamente importante para construir graduaciones en otras estructuras, como se muestra en secciones posteriores. Este capítulo no contiene resultados originales del autor.

Las definiciones básicas sobre graduaciones (de grupo abeliano) en sistemas de Jordan son explicadas en el Capítulo 2, y posteriormente se demuestran algunos resultados generales. Recordamos las definiciones de los pares y sistemas triples de Jordan simples excepcionales, es decir, los pares y sistemas triples bi-Cayley y Albert. Se obtienen también algunos resultados relacionados con las órbitas y grupos de automorfismos del par y sistema triple bi-Cayley.

En los Capítulos 3 y 4 obtenemos clasificaciones de las clases de equivalencia de graduaciones finas (de grupo abeliano) en los pares y sistemas triples de Jordan simples excepcionales (sobre un cuerpo algebraicamente cerrado de característica diferente de 2). Determinamos el grupo de Weyl asociado a cada graduación. También estudiamos cuáles son las graduaciones finas inducidas en $\mathfrak{e}_6$ y $\mathfrak{e}_7$ mediante la construcción TKK.

Finalmente, en el Capítulo 5 damos una construcción de una $\mathbb{Z}_4^3$-graduación en el álgebra de Brown. Después calculamos el grupo de Weyl de esta graduación y estudiamos cómo se puede utilizar esta graduación para construir algunas graduaciones finas en $\mathfrak{e}_6$, $\mathfrak{e}_7$ y $\mathfrak{e}_8$.

Chapter 1

Preliminaries

We always assume, unless otherwise stated, that the ground field $\mathbb{F}$ is algebraically closed of characteristic different from 2. Recall that an *algebra over* $\mathbb{F}$, or $\mathbb{F}$-*algebra*, is a vector space over $\mathbb{F}$ with a bilinear multiplication.

In this chapter many known results and definitions needed later will be recalled.

1.1 Gradings on algebras

For details on the results in this section the reader may consult [EK13, Chapter 1].

Let $\mathcal{A}$ be an algebra (not necessarily associative) over a field $\mathbb{F}$ and let G be a group (written multiplicatively).

Definition 1.1.1. A G-*grading* on $\mathcal{A}$ is a vector space decomposition

$$\Gamma : \ \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$$

such that $\mathcal{A}_g \mathcal{A}_h \subseteq \mathcal{A}_{gh} \quad \text{for all} \quad g, h \in G$. If such a decomposition is fixed, $\mathcal{A}$ is referred to as a G-*graded algebra*. The nonzero elements $x \in \mathcal{A}_g$ are said to be *homogeneous of degree* g, and one writes $\deg_\Gamma x = g$ or just $\deg x = g$ if the grading is clear from the context. The *support* of Γ is the set

$$\operatorname{Supp} \Gamma := \{g \in G \mid \mathcal{A}_g \neq 0\}.$$

An *involution* σ of $\mathcal{A}$ is an antiautomorphism of order 2 of $\mathcal{A}$. If $(\mathcal{A}, \sigma)$ is an algebra with involution, then we will always assume $\sigma(\mathcal{A}_g) = \mathcal{A}_g$ for all $g \in G$.

There is a more general concept of grading: a decomposition $\Gamma : \mathcal{A} = \bigoplus_{s\in S} \mathcal{A}_s$ into nonzero subspaces indexed by a set S and having the property that, for any $s_1, s_2 \in S$ with $\mathcal{A}_{s_1}\mathcal{A}_{s_2} \neq 0$, there exists (a unique) $s_3 \in S$ such that $\mathcal{A}_{s_1}\mathcal{A}_{s_2} \subseteq \mathcal{A}_{s_3}$; this will be called a *set grading*. For such a decomposition Γ, there may or may not exist a group G containing S that makes Γ a G-grading. If such a group exists, Γ is said to be a *group grading*. However, G is usually not unique even if we require that it should be generated by S. The *universal grading group*, $\mathcal{U}(\Gamma)$, is the group generated by S and the defining relations $s_1s_2 = s_3$ for all $s_1, s_2, s_3 \in S$ such that $0 \neq \mathcal{A}_{s_1}\mathcal{A}_{s_2} \subseteq \mathcal{A}_{s_3}$ (see [EK13, Chapter 1] for details).

Suppose that a group grading Γ on $\mathcal{A}$ admits a realization as a G_0-grading for some group G_0. Then G_0 is isomorphic to the universal group of Γ if and only if for any other realization of Γ as a G-grading there is a unique homomorphism $G_0 \to G$ that restricts to the identity on $\operatorname{Supp}\Gamma$.

It is known that if Γ is a group grading on a simple Lie algebra, then $\operatorname{Supp}\Gamma$ always generates an abelian subgroup. In other words, the universal grading group is abelian. Here we will deal exclusively with abelian groups, and we will sometimes write them additively. Gradings by abelian groups often arise as eigenspace decompositions with respect to a family of commuting diagonalizable automorphisms. If $\mathbb{F}$ is algebraically closed and $\operatorname{char}\mathbb{F} = 0$ then all abelian group gradings on finite-dimensional algebras can be obtained in this way.

Definition 1.1.2. Let $\Gamma : \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ and $\Gamma' : \mathcal{B} = \bigoplus_{h\in H} \mathcal{B}_h$ be two group gradings, with supports S and T, respectively.

- We say that Γ and Γ' are *equivalent* if there exists an isomorphism of algebras $\varphi\colon \mathcal{A} \to \mathcal{B}$ and a bijection $\alpha\colon S \to T$ such that $\varphi(\mathcal{A}_s) = \mathcal{B}_{\alpha(s)}$ for all $s \in S$. If G and H are universal grading groups then α extends to an isomorphism $G \to H$.
- In the case $G = H$, the G-gradings Γ and Γ' are *isomorphic* if $\mathcal{A}$ and $\mathcal{B}$ are isomorphic as G-graded algebras, i.e., if there exists an isomorphism of algebras $\varphi\colon \mathcal{A} \to \mathcal{B}$ such that $\varphi(\mathcal{A}_g) = \mathcal{B}_g$ for all $g \in G$.

Definition 1.1.3. Given a grading Γ on $\mathcal{A}$,

- the *automorphism group*, $\operatorname{Aut}(\Gamma)$, is the group of self-equivalences of Γ.
- Each $\varphi \in \operatorname{Aut}(\Gamma)$ determines a permutation of the support, which extends to an automorphism of the universal grading group G. Thus we obtain a group homomorphism $\operatorname{Aut}(\Gamma) \to \operatorname{Aut}(G)$. The kernel of this homomorphism is called the *stabilizer*, $\operatorname{Stab}(\Gamma)$. In other words, $\operatorname{Stab}(\Gamma)$ consists of the automorphisms of the G-graded algebra $\mathcal{A}$.

- The quotient group, $\mathrm{Aut}(\Gamma)/\mathrm{Stab}(\Gamma)$, can be regarded as a subgroup of $\mathrm{Aut}(G)$ and is called the *Weyl group*, $\mathcal{W}(\Gamma)$.

Definition 1.1.4. If $\Gamma : \mathcal{A} = \bigoplus_{g\in G}\mathcal{A}_g$ and $\Gamma' : \mathcal{A} = \bigoplus_{h\in H}\mathcal{A}'_h$ are two gradings on the same algebra, with supports S and T, respectively, then we will say that Γ' is a *refinement* of Γ (or Γ is a *coarsening* of Γ') if for any $t \in T$ there exists (a unique) $s \in S$ such that $\mathcal{A}'_t \subseteq \mathcal{A}_s$. If, moreover, $\mathcal{A}'_t \neq \mathcal{A}_s$ for at least one $t \in T$, then the refinement is said to be *proper*. Finally, Γ is said to be *fine* if it does not admit any proper refinements.

Definition 1.1.5. If Γ is a grading on a finite-dimensional algebra $\mathcal{A}$, a sequence of natural numbers $(n_1, n_2, \dots)$ is called the *type* of the grading Γ if there are exactly n_i homogeneous components of dimension i, for $i \in \mathbb{N}$. Note that $\dim\mathcal{A} = \sum_i i\cdot n_i$.

1.2 Lie algebras

Recall that a Lie algebra is a vector space $\mathfrak{g}$ over $\mathbb{F}$ with a product $[\cdot,\cdot]\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$, $(x,y)\mapsto[x,y]$, satisfying the identities

$$[x,x]=0 \quad \text{(anticommutativity)},$$
$$[[x,y],z]+[[y,z],x]+[[z,x],y]=0 \quad \text{(Jacobi identity)},$$

for any $x,y,z\in\mathfrak{g}$. We assume that the reader is familiar with Lie algebras (the reader may consult [H78]). Recall that any finite dimensional simple Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0 is isomorphic to a unique algebra of the following list: $\mathfrak{a}_n = \mathfrak{sl}_{n+1}(\mathbb{F})$ with $n\geq 1$, $\mathfrak{b}_n = \mathfrak{so}_{2n+1}(\mathbb{F})$ with $n\geq 2$, $\mathfrak{c}_n = \mathfrak{sp}_{2n}(\mathbb{F})$ with $n\geq 3$, $\mathfrak{d}_n = \mathfrak{so}_{2n}(\mathbb{F})$ with $n\geq 4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$, $\mathfrak{f}_4$, $\mathfrak{g}_2$. The Lie algebras of types A_n, B_n, C_n and D_n are called *classical*, and the ones of types E_6, E_7, E_8, F_4 and G_2 are called *exceptional*.

We also recall that (semi)simple Lie algebras over an algebraically closed field of characteristic 0 were classified using gradings by root systems, which are called *Cartan gradings*. If $\mathfrak{g}$ is a semisimple Lie algebra, its Cartan grading is unique up to automorphisms of the algebra and has the form

$$\mathfrak{g} = \mathfrak{h}\oplus\bigoplus_{\alpha\in\Phi}\mathfrak{g}_\alpha,$$

where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and Φ is the associated root system. Note that these gradings satisfy $[\mathfrak{g}_\alpha,\mathfrak{g}_\beta]\subseteq\mathfrak{g}_{\alpha+\beta}$. The Cartan grading of $\mathfrak{g}$ is a particular case of group grading and its universal group is $\mathbb{Z}^r$, where r is the rank of $\mathfrak{g}$. Also, recall that the *extended Weyl group* of $\mathfrak{g}$, denoted by $\widetilde{\mathcal{W}}(\mathfrak{g})$, which is isomorphic to the Weyl group of its Cartan grading, is isomorphic to the semidirect product of $\mathcal{W}(\mathfrak{g})$ (the Weyl group of the root system of $\mathfrak{g}$) and the automorphism group of the Dynkin diagram of the root system of $\mathfrak{g}$.

1.3 Affine group schemes

For the basic definitions and facts on affine group schemes the reader may consult [W79]. A nice introduction to affine group schemes, including the relation between gradings on algebras and their automorphism group schemes, can be found in [EK13, Section 1.4 and Appendix A] (note that these results also hold for Jordan pairs, and we will use them in subsequent sections without further mention). Recall that a G-grading on an algebra $\mathcal{A}$ is produced by a morphism $\eta_\Gamma\colon G^D \to \mathbf{Aut}(\mathcal{A})$.

Let Set and Grp denote the categories of sets and groups, respectively. Let $\mathbb{F}$ be a field and $\mathrm{Alg}_\mathbb{F}$ the category of commutative associative unital $\mathbb{F}$-algebras.

Let $\mathbf{F}$ and $\mathbf{G}$ be functors from $\mathrm{Alg}_\mathbb{F}$ to Set. Recall that a *natural map* $\theta\colon \mathbf{F} \to \mathbf{G}$ is a family of maps $\theta_\mathcal{R}\colon \mathbf{F}(\mathcal{R}) \to \mathbf{G}(\mathcal{R})$, where $\mathcal{R}$ denotes an object of $\mathrm{Alg}_\mathbb{F}$, such that for any $\mathcal{R}, \mathcal{S} \in \mathrm{Alg}_\mathbb{F}$ and any homomorphism $\varphi\colon \mathcal{R} \to \mathcal{S}$, the following diagram commutes:

$$\begin{array}{ccc} \mathbf{F}(\mathcal{R}) & \xrightarrow{\theta_\mathcal{R}} & \mathbf{G}(\mathcal{R}) \\ {\scriptstyle \mathbf{F}(\varphi)}\big\downarrow & & \big\downarrow{\scriptstyle \mathbf{G}(\varphi)} \\ \mathbf{F}(\mathcal{S}) & \xrightarrow{\theta_\mathcal{S}} & \mathbf{G}(\mathcal{S}) \end{array}$$

Definition 1.3.1.

- An *affine scheme* is a functor $\mathbf{F}\colon \mathrm{Alg}_\mathbb{F} \to \mathrm{Set}$ that is *representable*, i.e., there is an object $\mathcal{A}$ in $\mathrm{Alg}_\mathbb{F}$ such that $\mathbf{F}$ is naturally isomorphic to $\mathrm{Hom}_{\mathrm{Alg}_\mathbb{F}}(\mathcal{A}, -)$, that is, for each object $\mathcal{R}$ in $\mathrm{Alg}_\mathbb{F}$ there is a bijection from $\mathbf{F}(\mathcal{R})$ to $\mathrm{Alg}(\mathcal{A}, \mathcal{R})$ that preserves morphisms. In that case, the object $\mathcal{A}$ representing $\mathbf{F}$ is unique up to isomorphism, and is called a *representing object* of $\mathbf{F}$.

- An *affine group scheme* over $\mathbb{F}$ is a functor $\mathbf{G}\colon \mathrm{Alg}_\mathbb{F} \to \mathrm{Grp}$ such that its composition with the forgetful functor $\mathrm{Grp} \to \mathrm{Set}$ is representable. Recall that the representing object of an affine group scheme $\mathbf{G}$ is a commutative Hopf algebra, and conversely, each commutative Hopf algebra is the representing object of an affine group scheme.

Example 1.3.2. Let $\mathcal{U}$ denote a (not necessarily associative) algebra over $\mathbb{F}$ of finite dimension. For any $\mathcal{R}$ in $\mathrm{Alg}_{\mathbb{F}}$, the tensor product $\mathcal{U}\otimes\mathcal{R}$ is an $\mathcal{R}$-algebra, so we can define

$$\mathbf{Aut}(\mathcal{U})(\mathcal{R}) := \mathrm{Aut}_{\mathcal{R}}(\mathcal{U}\otimes\mathcal{R}).$$

Then, $\mathbf{Aut}(\mathcal{U})$ is an affine group scheme, which is called the *automorphism group scheme* of the algebra $\mathcal{U}$.

We recall now the relation between gradings on algebras by abelian groups and the associated automorphism group schemes. Fix a grading $\mathcal{U}=\bigoplus_{g\in G}\mathcal{U}_g$ by an abelian group G on the algebra $\mathcal{U}$ (that is, $\mathcal{U}_g\mathcal{U}_h\subseteq\mathcal{U}_{gh}$ for all $g,h\in G$). Let $\mathbb{F}G$ denote the group algebra of G, which is also a Hopf algebra with the comultiplication $\Delta(g)=g\otimes g$ for $g\in G$. Then, the map

$$\begin{aligned}\rho\colon \mathcal{U} &\longrightarrow \mathcal{U}\otimes\mathbb{F}G\\ u_g &\longmapsto u_g\otimes g\end{aligned}$$

where $u_g\in\mathcal{U}_g$, is an algebra homomorphism and it endows $\mathcal{U}$ with the structure of a right comodule for $\mathbb{F}G$. We can define a morphism of affine group schemes

$$\theta\colon\ \mathrm{Hom}_{\mathrm{Alg}_{\mathbb{F}}}(\mathbb{F}G,-)\to\mathbf{Aut}(\mathcal{U})$$

determined by

$$\theta_{\mathcal{R}}(f)(u\otimes 1)=(\mathrm{id}\otimes f)\rho(u)$$

for each $\mathcal{R}\in\mathrm{Alg}_{\mathbb{F}}$, $f\in\mathrm{Hom}_{\mathrm{Alg}_{\mathbb{F}}}(\mathbb{F}G,\mathcal{R})$, $u\in\mathcal{U}$. Conversely, given a morphism $\theta\colon\ \mathrm{Hom}_{\mathrm{Alg}_{\mathbb{F}}}(\mathbb{F}G,-)\to\mathbf{Aut}(\mathcal{U})$ we can define a G-grading

$$\mathcal{U}=\bigoplus_{g\in G}\mathcal{U}_g$$

where

$$\mathcal{U}_g=\{x\in\mathcal{U}\mid\theta_{\mathbb{F}G}(\mathrm{id})(x\otimes 1)=x\otimes g\}.$$

Hence, there is a correspondence between abelian group gradings on $\mathcal{U}$ and the associated morphisms of the automorphism group scheme.

In particular, the classifications (up to equivalence and up to isomorphism) of gradings on $\mathcal{U}$ by abelian groups are determined by $\mathbf{Aut}(\mathcal{U})$. Also, two algebras with isomorphic automorphism group schemes must have the same classifications of gradings, in certain sense (note that this correspondence preserves universal groups, Weyl groups, etc).

1.4 Gradings on Cayley algebras

We assume that the reader is familiar with Hurwitz algebras (for the basic facts about Hurwitz algebras the reader may consult [ZSSS82, Chap. 2] or [KMRT98, Chap. 8]). A classification of gradings on Hurwitz algebras was given in [Eld98]. The reader may consult [EK13, Chapter 4]. We will now recall the basic definitions.

Definition 1.4.1. A *composition algebra* is an algebra C (not necessarily associative) with a quadratic form $n\colon C \to \mathbb{F}$, called the *norm*, which is nondegenerate (i.e., $n(x,y)=0$ for all $y \in C$ implies $x=0$) and multiplicative. The dimension of any composition algebra is restricted to 1, 2, 4 or 8. Unital composition algebras are called *Hurwitz algebras*. The 4-dimensional Hurwitz algebras are the *quaternion algebras*, and the 8-dimensional Hurwitz algebras are called *Cayley algebras* or *octonion algebras*. Since our field $\mathbb{F}$ is algebraically closed, there is only one Cayley algebra up to isomorphism, which will be denoted by $\mathcal{C}$.

Recall that the polar form of the norm is given by

$$n(x,y) = n(x+y) - n(x) - n(y). \tag{1.4.1}$$

Any element of a Hurwitz algebra satisfies the quadratic equation

$$x^2 - n(x,1)x + n(x)1 = 0, \tag{1.4.2}$$

which can be written as $x\bar{x} = \bar{x}x = n(x)1$, where

$$\bar{x} = n(x,1)1 - x \tag{1.4.3}$$

is called the conjugate of x. The *trace* is the linear form $\operatorname{tr}\colon C \to \mathbb{F}$ given by $\operatorname{tr}(x) = n(x,1)$. The map $x \mapsto \bar{x}$ is an involution, called the *standard involution*, that satisfies

$$n(\bar{x},\bar{y}) = n(x,y), \quad n(xy,z) = n(y,\bar{x}z) = n(x,z\bar{y}) \tag{1.4.4}$$

for any $x,y,z \in C$.

Also, recall that if C is a Hurwitz algebra, then C with the new product $x * y := \bar{x}\bar{y}$ and the same norm is called the associated *para-Hurwitz algebra*, which will be denoted by $\bar{C}$. Para-Hurwitz algebras of dimension bigger than 1 are nonunital composition algebras. They are also well-known examples of symmetric composition algebras. The para-Hurwitz algebra $\bar{\mathcal{C}}$ is called the *para-Cayley algebra*.

Recall that, since char $\mathbb{F} \neq 2$, there are two fine gradings up to equivalence on $\mathcal{C}$, which are a $\mathbb{Z}^2$-grading (Cartan grading) and a $\mathbb{Z}_2^3$-grading. We will recall them now. (In case that char $\mathbb{F} = 2$, the Cartan grading is the only fine grading up to equivalence on $\mathcal{C}$. But we always assume that char $\mathbb{F} \neq 2$.)

A *split Hurwitz algebra* is a Hurwitz algebra C with a nonzero isotropic element: $0 \neq x \in C$ such that $n(x) = 0$. Note that any Hurwitz algebra of dimension ≥ 2 over an algebraically closed field is split. Let $\mathcal{C}$ be a split Cayley algebra and let a be a nonzero isotropic element. In that case, since n is nondegenerate, we can take $b \in \mathcal{C}$ such that $n(a, \bar{b}) = 1$. Let $e_1 := ab$. We have $n(e_1) = 0$ and $n(e_1, 1) = 1$, so $e_1^2 = e_1$. Let $e_2 := \bar{e_1} = 1 - e_1$, so $n(e_2) = 0$, $e_2^2 = e_2$, $e_1e_2 = 0 = e_2e_1$ and $n(e_1, e_2) = n(e_1, 1) = 1$. Then $K = \mathbb{F}e_1 \oplus \mathbb{F}e_2$ is a Hurwitz subalgebra of $\mathcal{C}$. For any $x \in K^\perp$, $xe_1 + \overline{xe_1} = n(xe_1, 1)1 = n(x, \bar{e}_1)1 = n(x, e_2)1 = 0$. Hence $xe_1 = -\bar{e}_1\bar{x} = e_2x$, and we get $xe_1 = e_2x$, $xe_2 = e_1x$. Also, $x = 1x = e_1x + e_2x$, and $e_2(e_1x) = (1 - e_1)(e_1x) = ((1 - e_1)e_1)x = 0 = e_1(e_2x)$. Therefore, $K^\perp = U \oplus V$, with

$$U = \{x \in \mathcal{C} \mid e_1x = x = xe_2,\ e_2x = 0 = xe_1\} = (e_1\mathcal{C})e_2,$$
$$V = \{x \in \mathcal{C} \mid e_2x = x = xe_1,\ e_1x = 0 = xe_2\} = (e_2\mathcal{C})e_1.$$

For any $u \in U$, $n(u) = n(e_1u) = n(e_1)n(u) = 0$, so U and V are isotropic subspaces of $\mathcal{C}$. Since n is nondegenerate, U and V are paired by n and $\dim U = \dim V = 3$. Take $u_1, u_2 \in U$ and $v \in V$. Then,

$$n(u_1u_2, K) \subseteq n(u_1, Ku_2) \subseteq n(U, U) = 0,$$
$$n(u_1u_2, v) = n(u_1u_2, e_2v) = -n(e_2u_2, u_1v) + n(u_1, e_2)n(u_2, v) = 0.$$

Hence U^2 is orthogonal to K and V, so $U^2 \subseteq V$. Also $V^2 \subseteq U$. Besides,

$$n(U, UV) \subseteq n(U^2, V) \subseteq n(V, V) = 0,$$
$$n(UV, V) \subseteq n(U, V^2) \subseteq n(U, U) = 0,$$

so $UV + VU \subseteq K$. Moreover, $n(UV, e_1) \subseteq n(U, e_1V) = 0$, so that $UV \subseteq \mathbb{F}e_1$ and $VU \subseteq \mathbb{F}e_2$. More precisely, for $u \in U$ and $v \in V$, $n(uv, e_2) = -n(u, e_2v) = -n(u, v)$, so that $uv = -n(u, v)e_1$, and $vu = -n(u, v)e_2$. Then, the decomposition $\mathcal{C} = K \oplus U \oplus V$ defines a $\mathbb{Z}_3$-grading on $\mathcal{C}$ where $\mathcal{C}_{\bar{0}} = K$, $\mathcal{C}_{\bar{1}} = U$, $\mathcal{C}_{\bar{2}} = V$.

For linearly independent elements $u_1, u_2 \in U$, take $v \in V$ with $n(u_1, v) \neq 0 = n(u_2, v)$. Then $(u_1u_2)v = -(u_1v)u_2 = n(u_1, v)u_2 \neq 0$, and so $U^2 \neq 0$. Moreover, the trilinear map

$$U \times U \times U \to \mathbb{F}, \quad (x, y, z) \mapsto n(xy, z),$$

is alternating (for any $x \in U$, $n(x) = 0 = n(x, 1)$, so $x^2 = 0$ and hence $n(x^2, z) = 0$; and $n(xy, y) = -n(x, y^2) = 0$ too). Take a basis $\{u_1, u_2, u_3\}$ of U with $n(u_1u_2, u_3) = 1$ (this is always possible because $n(U^2, U) \neq 0$ since n is nondegenerate). Then $\{v_1 = u_2u_3, v_2 = u_3u_1, v_3 = u_1u_2\}$ is the dual basis in V relative to n. We will say that $\{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$ is a *Cartan basis* of the split Cayley algebra $\mathcal{C}$, and its multiplication table is:

	e_1	e_2	u_1	u_2	u_3	v_1	v_2	v_3
e_1	e_1	0	u_1	u_2	u_3	0	0	0
e_2	0	e_2	0	0	0	v_1	v_2	v_3
u_1	0	u_1	0	v_3	$-v_2$	$-e_1$	0	0
u_2	0	u_2	$-v_3$	0	v_1	0	$-e_1$	0
u_3	0	u_3	v_2	$-v_1$	0	0	0	$-e_1$
v_1	v_1	0	$-e_2$	0	0	0	u_3	$-u_2$
v_2	v_2	0	0	$-e_2$	0	$-u_3$	0	u_1
v_3	v_3	0	0	0	$-e_2$	u_2	$-u_1$	0

Other authors may refer to a Cartan basis as a *canonical basis* or a *good basis*. The decomposition $\mathcal{C} = \mathbb{F}e_1 \oplus \mathbb{F}e_2 \oplus U \oplus V$, with $U = \text{span}\{u_1, u_2, u_3\}$ and $V = \text{span}\{v_1, v_2, v_3\}$, is the *Peirce decomposition* associated to the idempotents e_1 and e_2. Note that the elements of the Cartan basis are isotropic for the norm, and paired as follows: $n(e_1, e_2) = 1 = n(u_i, v_i)$, $n(e_i, u_j) = n(e_i, v_j) = n(u_j, v_k) = 0$ for any $i = 1, 2$ and $j \neq k = 1, 2, 3$, and $n(u_i, u_j) = n(v_i, v_j) = 0$ for any $i, j = 1, 2, 3$. On the split quaternion algebra, we have the Cartan basis given by $\{e_1, e_2, u_1, v_1\}$, with the multiplication as in the table above. On the 2-dimensional split Hurwitz algebra, the Cartan basis is given by the orthogonal idempotents $\{e_1, e_2\}$.

The Cartan basis determines a fine $\mathbb{Z}^2$-grading on $\mathcal{C}$, that is given by

$$\begin{aligned} &\deg(e_1) = 0 = \deg(e_2), \quad \deg(u_1) = (1, 0) = -\deg(v_1), \\ &\deg(u_2) = (0, 1) = -\deg(v_2), \quad \deg(v_3) = (1, 1) = -\deg(u_3), \end{aligned} \tag{1.4.5}$$

which is called the *Cartan grading* on $\mathcal{C}$ (see [EK12a]). The Cartan grading on $\mathcal{C}$ restricts to a fine $\mathbb{Z}$-grading on the quaternion algebra $Q = \text{span}\{e_1, e_2, u_1, v_1\}$, that we call the *Cartan grading* on Q.

We will now recall the *Cayley-Dickson doubling process* for Hurwitz algebras. Let Q be a Hurwitz algebra with norm n and involution $x \mapsto \bar{x}$, and fix $0 \neq \alpha \in \mathbb{F}$. Then, we define a new algebra as the vector space $C = \mathfrak{CD}(Q, \alpha) := Q \oplus Qu$ (the direct sum of two copies of Q) with multiplication

$$(a + bu)(c + du) = (ac + \alpha \bar{d}b) + (da + b\bar{c})u, \tag{1.4.6}$$

and quadratic form

$$n(a + bu) = n(a) - \alpha n(y). \tag{1.4.7}$$

Then, C is a Hurwitz algebra if and only if Q is associative. The decomposition $C = C_{\bar{0}} \oplus C_{\bar{1}}$ with $C_{\bar{0}} = Q$ and $C_{\bar{1}} = Qu$ is actually a $\mathbb{Z}_2$-grading.

A fine $\mathbb{Z}_2^3$-grading on $\mathcal{C}$ can be obtained by applying three times the Cayley-Dickson doubling process (see [Eld98] for the construction), because $\mathcal{C} = \mathfrak{CD}(\mathfrak{CD}(\mathfrak{CD}(\mathbb{F}, -1), -1), -1)$. (This grading only exists if char $\mathbb{F} \neq 2$.) Any homogeneous orthonormal basis $\{x_i\}_{i=1}^8$ of $\mathcal{C}$ associated to the $\mathbb{Z}_2^3$-grading will be called a *Cayley-Dickson basis* of $\mathcal{C}$, and we will usually assume that $x_1 = 1$. Note that, if $\{x_i\}_{i=1}^8$ and $\{y_i\}_{i=1}^8$ are Cayley-Dickson bases, then there exist some $\varphi \in \operatorname{Aut} \mathcal{C}$, signs $s_i \in \{\pm 1\}$ and permutation σ of the indices such that $\varphi(x_i) = s_i y_{\sigma(i)}$.

1.5 Gradings on Albert algebras

We assume that the reader is familiar with the exceptional Jordan algebra, usually called the Albert algebra (see [Jac68]). A classification of gradings on the Albert algebra over an algebraically closed field of characteristic different from 2 was obtained in [EK12a]. The reader may consult [EK13, Chapter 5]. We will now recall the basic definitions.

Definition 1.5.1. A *Jordan algebra* is a commutative algebra satisfying the identity $(x^2 y)x = x^2(yx)$. A Jordan algebra is called *special* if it is a subalgebra of $\mathcal{A}^{(+)}$, where $\mathcal{A}^{(+)}$ denotes an associative algebra $\mathcal{A}$ with the symmetric product $XY = \frac{1}{2}(X \cdot Y + Y \cdot X)$. A Jordan algebra which is not special is called *exceptional*.

Note that the algebras $\mathcal{A}^{(+)}$, where $\mathcal{A}$ is an associative algebra, are Jordan algebras. In particular, if $(\mathcal{A}, ^-)$ is an associative algebra with involution, then the subspace of symmetric elements $\mathcal{H}(\mathcal{A}, ^-)$ is a Jordan subalgebra of $\mathcal{A}^{(+)}$.

If V is a vector space over $\mathbb{F}$ endowed with a symmetric bilinear form $b \colon V \times V \to \mathbb{F}$, then the vector space $\mathcal{J}(V, b) := \mathbb{F}1 \oplus V$, with multiplication given by $1 \circ x = x = x \circ 1$ and $u \circ v = b(u, v)1$ for any $x \in \mathcal{J}(V, b)$ and $u, v \in V$, is a Jordan algebra called *the Jordan algebra of the bilinear form b*.

We now recall the classification of the finite-dimensional simple Jordan algebras over an algebraically closed field $\mathbb{F}$ with char $\mathbb{F} \neq 2$ (see [Jac68, Ch. V]). Each of them is isomorphic to one and only one algebra of the following list:

(1) $M_n(\mathbb{F})^{(+)}$, $n = 1$ or $n \geq 3$,

(2) $\mathcal{H}(M_n(\mathbb{F}), t)$, where t is the matrix transpose, $n \geq 3$,

(3) $\mathcal{H}(M_{2n}(\mathbb{F}), t_s)$, where t_s is the symplectic involution $X \mapsto S^{-1}(X^t)S$, $S = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$, $n \geq 3$,

(4) $\mathcal{J}(\mathbb{F}^n, b_n)$, with b_n the standard scalar product of $\mathbb{F}^n$, $n \geq 2$,

(5) the Albert algebra $\mathbb{A}$.

The only exceptional simple Jordan algebra is the Albert algebra $\mathbb{A}$, which is defined as the algebra of hermitian 3×3-matrices over $\mathcal{C}$, that is

$$\mathbb{A} = \mathcal{H}_3(\mathcal{C}, {}^-) = \left\{ \begin{pmatrix} \alpha_1 & \bar{a}_3 & a_2 \\ a_3 & \alpha_2 & \bar{a}_1 \\ \bar{a}_2 & a_1 & \alpha_3 \end{pmatrix} \mid \alpha_i \in \mathbb{F},\ a_i \in \mathcal{C} \right\}$$

$$= \mathbb{F}E_1 \oplus \mathbb{F}E_2 \oplus \mathbb{F}E_3 \oplus \iota_1(\mathcal{C}) \oplus \iota_2(\mathcal{C}) \oplus \iota_3(\mathcal{C}),$$

where

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\iota_1(a) = 2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \bar{a} \\ 0 & a & 0 \end{pmatrix}, \quad \iota_2(a) = 2\begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ \bar{a} & 0 & 0 \end{pmatrix}, \quad \iota_3(a) = 2\begin{pmatrix} 0 & \bar{a} & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

for all $a \in \mathcal{C}$, with multiplication $XY = \frac{1}{2}(X \cdot Y + Y \cdot X)$, where $X \cdot Y$ denotes the usual product of matrices. Then, the E_i's are orthogonal idempotents with $\sum E_i = 1$, and

$$\begin{aligned} &E_i\iota_i(a) = 0, \quad E_{i+1}\iota_i(a) = \frac{1}{2}\iota_i(a) = E_{i+2}\iota_i(a), \\ &\iota_i(a)\iota_{i+1}(b) = \iota_{i+2}(\bar{a}\bar{b}), \quad \iota_i(a)\iota_i(b) = 2n(a,b)(E_{i+1} + E_{i+2}), \end{aligned} \tag{1.5.1}$$

where the subindices are taken modulo 3.

Any element $X = \sum_{i=1}^{3}(\alpha_i E_i + \iota_i(a_i))$ of the Albert algebra satisfies the degree 3 equation

$$X^3 - T(X)X^2 + S(X)X - N(X)1 = 0, \tag{1.5.2}$$

where the linear form T (called the *trace*), the quadratic form S, and the cubic form N (called the *norm*) are given by:

$$\begin{aligned} T(X) &= \alpha_1 + \alpha_2 + \alpha_3, \\ S(X) &= \frac{1}{2}(T(X)^2 - T(X^2)) = \sum_{i=1}^{3}(\alpha_{i+1}\alpha_{i+2} - 4n(a_i)), \\ N(X) &= \alpha_1\alpha_2\alpha_3 + 8n(a_1, \bar{a}_2\bar{a}_3) - 4\sum_{i=1}^{3}\alpha_i n(a_i). \end{aligned} \tag{1.5.3}$$

The Albert algebra $\mathbb{A}$ has a *Freudenthal adjoint* given by

$$x^{\#} := x^2 - T(x)x + S(x)1,$$

with linearization

$$x \times y := (x+y)^{\#} - x^{\#} - y^{\#}.$$

There are four fine gradings up to equivalence on $\mathbb{A}$, with universal groups: $\mathbb{Z}^4$ (the Cartan grading), $\mathbb{Z}_2^5$, $\mathbb{Z} \times \mathbb{Z}_2^3$ and $\mathbb{Z}_3^3$ (the last one does not occur if char $\mathbb{F} = 3$). We recall the construction of these gradings now.

Let $B_1 = \{e_i, u_j, v_j \mid i = 1, 2,\ j = 1, 2, 3\}$ be a Cartan basis of $\mathcal{C}$. We will call the basis $\{E_i, \iota_i(x) \mid x \in B_1,\ i = 1, 2, 3\}$ a *Cartan basis* of $\mathbb{A}$. The $\mathbb{Z}^4$-grading on $\mathbb{A}$ is defined using this basis as follows:

$$\begin{aligned} &\deg E_i = 0, \quad \deg \iota_i(e_1) = a_i = -\deg \iota_i(e_2), \\ &\deg \iota_i(u_i) = g_i = -\deg \iota_i(v_i), \\ &\deg \iota_i(u_{i+1}) = a_{i+2} + g_{i+1} = -\deg \iota_i(v_{i+1}), \\ &\deg \iota_i(u_{i+2}) = -a_{i+1} + g_{i+2} = -\deg \iota_i(v_{i+2}), \end{aligned} \tag{1.5.4}$$

for $i = 1, 2, 3 \bmod 3$, and where

$$\begin{aligned} a_1 &= (1, 0, 0, 0), \quad a_2 = (0, 1, 0, 0), \quad a_3 = (-1, -1, 0, 0), \\ g_1 &= (0, 0, 1, 0), \quad g_2 = (0, 0, 0, 1), \quad g_3 = (0, 0, -1, -1). \end{aligned}$$

Let now $B_2 = \{x_i\}_{i=1}^8$ be a Cayley-Dickson basis of $\mathcal{C}$ with degree map $\deg_{\mathcal{C}}$. The $\mathbb{Z}_2^5$-grading on $\mathbb{A}$ is constructed by imposing that the elements of the basis $\{E_i, \iota_i(x) \mid x \in B_2,\ i = 1, 2, 3\}$ are homogeneous with:

$$\begin{aligned} &\deg E_i = 0, \quad \deg \iota_1(x) = (\bar{1}, \bar{0}, \deg_{\mathcal{C}} x), \\ &\deg \iota_2(x) = (\bar{0}, \bar{1}, \deg_{\mathcal{C}} x), \quad \deg \iota_3(x) = (\bar{1}, \bar{1}, \deg_{\mathcal{C}} x). \end{aligned} \tag{1.5.5}$$

Take $\mathbf{i} \in \mathbb{F}$ with $\mathbf{i}^2 = -1$. The $\mathbb{Z} \times \mathbb{Z}_2^3$-grading on $\mathbb{A}$ is constructed using the following elements of $\mathbb{A}$:

$$\begin{aligned} &E = E_1, \quad \widetilde{E} = 1 - E = E_2 + E_3, \quad S_\pm = E_3 - E_2 \pm \frac{\mathbf{i}}{2}\iota_1(1), \\ &\nu(a) = \mathbf{i}\iota_1(a), \quad \nu_\pm(x) = \iota_2(x) \pm \mathbf{i}\iota_3(\bar{x}), \end{aligned} \tag{1.5.6}$$

for any $a \in \mathcal{C}_0 = \{y \in \mathcal{C} \mid \operatorname{tr}(y) = 0\}$ and $x \in \mathcal{C}$, where the product is:

$$\begin{aligned} &E\widetilde{E} = 0, \quad ES_\pm = 0, \quad E\nu(a) = 0, \quad E\nu_\pm(x) = \frac{1}{2}\nu_\pm(x), \\ &\widetilde{E}S_\pm = S_\pm, \quad \widetilde{E}\nu(a) = \nu(a), \quad \widetilde{E}\nu_\pm(x) = \frac{1}{2}\nu_\pm(x), \\ &S_\pm S_\pm = 0, \quad S_+S_- = 2\widetilde{E}, \quad S_\pm\nu(a) = 0, \\ &S_\pm\nu_\mp(x) = \nu_\pm(x), \quad S_\pm\nu_\pm(x) = 0, \\ &\nu(a)\nu(b) = -2n(a,b)\widetilde{E}, \quad \nu(a)\nu_\pm(x) = \pm\nu_\pm(xa), \\ &\nu_\pm(x)\nu_\pm(y) = 2n(x,y)S_\pm, \quad \nu_+(x)\nu_-(y) = 2n(x,y)(2E + \widetilde{E}) - \nu(\bar{x}y - \bar{y}x), \end{aligned} \tag{1.5.7}$$

for any $x, y \in \mathcal{C}$ and $a, b \in \mathcal{C}_0$.

Fix a Cayley-Dickson basis B_2 of $\mathcal{C}$, with degree map $\deg_{\mathcal{C}}$. The degree map of the $\mathbb{Z} \times \mathbb{Z}_2^3$-grading is given by:

$$\begin{aligned} &\deg S_\pm = (\pm 2, \bar{0}, \bar{0}, \bar{0}), \quad \deg \nu_\pm(x) = (\pm 1, \deg_{\mathcal{C}} x), \\ &\deg E = 0 = \deg \widetilde{E}, \quad \deg \nu(a) = (0, \deg_{\mathcal{C}} a), \end{aligned} \tag{1.5.8}$$

for homogeneous $x \in \mathcal{C}$ and $a \in \mathcal{C}_0$.

Finally, we recall the construction of the $\mathbb{Z}_3^3$-grading on $\mathbb{A}$. Let ω be a cubic root of 1 in $\mathbb{F}$. Consider the order 3 automorphism τ of $\mathcal{C}$ given by $e_i \mapsto e_i$ for $i = 1, 2$ and $u_j \mapsto u_{j+1}$, $v_j \mapsto v_{j+1}$ for $j = 1, 2, 3$. Write $\tilde{\iota}_i(x) = \iota_i(\tau^i(x))$ for $i = 1, 2, 3$, and $x \in \mathcal{C}$. Then, the $\mathbb{Z}_3^3$-grading is determined by the homogeneous generators

$$X_1 = \sum_{i=1}^{3} \tilde{\iota}_i(e_1), \quad X_2 = \sum_{i=1}^{3} \tilde{\iota}_i(u_1), \quad X_3 = \sum_{i=1}^{3} \omega^{-i} E_i,$$

with degrees

$$\deg X_1 = (\bar{1}, \bar{0}, \bar{0}), \quad \deg X_2 = (\bar{0}, \bar{1}, \bar{0}), \quad \deg X_3 = (\bar{0}, \bar{0}, \bar{1}). \tag{1.5.9}$$

1.6 Jordan pairs and triple systems

We will now recall from [L75] some basic definitions about Jordan pairs and Jordan triple systems.

The index σ will always take the values $+$ and $-$, and will be omitted when there is no ambiguity.

Let $\mathcal{V}^+$ and $\mathcal{V}^-$ be vector spaces over $\mathbb{F}$, and let $Q^\sigma\colon \mathcal{V}^\sigma \to \mathrm{Hom}(\mathcal{V}^{-\sigma}, \mathcal{V}^\sigma)$ be quadratic maps. Define trilinear maps, $\mathcal{V}^\sigma \times \mathcal{V}^{-\sigma} \times \mathcal{V}^\sigma \to \mathcal{V}^\sigma$, $(x, y, z) \mapsto \{x, y, z\}^\sigma$, and bilinear maps, $D^\sigma\colon \mathcal{V}^\sigma \times \mathcal{V}^{-\sigma} \to \mathrm{End}(\mathcal{V}^\sigma)$, by the formulas

$$\{x, y, z\}^\sigma = D^\sigma(x, y)z := Q^\sigma(x, z)y \qquad (1.6.1)$$

where $Q^\sigma(x, z) = Q^\sigma(x + z) - Q^\sigma(x) - Q^\sigma(z)$. Note that $\{x, y, z\} = \{z, y, x\}$ and $\{x, y, x\} = 2Q(x)y$.

We will write x^σ to emphasize that x is an element of $\mathcal{V}^\sigma$. Alternatively, we may write $(x, y) \in \mathcal{V}$ to mean $x \in \mathcal{V}^+$ and $y \in \mathcal{V}^-$. The map $Q^\sigma(x)$ is also denoted by Q^σ_x.

Definition 1.6.1. A *(quadratic) Jordan pair* is a pair $\mathcal{V} = (\mathcal{V}^+, \mathcal{V}^-)$ of vector spaces and a pair (Q^+, Q^-) of quadratic maps $Q^\sigma\colon \mathcal{V}^\sigma \to \mathrm{Hom}(\mathcal{V}^{-\sigma}, \mathcal{V}^\sigma)$ such that the following identities hold in all scalar extensions:

(QJP1) $D^\sigma(x, y)Q^\sigma(x) = Q^\sigma(x)D^{-\sigma}(y, x)$,

(QJP2) $D^\sigma(Q^\sigma(x)y, y) = D^\sigma(x, Q^{-\sigma}(y)x)$,

(QJP3) $Q^\sigma(Q^\sigma(x)y) = Q^\sigma(x)Q^{-\sigma}(y)Q^\sigma(x)$.

Definition 1.6.2. A *(linear) Jordan pair* is a pair $\mathcal{V} = (\mathcal{V}^+, \mathcal{V}^-)$ of vector spaces with trilinear products $\mathcal{V}^\sigma \times \mathcal{V}^{-\sigma} \times \mathcal{V}^\sigma \to \mathcal{V}^\sigma$, $(x, y, z) \mapsto \{x, y, z\}^\sigma$, satisfying the following identities:

(LJP1) $\{x, y, z\}^\sigma = \{z, y, x\}^\sigma$,

(LJP2) $[D^\sigma(x, y), D^\sigma(u, v)] = D^\sigma(D^\sigma(x, y)u, v) - D^\sigma(u, D^{-\sigma}(y, x)v)$,

where $D^\sigma(x, y)z = \{x, y, z\}^\sigma$.

Note that, under the assumption char $\mathbb{F} \neq 2$, the definitions of quadratic and linear Jordan pairs are equivalent.

A pair $\mathcal{W} = (\mathcal{W}^+, \mathcal{W}^-)$ of subspaces of a Jordan pair $\mathcal{V}$ is called a *subpair* (respectively an *ideal*) if $Q^\sigma(\mathcal{W}^\sigma)\mathcal{W}^{-\sigma} \subseteq \mathcal{W}^\sigma$ (respectively $Q^\sigma(\mathcal{W}^\sigma)\mathcal{V}^{-\sigma} + Q^\sigma(\mathcal{V}^\sigma)\mathcal{W}^{-\sigma} + \{\mathcal{V}^\sigma, \mathcal{V}^{-\sigma}, \mathcal{W}^\sigma\} \subseteq \mathcal{W}^\sigma$). We say that $\mathcal{V}$ is *simple* if its only ideals are the trivial ones and the maps Q^σ are nonzero.

A *homomorphism* $h\colon \mathcal{V} \to \mathcal{W}$ of Jordan pairs is a pair $h = (h^+, h^-)$ of $\mathbb{F}$-linear maps $h^\sigma\colon \mathcal{V}^\sigma \to \mathcal{W}^\sigma$ such that $h^\sigma(Q^\sigma(x)y) = Q^\sigma(h^\sigma(x))h^{-\sigma}(y)$ for all $x \in \mathcal{V}^\sigma$, $y \in \mathcal{V}^{-\sigma}$. By linearization, this implies $h^\sigma(\{x, y, z\}) = \{h^\sigma(x), h^{-\sigma}(y), h^\sigma(z)\}$ for all $x \in \mathcal{V}^\sigma$, $y \in \mathcal{V}^{-\sigma}$. *Isomorphisms* and *automorphisms* are defined in the obvious way. The ideals are precisely the kernels of homomorphisms.

A *derivation* is a pair $\Delta = (\Delta^+, \Delta^-) \in \mathrm{End}(\mathcal{V}^+) \times \mathrm{End}(\mathcal{V}^-)$ such that $\Delta^\sigma(Q^\sigma(x)y) = \{\Delta^\sigma(x), y, x\} + Q^\sigma(x)\Delta^{-\sigma}(y)$ for all $x \in \mathcal{V}^\sigma$, $y \in \mathcal{V}^{-\sigma}$. For $(x, y) \in \mathcal{V}$, the pair $\nu(x, y) := (D(x, y), -D(y, x)) \in \mathfrak{gl}(\mathcal{V}^+) \oplus \mathfrak{gl}(\mathcal{V}^-)$ is a derivation, which is usually called the *inner derivation* defined by (x, y). It is well-known that $\mathrm{Innder}\,(\mathcal{V}) := \mathrm{span}\{\nu(x, y) \mid (x, y) \in \mathcal{V}\}$ is an ideal of $\mathrm{Der}(\mathcal{V})$.

Definition 1.6.3. A *(quadratic) Jordan triple system* is a vector space $\mathcal{T}$ with a quadratic map $P\colon \mathcal{T} \to \mathrm{End}(\mathcal{T})$ such that the following identities hold in all scalar extensions:

(QJT1) $\quad L(x, y)P(x) = P(x)L(y, x),$

(QJT2) $\quad L(P(x)y, y) = L(x, P(y)x),$

(QJT3) $\quad P(P(x)y) = P(x)P(y)P(x),$

where $L(x, y)z = P(x, z)y$ and $P(x, z) = P(x + z) - P(x) - P(z)$.

Definition 1.6.4. A *(linear) Jordan triple system* is a vector space $\mathcal{T}$ with a trilinear product $\mathcal{T} \times \mathcal{T} \times \mathcal{T} \to \mathcal{T}$, $(x, y, z) \mapsto \{x, y, z\}$, satisfying the following identities:

(LJT1) $\quad \{x, y, z\} = \{z, y, x\},$

(LJT2) $\quad [D(x, y), D(u, v)] = D(D(x, y)u, v) - D(u, D(y, x)v),$

where $D(x, y)z = \{x, y, z\}$.

Note that, under the assumption $\mathrm{char}\,\mathbb{F} \neq 2$, the definitions of quadratic and linear Jordan triple systems are equivalent. In a quadratic Jordan triple system, the triple product is given by $\{x, y, z\} = L(x, y)z$.

A *homomorphism* of Jordan triple systems is an $\mathbb{F}$-linear map $f\colon \mathcal{T} \to \mathcal{T}'$ such that $f(P(x)y) = P(f(x))f(y)$ for all $x, y \in \mathcal{T}$. The rest of basic concepts, including isomorphisms and automorphisms, are defined in the obvious way. Recall that a linear Jordan algebra J has an associated Jordan triple system $\mathcal{T}$ with quadratic product $P(x) = U_x := 2L_x^2 - L_{x^2}$, and similarly, a Jordan triple system $\mathcal{T}$ has an associated Jordan pair $\mathcal{V} = (\mathcal{T}, \mathcal{T})$ with quadratic products $Q^\sigma = P$.

Finite-dimensional simple Jordan pairs were classified by Ottmar Loos (see [L75]) as follows:

I) $(M_{p,q}(\mathbb{F}), M_{p,q}(\mathbb{F}))$, $p \times q$ matrices over $\mathbb{F}$,

II) $(A_n(\mathbb{F}), A_n(\mathbb{F}))$, alternating $n \times n$ matrices over $\mathbb{F}$,

III) $(\mathcal{H}_n(\mathbb{F}), \mathcal{H}_n(\mathbb{F}))$, symmetric $n \times n$ matrices over $\mathbb{F}$,

IV) $(\mathbb{F}^n, \mathbb{F}^n)$,

V) the bi-Cayley pair,

VI) the Albert pair.

In the first three cases the products are given by $Q(x)y = x^t \cdot y \cdot x$. In the fourth case, the products are given by $Q(x)y = q(x, y)x - q(x)y$, where q is the standard quadratic form on $\mathbb{F}^n$. We will describe the last two cases in subsequent sections, where we classify their fine gradings.

1.7 Peirce decompositions and orbits of Jordan pairs

We will now recall some well-known definitions related to Jordan pairs (for more details, see [L75], [L91a], [L91b], [ALM05]).

An element $x \in \mathcal{V}^\sigma$ is called *invertible* if $Q^\sigma(x)$ is invertible, and in this case, $x^{-1} := Q^\sigma(x)^{-1}x$ is said to be the *inverse* of x. The set of invertible elements of $\mathcal{V}^\sigma$ is denoted by $(\mathcal{V}^\sigma)^\times$. A Jordan pair $\mathcal{V}$ is called a *division pair* if $\mathcal{V} \neq 0$ and every nonzero element is invertible. The pair $\mathcal{V}$ is said to be *local* if the noninvertible elements of $\mathcal{V}$ form a proper ideal, say $\mathcal{N}$; in this case, $\mathcal{V}/\mathcal{N}$ is a division pair.

For a fixed $y \in \mathcal{V}^{-\sigma}$, the vector space $\mathcal{V}^\sigma$ with the operators

$$x^2 = x^{(2,y)} := Q^\sigma(x)y, \qquad U_x = U_x^{(y)} := Q^\sigma(x)Q^{-\sigma}(y), \tag{1.7.1}$$

becomes a Jordan algebra, which is denoted by $\mathcal{V}^\sigma_y$. An element $(x, y) \in \mathcal{V}$ is called *quasi-invertible* if x is quasi-invertible in the Jordan algebra $\mathcal{V}^\sigma_y$, i.e., if $1 - x$ is invertible in the unital Jordan algebra $\mathbb{F}1 + \mathcal{V}^\sigma_y$ obtained from $\mathcal{V}^\sigma_y$ by adjoining a unit element. In that case, $(1 - x)^{-1} = 1 + z$ for some

$z \in \mathcal{V}^\sigma$, and $x^y := z$ is called the *quasi-inverse* of (x, y). An element $x \in \mathcal{V}^\sigma$ is called *properly quasi-invertible* if (x, y) is quasi-invertible for all $y \in \mathcal{V}^{-\sigma}$. The *(Jacobson) radical* of $\mathcal{V}$ is Rad $\mathcal{V} := (\text{Rad}\ \mathcal{V}^+, \text{Rad}\ \mathcal{V}^-)$, where Rad $\mathcal{V}^\sigma$ is the set of properly quasi-invertible elements of $\mathcal{V}^\sigma$. It is well-known that Rad $\mathcal{V}$ is an ideal of $\mathcal{V}$. We say that $\mathcal{V}$ is *semisimple* if Rad $\mathcal{V} = 0$, and $\mathcal{V}$ is *quasi-invertible* or *radical* if $\mathcal{V} = \text{Rad}\ \mathcal{V}$. Of course, finite-dimensional simple Jordan pairs are semisimple, and finite-dimensional semisimple Jordan pairs are a direct sum of simple Jordan pairs.

An element $x \in \mathcal{V}^\sigma$ is called *von Neumann regular* (or vNr, for short) if there exists $y \in \mathcal{V}^{-\sigma}$ such that $Q(x)y = x$. A Jordan pair is called *von Neumann regular* if $\mathcal{V}^+$ and $\mathcal{V}^-$ consist of vNr elements. A pair $e = (x, y) \in \mathcal{V}$ is called *idempotent* if $Q(x)y = x$ and $Q(y)x = y$. Recall from [L75, Lemma 5.2] that if $x \in \mathcal{V}^+$ is vNr and $Q(x)y = x$, then $(x, Q(y)x)$ is an idempotent; therefore, every vNr element can be completed to an idempotent. An element $x \in \mathcal{V}^\sigma$ is called *trivial* if $Q(x) = 0$. A Jordan pair $\mathcal{V}$ is called *nondegenerate* if it contains no nonzero trivial elements.

Given a Jordan pair $\mathcal{V}$, a subspace $\mathcal{I} \subseteq \mathcal{V}^\sigma$ is called an *inner ideal* if $Q^\sigma(\mathcal{I})(\mathcal{V}^{-\sigma}) \subseteq \mathcal{I}$. Given an element $x \in \mathcal{V}^\sigma$, the *principal inner ideal* generated by x is defined by $[x] := Q(x)\mathcal{V}^{-\sigma}$. The *inner ideal* generated by $x \in \mathcal{V}^\sigma$ is defined by $(x) := \mathbb{F}x + [x]$.

Theorem 1.7.1 ([L75, Th. 10.17])**.** *The following conditions on a Jordan pair $\mathcal{V}$ with dcc on principal inner ideals are equivalent:*

i) $\mathcal{V}$ is von Neumann regular;

ii) $\mathcal{V}$ is semisimple;

iii) $\mathcal{V}$ is nondegenerate.

For any $x \in \mathcal{V}^\sigma$ and $y \in \mathcal{V}^{-\sigma}$, the Bergmann operator is defined by

$$B(x, y) = \text{id}_{\mathcal{V}^\sigma} - D(x, y) + Q(x)Q(y).$$

In case $(x, y) \in \mathcal{V}$ is quasi-invertible, the map

$$\beta(x, y) := (B(x, y), B(y, x)^{-1})$$

is an automorphism, called the *inner automorphism* defined by (x, y). The *inner automorphism group*, Inn$(\mathcal{V})$, is the group generated by the inner automorphisms.

Recall ([L75, Th. 5.4]) that given an idempotent $e = (e^+, e^-)$ of $\mathcal{V}$, the linear operators

$$E_2^\sigma = Q(e^\sigma)Q(e^{-\sigma}), \quad E_1^\sigma = D(e^\sigma, e^{-\sigma}) - 2E_2^\sigma, \quad E_0^\sigma = B(e^\sigma, e^{-\sigma}), \quad (1.7.2)$$

are orthogonal idempotents of $\mathrm{End}(\mathcal{V}^\sigma)$ whose sum is the identity, and we have the so-called *Peirce decomposition*: $\mathcal{V}^\sigma = \mathcal{V}_2^\sigma \oplus \mathcal{V}_1^\sigma \oplus \mathcal{V}_0^\sigma$, where $\mathcal{V}_i^\sigma = \mathcal{V}_i^\sigma(e) := E_i^\sigma(\mathcal{V}^\sigma)$. Moreover, this decomposition satisfies

$$Q(\mathcal{V}_i^\sigma)\mathcal{V}_j^{-\sigma} \subseteq \mathcal{V}_{2i-j}^\sigma, \qquad \{\mathcal{V}_i^\sigma, \mathcal{V}_j^{-\sigma}, \mathcal{V}_k^\sigma\} \subseteq \mathcal{V}_{i-j+k}^\sigma, \tag{1.7.3}$$

for any $i, j, k \in \{0, 1, 2\}$ (note that we use the convention $\mathcal{V}_i^\sigma = 0$ for $i \notin \{0, 1, 2\}$). In particular, $\mathcal{V}_i = (\mathcal{V}_i^+, \mathcal{V}_i^-)$ is a subpair of $\mathcal{V}$ for $i = 0, 1, 2$.

We recall a few more definitions related to idempotents. Two nonzero idempotents e and f are called *orthogonal* if $f \in \mathcal{V}_0(e)$; this is actually a symmetric relation. An *orthogonal system of idempotents* is an ordered set of pairwise orthogonal idempotents; it is usually denoted by $(e_1, \ldots, e_r)$ in case it is finite, and there is an associated Peirce decomposition (but we will not use this more general version). An orthogonal system of idempotents is called *maximal* if it is not properly contained in a larger orthogonal system of idempotents. It is known that a finite sum of pairwise orthogonal idempotents is again an idempotent. A nonzero idempotent e is called *primitive* if it cannot be written as the sum of two nonzero orthogonal idempotents. We say that e is a *local idempotent* (respectively a *division idempotent*) if $\mathcal{V}_2(e)$ is a local pair (respectively a division pair). In general, division idempotents are local, and local idempotents are primitive. If $\mathcal{V}$ is semisimple, then the local idempotents are exactly the division idempotents. A *frame* is a maximal set among orthogonal systems of local idempotents. Two frames of a simple finite-dimensional Jordan pair have always the same number of idempotents; that number of idempotents is called the *rank* of $\mathcal{V}$ (see [L75, Def. 15.18]), and we have:

Theorem 1.7.2 ([L75, Th. 17.1])**.** *Let $\mathcal{V}$ be a simple finite-dimensional Jordan pair over an algebraically closed field $\mathbb{F}$. Let $(c_1, \ldots, c_r)$ and $(e_1, \ldots, e_r)$ be frames of $\mathcal{V}$. Then there exists an inner automorphism g of $\mathcal{V}$ such that $g(c_i) = e_i$ for $i = 1, \ldots, r$.*

Let $\mathcal{V}$ be a semisimple Jordan pair and $x \in \mathcal{V}^\sigma$. The *rank* of x, $\mathrm{rk}(x)$, is defined as the supremum of the lenghts of all finite chains $[x_0] \subseteq [x_1] \subseteq \cdots \subseteq [x_n]$ of principal inner ideals $[x_i] = Q(x_i)\mathcal{V}^{-\sigma}$ where each x_i belongs to the inner ideal $(x) = \mathbb{F}x + [x]$ generated by x, and the length of the chain is the number of strict inclusions (for more details, see [L91a]). Hence, given a chain of length $n = \mathrm{rk}(x)$, we have $x_0 = 0$ and $[x_n] = [x]$.

Two elements $x, z \in \mathcal{V}^\sigma$ are called *orthogonal* ($x \perp z$) if they are part of orthogonal idempotents, i.e., $x = e^\sigma$ and $z = c^\sigma$ for some orthogonal idempotents e and c. For any $x, z \in \mathcal{V}^\sigma$, $\mathrm{rk}(x + z) \leq \mathrm{rk}(x) + \mathrm{rk}(z)$; and in case that x and z have finite rank, the equality holds if and only if $x \perp z$

([L91a, Th. 3]). Recall from [L91b] that the *capacity* of a Jordan pair $\mathcal{V}$, $\kappa(\mathcal{V})$, is the infimum of the cardinalities of all finite sets of orthogonal division idempotents whose Peirce-0-space is zero (if there are no such idempotents, the capacity is $+\infty$).

Recall that if $e = (x, y)$ is an idempotent, then $\mathrm{rk}(x) = \mathrm{rk}(y)$ ([L91a, Cor. 1 of Th. 3]), and this common value will be called the *rank* of e. In general, if $\mathrm{rk}(x) < \infty$, then $\mathrm{rk}(x) = \kappa(\mathcal{V}_2(e))$ ([L91a, Proposition 3]); hence, x has rank 1 if and only if $\mathcal{V}_2(e)$ is a division pair (i.e., the division idempotents are exactly the rank 1 idempotents), and since $\mathbb{F}$ is algebraically closed, this is equivalent to the condition $\mathrm{im}\, Q_x = \mathbb{F}x$ (see [L75, Lemma 15.5]).

An element $x \in \mathcal{V}^\sigma$ is called *diagonalizable* if there exist orthogonal division idempotents $d_1, \ldots, d_t$ such that $x = d_1^\sigma + \cdots + d_t^\sigma$, and it is called *defective* if $Q_y x = 0$ for all rank one elements $y \in \mathcal{V}^{-\sigma}$. The only element which is both diagonalizable and defective is 0. If $\mathcal{V}$ is simple, every element is either diagonalizable or defective ([L91a, Cor. 1]). The *defect* of $\mathcal{V}$ is $\mathrm{Def}(\mathcal{V}) := (\mathrm{Def}(\mathcal{V}^+), \mathrm{Def}(\mathcal{V}^-))$, where $\mathrm{Def}(\mathcal{V}^\sigma)$ denotes the set of defective elements of $\mathcal{V}^\sigma$. For the definition of the generic trace of $\mathcal{V}$, which is a bilinear map $\mathcal{V}^+ \times \mathcal{V}^- \to \mathbb{F}$ usually denoted by m_1 or t, see [L75, Def. 16.2].

Lemma 1.7.3 ([ALM05, 1.2.b]). *Let $\mathcal{V}$ be a semisimple finite-dimensional Jordan pair over an algebraically closed field $\mathbb{F}$. The defect is the kernel of the generic trace t in the sense that*

$$x \in \mathrm{Def}(\mathcal{V}^+) \Leftrightarrow t(x, \mathcal{V}^-) = 0,$$
$$y \in \mathrm{Def}(\mathcal{V}^-) \Leftrightarrow t(\mathcal{V}^+, y) = 0.$$

Recall that we only consider the case with $\mathrm{char}\, \mathbb{F} \neq 2$, and in this case the defect of a semisimple Jordan pair is always zero (see [L91a, Theorem 2]).

Proposition 1.7.4 ([ALM05, 1.9.(a)]). *Let $\mathcal{V}$ be a simple finite-dimensional Jordan pair of rank r over an algebraically closed field and such that $\mathrm{Def}(\mathcal{V}) = 0$, and let $\sigma \in \{\pm\}$. Then the automorphism group $\mathrm{Aut}\, \mathcal{V}$ and the inner automorphism group $\mathrm{Inn}\, \mathcal{V}$ have the same orbits on $\mathcal{V}^\sigma$, and these orbits are described as follows: two elements $x, y \in \mathcal{V}^\sigma$ belong to the same orbit if and only if $\mathrm{rk}(x) = \mathrm{rk}(y)$. Hence there are $r + 1$ orbits, corresponding to the possible values $0, \ldots, r$ of the rank function.*

Proposition 1.7.5 ([ALM05, 1.10.(a)]). *Let $\mathcal{V}$ be a simple finite-dimensional Jordan pair containing invertible elements over an algebraically closed field and satisfying $\mathrm{Def}(\mathcal{V}) = 0$. Then $\mathrm{Aut}\, \mathcal{V}$ acts transitively on $(\mathcal{V}^\sigma)^\times$.*

Remark 1.7.6. Given a finite-dimensional semisimple Jordan pair $\mathcal{V}$, each idempotent e of rank r decomposes as a sum of r orthogonal idempotents of rank 1 (see [L91a, Cor. 2 of Th. 1]). By (1.7.3), we also have $\mathcal{V}_2^\sigma(e) = \mathrm{im}\ Q_{e^\sigma}$, so the rank of e in $\mathcal{V}$ coincides with the rank of e in $\mathcal{V}_2(e)$.

Furthermore, if $\mathcal{V}$ is simple, all sets consisting of n orthogonal idempotents $e_1, \dots, e_n$ of fixed ranks $r_1, \dots, r_n$, respectively, are in the same orbit under the automorphism group. Indeed, first note that the Peirce subspaces $\mathcal{V}_2(e_i)$ are semisimple Jordan pairs (because the vNr property is inherited by these subpairs and by Theorem 1.7.1); hence the idempotent e_i decomposes as sum of r_i orthogonal idempotents $e_{i,1}, \dots, e_{i,r_i}$ of rank 1 in the corresponding Peirce subspace $\mathcal{V}_2(e_i)$, and it suffices to apply Theorem 1.7.2. In particular, idempotents of rank r are in the same orbit.

1.8 Structurable algebras

Let $(\mathcal{A}, ^-)$ be an algebra with involution over a field $\mathbb{F}$, i.e., $a \mapsto \bar{a}$ is an $\mathbb{F}$-linear involutive antiautomorphism of $\mathcal{A}$. We will use the notation

$$\mathcal{H}(\mathcal{A}, ^-) = \{a \in \mathcal{A} \mid \bar{a} = a\} \text{ and } \mathcal{K}(\mathcal{A}, ^-) = \{a \in \mathcal{A} \mid \bar{a} = -a\}.$$

Then $\mathcal{A} = \mathcal{H}(\mathcal{A}, ^-) \oplus \mathcal{K}(\mathcal{A}, ^-)$. The dimension of the subspace $\mathcal{K}(\mathcal{A}, ^-)$ will be referred to as the *skew-dimension* of $(\mathcal{A}, ^-)$.

Definition 1.8.1. Suppose char $\mathbb{F} \neq 2, 3$. A unital $\mathbb{F}$-algebra with involution $(\mathcal{A}, ^-)$ is said to be *structurable* if

$$[V_{x,y}, V_{z,w}] = V_{V_{x,y}z,w} - V_{z,V_{y,x}w} \quad \text{for all } x, y, z \in \mathcal{A}, \tag{1.8.1}$$

where $V_{x,y}(z) = \{x, y, z\} := (x\bar{y})z + (z\bar{y})x - (z\bar{x})y$.

In the case char $\mathbb{F} \neq 2, 3$, it is shown in [All78] that identity (1.8.1) implies that $(\mathcal{A}, ^-)$ is *skew-alternative*, i.e.,

$$(z - \bar{z}, x, y) = -(x, z - \bar{z}, y) = (x, y, z - \bar{z}) \quad \text{for all } x, y, z \in \mathcal{A},$$

where $(a, b, c) := (ab)c - a(bc)$. In the case char $\mathbb{F} = 2$ or 3, skew-alternativity is taken as an additional axiom.

Denote by $Z(\mathcal{A})$ the associative center of $\mathcal{A}$ (i.e., the set of elements $z \in \mathcal{A}$ satisfying $xz = zx$ and $(z, x, y) = (x, z, y) = (x, y, z) = 0$ for all $x, y \in \mathcal{A}$). The *center* of $(\mathcal{A}, ^-)$ is defined by $Z(\mathcal{A}, ^-) = Z(\mathcal{A}) \cap \mathcal{H}(\mathcal{A}, ^-)$. A structurable algebra $\mathcal{A}$ is said to be *central* if $Z(\mathcal{A}, ^-) = \mathbb{F}1$.

Theorem 1.8.2 (Allison, Smirnov)**.** *If* char $\mathbb{F} \neq 2, 3, 5$, *then any central simple structurable* $\mathbb{F}$*-algebra belongs to one of the of the following six (non-disjoint) classes:*

(1) central simple associative algebras with involution,

(2) central simple Jordan algebras (with identity involution),

(3) structurable algebras constructed from a non-degenerate Hermitian form over a central simple associative algebra with involution,

(4) forms of the tensor product of two Hurwitz algebras,

(5) simple structurable algebras of skew-dimension 1 (forms of structurable matrix algebras),

(6) an exceptional 35-dimensional case (Kantor-Smirnov algebra), which can be constructed from an octonion algebra. □

The classification was given by Allison in the case of characteristic 0 (see [All78]), but case (6) was overlooked. Later, Smirnov completed the classification and gave the generalization for char $\mathbb{F} \neq 2, 3, 5$ (see [Smi92]).

1.9 Structurable matrix algebras

Assume char $\mathbb{F} \neq 2, 3$. Let J and J' be vector spaces over $\mathbb{F}$ and consider a triple (T, N, N') where N and N' are symmetric trilinear forms on J and J', respectively, and $T\colon J \times J' \to \mathbb{F}$ is a nondegenerate bilinear form. For any $x, y \in J$, $x', y' \in J'$, define $x \times y \in J'$ and $x' \times y' \in J$ by

$$T(z, x \times y) = N(x, y, z) \quad \text{and} \quad T(x' \times y', z') = N'(x', y', z')$$

for all $z \in J$, $z' \in J'$. For any $x \in J$ and $x' \in J'$, define $N(x) = \frac{1}{6}N(x, x, x)$, $N'(x') = \frac{1}{6}N'(x', x', x')$, $x^\# = \frac{1}{2}x \times x$ and $x'^\# = \frac{1}{2}x' \times x'$. If the triple (T, N, N') satisfies the identities

$$(x^\#)^\# = N(x)x \quad \text{and} \quad (x'^\#)^\# = N'(x')x'$$

for all $x \in J$, $x' \in J'$, then the algebra

$$\mathcal{A} = \left\{ \begin{pmatrix} \alpha & x \\ x' & \beta \end{pmatrix} \mid \alpha, \beta \in \mathbb{F}, x \in J, x' \in J' \right\},$$

with multiplication

$$\begin{pmatrix} \alpha & x \\ x' & \beta \end{pmatrix} \begin{pmatrix} \gamma & y \\ y' & \delta \end{pmatrix} = \begin{pmatrix} \alpha\gamma + T(x, y') & \alpha y + \delta x + x' \times y' \\ \gamma x' + \beta y' + x \times y & T(y, x') + \beta\delta \end{pmatrix}, \tag{1.9.1}$$

and involution

$$\begin{pmatrix} \alpha & x \\ x' & \beta \end{pmatrix} \overset{-}{\mapsto} \begin{pmatrix} \beta & x \\ x' & \alpha \end{pmatrix}, \tag{1.9.2}$$

is a central simple structurable algebra of skew-dimension 1, where the space of skew elements is spanned by $s_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. These are called *structurable matrix algebras* in [AF84], where it is shown (see Proposition 4.5) that, conversely, if $(\mathcal{A}, ^-)$ is a simple structurable algebra with $\mathcal{K}(\mathcal{A}, ^-) = \mathbb{F}s_0 \neq 0$, then $s_0^2 = \mu 1$ with $\mu \in \mathbb{F}^\times$, and $(\mathcal{A}, ^-)$ is isomorphic to a structurable matrix algebra if and only if μ is a square in $\mathbb{F}$.

The triples (T, N, N'), as above, that satisfy $N \neq 0$ (equivalently, $N' \neq 0$) are called *admissible triples* in [All78], where it is noted that the corresponding structurable algebras possess a nondegenerate symmetric bilinear form

$$\langle a, b\rangle = \operatorname{tr}(a\bar{b}), \quad \text{where} \quad \operatorname{tr}\begin{pmatrix} \alpha & x \\ x' & \beta \end{pmatrix} := \alpha + \beta, \tag{1.9.3}$$

which is *invariant* in the sense that $\langle \bar{a}, \bar{b}\rangle = \langle a, b\rangle$ and $\langle ca, b\rangle = \langle a, \bar{c}b\rangle$ for all a, b, c. The main source of admissible triples are Jordan algebras: if J is a separable Jordan algebra of degree 3 with generic norm N and generic trace T, then $(\zeta T, \zeta N, \zeta^2 N)$ is an admissible triple (with $J' = J$) for any nonzero $\zeta \in \mathbb{F}$. Note that the map $x \mapsto \lambda x$ and $x' \mapsto \lambda^2 x'$ is an isomorphism from $(\lambda^3 T, \lambda^3 N, \lambda^6 N)$ to (T, N, N), so over algebraically closed fields, we can get rid of ζ.

1.10 Cayley–Dickson doubling process for algebras with involution

Let $(\mathcal{B}, ^-)$ be a unital $\mathbb{F}$-algebra with involution, and let $\phi\colon \mathcal{B} \times \mathcal{B} \to \mathbb{F}$ be a symmetric bilinear form such that $\phi(1, 1) \neq 0$ and $\phi(b, 1) = \phi(\bar{b}, 1)$ for all $b \in \mathcal{B}$. Denote $\phi(b) = \phi(b, 1)$ and define $\theta\colon \mathcal{B} \to \mathcal{B}$ by

$$b^\theta = -b + \frac{2\phi(b)}{\phi(1)}1. \tag{1.10.1}$$

Then θ is a linear map that commutes with the involution and satisfies $\theta^2 = \mathrm{id}$ and $\phi(b_1^\theta, b_2^\theta) = \phi(b_1, b_2)$ for all $b_1, b_2 \in \mathcal{B}$. Given $0 \neq \mu \in \mathbb{F}$, define a new algebra with involution

$$\mathfrak{CD}(\mathcal{B}, \mu) := \mathcal{B} \oplus \mathcal{B}$$

where multiplication is given by

$$(b_1, b_2)(c_1, c_2) = (b_1c_1 + \mu(b_2c_2^\theta)^\theta, b_1^\theta c_2 + (b_2^\theta c_1^\theta)^\theta) \tag{1.10.2}$$

and involution is given by

$$\overline{(b_1, b_2)} = (\bar{b}_1, -(\bar{b}_2)^\theta). \tag{1.10.3}$$

Note that $b \in \mathcal{B}$ can be identified with $(b, 0)$, that $(0, b) = vb$ for $v := (0, 1)$, and $v^2 = \mu 1$. Thus $(b_1, b_2) = b_1 + vb_2$ and $\mathfrak{CD}(\mathcal{B}, \mu) = \mathcal{B} \oplus v\mathcal{B}$. Moreover, the symmetric bilinear form ϕ can be extended to $\mathcal{B} \oplus v\mathcal{B}$ by setting

$$\phi(b_1 + vb_2, c_1 + vc_2) = \phi(b_1, c_1) - \mu\phi(b_2, c_2);$$

the extended ϕ satisfies $\phi(1, 1) \neq 0$ and $\phi(a, 1) = \phi(\bar{a}, 1)$ for all $a \in \mathfrak{CD}(\mathcal{B}, \mu)$.

This construction was introduced in [AF84] and called the *(generalized) Cayley–Dickson process* because it reduces to the classical doubling process for a Hurwitz algebra $\mathcal{B}$ if ϕ is the polar form of the norm and hence $b^\theta = \bar{b}$ for all $b \in \mathcal{B}$.

It is shown in [AF84] assuming char $\mathbb{F} \neq 2, 3$ (see [AF84, Theorem 6.6], where a slightly more general situation is considered) that if $\mathcal{B}$ is a separable Jordan algebra of degree 4, the involution is trivial and ϕ is the generic trace form, then $\mathfrak{CD}(\mathcal{B}, \mu)$ is a simple structurable algebra of skew-dimension 1. In fact, if μ is a square in $\mathbb{F}$ then such $\mathfrak{CD}(\mathcal{B}, \mu)$ is isomorphic to the structurable matrix algebra corresponding to a certain admissible triple defined on the space $\mathcal{B}_0 \subseteq \mathcal{B}$ of elements with generic trace 0 ([AF84, Proposition 6.5]).

So let $\mathcal{B}$ be a separable Jordan algebra of degree 4 and let $\mathcal{A} = \mathfrak{CD}(\mathcal{B}, \mu)$ as above. We state some basic properties of $\mathcal{A}$ for future use: $\mathcal{B}$ is a subalgebra of $\mathcal{A}$, there is an element $v \in \mathcal{B}$ such that $\mathcal{A} = \mathcal{B} \oplus v\mathcal{B}$, and the involution of $\mathcal{A}$ is given by $\overline{a + vb} = a - vb^\theta$ where $\theta\colon \mathcal{B} \to \mathcal{B}$ is a linear map defined by $1^\theta = 1$ and $b^\theta = -b$ for all $b \in \mathcal{B}_0$. The operators L_v and R_v of left and right multiplication by v, respectively, satisfy the relations $L_v^2 = R_v^2 = \mu\mathrm{id}$ and $L_v R_v = R_v L_v = \mu\theta$ where we extended θ to an operator on $\mathcal{A}$ by the rule $(a + vb)^\theta = a^\theta + vb^\theta$. The multiplication of $\mathcal{A}$ is determined by the formulas

$$a(vb) = v(a^\theta b),\ (va)b = v(a^\theta b^\theta)^\theta,\ (va)(vb) = \mu(ab^\theta)^\theta, \tag{1.10.4}$$

for all $a, b \in \mathcal{A}$. (This is equivalent to (1.10.2) if $a, b \in \mathcal{B}$, but a straightworward computation shows that the formulas continue to hold if we allow a and b to range over $\mathcal{A}$.)

Since $\mathcal{K}(\mathcal{A}, ^-) = \mathbb{F}v$ and $v^2 = \mu 1$, all automorphisms of $(\mathcal{A}, ^-)$ send v to $\pm v$ and all derivations of $(\mathcal{A}, ^-)$ annihilate v. Every automorphism (or derivation) φ of $\mathcal{B}$ extends to $\mathcal{A}$ in the natural way: $a + vb \mapsto \varphi(a) + v\varphi(b)$. We will denote this extended map by the same symbol. Similarly, any G-grading $\mathcal{B} = \bigoplus_{g\in G} \mathcal{B}_g$ gives rise to a G-grading on $\mathcal{A}$, namely, $\mathcal{A} = \bigoplus_{g\in G}(\mathcal{B}_g \oplus v\mathcal{B}_g)$.

1.11 Kantor systems and Kantor construction

We will now recall the TKK construction of a Jordan pair (the Kantor construction of a Jordan pair) and the notation that we will use for it. Let $\mathcal{V}$ be a Jordan pair. Recall that the inner derivations are defined by

$$\nu(x,y) := (D(x,y), -D(y,x)) \in \mathfrak{gl}(\mathcal{V}^+) \oplus \mathfrak{gl}(\mathcal{V}^-), \tag{1.11.1}$$

where $(x,y) \in \mathcal{V}$. Consider the 3-graded Lie algebra

$$L = \mathrm{TKK}(\mathcal{V}) = L^{-1} \oplus L^0 \oplus L^1 \tag{1.11.2}$$

defined by the TKK construction, due to Tits, Kantor and Koecher (see [CS11] and references therein). That is,

$$L^{-1} = \mathcal{V}^-, \quad L^1 = \mathcal{V}^+, \quad L^0 = \mathrm{span}\{\nu(x,y) \mid (x,y) \in \mathcal{V}\},$$

and the multiplication is given by

$$[a+X+b, c+Y+d] := (Xc - Ya) + ([X,Y] + \nu(a,d) - \nu(c,b)) + (Xd - Yb) \tag{1.11.3}$$

for each $X, Y \in L^0$, $a, c \in L^1$, $b, d \in L^{-1}$. This 3-grading will be called the TKK-*grading*. The TKK construction of a Jordan algebra J is defined as the TKK construction of its associated Jordan pair (J, J).

Definition 1.11.1. A *Kantor pair* (or *generalized Jordan pair of second order* [F94, AF99]) is a pair of vector spaces $\mathcal{V} = (\mathcal{V}^+, \mathcal{V}^-)$ and a pair of trilinear products $\mathcal{V}^\sigma \times \mathcal{V}^{-\sigma} \times \mathcal{V}^\sigma \to \mathcal{V}^\sigma$, denoted by $\{x,y,z\}^\sigma$, satisfying the identities:

$$[V^\sigma_{x,y}, V^\sigma_{z,w}] = V^\sigma_{V^\sigma_{x,y}z,w} - V^\sigma_{z,V^{-\sigma}_{y,x}w}, \tag{1.11.4}$$

$$K^\sigma_{K^\sigma_{x,y}z,w} = K^\sigma_{x,y} V^{-\sigma}_{z,w} + V^\sigma_{w,z} K^\sigma_{x,y}, \tag{1.11.5}$$

where $V^\sigma_{x,y} z = U^\sigma_{x,z}(y) := \{x,y,z\}^\sigma$, $U^\sigma_x := U^\sigma_{x,x}$ and $K^\sigma_{x,y} z = K^\sigma(x,y)z := \{x,z,y\}^\sigma - \{y,z,x\}^\sigma$. The map $V^\sigma_{x,y}$ is also denoted by $D^\sigma_{x,y}$ or $D^\sigma(x,y)$ (because $(V^+_{x,y}, -V^-_{y,x})$ is a derivation of the Kantor pair).

The superindex σ will always take the values $+$ and $-$, and will be omitted when there is no ambiguity.

Definition 1.11.2. A *Kantor triple system* (or *generalized Jordan triple system of second order* [K72, K73]) is a vector space $\mathcal{T}$ with a trilinear product $\mathcal{T} \times \mathcal{T} \times \mathcal{T} \to \mathcal{T}$, denoted by $\{x, y, z\}$, which satisfies:

$$[V_{x,y}, V_{z,w}] = V_{V_{x,y}z,w} - V_{z,V_{y,x}w}, \tag{1.11.6}$$

$$K_{K_{x,y}z,w} = K_{x,y}V_{z,w} + V_{w,z}K_{x,y}, \tag{1.11.7}$$

where $V_{x,y}z = U_{x,z}(y) := \{x, y, z\}$, $U_x := U_{x,x}$ and $K_{x,y}z := \{x, z, y\} - \{y, z, x\}$.

Given a structurable algebra $\mathcal{A}$, we can define its associated Kantor triple system as the vector space $\mathcal{A}$ endowed with the triple product $\{x, y, z\}$ of $\mathcal{A}$. Similarly, with two copies of a Kantor triple system $\mathcal{T}$ and two copies of its triple product we can define the associated Kantor pair $\mathcal{V} = (\mathcal{T}, \mathcal{T})$. In particular, a structurable algebra $\mathcal{A}$ with its triple product defines a Kantor pair $(\mathcal{A}, \mathcal{A})$. Note that Jordan pairs (respectively, Jordan triple systems) are a particular case of Kantor pairs (respectively, Kantor triple systems), those where $K_{x,y} = 0$ for all x, y.

We will now recall from [AF99, §3–4] the 5-graded Lie algebra obtained with the Kantor construction from a Kantor pair. The Kantor construction is a generalization of the Tits-Kantor-Koecher (TKK) construction from Jordan pairs. Consider the vector space

$$\mathfrak{K}(\mathcal{V}) := \mathfrak{K}(\mathcal{V})^{-2} \oplus \mathfrak{K}(\mathcal{V})^{-1} \oplus \mathfrak{K}(\mathcal{V})^{0} \oplus \mathfrak{K}(\mathcal{V})^{1} \oplus \mathfrak{K}(\mathcal{V})^{2}, \tag{1.11.8}$$

where

$$\mathfrak{K}(\mathcal{V})^{-2} = \begin{pmatrix} 0 & K(\mathcal{V}^-, \mathcal{V}^-) \\ 0 & 0 \end{pmatrix}, \quad \mathfrak{K}(\mathcal{V})^{-1} = \begin{pmatrix} \mathcal{V}^- \\ 0 \end{pmatrix},$$

$$\mathfrak{K}(\mathcal{V})^{0} = \left\{ \begin{pmatrix} D(x^-, x^+) & 0 \\ 0 & -D(x^+, x^-) \end{pmatrix} \mid x^\sigma \in \mathcal{V}^\sigma \right\},$$

$$\mathfrak{K}(\mathcal{V})^{1} = \begin{pmatrix} 0 \\ \mathcal{V}^+ \end{pmatrix}, \quad \mathfrak{K}(\mathcal{V})^{2} = \begin{pmatrix} 0 & 0 \\ K(\mathcal{V}^+, \mathcal{V}^+) & 0 \end{pmatrix}.$$

The vector space

$$\begin{aligned} \mathfrak{S}(\mathcal{V}) :=& \mathfrak{K}(\mathcal{V})^{-2} \oplus \mathfrak{K}(\mathcal{V})^{0} \oplus \mathfrak{K}(\mathcal{V})^{2} \\ =& \operatorname{span} \left\{ \begin{pmatrix} D(x^-, x^+) & K(y^-, z^-) \\ K(y^+, z^+) & -D(x^+, x^-) \end{pmatrix} \mid x^\sigma, y^\sigma, z^\sigma \in \mathcal{V}^\sigma \right\} \end{aligned}$$

is a subalgebra of the Lie algebra

$$\operatorname{End} \begin{pmatrix} \mathcal{V}^- \\ \mathcal{V}^+ \end{pmatrix} = \begin{pmatrix} \operatorname{End}(\mathcal{V}^-) & \operatorname{Hom}(\mathcal{V}^+, \mathcal{V}^-) \\ \operatorname{Hom}(\mathcal{V}^-, \mathcal{V}^+) & \operatorname{End}(\mathcal{V}^+) \end{pmatrix},$$

with the commutator product. Define an anti-commutative product on $\mathfrak{K}(\mathcal{V})$ by means of

$$[A,B] = AB - BA, \qquad [A, \begin{pmatrix} x^- \\ x^+ \end{pmatrix}] = A \begin{pmatrix} x^- \\ x^+ \end{pmatrix},$$
$$[\begin{pmatrix} x^- \\ x^+ \end{pmatrix}, \begin{pmatrix} y^- \\ y^+ \end{pmatrix}] = \begin{pmatrix} D(x^-,y^+) - D(y^-,x^+) & K(x^-,y^-) \\ K(x^+,y^+) & -D(y^+,x^-) + D(x^+,y^-) \end{pmatrix}.$$

Then, $\mathfrak{K}(\mathcal{V})$ becomes a Lie algebra, called the *Kantor Lie algebra* of $\mathcal{V}$. The 5-grading is a $\mathbb{Z}$-grading which is called the *standard grading* of $\mathfrak{K}(\mathcal{V})$. We will also refer to it as the *main grading* of $\mathfrak{K}(\mathcal{V})$. The subspaces $\mathfrak{K}(\mathcal{V})^1$ and $\mathfrak{K}(\mathcal{V})^{-1}$ are usually identified with $\mathcal{V}^+$ and $\mathcal{V}^-$, respectively. The Kantor construction of a structurable algebra or Kantor triple system is defined as the Kantor construction of the associated Kantor pair.

Let $\mathcal{A}$ be a structurable algebra and let $\mathcal{V} = (\mathcal{V}^+, \mathcal{V}^-) = (\mathcal{A}, \mathcal{A})$ be the associated Kantor pair. Recall that $\nu(x^-, x^+) := (D_{x^-,x^+}, -D_{x^+,x^-})$ is a derivation called *inner derivation* associated to $(x^-, x^+) \in \mathcal{V}^- \times \mathcal{V}^+$. The *inner structure algebra* of $\mathcal{A}$ is the Lie algebra $\mathfrak{innstr}(\mathcal{A}) = \mathrm{span}\{\nu(x,y) \mid x, y \in \mathcal{A}\}$. Let L_x denote the left multiplication by $x \in \mathcal{A}$ and write $\mathcal{S} = \mathcal{S}(\mathcal{A})$. Then, the map $\mathcal{S} \to L_{\mathcal{S}}$, $s \mapsto L_s$, is a linear monomorphism, so we can identify $\mathcal{S}$ with $L_{\mathcal{S}}$. Also, note that the map $\mathcal{A} \times \mathcal{A} \to \mathcal{S}$ given by $\psi(x,y) := x\bar{y} - y\bar{x}$ is an epimorphism (because $\psi(s,1) = 2s$ for $s \in \mathcal{S}$). By [AF84, (1.3)], we have the identity $L_{\psi(x,y)} = U_{x,y} - U_{y,x} = K(x,y)$ for all $x, y \in \mathcal{A}$. As a consequence of this, in the Kantor construction of $\mathcal{V}$ we can identify the subspaces $\mathfrak{K}(\mathcal{V})^2$ and $\mathfrak{K}(\mathcal{V})^{-2}$ with $L_{\mathcal{S}}$, or also with $\mathcal{S}$. Hence, the main grading of $\mathfrak{K}(\mathcal{A})$ can be written as follows:

$$\mathfrak{K}(\mathcal{A}) = \mathcal{S}^- \oplus \mathcal{A}^- \oplus \mathfrak{innstr}(\mathcal{A}) \oplus \mathcal{A}^+ \oplus \mathcal{S}^+. \tag{1.11.9}$$

This construction can be used to induce gradings on the Lie algebra constructed from a structurable algebra or Kantor pair, as we will see in subsequent sections.

Chapter 2

Gradings on Jordan systems

In this chapter we extend the natural definitions of gradings on algebras to the setting of Jordan pairs and triple systems. Some general results of gradings on (semi)simple Jordan pairs and triple systems are proven; some of the main results are Theorem 2.1.21, that relates gradings on a Jordan pair and on the Lie algebra obtained by the Tits-Kantor-Koecher construction, and Theorem 2.1.24, that proves that the nonzero homogeneous components of the fine gradings on finite-dimensional semisimple Jordan pairs are always one-dimensional.

Later on, we recall the definitions of Jordan pairs and triple systems of types bi-Cayley and Albert.

We also give an explicit description of generators of the automorphism groups of bi-Cayley systems (see Theorem 2.2.27, which is one of our main results in this chapter), and also of the orbits for the actions of these groups on the corresponding Jordan systems; note that these automorphism groups and the orbits of their actions are well-known, but our contribution here is the explicit description of these groups and their generators, and some characterization of the orbits.

2.1 Generalities about gradings

In this section, the definitions in Section 1.1 will be extended to Jordan pairs and triple systems.

2.1.1 Gradings on Jordan pairs and triple systems

Definition 2.1.1. Let $\mathcal{V} = (\mathcal{V}^+, \mathcal{V}^-)$ be a Jordan pair and let S be a set. Given two decompositions of vector spaces

$$\Gamma^\sigma : \mathcal{V}^\sigma = \bigoplus_{s\in S} \mathcal{V}^\sigma_s,$$

we will say that $\Gamma = (\Gamma^+, \Gamma^-)$ is an *S-grading* on $\mathcal{V}$ if for any $s_1, s_2, s_3 \in S$ there is $s \in S$ such that $\{\mathcal{V}^\sigma_{s_1}, \mathcal{V}^{-\sigma}_{s_2}, \mathcal{V}^\sigma_{s_3}\} \subseteq \mathcal{V}^\sigma_s$ for all $\sigma \in \{+,-\}$. In this case, we also say that Γ is a *set grading* on $\mathcal{V}$. The set

$$\operatorname{Supp}\Gamma = \operatorname{Supp}\Gamma^+ \cup \operatorname{Supp}\Gamma^-$$

is called the *support* of the grading, where $\operatorname{Supp}\Gamma^\sigma := \{s \in S \mid \mathcal{V}^\sigma_s \neq 0\}$. The vector space $\mathcal{V}^+_s \oplus \mathcal{V}^-_s$ is called the *homogeneous component of degree* s. If $0 \neq x \in \mathcal{V}^\sigma_s$, we say that x is *homogeneous of degree* s, and we write $\deg(x) = s$.

Definition 2.1.2. Let $\Gamma^\sigma : \mathcal{V}^\sigma = \bigoplus_{s\in S} \mathcal{V}^\sigma_s$ and $\widetilde{\Gamma}^\sigma : \mathcal{V}^\sigma = \bigoplus_{t\in T} \mathcal{V}^\sigma_t$ be two set gradings on a Jordan pair $\mathcal{V}$. We say that Γ is a *refinement* of $\widetilde{\Gamma}$, or that $\widetilde{\Gamma}$ is a *coarsening* of Γ, if for any $s \in S$ there is $t \in T$ such that $\mathcal{V}^\sigma_s \subseteq \mathcal{V}^\sigma_t$ for $\sigma \in \{+,-\}$. The refinement is said to be *proper* if some containment $\mathcal{V}^\sigma_s \subseteq \mathcal{V}^\sigma_t$ is strict. A set grading with no proper refinement is called a *fine* grading.

Let G be an abelian group. Given two decompositions

$$\Gamma^\sigma : \mathcal{V}^\sigma = \bigoplus_{g\in G} \mathcal{V}^\sigma_g,$$

we will say that $\Gamma = (\Gamma^+, \Gamma^-)$ is a *G-grading* on $\mathcal{V}$ if

$$\{\mathcal{V}^\sigma_g, \mathcal{V}^{-\sigma}_h, \mathcal{V}^\sigma_k\} \subseteq \mathcal{V}^\sigma_{g+h+k}$$

for any $g, h, k \in G$ and $\sigma \in \{+,-\}$. A set grading by a set S on $\mathcal{V}$ will be called *realizable as a group grading*, or a *group grading*, if S is contained in some abelian group G such that the subspaces $\mathcal{V}^\sigma_g := \mathcal{V}^\sigma_s$ for $g = s \in S$ and $\mathcal{V}^\sigma_g := 0$ for $g \notin S$ define a G-grading. By a *grading* we will mean a group grading. In particular, a grading is called *fine* if it has no proper refinements in the class of group gradings. We will not consider gradings by nonabelian groups.

Definition 2.1.3. Let Γ be a set grading on $\mathcal{V}$. The *universal group* of Γ, which is denoted by $\mathcal{U}(\Gamma)$, is defined as the abelian group generated by $\operatorname{Supp}\Gamma$ with the relations $s_1 + s_2 + s_3 = s$ when $0 \neq \{\mathcal{V}^\sigma_{s_1}, \mathcal{V}^{-\sigma}_{s_2}, \mathcal{V}^\sigma_{s_3}\} \subseteq \mathcal{V}^\sigma_s$ for some $\sigma \in \{+,-\}$. Note that this defines a group grading Γ' by $\mathcal{U}(\Gamma)$, which is a coarsening of Γ, and it is clear that Γ and Γ' have the same homogeneous components if and only if Γ is realizable as a group grading.

Suppose that a group grading Γ on $\mathcal{V}$ admits a realization as a G_0-grading for some abelian group G_0. Then G_0 is isomorphic to the universal group of Γ if and only if for any other realization of Γ as a G-grading there is a unique homomorphism $G_0 \to G$ that restricts to the identity on $\operatorname{Supp}\Gamma$.

Definition 2.1.4. Given a G-grading Γ on $\mathcal{V}$ and a group homomorphism $\alpha\colon G \to H$, we define the *induced* H-grading ${}^\alpha\Gamma$ determined by setting $\mathcal{V}^\sigma_h := \bigoplus_{g\in\alpha^{-1}(h)} \mathcal{V}^\sigma_g$. Then ${}^\alpha\Gamma$ is a coarsening of Γ. In case Γ is given by its universal group, i.e., $G = \mathcal{U}(\Gamma)$, then any coarsening of Γ (in the class of group gradings) is of the form ${}^\alpha\Gamma$ for some homomorphism $\alpha\colon \mathcal{U}(\Gamma) \to H$.

Example 2.1.5. Consider the Jordan pair $\mathcal{V} = (\mathbb{F},\mathbb{F})$ associated to the Jordan algebra $J = \mathbb{F}$, i.e., with products $U_x(y) = x^2y$ for $x, y \in \mathbb{F}$. Then, the trivial grading on $\mathcal{V}$ has universal group $\mathbb{Z}_2$ and support $\{\bar{1}\}$. On the other hand, for a nonzero Jordan pair with zero product, the trivial grading has universal group $\mathbb{Z}$ and support $\{1\}$.

Definition 2.1.6. Let $\Gamma^\sigma_1 : \mathcal{V}^\sigma = \bigoplus_{s\in S} \mathcal{V}^\sigma_s$ and $\Gamma^\sigma_2 : \mathcal{W}^\sigma = \bigoplus_{t\in T} \mathcal{W}^\sigma_t$ be graded Jordan pairs. An isomorphism of Jordan pairs $\varphi = (\varphi^+, \varphi^-)\colon \mathcal{V} \to \mathcal{W}$ is said to be an *equivalence* of graded Jordan pairs if, for each $s \in S$, there is (a unique) $t \in T$ such that $\varphi^\sigma(\mathcal{V}^\sigma_s) = \mathcal{W}^\sigma_t$ for all $\sigma \in \{+,-\}$. In that case, Γ_1 and Γ_2 are said to be *equivalent*.

Definition 2.1.7. Given a G-grading Γ on $\mathcal{V}$, the *automorphism group of* Γ, $\operatorname{Aut}(\Gamma)$, is the group of self-equivalences of Γ. The *stabilizer of* Γ, $\operatorname{Stab}(\Gamma)$, is the group of G-automorphisms of Γ, i.e., the group of automorphisms of $\mathcal{V}$ that fix the homogeneous components. The *diagonal group of* Γ, $\operatorname{Diag}(\Gamma)$, is the subgroup of $\operatorname{Stab}(\Gamma)$ consisting of the automorphisms that act by multiplication by a nonzero scalar on each homogeneous component. The *Weyl group of* Γ is the quotient group $\mathcal{W}(\Gamma) = \operatorname{Aut}(\Gamma)/\operatorname{Stab}(\Gamma)$, which can be regarded as a subgroup of $\operatorname{Sym}(\operatorname{Supp}\Gamma)$ and also of $\operatorname{Aut}(\mathcal{U}(\Gamma))$.

Proposition 2.1.8. *Let Γ be a fine grading on a Jordan pair $\mathcal{V}$ and let G be its universal group. Then, there is a group homomorphism $\pi\colon G \to \mathbb{Z}$ such that $\pi(g) = \sigma 1$ if $\mathcal{V}^\sigma_g \neq 0$ for some $\sigma \in \{+,-\}$. In particular,* $\operatorname{Supp}\Gamma^+$ *and* $\operatorname{Supp}\Gamma^-$ *are disjoint.*

Proof. Define a $G \times \mathbb{Z}$-grading Γ' on $\mathcal{V}$ by means of $\mathcal{V}^{\sigma}_{(g,\sigma 1)} := \mathcal{V}^{\sigma}_g$ and $\mathcal{V}^{\sigma}_{(g,-\sigma 1)} := 0$. Then, Γ' is a refinement of Γ. But Γ is fine, so Γ' and Γ have the same homogeneous components. Since $G = \mathcal{U}(\Gamma)$, there is a (unique) homomorphism $\phi\colon G \to G \times \mathbb{Z}$ such that $\mathcal{V}^{\sigma}_{\phi(g)} = \mathcal{V}^{\sigma}_g$ for all $g \in G$ and $\sigma \in \{+,-\}$. Therefore, ϕ has the form $\phi(g) = (g, \pi(g))$ for some homomorphism $\pi\colon G \to \mathbb{Z}$. By definition of Γ', π satisfies $\pi(g) = \sigma 1$ if $\mathcal{V}^{\sigma}_g \neq 0$. In particular, $\operatorname{Supp}\Gamma^+ = \pi^{-1}(1)$ and $\operatorname{Supp}\Gamma^- = \pi^{-1}(-1)$ are disjoint. □

Definition 2.1.9. Let $\mathcal{T}$ be a Jordan triple system and S a set. Consider a decomposition $\Gamma : \mathcal{T} = \bigoplus_{s\in S} \mathcal{T}_s$. We call Γ an *S-grading* if, for any $s_1, s_2, s_3 \in S$, there is $s \in S$ such that $\{\mathcal{T}_{s_1}, \mathcal{T}_{s_2}, \mathcal{T}_{s_3}\} \subseteq \mathcal{T}_s$.

Let G be an abelian group and consider a decomposition $\Gamma : \mathcal{T} = \bigoplus_{g\in G} \mathcal{T}_g$. We say that Γ is a *G-grading* if $\{\mathcal{T}_g, \mathcal{T}_h, \mathcal{T}_k\} \subseteq \mathcal{T}_{g+h+k}$ for any $g, h, k \in G$. A set grading is said to be *realizable as a group grading*, or a *group grading*, if S is contained in some abelian group G such that the subspaces $\mathcal{T}_g := \mathcal{T}_s$ for $g = s \in S$ and $\mathcal{T}_g := 0$ for $g \notin S$ define a G-grading.

The rest of definitions about gradings on Jordan triple systems are analogous to those given for graded Jordan pairs.

Definition 2.1.10. Given a graded algebra A, a bilinear form $b\colon A \times A \to \mathbb{F}$ will be called *homogeneous of degree* 0, or simply *homogeneous*, if we have $g + h = 0$ whenever $b(A_g, A_h) \neq 0$. (Analogous definition for a bilinear form on a graded Jordan triple system.) Similarly, given a graded Jordan pair $\mathcal{V}$, a bilinear form $b\colon \mathcal{V}^+ \times \mathcal{V}^- \to \mathbb{F}$ will be called *homogeneous* if we have $g + h = 0$ whenever $b(\mathcal{V}^+_g, \mathcal{V}^-_h) \neq 0$.

Let J be a Jordan algebra. Consider its associated Jordan pair $\mathcal{V} = (J, J)$ and Jordan triple system $\mathcal{T} = J$. Then, any G-grading Γ on J is a G-grading on $\mathcal{T}$. In the same way, any G-grading Γ on $\mathcal{T}$ (or on J) induces a G-grading on $\mathcal{V}$, given by (Γ, Γ). We say that a G-grading $\widetilde{\Gamma}$ on $\mathcal{V}$ is a G-grading on J (respectively on $\mathcal{T}$) when $\widetilde{\Gamma}$ equals (Γ, Γ) for some G-grading Γ on J (respectively on $\mathcal{T}$). If $\varphi = (\varphi^+, \varphi^-) \in \operatorname{Aut}(\mathcal{V})$, denote $\widehat{\varphi} := (\varphi^-, \varphi^+) \in \operatorname{Aut}(\mathcal{V})$. Notice that $\widehat{\varphi_1\varphi_2} = \widehat{\varphi}_1\widehat{\varphi}_2$ and $\widehat{1}_{\mathcal{V}} = 1_{\mathcal{V}}$, so $\widehat{\ } \in \operatorname{Aut}(\operatorname{Aut}(\mathcal{V}))$. Moreover, $\widehat{\widehat{\varphi}} = \varphi$ and $\operatorname{Aut}(\mathcal{T}) = \{\varphi \in \operatorname{Aut}(\mathcal{V}) \mid \widehat{\varphi} = \varphi\}$. We can consider, with natural identifications, that $\operatorname{Aut}(J) \leq \operatorname{Aut}(\mathcal{T}) \leq \operatorname{Aut}(\mathcal{V})$.

Let Γ_J be a G-grading on a Jordan algebra J and $\Gamma_{\mathcal{T}}$ the same G-grading on the Jordan triple system $\mathcal{T} = J$. Since $\operatorname{Aut}(J) \leq \operatorname{Aut}(\mathcal{T})$, we have $\operatorname{Aut}(\Gamma_J) \leq \operatorname{Aut}(\Gamma_{\mathcal{T}})$ and $\operatorname{Stab}(\Gamma_J) \leq \operatorname{Stab}(\Gamma_{\mathcal{T}})$. Thus,

$$\begin{aligned}\mathcal{W}(\Gamma_J) &= \operatorname{Aut}(\Gamma_J)/\operatorname{Stab}(\Gamma_J) = \operatorname{Aut}(\Gamma_J)/(\operatorname{Stab}(\Gamma_{\mathcal{T}}) \cap \operatorname{Aut}(\Gamma_J))\\ &\cong (\operatorname{Aut}(\Gamma_J)\cdot \operatorname{Stab}(\Gamma_{\mathcal{T}}))/\operatorname{Stab}(\Gamma_{\mathcal{T}}) \leq \operatorname{Aut}(\Gamma_{\mathcal{T}})/\operatorname{Stab}(\Gamma_{\mathcal{T}}) = \mathcal{W}(\Gamma_{\mathcal{T}}).\end{aligned}$$

In the same manner, if $\Gamma_{\mathcal{T}}$ is G-grading on a Jordan triple system $\mathcal{T}$ and $\Gamma_{\mathcal{V}} = (\Gamma_{\mathcal{T}}, \Gamma_{\mathcal{T}})$ is the induced G-grading on the associated Jordan pair $\mathcal{V}$, we have natural identifications: $\mathrm{Aut}(\Gamma_{\mathcal{T}}) \leq \mathrm{Aut}(\Gamma_{\mathcal{V}})$, $\mathrm{Stab}(\Gamma_{\mathcal{T}}) \leq \mathrm{Stab}(\Gamma_{\mathcal{V}})$ and $\mathcal{W}(\Gamma_{\mathcal{T}}) \leq \mathcal{W}(\Gamma_{\mathcal{V}})$.

Let Γ be a G-grading on a Jordan pair $\mathcal{V}$ with degree deg. Fix $g \in G$. For any homogeneous elements $x^+ \in \mathcal{V}^+$ and $y^- \in \mathcal{V}^-$, set

$$\deg_g(x^+) := \deg(x^+) + g, \quad \deg_g(y^-) := \deg(y^-) - g. \tag{2.1.1}$$

This defines a new G-grading, which will be denoted by $\Gamma^{[g]}$ and called the *g-shift of* Γ. Note that, although Γ and $\Gamma^{[g]}$ may fail to be equivalent (because the shift may collapse or split a homogeneous subspace of $\mathcal{V}^+$ with another of $\mathcal{V}^-$), the intersection of their homogeneous components with $\mathcal{V}^\sigma$ coincide for each σ. It is clear that $(\Gamma^{[g]})^{[h]} = \Gamma^{[g+h]}$. Similarly, if Γ is a G-grading on a Jordan triple system $\mathcal{T}$ and $g \in G$ has order 2, we can define the *g-shift* $\Gamma^{[g]}$ with the new degree

$$\deg_g(x) := \deg(x) + g. \tag{2.1.2}$$

The following result is a generalization of [N85, Theorem 3.7(a), Eq. (1)] to the case of affine group schemes and char $\mathbb{F} \neq 2$.

Theorem 2.1.11. *Let J be a finite-dimensional central simple Jordan $\mathbb{F}$-algebra with associated Jordan triple system $\mathcal{T}$. There is an isomorphism of affine group schemes $\mathbf{Aut}(\mathcal{T}) \simeq \mathbf{Aut}(J) \times \boldsymbol{\mu}_2$.*

Proof. Recall that the product of $\mathcal{T}$ is given by

$$\{x, y, z\} := x(yz) + z(xy) - (xz)y.$$

Denote by $\mathcal{T}^-$ the Lie triple system associated to J, that is, $\mathcal{T}^- = J$ with the product

$$[x, y, z] := \{x, y, z\} - \{y, x, z\}.$$

Then we have that $[x, y, z] = -2((xz)y - x(zy)) = -2(x, z, y)$. The center of $\mathcal{T}^-$ is defined by

$$Z(\mathcal{T}^-) := \{x \in J \mid [x, J, J] = 0\} = \{x \in J \mid (x, J, J) = 0\}.$$

From the identities $(x, y, z) = -(z, y, x)$ and $(x, y, z) + (y, z, x) + (z, x, y) = 0$ we obtain that $Z(\mathcal{T}^-) = \{x \in J \mid (x, J, J) = (J, x, J) = (J, J, x) = 0\} = Z(J) = \mathbb{F}1$. In consequence, for each associative commutative unital $\mathbb{F}$-algebra R we have $Z((\mathcal{T}^-)_R) = R1$. Note that for each $\varphi \in \mathrm{Aut}_R(\mathcal{T}_R)$

we also have $\varphi \in \mathrm{Aut}_R((\mathcal{T}^-)_R)$, and hence $\varphi(R1) = R1$. In particular, $\varphi(1) = r1$ for some $r \in R$. Since φ is bijective, there is some $s \in R$ such that $1 = \varphi(s1) = s\varphi(1) = sr1$, which shows that $r \in R^\times$. On the other hand, we have $r1 = \varphi(1) = \varphi(\{1,1,1\}) = \{\varphi(1), \varphi(1), \varphi(1)\} = r^3 1$ with $r \in R^\times$, which implies that $r^2 = 1$, that is $r \in \mu_2(R)$.

Recall that the automorphisms of a Jordan algebra J are exactly the automorphisms of the associated Jordan triple system $\mathcal{T}_J$ that fix the unit 1 of J. Indeed, if $f \in \mathrm{Aut}(\mathcal{T}_J)$ with $f(1) = 1$, then, since $\{x, 1, z\} = 2xz$ for all $x, z \in J$, we have $2f(xz) = f(2xz) = f(\{x, 1, z\}) = \{f(x), f(1), f(z)\} = \{f(x), 1, f(z)\} = 2f(x)f(z)$, hence $f(xz) = f(x)f(z)$ and $f \in \mathrm{Aut}(J)$.

Note that the map $\delta_r \colon x \mapsto rx$ is an order 2 automorphism of $\mathcal{T}_R$ and $\delta_r\varphi(1) = 1$, so that $\delta_r\varphi \in \mathrm{Aut}_R(J_R)$. Hence $\varphi = \delta_r\psi = \psi\delta_r$ with $\psi \in \mathrm{Aut}_R(J_R)$. We conclude that $\mathrm{Aut}_R(\mathcal{T}_R) \cong \mathrm{Aut}_R(J_R) \times \mu_2(R)$ for each R. □

Corollary 2.1.12. *Let J be a finite-dimensional central simple Jordan $\mathbb{F}$-algebra with associated Jordan triple system $\mathcal{T}$. Then, the map that sends a G-grading on J to the same G-grading on $\mathcal{T}$ gives a bijective correspondence from the equivalence classes of gradings on J to the equivalence classes of gradings on $\mathcal{T}$.*

Proof. Consequence of Theorem 2.1.11 (note that the elements of $\mu_2(R)$ are identified with automorphisms of $\mathcal{T}_R$ of the form $r1$ with $r \in R^\times$ and $r^2 = 1$) and the fact that the automorphism group scheme determines the equivalence classes of gradings. □

Remark 2.1.13. Note that fine gradings on $\mathcal{T}$ correspond to maximal quasitori of $\mathbf{Aut}(\mathcal{T})$, which are the direct product of a maximal quasitorus of $\mathbf{Aut}(J)$ and $\boldsymbol{\mu}_2$.

Corollary 2.1.14. *Let J be a finite-dimensional central simple Jordan $\mathbb{F}$-algebra with associated Jordan triple system $\mathcal{T}$. Let Γ_J be a G-grading on J and $\Gamma_{\mathcal{T}}$ the same G-grading on $\mathcal{T}$. Then $\mathcal{W}(\Gamma_{\mathcal{T}}) = \mathcal{W}(\Gamma_J)$.*

Proof. From Theorem 2.1.11 we know that $\mathrm{Aut}(\mathcal{T}) \cong \mathrm{Aut}(J) \times \{\pm 1\}$. Hence $\mathrm{Aut}(\Gamma_{\mathcal{T}}) \cong \mathrm{Aut}(\Gamma_J) \times \{\pm 1\}$ and the result follows. □

Proposition 2.1.15. *Let J be a Jordan $\mathbb{F}$-algebra with unity* 1*, and let $\mathcal{T}$ be its associated Jordan triple system. Let Γ be a G-grading on $\mathcal{T}$. If J is central simple, then* 1 *is homogeneous. Moreover, if $G = \mathcal{U}(\Gamma)$ and* 1 *is homogeneous, then* $\deg(1)$ *has order* 2.

Proof. We know that 1 is invariant under $\mathbf{Aut}(J)$. Hence, if J is central simple, $\mathbb{F}1$ is invariant under $\mathbf{Aut}(\mathcal{T}) = \mathbf{Aut}(J) \times \boldsymbol{\mu}_2$, and also under G^D

for any G-grading (where G^D acts via the morphism $\eta_\Gamma\colon G^D \to \mathbf{Aut}(\mathcal{T})$ producing the grading). In consequence, 1 is homogeneous.

Suppose now that $G = \mathcal{U}(\Gamma)$ and that 1 is homogeneous. Note that the trivial grading on $\mathcal{T}$ has universal group $\mathbb{Z}_2$ and support $\{\bar{1}\}$. Since $G = \mathcal{U}(\Gamma)$, the trivial grading is induced from Γ by some epimorphism $\varphi\colon \mathcal{U}(\Gamma) \to \mathbb{Z}_2$. Since φ sends all elements of the support to $\bar{1}$, $\deg(1)$ has at least order 2. On the other hand, $U_1(1) = 1$ implies that $2\deg(1) = 0$, and we can conclude that $\deg(1)$ has order 2. □

Remark 2.1.16. Given a unital Jordan algebra J with associated Jordan triple system $\mathcal{T}$ and a grading Γ on $\mathcal{T}$, it is not true in general that 1 is homogeneous. For example, take $J = \mathcal{T} = \mathbb{F} \times \mathbb{F}$ and consider the $\mathbb{Z}_2^2$-grading on $\mathcal{T}$ given by $\mathcal{T}_{(\bar{1},\bar{0})} = \mathbb{F} \times 0$ and $\mathcal{T}_{(\bar{0},\bar{1})} = 0 \times \mathbb{F}$.

Proposition 2.1.17. *Let J be a Jordan $\mathbb{F}$-algebra with unity 1, and G an abelian group. Consider the associated Jordan pair $\mathcal{V} = (J, J)$. If Γ is a set grading on $\mathcal{V}$ such that 1^+ (or 1^-) is homogeneous, then the restriction of Γ to $J = \mathcal{V}^+$ induces a set grading on J. If Γ is a G-grading on $\mathcal{V}$ such that 1^+ (or 1^-) is homogeneous, then the restriction of the shift $\Gamma^{[g]}$, with $g = -\deg(1^+)$, to $J = \mathcal{V}^+$ induces a G-grading Γ_J on J. Moreover, if $G = \mathcal{U}(\Gamma)$ then the universal group $\mathcal{U}(\Gamma_J)$ is isomorphic to the subgroup of $\mathcal{U}(\Gamma)$ generated by* Supp $\Gamma^{[g]}$*, and if in addition Γ is fine we also have that $\mathcal{U}(\Gamma)$ is isomorphic to $\mathcal{U}(\Gamma_J) \times \mathbb{Z}$.*

Proof. Let Γ be a set grading on $\mathcal{V}$ with 1^+ homogeneous. Since $U_{1^+}(y^-) = y^+$ for any y, the homogeneous components of $\mathcal{V}^+$ and $\mathcal{V}^-$ coincide. But from $\{x, 1, z\} = 2xz$ with char $\mathbb{F} \neq 2$, it follows that Γ induces a set grading on $J = \mathcal{V}^+$, where $J_s = \mathcal{V}_s^+$.

Assume that Γ is also a G-grading. From $U_{1^+}(1^-) = 1^+$, we get $\deg(1^+) + \deg(1^-) = 0$. Take $g := -\deg(1^+) = \deg(1^-)$. The grading $\Gamma^{[g]}$ satisfies $\deg_g(1^+) = 0 = \deg_g(1^-)$. But since $U_{1^+}(x^-) = x^+$, we have $\deg_g(x^+) = \deg_g(x^-)$. From $\{x, 1, z\} = 2xz$ we obtain $\deg_g(x) + \deg_g(z) = \deg_g(xz)$, so $\Gamma^{[g]}$ induces a G-grading on $J = \mathcal{V}^+$.

Suppose now that $G = \mathcal{U}(\Gamma)$ and set $H = \langle \mathrm{Supp}\, \Gamma^{[g]} \rangle$. Note that $\Gamma^{[g]}$ can be regarded as a $\mathcal{U}(\Gamma)$-grading and also as an H-grading; and similarly Γ_J can be regarded as a $\mathcal{U}(\Gamma_J)$-grading and as an H-grading. By the universal property of the universal group, the H-grading Γ_J is induced from the $\mathcal{U}(\Gamma_J)$-grading Γ_J by an epimorphism $\varphi_1\colon \mathcal{U}(\Gamma_J) \to H$ that restricts to the identity in the support. On the other hand, the $\mathcal{U}(\Gamma_J)$-grading Γ_J induces a $\mathcal{U}(\Gamma_J)$-grading (Γ_J, Γ_J) on $\mathcal{V}$ that is a coarsening of Γ, and therefore (Γ_J, Γ_J) is induced from Γ by some epimorphism $\varphi\colon \mathcal{U}(\Gamma) \to \mathcal{U}(\Gamma_J)$. Let $\varphi_2\colon H \to \mathcal{U}(\Gamma_J)$ be the restriction of φ to H. Note that $g \in \ker \varphi$, which implies that the $\mathcal{U}(\Gamma_J)$-grading (Γ_J, Γ_J) is induced from the H-grading $\Gamma^{[g]}$ by φ_2, and also that φ_2 is an epimorphism which is the identity in the support. Since each epimorphism φ_i is the identity in the support, both compositions $\varphi_1\varphi_2$ and $\varphi_2\varphi_1$ must be the identity and hence $\mathcal{U}(\Gamma_J) = H$.

Assume now that Γ is fine and denote by Γ_H the grading $\Gamma^{[g]}$ regarded as an H-grading. Note that $\mathcal{U}(\Gamma) = \langle \mathrm{Supp}\,\Gamma^{[g]}, g\rangle = \langle \mathrm{Supp}\,\Gamma_H, g\rangle = \langle H, g\rangle$. Consider H as a subgroup of $H \times \langle g_0\rangle \cong H \times \mathbb{Z}$, where the element g_0 has infinite order. The H-grading Γ_H can be regarded as an $H \times \langle g_0\rangle$-grading, and in consequence the shift $(\Gamma_H)^{[g_0]}$ defines an $H \times \langle g_0\rangle$-grading where $\deg(1^+) = g_0$. Since the $H \times \langle g_0\rangle$-grading $(\Gamma_H)^{[g_0]}$ is a coarsening of the $\mathcal{U}(\Gamma)$-grading Γ (because Γ is fine), by the universal property there is an epimorphism $\mathcal{U}(\Gamma) = \langle H, g\rangle \to H \times \langle g_0\rangle$ that sends $-g \mapsto g_0$ and fixes the elements of H. In consequence, $H \cap \langle g\rangle = 0$, $\langle g\rangle \cong \mathbb{Z}$, and we can conclude that $\mathcal{U}(\Gamma) = \langle H, g\rangle \cong H \times \mathbb{Z} = \mathcal{U}(\Gamma_J) \times \mathbb{Z}$. □

Proposition 2.1.18. *Let J be a Jordan $\mathbb{F}$-algebra with unity 1, and G an abelian group. Consider the associated Jordan triple system $\mathcal{T}$. If Γ is a set grading on $\mathcal{T}$ such that 1 is homogeneous, then Γ induces a set grading on J. If Γ is a G-grading on $\mathcal{T}$ such that 1 is homogeneous, then the shift $\Gamma^{[g]}$ with $g = \deg(1)$ induces a G-grading Γ_J on J. Moreover, if $G = \mathcal{U}(\Gamma)$ then $\mathcal{U}(\Gamma_J)$ is isomorphic to the subgroup of $\mathcal{U}(\Gamma)$ generated by* $\mathrm{Supp}\,\Gamma^{[g]}$*, and if in addition Γ is fine we also have that $\mathcal{U}(\Gamma)$ is isomorphic to $\mathcal{U}(\Gamma_J) \times \mathbb{Z}_2$.*

Proof. Let Γ is a set grading on $\mathcal{T}$ with 1 homogeneous; since $\{x, 1, z\} = 2xz$ with $\mathrm{char}\,\mathbb{F} \neq 2$ it follows that Γ induces a set grading on J. Assume from now on that Γ is a G-grading on $\mathcal{T}$ with 1 homogeneous of degree g. Proposition 2.1.15 shows that g has order 1 or 2. Hence the shift $\Gamma^{[g]}$ defines a G-grading on $\mathcal{T}$ with degree $\deg_g(x) = \deg(x) + g$, where $\deg_g(1) = 0$. Set $H = \langle \mathrm{Supp}\,\Gamma^{[g]}\rangle$. The rest of the proof follows using the same arguments of the proof of Proposition 2.1.17, but using $\mathcal{T}$ instead of the Jordan pair $\mathcal{V} = (J, J)$. □

2.1.2 Gradings induced by the TKK construction

Definition 2.1.19. Let $\mathcal{V}$ be a Jordan pair and consider the associated 3-graded Lie algebra $L = \mathrm{TKK}(\mathcal{V}) = L^{-1} \oplus L^0 \oplus L^1$ defined by the TKK construction (see Section 1.11). A G-grading on L will be called TKK-*compatible* if L^1 and L^{-1} are G-graded subspaces (and, therefore, so is $L^0 = [L^1, L^{-1}]$). In this case, we denote $L^n_g = L^n \cap L_g$ for $n \in \{-1, 0, 1\}$ and $g \in G$.

Consider a finite-dimensional Jordan pair $\mathcal{V}$ with associated Lie algebra $L = \mathrm{TKK}(\mathcal{V})$. Let Γ be a G-grading on $\mathcal{V}$. Then $E = \mathfrak{gl}(\mathcal{V}^+) \oplus \mathfrak{gl}(\mathcal{V}^-)$ is G-graded. For each homogeneous $x \in \mathcal{V}^+_g$ and $y \in \mathcal{V}^-_h$, we have $[x, y] = \nu(x, y) \in$

L^0_{g+h} because the triple product of $\mathcal{V}$ is a graded map. Hence, the subspace $L^0 \subseteq E$ is also G-graded. Since the triple product of $\mathcal{V}$ is a graded map, we can extend Γ to a TKK-compatible G-grading $E_G(\Gamma) : L = \bigoplus_{g\in G} L_g$, where $L^1_g = \mathcal{V}^+_g$, $L^{-1}_g = \mathcal{V}^-_g$ and $L^0_g = \mathrm{span}\{\nu(x,y) \mid \deg(x^+) + \deg(y^-) = g\}$. Conversely, since char $\mathbb{F} \neq 2$, any TKK-compatible G-grading $\widetilde{\Gamma}$ on L restricts to a G-grading $R_G(\widetilde{\Gamma})$ on $\mathcal{V}$, because $\{x^+, y^-, z^+\} = [[x^+, y^-], z^+]$.

Definition 2.1.20. Denote by $\mathrm{Grad}_G(\mathcal{V})$ the set of G-gradings on $\mathcal{V}$, and by $\mathrm{TKKGrad}_G(L)$ the set of TKK-compatible G-gradings on L. We will call

$$E_G\colon\ \mathrm{Grad}_G(\mathcal{V}) \longrightarrow \mathrm{TKKGrad}_G(L)$$

the *extension* map and

$$R_G\colon\ \mathrm{TKKGrad}_G(L) \longrightarrow \mathrm{Grad}_G(\mathcal{V})$$

the *restriction* map.

Theorem 2.1.21. *Let $\mathcal{V}$ be a Jordan pair with associated Lie algebra $L = \mathrm{TKK}(\mathcal{V})$, and let G be an abelian group. Then, the maps E_G and R_G are inverses of each other. Coarsenings are preserved by the correspondence, i.e., given a G_i-grading Γ_i on $\mathcal{V}$ with extended G_i-grading $\widetilde{\Gamma}_i = E_{G_i}(\Gamma_i)$ on L, for $i = 1, 2$, and a homomorphism $\alpha\colon G_1 \to G_2$, then $\Gamma_2 = {}^{\alpha}\Gamma_1$ if and only if $\widetilde{\Gamma}_2 = {}^{\alpha}\widetilde{\Gamma}_1$. If $G = \mathcal{U}(\Gamma)$, then $G = \mathcal{U}(E_G(\Gamma))$. Moreover, Γ is fine and $G = \mathcal{U}(\Gamma)$ if and only if $E_G(\Gamma)$ is fine and $G = \mathcal{U}(E_G(\Gamma))$.*

Proof. By construction, E_G and R_G are inverses of each other.

Assume that $\Gamma_2 = {}^{\alpha}\Gamma_1$ for some homomorphism $\alpha\colon G_1 \to G_2$. Then, $\mathcal{V}^\sigma_g \subseteq \mathcal{V}^\sigma_{\alpha(g)}$ for any $\sigma = \pm$ and $g \in G_1$. Thus, $L^{\sigma 1}_g \subseteq L^{\sigma 1}_{\alpha(g)}$, which implies that

$$L^0_g = \sum_{g_1+g_2=g} [L^1_{g_1}, L^{-1}_{g_2}] \subseteq \sum_{g_1+g_2=g} [L^1_{\alpha(g_1)}, L^{-1}_{\alpha(g_2)}] \subseteq L^0_{\alpha(g)}.$$

Hence, $\widetilde{\Gamma}_1$ refines $\widetilde{\Gamma}_2$ and $\widetilde{\Gamma}_2 = {}^{\alpha}\widetilde{\Gamma}_1$. Conversely, if $\widetilde{\Gamma}_2 = {}^{\alpha}\widetilde{\Gamma}_1$, by restriction we obtain $\Gamma_2 = {}^{\alpha}\Gamma_1$. We have proved that coarsenings are preserved.

Consider $\widetilde{\Gamma} = E_G(\Gamma)$ with $G = \mathcal{U}(\Gamma)$. Note that $\mathcal{U}(\Gamma)$ and $\mathcal{U}(\widetilde{\Gamma})$ are generated by $\mathrm{Supp}\,\Gamma$. Since the $\mathcal{U}(\widetilde{\Gamma})$-grading $\widetilde{\Gamma}$ restricts to Γ as a $\mathcal{U}(\widetilde{\Gamma})$-grading, there is a unique homomorphism $G = \mathcal{U}(\Gamma) \to \mathcal{U}(\widetilde{\Gamma})$ that is the identity in $\mathrm{Supp}\,\Gamma$; conversely, Γ extends to $\widetilde{\Gamma}$ as a G-grading, so there is a unique homomorphism $\mathcal{U}(\widetilde{\Gamma}) \to G$ that is the identity in $\mathrm{Supp}\,\widetilde{\Gamma}$ (and in $\mathrm{Supp}\,\Gamma$); therefore the compositions $\mathcal{U}(\widetilde{\Gamma}) \to G \to \mathcal{U}(\widetilde{\Gamma})$ and $G \to \mathcal{U}(\widetilde{\Gamma}) \to G$ are the identity map, and $G = \mathcal{U}(\widetilde{\Gamma})$.

Suppose again that $\widetilde{\Gamma} = E_G(\Gamma)$. Now, note that Γ is a fine G-grading on $\mathcal{V}$ with $G = \mathcal{U}(\Gamma)$ if and only if Γ satisfies the following property: if $\Gamma = {}^{\alpha}\Gamma_0$ for some G_0-grading Γ_0, where G_0 is generated by $\operatorname{Supp}\Gamma_0$, and $\alpha\colon G_0 \to G$ is an epimorphism, then α is an isomorphism. The same is true for TKK-compatible gradings. Since the coarsenings are preserved in the correspondence, so does this property, and therefore, Γ is fine and $G = \mathcal{U}(\Gamma)$ if and only if $\widetilde{\Gamma}$ is fine and $G = \mathcal{U}(\widetilde{\Gamma})$. □

Remark 2.1.22. The fact that $\widetilde{\Gamma} = E_G(\Gamma)$ with $G = \mathcal{U}(\widetilde{\Gamma})$, in general, does not imply that $G = \mathcal{U}(\Gamma)$. We will show this now.

First, take a G-grading $\Gamma_{\mathcal{V}}$ on a Jordan pair $\mathcal{V}$ such that there are nonzero elements in $\operatorname{Supp} L^0$ for $\widetilde{\Gamma}_{\mathcal{V}} = E_G(\Gamma_{\mathcal{V}})$, and assume that $G = \mathcal{U}(\Gamma_{\mathcal{V}})$ (it is not hard to find examples satisfying this). By Theorem 2.1.21, $G = \mathcal{U}(\widetilde{\Gamma}_{\mathcal{V}})$. Now, consider the Jordan pair $\mathcal{W} = \mathcal{V} \oplus \mathcal{V}'$ given by two copies of $\mathcal{V}$. There is a $G \times G$-grading Γ on $\mathcal{W}$, where $\mathcal{W}^{\sigma}_{(g,0)} = \mathcal{V}^{\sigma}_g$, $\mathcal{W}^{\sigma}_{(0,g)} = \mathcal{V}'^{\sigma}_g$. Besides, $\mathcal{U}(\Gamma) = G \times G$, so by Theorem 2.1.21, we have $\mathcal{U}(\widetilde{\Gamma}) = G \times G$ too, where $\widetilde{\Gamma} = E_{G\times G}(\Gamma)$. It suffices to find a proper coarsening $\widetilde{\Gamma}_1$ of $\widetilde{\Gamma}$ such that the restricted grading on $\mathcal{W}$ has the same homogeneous components as Γ. Actually, if $G_1 = \mathcal{U}(\widetilde{\Gamma}_1)$, then $\widetilde{\Gamma}_1 = E_{G_1}R_{G_1}(\widetilde{\Gamma}_1) = E_{G_1}(\Gamma)$ (where Γ is regarded as a G_1-grading) would be a proper coarsening of $\widetilde{\Gamma} = E_{\mathcal{U}(\Gamma)}(\Gamma)$, and therefore $G_1 \ncong \mathcal{U}(\Gamma)$.

Consider the $G \times \mathbb{Z}$-grading Γ_1 on $\mathcal{W}$ given by $\mathcal{W}^{\sigma}_{(g,0)} = \mathcal{V}^{\sigma}_g$, $\mathcal{W}^{\sigma}_{(g,\sigma 1)} = \mathcal{V}'^{\sigma}_g$. Then, Γ_1 and Γ have the same homogeneous components. The extension $\widetilde{\Gamma}_1 = E_{G\times\mathbb{Z}}(\Gamma_1)$ is a proper coarsening of $\widetilde{\Gamma}$, because $\widetilde{\Gamma}_1$ satisfies $\operatorname{Supp} L^0 \cap \operatorname{Supp} L'^0 = \operatorname{Supp} L^0 \neq \{0\}$ (where $L' = \operatorname{TKK}(\mathcal{V}')$) and for $\widetilde{\Gamma}$ we had $\operatorname{Supp} L^0 \cap \operatorname{Supp} L'^0 = \{0\}$. This proves the claim of the Remark.

Any automorphism $\varphi = (\varphi^+, \varphi^-)$ of $\mathcal{V}$ extends in a unique way to an automorphism $\widetilde{\varphi}$ of L (that leaves L_{-1} and L_1 invariant, so L_0 is invariant too). Indeed, it must satisfy $\widetilde{\varphi}(\nu(x^+, y^-)) = \widetilde{\varphi}([x^+, y^-]) = [\varphi^+(x^+), \varphi^-(y^-)] = \nu(\varphi^+(x^+), \varphi^-(y^-)) = \varphi\nu(x^+, y^-)\varphi^{-1}$, and this formula indeed defines an automorphism $\widetilde{\varphi}$ of L. Then, we can identify $\operatorname{Aut}\mathcal{V}$ with a subgroup of $\operatorname{Aut}L$, and so we have $\operatorname{Aut}\Gamma \leq \operatorname{Aut}\widetilde{\Gamma}$ and $\operatorname{Stab}\Gamma \leq \operatorname{Stab}\widetilde{\Gamma}$. We can also identify $\mathcal{W}(\Gamma) \leq \mathcal{W}(\widetilde{\Gamma})$. Indeed,

$$
\begin{aligned}
\mathcal{W}(\Gamma) &= \operatorname{Aut}\Gamma / \operatorname{Stab}\Gamma = \operatorname{Aut}\Gamma / (\operatorname{Stab}\widetilde{\Gamma} \cap \operatorname{Aut}\Gamma) \\
&\cong (\operatorname{Aut}\Gamma \cdot \operatorname{Stab}\widetilde{\Gamma}) / \operatorname{Stab}\widetilde{\Gamma} \leq \operatorname{Aut}\widetilde{\Gamma} / \operatorname{Stab}\widetilde{\Gamma} = \mathcal{W}(\widetilde{\Gamma}).
\end{aligned}
$$

But although $\mathcal{W}(\Gamma) \leq \mathcal{W}(\widetilde{\Gamma})$, these Weyl groups do not coincide in general, at least for the bi-Cayley and Albert Jordan pairs. Actually, we will see that their gradings are, up to equivalence, of the form $\Gamma = (\Gamma^+, \Gamma^-)$, where Γ^+ and Γ^- have the same homogeneous components, and hence there is an order 2 automorphism of L that interchanges $\mathcal{V}^+ \leftrightarrow \mathcal{V}^-$ and belongs to $\operatorname{Aut}\widetilde{\Gamma} \setminus \operatorname{Aut}\Gamma$, so $\mathcal{W}(\Gamma) < \mathcal{W}(\widetilde{\Gamma})$.

2.1.3 Some facts about gradings on semisimple Jordan pairs

Remark 2.1.23. Notice that, as a consequence of Equation (1.7.3), the Peirce spaces associated to an idempotent e define a $\mathbb{Z}$-grading Γ where the subspace $\mathcal{V}_i^\sigma$ has degree $\sigma(i+1)$, and $\operatorname{Supp}\Gamma = \{\pm 1, \pm 2, \pm 3\}$. If a Jordan pair $\mathcal{V}$ has a G-grading Γ, an idempotent $e = (e^+, e^-)$ of $\mathcal{V}$ will be called *homogeneous* if e^σ is homogeneous in $\mathcal{V}^\sigma$ for each σ. In that case, we have $\deg(e^+) + \deg(e^-) = 0$, which implies that the projections $E_i^\sigma = E_i^\sigma(e)$ are homogeneous maps of degree 0, and therefore the Peirce spaces $\mathcal{V}_i^\sigma = E_i^\sigma(\mathcal{V}^\sigma)$ are graded. If in addition the graded Jordan pair $\mathcal{V}$ is semisimple, then any nonzero homogeneous element $x = e^\sigma \in \mathcal{V}_g^\sigma$ can be completed to a homogeneous idempotent $e = (e^+, e^-) \in \mathcal{V}$; indeed, we can take a homogeneous element $y \in \mathcal{V}_{-g}^{-\sigma}$ such that $Q_x y = x$ (because $\mathcal{V}$ is vNr and the quadratic products are graded maps), and in consequence $e = (e^+, e^-)$ with $e^{-\sigma} := Q_y x$ is a homogeneous idempotent.

Since homogeneous elements are completed to homogeneous idempotents and these produce graded Peirce subspaces, it follows that we can always choose a maximal orthogonal system of idempotents whose elements happen to be homogeneous.

Theorem 2.1.24. *Let $\mathcal{V}$ be a finite-dimensional semisimple Jordan pair and Γ a G-grading on $\mathcal{V}$. Then:*

1) If $g \in \operatorname{Supp}\Gamma^\sigma$, the subpair $(\mathcal{V}_g^\sigma, \mathcal{V}_{-g}^{-\sigma})$ is semisimple.

2) For any subgroup $H \leq G$ with $H \cap \operatorname{Supp}\Gamma \neq \emptyset$, the subpair given by $\mathcal{V}_H^\sigma := \bigoplus_{h \in H} \mathcal{V}_h^\sigma$ is semisimple.

3) If Γ is fine, the homogeneous components are 1-dimensional.

Proof. 1) Take $0 \neq x \in \mathcal{V}_g^\sigma$. By Remark 2.1.23, x can be completed to an idempotent of $\mathcal{W} = (\mathcal{V}_g^\sigma, \mathcal{V}_{-g}^{-\sigma})$, so x is vNr in $\mathcal{W}$. Hence, $\mathcal{W}$ is vNr too, and by Theorem 1.7.1, $\mathcal{W}$ is semisimple.

2) Consider the epimorphism $\alpha\colon G \to \bar{G} = G/H$ and the induced $\bar{G}$-grading $\bar{\Gamma} = {}^\alpha\Gamma$. Then, $\mathcal{V}_H$ coincides with the $\bar{G}$-graded subpair $(\mathcal{V}_{\bar{0}}^\sigma, \mathcal{V}_{\bar{0}}^{-\sigma})$, that is semisimple by 1).

3) Let Γ be fine and assume by contradiction that $\dim \mathcal{V}_g^\sigma > 1$, where we can assume without loss of generality that $\sigma = +$. Then, the subpair

$\mathcal{W} = (\mathcal{V}_g^+, \mathcal{V}_{-g}^-)$ is semisimple by 1). By Theorem 1.7.1, $\mathcal{W}$ is nondegenerate, so we can consider the rank function. We can take an element $x \in \mathcal{W}^+$ of rank 1 in $\mathcal{W}$ (but not necessarily in $\mathcal{V}$), and complete it to an idempotent $e = (x, y)$ of $\mathcal{W}$. As in Remark 2.1.23, the Peirce spaces of the Peirce decomposition associated to e are the homogeneous components of a $\mathbb{Z}$-grading on $\mathcal{V}$, which is compatible with Γ because the Peirce spaces are graded with respect to Γ. Thus, combining the $\mathbb{Z}$-grading with Γ we get a $G \times \mathbb{Z}$-grading that refines Γ, given by $\mathcal{V}^\sigma_{(g,i)} = \mathcal{V}^\sigma_g \cap \mathcal{V}^\sigma_i$. Since $\mathrm{rk}_{\mathcal{W}}(x) = 1$ and $\mathbb{F} = \bar{\mathbb{F}}$, we have $\mathbb{F}x = Q(x)\mathcal{W}^- =: \mathcal{W}_2^+(e)$. But then, $\mathcal{V}_g^+ \cap \mathcal{V}_2^+(e) = \mathcal{V}_g^+ \cap Q(x)\mathcal{V}^- = \mathcal{V}_g^+ \cap Q(x)\mathcal{V}_{-g}^- = \mathcal{V}_g^+ \cap Q(x)\mathcal{W}^- = \mathcal{W}_2^+(e) = \mathbb{F}x \subsetneq \mathcal{V}_g^+$ and the refinement is proper, which contradicts that Γ is fine. $\square$

The next Corollary is a nice application of the above results to the study of gradings on Jordan algebras.

Corollary 2.1.25. *Let J be a finite-dimensional semisimple Jordan algebra and Γ a fine G-grading on J with* $\dim J_0 = 1$*. Then, all the homogeneous components of* Γ *have dimension* 1.

Proof. Assume by contradiction that some homogeneous component has dimension bigger than 1. Consider the Jordan pair $\mathcal{V} = (J, J)$ and let $\widetilde{\Gamma} = (\Gamma, \Gamma)$ be the induced G-grading on $\mathcal{V}$. Since some component of $\widetilde{\Gamma}$ has dimension bigger than 1, we can refine $\widetilde{\Gamma}$ to a fine grading $\widetilde{\Gamma}'$ on $\mathcal{V}$, which will have all components of dimension 1. Then, by Proposition 2.1.17, the shift $\widetilde{\Gamma}'^{[g]}$ for $g = -\deg(1^+)$ restricts to a group grading on J, which is a proper refinement of Γ, a contradiction. $\square$

Some examples of homogeneous bilinear forms are given by trace forms: this is the case of gradings on Hurwitz algebras, matrix algebras, and the Albert algebra. Other well-known example is the Killing form of a graded semisimple Lie algebra. The generic trace plays the same role for graded Jordan pairs and graded triple systems.

Remark 2.1.26. Assume that we have a graded finite-dimensional simple Jordan pair $\mathcal{V}$ where the generic trace form t is homogeneous. If t is nondegenerate, $\mathcal{V}^\sigma_g$ and $\mathcal{V}^{-\sigma}_{-g}$ are dual relative to t and have the same dimension. For any homogeneous element $x \in \mathcal{V}^\sigma_g$, define $t_x : \mathcal{V}^{-\sigma} \to \mathbb{F}$, $y \mapsto t(x, y)$. Since the trace form is homogeneous, the subspace $\ker(t_x)$ is graded too; we will use this fact in some proofs later on. Also, recall that the generic minimal polynomial $m(T, X, Y)$ of a Jordan pair $\mathcal{V}$ is $\mathbf{Aut}(\mathcal{V})$-invariant (see [L75, 16.7]), and in consequence the generic trace form t is invariant too, i.e., $t(\varphi^+(x), \varphi^-(y)) = t(x, y)$ for all $\varphi \in \mathrm{Aut}_R \mathcal{V}_R$, $x \in \mathcal{V}_R^+$, $y \in \mathcal{V}_R^-$ and R an associative commutative unital $\mathbb{F}$-algebra.

Proposition 2.1.27. *Let $\mathcal{V}$ be a finite-dimensional simple Jordan pair. Then, the generic trace of $\mathcal{V}$ is homogeneous for any grading on $\mathcal{V}$.*

Proof. Suppose that $\mathcal{V}$ is G-graded. We know by [L75, Proposition 16.7] that the minimal polynomial is invariant by the automorphism group scheme $\mathbf{Aut}(\mathcal{V})$. Hence the generic trace t is $\mathbf{Aut}(\mathcal{V})$-invariant, i.e., for any associative commutative unital $\mathbb{F}$-algebra R and for any $(\varphi^+, \varphi^-) \in \mathrm{Aut}_R(\mathcal{V}_R)$ we have $t(\varphi^+(v^+), \varphi^-(v^-)) = t(v^+, v^-)$ for all $(v^+, v^-) \in \mathcal{V}_R = \mathcal{V} \otimes R$. In particular, if we take the group algebra $R = \mathbb{F}G$ we can consider the automorphism φ of $\mathcal{V}_R$ given by $\varphi^\sigma(v_g^\sigma \otimes 1) = v_g^\sigma \otimes g$ for each $\sigma = \pm$ and each homogeneous element $v_g^\sigma \in \mathcal{V}_g^\sigma$. In order to avoid confusion, the binary operation in G will be denoted multiplicatively here. Now, fix homogeneous elements $v_g^+ \in \mathcal{V}_g^+$ and $v_h^- \in \mathcal{V}_h^-$. On the one hand, we know that

$$t(\varphi^+(v_g^+ \otimes 1), \varphi^-(v_h^- \otimes 1)) = t(v_g^+ \otimes 1, v_h^- \otimes 1) = t(v_g^+, v_h^-) \otimes 1$$

by $\mathrm{Aut}_R(\mathcal{V}_R)$-invariance. On the other hand,

$$t(\varphi^+(v_g^+ \otimes 1), \varphi^-(v_h^- \otimes 1)) = t(v_g^+ \otimes g, v_h^- \otimes h) = t(v_g^+, v_h^-) \otimes gh$$

by definition of φ. Therefore, $t(v_g^+, v_h^-) \otimes 1 = t(v_g^+, v_h^-) \otimes gh$. We conclude that $gh = 1$ whenever $t(v_g^+, v_h^-) \neq 0$. □

2.2 Exceptional Jordan pairs and triple systems

In this section we first recall the definitions of the well-known Jordan pairs and triple systems of types bi-Cayley and Albert (see [L75] for the definition of the mentioned Jordan pairs). We also give examples of automorphisms of these Jordan systems and prove some results related to the orbits of bi-Cayley systems; this will be used in subsequent chapters to classify gradings up to equivalence. Furthermore, we give an explicit description of the automorphism groups of bi-Cayley systems.

2.2.1 Jordan pairs and triple systems of types bi-Cayley and Albert

Definition 2.2.1. The *Albert pair* is the Jordan pair associated to the Albert algebra, that is, $\mathcal{V}_\mathbb{A} := (\mathbb{A}, \mathbb{A})$ with the products $Q_x^\sigma(y) = U_x(y) := 2L_x^2(y) - L_{x^2}(y)$. Its associated Jordan triple system $\mathcal{T}_\mathbb{A} := \mathbb{A}$, with the product $Q_x =$

U_x, will be called the *Albert triple system*. It is well-known (see [McC70, Theorem 1] and [McC69, Theorem 1]) that the U-operator can be written as

$$U_x(y) = T(x,y)x - x^\# \times y.$$

Let $\mathcal{C}$ be the Cayley algebra. We will write for short the vector spaces

$$\mathcal{B} := \mathcal{C} \oplus \mathcal{C}, \quad \mathcal{C}_1^\sigma := (\mathcal{C} \oplus 0)^\sigma \quad \text{and} \quad \mathcal{C}_2^\sigma := (0 \oplus \mathcal{C})^\sigma.$$

Let n be the norm of $\mathcal{C}$. The quadratic form $q\colon \mathcal{B} \to \mathbb{F}$, $q((x_1,x_2)) := n(x_1)+n(x_2)$, will be called the *norm* of $\mathcal{B}$. The nondegenerate bilinear form defined by $t\colon \mathcal{B}\times\mathcal{B} \to \mathbb{F}$, $t(x,y) = n(x_1,y_1) + n(x_2,y_2)$ for $x = (x_1,x_2)$, $y = (y_1,y_2) \in \mathcal{B}$ (i.e., t is the linearization of q) will be called the *trace* of $\mathcal{B}$.

Denote by $\mathcal{V}_{\mathbb{A}}^{12}$ the Jordan subpair $(\mathcal{V}_{\mathbb{A}})_1(e)$ of the Peirce decomposition $\mathcal{V}_{\mathbb{A}} = (\mathcal{V}_{\mathbb{A}})_0(e) \oplus (\mathcal{V}_{\mathbb{A}})_1(e) \oplus (\mathcal{V}_{\mathbb{A}})_2(e)$ relative to the idempotent $e = (E_3, E_3)$ (notation as in Section 1.5); that is, $(\mathcal{V}_{\mathbb{A}}^{12})^+ = (\mathcal{V}_{\mathbb{A}}^{12})^- := \iota_1(\mathcal{C}) \oplus \iota_2(\mathcal{C})$. Identifying $\iota_1(\mathcal{C}) \oplus \iota_2(\mathcal{C}) \equiv \mathcal{C} \oplus \mathcal{C} = \mathcal{B}$, the trace t of $\mathcal{B}$ is also defined as a map $(\mathcal{V}_{\mathbb{A}}^{12})^+ \times (\mathcal{V}_{\mathbb{A}}^{12})^- \to \mathbb{F}$.

Proposition 2.2.2. *The quadratic and triple products of* $\mathcal{V}_{\mathbb{A}}^{12}$ *are given by:*

$$U_x(y) = 4t(x,y)x - 4n(x_1)\iota_1(y_1) - 4n(x_2)\iota_2(y_2) - 4\iota_1\big(\bar{y}_2(x_2x_1)\big) - 4\iota_2\big((x_2x_1)\bar{y}_1\big),$$

and

$$\begin{aligned}\{x,y,z\} = U_{x,z}(y) = {}& 4t(x,y)z + 4t(z,y)x - 4n(x_1,z_1)\iota_1(y_1) - 4n(x_2,z_2)\iota_2(y_2)\\ &- 4\iota_1\big(\bar{y}_2(x_2z_1 + z_2x_1)\big) - 4\iota_2\big((x_2z_1 + z_2x_1)\bar{y}_1\big),\end{aligned}$$

for all $x = \iota_1(x_1) + \iota_2(x_2)$, $y = \iota_1(y_1) + \iota_2(y_2) \in \iota_1(\mathcal{C}) \oplus \iota_2(\mathcal{C})$.

Proof. Take $x, y \in \iota_1(\mathcal{C}) \oplus \iota_2(\mathcal{C})$. Then,

$$\begin{aligned} xy =& \big(\iota_1(x_1) + \iota_2(x_2)\big)\big(\iota_1(y_1) + \iota_2(y_2)\big)\\ =& 2n(x_1,y_1)(E_2+E_3) + 2n(x_2,y_2)(E_3+E_1) + \iota_3(\bar{x}_1\bar{y}_2 + \bar{y}_1\bar{x}_2),\end{aligned}$$

$$\begin{aligned} L_x^2(y) =& 2n(x_1,y_1)\iota_1(x_1) + n(x_2,y_2)\iota_1(x_1) + \iota_2\big((y_2x_1)\bar{x}_1 + (x_2y_1)\bar{x}_1\big)\\ &+ n(x_1,y_1)\iota_2(x_2) + 2n(x_2,y_2)\iota_2(x_2) + \iota_1\big(\bar{x}_2(y_2x_1) + \bar{x}_2(x_2y_1)\big)\\ =& 2t(x,y)x + n(x_1)\iota_2(y_2) + n(x_2)\iota_1(y_1) - \iota_1\big(\bar{y}_2(x_2x_1)\big) - \iota_2\big((x_2x_1)\bar{y}_1\big),\end{aligned}$$

so we get

$$x^2 = 4n(x_1)(E_2+E_3) + 4n(x_2)(E_3+E_1) + 2\iota_3(\bar{x}_1\bar{x}_2),$$

$$\begin{aligned} L_{x^2}(y) =& 4n(x_1)\iota_1(y_1) + 2n(x_2)\iota_1(y_1) + 2\iota_2\big((x_2x_1)\bar{y}_1\big)\\ &+ 2n(x_1)\iota_2(y_2) + 4n(x_2)\iota_2(y_2) + 2\iota_1\big(\bar{y}_2(x_2x_1)\big)\\ =& \big(4n(x_1) + 2n(x_2)\big)\iota_1(y_1) + \big(2n(x_1) + 4n(x_2)\big)\iota_2(y_2)\\ &+ 2\iota_2\big((x_2x_1)\bar{y}_1\big) + 2\iota_1\big(\bar{y}_2(x_2x_1)\big).\end{aligned}$$

Then, by substituting $U_x(y) := 2L_x^2(y) - L_{x^2}(y)$ we obtain the first expression, and its linearization is the second one. $\square$

Definition 2.2.3. Define the *bi-Cayley pair* as the Jordan pair $\mathcal{V}_{\mathcal{B}} := (\mathcal{B}, \mathcal{B})$ with products:

$$\begin{aligned}
Q_x^\sigma(y) =& Q_x(y) \\
=& t(x,y)x - \Big(n(x_1)y_1 + \bar{y}_2(x_2x_1), n(x_2)y_2 + (x_2x_1)\bar{y}_1\Big) \\
=& \Big(x_1\bar{y}_1x_1 + \bar{x}_2(y_2x_1), x_2\bar{y}_2x_2 + (x_2y_1)\bar{x}_1\Big), \\
\{x,y,z\}^\sigma =& \{x,y,z\} = Q_{x,z}(y) \\
=& t(x,y)z + t(z,y)x \\
&- \Big(n(x_1,z_1)y_1 + \bar{y}_2(x_2z_1 + z_2x_1), n(x_2,z_2)y_2 + (x_2z_1 + z_2x_1)\bar{y}_1\Big) \\
=& \Big(x_1(\bar{y}_1z_1) + z_1(\bar{y}_1x_1) + \bar{x}_2(y_2z_1) + \bar{z}_2(y_2x_1), \\
& \qquad (x_2\bar{y}_2)z_2 + (z_2\bar{y}_2)x_2 + (x_2y_1)\bar{z}_1 + (z_2y_1)\bar{x}_1\Big).
\end{aligned} \tag{2.2.1}$$

Since the products of $\mathcal{V}_{\mathcal{B}}$ and $\mathcal{V}_{\mathbb{A}}^{12}$ are proportional, it is clear that $\mathcal{V}_{\mathcal{B}}$ is a Jordan pair and the map $\mathcal{V}_{\mathbb{A}}^{12} \to \mathcal{V}_{\mathcal{B}}$, $\iota_1(x_1) + \iota_2(x_2) \mapsto (2x_1, 2x_2)$ is an isomorphism of Jordan pairs if char $\mathbb{F} \neq 2$. We also define the *bi-Cayley triple system* as the Jordan triple system $\mathcal{T}_{\mathcal{B}} := \mathcal{B}$ associated to the bi-Cayley pair $\mathcal{V}_{\mathcal{B}}$, so its quadratic and triple products are defined as for $\mathcal{V}_{\mathcal{B}}$.

The above definition is not the one in [L75], which is given below:

Definition 2.2.4. Consider $\mathcal{M}_{1\times 2} := (\mathcal{M}_{1\times 2}(\mathcal{C}), \mathcal{M}_{1\times 2}(\mathcal{C}^{\mathrm{op}}))$, which is known to be a simple Jordan pair (see [L75]). The quadratic products are given by $Q_x(y) = x(y^*x)$, where y^* denotes y traspósed with coefficients in the opposite algebra. Considering elements in $\mathcal{C}$, we can write:

$$\begin{aligned}
Q_x^+(y) =& x(yx) = \Big(x_1y_1x_1 + x_2(y_2x_1), x_1(y_1x_2) + x_2y_2x_2\Big), \\
Q_y^-(x) =& (yx)y = \Big(y_1x_1y_1 + (y_1x_2)y_2, (y_2x_1)y_1 + y_2x_2y_2\Big),
\end{aligned} \tag{2.2.2}$$

where we have omitted some parentheses using the alternativity of $\mathcal{C}$.

Although the following result is probably known, the author does not know of a reference, so we include the proof.

Proposition 2.2.5. *The Jordan pairs $\mathcal{V}_{\mathcal{B}}$ and $\mathcal{M}_{1\times 2}$ are isomorphic.*

Proof. There is an isomorphism $\varphi = (\varphi^+, \varphi^-) \colon \mathcal{V}_{\mathcal{B}} \to \mathcal{M}_{1\times 2}$ given by:

$$\varphi^+ \colon (x_1, x_2) \mapsto (\bar{x}_2, x_1), \qquad \varphi^- \colon (y_1, y_2) \mapsto (y_2, \bar{y}_1).$$

Indeed,

$$\begin{aligned}
\varphi^+(Q_x y) =& \varphi^+\big(x_1\bar{y}_1 x_1 + \bar{x}_2(y_2 x_1), (x_2\bar{y}_2)x_2 + (x_2 y_1)\bar{x}_1\big)\\
=& \big(\bar{x}_2 y_2 \bar{x}_2 + x_1(\bar{y}_1\bar{x}_2), x_1\bar{y}_1 x_1 + \bar{x}_2(y_2 x_1)\big),\\
Q^+_{\varphi^+(x)}\big(\varphi^-(y)\big) =& Q^+_{(\bar{x}_2, x_1)}(y_2, \bar{y}_1) = \big(\bar{x}_2 y_2 \bar{x}_2 + x_1(\bar{y}_1\bar{x}_2), \bar{x}_2(y_2 x_1) + x_1\bar{y}_1 x_1\big),\\
\varphi^-(Q_y x) =& \varphi^-\big(y_1\bar{x}_1 y_1 + \bar{y}_2(x_2 y_1), (y_2\bar{x}_2)y_2 + (y_2 x_1)\bar{y}_1\big)\\
=& \big(y_2\bar{x}_2 y_2 + (y_2 x_1)\bar{y}_1, \bar{y}_1 x_1 \bar{y}_1 + (\bar{y}_1\bar{x}_2)y_2\big),\\
Q^-_{\varphi^-(y)}\big(\varphi^+(x)\big) =& Q^-_{(y_2, \bar{y}_1)}(\bar{x}_2, x_1) = \big(y_2\bar{x}_2 y_2 + (y_2 x_1)\bar{y}_1, (\bar{y}_1\bar{x}_2)y_2 + \bar{y}_1 x_1 \bar{y}_1\big),
\end{aligned}$$

so we get $\varphi^+(Q_x y) = Q^+_{\varphi^+(x)}\big(\varphi^-(y)\big)$ and $\varphi^-(Q_y x) = Q^-_{\varphi^-(y)}\big(\varphi^+(x)\big)$. □

The generic trace form of $\mathcal{V}_{\mathbb{A}}$ is given by $T(x, y) := T(xy)$ where T is the trace form of $\mathbb{A}$ ([L75, 17.10]), and the generic trace form of $\mathcal{M}_{1\times 2}$ is given by $t(xy^*) = \mathrm{tr}(x_1 y_1 + x_2 y_2)$ ([L75, 17.9]), where tr denotes the trace of $\mathcal{C}$. Thus, applying the isomorphism in Proposition 2.2.5, we get that the generic trace of $\mathcal{V}_{\mathcal{B}}$ is the bilinear form $t = n \perp n$, that is, $t(x^+, y^-) = n(x_1, y_1) + n(x_2, y_2)$ for $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathcal{B}$. (Note that $t = \frac{1}{4} T|_{\mathcal{V}_{\mathcal{B}}}$.) Also, we will refer to t, respectively to T, as the trace of $\mathcal{T}_{\mathcal{B}}$, respectively of $\mathcal{T}_{\mathbb{A}}$.

Lemma 2.2.6. *For any grading on the Jordan pairs and triple systems of types bi-Cayley or Albert, the trace is homogeneous.*

Proof. Consequence of Proposition 2.1.27 and the fact that gradings on a triple system extend to gradings on the associated Jordan pair. □

2.2.2 Some automorphisms

In order to study the gradings on the Jordan pairs and triple systems under consideration, we will need to use some automorphisms defined in this section.

Notation 2.2.7. Recall that for any automorphism $\varphi = (\varphi^+, \varphi^-)$ of $\mathcal{V}_{\mathcal{B}}$ or $\mathcal{V}_{\mathbb{A}}$, the pair (φ^-, φ^+) is also an automorphism, which we denote by $\widehat{\varphi}$.

Denote by $\bar{\tau}_{12}$ the order 2 automorphism of $\mathbb{A}$ (and therefore of $\mathcal{T}_{\mathbb{A}}$ and $\mathcal{V}_{\mathbb{A}}$) given by $E_1 \leftrightarrow E_2$, $E_3 \mapsto E_3$, $\iota_1(x) \leftrightarrow \iota_2(\bar{x})$, $\iota_3(x) \mapsto \iota_3(\bar{x})$. Similarly, we define $\bar{\tau}_{23}$ and $\bar{\tau}_{13}$.

Identifying $\mathcal{B}$ with $\iota_1(\mathcal{C}) \oplus \iota_2(\mathcal{C})$, the automorphism $\bar{\tau}_{12}$ of $\mathbb{A}$ restricts to one of $\mathcal{T}_{\mathcal{B}}$ (and therefore of $\mathcal{V}_{\mathcal{B}}$), denoted also by $\bar{\tau}_{12}$, and given by:

$$\bar{\tau}_{12} \colon \mathcal{B} \to \mathcal{B}, \quad (x_1, x_2) \mapsto (\bar{x}_2, \bar{x}_1).$$

Take $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{F}^\times$ and $\mu_i := \lambda_i^{-1}\lambda_{i+1}\lambda_{i+2}$. Define $c_{\lambda_1,\lambda_2,\lambda_3}$ by

$$\begin{aligned} &\iota_i(x)^+ \mapsto \iota_i(\lambda_i x)^+, \quad \iota_i(x)^- \mapsto \iota_i(\lambda_i^{-1} x)^-, \\ &E_i^+ \mapsto \mu_i E_i^+, \quad E_i^- \mapsto \mu_i^{-1} E_i^-. \end{aligned} \tag{2.2.3}$$

One checks that $c_{\lambda_1,\lambda_2,\lambda_3}$ is an automorphism of $\mathcal{V}_{\mathbb{A}}$ (these were considered before, for example in [Gar01, 1.6]). If $\lambda \in \mathbb{F}^\times$, denote $c_\lambda := c_{\lambda,\lambda,\lambda}$.

The automorphisms $c_{\lambda_1,\lambda_2,\lambda_3}$ restrict to $\mathcal{V}_{\mathcal{B}}$. For $\lambda, \mu \in \mathbb{F}^\times$ define $c_{\lambda,\mu} \in \operatorname{Aut} \mathcal{V}_{\mathcal{B}}$ given by:

$$c^+_{\lambda,\mu} \colon (x_1, x_2) \mapsto (\lambda x_1, \mu x_2), \quad c^-_{\lambda,\mu} \colon (y_1, y_2) \mapsto (\lambda^{-1} y_1, \mu^{-1} y_2).$$

We also write $c_\lambda := c_{\lambda,\lambda}$ (which is consistent with notation introduced in the previous paragraph).

Proposition 2.2.8. *For each $a \in \mathcal{C}$, there is an automorphism φ_a of $\mathcal{V}_{\mathcal{B}}$ given by:*

$$\varphi_a^+ \colon (x_1, x_2) \mapsto (x_1 - \bar{x}_2 a, x_2), \quad \varphi_a^- \colon (y_1, y_2) \mapsto (y_1, a\bar{y}_1 + y_2),$$

for any $x_1, x_2, y_1, y_2 \in \mathcal{C}$.

Proof. It suffices to check that φ_a is the inner automorphism $\beta((a, 0), (0, 1))$. (Notice that φ_a is the exponential of the derivation $d_a = -\nu((a, 0), (0, 1))$, which is nilpotent of order 2 and given by $d_a^+(x_1, x_2) = (-\bar{x}_2 a, 0)$, $d_a^-(y_1, y_2) = (0, a\bar{y}_1)$.) □

Remark 2.2.9. Since $\varphi_a\varphi_b = \varphi_{a+b}$ for any $a, b \in \mathcal{C}$, these automorphisms generate an abelian subgroup of $\operatorname{Aut} \mathcal{V}_{\mathcal{B}}$ isomorphic to $(\mathcal{C}, +)$. The same is true for $\widehat{\varphi}_a := (\varphi_a^-, \varphi_a^+)$, $a \in \mathcal{C}$. Note that, since $\mathcal{B} = \mathcal{C} \oplus \mathcal{C}$, we can write

$$\varphi_a^+ = \begin{pmatrix} 1 & -r_{\bar{a}} \\ 0 & 1 \end{pmatrix}, \quad \varphi_a^- = \begin{pmatrix} 1 & 0 \\ l_{\bar{a}} & 1 \end{pmatrix},$$

where l_a, r_a denote the left and right multiplications by a in the para-Cayley algebra $\bar{\mathcal{C}}$. This matrix notation is useful to make computations with these automorphisms.

Proposition 2.2.10. *Let $\lambda \in \mathbb{F}$ and $a \in \mathcal{C}$ be such that $n(a)+\lambda^2 = 1$. There is an automorphism $\phi_1(a,\lambda)$ of $\mathbb{A}$ given by:*

$$
\begin{aligned}
x = \sum_{i=1}^{3} \big(\alpha_i E_i + \iota_i(x_i)\big) \;\mapsto\; & \alpha_1 E_1 + \big(\alpha_2\lambda^2 + \alpha_3 n(a) + 2\lambda n(\bar a, x_1)\big) E_2 \\
& + \big(\alpha_2 n(a) + \alpha_3\lambda^2 - 2\lambda n(\bar a, x_1)\big) E_3 \\
& + \iota_1\Big(x_1 + \big(\tfrac{1}{2}\alpha_3\lambda - \tfrac{1}{2}\alpha_2\lambda - n(\bar a, x_1)\big)\bar a\Big) \\
& + \iota_2(\lambda x_2 - \bar x_3 a) + \iota_3(\lambda x_3 + a\bar x_2).
\end{aligned}
$$

Proof. Straightforward. □

Proposition 2.2.11. *Let $a \in \mathcal{C}$ and $\lambda \in \mathbb{F}$ be such that $n(a)+\lambda^2 = 1$. There is an automorphism of $\mathcal{T}_{\mathcal{B}}$ given by:*

$$\varphi_{a,\lambda}\colon (x_1, x_2) \mapsto (\lambda x_1 - \bar x_2 a, a\bar x_1 + \lambda x_2).$$

Moreover, $\varphi_{a,\lambda} \in \mathrm{O}^+(\mathcal{B}, q)$.

Proof. Note that, if we identify $\mathcal{B}$ with $\iota_2(\mathcal{C}) \oplus \iota_3(\mathcal{C}) \subseteq \mathbb{A}$, then $\varphi_{a,\lambda}$ is the restriction of $\phi_1(a,\lambda)$ to $\mathcal{B}$, so it is an automorphism. We will give a different proof now. In case $n(a) = 0$, $\lambda = \pm 1$, define $\varphi := \lambda\widehat{\varphi}_{\lambda a}\varphi_{\lambda a} \in \operatorname{Aut}\mathcal{V}_{\mathcal{B}}$, and in case $n(a) \neq 0$, define $\varphi := \widehat{\varphi}_{\mu a}\varphi_a\widehat{\varphi}_{\mu a} \in \operatorname{Aut}\mathcal{V}_{\mathcal{B}}$ with $\mu = \frac{1-\lambda}{n(a)}$. In both cases, it is checked that $\varphi_{a,\lambda} = \varphi \in \operatorname{Aut}\mathcal{T}_{\mathcal{B}} \leq \mathrm{O}(\mathcal{B}, q)$. Since $\det(\varphi_a^{\pm}) = 1 = \det(\widehat{\varphi}_a^{\pm})$ for any $a \in \mathcal{C}$, we also have $\det(\varphi_{a,\lambda}) = 1$, and so $\varphi_{a,\lambda} \in \mathrm{O}^+(\mathcal{B}, q)$. □

Remark 2.2.12. In $\mathcal{T}_{\mathcal{B}}$ we have $Q_x(x) = q(x)x$ for any $x \in \mathcal{B}$ and, as a consequence, $\operatorname{Aut}\mathcal{T}_{\mathcal{B}} \leq \mathrm{O}(\mathcal{B}, q)$. Since $\mathcal{B} = \mathcal{C} \oplus \mathcal{C}$, we can write

$$\varphi_{a,\lambda} = \begin{pmatrix} \lambda & -r_{\bar a} \\ l_{\bar a} & \lambda \end{pmatrix},$$

where l_a, r_a are the left and right multiplications by a in the para-Cayley algebra $\bar{\mathcal{C}}$.

Definition 2.2.13. (See, for instance, [Jac89, Chapter 4].) Let V be a finite-dimensional vector space and $q\colon V \to F$ a nondegenerate quadratic form. Recall that the map $\tau(a) = a$, $a \in V$, is extended to an involution of the Clifford algebra $\mathfrak{Cl}(V, q)$, called the *standard involution*. The map $\alpha(a) = -a$, $a \in V$, extended to an automorphism of $\mathfrak{Cl}(V, q)$, produces the standard $\mathbb{Z}_2$-grading $\mathfrak{Cl}(V, q) = \mathfrak{Cl}(V, q)_{\bar 0} \oplus \mathfrak{Cl}(V, q)_{\bar 1}$. The *Clifford group* of $\mathfrak{Cl}(V, q)$ is defined as $\Gamma = \Gamma(V, q) := \{x \in \mathfrak{Cl}(V, q)^\times \mid x \cdot V \cdot x^{-1} \subseteq V\}$. Here $\cdot$ denotes the product of $\mathfrak{Cl}(V, q)$. The subgroup $\Gamma^+ := \Gamma \cap \mathfrak{Cl}(V, q)_{\bar 0}$ is called the *even Clifford group*. The *spin group* is defined by $\mathrm{Spin}(V,q) := \{x \in \Gamma^+ \mid x \cdot \tau(x) = 1\}$. Note that $\mathrm{Spin}(V,q)$ is generated by the elements of the form $x \cdot y$ where $x, y \in V$ and $q(x)q(y) = 1$.

For each $u \in \mathrm{Spin}(V,q)$, define the map $\chi_u \colon V \to V$, $x \mapsto u \cdot x \cdot u^{-1}$. It is well-known that χ_u belongs to the special orthogonal group $\mathrm{O}^+(V,q)$, and $\mathrm{O}'(V,q) := \{\chi_u \mid u \in \mathrm{Spin}(V,q)\}$ is called the *reduced orthogonal group*. Moreover, $\mathrm{O}'(V,q) \trianglelefteq \mathrm{O}^+(V,q)$ and $\mathrm{O}^+(V,q)/\mathrm{O}'(V,q) \cong \mathbb{F}^\times/(\mathbb{F}^\times)^2$ (see [Jac89, 4.8]), where $(\mathbb{F}^\times)^2$ is the multiplicative group of squares of $\mathbb{F}^\times$. Here $\mathbb{F}$ is assumed to be algebraically closed, so we have $\mathrm{O}^+(V,q) = \mathrm{O}'(V,q)$. A triple $(f_1, f_2, f_3) \in \mathrm{O}(\mathcal{C}, n)^3$ is said to be *related* if $f_1(\bar{x}\bar{y}) = \overline{f_2(x)}\ \overline{f_3(y)}$ for any $x, y \in \mathcal{C}$. Note that if (f_1, f_2, f_3) is a related triple, then (f_2, f_3, f_1) is also a related triple. Related triples have the property that $f_i \in \mathrm{O}'(\mathcal{C}, n)$, and there is a group isomorphism

$$\mathrm{Spin}(\mathcal{C}, n) \longrightarrow \{\text{related triples in } \mathrm{O}(\mathcal{C}, n)^3\}, \quad u \mapsto (\chi_u, \rho_u^+, \rho_u^-),$$

for certain associated maps ρ_u^+ and ρ_u^- (see e.g. [Eld00] for more details).

Remark 2.2.14. Note that, if $(f_1, f_2, f_3) \in \mathrm{O}(\mathcal{C}, n)^3$ is a related triple, then it is easy to check that (f_1, f_2) is an automorphism of the bi-Cayley triple system. It is well-known that the map $\mathbb{A} \to \mathbb{A}$, $E_i \mapsto E_i$, $\iota_i(x) \mapsto \iota_i(f_i(x))$ for $i = 1, 2, 3$, is an automorphism of the Albert algebra (see e.g. [EK13, Corollary 5.6]).

Lemma 2.2.15. *For any $x_1, x_2 \in \mathcal{C}$ of norm 1, there is a related triple (f_1, f_2, f_3) in $\mathrm{O}(\mathcal{C}, n)^3$ such that $f_i(x_i) = 1$ for $i = 1, 2$. Besides, for any $f_1 \in \mathrm{O}^+(\mathcal{C}, n)$, there are $f_2, f_3 \in \mathrm{O}^+(\mathcal{C}, n)$ such that (f_1, f_2, f_3) is a related triple in $\mathrm{O}(\mathcal{C}, n)^3$.*

Proof. The first statement was proved in [EK13, Lemma 5.25]. For the second part, since $\chi \colon \mathrm{Spin}(\mathcal{C}, n) \to \mathrm{O}'(\mathcal{C}, n) = \mathrm{O}^+(\mathcal{C}, n)$ is onto, we can write $f_1 = \chi_u$ for some $u \in \mathrm{Spin}(\mathcal{C}, n)$, and $(\chi_u, \rho_u^+, \rho_u^-)$ is a related triple. □

Consider $\mathfrak{Cl}(\mathcal{C}, n)$ with the the $\mathbb{Z}_2$-grading given by $\deg(x) = \bar{1}$ for each $x \in \mathcal{C}$, and the *standard involution* defined by setting $x \mapsto x$ for $x \in \mathcal{C}$. Consider $\mathrm{End}(\mathcal{C} \oplus \mathcal{C})$ with the $\mathbb{Z}_2$-grading that has degree $\bar{0}$ on the endomorphisms that preserve the two copies of $\mathcal{C}$ and degree $\bar{1}$ on the endomorphisms that swap these two copies, and the involution given by the adjoint relative to the quadratic form $n \perp n$ on $\mathcal{C} \oplus \mathcal{C}$.

The next result is a slight modification of [KMRT98, Proposition (35.1)]:

Proposition 2.2.16. *Denote by l_x, r_x the left and right multiplications in the para-Cayley algebra $\bar{\mathcal{C}} = (\mathcal{C}, *)$. Then, the map*

$$\Phi\colon \mathcal{C} \to \mathrm{End}(\mathcal{C} \oplus \mathcal{C}), \quad x \mapsto \begin{pmatrix} 0 & r_{\bar{x}} \\ l_{\bar{x}} & 0 \end{pmatrix},$$

defines an isomorphism of superalgebras $\Phi\colon \mathfrak{Cl}(\mathcal{C}, n) \to \mathrm{End}(\mathcal{C} \oplus \mathcal{C})$ that preserves the involution.

Proof. Since $\Phi(x)^2 = n(x)\mathrm{id}$ for $x \in \mathcal{C}$, it follows that Φ extends to a homomorphism of superalgebras. But since $\mathfrak{Cl}(\mathcal{C}, n)$ is simple and has the same dimension as $\mathrm{End}(\mathcal{C} \oplus \mathcal{C})$, we have that Φ is an isomorphism. From $l_x^* = r_x$, we deduce that Φ is an isomorphism of algebras with involution. □

Remark 2.2.17. Given $a \in \mathcal{C}$ with $n(a) = 1$, we have

$$\Phi(a) = \begin{pmatrix} 0 & r_{\bar{a}} \\ l_{\bar{a}} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -r_{\bar{a}} \\ l_{\bar{a}} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \varphi_{a,0} c_{1,-1} \in \mathrm{Aut}\, \mathcal{T}_{\mathcal{B}},$$

and in particular, $\mathrm{RT} := \Phi(\mathrm{Spin}(\mathcal{C}, n)) \leq \mathrm{Aut}\, \mathcal{T}_{\mathcal{B}}$.

For any $u \in \mathrm{Spin}(\mathcal{C}, n)$, $\Phi(u) = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right)$ if and only if $(\bar{\chi}_u, \alpha, \beta)$ is a related triple (see [EK13, Theorem 5.5]), with $\bar{\chi}_u(a) = \overline{\chi_u(\bar{a})}$, so

$$\mathrm{RT} = \Phi(\mathrm{Spin}(\mathcal{C}, n)) = \Big\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} : \alpha, \beta \in \mathrm{O}(\mathcal{C}, n) \text{ and there is } \gamma \in \mathrm{O}(\mathcal{C}, n) \text{ such that } (\gamma, \alpha, \beta) \text{ is a related triple} \Big\},$$

and this explains our notation RT. The subgroup $\mathrm{RT} \cong \mathrm{Spin}(\mathcal{C}, n)$ is generated by the elements of the form

$$\Phi(a)\Phi(b) = \begin{pmatrix} r_{\bar{a}} l_{\bar{b}} & 0 \\ 0 & l_{\bar{a}} r_{\bar{b}} \end{pmatrix} = \varphi_{a,0} c_{1,-1} \varphi_{b,0} c_{1,-1} = \varphi_{a,0} \varphi_{-b,0},$$

with $n(a) = n(b) = 1$. Note that the group $\mathrm{Aut}\, \mathcal{C}$ embeds in RT because for any automorphism f of $\mathcal{C}$, (f, f, f) is a related triple.

Remark 2.2.18. Consider the subgroup $G_{\mathcal{V}} = \langle \varphi_a, \widehat{\varphi}_a, c_\lambda \mid a \in \mathcal{C}, \lambda \in \mathbb{F}^\times \rangle$ of $\mathrm{Aut}\, \mathcal{V}_{\mathcal{B}}$ and the subgroup $G_{\mathcal{T}} = \langle \varphi_{a,\lambda} \mid a \in \mathcal{C}, \lambda \in \mathbb{F}, n(a) + \lambda^2 = 1 \rangle$ of $\mathrm{Aut}\, \mathcal{T}_{\mathcal{B}}$. (We will prove later that $G_{\mathcal{V}} = \mathrm{Aut}\, \mathcal{V}_{\mathcal{B}}$ and $G_{\mathcal{T}} = \mathrm{Aut}\, \mathcal{T}_{\mathcal{B}}$.) It follows from the proof of Proposition 2.2.11 that $G_{\mathcal{T}} \leq G_{\mathcal{V}}$. The group RT of related triples is contained in the subgroup generated by the automorphisms $\varphi_{a,0}$ with $n(a) = 1$, so we have $\mathrm{RT} \leq G_{\mathcal{T}}$. Also, $(-\mathrm{id}, \mathrm{id}, -\mathrm{id})$ is a related triple, so $c_{1,-1} = (\mathrm{id}, -\mathrm{id}) \in \mathrm{RT} \leq G_{\mathcal{T}}$ and hence $\bar{\tau}_{12} = \varphi_{1,0} c_{1,-1} \in G_{\mathcal{T}}$.

We claim that $c_{\lambda,\mu} \in G_{\mathcal{V}}$ for any $\lambda, \mu \in \mathbb{F}^\times$. For any $\lambda \in \mathbb{F}^\times$ and $a \in \mathcal{C}$ such that $\lambda n(a) = 1$, we have $c_{\lambda,1} = c_{-\sqrt{\lambda}}\varphi_{\sqrt{\lambda}a,0}\varphi_a\widehat{\varphi}_{\lambda a}\varphi_a \in G_{\mathcal{V}}$. But since c_μ belongs to $G_{\mathcal{V}}$ for any $\mu \in \mathbb{F}^\times$, we deduce that $c_{\lambda,\mu} = c_{\lambda\mu^{-1},1}c_\mu \in G_{\mathcal{V}}$ for any $\lambda, \mu \in \mathbb{F}^\times$.

Remark 2.2.19. Let J be a unital Jordan algebra with associated Jordan pair $\mathcal{V} = (J, J)$. Let $\mathrm{Str}(J)$ denote the *structure group* of J, i.e., the group consisting of all the *autotopies*, that is, the elements $g \in \mathrm{GL}(J)$ such that $U_{g(x)} = gU_xg^\#$ for some $g^\# \in \mathrm{GL}(J)$ and all $x \in J$. The structure group functor $\mathbf{Str}(J)$ is defined by $\mathbf{Str}(J)(R) = \mathrm{Str}_R(J_R)$. There is an isomorphism of group schemes $\mathbf{Aut}(\mathcal{V}) \to \mathbf{Str}(J)$, which is given by $\mathrm{Aut}_R(\mathcal{V}_R) \to \mathrm{Str}_R(J_R)$, $(\varphi^+, \varphi^-) \mapsto \varphi^+$ for each associative commutative unital $\mathbb{F}$-algebra R (see [L79, Proposition 2.6] and [L75, Proposition 1.8] for more details).

Let $M(\mathbb{A})$ and $M_1(\mathbb{A})$ be the groups of similarities and isometries for the norm of $\mathbb{A}$ (notation as in [Jac68, Chap.IX]). By [Jac68, Chap.V, Th.4], $\mathbb{A}$ is reduced, so by [Jac68, Chap.IX, Ex.2], a linear map $\mathbb{A} \to \mathbb{A}$ is a norm similarity if and only if it is an isotopy; that is, $M(\mathbb{A}) = \mathrm{Str}(\mathbb{A})$. Also, if we identify $\langle c_\lambda \mid \lambda \in \mathbb{F}^\times \rangle \cong \mathbb{F}^\times$, we have $M(\mathbb{A}) = \mathbb{F}^\times \cdot M_1(\mathbb{A})$.

For each norm similarity φ of $\mathbb{A}$, denote $\varphi^\dagger := (\varphi^{-1})^*$, where $*$ denotes the adjoint relative to the trace form T of $\mathbb{A}$. Since the trace is invariant under automorphisms, it follows that the automorphisms of $\mathcal{V}_{\mathbb{A}}$ are exactly the pairs $(\varphi, \varphi^\dagger)$ where φ is a norm similarity of $\mathbb{A}$. We know from [Gar01, Lemma 1.7] that, if $\varphi = (\varphi^+, \varphi^-)$ is an automorphism of $\mathcal{V}_{\mathbb{A}}$ where the norm similarity φ^σ has multiplier λ_σ then $\lambda_+\lambda_- = 1$; also $\varphi^\sigma(x^\#) = \lambda_\sigma\varphi^{-\sigma}(x)^\#$. Moreover, we have $\varphi^\sigma(x^{-1}) = \varphi^{-\sigma}(x)^{-1}$ for each $x \in \mathbb{A}^\times$ (because $U_{\varphi^{-\sigma}(x)}\varphi^\sigma(x^{-1}) = \varphi^{-\sigma}(U_xx^{-1}) = \varphi^{-\sigma}(x)$ for each $x \in \mathbb{A}^\times$).

2.2.3 Orbits of the automorphism groups of bi-Cayley systems

The trace forms of the bi-Cayley and Albert pairs are nondegenerate, so by Proposition 1.7.4, there are exactly three orbits in $\mathcal{B}^\sigma$ for the bi-Cayley pair, and four orbits in $\mathbb{A}^\sigma$ for the Albert pair, all of them determined by the rank function.

Notation 2.2.20. Recall that the norm of the vector space $\mathcal{B}$ is the quadratic form $q = n \perp n\colon \mathcal{B} \to \mathbb{F}$, given by $q(x) = n(x_1) + n(x_2)$ for $x = (x_1, x_2) \in \mathcal{B}$. For $i = 0, 1, 2$, denote by $\mathcal{O}_i$ the subset of $\mathcal{B}$ of elements of rank i for the bi-Cayley pair. For each $\lambda \in \mathbb{F}$, set $\mathcal{O}_2(\lambda) := \{x \in \mathcal{O}_2 \mid q(x) = \lambda\}$. Thus $\mathcal{O}_2 = \dot{\bigcup}_{\lambda\in\mathbb{F}}\mathcal{O}_2(\lambda)$.

Lemma 2.2.21. *The different orbits for the action of* $\operatorname{Aut}\mathcal{T}_{\mathcal{B}}$ *on* $\mathcal{B}$ *are exactly* $\mathcal{O}_0 = \{0\}$, $\mathcal{O}_1$ *and* $\mathcal{O}_2(\lambda)$ *with* $\lambda \in \mathbb{F}$. *Moreover, for* $0 \neq x = (x_1, x_2) \in \mathcal{B}$ *we have* $x \in \mathcal{O}_1$ *if and only if* $x_2x_1 = 0$ *and* $n(x_1) = n(x_2) = 0$. *The orbits are the same if we consider the action of the subgroup* $G_{\mathcal{T}} = \langle \varphi_{a,\lambda} \mid a \in \mathcal{C}, \lambda \in \mathbb{F}, n(a) + \lambda^2 = 1\rangle$.

Proof. Recall that $\mathcal{O}_i$, for $i = 0, 1, 2$, are the orbits of the bi-Cayley pair. Also, note that $\operatorname{Aut}\mathcal{T}_{\mathcal{B}} \leq \mathrm{O}(\mathcal{B}, q)$. Hence, the sets $\mathcal{O}_0$, $\mathcal{O}_1$, and $\mathcal{O}_2(\lambda)$ for $\lambda \in \mathbb{F}$, are disjoint unions of orbits of the bi-Cayley triple system.

First, we will check that $\mathcal{O}_2(\lambda)$ is an orbit for each $\lambda \neq 0$. Take $x \in \mathcal{O}_2(\lambda^2)$ with $\lambda \neq 0$. We claim that x belongs to the orbit of $(\lambda 1, 0)$. By applying $\bar\tau_{12}$ if necessary, we can assume that $n(x_1) \neq 0$. Since $q(x) = \lambda^2 \neq 0$, $n(x_1) \neq -n(x_2)$ and we can take $\mu \in \mathbb{F}^\times$ such that $\mu^{-2} = 1 + \frac{n(x_2)}{n(x_1)}$. The element $a = -\mu n(x_1)^{-1}x_2x_1$ satisfies $n(a) + \mu^2 = 1$, so we can consider the automorphism $\varphi_{a,\mu}$ (see Proposition 2.2.11). Then, $\varphi_{a,\mu}(x) = (\mu(1 - \frac{n(x_2)}{n(x_1)})x_1, 0)$, and by Lemma 2.2.15, this element is in the orbit of $(\lambda 1, 0)$. Hence, $\mathcal{O}_2(\lambda^2)$ is an orbit for each $\lambda \neq 0$. Since $\mathbb{F}$ is algebraically closed, $\mathcal{O}_2(\lambda)$ is an orbit too.

Second, given $0 \neq x \in \mathcal{B}$ we claim that $x \in \mathcal{O}_1$ if and only if $x_2x_1 = 0$ and $n(x_1) = n(x_2) = 0$. Indeed, $x \in \mathcal{O}_1$ means that $Q_x\mathcal{B} = \mathbb{F}x$, i.e., $(n(x_1)y_1 + \bar y_2(x_2x_1), n(x_2)y_2 + (x_2x_1)\bar y_1) = t(x,y)x - Q_x(y)$ must belong to $\mathbb{F}x$ for any $y \in \mathcal{B}$, which is equivalent to saying that $x_2x_1 = 0$ and $n(x_1) = n(x_2) = 0$.

Third, we will prove that $\mathcal{O}_1$ is an orbit. Take $x = (x_1, x_2) \in \mathcal{O}_1$. We know that $n(x_1) = n(x_2) = 0$ and $x_2x_1 = 0$. Then, using $\bar\tau_{12}$ if necessary, we can assume that $x_1 \neq 0$ and $n(x_1) = 0$, and by Lemma 2.2.15 we can also assume that $x_1 = e_1$ is a nontrivial idempotent. Take $e_2 := 1 - e_1$, and consider the Peirce decomposition of $\mathcal{C}$ associated to the idempotents e_i as always. Since $x_2x_1 = 0$, we have $x_2 = \lambda e_2 + u$ with $\lambda \in \mathbb{F}$, $u \in U$ (see Subsection 1.4). Thus, $x = (e_1, \lambda e_2 + u)$. But taking $a = -\lambda e_2 - u$ and $\mu = 1$ we have $n(a) + \mu^2 = 1$, so $\varphi_{a,1}$ is an automorphism. Therefore, $\varphi_{a,1}(x) = (e_1 + (\lambda e_1 + \bar u)(\lambda e_2 + u), \lambda e_2 + u - (\lambda e_2 + u)e_2) = (e_1, 0)$. This proves that $\mathcal{O}_1$ is an orbit.

Finally, we claim that $\mathcal{O}_2(0)$ is an orbit. Take $x \in \mathcal{O}_2(0)$, and fix $\mathbf{i} \in \mathbb{F}$ with $\mathbf{i}^2 = -1$. It suffices to prove that x is in the orbit of $(1, \mathbf{i}1)$. But we will prove first that if $n(x_1) = n(x_2) = 0$, then there is an automorphism φ of $\mathcal{T}_{\mathcal{B}}$ such that the two components of $\varphi(x)$ are nonisotropic. Indeed, since $x \notin \mathcal{O}_1$ and $n(x_1) = n(x_2) = 0$, we must have $x_2x_1 \neq 0$, and hence $x_1, x_2 \neq 0$. If $n(x_1, \bar x_2) \neq 0$, it suffices to take $\mu = \frac{1}{\sqrt{2}}$ and apply $\varphi = \varphi_{\mu 1,\mu}$ to x to obtain an element with nonisotropic components. Otherwise, $n(x_1, \bar x_2) = 0 = n(x_i)$ and by Lemma 2.2.15, we can assume that $x_1 = e_1$ is a nontrivial idempotent. Consider the idempotents e_1, $e_2 := 1 - e_1$ with their Peirce decomposition $\mathcal{C} = \mathbb{F}e_1 \oplus \mathbb{F}e_2 \oplus U \oplus V$, so we have $x_2 = \gamma e_2 + u + v$ for some $\gamma \in \mathbb{F}$, $u \in U$,

$v \in V$. Since $x_2x_1 \neq 0$, we have $v \neq 0$. Take $u_1 \in U$ with $vu_1 = e_2$, so we obtain $\varphi_{u_1,1}(x) = (1 - \gamma u_1 + uu_1, \gamma e_2 + u + u_1 + v)$, which has the first component nonisotropic. In conclusion, there is an automorphism φ of $\mathcal{T}_\mathcal{B}$ such that $\varphi(x)$ has both components nonisotropic. By Lemma 2.2.15, we can assume that $x = (\lambda 1, \mathbf{i}\lambda 1)$, for certain $0 \neq \lambda \in \mathbb{F}$. Take $a \in \mathcal{C}$ with $\mathrm{tr}(a) = 0$ and $n(a) = \frac{\lambda^2-1}{2\lambda^2}$, and $\mu \in \mathbb{F}$ such that $n(a) + \mu^2 = 1$. Then, $y := \varphi_{a,\mu}(x) = (\lambda\mu 1 - \lambda\mathbf{i}a, \lambda a + \lambda\mu\mathbf{i}1)$, and we have $n(y_1) = \lambda^2 n(\mu 1 - \mathbf{i}a) = \lambda^2(\mu^2 - n(a)) = \lambda^2(1 - 2n(a)) = 1$; since $\varphi_{a,\mu} \in \mathrm{O}(\mathcal{B}, q)$, we obtain $n(y_2) = -1$. By Lemma 2.2.15 again, we can assume that $x = (1, \mathbf{i}1)$, and therefore $\mathcal{O}_2(0)$ is an orbit. □

We have a similar result for the orbits of the bi-Cayley pair:

Lemma 2.2.22. *The orbits of $\mathcal{B}^+$ under the action of the group $\mathrm{Aut}\,\mathcal{V}_\mathcal{B}$ coincide with the orbits under the action of $G_\mathcal{V} = \langle \varphi_a, \widehat{\varphi}_a, c_\lambda \mid a \in \mathcal{C},\ \lambda \in \mathbb{F}^\times \rangle$.*

Proof. First, recall that $\mathrm{Aut}\,\mathcal{V}_\mathcal{B}$ has 3 orbits on $\mathcal{B}^+$, determined by the rank function, that can take values 0, 1 and 2 (see Proposition 1.7.4). From now on, consider the action of $G_\mathcal{V}$ on $\mathcal{B}^+$. We have to prove that the orbits under the action of $G_\mathcal{V}$ are $\mathcal{O}_0$, $\mathcal{O}_1$ and $\mathcal{O}_2$. Clearly, $\mathcal{O}_0 = \{0\}$ is an orbit of this action. Recall from Remark 2.2.18 that $G_\mathcal{V}$ contains the subgroup of related triples and $\bar{\tau}_{12}$. By Lemma 2.2.15, two nonzero elements of $\mathcal{C}_1 = \mathcal{C} \oplus 0$ of the same norm are in the same orbit under the action of $G_\mathcal{V}$ (because $G_\mathcal{V}$ contains the subgroup of related triples). Using automorphisms of type c_λ and the fact that $\mathbb{F} = \bar{\mathbb{F}}$, we also deduce that two nonisotropic elements of $\mathcal{C}_1$ belong to the same orbit; a representative element of this orbit is $(1, 0)$. Note that $\dim \mathrm{im}\, Q_x$ is an invariant of the orbit of each element $x \in \mathcal{B}$. Given $0 \neq z \in \mathcal{C}$ with $n(z) = 0$, we have $\dim \mathrm{im}\, Q_0 = 0$, $\dim \mathrm{im}\, Q_{(z,0)} = 1$ and $\dim \mathrm{im}\, Q_{(1,0)} = 8$; consequently, there are exactly 3 orbits on $\mathcal{C}_1$. It suffices to prove that each element of $\mathcal{B}$ belongs to an orbit of $\mathcal{C}_1$. Fix $x = (x_1, x_2) \in \mathcal{B}$ with $x_1, x_2 \neq 0$; we claim that there is an automorphism φ in $G_\mathcal{V}$ such that $\varphi^+(x) \in \mathcal{C}_1$.

Assume that $n(x_i) \neq 0$ for some $i = 1, 2$. We can apply $\bar{\tau}_{12}$ if necessary to assume that $n(x_1) \neq 0$. Then, take $a = -n(x_1)^{-1}x_2x_1$, so we have $\widehat{\varphi}_a^+(x) = (x_1, 0) \in \mathcal{C}_1$.

Now, consider the case with $n(x_1) = 0 = n(x_2)$. In the case that $n(x_1, \bar{x}_2) \neq 0$, take $a = 1$, so we get that $\varphi_a^+(x)$ has a nonisotropic component, which is the case that we have considered above. Otherwise, we are in the case that $n(x_i) = 0 = n(x_1, \bar{x}_2)$. By Lemma 2.2.15, without loss of generality we can assume that $e_1 := x_1$ is a nontrivial idempotent of $\mathcal{C}$. Consider the associated Peirce decomposition $\mathcal{C} = \mathbb{F}e_1 \oplus \mathbb{F}e_2 \oplus U \oplus V$ associated to the idempotents e_1 and $e_2 = 1 - e_1$. Since $n(e_2, x_2) = n(\bar{x}_1, x_2) = 0$, we have $x_2 = \lambda e_2 + u + v$ for certain elements $\lambda \in \mathbb{F}$, $u \in U$, $v \in V$. There are two cases now:

• In case $v \neq 0$, we can take $u_1 \in U$ such that $vu_1 = e_2$, so $\varphi^+_{u_1}(x) = (e_1 - (\lambda\bar{e}_2 + \bar{u} + \bar{v})u_1, x_2) = (1 - \lambda u_1 + uu_1, x_2)$, where the first component is nonisotropic (it has norm 1), which is the case considered above.

• In case $v = 0$, we have that $\widehat{\varphi}^+_{-\lambda 1}(x) = \varphi^-_{-\lambda 1}(x) = (e_1, u)$, and we can assume that $x = (e_1, u)$. But if $u \neq 0$, there is $v \in V$ such that $uv = -e_1$, so $\bar{\tau}_{12}\varphi^+_v(x) = \bar{\tau}_{12}(e_1 - \bar{u}v, u) = \bar{\tau}_{12}(0, u) = (-u, 0) \in \mathcal{C}_1$, and we are done. □

2.2.4 Automorphism groups of bi-Cayley systems

In this subsection, we will give an explicit description of the automorphism groups of the bi-Cayley pair and triple system.

Theorem 2.2.23. *The group* $\operatorname{Aut}\mathcal{V}_{\mathcal{B}}$ *is generated by the automorphisms of the form* φ_a, $\widehat{\varphi}_a$ *and* c_λ *(with* $a \in \mathcal{C}$, $\lambda \in \mathbb{F}^\times$*).*

Proof. Take $\varphi \in \operatorname{Aut}\mathcal{V}_{\mathcal{B}}$ and call $G_{\mathcal{V}} = \langle \varphi_a, \widehat{\varphi}_a, c_\lambda \mid a \in \mathcal{C}, \lambda \in \mathbb{F}^\times \rangle$. We have to prove that $\varphi \in G_{\mathcal{V}}$. Recall from Remark 2.2.18 that related triples and automorphisms of type $c_{\lambda,\mu}$ belong to $G_{\mathcal{V}}$.

By Lemma 2.2.22, there is some element φ' of $G_{\mathcal{V}}$ such that $\varphi'\varphi(1,0)^+ = (1,0)^+$. Thus, without loss of generality (changing φ with $\varphi'\varphi$) we can assume that $\varphi(1,0)^+ = (1,0)^+$. Since the image of the idempotent $((1,0)^+, (1,0)^-)$ is an idempotent of the form $((1,0)^+, (a,b)^-)$, it must be $(1,0)^+ = Q_{(1,0)^+}(a,b)^- = (\bar{a}, 0)^+$, hence $a = 1$. The composition $\varphi_{-b}\varphi$ fixes $(1,0)^\pm$, so we can assume (changing φ with $\varphi_{-b}\varphi$) that the same holds for φ. In consequence, the subspaces $\mathcal{C}^\sigma_1 = \operatorname{im} Q_{(1,0)^\sigma}$ and $\mathcal{C}^\sigma_2 = \ker Q_{(1,0)^{-\sigma}}$ must be φ-invariant. Write $\varphi(0,1)^+ = (0,a)^+$ with $a \in \mathcal{C}$. Since the element $\varphi(0,1)^+$ has rank 2, we have $n(a) \neq 0$, and composing with an automorphism of type $c_{1,\lambda}$ if necessary we can also assume that $n(a) = 1$. Then, by Lemma 2.2.15, composing with a related triple we can assume that φ fixes $(1,0)^+$ and $(0,1)^+$. Note that the subspaces $\mathcal{C}^\sigma_i$ are still φ-invariant and we can write $\varphi^\sigma = \phi^\sigma_1 \times \phi^\sigma_2$ with $\phi^\sigma_i \in \mathrm{GL}(\mathcal{C}^\sigma_i)$. Then, since $(1,0)^+ = \varphi(1,0)^+ = \varphi(Q_{(1,0)^+}(1,0)^-) = Q_{(1,0)^+}(\phi^-_1(1), 0) = (\overline{\phi^-_1(1)}, 0)^+$, we have $\phi^-_1(1) = 1$, and similarly $\phi^-_2(1) = 1$. Therefore, φ fixes the elements $(1,0)^\pm$ and $(0,1)^\pm$.

Denote $\mathcal{C}_0 = \{a \in \mathcal{C} \mid \operatorname{tr}(a) = 0\}$. Since the trace t of $\mathcal{V}_{\mathcal{B}}$ is invariant by automorphisms and $(1,0)^\pm$ are fixed by φ, we obtain that the subspaces $(\mathcal{C}_0 \oplus 0)^\pm$ are φ-invariant (note that $\operatorname{tr}(a) = t((a,0),(1,0)))$, and the same holds for $(0 \oplus \mathcal{C}_0)^\pm$. For each $z \in \mathcal{C}_0$, we have $Q_{(1,0)^+}(z,0)^- = (-z,0)^+$, which implies that $(-\phi^+_1(z), 0)^+ = \varphi Q_{(1,0)^+}(z,0)^- = Q_{(1,0)^+}(\phi^-_1(z), 0)^- = (\overline{\phi^-_1(z)}, 0)^+ = (-\phi^-_1(z), 0)^+$. Hence $\phi^+_1 = \phi^-_1$ and, in the same manner, $\phi^+_2 =$

ϕ_2^-. With abuse of notation, we can omit the index σ and write $\varphi = \phi_1 \times \phi_2$. On the other hand, for each $z \in \mathcal{C}_0$ we have $\{(1,0)^+, (0,1)^-, (0,z)^+\} = (-z,0)^+$, from where we get that $(-\phi_1(z),0)^+ = \varphi\{(1,0)^+, (0,1)^-, (0,z)^+\} = \{(1,0)^+, (0,1)^-, (0,\phi_2(z))^+\} = (-\phi_2(z),0)^+$. Thus, $\phi_1 = \phi_2$ and, with more abuse of notation we can omit the subindex $i = 1,2$ and write $\varphi = \phi \times \phi$, where $\phi \in \mathrm{GL}(\mathcal{C})$. Moreover, applying φ to the equality $\{(0,1),(0,x),(y,0)\} = (xy,0)$ we obtain $\phi(xy) = \phi(x)\phi(y)$, which shows that $\phi \in \operatorname{Aut}\mathcal{C}$. Since $\operatorname{Aut}\mathcal{C} \leq \mathrm{RT} \leq G_{\mathcal{V}}$ (with the obvious identifications), we have $\varphi \in G_{\mathcal{V}}$ and we are done. $\square$

Theorem 2.2.24. *The group* $\operatorname{Aut}\mathcal{T}_{\mathcal{B}}$ *is generated by the automorphisms of the form* $\varphi_{a,\lambda}$ *(with* $a \in \mathcal{C}$ *and* $\lambda \in \mathbb{F}$ *such that* $n(a) + \lambda^2 = 1$*).*

Proof. Take $\varphi \in \operatorname{Aut}\mathcal{T}_{\mathcal{B}}$ and call $G_{\mathcal{T}} = \langle \varphi_{a,\lambda} \mid a \in \mathcal{C}, \lambda \in \mathbb{F}, n(a) + \lambda^2 = 1\rangle$. We have to prove that $\varphi \in G_{\mathcal{T}}$. By Lemma 2.2.21, there is some element φ' of $G_{\mathcal{T}}$ such that $\varphi'\varphi(1,0) = (1,0)$. Thus, without loss of generality (changing φ with $\varphi'\varphi$) we can assume that $\varphi(1,0) = (1,0)$. Now, the subspaces $\mathcal{C}_1 = \operatorname{im} Q_{(1,0)}$ and $\mathcal{C}_2 = \ker Q_{(1,0)}$ must be φ-invariant. Write $\varphi(0,1) = (0,a)$ with $a \in \mathcal{C}$. We know from Remark 2.2.12 that $\operatorname{Aut}\mathcal{T}_{\mathcal{B}} \leq \mathrm{O}(\mathcal{B}, q)$, so we have $n(a) = q(0,a) = q(0,1) = 1$. Then, by Lemma 2.2.15, composing with a related triple we can assume without loss of generality that φ fixes $(1,0)$ and $(0,1)$. Since the subspaces $\mathcal{C}_i$ are φ-invariant, we can write $\varphi = \phi_1 \times \phi_2$ with $\phi_i \in \mathrm{GL}(\mathcal{C}_i)$. With the same arguments as in the proof of Theorem 2.2.23 we deduce that $\phi_1 = \phi_2 \in \operatorname{Aut}\mathcal{C}$, and therefore φ belongs to $G_{\mathcal{T}}$. $\square$

We introduce now some notation that will be used in the following results of this section:

Notation 2.2.25. We extend the norm n on $\mathcal{C}$ to a ten-dimensional vector space

$$W = \mathcal{C} \perp (\mathbb{F}e \oplus \mathbb{F}f)$$

with $n(e) = n(f) = 0$ and $n(e,f) = 1$. Fix $\mathbf{i} \in \mathbb{F}$ with $\mathbf{i}^2 = -1$ and note that the elements $x = e + f$ and $y = \mathbf{i}(e - f)$ are orthogonal of norm 1. Then, $e = (x - \mathbf{i}y)/2$, $f = (x + \mathbf{i}y)/2$. Also, denote

$$V = \mathcal{C} \perp \mathbb{F}x \subseteq W.$$

Lemma 2.2.26. *With notation as above, we have*

$$\operatorname{Spin}(W,n) = \langle 1 + a \cdot e, 1 + a \cdot f \mid a \in \mathcal{C}\rangle$$

and

$$\operatorname{Spin}(V,n) = \langle \lambda 1 + a \cdot x \mid \lambda \in \mathbb{F}, a \in \mathcal{C}, n(a) + \lambda^2 = 1\rangle.$$

Proof. First, note that $e \cdot f + f \cdot e = 1$, hence $e \cdot f \cdot e = e$ and $f \cdot e \cdot f = f$ in the Clifford algebra $\mathfrak{Cl}(W, n)$. Besides, $x \cdot x = 1$. For each $a \in \mathcal{C}$, it is easily checked that $(1 + a \cdot e) \cdot \tau(1 + a \cdot e) = (1 + a \cdot e) \cdot (1 + e \cdot a) = 1$, and also $(1 + a \cdot e) \cdot W \cdot (1 + e \cdot a) \subseteq W$, so $1 + a \cdot e, 1 + a \cdot f \in \mathrm{Spin}(W, n)$. Then $G_W := \langle 1 + a \cdot e, 1 + a \cdot f \mid a \in \mathcal{C} \rangle \leq \mathrm{Spin}(W, n)$. Similarly, $G_V := \langle \lambda 1 + a \cdot x \mid \lambda \in \mathbb{F}, a \in \mathcal{C}, n(a) + \lambda^2 = 1 \rangle \leq \mathrm{Spin}(V, n)$.

Now, note that $\mathrm{Spin}(V, n)$ is generated by elements of the form $(\lambda_1 x + a_1) \cdot (\lambda_2 x + a_2) = (\lambda_1 1 + a_1 \cdot x) \cdot (\lambda_2 1 - a_2 \cdot x)$ with $\lambda_i \in \mathbb{F}$, $a_i \in \mathcal{C}$ such that $\lambda_i^2 + n(a_i) = 1$. Therefore, $\mathrm{Spin}(V, n) = G_V$.

Since $(a_1 + \lambda_1 e + \mu_1 f) \cdot (a_2 + \lambda_2 e + \mu_2 f) = (a_1 + \lambda_1 e + \mu_1 f) \cdot x \cdot (-a_2 + \mu_2 e + \lambda_2 f) \cdot x$, it is clear that $\mathrm{Spin}(W, n)$ is generated by the elements of the form $g = (a + \lambda e + \mu f) \cdot x$ with $a \in \mathcal{C}$, $\lambda, \mu \in \mathbb{F}$ and $n(a) + \lambda\mu = 1$, so it suffices to prove that these generators belong to G_W.

• Case $\lambda = \mu$. The generator has the form $g = (a + \lambda x) \cdot x = \lambda 1 + a \cdot x$ (i.e., a generator of G_V). If $n(a) \neq 0$ we can write $\lambda 1 + a \cdot x = (1 + \nu a \cdot f) \cdot (1 + a \cdot e) \cdot (1 + \nu a \cdot f) \in G_W$ with $\nu = \frac{1-\lambda}{n(a)} = \frac{1}{1+\lambda}$ (because $n(a) + \lambda^2 = 1$. This implies in particular that $-1 \in G_W$, because if $a \in \mathcal{C}$ satisfies $n(a) = 1$ and we take $\lambda = 0$, then $-1 = (a \cdot x) \cdot (a \cdot x) = (0 + a \cdot x) \cdot (0 + a \cdot x) \in G_W$. On the other hand, if $n(a) = 0$, then $\lambda \in \{\pm 1\}$ and we can write $\lambda 1 + a \cdot x = \lambda 1 \cdot (1 + \nu a \cdot f) \cdot (1 + \lambda a \cdot e) \cdot (1 + \nu a \cdot f) \in G_W$ with $\nu = \lambda/2$.

• Case $\lambda \neq \mu$. The generator has the form $g = (a + \lambda e + \mu f) \cdot x$. Without loss of generality, we can assume that $\mu \neq 0$ (the case $\lambda \neq 0$ is similar).

Take $\alpha = \mu^2 \in \mathbb{F}^\times$ and $b \in \mathcal{C}$ with $n(b)\alpha = 1$. Then, $(1 + b \cdot e) \cdot (1 + \alpha b \cdot f) \cdot (1 + b \cdot e) = b \cdot (e + \alpha f) = \mu b \cdot (\mu^{-1} e + \mu f) = (\mu^{-1} e + \mu f) \cdot (-\mu b)$, and note that $n(-\mu b) = 1$. In consequence, for any $b \in \mathcal{C}$ with $n(b) = 1$ we have $(\mu^{-1} e + \mu f) \cdot b \in G_W$, and therefore $(\mu^{-1} e + \mu f) \cdot x = ((\mu^{-1} e + \mu f) \cdot b) \cdot (b \cdot x) \in G_W$ (because $b \cdot x \in G_W$ by the case $\lambda = \mu$). Then, $(1 - \mu a \cdot f) \cdot (1 - \mu^{-1} a \cdot e) \cdot g = (\mu^{-1} e + \mu f) \cdot x \in G_W$, and therefore $g \in G_W$. □

The groups $\mathrm{Aut}\, \mathcal{V}_\mathcal{B}$ and $\mathrm{Aut}\, \mathcal{T}_\mathcal{B}$ are explicitly described by the following Theorem.

Theorem 2.2.27. *With the same notation as above, define the linear maps* $\Phi^\pm \colon W \to \mathrm{End}(\mathcal{C} \oplus \mathcal{C})$ *by*

$$\Phi^\pm(a) = \begin{pmatrix} 0 & r_{\bar{a}} \\ l_{\bar{a}} & 0 \end{pmatrix}, \quad \Phi^\pm(x) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Phi^\pm(y) = \begin{pmatrix} \pm\mathbf{i} & 0 \\ 0 & \pm\mathbf{i} \end{pmatrix},$$

where $a \in \mathcal{C}$. *Then, the linear map*

$$\Psi \colon W \to \mathrm{End}(\mathcal{B} \oplus \mathcal{B}), \quad w \mapsto \begin{pmatrix} 0 & \Phi^+(w) \\ \Phi^-(w) & 0 \end{pmatrix},$$

defines an algebra isomorphism $\Psi\colon \mathfrak{Cl}(W,n) \to \mathrm{End}(\mathcal{B}\oplus\mathcal{B})$. *Moreover, if we identify each* $\varphi \in \mathrm{Aut}\,\mathcal{V}_{\mathcal{B}}$ *with*

$$\begin{pmatrix} \varphi^+ & 0 \\ 0 & \varphi^- \end{pmatrix} \in \mathrm{End}(\mathcal{B}\oplus\mathcal{B}),$$

then Ψ *restricts to a group isomorphism* $\mathrm{Spin}(W,n) \to \langle \varphi_a, \widehat{\varphi}_a \mid a \in \mathcal{C}\rangle \leq \mathrm{Aut}\,\mathcal{V}_{\mathcal{B}}$*, which in turn restricts to a group isomorphism* $\mathrm{Spin}(V,n) \to \mathrm{Aut}\,\mathcal{T}_{\mathcal{B}}$*. Furthermore,* $\mathrm{Aut}\,\mathcal{V}_{\mathcal{B}} \cong \Gamma^+(W,n)/\langle -\mathbf{i}z\rangle$ *with* $\langle -\mathbf{i}z\rangle \cong \mathbb{Z}_2$*, where* $z = \Psi^{-1}(c_{\mathbf{i}})$*. (Recall that* $\Gamma^+(W,n)$ *is the even Clifford group.)*

Proof. Fix $a \in \mathcal{C}$. First, note that $\Psi(a)^2 = n(a)\mathrm{id}$, $\Psi(x)^2 = \Psi(y)^2 = \mathrm{id}$. Also, the matrices $\Psi(a)$, $\Psi(x)$ and $\Psi(y)$ anticommute, so we have $\Psi(w)^2 = n(w)\mathrm{id}$ for each $w \in W$. Therefore, the linear map $W \to \mathrm{End}(\mathcal{B}\oplus\mathcal{B})$, $w \mapsto \Psi(w)$, extends to an algebra homomorphism $\mathfrak{Cl}(W,n) \to \mathrm{End}(\mathcal{B}\oplus\mathcal{B})$. Since $\mathfrak{Cl}(W,n)$ is simple and has the same dimension as $\mathrm{End}(\mathcal{B}\oplus\mathcal{B})$, it follows that Ψ is an isomorphism.

It can be checked that Ψ sends $\lambda 1 + a\cdot x \mapsto \varphi_{a,\lambda}$ (where $n(a)+\lambda^2 = 1$). We know by Theorem 2.2.24 that $\mathrm{Aut}\,\mathcal{T}_{\mathcal{B}} = \langle \varphi_{a,\lambda} \mid a \in \mathcal{C}, \lambda \in \mathbb{F}, n(a)+\lambda^2 = 1\rangle$, and on the other hand, by Lemma 2.2.26 we have $\mathrm{Spin}(V,n) = \langle \lambda 1 + a\cdot x \mid a \in \mathcal{C}, \lambda \in \mathbb{F}, n(a)+\lambda^2 = 1\rangle$, so that Ψ restricts to an isomorphism $\mathrm{Spin}(V,n) \to \mathrm{Aut}\,\mathcal{T}_{\mathcal{B}}$.

Furthermore, Ψ sends $1 + a\cdot e \mapsto \varphi_a$, $1 + a\cdot f \mapsto \widehat{\varphi}_a$. By Theorem 2.2.23 we have that $\mathrm{Aut}\,\mathcal{V}_{\mathcal{B}} = \langle \varphi_a, \widehat{\varphi}_a, c_\lambda \mid a \in \mathcal{C},\ \lambda \in \mathbb{F}^\times\rangle$ and, by Lemma 2.2.26, we have $\mathrm{Spin}(W,n) = \langle 1 + a\cdot e, 1 + a\cdot f \mid a \in \mathcal{C}\rangle$. Consequently, Ψ restricts to a group isomorphism $\mathrm{Spin}(W,n) \to \langle \varphi_a, \widehat{\varphi}_a \mid a \in \mathcal{C}\rangle$. Moreover, we obtain a group epimorphism

$$\Lambda\colon \mathbb{F}^\times \times \mathrm{Spin}(W,n) \to \mathrm{Aut}\,\mathcal{V}_{\mathcal{B}}, \quad (\lambda, x) \mapsto c_\lambda \circ \Psi(x). \tag{2.2.4}$$

It is well-known that $Z(\mathrm{Spin}(W,n)) = \langle z\rangle \cong \mathbb{Z}_4$, with $z^2 = -1$ and $z \notin \mathbb{F}$ (also $Z(\mathfrak{Cl}(W,n)_{\bar 0}) = \mathbb{F}1 + \mathbb{F}z$). Since Ψ restricts to an isomorphism $\mathrm{Spin}(W,n) \to \langle \varphi_a, \widehat{\varphi}_a\rangle \leq \mathrm{Aut}\,\mathcal{V}_{\mathcal{B}}$, replacing z by $-z$ if necessary, we have $\Psi(z)^\pm = \pm\mathbf{i}\mathrm{id}_{\mathcal{B}}$ (because $\Psi(z) \in Z(\mathrm{Aut}\,\mathcal{V}_{\mathcal{B}}) = \langle c_\lambda \mid \lambda \in \mathbb{F}^\times\rangle$ and $z^4 = 1$). Hence $\Psi(z) = c_{\mathbf{i}}$ (note that this implies that $\Psi^{-1}(c_\lambda) = \frac{1}{2}(\lambda+\lambda^{-1})1 + \frac{1}{2\mathbf{i}}(\lambda - \lambda^{-1})z$).

We claim that $\ker\Lambda = \langle(-\mathbf{i}, z)\rangle$. It is clear that $\langle(-\mathbf{i}, z)\rangle \leq \ker\Lambda$. Fix $(\lambda, x) \in \ker\Lambda$, so that $\Lambda(\lambda, x) = c_\lambda \circ \Psi(x) = 1$, i.e.,

$$\begin{pmatrix} \lambda\mathrm{id}^+ & 0 \\ 0 & \lambda^{-1}\mathrm{id}^- \end{pmatrix}\begin{pmatrix} \Psi(x)^+ & 0 \\ 0 & \Psi(x)^- \end{pmatrix} = \begin{pmatrix} \mathrm{id}^+ & 0 \\ 0 & \mathrm{id}^- \end{pmatrix},$$

which in turn implies that $\Psi(x)^\pm \in \mathbb{F}^\times\mathrm{id}$ and $\Psi(x) \in Z(\mathrm{Aut}\,\mathcal{V}_{\mathcal{B}})$. Recall again that Ψ restricts to an isomorphism $\mathrm{Spin}(W,n) \to \langle \varphi_a, \widehat{\varphi}_a \mid a \in \mathcal{C}\rangle \leq$

$\operatorname{Aut}\mathcal{V}_{\mathcal{B}}$, so that $x \in Z(\mathrm{Spin}(W,n)) = \langle z\rangle$ and therefore $\ker\Lambda = \langle(-\mathbf{i},z)\rangle \cong \mathbb{Z}_4$. Therefore, we obtain $(\mathbb{F}^\times \times \mathrm{Spin}(W,n))/\langle(-\mathbf{i},z)\rangle \cong \operatorname{Aut}\mathcal{V}_{\mathcal{B}}$.

Define a new epimorphism by means of

$$\widetilde{\Lambda}\colon \mathbb{F}^\times \times \mathrm{Spin}(W,n) \to \Gamma^+(W,n), \quad (\lambda,x) \mapsto \lambda x. \tag{2.2.5}$$

Then, $\ker\widetilde{\Lambda} = \langle(-1,-1)\rangle \cong \mathbb{Z}_2$ and $(\mathbb{F}^\times \times \mathrm{Spin}(W,n))/\langle(-1,-1)\rangle \cong \Gamma^+(W,n)$.

Finally, note that $(-1,-1) \in \ker\Lambda$. Hence, the epimorphism Λ factors through $\widetilde{\Lambda}$, and we obtain an epimorphism $\Gamma^+(W,n) \to \operatorname{Aut}\mathcal{V}_{\mathcal{B}}$ with kernel $\widetilde{\Lambda}\langle(-\mathbf{i},z)\rangle = \langle -\mathbf{i}z\rangle \cong \mathbb{Z}_2$. □

Chapter 3

Gradings on bi-Cayley systems

In this Chapter we give classifications of the fine gradings on the bi-Cayley pair and on the bi-Cayley triple system, which are some of the main original results of this thesis. Also, all these fine gradings are described given by their universal grading groups, their Weyl groups are computed, and we determine the (fine) gradings on $\mathfrak{e}_6$ induced by the fine gradings on the bi-Cayley pair.

The main results in this chapter are Theorem 3.3.3 and Theorem 3.4.4, that classify the fine gradings, up to equivalence, on the bi-Cayley Jordan pair and triple system, respectively.

3.1 Construction of fine gradings on the bi-Cayley pair

Given a grading on $\mathcal{V}_{\mathcal{B}}$ such that $\mathcal{C}_i^\sigma$ are graded subspaces for $i = 1, 2$, and $\sigma = \pm$, we will denote by $\deg_i^\sigma$ the restriction of deg to $\mathcal{C}_i^\sigma$.

Recall that the trace of $\mathcal{V}_{\mathcal{B}}$ is homogeneous for each grading, i.e., we have that $t(x^+, y^-) \neq 0$ implies $\deg(y^-) = -\deg(x^+)$ for x^+, y^- homogeneous elements of the grading. Hence, to give a grading on $\mathcal{V}_{\mathcal{B}}$ it suffices to give the degree map on $\mathcal{V}_{\mathcal{B}}^+$.

Example 3.1.1. Since $\operatorname{char} \mathbb{F} \neq 2$, we can take a Cayley-Dickson basis $\{x_i\}_{i=0}^7$ of $\mathcal{C}$, as in Section 1.4. Let $\deg_{\mathcal{C}}$ denote the associated degree of the $\mathbb{Z}_2^3$-grading on $\mathcal{C}$. Then, we will call the set $\{(x_i, 0)^\sigma, (0, x_i)^\sigma \mid \sigma = \pm\}_{i=0}^7$ a *Cayley-Dickson basis* of $\mathcal{V}_{\mathcal{B}}$. It is checked directly that we have a fine $\mathbb{Z}^2 \times \mathbb{Z}_2^3$-grading on $\mathcal{V}_{\mathcal{B}}$ that is given by $\deg_1^+(x_i) = (1, 0, \deg_{\mathcal{C}}(x_i)) = -\deg_1^-(x_i)$ and $\deg_2^+(x_i) = (0, 1, \deg_{\mathcal{C}}(x_i)) = -\deg_2^-(x_i)$. This grading will be called the *Cayley-Dickson grading* on $\mathcal{V}_{\mathcal{B}}$ (and is fine because its homogeneous components have dimension 1).

Note that, for the Cayley-Dickson basis, the triple product is determined by:

i) $\{(x_i,0),(x_j,0),(x_k,0)\} = (2\delta_{ij}x_k + 2\delta_{jk}x_i - 2\delta_{ik}x_j, 0)$,

ii) $\{(x_i,0),(0,x_j),(x_k,0)\} = 0$,

iii) $\{(x_i,0),(x_j,0),(0,x_k)\} = (0, 2\delta_{ij}x_k - (x_k x_i)\bar{x}_j)$.

The rest of the cases are obtained by symmetry in the first and third components of the triple product, and using the automorphism $\bar{\tau}_{12}\colon \mathcal{C}\oplus 0 \leftrightarrow 0\oplus\mathcal{C}$, $(x_1,x_2)^\sigma \mapsto (\bar{x}_2,\bar{x}_1)^\sigma$.

Example 3.1.2. Let $\{z_i\}_{i=1}^8$ be a Cartan basis of $\mathcal{C}$, as in Section 1.4. Then, $\{(z_i,0)^\sigma,(0,z_i)^\sigma \mid \sigma=\pm\}_{i=1}^8$ will be called a *Cartan basis* of $\mathcal{V}_\mathcal{B}$. It is checked directly that we have a fine $\mathbb{Z}^6$-grading on $\mathcal{V}_\mathcal{B}$ determined by

deg	$\mathcal{C}_1^+$	$\mathcal{C}_2^+$
e_1	$(0,0,1,0,0,0)$	$(0,0,0,0,1,0)$
e_2	$(0,0,0,1,0,0)$	$(0,0,0,0,0,1)$
u_1	$(1,0,0,1,0,0)$	$(1,0,0,0,1,0)$
u_2	$(0,1,0,1,0,0)$	$(0,1,0,0,1,0)$
u_3	$(-1,-1,1,0,-1,1)$	$(-1,-1,1,-1,0,1)$
v_1	$(-1,0,1,0,0,0)$	$(-1,0,0,0,0,1)$
v_2	$(0,-1,1,0,0,0)$	$(0,-1,0,0,0,1)$
v_3	$(1,1,0,1,1,-1)$	$(1,1,-1,1,1,0)$

and $\deg(x^+)+\deg(y^-)=0$ for any elements x^+, y^- of the Cartan basis such that $t(x^+,y^-)\neq 0$. (Notice that the projection on the two first coordinates of the group coincides with the Cartan $\mathbb{Z}^2$-grading on $\mathcal{C}$, which behaves well with respect to the product on $\mathcal{V}_\mathcal{B}$, so it suffices to show that the projection on the last four coordinates behaves well with respect to the product.) This grading will be called the *Cartan grading* on $\mathcal{V}_\mathcal{B}$ (and is fine because its homogeneous components have dimension 1).

We will prove now that the grading groups of these gradings are their universal groups.

Proposition 3.1.3. *The Cayley-Dickson grading on the bi-Cayley pair has universal group* $\mathbb{Z}^2\times\mathbb{Z}_2^3$.

Proof. Let $\{x_i\}_{i=0}^7$ be a Cayley-Dickson basis of $\mathcal{C}$ with $x_0=1$. Let Γ be a realization as a G-grading of the associated Cayley-Dickson grading on $\mathcal{V}_\mathcal{B}$, for some abelian group G. For each element x of the Cayley-Dickson basis of

$\mathcal{V}_{\mathcal{B}}$ we have $t(x^+, x^-) \neq 0$, and since the trace is homogeneous, it has to be $\deg(x^+)+\deg(x^-) = 0$. Define $g_i = \deg_1^+(x_i) = -\deg_1^-(x_i)$, $a = g_0 = \deg_1^+(1)$ and $b = \deg_2^+(1)$. If $i \neq j$, then $Q^+_{(x_i,0)^+}(x_j, 0)^- = (-x_j, 0)^+$, so that we have $2g_i = 2g_j$. Thus, $a_i := g_i - g_0$ has order ≤ 2, and we have $\deg_1^+(x_i) = a_i + a$. If $i \neq 0$, $\{(x_i, 0)^+, (1, 0)^-, (0, 1)^+\} = (0, -x_i)^+$, so $\deg_2^+(x_i) = a_i + b$. If $0 \neq i \neq j \neq 0$, we have $\{(x_i, 0)^+, (x_j, 0)^-, (0, 1)^+\} = (0, -x_i\bar{x}_j)^+$, and we get $\deg_2^+(x_ix_j) = (a_i + a_j) + b$, and also $\{(0, x_i)^+, (0, 1)^-, (x_j, 0)^+\} = (-x_ix_j, 0)^+$, so $\deg_1^+(x_ix_j) = (a_i + a_j) + a$. Therefore, $\deg_{\mathcal{C}}(x_i) := a_i$ defines a group grading by $\langle a_i \rangle$ on $\mathcal{C}$ that is a coarsening of the $\mathbb{Z}_2^3$-grading on $\mathcal{C}$. Therefore, there is an epimorphism $\mathbb{Z}^2 \times \mathbb{Z}_2^3 \to G$ that sends $(1, 0, \bar{0}, \bar{0}, \bar{0}) \mapsto a$, $(0, 1, \bar{0}, \bar{0}, \bar{0}) \mapsto b$, and restricts to an epimorphism $0 \times \mathbb{Z}_2^3 \to \langle a_i \rangle$, so we conclude that $\mathbb{Z}^2 \times \mathbb{Z}_2^3$ is the universal group. □

Proposition 3.1.4. *The Cartan grading on the bi-Cayley pair has universal group* $\mathbb{Z}^6$.

Proof. Let $\{e_i, u_j, v_j \mid i = 1, 2; j = 1, 2, 3\}$ be a Cartan basis of $\mathcal{C}$. Let Γ be a realization as a G-grading of the associated Cartan grading on $\mathcal{V}_{\mathcal{B}}$, for some abelian group G. Recall that if $t(x^+, y^-) \neq 0$ for homogeneous elements x^+, y^-, since the trace is homogeneous we have $\deg(x^+) + \deg(y^-) = 0$, and therefore the degree is determined by its values in $\mathcal{V}_{\mathcal{B}}^+$. Put $a_1 = \deg_1^+(e_1)$, $a_2 = \deg_1^+(e_2)$, $b_1 = \deg_2^+(e_1)$, $b_2 = \deg_2^+(e_2)$. To simplify the degree map, define g_i $(i = 1, 2)$ by means of $\deg_1^+(u_1) = g_1 + a_2$, $\deg_1^+(u_2) = g_2 + a_2$, $g_3 = -g_1 - g_2$. Then we deduce:

$$\begin{aligned}
&\{(e_1, 0)^-, (u_i, 0)^+, (0, e_1)^-\} = (0, u_i)^- \quad (i = 1, 2)\\
&\qquad \Rightarrow \deg_2^+(v_i) = -g_i + b_2 \quad (i = 1, 2),\\
&\{(v_i, 0)^+, (e_2, 0)^-, (0, e_2)^+\} = (0, -v_i)^+ \quad (i = 1, 2)\\
&\qquad \Rightarrow \deg_1^+(v_i) = -g_i + a_1 \quad (i = 1, 2),\\
&\{(u_i, 0)^+, (e_1, 0)^-, (0, e_1)^+\} = (0, -u_i)^+ \quad (i = 1, 2)\\
&\qquad \Rightarrow \deg_2^+(u_i) = g_i + b_1 \quad (i = 1, 2),\\
&\{(u_2, 0)^+, (e_2, 0)^-, (0, u_1)^+\} = (0, -v_3)^+ \Rightarrow \deg_2^+(v_3) = -g_3 - a_1 + a_2 + b_1,\\
&\{(v_2, 0)^+, (e_1, 0)^-, (0, v_1)^+\} = (0, -u_3)^+ \Rightarrow \deg_2^+(u_3) = g_3 + a_1 - a_2 + b_2,\\
&\{(0, u_3)^+, (0, e_2)^-, (e_2, 0)^+\} = (-u_3, 0)^+ \Rightarrow \deg_1^+(u_3) = g_3 + a_1 - b_1 + b_2,\\
&\{(0, v_3)^+, (0, e_1)^-, (e_1, 0)^+\} = (-v_3, 0)^+ \Rightarrow \deg_1^+(v_3) = -g_3 + a_2 + b_1 - b_2.
\end{aligned}$$

The relations above show that the set $\{a_1, a_2, b_1, b_2, g_1, g_2\}$ generates G. Hence, there is an epimorphism $\mathbb{Z}^6 \to G$ determined by

$$\begin{aligned}
&(1, 0, 0, 0, 0, 0) \mapsto g_1, \quad (0, 0, 1, 0, 0, 0) \mapsto a_1, \quad (0, 0, 0, 0, 1, 0) \mapsto b_1,\\
&(0, 1, 0, 0, 0, 0) \mapsto g_2, \quad (0, 0, 0, 1, 0, 0) \mapsto a_2, \quad (0, 0, 0, 0, 0, 1) \mapsto b_2,
\end{aligned}$$

and therefore the universal group is $\mathbb{Z}^6$. □

3.2 Construction of fine gradings on the bi-Cayley triple system

Example 3.2.1. Consider a Cayley-Dickson basis $\{x_i\}_{i=0}^7$ of $\mathcal{C}$ and denote by $\deg_{\mathcal{C}}$ the degree map of the associated $\mathbb{Z}_2^3$-grading. Then $\{(x_i, 0), (0, x_i)\}_{i=0}^7$ will be called a *nonisotropic Cayley-Dickson basis* of $\mathcal{T_B}$. It can be checked that we have a fine $\mathbb{Z}_2^5$-grading on $\mathcal{T_B}$ given by $\deg(x_i, 0) = (\bar{1}, \bar{0}, \deg_{\mathcal{C}}(x_i))$ and $\deg(0, x_i) = (\bar{0}, \bar{1}, \deg_{\mathcal{C}}(x_i))$. This grading will be called the *nonisotropic Cayley-Dickson grading* on $\mathcal{T_B}$. (The isotropy is relative to the quadratic form $q = n \perp n$ of $\mathcal{B}$.)

Example 3.2.2. Let $\{x_i\}_{i=0}^7$ be as above. Fix $\mathbf{i} \in \mathbb{F}$ with $\mathbf{i}^2 = -1$. Then, $\{(x_i, \pm\mathbf{i}\bar{x}_i)\}_{i=0}^7$ will be called an *isotropic Cayley-Dickson basis* of $\mathcal{T_B}$. It can be checked that we have a fine $\mathbb{Z} \times \mathbb{Z}_2^3$-grading on $\mathcal{T_B}$ given by $\deg(x_i, \pm\mathbf{i}\bar{x}_i) = (\pm 1, \deg_{\mathcal{C}}(x_i))$. This grading will be called the *isotropic Cayley-Dickson grading* on $\mathcal{T_B}$. (The isotropy is relative to the quadratic form $q = n \perp n$ of $\mathcal{B}$.)

Example 3.2.3. Let $\{z_i\}_{i=1}^8$ be a Cartan basis of $\mathcal{C}$. Then, we will say that $\{(z_i, 0), (0, z_i)\}_{i=1}^8$ is a *Cartan basis* of $\mathcal{T_B}$. It can be checked that we have a fine $\mathbb{Z}^4$-grading where the degree map is given by the following table:

deg	$\mathcal{C}_1$	$\mathcal{C}_2$
e_1	$(0,0,1,0)$	$(0,0,0,-1)$
e_2	$(0,0,-1,0)$	$(0,0,0,1)$
u_1	$(1,0,-1,0)$	$(1,0,0,-1)$
u_2	$(0,1,-1,0)$	$(0,1,0,-1)$
u_3	$(-1,-1,1,2)$	$(-1,-1,2,1)$
v_1	$(-1,0,1,0)$	$(-1,0,0,1)$
v_2	$(0,-1,1,0)$	$(0,-1,0,1)$
v_3	$(1,1,-1,-2)$	$(1,1,-2,-1)$

(Notice that the projection $\mathbb{Z}^4 \to \mathbb{Z}^2$ of the degree on the first two coordinates induces the Cartan $\mathbb{Z}^2$-grading on $\mathcal{C}$, so it suffices to show that the last two coordinates behave well with respect to the product, and this is easily checked). This grading will be called the *Cartan grading* on $\mathcal{T_B}$.

Note that the homogeneous elements of the nonisotropic Cayley-Dickson grading on $\mathcal{T_B}$ are in the orbits $\mathcal{O}_2(\lambda)$ with $\lambda \in \mathbb{F}^\times$, the ones of the isotropic

Cayley-Dickson grading on $\mathcal{T}_{\mathcal{B}}$ are in the orbit $\mathcal{O}_2(0)$, and the ones of the Cartan grading on $\mathcal{T}_{\mathcal{B}}$ are in the orbit $\mathcal{O}_1$ (see Lemma 2.2.21); hence these three gradings cannot be equivalent. This can also be seen from their universal groups, as follows:

Proposition 3.2.4. *The nonisotropic Cayley-Dickson grading on the bi-Cayley triple system has universal group $\mathbb{Z}_2^5$.*

Proof. Let $\{x_i\}_{i=0}^7$ be a Cayley-Dickson basis of $\mathcal{C}$ with $x_0 = 1$. Consider a realization of the nonisotropic Cayley-Dickson grading on $\mathcal{T}_{\mathcal{B}}$ as G-grading for some abelian group G. Since the trace t is homogeneous and $t((x_i,0),(x_i,0)) \neq 0 \neq t((0,x_i),(0,x_i))$, it follows that all the elements of G have order ≤ 2. Call $a = \deg(1,0)$, $b = \deg(0,1)$, $g_i = \deg(x_i,0)$ and $a_i = a + g_i$ for $0 \leq i \leq 7$. Note that we have $\deg(x_i,0) = g_i = a_i + a$. Since $\{(1,0),(x_i,0),(0,1)\} = (0,x_i)$ for each i, we have $\deg(0,x_i) = a + g_i + b = a_i + b$. If $i \neq j$ with $i,j \neq 0$, we have $\{(x_i,0),(0,1),(0,x_j)\} = (x_ix_j,0)$; hence $\deg(x_ix_j,0) = g_i + b + (a_j + b) = (a_i + a_j) + a$, and it follows that $\deg_{\mathcal{C}}(x_i) := a_i$ defines a coarsening of the $\mathbb{Z}_2^3$-grading on $\mathcal{C}$. It is clear that the G-grading is induced from the $\mathbb{Z}_2^5$-grading by an epimorphism $\mathbb{Z}_2^5 \to G$ that sends $(\bar{1},\bar{0},\bar{0},\bar{0},\bar{0}) \mapsto a$, $(\bar{0},\bar{1},\bar{0},\bar{0},\bar{0}) \mapsto b$ and restricts to some epimorphism $0 \times \mathbb{Z}_2^3 \to \langle a_i \rangle$. We can conclude that $\mathbb{Z}_2^5$ is (isomorphic to) the universal group of the nonisotropic Cayley-Dickson grading on $\mathcal{T}_{\mathcal{B}}$. □

Proposition 3.2.5. *The isotropic Cayley-Dickson grading on the bi-Cayley triple system has universal group $\mathbb{Z} \times \mathbb{Z}_2^3$.*

Proof. Let $\{x_i\}_{i=0}^7$ be a Cayley-Dickson basis of $\mathcal{C}$ with $x_0 = 1$. Consider a realization of the isotropic Cayley-Dickson grading on $\mathcal{T}_{\mathcal{B}}$ as G-grading for some abelian group G. Call $g_i = \deg(x_i, \mathbf{i}\bar{x}_i)$ and $a_i = g_i - g_0$. Since the trace t is homogeneous and $t((x_i,\mathbf{i}\bar{x}_i),(x_i,-\mathbf{i}\bar{x}_i)) \neq 0$, it follows that $\deg(x_i,-\mathbf{i}\bar{x}_i) = -g_i$. For each $i \neq 0$ we have $Q_{(1,\mathbf{i}1)}(x_i,-\mathbf{i}\bar{x}_i) = -2(x_i,\mathbf{i}\bar{x}_i)$, so $2g_0 = 2g_i$. Thus, a_i has order ≤ 2. Moreover, $\deg(x_i,\mathbf{i}\bar{x}_i) = a_i + g_0$, $\deg(x_i,-\mathbf{i}\bar{x}_i) = a_i - g_0$. But also, for each $i \neq j$ with $i,j \neq 0$, we have $\bar{x}_i = -x_i$, $\bar{x}_j = -x_j$, $x_ix_j = -x_jx_i$, from where we get $\{(1,\mathbf{i}1),(x_i,\mathbf{i}\bar{x}_i),(x_j,-\mathbf{i}\bar{x}_j)\} = -2(x_ix_j,\mathbf{i}\overline{x_ix_j})$, and taking degrees we obtain $\deg(x_ix_j,\mathbf{i}\overline{x_ix_j}) = (a_i + a_j) + g_0$. In consequence, $\deg_{\mathcal{C}}(x_i) := a_i$ defines a coarsening of the $\mathbb{Z}_2^3$-grading on $\mathcal{C}$. Therefore, the G-grading is induced from the $\mathbb{Z} \times \mathbb{Z}_2^3$-grading by an epimorphism $\mathbb{Z} \times \mathbb{Z}_2^3 \to G$ that sends $(1,\bar{0},\bar{0},\bar{0}) \mapsto g_0$ and restricts to some epimorphism $0 \times \mathbb{Z}_2^3 \to \langle a_i \rangle$. We can conclude that $\mathbb{Z} \times \mathbb{Z}_2^3$ is (isomorphic to) the universal group of the isotropic Cayley-Dickson grading on $\mathcal{T}_{\mathcal{B}}$. □

Proposition 3.2.6. *The Cartan grading on the bi-Cayley triple system has universal group $\mathbb{Z}^4$.*

Proof. Let $\{e_i, u_j, v_j | i = 1, 2; j = 1, 2, 3\}$ be a Cartan basis of $\mathcal{C}$. Consider a realization of the Cartan grading on $\mathcal{T}_{\mathcal{B}}$ as G-grading for some abelian group G. Call $a = \deg(e_1, 0)$, $b = \deg(0, e_2)$, and $h_i = \deg(u_i, 0)$ for $i = 1, 2$. We claim that $\{a, b, g_1, g_2\}$ generate G. Indeed, since the trace is homogeneous, we get $\deg(e_2, 0) = -a$, $\deg(0, e_1) = -b$, and $\deg(v_i, 0) = -h_i$ for $i = 1, 2$. Since $\{(v_1, 0), (v_2, 0), (0, e_2)\} = (0, u_3)$, we deduce that $\deg(0, u_3) = -h_1 - h_2 + b = -\deg(0, v_3)$. Also, from $\{(0, e_2), (0, u_3), (e_2, 0)\} = (u_3, 0)$, we obtain $\deg(u_3, 0) = -h_1 - h_2 - a + 2b = -\deg(v_3, 0)$. We have proved the claim. It is clear that the G-grading is induced from the $\mathbb{Z}^4$-grading by an epimorphism $\mathbb{Z}^4 \to G$ that sends $(1, 0, 0, 0) \mapsto g_1$, $(0, 1, 0, 0) \mapsto g_2$, $(0, 0, 1, 0) \mapsto a$ and $(0, 0, 0, 1) \mapsto b$, and we can conclude that $\mathbb{Z}^4$ is (isomorphic to) the universal group of the Cartan grading on $\mathcal{T}_{\mathcal{B}}$. □

3.3 Classification of fine gradings on the bi-Cayley pair

Given a grading on a semisimple Jordan pair, by Remark 2.1.23, any homogeneous element can be completed to a maximal orthogonal system of homogeneous idempotents. In the case of the bi-Cayley pair, since the capacity is 2, it will consist either of two idempotents of rank 1, or one idempotent of rank 2. We will cover these possibilities with the following Lemmas.

Lemma 3.3.1. *Let Γ be a fine grading on the bi-Cayley pair such that there is some homogeneous element of rank 1. Then Γ is equivalent to the Cartan grading (Example 3.1.2).*

Proof. Write $\mathcal{V} = \mathcal{V}_{\mathcal{B}}$ for short. First, we complete the homogeneous element to a set consisting of two homogeneous orthogonal idempotents of rank 1. By Theorem 1.7.2, we can assume without loss of generality that the homogeneous orthogonal idempotents are (c_1^+, c_2^-) and (c_2^+, c_1^-), where $c_i = (e_i, 0) \in \mathcal{B}$ and e_i are nontrivial orthogonal idempotents of $\mathcal{C}$ with $e_1 + e_2 = 1$. We will consider the Peirce decomposition $\mathcal{C} = \mathbb{F}e_1 \oplus \mathbb{F}e_2 \oplus U \oplus V$ associated to the idempotents e_1 and e_2. Since the generic trace is homogeneous,

$$\begin{aligned} &f(x^\sigma, y^{-\sigma}, z^\sigma) := t(x, y)z + t(z, y)x - \{x, y, z\} \\ &= (n(x_1, z_1)y_1 + \bar{y}_2(x_2 z_1 + z_2 x_1), n(x_2, z_2)y_2 + (x_2 z_1 + z_2 x_1)\bar{y}_1) \end{aligned} \tag{3.3.1}$$

is a homogeneous map too. By Remark 2.1.26, $K^\sigma = \ker(t_{c_1}) \cap \ker(t_{c_2})$ is a graded subspace of $\mathcal{V}^\sigma$. For each homogeneous $z^+ \in K^+$, we have $n(e_1, z_1) = t(c_1, z) = 0$ and $f(c_1^+, c_2^-, z^+) = (0, z_2 e_1)^+$ is homogeneous. Note that $(0 \oplus \mathcal{C})^\sigma \subseteq K^\sigma$, so there are homogeneous elements $\{(x_i, y_i)\}_{i=1}^8$ of

$\mathcal{B}^+$ such that $\{y_i\}_{i=1}^8$ is a basis of $\mathcal{C}$. Thus, the subspace $(0 \oplus \mathcal{C}e_1)^+ = \sum_{i=1}^8 \mathbb{F}(0, y_i e_1)^+$ is graded. Similarly, $(0 \oplus \mathcal{C}e_1)^\sigma$ and $(0 \oplus \mathcal{C}e_2)^\sigma$ are graded for $\sigma = \pm$. Hence, $\mathcal{C}_2^\sigma = (0 \oplus \mathcal{C})^\sigma$ is graded, and in consequence $\mathcal{C}_1^\sigma = (\mathcal{C} \oplus 0)^\sigma = \bigcap_{x \in \mathcal{C}_2^{-\sigma}} \ker(t_x)$ is graded too.

We claim that the homogeneous elements of $\mathcal{C}_i^+$ and $\mathcal{C}_i^-$ coincide. Indeed, take homogeneous elements $x^+ = (x_1, 0)$ and $z^+ = (z_1, 0)$ of $\mathcal{C}_1^+$ such that $n(x_1, z_1) = 1$. Then, for any homogeneous element $y^- = (y_1, 0)$ of $\mathcal{C}_1^-$, $f(x^+, y^-, z^+) = y^+$ is homogeneous too, and hence the homogeneous elements of $\mathcal{C}_1^+$ and $\mathcal{C}_1^-$ coincide; and similarly this is true for $\mathcal{C}_2^+$ and $\mathcal{C}_2^-$. Since Γ is fine, the supports Supp $\mathcal{C}_i^\sigma$ are disjoint (because otherwise we could obtain a refinement of Γ combining it with the $\mathbb{Z}^2$-grading: $\mathcal{V}_{(\sigma 1,0)} = \mathcal{C}_1^\sigma$, $\mathcal{V}_{(0,\sigma 1)} = \mathcal{C}_2^\sigma$.

From now on, we can omit the index σ, because the homogeneous components of $\mathcal{V}^+$ coincide with those of $\mathcal{V}^-$. The rest of this proof will be used in the proof of Lemma 3.4.1.

Recall that $(0 \oplus \mathcal{C}e_i)$ are graded subspaces, where $\mathcal{C}e_1$ and $\mathcal{C}e_2$ are isotropic subspaces of $\mathcal{C}$. Since the trace is homogeneous, there is a homogeneous basis $\{(0, x_i), (0, y_i)\}_{i=1}^4$ of $\mathcal{C}_2$ such that $\{x_i, y_i\}_{i=1}^4$ is a basis of $\mathcal{C}$ consisting of four orthogonal hyperbolic pairs, that is, such that $n(x_i, y_j) = \delta_{ij}$, $n(x_i, x_j) = 0 = n(y_i, y_j)$. It is not hard to see that there is an element of $\mathrm{O}^+(\mathcal{C}, n)$ that sends the elements $\{x_i, y_i\}_{i=1}^4$ to a Cartan basis $\{e_i, u_j, v_j \mid i = 1, 2; j = 1, 2, 3\}$ of $\mathcal{C}$, and by Lemma 2.2.15, that can be done in $\mathcal{C}_2$ with an automorphism given by a related triple (as in Remark 2.2.14). Hence, we can assume that we have a homogeneous Cartan basis of $\mathcal{C}_2$ (and the subspace $\mathcal{C}_1$ is still graded). Then we have the following graded subspaces:

$$
\begin{aligned}
&f((0, e_1), (0, e_2), \mathcal{C} \oplus 0) = \bar{e}_2(e_1\mathcal{C}) \oplus 0 = (\mathbb{F}e_1 + U) \oplus 0, \\
&f((0, e_2), (0, e_1), \mathcal{C} \oplus 0) = \bar{e}_1(e_2\mathcal{C}) \oplus 0 = (\mathbb{F}e_2 + V) \oplus 0, \\
&f((0, u_1), (0, e_1), (\mathbb{F}e_1 + U) \oplus 0) = (\bar{e}_1(u_1(\mathbb{F}e_1 + U))) \oplus 0 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad = (\mathbb{F}v_2 + \mathbb{F}v_3) \oplus 0, \\
&f((0, v_2), (0, e_2), (\mathbb{F}v_2 + \mathbb{F}v_3) \oplus 0) = (\bar{e}_2(v_2(\mathbb{F}v_2 + \mathbb{F}v_3))) \oplus 0 = (\mathbb{F}u_1) \oplus 0,
\end{aligned}
$$

so $(u_1, 0)$ is homogeneous, and similarly $(u_i, 0)$, $(v_i, 0)$ are homogeneous for $i = 1, 2, 3$. Furthermore, $f((0, u_1), (0, e_2), (v_1, 0)) = (\bar{e}_2(u_1 v_1), 0) = (-e_1, 0)$, so $(e_1, 0)$ and $(e_2, 0)$ are homogeneous. Since Γ is fine, we conclude that Γ is the Cartan grading. □

Lemma 3.3.2. *Let Γ be a fine grading on the bi-Cayley pair such that the nonzero homogeneous elements have rank 2. Then Γ is equivalent to the Cayley-Dickson grading (Example 3.1.1).*

Proof. Write for short $\mathcal{V} = \mathcal{V}_{\mathcal{B}}$. Take a homogeneous element and complete it to a homogeneous idempotent of rank 2. By Remark 1.7.6, we can assume without loss of generality that our homogeneous idempotent is $c_1 = ((1,0)^+, (1,0)^-)$. The subpaces $\mathcal{C}_1^\sigma = \operatorname{im} Q_{(1,0)^\sigma}$ and $\mathcal{C}_2^\sigma = \ker Q_{(1,0)^{-\sigma}}$ are graded. With the same arguments given in the proof of Lemma 3.3.1, we can deduce that the supports $\operatorname{Supp} \mathcal{C}_i^\sigma$ are disjoint and the homogeneous components of $\mathcal{V}^+$ coincide with those of $\mathcal{V}^-$. From now on, we can omit the index σ. The arguments of the rest of this proof will be used in the proof of Lemma 3.4.2.

We can take a homogeneous element $(0,x)$ with $n(x) = 1$ (otherwise, $n(x) = 0$ and $(0,x)$ would have rank 1, a contradiction). By Lemma 2.2.15 and Remark 2.2.14, there is an automorphism of $\mathcal{V}$, given by a related triple, that maps $(1,0) \mapsto (1,0)$, $(0,x) \mapsto (0,1)$. In consequence, we can assume that $(1,0)$ and $(0,1)$ are homogeneous. Recall that the map f in Equation (3.3.1) is homogeneous. From $f((x,0),(1,0),(0,1)) = (0,x)$, it follows that $(x,0)$ is homogeneous if and only if $(0,x)$ is homogeneous, i.e., the homogeneous components coincide in both $\mathcal{C}_1$ and $\mathcal{C}_2$. We can take a homogeneous basis $B = \{(x_i,0),(0,x_i)\}_{i=1}^8$ where $x_1 = 1$ and $n(x_i) = 1$ for all i. Since the trace is homogeneous and the homogeneous components are 1-dimensional (by Theorem 2.1.24), we also have $n(x_i,x_j) = t((x_i,0),(x_j,0)) = 0$, i.e., $\{x_i\}$ is an orthonormal basis of $\mathcal{C}$. Using the map f, it is easy to deduce that $(x_ix_j,0)$ and $(0,x_ix_j)$ are homogeneous for any $1 \le i,j \le 8$, so actually we can assume, without loss of generality, that B is a Cayley-Dickson basis of $\mathcal{V}$. Since Γ is fine, we conclude that Γ is the Cayley-Dickson grading. □

Theorem 3.3.3. *Let Γ be a fine grading on the bi-Cayley pair. Then, Γ is equivalent to either*

- *the Cartan grading, with universal group $\mathbb{Z}^6$,*
- *the Cayley-Dickson grading, with universal group $\mathbb{Z}^2 \times \mathbb{Z}_2^3$.*

Proof. Consequence of Lemmas 3.3.1 and 3.3.2 (and Propositions 3.1.3 and 3.1.4), since they cover all the possibilities. □

3.4 Classification of fine gradings on the bi-Cayley triple system

Recall that we defined the norm of $\mathcal{B}$ as the quadratic form $q\colon \mathcal{B} \to \mathbb{F}$, $q(x,y) := n(x) + n(y)$. Also, we already know (see Section 2.2.3) that $\operatorname{Aut} \mathcal{T}_{\mathcal{B}} \le \mathrm{O}(\mathcal{B}, q)$, and the nonzero isotropic elements of $\mathcal{B}$ are exactly the ones contained in the orbits $\mathcal{O}_1$ and $\mathcal{O}_2(0)$ of $\mathcal{T}_{\mathcal{B}}$.

Lemma 3.4.1. *Let Γ be a fine grading on $\mathcal{T}_{\mathcal{B}}$ with some homogeneous element in the orbit $\mathcal{O}_1$. Then Γ is, up to equivalence, the Cartan grading on $\mathcal{T}_{\mathcal{B}}$ (Example 3.2.3).*

Proof. Let x be homogeneous in the orbit $\mathcal{O}_1$. We claim that we can take a homogeneous element y in the orbit $\mathcal{O}_1$ and such that $t(x, y) = 1$. Indeed, it suffices to consider the grading (Γ, Γ) on the bi-Cayley pair and complete the element x to a homogeneous idempotent (x, y) of the pair (recall that we have $\mathrm{rk}(e^+) = \mathrm{rk}(e^-)$ for any idempotent). Since the trace form is invariant for automorphisms of the pair and all idempotents of rank 1 of the pair are in the same orbit, it follows that $t(x, y) = 1$ (it suffices to check this for an idempotent of rank 1 of the pair).

Up to automorphism, by Lemma 2.2.21, we can assume that $x = (e_1, 0)$ with e_1 a nontrivial idempotent of $\mathcal{C}$. Consider, as usual, the Peirce decomposition of $\mathcal{C}$ relative to the idempotents e_1 and $e_2 := \bar{e}_1$. By Lemma 2.2.21, we know that $n(y_1) = n(y_2) = 0$ and $y_2 y_1 = 0$. Since $n(y_1) = 0$ and $n(e_1, y_1) = t(x, y) = 1$, there is an automorphism given by a related triple (see Lemma 2.2.15) that sends $(e_1, 0) \mapsto (e_1, 0)$, $y \mapsto (e_2, y_2)$. Thus, we can also assume that $y = (e_2, y_2)$. Since $y_2 y_1 = 0$, it follows that $y_2 = \lambda e_1 + v$ with $\lambda \in \mathbb{F}$, $v \in V$. Take $a = -y_2$ and $\mu = 1$ (so $n(a) + \mu^2 = 1$). We have $\varphi_{a,\mu}(e_1, 0) = (e_1, 0)$ and $\varphi_{a,\mu}(e_2, y_2) = (e_2, 0)$. Therefore, we can assume that $(e_i, 0)$ are homogeneous for $i = 1, 2$.

Since the trace is homogeneous, $f(x, y, z) := t(x, y)z + t(z, y)x - \{x, y, z\}$ is a homogeneous map and $\ker(t_x)$ is graded. For any homogeneous $z \in \ker(t_x)$, we have $n(e_1, z_1) = t(x, z) = 0$, and so $f((e_1, 0), (e_2, 0), z) = (n(e_1, z_1)e_2, z_2 e_1) = (0, z_2 e_1)$ is homogeneous. In consequence $(0 \oplus \mathcal{C}e_1)$ is graded. Similarly, $(0 \oplus \mathcal{C}e_2)$ is graded, and hence $\mathcal{C}_2$ is graded. Since the trace is homogeneous, the subspace orthogonal (for the trace) to $\mathcal{C}_2$, which is $\mathcal{C}_1$, is graded too. We can conclude the proof with the same arguments given in the end of the proof of Lemma 3.3.1. □

Lemma 3.4.2. *Let Γ be a fine grading on $\mathcal{T}_{\mathcal{B}}$ with some homogeneous element in some orbit $\mathcal{O}_2(\lambda)$ with $\lambda \neq 0$. Then Γ is, up to equivalence, the nonisotropic Cayley-Dickson grading (Example 3.2.1).*

Proof. It is clear that Γ cannot be equivalent to the Cartan grading, because there is a homogeneous element x in the orbit $\mathcal{O}_2(\lambda)$ with $\lambda \neq 0$ and in the Cartan grading all the homogeneous elements have rank 1. In particular, by Lemma 3.4.1, all nonzero homogeneous elements of Γ must have rank 2. Up to automorphism and up to scalars, we can assume by Lemma 2.2.21 that $x = (1, 0)$. Then, $\mathcal{C}_1 = \operatorname{im} Q_x$ and $\mathcal{C}_2 = \ker Q_x$ are graded subspaces, and we can conclude with the same arguments given in the end of the proof of Lemma 3.3.2. □

Lemma 3.4.3. *Let Γ be a fine grading on $\mathcal{T}_{\mathcal{B}}$ where all the nonzero homogeneous elements are in the orbit $\mathcal{O}_2(0)$. Then Γ is, up to equivalence, the isotropic Cayley-Dickson grading (Example 3.2.2).*

Proof. Take a nonzero homogeneous element $x \in \mathcal{B}$. Since $x \in \mathcal{O}_2(0)$, up to automorphism we can assume that $x = (1, \mathbf{i}1)$ for some $\mathbf{i} \in \mathbb{F}$ with $\mathbf{i}^2 = -1$. Then, $W := \operatorname{im} Q_x = \ker Q_x = \{(z, \mathbf{i}\bar{z}) \mid z \in \mathcal{C}\}$ is a graded subspace. Let $\mathcal{C}_0$ denote the traceless octonions and set $V := \{(z_0, \mathbf{i}\bar{z}_0) \mid z_0 \in \mathcal{C}_0\}$, $W' := \{(z, -\mathbf{i}\bar{z}) \mid z \in \mathcal{C}\}$, $V' := \{(z_0, -\mathbf{i}\bar{z}_0) \mid z_0 \in \mathcal{C}_0\}$, $x' := (1, -\mathbf{i}1)$. Consider the map $t_x : \mathcal{B} \to \mathcal{B}$, $z \mapsto t(x, z)$. Since the trace is homogeneous, $\ker t_x = W \oplus V' = \mathbb{F}x \oplus V \oplus V'$ is a graded subspace. Hence $Q_x(\ker t_x) = V$ is graded too. (Note that V and V' are isotropic subspaces which are paired relative to the trace form, and x is paired with x' too. But in general, $\mathbb{F}x'$, W' and V' are not graded subspaces.) The subspace $V^{\perp} = \mathbb{F}x' \oplus W$ is graded because the trace is homogeneous, so we can take a homogeneous element $\widetilde{x} = x' + \lambda x + v$ with $\lambda \in \mathbb{F}$, $v \in V$. Since $\widetilde{x} \in \mathcal{O}_2(0)$, we have $q(\widetilde{x}) = 0$, so $\lambda = 0$ and $\widetilde{x} = x' + v$. Put $v = (w, \mathbf{i}\bar{w})$ with $w \in \mathcal{C}_0$, so $\widetilde{x} = (1 + w, -\mathbf{i}1 + \mathbf{i}\bar{w})$.

We claim that there is an automorphism such that $\varphi(x) \in \mathbb{F}x$ and $\varphi(\widetilde{x}) \in \mathbb{F}x'$. If $v = 0$ there is nothing to prove, so we can assume $w \neq 0$. We consider two cases.

First, consider the case $n(w) = 0$. Set $\mu = \frac{1}{2}(1 + \mathbf{i})$, $a = \mu w$, $\lambda = 1$. Then $\lambda^2 + n(a) = 1$, and hence $\varphi_{a,\lambda}$ is an automorphism. It is not hard to check that $\varphi_{a,\lambda}(x) = (b, \mathbf{i}b)$ and $\varphi_{a,\lambda}(\widetilde{x}) = (b, -\mathbf{i}b)$, where $b = 1 + \frac{1}{2}(1 - \mathbf{i})w$. Since $n(b) = 1$, by Lemma 2.2.15 we can apply an automorphism given by a related triple that sends $(b, \mathbf{i}b) \mapsto x = (1, \mathbf{i}1)$ and $(b, -\mathbf{i}b) \mapsto x' = (1, -\mathbf{i}1)$, so we are done with this case.

Second, consider the case $n(w) \neq 0$. Take $\lambda, \mu \in \mathbb{F}$ such that $\lambda^2 + \mu^2 n(w) = 1$ and $\mu = \frac{1-2\lambda^2}{2\mathbf{i}\lambda}$. (Replace the expression of μ of the second equation in the first one, multiply by λ^2 to remove denominators, take a solution λ of this new equation, which exists because $\mathbb{F}$ is algebraically closed and is nonzero because $n(w) \neq 0$. Then take μ as in the second equation, which is well defined because $\lambda \neq 0$.) Moreover, it is clear that $2\lambda^2 - 1 \neq 0$, because otherwise we would have $\mu = 0$ and the first equation would not be satisfied. Set $a = \mu w$, so we have $\lambda^2 + n(a) = 1$ and therefore $\varphi_{a,\lambda}$ is an automorphism, that sends $x \mapsto (b, \mathbf{i}b)$, $\widetilde{x} \mapsto (\gamma b, -\mathbf{i}\gamma b)$, where $b = \lambda 1 - \mathbf{i}\mu w$ and $\gamma = (\lambda + \mathbf{i}\mu n(w))\lambda^{-1}$ (this is easy to check using the two equations satisfied by λ and μ). Note that $n(b) = 2\lambda^2 - 1 \neq 0$, so again we can compose with an automorphism given by a related triple to obtain $\varphi(x) \in \mathbb{F}x$ and $\varphi(\widetilde{x}) \in \mathbb{F}x'$.

By the last paragraphs, we can assume that $x = (1, \mathbf{i}1)$ and $x' = (1, -\mathbf{i}1)$ are homogeneous elements. Therefore, $\operatorname{im} Q_x = W$, $Q_x(\ker t_x) = V$, $\operatorname{im} Q_{x'} =$

W' and $Q_{x'}(\ker t_{x'}) = V'$ are graded subspaces (where V, V', W and W' are defined as above). Note that for each $z \in \mathcal{C}_0$, $(z, \mathbf{i}\bar{z}) \in V$ is homogeneous if and only if $(z, -\mathbf{i}\bar{z}) \in V'$ is homogeneous because $Q_x(z, -\mathbf{i}\bar{z}) = -2(z, \mathbf{i}\bar{z})$ and $Q_{x'}(z, \mathbf{i}\bar{z}) = -2(z, -\mathbf{i}\bar{z})$ for any $z \in \mathcal{C}_0$. On the other hand, if $Z = (z, \mathbf{i}\bar{z})$ is homogeneous for some $z \in \mathcal{C}$, then $n(z) \neq 0$, because otherwise we would have $Z \in \mathcal{O}_1$ by Lemma 2.2.21, which is not possible.

Take a homogeneous element $x_1 = (z_1, \mathbf{i}\bar{z}_1) \in V$. Since $n(z_1) \neq 0$, scaling x_1 we can assume that $n(z_1) = 1$. Also, $x_1' := Q_{x'}(x_1) = (z_1, -\mathbf{i}\bar{z}_1) \in V'$ is homogeneous. Since the trace is homogeneous, we can take a homogeneous element $x_2 = (z_2, \mathbf{i}\bar{z}_2) \in V \cap \ker t_x \cap \ker t_{x'} \cap \ker t_{x_1} \cap \ker t_{x_1'}$. Note that $n(z_1, z_2) = 0 = n(1, z_2)$, and scaling x_2 if necessary, we will assume that $n(z_2) = 1$. Then $x_2' = (z_2, -\mathbf{i}\bar{z}_2) \in V'$ is homogeneous. Furthermore, for any homogeneous elements $(y_i, \pm\mathbf{i}\bar{y}_i)$, $i = 1, 2$, we have that $\{(y_1, \mathbf{i}\bar{y}_1), (1, \mathbf{i}1), (y_2, -\mathbf{i}\bar{y}_2)\} = 2(y_1y_2, \mathbf{i}\overline{y_1y_2})$ is homogeneous too. Thus, in our case, $(x_1x_2, \pm\mathbf{i}\overline{x_1x_2})$ are homogeneous. Again, since the trace is homogeneous, we can take homogeneous elements $x_3 = (z_3, \mathbf{i}\bar{z}_3)$ and $x_3' = (z_3, -\mathbf{i}\bar{z}_3)$, with $n(z_3) = 1$ and z_3 orthogonal to $\operatorname{span}\{1, z_1, z_2, z_1z_2\}$. Notice that $\{z_1, z_2, z_3\}$ are homogeneous elements generating a $\mathbb{Z}_2^3$-grading on $\mathcal{C}$, and the elements $\{x, x', x_i, x_i' \mid i = 1, 2, 3\}$ generate an isotropic Cayley-Dickson grading on the bi-Cayley triple system. Note that there is only one orbit of isotropic Cayley-Dickson bases (up to constants) on $\mathcal{T}_{\mathcal{B}}$, because the same is true for Cayley-Dickson bases (up to constants) on $\mathcal{C}$. We can conclude the proof since Γ is fine. □

Theorem 3.4.4. *Any fine grading on the bi-Cayley triple system is equivalent to one of the three following nonequivalent gradings:*

- *the nonisotropic Cayley-Dickson $\mathbb{Z}_2^5$-grading (Example 3.2.1),*
- *the isotropic Cayley-Dickson $\mathbb{Z} \times \mathbb{Z}_2^3$-grading (Example 3.2.2),*
- *the Cartan $\mathbb{Z}^4$-grading (Example 3.2.3).*

Proof. This is a consequence of Lemmas 3.4.1, 3.4.2 and 3.4.3. □

Remark 3.4.5. We already know that the isotropic and nonisotropic Cayley-Dickson gradings on the bi-Cayley triple system are not equivalent. However, the isotropic Cayley-Dickson grading on the bi-Cayley pair (defined in the obvious way) and the (nonisotropic) Cayley-Dickson grading on the bi-Cayley pair are equivalent. This equivalence is given by the restriction of the automorphism in Equation (4.1.1) to the bi-Cayley pair defined on $\mathcal{B} = \iota_2(\mathcal{C}) \oplus \iota_3(\mathcal{C})$.

3.5 Induced gradings on $\mathfrak{e}_6$

It is well-known that $\mathrm{TKK}(\mathcal{V}_{\mathcal{B}}) = \mathfrak{e}_6$ and $\mathrm{TKK}(\mathcal{V}_{\mathbb{A}}) = \mathfrak{e}_7$. Recall that $\dim \mathfrak{e}_6 = 78$ and $\dim \mathfrak{e}_7 = 133$. We will study now the gradings induced by the TKK construction from the fine gradings on $\mathcal{V}_{\mathcal{B}}$ and $\mathcal{V}_{\mathbb{A}}$. Note that the classification of fine gradings, up to equivalence, on all finite-dimensional simple Lie algebras over an algebraically closed field of characteristic 0 is complete ([EK13, Chapters 3-6], [Eld16], [Yu14]). A classification of the fine gradings on $\mathfrak{e}_6$, for the case $\mathbb{F} = \mathbb{C}$, can be found in [DV16].

Proposition 3.5.1. *The Cartan $\mathbb{Z}^6$-grading on the bi-Cayley pair extends to a fine grading with universal group $\mathbb{Z}^6$ and type $(72, 0, 0, 0, 0, 1)$ on $\mathfrak{e}_6$, that is, a Cartan grading on $\mathfrak{e}_6$.*

Proof. This is a consequence of Theorem 2.1.21 and the fact that the only gradings up to equivalence with these universal groups on the Lie algebras are the Cartan gradings. (Recall that Cartan gradings on simple Lie algebras are induced by maximal tori. By [H75, Section 21.3], the maximal tori of $\mathrm{Aut}(\mathfrak{e}_6)$ are conjugate, so their associated $\mathbb{Z}^6$-gradings on $\mathfrak{e}_6$ must be equivalent. □

Proposition 3.5.2. *The Cayley-Dickson $\mathbb{Z}^2 \times \mathbb{Z}_2^3$-grading on the bi-Cayley pair extends to a fine grading with universal group $\mathbb{Z}^2 \times \mathbb{Z}_2^3$ and type $(48, 1, 0, 7)$ on $\mathfrak{e}_6$. (The fourth grading in the list of the classification given in [DV16].)*

Proof. This is a consequence of Theorem 2.1.21, except for the type, which we will now compute. Set $e = (0, 0, \bar{0}, \bar{0}, \bar{0})$ and write $L = \mathfrak{e}_6$, $\mathcal{V} = \mathcal{V}_{\mathcal{B}}$. If $\nu(x, y) \in L_e^0$, it must be $\deg(x^+) + \deg(y^-) = e$ and hence $\mathbb{F}x = \mathbb{F}y$. For elements in the Cayley-Dickson basis of $\mathcal{V}_{\mathcal{B}}$ we have $\{(x_i, 0), (x_i, 0), \cdot\} = m_{2,1}$ and $\{(0, x_i), (0, x_i), \cdot\} = m_{1,2}$, where $m_{\lambda,\mu} : \mathcal{B} \to \mathcal{B}$, $(a, b) \mapsto (\lambda a, \mu b)$. It follows that L_e^0 is spanned by $(m_{2,1}, -m_{2,1})$ and $(m_{1,2}, -m_{1,2})$. In particular, $\dim L_e^0 = 2$.

Take $g = (0, 0, t) \in G = \mathbb{Z}^2 \times \mathbb{Z}_2^3$ with $0 \neq t \in \mathbb{Z}_2^3$. Given a homogeneous element $x \in L^1 = \mathcal{V}^+$ in the Cayley-Dickson basis of $\mathcal{V}$, there is a unique $y \in L^{-1} = \mathcal{V}^-$ in the Cayley-Dickson basis such that $\nu(x, y) = [x, y] \in L_g^0$, i.e., $\deg(x^+) + \deg(y^-) = g$, and in that case we always have $x \neq y$. Take different elements x_i, x_j in the Cayley-Dickson basis of $\mathcal{C}$ such that $\deg((x_i, 0)^+) + \deg((x_j, 0)^-) = g$. There are four such pairs $\{i, j\}$. Then, we have four linearly independent elements of L_g^0 such that their first components are given by:

$$\{(x_i, 0), (x_j, 0), \cdot\} = -\{(x_j, 0), (x_i, 0), \cdot\} = \begin{cases} (x_j, 0) \mapsto 2(x_i, 0) \\ (x_i, 0) \mapsto -2(x_j, 0) \\ (0, x_k) \mapsto (0, -(x_k x_i)\bar{x}_j) \text{ for any } k \\ 0 \text{ otherwise.} \end{cases}$$

It follows that $\dim L_g^0 \geq 4$, and there are seven homogeneous components of this type, one for each choice of t.

Take now $g = (1, -1, t)$ with $t \in \mathbb{Z}_2^3$ (the case $g = (-1, 1, t)$ is similar). Take elements $(x_i, 0)$ and $(0, x_j)$ in the Cayley-Dickson basis such that $\deg((x_i, 0)^+) + \deg((x_j, 0)^-) = g$. Note that, for elements in the Cayley-Dickson basis we have

$$\{(x_i, 0), (0, x_j), \cdot\} = \begin{cases} (0, x_k) \mapsto (\bar{x}_k(x_j x_i), 0) \\ (x_k, 0) \mapsto (0, 0) \end{cases},$$

which is a nonzero map. Hence $L_g^0 \neq 0$, and therefore $\dim L_g^0 \geq 1$. Note that there are 8 homogeneous components with degrees $g = (1, -1, t)$ for $t \in \mathbb{Z}_2^3$, and 8 more with degrees $g = (-1, 1, t)$ for $t \in \mathbb{Z}_2^3$.

Finally, the subspace $L_1 \oplus L_{-1} = V^+ \oplus V^-$ consists of other 32 homogeneous components of dimension 1. The sum of the subspaces already considered has dimension at least $2 + 4 \cdot 7 + 16 + 32 = 78 = \dim L$. Therefore, the inequalities above are actually equalities and the type of the grading is $(48, 1, 0, 7)$. □

Remark 3.5.3. For the TKK construction $L = \mathrm{TKK}(\mathcal{V}_\mathcal{B})$ of $\mathfrak{e}_6$, it can be proved that $L^0 = \mathfrak{innstr}(\mathcal{V}_\mathcal{B}) \cong \mathfrak{d}_5 \oplus Z$, where Z is a 1-dimensional center. The $\mathbb{Z}^6$-grading on L restricts to a $\mathbb{Z}^5$-grading of type $(40, 0, 0, 0, 0, 1)$ on L_0, which restricts to the Cartan $\mathbb{Z}^5$-grading on $\mathfrak{d}_5$. On the other hand, the $\mathbb{Z}^2 \times \mathbb{Z}_2^3$-grading on L restricts to a $\mathbb{Z} \times \mathbb{Z}_2^3$-grading of type $(16, 1, 0, 7)$ on L^0.

On the other hand, note that the fine gradings on $\mathcal{T}_\mathcal{B}$ induce three of the four fine gradings on $\mathfrak{f}_4 \cong \mathrm{Der}(\mathcal{T}_\mathcal{B}) \oplus \mathcal{T}_\mathcal{B}$.

3.6 Weyl groups of fine gradings on bi-Cayley systems

Now we will compute the Weyl groups of the fine gradings on the bi-Cayley pair and the bi-Cayley triple system.

Theorem 3.6.1. *Let Γ be either the Cayley-Dickson $\mathbb{Z}^2 \times \mathbb{Z}_2^3$-grading on the bi-Cayley pair, or the nonisotropic Cayley-Dickson $\mathbb{Z}_2^5$-grading on the bi-Cayley triple system. Then,*

$$\mathcal{W}(\Gamma) \cong \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline B & C \end{array} \right) \in \mathrm{GL}_5(\mathbb{Z}_2) \mid A \in \langle \tau \rangle, B \in \mathcal{M}_{3\times 2}(\mathbb{Z}_2), C \in \mathrm{GL}_3(\mathbb{Z}_2) \right\},$$

where $\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_2)$.

Proof. We will prove now the first case. Identify $\mathbb{Z}^2$ and $\mathbb{Z}_2^3$ with the subgroups $\mathbb{Z}^2 \times 0$ and $0 \times \mathbb{Z}_2^3$ of $G = \mathbb{Z}^2 \times \mathbb{Z}_2^3$. Let $\{a, b\}$ and $\{a_i\}_{i=1}^3$ denote the canonical bases of the subgroups $\mathbb{Z}^2$ and $\mathbb{Z}_2^3$. Let $\Gamma_{\mathcal{C}}$ be the $\mathbb{Z}_2^3$-grading on $\mathcal{C}$. It is well-known (see [EK13]) that $\mathcal{W}(\Gamma_{\mathcal{C}}) \cong \mathrm{Aut}(\mathbb{Z}_2^3) \cong \mathrm{GL}_3(\mathbb{Z}_2)$. If $f \in \mathrm{Aut}\,\Gamma_{\mathcal{C}}$, then $f \times f \in \mathrm{Aut}\,\Gamma$ (notation as in the proof of Theorem 2.2.23), and with an abuse of notation we have $\mathcal{W}(\Gamma_{\mathcal{C}}) \leq \mathcal{W}(\Gamma) \leq \mathrm{Aut}(G)$.

Since $\bar{\tau}_{12}$ induces an element τ of $\mathcal{W}(\Gamma)$ of order 2, given by $a \leftrightarrow b$, that commutes with $\mathcal{W}(\Gamma_{\mathcal{C}})$, we have $\langle \tau \rangle \times \mathrm{GL}_3(\mathbb{Z}_2) \leq \mathcal{W}(\Gamma)$. Furthermore, from Lemma 2.2.15 we can deduce that there is a related triple φ that induces an element $\bar{\varphi}$ of $\mathcal{W}(\Gamma)$ of the form $a \mapsto a + c_1$, $b \mapsto b + c_2$ with $c_i \in \mathbb{Z}_2^3$, $c_1 \neq 0$. Without loss of generality, composing φ with some element of $\mathcal{W}(\Gamma_{\mathcal{C}})$ if necessary, we can also assume that $\bar{\varphi}$ fixes a_i for $i = 1, 2, 3$. It is clear that $\bar{\varphi}$ and $\langle \tau \rangle \times \mathrm{GL}_3(\mathbb{Z}_2)$ generate a subgroup $\mathcal{W}$ of $\mathcal{W}(\Gamma)$ isomorphic to the one stated in the result. It remains to show that $\mathcal{W}(\Gamma) \leq \mathcal{W}$.

Take $\phi \in \mathcal{W}(\Gamma)$; we claim that $\phi \in \mathcal{W}$. By ϕ-invariance of Supp Γ, either $\phi(a) = a + c$, or $\phi(a) = b + c$, for some $c \in \mathbb{Z}_2^3$, so if we compose ϕ with elements of $\mathcal{W}$ we can assume that $\phi(a) = a$. Since the torsion subgroup $\mathbb{Z}_2^3$ is ϕ-invariant, if we compose with elements of $\mathcal{W}(\Gamma_{\mathcal{C}})$ we can also assume that $\phi(a_i) = a_i$ $(i = 1, 2, 3)$. Finally, by ϕ-invariance of Supp Γ and $\mathbb{Z}_2^3$, it must be $\phi(b) = b + c$ for some $c \in \mathbb{Z}_2^3$, and composing again with elements of $\mathcal{W}$ we can assume in addition that $\phi(b) = b$. Hence $\phi = 1$ and $\mathcal{W}(\Gamma) = \mathcal{W}$.

Finally, let Γ' denote the $\mathbb{Z}_2^5$ grading on the bi-Cayley triple system. Note that Γ' induces a coarsening (Γ', Γ') of Γ on $\mathcal{V}_{\mathcal{B}}$, where we can identify the set Supp Γ' with Supp Γ^+. Since $\mathcal{W}(\Gamma)$ is determined by the action of $\mathrm{Aut}(\Gamma)$ on Supp $\Gamma^+ \equiv$ Supp Γ', it follows that we can identify $\mathrm{Aut}(\Gamma')$ with a subgroup of $\mathrm{Aut}(\Gamma)$ and hence $\mathcal{W}(\Gamma') \leq \mathcal{W}(\Gamma)$. Recall that $\bar{\tau}_{12}$, $\mathrm{Aut}(\Gamma_{\mathcal{C}})$ and φ induce the generators of $\mathcal{W}(\Gamma)$, and on the other hand these are given by elements of $\mathrm{Aut}(\Gamma')$, so we also have $\mathcal{W}(\Gamma) \leq \mathcal{W}(\Gamma')$ with the previous identification, and therefore $\mathcal{W}(\Gamma) = \mathcal{W}(\Gamma')$. □

Theorem 3.6.2. *Let Γ be the isotropic Cayley-Dickson $\mathbb{Z} \times \mathbb{Z}_2^3$-grading on the bi-Cayley triple system. Then $\mathcal{W}(\Gamma)$ is the whole $\mathrm{Aut}(\mathbb{Z} \times \mathbb{Z}_2^3)$.*

With the natural identification we can express this result as follows:

$$\mathcal{W}(\Gamma) \cong \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline B & C \end{array} \right) \mid A \in \{\pm 1\}, B \in \mathcal{M}_{3\times 1}(\mathbb{Z}_2), C \in \mathrm{GL}_3(\mathbb{Z}_2) \right\}.$$

Proof. The proof is similar to the proof of Theorem 3.6.1, so we do not give all the details. The block with $\mathrm{GL}_3(\mathbb{Z}_2)$ is induced by automorphisms of $\mathcal{C}$ extended to $\mathcal{T}_{\mathcal{B}}$. The blocks with 0 and $\{\pm 1\}$ are obtained by $\mathcal{W}(\Gamma)$-invariance of the torsion subgroup and the support of Γ. (Both automorphisms $\bar{\tau}_{12}$ and

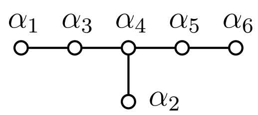
$c_{1,-1}$ induce the element that generates the block $\{\pm 1\}$ in $\mathcal{W}(\Gamma)$.) Take a homogeneous $a \in \mathcal{C}$ with nonzero degree in the associated Cayley-Dickson grading on $\mathcal{C}$. Hence, $\mathrm{tr}(a) = 0$, and scaling we can assume that $n(a) = 1$. Consider the automorphism $\varphi = \Phi(a)$ of $\mathcal{T}_{\mathcal{B}}$, with Φ as in Proposition 2.2.16. It is checked that $\varphi(x_i, \pm\mathbf{i}\bar{x}_i) = \pm\mathbf{i}(x_i a, \pm\mathbf{i}\overline{x_i a})$, and therefore φ belongs to $\mathrm{Aut}(\Gamma)$ and induces a nonzero element of the block $\mathcal{M}_{3\times 1}(\mathbb{Z}_2)$. We conclude that all the block $\mathcal{M}_{3\times 1}(\mathbb{Z}_2)$ appears, which finishes the proof. □

Remark 3.6.3. The Weyl group in the result above is isomorphic to the Weyl group of the $\mathbb{Z} \times \mathbb{Z}_2^3$-grading on the Albert algebra.

We will now compute the Weyl group of the Cartan grading on $\mathcal{V}_{\mathcal{B}}$.

Let $\mathcal{V}$ denote the bi-Cayley pair. Let Γ be the Cartan grading on $\mathcal{V}$ by $G = \mathcal{U}(\Gamma) = \mathbb{Z}^6$. Let $T := \mathrm{Diag}(\Gamma) \leq \mathrm{Aut}(\mathcal{V}) \leq \mathrm{Aut}(L)$, where $L = \mathrm{TKK}(\mathcal{V}) = L^{-1} \oplus L^0 \oplus L^1$. Then, T is a maximal torus of $\mathrm{Aut}(L)$ that preserves L^i for $i = -1, 0, 1$. Consider the extended grading $\widetilde{\Gamma} = E_G(\Gamma)$ on L (the Cartan grading) and let Φ be the root system associated to $\widetilde{\Gamma}$. We have the corresponding root space decomposition $L = H \oplus (\bigoplus_{\alpha\in\Phi} L_\alpha)$, where the Cartan subalgebra H is contained in L^0, Φ splits as a disjoint union $\Phi = \Phi^{-1} \cup \Phi^0 \cup \Phi^1$, and $\Gamma^\sigma \colon \mathcal{V}^\sigma = \bigoplus_{\alpha\in\Phi^{\sigma 1}} L_\alpha$. Also, $L^0 = Z(L^0) \oplus [L^0, L^0]$ with $\dim Z(L^0) = 1$, where $[L^0, L^0]$ is simple of type D_5.

Take a system of simple roots $\Delta = \{\alpha_1, \dots, \alpha_6\}$ of Φ with Dynkin diagram

E_6 

(3.6.1)

such that $\{\alpha_1, \dots, \alpha_5\}$ is a system of simple roots of Φ^0 and

$$\Phi^{\pm 1} = \{\sum_{i=1}^{6} m_i\alpha_i \in \Phi \mid m_6 = \pm 1\}.$$

Any $\varphi = (\varphi^+, \varphi^-) \in \mathrm{Aut}(\Gamma)$ induces an automorphism $\bar{\varphi} \in \mathrm{Aut}\,\mathcal{U}(\Gamma)$ which in turn induces an automorphism $\widehat{\varphi} \in \mathrm{Aut}\,\Phi$ preserving Φ^i for $i = -1, 0, 1$. Conversely, given any $\psi \in \mathrm{Aut}\,\Phi$ preserving Φ^i for $i = -1, 0, 1$, there is an automorphism $\varphi \in \mathrm{Aut}(L)$ such that $\varphi(L_\alpha) = L_{\psi(\alpha)}$ for each $\alpha \in \Phi$; in particular, $\varphi(L^{\pm 1}) = \varphi(\bigoplus_{\alpha\in\Phi^{\pm 1}} L_\alpha) = L_{\pm 1}$ because $\psi(\Phi^{\pm 1}) = \Phi^{\pm 1}$, so φ restricts to an automorphism of $\mathcal{V}$. Therefore we have proven:

Theorem 3.6.4. *Let Γ denote the Cartan grading on the bi-Cayley pair. Then, the Weyl group of Γ is isomorphic to the group*

$$\mathcal{W} := \{\psi \in \mathrm{Aut}\,\Phi \mid \psi(\Phi^i) = \Phi^i, i = -1, 0, 1\},$$

where Φ *is the root system of type* E_6.

Consider the restriction map

$$\Theta\colon \mathcal{W} \to \operatorname{Aut}\Phi^0. \tag{3.6.2}$$

Lemma 3.6.5. *The map* Θ *is injective and* $\operatorname{im}\Theta$ *is the Weyl group of type* D_5.

Proof. Recall the ordering of the roots in (3.6.1). Extend Θ to the restriction map

$$\widehat{\Theta}\colon \operatorname{Stab}_{\operatorname{Aut}\Phi}(\Phi^0) \to \operatorname{Aut}\Phi^0.$$

We claim that $\widehat{\Theta}$ is injective. If $\psi \in \ker\widehat{\Theta}$, then $\psi \in \operatorname{Aut}\Phi$ and $\psi(\alpha_i) = \alpha_i$ for $i = 1, \ldots, 5$. In the euclidean space $E = \mathbb{R}\otimes_{\mathbb{Z}} \mathbb{Z}\Phi$, ψ is an isometry. The orthogonal subspace to $\alpha_1, \ldots, \alpha_5$ is spanned by the fundamental dominant weight $w_6 = \frac{1}{3}(2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+5\alpha_5+4\alpha_6)$. Therefore either $\psi(w_6) = w_6$ and $\psi = \mathrm{id}$, or $\psi(w_6) = -w_6$. In the latter case,

$$\begin{aligned}&\frac{1}{3}(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\psi(\alpha_6))\\ &= -\frac{1}{3}(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6),\end{aligned}$$

so that $\psi(\alpha_6) = -\alpha_6 - \frac{1}{2}(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5)$, which is not a root, a contradiction.

Now, we claim that $\widehat{\Theta}$ is surjective. We have $\operatorname{Aut}\Phi^0 = \mathcal{W}_{D_5} \rtimes C_2$ with $C_2 = \langle\vartheta\rangle$, where $\mathcal{W}_{D_5}$ is the Weyl group of D_5 and ϑ is the 'outer' automorphism such that $\vartheta(\alpha_i) = \alpha_i$ for $i = 1, 3, 4$, $\vartheta(\alpha_2) = \alpha_5$, $\vartheta(\alpha_5) = \alpha_2$. The linear map ψ such that

$$\begin{aligned}&\psi(\alpha_i) = \alpha_i \text{ for } i = 1, 3, 4,\\ &\psi(\alpha_2) = \alpha_5, \quad \psi(\alpha_5) = \alpha_2,\\ &\psi(\alpha_6) = -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6) \quad \text{(the lowest root)},\end{aligned} \tag{3.6.3}$$

belongs to $\operatorname{Aut}\Phi$ and $\widehat{\Theta}(\psi) = \vartheta$. Also, $\mathcal{W}_{D_5}$ is generated by the reflections $s_{\alpha_1}, \ldots, s_{\alpha_5}$, which are the images under $\widehat{\Theta}$ of the corresponding reflections of Φ. We conclude that $\widehat{\Theta}$ is surjective.

Finally, the reflections $s_{\alpha_1}, \ldots, s_{\alpha_5}$ in Φ belong to $\mathcal{W}$, while ψ in (3.6.3) permutes Φ^1 and Φ^{-1}. Hence, since $\widehat{\Theta}$ is bijective, we get $\operatorname{Stab}_{\operatorname{Aut}\Phi}(\Phi^0) = \mathcal{W} \rtimes \langle\psi\rangle$ and $\operatorname{im}\Theta = \widetilde{\Theta}(\mathcal{W}) = \mathcal{W}_{D_5}$. □

Theorem 3.6.6. *The Weyl group of the Cartan grading on the bi-Cayley pair is isomorphic to the Weyl group of the root system of type* D_5.

Proof. Consequence of Lemma 3.6.5. □

Theorem 3.6.7. *The Weyl group of the Cartan grading on the bi-Cayley triple system is isomorphic to* $\mathbb{Z}_2^4 \rtimes \mathrm{Sym}(4)$*, i.e., the automorphism group of the root system of type* B_4 *(or* C_4*).*

Proof. Let Γ denote the Cartan grading on the bi-Cayley triple system. The automorphisms 2), 3) and 4) of the proof of [EK13, Th. 5.15] induce a subgroup of the Weyl group of the Cartan $\mathbb{Z}^4$-grading on $\mathbb{A}$ that is isomorphic to $\mathbb{Z}_2^4 \rtimes \mathrm{Sym}(4)$. It can be checked that these automorphisms of $\mathbb{A}$ restrict to automorphisms of the bi-Cayley triple system, which is identified with the subspace $\mathcal{B} = \iota_2(\mathcal{C}) \oplus \iota_3(\mathcal{C})$, and they induce a subgroup $\mathcal{W} \cong \mathbb{Z}_2^4 \rtimes \mathrm{Sym}(4)$ of $\mathcal{W}(\Gamma)$ (we omit the details here).

On the other hand, the pair (Γ, Γ) can be regarded as a coarsening of the Cartan grading $\widetilde{\Gamma}$ on $\mathcal{V}_{\mathcal{B}}$. Since $\mathrm{Supp}\, \widetilde{\Gamma}^+$ generates $\mathcal{U}(\widetilde{\Gamma})$, we have that the Weyl group of $\widetilde{\Gamma}$ can be identified with a subgroup of $\mathrm{Sym}(\mathrm{Supp}\, \widetilde{\Gamma}^+)$. Also, we can identify $\mathcal{W}(\Gamma)$ with a subgroup of $\mathrm{Sym}(\mathrm{Supp}\, \Gamma)$, and $\mathrm{Supp}\, \Gamma$ with $\mathrm{Supp}\, \widetilde{\Gamma}^+$. Therefore, with the previous identifications, we can identify $\mathcal{W}(\Gamma)$ with a subgroup of $\mathcal{W}(\widetilde{\Gamma})$. We recall from Theorem 3.6.6 that $\mathcal{W}(\widetilde{\Gamma})$ is isomorphic to the Weyl group of the root system of type D_5, which is known to be isomorphic to $\mathbb{Z}_2^4 \rtimes \mathrm{Sym}(5)$ (see [H78, Section 12.2]). Therefore, we have $\mathcal{W} \leq \mathcal{W}(\Gamma) \leq \mathcal{W}(\widetilde{\Gamma})$, where $\mathcal{W}$ has index 5 in $\mathcal{W}(\widetilde{\Gamma})$. We need to prove that $\mathcal{W} = \mathcal{W}(\Gamma)$, so it suffices to prove that $\mathcal{W}(\Gamma) \neq \mathcal{W}(\widetilde{\Gamma})$.

Note that any element $\varphi \in \mathcal{W}(\Gamma)$ satisfies $\varphi(\deg(e_1, 0)) = -\varphi(\deg(e_2, 0))$ because $\deg(e_1, 0) = -\deg(e_2, 0)$. Consider the automorphism $\phi = \widehat{\varphi}_{e_2} \varphi_{e_1} \widehat{\varphi}_{e_2}$ of $\mathcal{V}_{\mathcal{B}}$. Then, it can be checked that ϕ^+ is given by

$$\phi^+ \colon \begin{cases} (e_1, 0) \mapsto (0, e_2), & (0, e_1) \mapsto (0, e_1), \\ (e_2, 0) \mapsto (e_2, 0), & (0, e_2) \mapsto (-e_1, 0), \\ (u_i, 0) \mapsto (u_i, 0), & (0, u_i) \mapsto (0, u_i), \\ (v_i, 0) \mapsto (0, -v_i), & (0, v_i) \mapsto (v_i, 0), \end{cases} \tag{3.6.4}$$

and similarly for ϕ^-, so ϕ belongs to $\mathrm{Aut}(\widetilde{\Gamma})$ and induces an element $\bar{\phi}$ in $\mathcal{W}(\widetilde{\Gamma})$. Notice that $\bar{\phi}$ satisfies $\bar{\phi}(\deg(e_1, 0)^+) \neq -\bar{\phi}(\deg(e_2, 0)^+)$, and this implies (with the previous identifications of supports $\mathrm{Supp}\, \Gamma \equiv \mathrm{Supp}\, \widetilde{\Gamma}^+$) that $\bar{\phi} \notin \mathcal{W}(\Gamma)$, which concludes the proof. □

Chapter 4

Gradings on Albert systems

In this Chapter we give classifications of the fine gradings on the Albert pair and on the Albert triple system, which are some of the main original results of this thesis. Also, all these fine gradings are described given by their universal grading groups, their Weyl groups are computed, and we determine the (fine) gradings on $\mathfrak{e}_7$ induced by the fine gradings on the Albert pair.

The main results in this chapter are Theorem 4.2.4 and Theorem 4.3.1, that classify the fine gradings, up to equivalence, on the Albert Jordan pair and triple system, respectively.

4.1 Construction of fine gradings on the Albert pair

Recall that the trace of $\mathcal{V}_{\mathbb{A}}$ is homogeneous for each grading, i.e., we have that $T(x^+, y^-) \neq 0$ implies $\deg(y^-) = -\deg(x^+)$ for x^+, y^- homogeneous elements of the grading. Hence, to give a grading on $\mathcal{V}_{\mathbb{A}}$ it suffices to give the degree map on $\mathbb{A}^+$.

Example 4.1.1. Consider a Cayley-Dickson basis $\{x_i\}_{i=0}^7$ on $\mathcal{C}$ with associated $\mathbb{Z}_2^3$-grading and degree map $\deg_{\mathcal{C}}$. It can be checked directly that we have a fine $\mathbb{Z}^3 \times \mathbb{Z}_2^3$-grading on $\mathcal{V}_{\mathbb{A}}$, with homogeneous basis $\{E_j^\sigma, \iota_j(x_i)^\sigma\}$, and determined by

$$\begin{aligned}
\deg(E_1^+) &= (-1, 1, 1, \bar{0}, \bar{0}, \bar{0}), & \deg(\iota_1(x_i)^+) &= (1, 0, 0, \deg_{\mathcal{C}}(x_i)),\\
\deg(E_2^+) &= (1, -1, 1, \bar{0}, \bar{0}, \bar{0}), & \deg(\iota_2(x_i)^+) &= (0, 1, 0, \deg_{\mathcal{C}}(x_i)),\\
\deg(E_3^+) &= (1, 1, -1, \bar{0}, \bar{0}, \bar{0}), & \deg(\iota_3(x_i)^+) &= (0, 0, 1, \deg_{\mathcal{C}}(x_i)),
\end{aligned}$$

and $\deg(x^+) + \deg(y^-) = 0$ for any elements x^+, y^- of the homogeneous basis such that $t(x^+, y^-) \neq 0$. This grading will be called the *Cayley-Dickson grading* on $\mathcal{V}_{\mathbb{A}}$.

Example 4.1.2. Consider the $\mathbb{Z}_3^3$-grading on $\mathbb{A}$ as a grading on $\mathcal{V}_{\mathbb{A}}$ and denote its degree map by $\deg_{\mathbb{A}}$. Then, we can define a fine $\mathbb{Z} \times \mathbb{Z}_3^3$-grading on $\mathcal{V}_{\mathbb{A}}$ by $\deg(x^\sigma) = (\sigma 1, \deg_{\mathbb{A}}(x))$. (Note that, if we identify $\mathbb{Z}_3^3$ with a subgroup of $\mathbb{Z} \times \mathbb{Z}_3^3$, our new grading is just the g-shift of the $\mathbb{Z}_3^3$-grading on $\mathcal{V}_{\mathbb{A}}$ with $g = (1, \bar{0}, \bar{0}, \bar{0})$.)

Example 4.1.3. Using the triple product, one can check directly that we have a fine $\mathbb{Z}^7$-grading on $\mathcal{V}_{\mathbb{A}}$, where the degree map on $\mathbb{A}^+$ is given by

deg	$\iota_1(\mathcal{C})^+$	$\iota_2(\mathcal{C})^+$	$\iota_3(\mathcal{C})^+$
e_1	$(1,0,0,0,0,0,0)$	$(0,0,0,0,0,1,0)$	$(0,0,0,0,0,0,1)$
e_2	$(0,1,0,0,0,0,0)$	$(-1,-2,1,1,1,1,0)$	$(2,1,-1,-1,-1,0,1)$
u_1	$(0,0,1,0,0,0,0)$	$(0,-1,1,0,0,1,0)$	$(1,1,0,-1,-1,0,1)$
u_2	$(0,0,0,1,0,0,0)$	$(0,-1,0,1,0,1,0)$	$(1,1,-1,0,-1,0,1)$
u_3	$(0,0,0,0,1,0,0)$	$(0,-1,0,0,1,1,0)$	$(1,1,-1,-1,0,0,1)$
v_1	$(1,1,-1,0,0,0,0)$	$(-1,-1,0,1,1,1,0)$	$(1,0,-1,0,0,0,1)$
v_2	$(1,1,0,-1,0,0,0)$	$(-1,-1,1,0,1,1,0)$	$(1,0,0,-1,0,0,1)$
v_3	$(1,1,0,0,-1,0,0)$	$(-1,-1,1,1,0,1,0)$	$(1,0,0,0,-1,0,1)$

E_1^+	$(0,-1,0,0,0,1,1)$
E_2^+	$(2,2,-1,-1,-1,-1,1)$
E_3^+	$(-1,-1,1,1,1,1,-1)$

and $\deg(y^-) := -\deg(x^+)$ if $T(x^+, y^-) \neq 0$, where $y \in \{E_i, \iota_i(z) \mid i = 1,2,3,\ z \in B_{\mathcal{C}}\}$ and $B_{\mathcal{C}}$ denotes the associated Cartan basis on $\mathcal{C}$. (The proof is similar to the proof of Proposition 3.1.4, and using the fact that the trace is homogeneous.) This $\mathbb{Z}^7$-grading will be called the *Cartan grading* on $\mathcal{V}_{\mathbb{A}}$.

Proposition 4.1.4. *The Cayley-Dickson grading on the Albert pair has universal group* $\mathbb{Z}^3 \times \mathbb{Z}_2^3$.

Proof. Consider a realization as G-grading, with G an abelian group, of the Cayley-Dickson grading on $\mathcal{V}_{\mathbb{A}}$. Identify $\iota_1(\mathcal{C}) \oplus \iota_2(\mathcal{C})$ with $\mathcal{B}$, and notice that the restriction of the grading to these homogeneous components is the Cayley-Dickson grading on $\mathcal{V}_{\mathcal{B}}$. Call $g_i = \deg(\iota_1(x_i)^+) = -\deg(\iota_1(x_i)^-)$, $a = g_0 = \deg(\iota_1(1)^+)$, $b = \deg(\iota_2(1)^+)$, $c = \deg(\iota_3(1)^+)$ and $a_i = g_i - g_0$. Using the same arguments of the proof of Proposition 3.1.3, we deduce that $\deg(\iota_1(x_i)^+) = a + a_i$, $\deg(\iota_2(x_i)^+) = b + a_i$, $\deg(\iota_3(x_i)^+) = c + a_i$, and also that $\deg_{\mathcal{C}}(x_i) := a_i$ defines a group grading which is a coarsening of the $\mathbb{Z}_2^3$-grading on $\mathcal{C}$. Therefore, there is an epimorphism $\mathbb{Z}^3 \times \mathbb{Z}_2^3 \to G$ that sends

$(1,0,0,\bar{0},\bar{0},\bar{0}) \mapsto a$, $(0,1,0,\bar{0},\bar{0},\bar{0}) \mapsto b$, $(0,0,1,\bar{0},\bar{0},\bar{0}) \mapsto c$ and restricts to an epimorphism $0 \times \mathbb{Z}_2^3 \to \langle a_i \rangle$. We conclude that $\mathbb{Z}^3 \times \mathbb{Z}_2^3$ is the universal group. □

Proposition 4.1.5. *The fine $\mathbb{Z} \times \mathbb{Z}_2^3$-grading on $\mathbb{A}$ in* (1.5.8)*, considered as a grading on $\mathcal{V}_{\mathbb{A}}$, admits a unique fine refinement, up to relabeling, which has universal group $\mathbb{Z}^3 \times \mathbb{Z}_2^3$ and is equivalent to the Cayley-Dickson grading.*

Proof. With the notation in (1.5.6), since

$$U_{\nu_+(1)}(S_-) = 16E \quad \text{and} \quad \{\nu_+(1), E, \nu_-(1)\} = 8\widetilde{E},$$

it follows that E^σ and $\widetilde{E}^\sigma$ are homogeneous for any refinement of this grading. Set $a = \frac{-1}{\sqrt{2}}1$, $\lambda = \frac{1}{\sqrt{2}}$. It suffices to prove that the automorphism $\varphi = c_{\mathbf{i},1,\mathbf{i}}\ \phi_1(a,\lambda)$ of the Albert pair (see Proposition 2.2.10 and (2.2.3)) is an equivalence between the Cayley-Dickson grading and any fine refinement of the $\mathbb{Z} \times \mathbb{Z}_2^3$-grading. A straightforward computation shows that:

$$\begin{aligned}
\varphi^+ :\quad & E_1 \mapsto E_1, \quad E_2 \mapsto \frac{1}{2}S_+, \quad E_3 \mapsto \frac{1}{2}S_-,\\
& \iota_1(1) \mapsto 2\widetilde{E}, \quad \iota_1(a) \mapsto \mathbf{i}\iota_1(a) = \nu(a),\\
& \iota_2(x) \mapsto \frac{1}{\sqrt{2}}\nu_-(x), \quad \iota_3(\bar{x}) \mapsto \frac{1}{\sqrt{2}}\nu_+(x),\\
\varphi^- :\quad & E_1 \mapsto E_1, \quad E_2 \mapsto \frac{1}{2}S_-, \quad E_3 \mapsto \frac{1}{2}S_+,\\
& \iota_1(1) \mapsto 2\widetilde{E}, \quad \iota_1(a) \mapsto -\mathbf{i}\iota_1(a) = -\nu(a),\\
& \iota_2(x) \mapsto \frac{1}{\sqrt{2}}\nu_+(x), \quad \iota_3(\bar{x}) \mapsto \frac{1}{\sqrt{2}}\nu_-(x)
\end{aligned} \tag{4.1.1}$$

so that φ takes the homogeneous components of the Cayley-Dickson grading to homogeneous components in any refinement of the $\mathbb{Z} \times \mathbb{Z}_2^3$-grading, as required. □

Proposition 4.1.6. *The fine $\mathbb{Z}_3^3$-grading on $\mathbb{A}$, considered as a grading on $\mathcal{V}_{\mathbb{A}}$, admits a unique fine refinement, up to relabeling, which has universal group $\mathbb{Z} \times \mathbb{Z}_3^3$.*

Proof. This is a consequence of Proposition 2.1.17. The degree map can be given by $\deg(x^\sigma) = (\sigma 1, \deg_{\mathbb{A}}(x))$, where $\deg_{\mathbb{A}}(x)$ denotes the degree of the $\mathbb{Z}_3^3$-grading on $\mathbb{A}$. □

Proposition 4.1.7. *The fine $\mathbb{Z}^4$-grading on $\mathbb{A}$, considered as a grading on $\mathcal{V}_{\mathbb{A}}$, admits a unique fine refinement, up to relabeling, which has universal group $\mathbb{Z}^7$ and is the Cartan grading on $\mathcal{V}_{\mathbb{A}}$.*

Proof. The proof is arduous but straightforward, so we do not give all the details. Note that, since $\{\iota_{i+1}(e_1)^\sigma, \iota_i(e_1)^{-\sigma}, \iota_{i+2}(e_1)^\sigma\} = 8E_i^\sigma$, the elements E_i^σ must be homogeneous, so the fine refinement is unique. Consider a realization as G-grading, with G an abelian group, of the Cartan grading on $\mathcal{V}_{\mathbb{A}}$. One can check directly that the degrees of the elements $\iota_1(e_1)^+$, $\iota_1(e_2)^+$, $\iota_1(u_1)^+$, $\iota_1(u_2)^+$, $\iota_1(u_3)^+$, $\iota_2(e_1)^+$, $\iota_3(e_1)^+$ generate G. We conclude that the G-grading is induced from the Cartan grading by some epimorphism $\mathbb{Z}^7 \to G$ that is the identity on the support (and sends the canonical basis of $\mathbb{Z}^7$ to the degrees of the mentioned elements), and so $\mathbb{Z}^7$ is the universal group of the Cartan grading on $\mathcal{V}_{\mathbb{A}}$. □

4.2 Classification of fine gradings on the Albert pair

Given a grading on a semisimple Jordan pair, by Remark 2.1.23, any homogeneous element can be completed to a maximal orthogonal system of homogeneous idempotents. In the case of the Albert pair, since the capacity is 3, it will consist either of three idempotents of rank 1, or one idempotent of rank 2 and another of rank 1, or one of rank 3. We will cover these possibilities with the following Lemmas.

Lemma 4.2.1. *Let Γ be a fine grading on $\mathcal{V}_{\mathbb{A}}$ such that all nonzero homogeneous idempotents have rank 1. Then, Γ is equivalent to the Cartan grading (Example 4.1.3).*

Proof. We can take a set of three orthogonal homogeneous idempotents $F = \{e_1, e_2, e_3\}$, so F is a frame, and up to automorphism (by Theorem 1.7.2 or Remark 1.7.6), we can assume that $e_i = (E_i^+, E_i^-)$. Hence, for any permutation $\{i, j, k\} = \{1, 2, 3\}$, the associated Peirce subspaces,

$$(\mathcal{V}_{\mathcal{B}})^\sigma_{jk} = \{x \in \mathbb{A} \mid D(e_j^\sigma, e_j^{-\sigma})x = x = D(e_k^\sigma, e_k^{-\sigma})x\} = \iota_i(\mathcal{C})^\sigma,$$

are graded. It is clear that Γ restricts to a grading $\Gamma_{\mathcal{B}}$ on the bi-Cayley pair $\mathcal{V}_{\mathcal{B}} := (\mathcal{B}, \mathcal{B})$, where $\mathcal{B} := \iota_1(\mathcal{C}) \oplus \iota_2(\mathcal{C})$. By [S87], we know that each automorphism of the bi-Cayley pair has a unique extension to the Albert pair that fixes E_3^+ and E_3^-, and hence we can identify $\operatorname{Aut}\mathcal{V}_{\mathcal{B}}$ with the stabilizer of e_3 in $\operatorname{Aut}\mathcal{V}_{\mathbb{A}}$. The nonzero homogeneous elements of $\Gamma_{\mathcal{B}}$ must have rank one, and therefore $\Gamma_{\mathcal{B}}$ is equivalent to the Cartan $\mathbb{Z}^6$-grading. We can apply an automorphism of $\mathcal{V}_{\mathcal{B}}$ extended to $\operatorname{Aut}\mathcal{V}_{\mathbb{A}}$ and assume that we have the Cartan basis on $\mathcal{V}_{\mathcal{B}}$ as in Example 3.1.2. Then, it is easy to check that we have the homogeneous basis of the Cartan grading on the Albert pair, and consequently, Γ is the Cartan $\mathbb{Z}^7$-grading on the Albert pair. □

Lemma 4.2.2. *Let Γ be a fine grading on $\mathcal{V}_{\mathbb{A}}$ such that there are two orthogonal homogeneous idempotents, one of rank 1 and the other of rank 2. Then, Γ is equivalent to the Cayley-Dickson $\mathbb{Z}^3 \times \mathbb{Z}_2^3$-grading (Example 4.1.1).*

Proof. Denote by e_1 and e_2 the orthogonal homogeneous idempotents, with $\mathrm{rk}(e_1) = 1$ and $\mathrm{rk}(e_2) = 2$. By Remark 1.7.6 we can assume that $e_1^\sigma = E := E_1$ and $e_2^\sigma = \widetilde{E} := E_2 + E_3$. The Peirce subspace $\mathcal{B}^\sigma := \{x \in \mathbb{A} \mid D(e_1^\sigma, e_1^{-\sigma})x = x = D(e_2^\sigma, e_2^{-\sigma})x\} = \iota_2(\mathcal{C}) \oplus \iota_3(\mathcal{C})$ is graded, and we can identify it with the bi-Cayley pair $\mathcal{V}_{\mathcal{B}}$. The grading $\Gamma_{\mathcal{B}}$ induced on $\mathcal{V}_{\mathcal{B}}$ must be equivalent to the Cayley-Dickson $\mathbb{Z}^2 \times \mathbb{Z}_2^3$-grading (because the Cartan grading on $\mathcal{V}_{\mathcal{B}}$ can only be extended to the Cartan grading on $\mathcal{V}_{\mathbb{A}}$, which does not have homogeneous elements of rank 2). By the same arguments used in the proof of Lemma 4.2.1, we can apply an automorphism of the bi-Cayley pair extended to $\mathcal{V}_{\mathbb{A}}$ to assume that we have a homogeneous basis of $\mathcal{V}_{\mathbb{A}}$ as in Proposition 4.1.5 (the elements of $\mathcal{V}_{\mathcal{B}}$ are of the form $\nu_\pm(x)$). We conclude that Γ is equivalent to the Cayley-Dickson $\mathbb{Z}^3 \times \mathbb{Z}_2^3$-grading. □

Lemma 4.2.3. *Let Γ be a fine grading on $\mathcal{V}_{\mathbb{A}}$ with some homogeneous idempotent of rank 3. Then,* char $\mathbb{F} \neq 3$ *and Γ is equivalent to the $\mathbb{Z} \times \mathbb{Z}_3^3$-grading (Example 4.1.2).*

Proof. Let e be a homogeneous idempotent of rank 3. By Remark 1.7.6, we can assume, up to automorphism, that $e = (1^+, 1^-)$, where 1 is the identity of $\mathbb{A}$. By Theorem 2.1.24, the homogeneous components are 1-dimensional, and on the other hand the trace is homogeneous and nondegenerate, so the restriction of the trace to the subpair $(\mathbb{F}1^+, \mathbb{F}1^-)$ must be nondegenerate, which forces char $\mathbb{F} \neq 3$.

By Proposition 2.1.17, if $g = -\deg(1^+)$ and $\deg_g$ is the degree map of the shift $\Gamma^{[g]}$ of Γ, then $\deg_g(x^+) = \deg_g(x^-)$ for any homogeneous element $x \in \mathbb{A}$, and $\deg_g$ restricts to a grading $\Gamma_{\mathbb{A}}$ on $\mathbb{A}$. Since Γ is fine, its homogeneous components are 1-dimensional by Proposition 2.1.24, and this is also true for $\Gamma_{\mathbb{A}}$. Therefore, $\Gamma_{\mathbb{A}}$ must be, up to equivalence, the $\mathbb{Z}_3^3$-grading, because this is the only grading on $\mathbb{A}$ with 1-dimensional homogeneous components. Finally, since Γ is a shift of the $\mathbb{Z}_3^3$-grading $(\Gamma_{\mathbb{A}}, \Gamma_{\mathbb{A}})$ on $\mathcal{V}_{\mathbb{A}}$, this forces Γ to be the $\mathbb{Z} \times \mathbb{Z}_3^3$-grading (see Proposition 2.1.17). □

Theorem 4.2.4. *The fine gradings on the Albert pair are, up to equivalence,*

- *the Cartan $\mathbb{Z}^7$-grading (Example 4.1.3),*
- *the Cayley-Dickson $\mathbb{Z}^3 \times \mathbb{Z}_2^3$-grading (Example 4.1.1),*
- *the $\mathbb{Z} \times \mathbb{Z}_3^3$-grading (Example 4.1.2).*

The $\mathbb{Z}\times\mathbb{Z}_3^3$*-grading only occurs if* $\operatorname{char}\mathbb{F}\neq 3$.

Proof. This result follows since Lemmas 4.2.1, 4.2.2 and 4.2.3 cover all possible cases. □

4.3 Classification of fine gradings on the Albert triple system

Theorem 4.3.1. *There are four gradings, up to equivalence, on the Albert triple system. Their universal groups are:* $\mathbb{Z}^4\times\mathbb{Z}_2$, $\mathbb{Z}_2^6$, $\mathbb{Z}\times\mathbb{Z}_2^4$ *and* $\mathbb{Z}_3^3\times\mathbb{Z}_2$.

Proof. This is a consequence of Corollary 2.1.12, Proposition 2.1.18 and the classification of fine gradings on the Albert algebra ([EK12a]). □

4.4 Induced gradings on $\mathfrak{e}_7$

Proposition 4.4.1. *The Cartan* $\mathbb{Z}^7$*-grading on the Albert pair extends to a fine grading with universal group* $\mathbb{Z}^7$ *and type* $(126,0,0,0,0,0,1)$ *on* $\mathfrak{e}_7$*, that is, a Cartan grading on* $\mathfrak{e}_7$.

Proof. This follows by the same arguments as in the proof of Proposition 3.5.1. □

Proposition 4.4.2. *The Cayley-Dickson* $\mathbb{Z}^3\times\mathbb{Z}_2^3$*-grading on the Albert pair extends to a fine grading with universal group* $\mathbb{Z}^3\times\mathbb{Z}_2^3$ *and type* $(102,0,1,7)$ *on* $\mathfrak{e}_7$.

Proof. This is a consequence of Theorem 2.1.21, except for the type, which we will now compute. Notice that $L^1\oplus L^{-1}=\mathbb{A}^+\oplus\mathbb{A}^-$ consists of 54 homogeneous components of dimension 1.

Set $e=(0,0,0,\bar 0,\bar 0,\bar 0)\in\mathbb{Z}^3\times\mathbb{Z}_2^3$. Note that $\{E_i,E_i,\cdot\}$ acts multiplying by 2 on E_i, and multiplying by 0 on E_{i+1} and E_{i+2}. Therefore, $\dim L_e^0\geq 3$.

Recall from the proof of Proposition 3.5.2 that the $\mathbb{Z}^2\times\mathbb{Z}_2^3$-grading on $\mathfrak{e}_6$ (induced from $\mathcal{V}_{\mathcal{B}}$) has 16 components of dimension 1 with associated degrees $(\pm1,\mp1,g)$ with $g\in\mathbb{Z}_2^3$. Therefore, by symmetry for our grading on $\mathfrak{e}_7$, there must be $16\times3=48$ homogeneous components of at least dimension 1 (the dimension may increase on $\mathfrak{e}_7$), with associated degrees $(\pm1,\mp1,0,g)$, $(\pm1,0,\mp1,g)$, $(0,\pm1,\mp1,g)$, where $g\in\mathbb{Z}_2^3$. These components span a subspace of dimension at least 48.

Recall also that the $\mathbb{Z}^2\times\mathbb{Z}_2^3$-grading on $\mathfrak{e}_6$ has 7 components of dimension 4 and degrees $(0,0,g)$ with $e\neq g\in\mathbb{Z}_2^3$, so the $\mathbb{Z}^3\times\mathbb{Z}_2^3$-grading on $\mathfrak{e}_7$ has at least 7 components, with degrees $(0,0,0,g)$ with $e \neq g \in \mathbb{Z}_2^3$, of dimension at least 4, whose sum spans a subspace of dimension at least 28.

Finally, note that de sum of the previous subspaces has dimension at least $54+3+48+28 = 133 = \dim \mathfrak{e}_7$. Hence, the inequalities in the dimensions above are equalities, and the result follows. □

Proposition 4.4.3. *The $\mathbb{Z} \times \mathbb{Z}_3^3$-grading on the Albert pair extends to a fine grading with universal group $\mathbb{Z} \times \mathbb{Z}_3^3$ and type $(55, 0, 26)$ on $\mathfrak{e}_7$.*

Proof. This is a consequence of Theorem 2.1.21, except for the type, which we will now compute. We know that our grading satisfies that, if $\mathbb{A}_g^\sigma = \mathbb{F}x$ for some $0 \neq x \in \mathbb{A}$, then $\mathbb{A}_{-g}^{-\sigma} = \mathbb{F}x^{-1}$. Hence L_e^0 is spanned by elements of the form $\nu(x, x^{-1})$. But it is well-known that, if an element x is invertible in a Jordan algebra, then $\{x, x^{-1}, \cdot\} = 2\mathrm{id}$. Therefore, $L_e^0 = \mathbb{F}(\mathrm{id}, -\mathrm{id})$ has dimension 1. (Actually, L_e^0 is the center of L^0.) Moreover, the subspace $L^1 \oplus L^{-1} = \mathbb{A}^+ \oplus \mathbb{A}^-$ consists of 54 homogeneous components of dimension 1.

The rest of homogeneous components span a subspace of dimension $133 - 55 = 78$ (actually, a subalgebra isomorphic to $\mathfrak{e}_6$) and support $\{(0, g) \mid 0 \neq g \in \mathbb{Z}_3^3\}$, and since its homogeneous components are clearly in the same orbit under the action of $\mathrm{Aut}\,\Gamma$ (see Theorem 4.5.1 and its proof for more details), each of them must have dimension $78/26 = 3$. □

4.5 Weyl groups of fine gradings on Albert systems

Now we will compute the Weyl groups of the fine gradings on the Albert pair and Albert triple system.

As a consequence of Corollary 2.1.14 and the classification of the Weyl groups of fine gradings on $\mathbb{A}$ (see [EK13]), we already know the Weyl groups of the fine gradings on the Albert triple system.

Theorem 4.5.1. *Let Γ be the fine grading on $\mathcal{V}_\mathbb{A}$ with universal group $\mathbb{Z} \times \mathbb{Z}_3^3$. Then, with the natural identification of $\mathrm{Aut}(\mathbb{Z} \times \mathbb{Z}_3^3)$ with a group of 4×4-matrices,*

$$\mathcal{W}(\Gamma) \cong \left\{ \left(\begin{array}{c|c} 1 & 0 \\ \hline A & B \end{array} \right) \mid A \in \mathcal{M}_{3\times 1}(\mathbb{Z}_3), B \in \mathrm{SL}_3(\mathbb{Z}_3) \right\}.$$

Proof. Set $G = \mathbb{Z} \times \mathbb{Z}_3^3$ and identify the subgroups $\mathbb{Z}$ and $\mathbb{Z}_3^3$ with $\mathbb{Z} \times 0$ and $0 \times \mathbb{Z}_3^3$. Let $\Gamma_\mathbb{A}$ be the $\mathbb{Z}_3^3$-grading on $\mathbb{A}$ as given in Equation (1.5.9), and $\deg_{\mathbb{A}}$ its degree map. Let a_1, a_2, a_3 denote the canonical generators of $\mathbb{Z}_3^3$ (hence $\deg_{\mathbb{A}}(X_i) = a_i$), and write a for the generator 1 of $\mathbb{Z}$. Therefore, for each homogeneous element $x \in \mathbb{A}$, we have $\deg(x^{\pm}) = (\pm 1, \deg_{\mathbb{A}}(x))$ in Γ. By [EK12b], $\mathcal{W}(\Gamma_{\mathbb{A}}) \cong \mathrm{SL}_3(\mathbb{Z}_3)$. With the identification $\mathrm{Aut}\,\mathbb{A} \leq \mathrm{Aut}\,\mathcal{V}_{\mathbb{A}}$, we have $\mathrm{Aut}\,\Gamma_{\mathbb{A}} \leq \mathrm{Aut}\,\Gamma$, and we can also identify $\mathcal{W}(\Gamma_{\mathbb{A}})$ with a subgroup of $\mathcal{W}(\Gamma)$ which acts on $\mathbb{Z}_3^3$ and fixes the generator a of $\mathbb{Z}$. Since $\mathrm{Supp}\,\Gamma^+$ and the torsion subgroup $\mathbb{Z}_3^3$ are $\mathcal{W}(\Gamma)$-invariant, we deduce that $\mathrm{SL}_3(\mathbb{Z}_3)$, 0 and 1 appear in the block structure of $\mathcal{W}(\Gamma)$, as it is described above. We claim now that $\mathcal{M}_{3\times 1}(\mathbb{Z}_3)$ appears in the block structure, and for this purpose, it suffices to find an element of $\mathcal{W}(\Gamma)$ given by $a \mapsto a + a_3$ and that fixes each a_i. Take $\varphi = c_{1,\omega^2,\omega}$ with ω a primitive cubic root of 1, and the induced automorphism τ in $\mathcal{W}(\Gamma)$ (notation $c_{\lambda_1,\lambda_2,\lambda_3}$ as in Subsection 2.2.2). It is clear that $\tau(a_1) = a_1$ and $\tau(a_2) = a_2$. Since $\varphi(1) = \sum_{i=1}^3 \omega^{2i} E_i$, we have $\tau(a) = a + a_3$. Also, $\varphi(\sum_{i=1}^3 \omega^{2i} E_i) = \sum_{i=1}^3 \omega^{i} E_i$, from where we get $\tau(a + a_3) = a + 2a_3$, and so $\tau(a_3) = a_3$. We conclude that τ is the element of $\mathcal{W}(\Gamma)$ that we were looking for, and hence a subgroup $\mathcal{W}$ of $\mathcal{W}(\Gamma)$ as in the statement appears. It remains to prove that $\mathcal{W}(\Gamma) \leq \mathcal{W}$.

Take $\phi \in \mathcal{W}(\Gamma)$; we claim that $\phi \in \mathcal{W}$. Without loss of generality for our purpose, if we compose with elements of $\mathcal{W}$ we can assume that $\phi(a) = a$. It suffices to show that ϕ acts on $\mathbb{Z}_3^3$ as an element of $\mathrm{SL}_3(\mathbb{Z}_3)$. We know by [EK12b] that there are two equivalent but nonisomorphic G-gradings on $\mathbb{A}$, that we denote by $\Gamma^+ = \Gamma_{\mathbb{A}}$ and Γ^-. (The nonisomorphy is due to the existence of homogeneous elements X_i in Γ^+ and X_i' in Γ^-, with X_i and X_i' of the same degree, and such that $(X_1X_2)X_3 = \omega X_1(X_2X_3)$ and $(X_1'X_2')X_3' = \omega^2 X_1'(X_2'X_3')$.) Notice that the product in the algebra is determined by the triple product and the elements $1^{\pm}$ (because $\{x, 1, y\} = 2xy$), so it follows that Γ^+ and Γ^- remain nonisomorphic when they are considered as $\mathbb{Z}_3^3$-gradings on $\mathcal{V}_{\mathbb{A}}$. Thus, the whole $\mathrm{GL}_3(\mathbb{Z}_3)$ cannot appear in the block structure. Since $\mathrm{SL}_3(\mathbb{Z}_3)$ has index 2 in $\mathrm{GL}_3(\mathbb{Z}_3)$, we deduce that $\mathrm{SL}_3(\mathbb{Z}_3)$ is exactly what appears in the block structure of $\mathcal{W}(\Gamma)$. We conclude that ϕ acts on $\mathbb{Z}_3^3$ as an element of $\mathrm{SL}_3(\mathbb{Z}_3)$, and so $\phi \in \mathcal{W}$. □

Theorem 4.5.2. *Let Γ be the Cayley-Dickson $\mathbb{Z}^3 \times \mathbb{Z}_2^3$-grading on $\mathcal{V}_{\mathbb{A}}$. Then,*

$$\mathcal{W}(\Gamma) \cong \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline B & C \end{array} \right) \mid A \in \mathrm{Sym}(3) = \langle \tau, \sigma \rangle, B \in \mathcal{M}_3(\mathbb{Z}_2), C \in \mathrm{GL}_3(\mathbb{Z}_2) \right\},$$

with

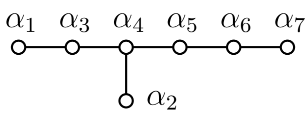
$$\tau = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \in \mathrm{GL}_3(\mathbb{Z}_2).$$

Proof. Set $G = \mathbb{Z}^3 \times \mathbb{Z}_2^3$ and identify $\mathbb{Z}^3$ and $\mathbb{Z}_2^3$ with the subgroups $\mathbb{Z}^3 \times 0$ and $0 \times \mathbb{Z}_2^3$. Let a_1, a_2, a_3 be the canonical generators of $\mathbb{Z}_2^3$, and $b_i = \deg^+(E_i)$, for $i = 1, 2, 3$, the generators of $\mathbb{Z}^3$. Let $\Gamma_{\mathcal{C}}$ be the $\mathbb{Z}_2^3$-grading on $\mathcal{C}$. It is well-known (see [EK13, Th. 4.19]) that $\mathcal{W}(\Gamma_{\mathcal{C}}) \cong \mathrm{Aut}(\mathbb{Z}_2^3) \cong \mathrm{GL}_3(\mathbb{Z}_2)$, and the automorphisms of $\mathcal{C}$ are extended to related triples, which are also extended to $\mathrm{Aut}(\mathcal{V}_{\mathbb{A}})$, and hence $\mathrm{GL}_3(\mathbb{Z}_2)$ appears in the block structure. Since the torsion subgroup $\mathbb{Z}_2^3$ is $\mathcal{W}(\Gamma)$-invariant, the zero block must appear. The homogeneous components consisting of elements of rank 1 are exactly the ones of the idempotents E_i, and therefore the set $\{b_1, b_2, b_3\}$ is $\mathcal{W}(\Gamma)$-invariant. This implies that the $(1,1)$-block is, up to isomorphism, a subgroup of $\mathrm{Sym}(3)$; since there are elements of $\mathrm{Aut}\,\Gamma$ that permute the idempotents E_i, the group $\mathrm{Sym}(3)$ must be what appears in the block. On the other hand, for the Cayley-Dickson grading Γ' of $\mathcal{V}_{\mathcal{B}}$, we know that there are related triples in $\mathrm{Aut}\,\Gamma'$ that do not fix the subgroup $\mathbb{Z}^3$ of the universal group, and these are obtained as restriction of elements of $\mathrm{Aut}\,\Gamma$ that do not fix $\mathbb{Z}^3$, so it follows that $\mathcal{M}_3(\mathbb{Z}_2)$ must appear in the block structure. This concludes the proof. □

We will now compute the Weyl group of the Cartan grading on $\mathcal{V}_{\mathbb{A}}$.

Let $\mathcal{V}$ denote the Albert pair. Let Γ be the Cartan grading on $\mathcal{V}$ by $G = \mathcal{U}(\Gamma) = \mathbb{Z}^7$. Let $T := \mathrm{Diag}(\Gamma) \leq \mathrm{Aut}(\mathcal{V}) \leq \mathrm{Aut}(L)$, where $L = \mathrm{TKK}(\mathcal{V}) = L^{-1} \oplus L^0 \oplus L^1$. Then, T is a maximal torus of $\mathrm{Aut}(L)$ that preserves L^i for $i = -1, 0, 1$. Consider the extended grading $\widetilde{\Gamma} = E_G(\Gamma)$ on L (the Cartan grading) and let Φ be the root system associated to $\widetilde{\Gamma}$. We have the corresponding root space decomposition $L = H \oplus (\bigoplus_{\alpha\in\Phi} L_\alpha)$, where the Cartan subalgebra H is contained in L^0, Φ splits as a disjoint union $\Phi = \Phi^{-1} \cup \Phi^0 \cup \Phi^1$, and $\Gamma^\sigma \colon \mathcal{V}^\sigma = \bigoplus_{\alpha\in\Phi^{\sigma 1}} L_\alpha$. Also, $L^0 = Z(L^0) \oplus [L^0, L^0]$ with $\dim Z(L^0) = 1$, where $[L^0, L^0]$ is simple of type E_6.

Take a system of simple roots $\Delta = \{\alpha_1, \dots, \alpha_7\}$ of Φ with Dynkin diagram

E_7 

(4.5.1)

such that $\{\alpha_1, \dots, \alpha_6\}$ is a system of simple roots of Φ^0 and $\Phi^{\pm 1} = \{\sum_{i=1}^{7} m_i\alpha_i \in \Phi \mid m_7 = \pm 1\}$.

Any $\varphi = (\varphi^+, \varphi^-) \in \mathrm{Aut}(\Gamma)$ induces an automorphism $\bar{\varphi} \in \mathrm{Aut}\,\mathcal{U}(\Gamma)$ which in turn induces an automorphism $\widehat{\varphi} \in \mathrm{Aut}\,\Phi$ preserving Φ^i for $i = -1, 0, 1$. Conversely, given any $\psi \in \mathrm{Aut}\,\Phi$ preserving Φ^i for $i = -1, 0, 1$, there is an automorphism $\varphi \in \mathrm{Aut}(L)$ such that $\varphi(L_\alpha) = L_{\psi(\alpha)}$ for each

$\alpha \in \Phi$; in particular, $\varphi(L^{\pm 1}) = \varphi(\bigoplus_{\alpha\in\Phi^{\pm 1}} L_\alpha) = L_{\pm 1}$ because $\psi(\Phi^{\pm 1}) = \Phi^{\pm 1}$, so φ restricts to an automorphism of $\mathcal{V}$. Therefore we have proven:

Theorem 4.5.3. *Let Γ denote the Cartan grading on the Albert pair. Then, the Weyl group of Γ is isomorphic to the group*

$$\mathcal{W} := \{\psi \in \operatorname{Aut}\Phi \mid \psi(\Phi^i) = \Phi^i, i = -1, 0, 1\},$$

where Φ is the root system of type E_7.

Consider the restriction map

$$\Theta\colon \mathcal{W} \to \operatorname{Aut}\Phi^0. \tag{4.5.2}$$

Lemma 4.5.4. *The map Θ is injective and* im Θ *is the Weyl group of type E_6.*

Proof. This proof is analogous to the proof of Lemma 3.6.5, so we will not give all the details. Recall the ordering of the roots in (4.5.1). Extend Θ to the restriction map

$$\widehat{\Theta}\colon \operatorname{Stab}_{\operatorname{Aut}\Phi}(\Phi^0) \to \operatorname{Aut}\Phi^0.$$

The proof of the injectivity of $\widehat{\Theta}$ is similar to the one given in the proof of Lemma 3.6.5. In this case, this is proven with the same arguments and using that the orthogonal subspace to $\alpha_1, \dots, \alpha_6$ is spanned by the fundamental dominant weight $w_7 = \frac{1}{2}(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7)$.

The proof of the surjectivity of $\widehat{\Theta}$ is similar to the one given in the proof of Lemma 3.6.5. In this case, this is proven using the automorphism ψ of Φ that fixes α_2 and α_4, interchanges $\alpha_1 \leftrightarrow \alpha_6$, $\alpha_3 \leftrightarrow \alpha_5$, and sends α_7 to the lowest root $-(2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7))$.

Finally, the reflections $s_{\alpha_1}, \dots, s_{\alpha_6}$ in Φ belong to $\mathcal{W}$, while ψ in (3.6.3) permutes Φ^1 and Φ^{-1}. Hence, since $\widehat{\Theta}$ is bijective, we get $\operatorname{Stab}_{\operatorname{Aut}\Phi}(\Phi^0) = \mathcal{W} \rtimes \langle\psi\rangle$ and im $\Theta = \widetilde{\Theta}(\mathcal{W}) = \mathcal{W}_{E_6}$. □

Theorem 4.5.5. *The Weyl group of the Cartan grading on the Albert pair is isomorphic to the Weyl group of the root systems of type E_6.*

Proof. Consequence of Lemma 4.5.4. □

Chapter 5

A $\mathbb{Z}_4^3$-grading on the Brown algebra

This Chapter is part of a joint work with A. Elduque and M. Kochetov ([AEK14]). In Section 5.1 we recall some well-known constructions of the Brown algebra. A construction of a fine $\mathbb{Z}_4^3$-grading on the Brown algebra is given in Section 5.2, which is one of the main original contributions of the author in this thesis. (Another different construction of this grading was given in [AEK14], which is not included in this thesis since the author did not contribute to it.) In the latter sections, which are a joint work with the authors mentioned above, we give a recognition Theorem of the $\mathbb{Z}_4^3$-grading, the Weyl group of the grading is computed, and we study some related fine gradings induced on the exceptional simple Lie algebras of type E.

The main results in this chapter are given in Section 5.2, where the $\mathbb{Z}_4^3$-grading is constructed explicitly, and Theorem 5.3.1, that is the recognition theorem.

5.1 Brown algebras

5.1.1 Brown algebras via Cayley–Dickson process

Recall that we assume char $\mathbb{F} \neq 2$. The split Brown algebra can be obtained as the Cayley–Dickson double of two different separable Jordan algebras of degree 4. We will consider a more general situation. For the basic definitions of structurable algebras and the Cayley-Dickson doubling process, the reader may consult Chapter 1.

Let $\mathcal{Q}$ be a quaternion algebra over $\mathbb{F}$ with its standard involution, $q \mapsto \bar{q}$. The algebra $M_4(\mathcal{Q})$ is associative and has a natural involution $(q_{ij})^* = (\bar{q}_{ji})$, so $\mathcal{H}_4(\mathcal{Q}) := \{x \in M_4(\mathcal{Q}) \mid x^* = x\}$ is a Jordan algebra with respect to the symmetrized product $(x, y) \mapsto \frac{1}{2}(xy + yx)$. This is a simple Jordan algebra of degree 4 and dimension 28, so $\mathfrak{CD}(\mathcal{H}_4(\mathcal{Q}), \mu)$ is a structurable algebra of dimension 56, for any $\mu \in \mathbb{F}^\times$.

Remark 5.1.1. If char $\mathbb{F} = 3$, we cannot apply the results in [AF84] directly, but $\mathcal{Q}$ can be obtained by "extension of scalars" from the "generic" quaternion algebra $\tilde{\mathcal{Q}}$ over the polynomial ring $\mathbb{Z}[X, Y]$, hence $\mathcal{H}_4(\mathcal{Q})$ can be obtained from the Jordan algebra $\mathcal{H}_4(\tilde{\mathcal{Q}})$ over $\mathbb{Z}[\frac{1}{2}][X, Y]$, and $\mathfrak{CD}(\mathcal{H}_4(\mathcal{Q}), \mu)$ can be obtained from the algebra $\mathfrak{CD}(\mathcal{H}_4(\tilde{\mathcal{Q}}), Z)$ over $\mathbb{Z}[\frac{1}{2}][X, Y, Z]$, which satisfies the required identities because it is a subring with involution in a structurable algebra over the field $\mathbb{Q}(X, Y, Z)$.

Let $\mathcal{C}$ be an octonion algebra over $\mathbb{F}$ and consider the associated Albert algebra $\mathbb{A} = \mathcal{H}_3(\mathcal{C})$. Then, $\mathbb{A} \times \mathbb{F}$ is a separable Jordan algebra of degree 4 and dimension 28, so $\mathfrak{CD}(\mathbb{A} \times \mathbb{F}, \mu)$ is a structurable algebra of dimension 56, for any $\mu \in \mathbb{F}^\times$.

The connection between the above two Cayley–Dickson doubles is the following: if $\mathcal{C} = \mathfrak{CD}(\mathcal{Q}, \mu)$, then $\mathfrak{CD}(\mathcal{H}_4(\mathcal{Q}), \mu)$ is isomorphic to $\mathfrak{CD}(\mathbb{A} \times \mathbb{F}, \mu)$. Indeed, we have $\mathfrak{CD}(\mathcal{H}_4(\mathcal{Q}), \mu) = \mathcal{H}_4(\mathcal{Q}) \oplus v\mathcal{H}_4(\mathcal{Q})$ and $\mathfrak{CD}(\mathbb{A} \times \mathbb{F}, \mu) = (\mathbb{A} \times \mathbb{F}) \oplus v'(\mathbb{A} \times \mathbb{F})$ with $v^2 = \mu 1 = v'^2$. For any $a \in \mathcal{Q}$, define the elements of $\mathcal{H}_4(\mathcal{Q})$:

$$\iota_1'(a) = \begin{pmatrix} 0 & 0 & 0 & 2a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2\bar{a} & 0 & 0 & 0 \end{pmatrix}, \ \iota_2'(a) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2a \\ 0 & 0 & 0 & 0 \\ 0 & 2\bar{a} & 0 & 0 \end{pmatrix}, \ \iota_3'(a) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2a \\ 0 & 0 & 2\bar{a} & 0 \end{pmatrix}.$$

Then we have a $\mathbb{Z}_2$-grading on $\mathcal{H}_4(\mathcal{Q})$ given by $\mathcal{H}_4(\mathcal{Q})_{\bar{0}} = \operatorname{diag}(\mathcal{H}_3(\mathcal{Q}), \mathbb{F})$ and $\mathcal{H}_4(\mathcal{Q})_{\bar{1}} = \bigoplus_{j=1}^3 \iota_j'(\mathcal{Q})$. The automorphism of order 2 producing this grading can be extended to an automorphism of $\mathcal{A} = \mathfrak{CD}(\mathcal{H}_4(\mathcal{Q}), \mu)$ sending v to $-v$, which also has order 2 and will be denoted by Υ. The fixed subalgebra of Υ is $\mathcal{B} = \operatorname{diag}(\mathcal{H}_3(\mathcal{Q}), \mathbb{F}) \oplus \bigoplus_{j=1}^3 v\iota_j'(\mathcal{Q})$. The involution is trivial on $\mathcal{B}$, so it is a Jordan algebra. Since L_v is an invertible operator, the $\mathbb{Z}_2$-grading produced by Υ is $\mathcal{A} = \mathcal{B} \oplus v\mathcal{B}$. Write $\mathcal{C} = \mathcal{Q} \oplus u\mathcal{Q}$ with $u^2 = \mu 1$. Then it is straightforward to verify that the mapping $\varphi_{\mathfrak{CD}} \colon \mathcal{B} \to \mathbb{A} \times \mathbb{F}$ defined by $\operatorname{diag}(x, \lambda) \mapsto (x, \lambda)$, for $x \in \mathcal{H}_3(\mathcal{Q})$, $\lambda \in \mathbb{F}$, and $v\iota_j'(a) \mapsto (\iota_j(ua), 0)$, for $a \in \mathcal{Q}$, is an isomorphism of algebras. Moreover, we have $\varphi_{\mathfrak{CD}}(b^\theta) = \varphi_{\mathfrak{CD}}(b)^\theta$ for all $b \in \mathcal{B}$, so identities (1.10.4) for the algebra $\mathcal{A}$ imply that $\varphi_{\mathfrak{CD}}$ can be extended to an isomorphism $\varphi_{\mathfrak{CD}} \colon \mathfrak{CD}(\mathcal{H}_4(\mathcal{Q}), \mu) \to \mathfrak{CD}(\mathbb{A} \times \mathbb{F}, \mu)$ sending v to v'.

Definition 5.1.2. Let $\mathcal{Q}$ be a quaternion algebra over $\mathbb{F}$ and let $\mathcal{C} = \mathfrak{CD}(\mathcal{Q}, 1)$, so $\mathcal{C}$ is the split octonion algebra and $\mathbb{A} = \mathcal{H}_3(\mathcal{C})$ is the split Albert algebra.

Then the structurable algebra $\mathfrak{CD}(\mathcal{H}_4(\mathcal{Q}), 1) \cong \mathfrak{CD}(\mathbb{A} \times \mathbb{F}, 1)$ will be referred to as the *split Brown algebra*.

5.1.2 Brown algebras as structurable matrix algebras

It is shown in [AF84], assuming char $\mathbb{F} \neq 2, 3$, that the admissible triple (T, N, N) arising from a separable Jordan algebra $\mathcal{J}$ of degree 3 can be realized on the space of elements with generic trace 0 in the separable Jordan algebra $\mathcal{J} \times \mathbb{F}$ of degree 4 (see Propositions 5.6 and 6.5) so that $\mathfrak{CD}(\mathcal{J} \times \mathbb{F}, 1)$ is isomorphic to the structurable matrix algebra defined by (T, N, N). We will now exhibit this isomorphism for the case $\mathcal{J} = \mathbb{A}$ and see that it also works in the case char $\mathbb{F} = 3$.

Remark 5.1.3. If char $\mathbb{F} = 3$, we can still define "structurable matrix algebras" starting from the cubic form $N(x)$ and taking its polarization for the symmetric trilinear form $N(x, y, z)$.

For the admissible triple (T, N, N) on $\mathbb{A}$, we have

$$\begin{aligned} x^{\#} &= x^2 - T(x)x + S(x)1 \quad (\textit{Freudenthal adjoint}), \\ x \times y &= (x+y)^{\#} - x^{\#} - y^{\#} \quad (\textit{Freudenthal cross product}), \\ S(x) &= \frac{1}{2}(T(x)^2 - T(x^2)), \end{aligned}$$

for any $x, y \in \mathbb{A}$, hence we have the identities

$$\begin{aligned} x \times x &= 2x^2 - 2T(x)x + \big(T(x)^2 - T(x^2)\big)1, \\ x \times 1 &= T(x)1 - x. \end{aligned} \tag{5.1.1}$$

Let $\tilde{\mathcal{A}}$ be the corresponding structurable matrix algebra (defined as in Section 1.9) and let $s_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, so s_0 spans the space of skew elements and $s_0^2 = 1$. For $x \in \mathbb{A}$, denote $\eta(x) = \left(\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right)$ and $\eta'(x) = \left(\begin{smallmatrix} 0 & 0 \\ x & 0 \end{smallmatrix}\right)$. The subalgebra $\tilde{\mathcal{B}} := \{\eta(x) + \eta'(x) + \lambda 1 \mid x \in \mathbb{A}, \lambda \in \mathbb{F}\}$ of $\tilde{\mathcal{A}}$ consists of symmetric elements, so it is a Jordan algebra. We claim that it is isomorphic to $\mathbb{A} \times \mathbb{F}$. Indeed, define a linear injection $\iota \colon \mathbb{A} \to \tilde{\mathcal{B}}$ by setting $\iota(x) = \frac{1}{4}\big(\eta(2x - T(x)1) + \eta'(2x - T(x)1) + T(x)1\big)$ for all $x \in \mathbb{A}$. Using identities (5.1.1), one verifies that $\iota(x)^2 = \iota(x^2)$, so ι is a nonunital monomorphism of algebras. Then $e_{\mathbb{A}} = \iota(1)$ and $e_{\mathbb{F}} = 1 - e_{\mathbb{A}}$ are orthogonal idempotents and $\tilde{\mathcal{B}} = \iota(\mathbb{A}) \oplus \mathbb{F}e_{\mathbb{F}}$. We conclude that $\mathbb{A} \times \mathbb{F} \to \tilde{\mathcal{B}}$, $(x, \lambda) \mapsto \iota(x) + \lambda e_{\mathbb{F}}$, is an isomorphism of algebras. This isomorphism extends to an isomorphism $\mathfrak{CD}(\mathbb{A} \times \mathbb{F}, 1) = (\mathbb{A} \times \mathbb{F}) \oplus v'(\mathbb{A} \times \mathbb{F}) \to \tilde{\mathcal{A}}$ sending v' to s_0. (The bilinear form $\phi(a, b)$ in the Cayley–Dickson construction corresponds to $2\mathrm{tr}(a\bar{b})$ on the structurable matrix algebra.)

5.2 A construction of a fine $\mathbb{Z}_4^3$-grading

In this section, we will construct a $\mathbb{Z}_4^3$-grading for the model of the split Brown algebra as in Subsection 5.1.2, assuming $\mathbb{F}$ contains a 4-th root of unity $\mathbf{i}$. Let $\mathcal{C}$ be the split octonion algebra and let $\mathbb{A} = \mathcal{H}_3(\mathcal{C})$ be the split Albert algebra, with generic trace T and generic norm N. Consider the structurable matrix algebra $\mathcal{A}$ associated to the admissible triple (T, N, N), i.e., the product is given by (1.9.1) and the involution is given by (1.9.2). Note that the Freudenthal cross product on $\mathbb{A}$, which appears in (1.9.1), is given by:

i) $E_i \times E_{i+1} = E_{i+2},\ \ E_i \times E_i = 0,$

ii) $E_i \times \iota_i(x) = -\iota_i(x),\ \ E_i \times \iota_{i+1}(x) = 0 = E_i \times \iota_{i+2}(x),$

iii) $\iota_i(x) \times \iota_i(y) = -4n(x,y)E_i,\ \ \iota_i(x) \times \iota_{i+1}(y) = 2\iota_{i+2}(\bar{x}\bar{y}).$

As shown in [AM99], for the $\mathbb{Z}_2^3$-grading on the split Cayley algebra $\mathcal{C}$ one can choose a homogeneous basis $\{x_g \mid g \in \mathbb{Z}_2^3\}$ such that the product is given by $x_g x_h = \sigma(g,h)x_{g+h}$ where

$$\begin{aligned} \sigma(g,h) &= (-1)^{\psi(g,h)}, \\ \psi(g,h) &= h_1g_2g_3 + g_1h_2g_3 + g_1g_2h_3 + \sum_{i\le j} g_ih_j. \end{aligned}$$

Consider the para-Cayley algebra associated to $\mathcal{C}$, i.e., the same vector space with the new product $x * y = \bar{x}\bar{y}$. Note that $x_g * x_h = \gamma(g,h)x_{g+h}$ where

$$\begin{aligned} \gamma(g,h) &= s(g)s(h)\sigma(g,h), \\ s(g) &= (-1)^{\phi(g)}, \\ \phi(g) &= \sum_i g_i + \sum_{i<j} g_ig_j + g_1g_2g_3, \end{aligned}$$

because $s(g) = -1$ if $g \neq 0$ and $s(0) = 1$, so $\bar{x}_g = s(g)x_g$ for all $g \in \mathbb{Z}_2^3$.

Denote $a_0 = 0$, $a_1 = (\bar{0},\bar{1},\bar{0})$, $a_2 = (\bar{1},\bar{0},\bar{0})$, $a_3 = a_1 + a_2$, $g_0 = (\bar{0},\bar{0},\bar{1})$ in $\mathbb{Z}_2^3$. We will consider the quaternion algebra $\mathcal{Q} = \mathrm{span}\{x_{a_i} \mid i = 0,1,2,3\}$ with the ordered basis $B_{\mathcal{Q}} = \{x_{a_i} \mid i = 0,1,2,3\}$, and $\mathcal{Q}^\perp = \mathrm{span}\{x_{g_0+a_i} \mid i = 0,1,2,3\}$ with the ordered basis $B_{\mathcal{Q}^\perp} = \{x_{g_0+a_i} \mid i = 0,1,2,3\}$. Thus, $B_{\mathcal{C}} = B_{\mathcal{Q}} \cup B_{\mathcal{Q}^\perp}$ is an ordered basis of $\mathcal{C}$. It will be convenient to write the values $\gamma(g,h)$ as an 8×8 matrix according to this ordering and split this matrix into 4×4 blocks: $\gamma = \left(\begin{smallmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{smallmatrix}\right)$, so γ_{11} records the values for the support of $\mathcal{Q}$, etc.

A straightforward calculation shows that

$$\gamma_{11} = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 \end{bmatrix}, \quad \gamma_{12} = \begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{bmatrix},$$
$$\gamma_{21} = \begin{bmatrix} -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \end{bmatrix}, \quad \gamma_{22} = \begin{bmatrix} -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}. \tag{5.2.1}$$

Define $\sigma_j(h) := \sigma(a_j, g_0 + h)$ for any $h \in \text{Supp } \mathcal{Q}^\perp$, $j = 1, 2, 3$. Note that the matrix of $\sigma(a_j, g_0 + h)$, $h \in B_{\mathcal{Q}^\perp}$, coincides with the matrix σ_{11}, which is given by

$$\sigma_{11} = (\sigma(a_j, a_k))_{j,k} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix}. \tag{5.2.2}$$

We will need the following result in our construction of the $\mathbb{Z}_4^3$-grading on $\mathcal{A}$.

Lemma 5.2.1. *The basis $B_{\mathcal{C}} = B_{\mathcal{Q}} \cup B_{\mathcal{Q}^\perp}$ of $\mathcal{C}$ has the following properties:*

(P_{11}) $\quad \gamma(g, g') = \gamma(g + a_j, g' + a_{j+1})$,

(P_{22}) $\quad \gamma(h, h') = \sigma_j(h)\sigma_{j+1}(h')\gamma(h + a_j, h' + a_{j+1})$,

(P_{12}) $\quad \gamma(g, h) = \sigma_{j+1}(h)\sigma_{j+2}(g + h)\gamma(g + a_j, h + a_{j+1})$,

(P_{21}) $\quad \gamma(h, g) = \sigma_j(h)\sigma_{j+2}(g + h)\gamma(h + a_j, g + a_{j+1})$,

for all $g, g' \in \text{Supp } \mathcal{Q}$, $h, h' \in \text{Supp } \mathcal{Q}^\perp$ and $j \in \{1, 2, 3\}$.

Proof. To shorten the proof, we will use matrices, but we need to introduce some notation. For $j = 1, 2, 3$, let σ_j be the column of values $\sigma_j(h)$, $h \in \text{Supp } \mathcal{Q}^\perp$, i.e., σ_j is the traspose of the corresponding row of matrix σ_{11}. We will denote by $\cdot$ the entry-wise product of matrices. (It is interesting to note that the rows and columns of σ_{11} are the characters of $\mathbb{Z}_2^2$, which is related to the obvious fact $\sigma_j \cdot \sigma_{j+1} = \sigma_{j+2}$.) Denote $M_{\sigma_j} = [\sigma_j|\sigma_j|\sigma_j|\sigma_j]$ (the column σ_j repeated 4 times), and define the permutation matrices

$$P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Note that the properties asserted in this lemma possess a cyclic symmetry in $j = 1, 2, 3$ (as can be checked in the four blocks of γ), so it suffices to verify them for $j = 1$. Then, property (P_{11}) can be written as $\gamma_{11} = P_1\gamma_{11}P_2$, because $P_1\gamma_{11}P_2$ is the matrix associated to $\gamma(g + a_1, g' + a_2)$. Similarly, property (P_{22}) can be writen as $\gamma_{22} = M_{\sigma_1} \cdot (P_1\gamma_{22}P_2) \cdot M^t_{\sigma_2}$. Note that σ is multiplicative in the second variable (because ψ is linear in the second variable), so $\sigma_3(g + h) = \sigma(a_3, g + g_0 + h) = \sigma(a_3, g)\sigma(a_3, g_0 + h) = \sigma_3(g_0 + g)\sigma_3(h)$. Therefore, (P_{12}) and (P_{21}) can be written as $\gamma_{12} = M_{\sigma_3} \cdot (P_1\gamma_{12}P_2) \cdot M^t_{\sigma_2\cdot\sigma_3}$ and $\gamma_{21} = M_{\sigma_1\cdot\sigma_3} \cdot (P_1\gamma_{21}P_2) \cdot M^t_{\sigma_3}$. It is straightforward to check these four matrix equations. □

We will consider $\mathbb{Z}_2^3$ as a subgroup of $\mathbb{Z}_4^3$ via the embedding $a_1 \mapsto (\bar{0}, \bar{2}, \bar{0})$, $a_2 \mapsto (\bar{0}, \bar{0}, \bar{2})$, $a_3 \mapsto (\bar{0}, \bar{2}, \bar{2})$ and $g_0 \mapsto (\bar{2}, \bar{0}, \bar{0})$, so we can assume that γ and σ are defined on a subgroup of $\mathbb{Z}_4^3$ and take values as recorded in matrices (5.2.1) and (5.2.2). Define $b_1 = (\bar{0}, \bar{1}, \bar{0})$, $b_2 = (\bar{0}, \bar{0}, \bar{1})$ and $b_3 = -b_1 - b_2$ in $\mathbb{Z}_4^3$. Note that $\sum b_j = 0$ and $a_j \mapsto 2b_j$ under the embedding.

Now we will define a $\mathbb{Z}_4^3$-grading on $\mathcal{A}$ by specifying a homogeneous basis. For each $g \in \text{Supp } \mathcal{Q}$, $h \in \text{Supp } \mathcal{Q}^\perp$ and $j \in \{1, 2, 3\}$, consider the elements of $\mathcal{A}$:

$$\begin{aligned} \alpha_{j,g} &:= \begin{pmatrix} 0 & \iota_j(x_g) \\ \iota_j(x_{g+a_j}) & 0 \end{pmatrix}, \quad \alpha'_{j,h} := \begin{pmatrix} 0 & \sigma_j(h)\mathbf{i}\iota_j(x_h) \\ \iota_j(x_{h+a_j}) & 0 \end{pmatrix}, \\ \varepsilon_j &:= \begin{pmatrix} 0 & E_j \\ E_j & 0 \end{pmatrix}, \qquad\qquad \varepsilon'_j := \varepsilon_j s_0 = \begin{pmatrix} 0 & -E_j \\ E_j & 0 \end{pmatrix}. \end{aligned} \tag{5.2.3}$$

Then $B_\mathcal{A} = \{1, s_0, \alpha_{j,g}, \alpha_{j,g}s_0, \alpha'_{j,h}, \alpha'_{j,h}s_0, \varepsilon_j, \varepsilon'_j\}$ is a basis of $\mathcal{A}$. Set

$$\begin{array}{ll} \deg(1) := 0, & \deg(\varepsilon_j) := a_j, \\ \deg(\alpha_{j,g}) := b_j + g, & \deg(\alpha'_{j,h}) := (\bar{1}, \bar{0}, \bar{0}) + b_j + h, \\ \deg(xs_0) := \deg(x) + g_0 & \text{for } x \in \{1, \alpha_{j,g}, \alpha'_{j,h}, \varepsilon_j\}. \end{array} \tag{5.2.4}$$

To check that (5.2.4) defines a $\mathbb{Z}_4^3$-grading, we compute the products of basis elements.

Proposition 5.2.2. *For any elements $x, y \in B_\mathcal{A} \backslash \{1, s_0\}$, if $\deg(x) + \deg(y) \neq g_0$ then $xy = yx$, and otherwise $xy = -yx$. The products of the elements of $B_\mathcal{A}$ are then determined as follows:*

i) $s_0^2 = \varepsilon_j^2 = 1 = -\varepsilon_j'^2$, $\varepsilon_j\varepsilon_{j+1} = \varepsilon_{j+2}$,

ii) $\varepsilon_j\varepsilon'_j = s_0$, $\varepsilon'_j\varepsilon'_{j+1} = \varepsilon_{j+2}$, $\varepsilon_j\varepsilon'_{j+1} = -\varepsilon'_{j+2}$, $\varepsilon_{j+1}\varepsilon'_j = -\varepsilon'_{j+2}$,

iii) $\varepsilon_j\alpha_{j,g} = \varepsilon'_j(\alpha_{j,g}s_0) = -\alpha_{j,g+a_j}$, $\varepsilon_j(\alpha_{j,g}s_0) = \varepsilon'_j\alpha_{j,g} = \alpha_{j,g+a_j}s_0$, *iv)* $\varepsilon_j\alpha'_{j,h} = \varepsilon'_j(\alpha'_{j,h}s_0) = -\mathbf{i}\sigma_j(h)\alpha'_{j,h+a_j},\ \varepsilon_j(\alpha'_{j,h}s_0) = \varepsilon'_j\alpha'_{j,h} = \mathbf{i}\sigma_j(h)\alpha'_{j,h+a_j}s_0,$

v) $\varepsilon_j\alpha_{k,g} = \varepsilon_j\alpha'_{k,h} = \varepsilon'_j\alpha_{k,g} = \varepsilon'_j\alpha'_{k,h} = 0$ *if* $j \neq k$,

vi) $\alpha_{j,g}^2 = (\alpha_{j,g}s_0)^2 = -8\varepsilon_j,\ \alpha_{j,g}\alpha_{j,g+a_j} = -(\alpha_{j,g}s_0)(\alpha_{j,g+a_j}s_0) = 8,$

vii) $\alpha_{j,g}(\alpha_{j,g}s_0) = 8\varepsilon'_j,\ \alpha_{j,g}(\alpha_{j,g+a_j}s_0) = 8s_0,$

viii) $\alpha'^2_{j,h} = (\alpha'_{j,h}s_0)^2 = 8\varepsilon'_j,\ \alpha'_{j,h}\alpha'_{j,h+a_j} = -(\alpha'_{j,h}s_0)(\alpha'_{j,h+a_j}s_0) = 8\mathbf{i}\sigma_j(h)s_0,$

ix) $\alpha'_{j,h}(\alpha'_{j,h}s_0) = -8\varepsilon_j,\ \alpha'_{j,h}(\alpha'_{j,h+a_j}s_0) = 8\mathbf{i}\sigma_j(h),$

x) $\alpha_{j,g}\alpha_{j,g'} = \alpha_{j,g}(\alpha_{j,g'}s_0) = (\alpha_{j,g}s_0)(\alpha_{j,g'}s_0) = 0$ *if* $g' \notin \{g, g+a_j\}$,

xi) $\alpha'_{j,h}\alpha'_{j,h'} = \alpha'_{j,h}(\alpha'_{j,h'}s_0) = (\alpha'_{j,h}s_0)(\alpha'_{j,h'}s_0) = 0$ *if* $h' \notin \{h, h+a_j\}$,

xii) $\alpha_{j,g}\alpha'_{j,h} = (\alpha_{j,g}s_0)\alpha'_{j,h} = \alpha_{j,g}(\alpha'_{j,h}s_0) = (\alpha_{j,g}s_0)(\alpha'_{j,h}s_0) = 0,$

xiii) $\alpha_{j,g}\alpha_{j+1,g'} = (\alpha_{j,g}s_0)(\alpha_{j+1,g'}s_0) = 2\gamma(g,g')\alpha_{j+2,g+g'+a_{j+2}},$

xiv) $(\alpha_{j,g}s_0)\alpha_{j+1,g'} = \alpha_{j,g}(\alpha_{j+1,g'}s_0) = -2\gamma(g,g')\alpha_{j+2,g+g'+a_{j+2}}s_0,$

xv) $\alpha_{j,g}\alpha'_{j+1,h} = (\alpha_{j,g}s_0)(\alpha'_{j+1,h}s_0) = 2\mathbf{i}\sigma_{j+1}(h)\gamma(g,h)\alpha'_{j+2,g+h+a_{j+2}},$

xvi) $(\alpha_{j,g}s_0)\alpha'_{j+1,h} = \alpha_{j,g}(\alpha'_{j+1,h}s_0) = -2\mathbf{i}\sigma_{j+1}(h)\gamma(g,h)\alpha'_{j+2,g+h+a_{j+2}}s_0,$

xvii) $\alpha'_{j,h}\alpha_{j+1,g} = (\alpha'_{j,h}s_0)(\alpha_{j+1,g}s_0) = 2\mathbf{i}\sigma_j(h)\gamma(h,g)\alpha'_{j+2,h+g+a_{j+2}},$

xviii) $(\alpha'_{j,h}s_0)\alpha_{j+1,g} = \alpha'_{j,h}(\alpha_{j+1,g}s_0) = -2\mathbf{i}\sigma_j(h)\gamma(h,g)\alpha'_{j+2,h+g+a_{j+2}}s_0,$

xix) $\alpha'_{j,h}\alpha'_{j+1,h'} = (\alpha'_{j,h}s_0)(\alpha'_{j+1,h'}s_0) = -2\gamma(h+a_j, h'+a_{j+1})\alpha_{j+2,h+h'+a_{j+2}}s_0,$

xx) $(\alpha'_{j,h}s_0)\alpha'_{j+1,h'} = \alpha'_{j,h}(\alpha'_{j+1,h'}s_0) = 2\gamma(h+a_j, h'+a_{j+1})\alpha_{j+2,h+h'+a_{j+2}}.$

Proof. For the first assertion, observe that x and y are symmetric with respect to the involution, while xy is symmetric if $\deg(x)+\deg(y) \neq g_0$ and skew otherwise.

Equations from i) to xii) are easily checked. For iv), viii) and ix), we use the property $\sigma_j(h+a_j) = -\sigma_j(h)$, which is a consequence of $\sigma(a_j,a_j) = -1$ and the multiplicativity of σ in the second variable.

The first equation in all cases from xiii) to xx) is easy to check, too. Also note that $(\alpha_{j,g}s_0)\alpha_{j+1,g'} = -(\alpha_{j,g}\alpha_{j+1,g'})s_0$, so case xiv) is a consequence of xiii). Similarly, cases xvi), xviii) and xx) are consequences of xv), xvii) and xix), respectively. It remains to check the second equation for the cases xiii), xv), xvii) and xix).

In xiii), equation $\alpha_{j,g}\alpha_{j+1,g'} = 2\gamma(g,g')\alpha_{j+2,g+g'+a_{j+2}}$ can be established using property (P_{11}). Indeed,

$$\begin{aligned}\alpha_{j,g}\alpha_{j+1,g'} &= \eta(2\iota_{j+2}(\bar{x}_{g+a_j}\bar{x}_{g'+a_{j+1}})) + \eta'(2\iota_{j+2}(\bar{x}_g\bar{x}_{g'})) \\ &= 2\gamma(g,g')\alpha_{j+2,g+g'+a_{j+2}},\end{aligned}$$

because

$$\bar{x}_{g+a_j}\bar{x}_{g'+a_{j+1}} = \gamma(g+a_j, g'+a_{j+1})x_{g+g'+a_{j+2}} = \gamma(g,g')x_{g+g'+a_{j+2}}$$

and $\bar{x}_g\bar{x}_{g'} = \gamma(g,g')x_{g+g'}$.

In xix), we use property (P_{22}) to obtain

$$\begin{aligned}\alpha'_{j,h}\alpha'_{j+1,h'} =&\eta(2\iota_{j+2}(\bar{x}_{h+a_j}\bar{x}_{h'+a_{j+1}})) + \eta'(2\iota_{j+2}(-\sigma_j(h)\sigma_{j+1}(h')\bar{x}_h\bar{x}_{h'})) \\ =&-2\gamma(h+a_j, h'+a_{j+1})[\eta(\iota_{j+2}(-x_{h+h'+a_{j+2}})) + \eta'(\iota_{j+2}(x_{h+h'}))] \\ =&-2\gamma(h+a_j, h'+a_{j+1})\alpha_{j+2,h+h'+a_{j+2}}s_0.\end{aligned}$$

Finally, to complete cases xv) and xvii), we use property (P_{12}), respectively (P_{21}), and the fact $\sigma_j(h+a_j) = -\sigma_j(h)$ to deduce, with the same arguments as above, that

$$\alpha_{j,g}\alpha'_{j+1,h} = 2\mathbf{i}\sigma_{j+1}(h)\gamma(g,h)\alpha'_{j+2,g+h+a_{j+2}}$$

and

$$\alpha'_{j,h}\alpha_{j+1,g} = 2\mathbf{i}\sigma_j(h)\gamma(h,g)\alpha'_{j+2,h+g+a_{j+2}}.$$

□

Clearly, all products in Proposition 5.2.2 are either zero or have the correct degree to make (5.2.4) a $\mathbb{Z}_4^3$-grading of the algebra $\mathcal{A}$. Moreover, $\mathbb{Z}_4^3$ is the universal grading group.

Remark 5.2.3. The grading given by (5.2.4) restricts to a $\mathbb{Z}_4^2$-grading on the subalgebra spanned by $\{1, \varepsilon_j, \alpha_{j,g} \mid g \in \operatorname{Supp} \mathcal{Q}\}$, which is isomorphic to $\mathcal{H}_4(\mathcal{K}) \cong M_4(\mathbb{F})^{(+)}$.

Remark 5.2.4. For any (finite-dimensional) structurable algebra $\mathcal{X}$, an element $x \in \mathcal{X}$ is said to be *(conjugate) invertible* if there exists $\hat{x} \in \mathcal{X}$ such that $V_{x,\hat{x}} = \mathrm{id}$ (equivalently, $V_{\hat{x},x} = \mathrm{id}$) — see [AF92] and references therein. If x is invertible then the operator $U_x \colon y \mapsto \{x, y, x\}$ is invertible and $\hat{x} = U_x^{-1}(x)$, so $\hat{x}$ is uniquely determined. In the case when $\mathcal{X}$ is a Jordan algebra, $\hat{x}$ coincides with the inverse of x in the Jordan sense, whereas in the case when $\mathcal{X}$ is a composition algebra, $\hat{x}$ coincides with the conjugate of the inverse of x in the sense of alternative algebras (hence the terminology). In the case of the Brown algebra, s_0 is invertible, with $(s_0)\hat{} = -s_0$, and any element x is invertible if and only if $\psi(x, U_x(s_0x)) \neq 0$, with $\hat{x} = 2\psi(x, U_x(s_0x))\hat{}\, U_x(s_0x)$, where $\psi\colon \mathcal{A} \to \mathbb{F}s_0$ is defined by $\psi(x, y) := x\bar{y} - y\bar{x}$ ([AF92, Proposition 5.4]). It is straightforward to verify that nonzero homogeneous elements in grading (5.2.4) are invertible, with $(\varepsilon_j)\hat{} = \varepsilon_j$, $(\alpha_{j,g})\hat{} = \frac{1}{8}\alpha_{j,g+a_j}$, $(\alpha'_{j,h})\hat{} = -\frac{\mathbf{i}\sigma_j(h)}{8}\alpha'_{j,h+a_j}s_0$, and $(xs_0)\hat{} = -\hat{x}s_0$.

Remark 5.2.5. Denote by Γ^+ the grading (5.2.4). With a slight modification, we can define a new $\mathbb{Z}_4^3$-grading Γ^-, determined by

$$\deg(\alpha_{1,0}) := b_1, \quad \deg(\alpha_{2,0}) := b_2, \quad \deg(\alpha'_{1,g_0}s_0) := (\bar{1}, \bar{0}, \bar{0}) + b_1 + g_0. \quad (5.2.5)$$

Although Γ^+ and Γ^- are equivalent, they are not isomorphic. Indeed, we can write

$$(\alpha_{1,0}\alpha_{2,0})\alpha'_{1,g_0} = \lambda_1\alpha_{1,0}(\alpha_{2,0}\alpha'_{1,g_0})$$

and

$$(\alpha_{1,0}\alpha_{2,0})(\alpha'_{1,g_0}s_0) = \lambda_2\alpha_{1,0}(\alpha_{2,0}(\alpha'_{1,g_0}s_0))$$

for some $\lambda_i \in \mathbb{F}$. If Γ^+ and Γ^- were isomorphic, we would have $\lambda_1 = \lambda_2$. But a straightforward computation shows that $\lambda_1 = -\mathbf{i}$ and $\lambda_2 = \mathbf{i}$, which implies that Γ^+ and Γ^- are not isomorphic.

5.3 A recognition theorem for the $\mathbb{Z}_4^3$-grading

The goal of this section is to prove the following result:

Theorem 5.3.1. *[AEK14, Th. 14] Let $\mathcal{A}$ be the Brown algebra over an algebraically closed field $\mathbb{F}$,* char $\mathbb{F} \neq 2, 3$. *Then, up to equivalence, there is a unique $\mathbb{Z}_4^3$-grading on $\mathcal{A}$ such that all nonzero homogeneous components have dimension* 1.

To this end, we will need some general results about gradings on $\mathcal{A}$ and the action of the group $\mathrm{Aut}(\mathcal{A}, {}^-)$, which contains an algebraic group of type E_6 as a subgroup of index 2 (see [Gar01]). The arguments in [Gar01] also give that $\mathrm{Der}(\mathcal{A}, {}^-)$ is the simple Lie algebra of type E_6 (see also [All79]). We will use the model of $\mathcal{A}$ described in Subsection 5.1.2.

5.3.1 Group gradings on $\mathcal{A}$

Recall from (1.9.3) the trace form on $\mathcal{A}$ and the bilinear form $\langle a, b\rangle = \mathrm{tr}(a\bar{b})$.

Lemma 5.3.2. *The trace form on $\mathcal{A}$ has the following properties:*

i) If $a^2 = 0$ *and* $\bar{a} = a$*, then* $\mathrm{tr}(a) = 0$*.*

ii) $\langle a, b\rangle$ *is a nondegenerate symmetric bilinear form.*

iii) $\langle a, b\rangle$ *is an invariant form:* $\langle \bar{a}, \bar{b}\rangle = \langle a, b\rangle$ *and* $\langle ca, b\rangle = \langle a, \bar{c}b\rangle$*.*

iv) For any group grading $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$*,* $gh \neq e$ *implies* $\langle \mathcal{A}_g, \mathcal{A}_h\rangle = 0$*.*

Proof. i) Since $\bar{a} = a$, we have $a = \left(\begin{smallmatrix} \alpha & x \\ x' & \alpha \end{smallmatrix}\right)$ and $\mathrm{tr}(a) = 2\alpha$. Moreover,

$$0 = a^2 = \begin{pmatrix} \alpha^2 + T(x, x') & 2\alpha x + x' \times x' \\ 2\alpha x' + x \times x & \alpha^2 + T(x, x') \end{pmatrix}, \tag{5.3.1}$$

so $\alpha^2 + T(x, x') = 0$, $\alpha x = -x'^{\#}$ and $\alpha x' = -x^{\#}$. In case $x = 0$ or $x' = 0$, we have $0 = \mathrm{tr}(a^2) = 2\alpha^2$, so $\alpha = 0$ and hence $\mathrm{tr}(a) = 0$. Now assume that $x \neq 0 \neq x'$ but $\alpha \neq 0$. Since $(x^{\#})^{\#} = N(x)x$ for any $x \in \mathbb{A}$ (see e.g. [McC69, Eq.(4)]), we get $\alpha x = -x'^{\#} = -(-\alpha^{-1}x^{\#})^{\#} = -\alpha^{-2}N(x)x$. Thus $-\alpha^3 x = N(x)x$, and similarly $-\alpha^3 x' = N(x')x'$, which implies $N(x) = N(x') = -\alpha^3 \neq 0$. But then $T(x, x') = T(x(-\alpha^{-1})x^{\#}) = -3\alpha^{-1}N(x) = 3\alpha^2$ and $\alpha^2 + T(x, x') = 4\alpha^2 \neq 0$, which contradicts the equation $\alpha^2 + T(x, x') = 0$. Therefore, $\alpha = 0$ and $\mathrm{tr}(a) = 0$.

ii) Since tr is invariant under the involution, $\langle a, b\rangle = \mathrm{tr}(a\bar{b}) = \mathrm{tr}(\overline{a\bar{b}}) = \mathrm{tr}(b\bar{a}) = \langle b, a\rangle$, so $\langle\cdot,\cdot\rangle$ is symmetric. The nondegeneracy of the bilinear form $\langle\cdot,\cdot\rangle$ is a consequence of the nondegeneracy of the trace form T of $\mathbb{A}$.

iii) It is easy to see that $\mathrm{tr}(ab) = \mathrm{tr}(ba)$. Hence $\langle \bar{a}, \bar{b}\rangle = \mathrm{tr}(\bar{a}b) = \mathrm{tr}(b\bar{a}) = \mathrm{tr}(\overline{b\bar{a}}) = \mathrm{tr}(a\bar{b}) = \langle a, b\rangle$. Using the fact that $T(x \times y, z) = N(x, y, z)$ is symmetric in the three variables, is is straightforward to check that $\langle ca, b\rangle = \langle a, \bar{c}b\rangle$.

iv) Observe that the restriction of tr to the subspace $\mathcal{A}_0 := \mathbb{F}s_0 \oplus \ker(\mathrm{id} + L_{s_0}R_{s_0})$ is zero, and $\mathcal{A} = \mathbb{F}1 \oplus \mathcal{A}_0$, so $\mathcal{A}_0$ equals the kernel of tr. Now, $\mathbb{F}s_0$ is a graded subspace and $s_0^2 = 1$, hence s_0 is a homogeneous element and its degree has order at most 2. It follows that $\mathcal{A}_0$ is a graded subspace. Therefore, $\mathcal{A}_g \subseteq \mathcal{A}_0$ for any $g \neq e$. The result follows. □

Lemma 5.3.3. *For any* G*-grading on* $\mathcal{A}$ *and a subgroup* $H \subseteq G$ *such that* $\deg(s_0) \notin H$*,* $\mathcal{B} = \bigoplus_{h\in H} \mathcal{A}_h$ *is a semisimple Jordan algebra of degree* ≤ 4*.*

Proof. Since $\deg(s_0) \notin H$, the involution is trivial on $\mathcal{B}$, so $\mathcal{B}$ is a Jordan algebra. By Lemma 5.3.2(ii), the symmetric form $\langle\cdot,\cdot\rangle$ is nondegenerate on $\mathcal{A}$. By (iv), the subspaces $\mathcal{A}_g$ and $\mathcal{A}_{g^{-1}}$ are paired by $\langle\cdot,\cdot\rangle$ for any $g \in G$. It follows that the restriction of $\langle\cdot,\cdot\rangle$ to $\mathcal{B}$ is nondegenerate. Moreover, (iii) implies that this restriction is associative in the sense that $\langle ab, c\rangle = \langle a, bc\rangle$ for all $a, b, c \in \mathcal{B}$.

Suppose $\mathcal{I}$ is an ideal of $\mathcal{B}$ satisfying $\mathcal{I}^2 = 0$. For any $a \in \mathcal{I}$ and $b \in \mathcal{B}$, we have $ab \in \mathcal{I}$ and hence $(ab)^2 = 0$. By Lemma 5.3.2(i), this implies $\mathrm{tr}(ab) = 0$. We have shown that $\langle \mathcal{I}, \mathcal{B} \rangle = 0$, so $\mathcal{I} = 0$. By Dieudonné's Lemma (see [Jac68, p.239]), we conclude that $\mathcal{B}$ is a direct sum of simple ideals.

The conjugate norm of a structurable algebra was defined in [AF92] as the exact denominator of the (conjugate) inversion map (i.e., the denominator of minimal degree), and it coincides with the generic norm in the case of a Jordan algebra. If $N_{\mathcal{B}}$ is the generic norm of $\mathcal{B}$, then it is the denominator of minimal degree for the inversion map, and therefore it divides any other denominator for the inversion map. Since the conjugate norm of $\mathcal{A}$ has degree 4, we conclude that the degree of $N_{\mathcal{B}}$ is at most 4. □

Lemma 5.3.4. *For any G-grading on $\mathcal{A}$ and a subgroup $H \subseteq G$ such that* $\deg(s_0) \in H$*, $\mathcal{B} = \bigoplus_{h \in H} \mathcal{A}_h$ is a simple structurable algebra of skew-dimension* 1*.*

Proof. If $\mathcal{I}$ is an ideal of $\mathcal{B}$ as an algebra with involution and $\mathcal{I}^2 = 0$, then $s_0 \notin \mathcal{I}$, so $\mathcal{I}$ is a Jordan algebra, and, as in the proof of Lemma 5.3.3, we obtain $\mathcal{I} = 0$. On the other hand, if $\mathcal{I} \neq 0$ is an ideal of $\mathcal{B}$ as an algebra (disregarding involution), $\mathcal{I}^2 = 0$, and $\mathcal{I}$ is of minimal dimension with this property, then either $\mathcal{I} = \bar{\mathcal{I}}$ or $\mathcal{I} \cap \bar{\mathcal{I}} = 0$. In the first case, $\mathcal{I}$ is an ideal of $\mathcal{B}$ as an algebra with involution, so we get a contradiction. In the second case, $\mathcal{I} \oplus \bar{\mathcal{I}}$ is an ideal of $\mathcal{B}$ as an algebra with involution and $(\mathcal{I} \oplus \bar{\mathcal{I}})^2 = 0$, again a contradiction. The bilinear form $(a|b) := \langle a, \bar{b} \rangle$ is symmetric, nondegenerate and associative on $\mathcal{A}$, and hence on $\mathcal{B}$. Therefore, Dieudonné's Lemma applies and tells us that $\mathcal{B}$ is a direct sum of simple ideals (as an algebra). The involution permutes these ideals so, adding each of them with its image under the involution, we write $\mathcal{B}$ as a direct sum of ideals, each of which is simple as an algebra with involution. Since $\dim \mathcal{K}(\mathcal{B}, ^-) = 1$, there is only one such ideal where the involution is not trivial, and it contains s_0. Since $s_0^2 = 1$, this ideal is the whole $\mathcal{B}$. □

5.3.2 Norm similarities of the Albert algebra

A linear bijection $f \colon \mathbb{A} \to \mathbb{A}$ is called a *norm similarity with multiplier* λ if $N(f(x)) = \lambda N(x)$ for all $x \in \mathbb{A}$. Norm similarities with multiplier 1 are called *(norm) isometries*. We will denote the group of norm similarities by $M(\mathbb{A})$ and the group of isometries by $M_1(\mathbb{A})$.

For $f \in \mathrm{End}(\mathbb{A})$, denote by f^* the adjoint with respect to the trace form T of $\mathbb{A}$, i.e., $T(f(x), y) = T(x, f^*(y))$ for all $x, y \in \mathbb{A}$. Following the notation of [Gar01], for any element $\varphi \in M(\mathbb{A})$, we denote the element $(\varphi^*)^{-1} = (\varphi^{-1})^*$

by $\varphi^\dagger$, so we have $T(\varphi(x), \varphi^\dagger(y)) = T(x, y)$ for all $x, y \in \mathbb{A}$. If the multiplier of φ is λ, then $\varphi^\dagger$ is a norm similarity with multiplier λ^{-1}, and also

$$\varphi(x) \times \varphi(y) = \lambda\varphi^\dagger(x \times y) \quad \text{and} \quad \varphi^\dagger(x) \times \varphi^\dagger(y) = \lambda^{-1}\varphi(x \times y) \qquad (5.3.2)$$

for all $x, y \in \mathbb{A}$ (see [Gar01, Lemma 1.7]). The U-operator $U_x(y) := \{x, y, x\} = 2x(xy) - x^2y$ can also be written as $U_x(y) = T(x, y)x - x^\# \times y$ (see [McC70, Theorem 1]; cf. [McC69, Theorem 1]). Therefore, $U_{\varphi(x)}\varphi^\dagger(y) = \varphi U_x(y)$ for any $\varphi \in M(\mathbb{A})$ and $x, y \in \mathbb{A}$. In other words, φ is a norm similarity of $\mathbb{A}$ if and only if $(\varphi, \varphi^\dagger)$ is an automorphism of the Albert pair, and we can identify $M(\mathbb{A})$ with $\operatorname{Aut}\mathcal{V}_\mathbb{A}$.

Also, it follows that the automorphisms of the Albert algebra are precisely the elements $\varphi \in M_1(\mathbb{A})$ such that $\varphi^\dagger = \varphi$. Moreover, any $\varphi \in M_1(\mathbb{A})$ defines an automorphism of the Brown algebra $\mathcal{A}$ given by

$$\begin{pmatrix} \alpha & x \\ x' & \beta \end{pmatrix} \mapsto \begin{pmatrix} \alpha & \varphi(x) \\ \varphi^\dagger(x') & \beta \end{pmatrix}.$$

Thus we can identify $M_1(\mathbb{A})$ with a subgroup of $\operatorname{Aut}(\mathcal{A}, ^-)$. In fact, this subgroup is precisely the stabilizer of the element s_0.

For $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{F}^\times$ and $\mu_i = \lambda_i^{-1}\lambda_{i+1}\lambda_{i+2}$, we can define a norm similarity $c_{\lambda_1,\lambda_2,\lambda_3}$, with multiplier $\lambda_1\lambda_2\lambda_3$, given by $\iota_i(x) \mapsto \iota_i(\lambda_i x)$, $E_i \mapsto \mu_i E_i$. Note that $c^\dagger_{\lambda_1,\lambda_2,\lambda_3}$ is given by $\iota_i(x) \mapsto \iota_i(\lambda_i^{-1}x)$, $E_i \mapsto \mu_i^{-1}E_i$. (These norm similarities appear e.g. in [Gar01, Eq. (1.6)].) For $\lambda \in \mathbb{F}^\times$, denote $c_\lambda := c_{\lambda,\lambda,\lambda}$.

We already know that there are four orbits of elements of $\mathbb{A}$ under the action of $\operatorname{Aut}\mathcal{V}_\mathbb{A} \cong M(\mathbb{A})$, which are determined by the rank (see Section 1.7). Denote by $\mathcal{O}_r := \{x \in \mathbb{A} \mid \operatorname{rank}(x) = r\}$ the orbit of elements of rank r, where $r \in \{0, 1, 2, 3\}$. Denote by $\mu_x(X)$ the minimal polynomial of x; it is a divisor of the generic minimal polynomial $m_x(X) = X^3 - T(x)X^2 + S(x)X - N(x)1$ in $\mathbb{A}$.

Proposition 5.3.5. *The orbits for the action of $M(\mathbb{A})$ on $\mathbb{A}$ are exactly $\mathcal{O}_0 = \{0\}$, $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$. The orbit $\mathcal{O}_3$ consists of all nonisotropic elements: $x \in \mathbb{A}$ with $N(x) \neq 0$. The orbit $\mathcal{O}_1$ consists of all $0 \neq x \in \mathbb{A}$ satisfying $N(x) = 0$, $S(x) = 0$ and $\deg \mu_x(X) = 2$; equivalently, $\mathcal{O}_1$ consists of all $0 \neq x \in \mathbb{A}$ satisfying $x^\# = 0$. Moreover, if $x \in \mathcal{O}_1$ we have $\mu_x(X) = X^2 - T(x)X$.*

Proof. The two first statements are already known by Section 1.7.

If $\varphi \in M(\mathbb{A})$ has multiplier λ, then by (5.3.2) we have $2\varphi(x)^\# = \varphi(x) \times \varphi(x) = \lambda\varphi^\dagger(x \times x) = 2\lambda\varphi^\dagger(x^\#)$. Therefore, since E_1 belongs to the orbit $\mathcal{O}_1$ and satisfies $E_1^\# = 0$, it follows that $x^\# = 0$ for any $x \in \mathcal{O}_1$. Similarly, since

$E_1 + E_2$ and 1 are representative elements of the orbits $\mathcal{O}_2$ and $\mathcal{O}_3$ and they satisfy $(E_1 + E_2)^\# \neq 0 \neq 1^\#$, it follows that $x^\# \neq 0$ for any $x \in \mathcal{O}_2 \cup \mathcal{O}_3$.

Fix $x \in \mathcal{O}_1$. We have $x \neq 0$, hence $\deg \mu_x(X) > 1$. Since $N(x) = 0$, we have $X|m_x(X)$; also $\mu_x(X)$ and $m_x(X)$ have the same irreducible factors by [Jac68, Chapter VI, Theorem 1], hence $X|\mu_x(X)$. On the other hand, we have $x^\# = 0$, i.e., $x^2 - T(x)x + S(x)1 = 0$. Thus $\mu_x(X)$ divides $X^2 - T(x)X + S(x)1$, which implies that $\mu_x(X) = X^2 - T(x)X + S(x)1$, and since $X|\mu_x(X)$ we have $S(x) = 0$. Conversely, if $0 \neq x \in \mathbb{A}$ satisfies the conditions $\deg \mu_x(X) = 2$ and $S(x) = 0 = N(x)$, then, since $\mu_x(X)$ divides $m_x(X) = X^3 - T(x)X^2 = X^2(X - T(x))$ and has the same irreducible factors, we have $\mu_x(X) = X^2 - T(x)X$ and also $x^\# = \mu_x(x) + S(x)1 = 0$. The conditions $x \neq 0$ and $x^\# = 0$ imply that x has rank 1. □

Remark 5.3.6. Note that in [Jac68, p.364], the elements of rank 1 of $\mathbb{A}$ are defined as the nonzero elements such that $x^\# = 0$, which is equivalent to our definition.

Corollary 5.3.7. *The orbits for the action of $M_1(\mathbb{A})$ on $\mathbb{A}$ are $\mathcal{O}_0$, $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3(\lambda) := \{x \in \mathcal{O}_3 \mid N(x) = \lambda\}$, $\lambda \in \mathbb{F}^\times$.*

Proof. Note that the elements E_1 and $E_2 + E_3$ can be scaled by any $\lambda \in \mathbb{F}^\times$ using some norm similarity $c_{\alpha,\beta,\gamma}$ with $\alpha\beta\gamma = 1$. Therefore, $\mathcal{O}_1$ and $\mathcal{O}_2$ are orbits for $M_1(\mathbb{A})$, too. Fix $x, y \in \mathcal{O}_3(\lambda)$. By Proposition 1.7.5 there is $\varphi \in M(\mathbb{A})$ such that $\varphi(x) = y$, and the multiplier associated to φ must be 1 because $N(x) = N(y)$, so $\varphi \in M_1(\mathbb{A})$. This proves the fact that $\mathcal{O}_3(\lambda)$ is an orbit for $M_1(\mathbb{A})$. □

Lemma 5.3.8. *The rank function on $\mathbb{A}$ has the following properties:*

i) If $x, y \in \mathbb{A}$ have rank 1, then $N(x + y) = 0$.

ii) If $x_1, x_2, x_3 \in \mathbb{A}$ have rank 1 and $N(x_1 + x_2 + x_3) \neq 0$, then $x_i + x_j$ has rank 2 for each $i \neq j$.

iii) If $x_1, x_2, x_3 \in \mathbb{A}$ have rank 1 and $N(x_1 + x_2 + x_3) = 1$, then there is an isometry sending x_i to E_i for all i.

iv) If $\operatorname{rank}(x) = 1$, then $\operatorname{rank}(x^\#) = 0$. If $\operatorname{rank}(x) = 2$, then $\operatorname{rank}(x^\#) = 1$. If $\operatorname{rank}(x) = 3$, then $\operatorname{rank}(x^\#) = 3$. In general, $\operatorname{rank}(x^\#) \leq \operatorname{rank}(x)$.

Proof. i) Assume, to the contrary, that $N(x + y) \neq 0$. By Proposition 1.7.5, applying a norm similarity, we may assume $x + y = 1$. We know by Proposition 5.3.5 that $x^2 = T(x)x$. If it were $T(x) = 0$, applying an automorphism of $\mathbb{A}$ we would have $x = \iota_1(a)$ with $n(a) = 0$, and therefore $N(y) =$

$N(1-\iota_1(a)) \neq 0$, which contradicts $\operatorname{rank}(y) = 1$. Thus $\lambda := T(x) \neq 0$. Hence, applying an automorphism of $\mathbb{A}$, we may assume $x = \lambda E_1$, and we still have $x+y = 1$. If $\lambda = 1$, then $S(y) = 1 \neq 0$; otherwise $N(y) = 1-\lambda \neq 0$. By Proposition 5.3.5, in both cases we get a contradiction: $y \notin \mathcal{O}_1$.

ii) Take k such that $\{i,j,k\} = \{1,2,3\}$. By i), $\operatorname{rank}(x_i + x_j) \neq 3$. We cannot have $\operatorname{rank}(x_i + x_j) = 0$, because this would imply $x_i + x_j = 0$ and $\operatorname{rank}(x_k) = 3$. We cannot have $\operatorname{rank}(x_i + x_j) = 1$, because this would imply $N(x_i + x_j + x_k) = 0$ by i). Therefore, $\operatorname{rank}(x_i + x_j) = 2$.

iii) Applying an isometry, we may assume $x_1 + x_2 + x_3 = 1$. By ii), we have $\operatorname{rank}(x_i + x_{i+1}) = 2$. By Proposition 5.3.5, we know that $x_i^2 = T(x_i)x_i$. If it were $T(x_1) = 0$, applying an automorphism of $\mathbb{A}$ we would have $x_1 = \iota_1(a)$ with $n(a) = 0$, and therefore $N(x_2 + x_3) = N(1 - \iota_1(a)) \neq 0$, which contradicts $\operatorname{rank}(x_2 + x_3) = 2$. Hence, $T(x_i) \neq 0$ for $i = 1, 2, 3$. Applying an automorphism of $\mathbb{A}$, we obtain $x_1 = \lambda E_1$ where $\lambda = T(x_1)$, and still $x_1 + x_2 + x_3 = 1$. If $\lambda \neq 1$, then $N(x_2 + x_3) = 1 - \lambda \neq 0$, which contradicts i). Therefore, $T(x_1) = 1$, and similarly $T(x_2) = T(x_3) = 1$. We have shown that the x_i are idempotents. Moreover, since $1 - x_i = x_{i+1} + x_{i+2}$ is an idempotent, we also have $x_{i+1}x_{i+2} = 0$, so the idempotents x_i are orthogonal with $\sum x_i = 1$. Now by [Jac68, Chapter IX, Theorem 10], there exists an automorphism of $\mathbb{A}$ sending x_i to E_i for $i = 1, 2, 3$.

iv) If $\operatorname{rank}(x) = 1$, we already know that $x^\# = 0$. It follows from (5.3.2) that $\varphi(x)^\# = \varphi^\dagger(x^\#)$ for any isometry φ. If $\operatorname{rank}(x) = 2$, then by Corollary 5.3.7 there is $\varphi \in M_1(\mathbb{A})$ such that $\varphi(x) = E_2 + E_3$, hence $\varphi^\dagger(x^\#) = \varphi(x)^\# = (E_2+E_3)^\# = E_1$, and so $\operatorname{rank}(x^\#) = 1$. If $\operatorname{rank}(x) = 3$, then $N(x) \neq 0$. Since $N(x^\#) = N(x)^2$ (see [McC69]), we obtain $N(x^\#) \neq 0$ and $\operatorname{rank}(x^\#) = 3$. □

5.3.3 Proof of the recognition Theorem

Suppose $\Gamma :\ \mathcal{A} = \bigoplus_{g\in\mathbb{Z}_4^3} \mathcal{A}_g$ is a grading such that $\dim \mathcal{A}_g \leq 1$ for all $g \in \mathbb{Z}_4^3$. Set $g_0 = \deg(s_0)$, so g_0 is an element of order 2.

Denote $W = \eta(\mathbb{A}) \oplus \eta'(\mathbb{A})$. Since $W = \ker(\mathrm{id} + L_{s_0}R_{s_0})$, it is a graded subspace. Hence, for any $g \neq 0, g_0$, we have $\mathcal{A}_g \subseteq W$. Also, for any $g \neq g_0$, the component $\mathcal{A}_g$ consists of symmetric elements.

Let $S_{g_0} = \{g \in \mathbb{Z}_4^3 \mid 2g \neq g_0\}$. We claim that $\operatorname{Supp}\Gamma = S_{g_0}$. Note that $|S_{g_0}| = 56 = \dim \mathcal{A}$, so it is sufficient to prove that $2g = g_0$ implies $\mathcal{A}_g = 0$. Assume, to the contrary, that $0 \neq a \in \mathcal{A}_g$. Then $b = as_0$ is a nonzero element in $\mathcal{A}_{-g}$. By Lemma 5.3.2, the components $\mathcal{A}_g$ and $\mathcal{A}_{-g}$ are in duality with respect to the form $\langle\cdot,\cdot\rangle$, hence $\langle a, b\rangle \neq 0$. But $a = \eta(x) + \eta'(x')$ for some $x, x' \in \mathbb{A}$, so $b = -\eta(x) + \eta'(x')$, which implies $\langle a, b\rangle = \operatorname{tr}(ab) = T(x, x') - T(x, x') = 0$, a contradiction.

Suppose H is a subgroup of $\mathbb{Z}_4^3$ isomorphic to $\mathbb{Z}_4^2$ and not containing g_0. Consider $\mathcal{B} = \bigoplus_{h\in H} \mathcal{A}_h$ and $\mathcal{D} = \mathcal{B} \oplus s_0\mathcal{B}$. Lemma 5.3.4 shows that $\mathcal{D}$ is a simple structurable algebra of skew-dimension 1 and dimension 32. Hence, by [All90, Example 1.9], $\mathcal{D}$ is the structurable matrix algebra corresponding to a triple (T, N, N) where either (a) N and T are the generic norm and trace form of a degree 3 semisimple Jordan algebra J, or (b) $N = 0$ and T is the generic trace form of the Jordan algebra $J = \mathcal{J}(V)$ of a vector space V with a nondegenerate symmetric bilinear form. In case (a), we have by dimension count that either $J = \mathcal{H}_3(\mathcal{Q})$ or $J = \mathbb{F}\times\mathcal{J}(V)$, where $\dim V = 13$. In case (a), as in Subsection 5.1.2, $\mathbb{F} \times J$ is a Jordan subalgebra of $\mathcal{D}$. If $J = \mathbb{F} \times \mathcal{J}(V)$, then $\mathcal{L} := \operatorname{span}\{D_{x,y} \mid x, y \in V\}$ (the operators $D_{x,y}$ are defined by (5.5.1) in the next section) is a subalgebra of $\operatorname{Der}(\mathcal{A}, ^-)$ isomorphic to the orthogonal Lie algebra $\mathfrak{so}(V)$. Indeed, the image of $\mathcal{L}$ in $\operatorname{End}(V)$ is $\mathfrak{so}(V)$, and $\dim \mathcal{L} \le \wedge^2 V = \dim \mathfrak{so}(V)$. But $\dim \mathfrak{so}(V) = 78 = \dim \operatorname{Der}(\mathcal{A}, ^-)$ and $\operatorname{Der}(\mathcal{A}, ^-)$ is simple of type E_6, so we obtain a contradiction. In case (b), $\mathcal{D}$ contains the Jordan algebra of a vector space of dimension 15 (the Jordan algebra J with its generic trace form), hence $\operatorname{Der}(\mathcal{A}, ^-)$ contains a Lie subalgebra isomorphic to $\mathfrak{so}_{15}(\mathbb{F})$, which has dimension larger than 78, so we again obtain a contradiction. Therefore, the only possibility is $J = \mathcal{H}_3(\mathcal{Q})$. Then, with the same arguments as for $(\mathcal{A}, ^-)$, it can be shown that $\operatorname{Der}(\mathcal{D}, ^-)$ is a simple Lie algebra of type A_5, so it has dimension 35.

By Lemma 5.3.3, $\mathcal{B}$ is a semisimple Jordan algebra of degree ≤ 4. Since $\dim \mathcal{B} = 16$, we have the following possibilities: (i) $\mathcal{J}(V)$ with $\dim V = 15$, (ii) $\mathbb{F} \times \mathcal{J}(V)$ with $\dim V = 14$, (iii) $\mathbb{F} \times \mathbb{F} \times \mathcal{J}(V)$ with $\dim V = 13$, (iv) $\mathcal{J}(V_1)\times\mathcal{J}(V_2)$ with $\dim V_1 + \dim V_2 = 14$ and $\dim V_i \ge 2$, (v) $\mathbb{F}\times\mathcal{H}_3(\mathcal{Q})$ and (vi) $M_4(\mathbb{F})^{(+)}$, where, as before, $\mathcal{J}(V)$ denotes the Jordan algebra of a vector space V with a nondegenerate symmetric bilinear form. Cases (ii), (iii) and (v) are impossible, because these algebras do not admit a $\mathbb{Z}_4^2$-grading with 1-dimensional components. Indeed, since $\operatorname{char} \mathbb{F} \neq 2$, such a grading would be the eigenspace decomposition with respect to a family of automorphisms, but in each case there is a subalgebra of dimension 2 whose elements are fixed by all automorphisms. The same argument applies in case (iv) unless $\dim V_1 = \dim V_2 = 7$. On the other hand, cases (i) and (iv) give, as in the previous paragraph, subalgebras of $\operatorname{Der}(\mathcal{D}, ^-)$ isomorphic to $\mathfrak{so}(V)$ or $\mathfrak{so}(V_1) \times \mathfrak{so}(V_2)$ of dimension larger than 35, so these cases are impossible too. We are left with case (vi), i.e., $\mathcal{B} \cong M_4(\mathbb{F})^{(+)}$. Then, up to equivalence, there is only one $\mathbb{Z}_4^2$-grading with 1-dimensional components, namely, the Pauli grading on the associative algebra $M_4(\mathbb{F})$. (For the classification of gradings on simple special Jordan algebras, we refer the reader to [EK13, §5.6].)

As a consequence of the above analysis, if $X \neq 0$ is a homogeneous element of $\mathcal{A}$ whose degree has order 4 then we have $0 \neq X^4 \in \mathbb{F}1$. Indeed, the degree of X is contained in a subgroup H as above, so X is an invertible matrix in $\mathcal{B} \cong M_4(\mathbb{F})^{(+)}$. Moreover, we can fix homogeneous elements X_1, X_2 and X_3 of $\mathcal{B}$ such that $X_i^2 = 1$ and $X_i X_{i+1} = X_{i+2}$ (indices modulo 3), because these elements exist in the $\mathbb{Z}_4^2$-grading on $M_4(\mathbb{F})^{(+)}$. We will now show that Γ is equivalent to the grading defined by (5.2.4) in Section 5.2. Denote $a_i = \deg(X_i)$, then the subgroup $\langle a_1, a_2\rangle$ is isomorphic to $\mathbb{Z}_2^2$ and does not contain g_0.

We can write $X_i = \eta(x_i) + \eta'(x_i')$ with $x, x' \in \mathbb{A}$. Since $X_i^2 = 1$, we get $x_i^{\#} = 0 = x_i'^{\#}$ and thus x_i and x_i' have rank 1 (see Remark 5.3.6). Set $Z = X_1 + X_2 + X_3$ and write $Z = \eta(z) + \eta(z')$ with $z, z' \in \mathbb{A}$. Then $Z^2 = 2Z + 3$, which implies $z^{\#} = z'$ and $z'^{\#} = z$. But, by Lemma 5.3.8(iv), $\mathrm{rank}(z^{\#}) \leq \mathrm{rank}(z)$ and $\mathrm{rank}(z'^{\#}) \leq \mathrm{rank}(z')$, so we get $\mathrm{rank}(z^{\#}) = \mathrm{rank}(z) = \mathrm{rank}(z') = \mathrm{rank}(z'^{\#})$. Since $Z \neq 0$, we have $z \neq 0$ or $z' \neq 0$, and hence by Lemma 5.3.8(iv), we obtain $\mathrm{rank}(z) = 3 = \mathrm{rank}(z')$. Then, by Lemma 5.3.8(iii), there is an isometry of $\mathbb{A}$ sending x_i to λE_i ($i = 1, 2, 3$), where λ is any element of $\mathbb{F}$ satisfying $\lambda^3 = N(z)$. Since isometries of $\mathbb{A}$ extend to automorphisms of $(\mathcal{A}, ^-)$, we may assume that $x_i = \lambda E_i$. Then $X_i X_{i+1} = X_{i+2}$ implies $x_i' = \lambda^2 E_i$ and hence $\lambda^3 = 1$. Therefore, $N(z) = 1$ and we may take $\lambda = 1$, so $x_i = E_i = x_i'$, i.e., $X_i = \varepsilon_i := \eta(E_i) + \eta'(E_i)$. Thus, ε_i and $\varepsilon_i' := \varepsilon_i s_0$ are homogeneous elements; their degrees are precisely the order 2 elements of $\mathbb{Z}_4^3$ different from g_0.

Since the subspaces $\ker(L_{\varepsilon_i}) = \eta(\iota_{i+1}(\mathcal{C}) \oplus \iota_{i+2}(\mathcal{C})) \oplus \eta'(\iota_{i+1}(\mathcal{C}) \oplus \iota_{i+2}(\mathcal{C}))$ are graded, so are $\eta(\iota_i(\mathcal{C})) \oplus \eta'(\iota_i(\mathcal{C}))$, $i = 1, 2, 3$. For any homogeneous element $X = \eta(\iota_j(x)) + \eta'(\iota_j(x'))$, we saw that $0 \neq X^4 \in \mathbb{F}1$, which forces $0 \neq X^2 \in \mathbb{F}\varepsilon_j \cup \mathbb{F}\varepsilon_j'$, and this implies $n(x, x') = 0$ and $n(x) = \pm n(x') \neq 0$. These facts will be used several times. Also note that automorphisms of $\mathcal{C}$ extend to automorphisms of $\mathbb{A}$ preserving E_i, and therefore to automorphisms of $\mathcal{A}$ preserving ε_i.

Fix homogeneous elements $Y_1 = \eta(\iota_1(y_1)) + \eta'(\iota_1(y_1'))$ and $Y_2 = \eta(\iota_2(y_2)) + \eta'(\iota_2(y_2'))$ such that $Y_i^2 \in \mathbb{F}\varepsilon_i$. Without loss of generality, we may assume $n(y_1) = 1 = n(y_2)$, and therefore $n(y_1') = 1 = n(y_2')$. Also, we have $n(y_i, y_i') = 0$. By [EK13, Lemma 5.25], there exists an automorphism of $\mathbb{A}$ that fixes E_i and sends y_1 and y_2 to 1. Thus we may assume $y_1 = 1 = y_2$ and hence $\overline{y_i'} = -y_i'$. Then $Y_1 Y_2 = \eta(2\iota_3(y_1' y_2')) + \eta'(2\iota_3(1))$, so we obtain $n(1, y_1' y_2') = 0$, which implies $n(y_1', y_2') = 0$. Thus the elements $1, y_1', y_2'$ are orthogonal of norm 1, and applying an automorphism of $\mathcal{C}$ (extended to $\mathcal{A}$) we may assume that $Y_1 = \alpha_{1,0} := \eta(\iota_1(1)) + \eta'(\iota_1(x_{a_1}))$ and $Y_2 = \alpha_{2,0} := \eta(\iota_2(1)) + \eta'(\iota_2(x_{a_2}))$, as in the grading (5.2.4). Consequently, the elements of the form $\alpha_{j,g}$, for $j =$

$1, 2, 3$ and $g \in \langle a_1, a_2\rangle$, will be homogeneous because they can be expressed in terms of $\alpha_{1,0}$ and $\alpha_{2,0}$.

Fix a new homogeneous element $Y_3 = \eta(\iota_3(y_3)) + \eta'(\iota_3(y_3'))$ such that $Y_3^2 \in \mathbb{F}\varepsilon_3'$. As before, we have $n(y_3, y_3') = 0$, but this time $n(y_3) = -n(y_3')$. Using again that the products of the form $Y_3\alpha_{1,g}$ and $Y_3\alpha_{2,g}$, with $g \in \langle a_1, a_2\rangle$, have orthogonal entries in $\mathcal{C}$, we deduce that $y_3, y_3' \in \mathcal{Q}^\perp$, where $\mathcal{Q} = \mathrm{span}\{1, x_{a_i} \mid i = 1, 2, 3\}$, and that $y_3' \in \mathbb{F}y_3x_{a_3}$. Hence, scaling Y_3, we obtain either $Y_3 = \alpha'_{3,h}$ or $Y_3 = \alpha'_{3,h}s_0$ for some $h \in g_0 + \langle a_1, a_2\rangle$. (Actually, applying another automorphism of $\mathcal{C}$ that fixes the subalgebra $\mathcal{Q}$ point-wise, we can make h any element we like in the indicated coset.) Replacing Y_3 by Y_3s_0 if necessary, we may assume $Y_3 = \alpha'_{3,h}$. Since the elements $\alpha_{1,0}$, $\alpha_{2,0}$ and $\alpha'_{3,h}$ determine the $\mathbb{Z}_4^3$-grading (5.2.4), the proof is complete. □

Remark 5.3.9. If char $\mathbb{F} = 3$, the arguments with derivations of $\mathcal{A}$ that we used to establish the existence of the elements X_i are not valid, but the remaining part of the proof still goes through. Hence, in this case, we obtain a weaker recognition theorem by adding the condition of the existence of X_i to the hypothesis.

5.4 Weyl group

The Weyl group of the $\mathbb{Z}_2^3$-grading on the octonions is $\mathrm{GL}_3(\mathbb{Z}_2)$, the entire automorphism group of $\mathbb{Z}_2^3$, whereas the Weyl group of the $\mathbb{Z}_3^3$-grading on the Albert algebra is $\mathrm{SL}_3(\mathbb{Z}_3)$, which has index 2 in the automorphism group of $\mathbb{Z}_3^3$ (see e.g. [EK13]). This means that in the case of the octonions, all gradings in the equivalence class of the $\mathbb{Z}_2^3$-grading are actually isomorphic to each other, while in the case of the Albert algebra, there are two isomorphism classes in the equivalence class of the $\mathbb{Z}_3^3$-grading.

Theorem 5.4.1. *Let Γ be a $\mathbb{Z}_4^3$-grading on the Brown algebra as in Theorem 5.3.1. Then the Weyl group of Γ is the subgroup $\mathrm{Stab}_{\mathrm{GL}_3(\mathbb{Z}_4)}(g_0) \cap \mathrm{SL}_3(\mathbb{Z}_4)$ of $\mathrm{GL}_3(\mathbb{Z}_4)$, where g_0 is the degree of nonzero skew elements.*

Proof. We will work in the model given by (5.2.4), where $g_0 = (\bar{2}, \bar{0}, \bar{0})$. Denote $H = \mathrm{Stab}_{\mathrm{GL}_3(\mathbb{Z}_4)}(g_0)$ and $\mathcal{W} = \mathcal{W}(\Gamma)$. It is clear that $\mathcal{W} \subseteq H$, and the proof of Theorem 5.3.1 shows that $\mathcal{W}$ has index at most 2 in H (because any ordered generator set of $\mathbb{Z}_4^3$ can be sent to either to (a_1, a_2, a_3) or to $(a_1, a_2, a_3 + g_0)$ by applying an element of $\mathcal{W}$, where $\{a_i\}$ denotes the canonical basis of $\mathbb{Z}_4^3$). On the other hand, by Remark 5.2.5, $\mathcal{W}$ is not the entire H, so we get $[H : \mathcal{W}] = 2$. The derived subgroup H' has index 4 in H (see below), hence there are three subgroups of index 2 in H, including $H \cap \mathrm{SL}_3(\mathbb{Z}_4)$.

The elements of H have the form $A = (a_{ij})$ where $a_{11} \equiv 1$ and $a_{21} \equiv a_{31} \equiv 0 \pmod 2$. Hence, the mapping $(a_{ij}) \mapsto (a_{ij})_{2\le i,j\le 3} \bmod 2$ is a homomorphism $H \to \mathrm{GL}_2(\mathbb{Z}_2)$. Since $\mathrm{GL}_2(\mathbb{Z}_2)$ is isomorphic to S_3, it has a unique nontrivial homomorphism to $\mathbb{Z}_2$. Composing these two, we obtain a nontrivial homomorphism $\varphi_1\colon H \to \mathbb{Z}_2$. Of course, another nontrivial homomorphism, $\varphi_2\colon H \to \mathbb{Z}_2$, is given by $\det A = (-\bar{1})^{\varphi_2(A)}$, and we want to show that $\mathcal{W} = \ker\varphi_2$. Clearly, $H' \subseteq \ker\varphi_1 \cap \ker\varphi_2$, and with elementary arguments (using the fact that the commutator of the elementary matrices $I + \alpha E_{ij}$ and $I + \beta E_{jk}$ is equal to $I + \alpha\beta E_{ik}$ if i, j, k are distinct) one shows that actually $H' = \ker\varphi_1 \cap \ker\varphi_2$. Therefore, it will be sufficient to find a matrix A in $\mathcal{W}$ that belongs to $\ker\varphi_2$ but not $\ker\varphi_1$. One such matrix is $A = \left(\begin{smallmatrix} \bar{3} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} \\ \bar{0} & \bar{1} & \bar{0} \end{smallmatrix}\right)$. Indeed, consider the automorphism $\psi_{(12)}$ of the Albert algebra $\mathbb{A}$ given by $\iota_1(x) \leftrightarrow \iota_2(\bar{x})$, $E_1 \leftrightarrow E_2$, $\iota_3(x) \mapsto \iota_3(\bar{x})$, $E_3 \mapsto E_3$. Also, there is an automorphism $f_{(12)}$ of $\mathcal{C}$ given by $x_{a_1} \leftrightarrow -x_{a_2}$, $x_{a_3} \mapsto -x_{a_3}$, and fixing x_{g_0}. Extend both to automorphisms of $\mathcal{A}$ and consider the composition $\phi = f_{(12)}\psi_{(12)}$. One checks that ϕ sends $\alpha_{1,0} \leftrightarrow \alpha_{2,0}$, $\alpha'_{3,g_0} \leftrightarrow \alpha'_{3,g_0}s_0$, thus inducing A in $\mathcal{W}$. □

Corollary 5.4.2. *The equivalence class of gradings characterized by Theorem 5.3.1 consists of* 14 *isomorphism classes: for each order* 2 *element of* $\mathbb{Z}_4^3$*, there are two nonisomorphic gradings (analogous to* Γ^+ *and* Γ^- *of Remark 5.2.5).* □

5.5 Fine gradings on the exceptional simple Lie algebras E_6, E_7, E_8

Gradings on the exceptional simple Lie algebras are quite often related to gradings on certain nonassociative algebras that coordinatize the Lie algebra in some way. The aim of this section is to indicate how the fine grading by $\mathbb{Z}_4^3$ on the split Brown algebra is behind certain fine gradings on the simple Lie algebras of types E_6, E_7 and E_8. Here we will assume that the ground field $\mathbb{F}$ is algebraically closed and $\operatorname{char}\mathbb{F} \neq 2, 3$.

Given a structurable algebra $(\mathcal{X}, ^-)$, there are several Lie algebras attached to it. To begin with, there is the Lie algebra of derivations $\mathrm{Der}(\mathcal{X}, ^-)$. For the Brown algebra, this coincides with the Lie algebra of inner derivations $\mathrm{IDer}(\mathcal{X}, ^-)$, which is the linear span of the operators $D_{x,y}$, for $x, y \in \mathcal{X}$, where

$$D_{x,y}(z) = \frac{1}{3}\,[[x,y] + [\bar{x},\bar{y}], z] + (z,y,x) - (z,\bar{x},\bar{y}) \qquad (5.5.1)$$

for $x, y, z \in \mathcal{X}$. (As usual, (x, y, z) denotes the associator $(xy)z - x(yz)$.) If $(\mathcal{X}, ^-)$ is G-graded, then $\mathrm{Der}(\mathcal{X}, ^-)$ is a graded Lie subalgebra of $\mathrm{End}(\mathcal{X})$, so we obtain an induced G-grading on $\mathrm{Der}(\mathcal{X}, ^-)$. For the Brown algebra $(\mathcal{A}, ^-)$, the Lie algebra of derivations is the simple Lie algebra of type E_6. The fine grading by $\mathbb{Z}_4^3$ on the Brown algebra induces the fine grading by $\mathbb{Z}_4^3$ on E_6 that appears in [DV16].

Another Lie subalgebra of $\mathrm{End}(\mathcal{X})$ is the *structure Lie algebra*

$$\mathfrak{str}(\mathcal{X}, ^-) = \mathrm{Der}(\mathcal{X}, ^-) \oplus T_{\mathcal{X}}$$

where $T_x := V_{x,1}$, $x \in \mathcal{X}$. The linear span of the operators $V_{x,y}$, $x, y \in \mathcal{X}$, is contained in $\mathfrak{str}(\mathcal{X}, ^-)$ and called the *inner structure Lie algebra* (as it actually equals $\mathrm{IDer}(\mathcal{X}, ^-) \oplus T_{\mathcal{X}}$). It turns out (see e.g. [All78, Corollaries 3 and 5]) that $\mathfrak{str}(\mathcal{X}, ^-)$ is graded by $\mathbb{Z}_2$, with $\mathfrak{str}(\mathcal{X}, ^-)_{\bar{0}} = \mathrm{Der}(\mathcal{X}, ^-) \oplus T_{\mathcal{K}}$ and $\mathfrak{str}(\mathcal{X}, ^-)_{\bar{1}} = T_{\mathcal{H}}$, where $\mathcal{K} = \mathcal{K}(\mathcal{X}, ^-)$ and $\mathcal{H} = \mathcal{H}(\mathcal{X}, ^-)$ denote, respectively, the spaces of symmetric and skew-symmetric elements for the involution. If $(\mathcal{X}, ^-)$ is G-graded then we obtain an induced grading by $\mathbb{Z}_2 \times G$ on $\mathfrak{str}(\mathcal{X}, ^-)$ and on its derived algebra. In the case of the Brown algebra $(\mathcal{A}, ^-)$, the inner structure Lie algebra coincides with the structure Lie algebra and is the direct sum of a one-dimensional center and the simple Lie algebra of type E_7. (The arguments in [All79, Corollary 7] work here because the Killing form of E_6 is nondegenerate.) Therefore, the $\mathbb{Z}_4^3$-grading on $(\mathcal{A}, ^-)$ induces a grading by $\mathbb{Z}_2 \times \mathbb{Z}_4^3$ on the simple Lie algebra of type E_7.

Recall from Section 1.11 that the Kantor Lie algebra $\mathcal{L} = \mathfrak{K}(\mathcal{X}, ^-)$ has a $\mathbb{Z}$-grading with support contained in $\{-2, -1, 0, 1, 2\}$. Any grading on $(\mathcal{X}, ^-)$ by a group G induces naturally a grading by $\mathbb{Z} \times G$ on $\mathfrak{K}(\mathcal{X}, ^-)$. For the Brown algebra, $\mathfrak{K}(\mathcal{A}, ^-)$ is the simple Lie algebra of type E_8 (see [All79] and note that, as for $\mathfrak{str}(\mathcal{A}, ^-)$, the arguments are valid in characteristic $\neq 2, 3$), and we obtain a grading by $\mathbb{Z} \times \mathbb{Z}_4^3$ on E_8, which is the grading that prompted this study of the $\mathbb{Z}_4^3$-gradings on the Brown algebra.

Conclusions and open problems

In this work we have given a classification of the equivalence classes of fine gradings by abelian groups on exceptional simple Jordan pairs and triple systems (i.e., the Jordan pairs and triple systems of types Albert and bi-Cayley) over an algebraically closed field of characteristic different from 2.

We gave an explicit construction of a grading of each equivalence class of fine gradings, and then we computed the Weyl group for each one. Each equivalence class of fine gradings that appear in the classification is determined by its universal grading group. We list them in the table below (note that some gradings do not occur if char $\mathbb{F} = 3$):

Jordan system	Universal groups of fine gradings
Bi-Cayley pair: $\mathcal{V}_{\mathcal{B}}$	$\mathbb{Z}^6$, $\mathbb{Z}^2 \times \mathbb{Z}_2^3$
Bi-Cayley triple: $\mathcal{T}_{\mathcal{B}}$	$\mathbb{Z}^4$, $\mathbb{Z} \times \mathbb{Z}_2^3$, $\mathbb{Z}_2^5$
Albert pair: $\mathcal{V}_{\mathbb{A}}$	$\mathbb{Z}^7$, $\mathbb{Z}^3 \times \mathbb{Z}_2^3$, $\mathbb{Z} \times \mathbb{Z}_3^3$ (if char $\mathbb{F} \neq 3$)
Albert triple: $\mathcal{T}_{\mathbb{A}}$	$\mathbb{Z}^4 \times \mathbb{Z}_2$, $\mathbb{Z}_2^6$, $\mathbb{Z} \times \mathbb{Z}_2^4$, $\mathbb{Z}_3^3 \times \mathbb{Z}_2$ (if char $\mathbb{F} \neq 3$)

The orbits under the automorphism groups of the Jordan systems of type bi-Cayley have been classified. In turn, the orbits have been used to classify the equivalence classes of fine gradings. (Note that the orbits were already classified for simple Jordan pairs in [ALM05].) These orbits, in the case of simple Jordan pairs, are determined by the ranks of the elements of the Jordan pair. We have obtained generators of the automorphism groups of both Jordan systems of type bi-Cayley, and then we have used them to determine their structure, i.e., we have proved that $\operatorname{Aut} \mathcal{T}_{\mathcal{B}} \cong \mathrm{Spin}_9(\mathbb{F})$ and that $\operatorname{Aut} \mathcal{V}_{\mathcal{B}}$ is a quotient of $\mathbb{F}^\times \times \mathrm{Spin}_{10}(\mathbb{F})$.

We also proved that the fine gradings by their universal group on the bi-Cayley and Albert pairs extend to fine gradings on their respectives TKK Lie algebras by the same universal groups. In particular, we obtained that the fine gradings on $\mathcal{V}_{\mathcal{B}}$ induce the following fine gradings on $\mathrm{TKK}(\mathcal{V}_{\mathcal{B}}) = \mathfrak{e}_6$:

- $\mathcal{U}(\Gamma) = \mathbb{Z}^6$ and type $(72, 0, 0, 0, 0, 1)$ (Cartan grading),
- $\mathcal{U}(\Gamma) = \mathbb{Z}^2 \times \mathbb{Z}_2^3$ and type $(48, 1, 0, 7)$,

and the fine gradings on $\mathcal{V}_{\mathbb{A}}$ induce the following fine gradings on $\text{TKK}(\mathcal{V}_{\mathbb{A}}) = \mathfrak{e}_7$:

- $\mathcal{U}(\Gamma) = \mathbb{Z}^7$ and type $(126, 0, 0, 0, 0, 0, 1)$ (Cartan grading),
- $\mathcal{U}(\Gamma) = \mathbb{Z}^3 \times \mathbb{Z}_2^3$ and type $(102, 0, 1, 7)$,
- $\mathcal{U}(\Gamma) = \mathbb{Z} \times \mathbb{Z}_3^3$ and type $(55, 0, 26)$.

It remains as an open problem to extend this classification to the remaining simple Jordan pairs, i.e., the four infinite families of non-exceptional simple Jordan pairs. A more ambitious open problem would be to extend the classification for all simple Kantor pairs, simple Kantor triple systems and simple structurable algebras.

We have also given in this work a construction of a fine $\mathbb{Z}_4^3$-grading on the Brown algebra. We have given a recognition Theorem of this grading and computed its Weyl group, which is isomorphic to $\text{Stab}_{\text{GL}_3(\mathbb{Z}_4)}(g_0) \cap \text{SL}_3(\mathbb{Z}_4)$ where g_0 denotes the degree of the skew homogeneous component. We have shown that this grading can be used to obtain gradings on the exceptional simple Lie algebras of types E_6, E_7 and E_8; in particular, we can use it to construct a fine $\mathbb{Z}_4^3$-grading on its derivation Lie algebra $\mathfrak{e}_6$, and a fine $\mathbb{Z} \times \mathbb{Z}_4^3$-grading on $\mathfrak{e}_8$ via the Kantor construction.

Many other (less complicated) fine gradings are known on the Brown algebra, but the complete classification remains an open problem that may be considered by the author in future work. Also, an interesting Kantor pair is the Brown pair (i.e., the Kantor pair associated to the Brown algebra), which contains subpairs isomorphic to the Albert pair. Note that the Kantor construction of the Brown pair is the exceptional simple Lie algebra $\mathfrak{e}_8$, so its gradings can be extended to gradings on $\mathfrak{e}_8$.

The author has considered for future work to extend each result of Section 2.1, whenever it is possible, to the Kantor case, i.e., for structurable algebras and Kantor pairs and triple systems. This could allow to use similar techniques to the study of gradings on Kantor systems.

In the present state of the art, fine gradings on simple Lie algebras over algebraically closed fields of characteristic 0 have already been classified up to equivalence by other authors (see [DE16b] and references therein); however, the problem remains open for the case of positive characteristic. (Note that the $\mathbb{Z}_4^3$-grading on the Brown algebra has been used by other authors in the classification of fine gradings, up to equivalence, on exceptional simple Lie algebras of type E over an algebraically closed field of characteristic 0.) Nonassociative systems appear frequently as coordinate algebras of Lie algebras in many different constructions, and in many cases their gradings can be extended to gradings on the Lie algebra. Since the results of this thesis, including the constructions of fine gradings, are proven for characteristic different from 2, they could be useful to extend known results on simple Lie algebras over a field of characteristic 0 to the case of positive characteristic.

Conclusiones y problemas abiertos

En este trabajo hemos dado clasificaciones de las clases de equivalencia de graduaciones finas de grupo abeliano en pares y sistemas triples de Jordan simples excepcionales (es decir, los pares y sistemas triples de Jordan de tipos Albert y bi-Cayley) sobre un cuerpo algebraicamente cerrado de característica distinta de 2.

Hemos dado una construcción explícita de una graduación de cada clase de equivalencia de graduaciones finas, y después hemos calculado el grupo de Weyl para cada una. Cada clase de equivalencia de graduaciones finas que aparece en la clasificación está determinada por su grupo de graduación universal. Damos una lista de ellas en la tabla siguiente (observamos que algunas de las graduaciones no aparecen si char $\mathbb{F} = 3$):

Sistema de Jordan	Grupos universales de graduaciones finas
Par bi-Cayley: $\mathcal{V}_{\mathcal{B}}$	$\mathbb{Z}^6$, $\mathbb{Z}^2 \times \mathbb{Z}_2^3$
Triple bi-Cayley: $\mathcal{T}_{\mathcal{B}}$	$\mathbb{Z}^4$, $\mathbb{Z} \times \mathbb{Z}_2^3$, $\mathbb{Z}_2^5$
Par de Albert: $\mathcal{V}_{\mathbb{A}}$	$\mathbb{Z}^7$, $\mathbb{Z}^3 \times \mathbb{Z}_2^3$, $\mathbb{Z} \times \mathbb{Z}_3^3$ (si char $\mathbb{F} \neq 3$)
Triple de Albert: $\mathcal{T}_{\mathbb{A}}$	$\mathbb{Z}^4 \times \mathbb{Z}_2$, $\mathbb{Z}_2^6$, $\mathbb{Z} \times \mathbb{Z}_2^4$, $\mathbb{Z}_3^3 \times \mathbb{Z}_2$ (si char $\mathbb{F} \neq 3$)

Las órbitas bajo el grupo de automorfismos de los sistemas de Jordan de tipo bi-Cayley han sido clasificadas. A su vez, las órbitas han sido usadas como herramienta para clasificar las clases de equivalencia de graduaciones finas. (Notemos que las órbitas fueron clasificadas para los pares de Jordan simples en [ALM05]). Estas órbitas, en el caso de pares de Jordan simples, están determinadas por el rango de los elementos del par de Jordan. Hemos obtenido generadores de los grupos de automorfismos de ambos sistemas de Jordan de tipo bi-Cayley, y luego los hemos usado para determinar su estructura, es decir, hemos probado que $\operatorname{Aut} \mathcal{T}_{\mathcal{B}} \cong \operatorname{Spin}_9(\mathbb{F})$ y que $\operatorname{Aut} \mathcal{V}_{\mathcal{B}}$ es un cociente de $\mathbb{F}^\times \times \operatorname{Spin}_{10}(\mathbb{F})$.

También hemos probado que las graduaciones finas sobre su grupo universal en los pares de tipos bi-Cayley y Albert se extienden a graduaciones finas en sus respectivas álgebras de Lie TKK dadas sobre los mismos grupos universales. En particular, obtuvimos que las graduaciones finas en $\mathcal{V}_{\mathcal{B}}$ inducen las siguientes graduaciones finas en $\mathrm{TKK}(\mathcal{V}_{\mathcal{B}}) = \mathfrak{e}_6$:

- $\mathcal{U}(\Gamma) = \mathbb{Z}^6$ de tipo $(72, 0, 0, 0, 0, 1)$ (graduación de Cartan),
- $\mathcal{U}(\Gamma) = \mathbb{Z}^2 \times \mathbb{Z}_2^3$ de tipo $(48, 1, 0, 7)$,

y las graduaciones finas en $\mathcal{V}_{\mathbb{A}}$ inducen las siguientes graduaciones finas en $\mathrm{TKK}(\mathcal{V}_{\mathbb{A}}) = \mathfrak{e}_7$:

- $\mathcal{U}(\Gamma) = \mathbb{Z}^7$ de tipo $(126, 0, 0, 0, 0, 0, 1)$ (graduación de Cartan),
- $\mathcal{U}(\Gamma) = \mathbb{Z}^3 \times \mathbb{Z}_2^3$ de tipo $(102, 0, 1, 7)$,
- $\mathcal{U}(\Gamma) = \mathbb{Z} \times \mathbb{Z}_3^3$ de tipo $(55, 0, 26)$.

Todavía es un problema abierto extender esta clasificación al resto de pares de Jordan simples, es decir, las cuatro familias infinitas de pares de Jordan simples no excepcionales. Un problema abierto aún más ambicioso sería extender la clasificación a todos los pares de Kantor simples, sistemas triples de Kantor simples y álgebras estructurables simples.

En este trabajo también hemos dado una construcción para una $\mathbb{Z}_4^3$-graduación fina en el álgebra de Brown. Hemos dado un Teorema de reconocimiento de esta graduación y calculado su grupo de Weyl, el cual es isomorfo a $\mathrm{Stab}_{\mathrm{GL}_3(\mathbb{Z}_4)}(g_0) \cap \mathrm{SL}_3(\mathbb{Z}_4)$ donde g_0 denota el grado de la componente homogénea antisimétrica. Hemos visto cómo esta graduación puede usarse para construir graduaciones en las álgebras de Lie simples excepcionales de tipos E_6, E_7 y E_8; en particular podemos usarla para construir una $\mathbb{Z}_4^3$-graduación fina en su álgebra de Lie de derivaciones $\mathfrak{e}_6$, y una $\mathbb{Z} \times \mathbb{Z}_4^3$-graduación fina en $\mathfrak{e}_8$ obtenida mediante la construcción de Kantor.

Muchas otras graduaciones (menos complicadas) son ya conocidas en el álgebra de Brown, pero la clasificación completa sigue siendo un problema abierto y podría ser considerado por el autor como trabajo futuro. Además, un par de Kantor interesante es el par de Brown (es decir, el par de Kantor asociado al álgebra de Brown), el cual contiene subpares isomorfos al par de Albert. Conviene recalcar que la construcción de Kantor del par de Brown es el álgebra de Lie simple excepcional $\mathfrak{e}_8$, lo que permite extender sus graduaciones a graduaciones en $\mathfrak{e}_8$.

El autor también ha considerado como trabajo futuro extender los resultados de la Sección 2.1, siempre que sea posible, al caso de Kantor, es decir, para álgebras estructurables y pares y sistemas triples de Kantor. Esto podría permitir usar técnicas similares para el estudio de las graduaciones en sistemas de Kantor.

En los últimos avances en esta área, otros autores han clasificado salvo equivalencia las graduaciones finas en álgebras de Lie simples sobre cuerpos algebraicamente cerrados de característica 0 (véase [DE16b] y referencias allí citadas); sin embargo, el problema sigue abierto para el caso de característica positiva. (Notemos que la $\mathbb{Z}_4^3$-graduación en el álgebra de Brown ha sido utilizada por otros autores en la clasificación de graduaciones finas, salvo equivalencia, de las álgebras de Lie simples excepcionales de tipo E sobre un cuerpo algebraicamente cerrado de característica 0.) Los sistemas no asociativos aparecen con frecuencia como álgebras coordenadas de álgebras de Lie en muchas construcciones diferentes, y en muchos casos sus graduaciones pueden ser extendidas a graduaciones en el álgebra de Lie. Como los resultados de esta tesis, incluyendo las construcciones de graduaciones finas, se han demostrado en el caso de característica distinta de 2, podrían ser útiles para extender resultados conocidos en álgebras de Lie simples sobre un cuerpo de característica 0 al caso de característica positiva.

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