A singularly perturbed convection-diffusion problem with a moving pulse

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Abstract

A singularly perturbed parabolic equation of convection-diffusion type is examined. Initially the solution approximates a concentrated source. This causes an interior layer to form within the domain for all future times. Using a suitable transformation, a layer adapted mesh is constructed to track the movement of the center of the interior layer. A parameter-uniform numerical method is then defined, by combining the backward Euler method and a simple upwinded finite difference operator with this layer-adapted mesh. Numerical results are presented to illustrate the theoretical error bounds established.

Keywords: Singular perturbation, Gaussian pulse, Shishkin mesh.
AMS subject classifications: 65L11,65L12.

1. Introduction

Singularly perturbed convection-diffusion parabolic problems can be viewed as simple mathematical models of some pollutant being transported through a fast flowing medium. In this paper, we consider a model, where the width of the initial profile of the pollutant concentration approximates a point source.

In the case of smooth data, boundary layers can appear in the solutions of singularly perturbed parabolic problems. Globally accurate parameter-uniform numerical approximations [2] to the solutions of these kind of problems can be constructed using layer-adapted meshes such as piecewise-uniform Shishkin meshes [11]. In the case of non-smooth data, additional interior layers can appear in the solution [1]. If the initial condition is discontinuous, then parameter-uniform globally accurate numerical methods do not exist for such problems [8, 4].

In [6] we constructed and analysed a numerical method for a problem with a regularised step-function as the initial profile. In the current paper, the methodology (in both the construction and the associated theoretical analysis) is extended to a singularly perturbed convection-diffusion parabolic problem with a regularised delta-function as the initial profile.

The width of the layer in the regularized initial condition in [6] was directly related to the scale of the singular perturbation parameter ε contained within the differential equation. In the current paper, we examine the case of a layer in the initial condition having a potentially different scale to, what we shall call, the *normal scale* of $\mathcal{O}(\sqrt{\varepsilon})$. The normal scale is the scale of an interior layer emanating from a singularly perturbed parabolic equation of the form $-\varepsilon z_{yy} + z_t = 0$ [8, 4], where the interior layer is moving (in time) along a direction orthogonal to the y-coordinate axis. Here we examine an initial condition involving a Gaussian profile with a standard deviation determined by two parameters ε and θ, where the value of θ determines how far the width of the pulse deviates from the normal layer width. We see that if the scale of the layer in the initial condition (of order $\mathcal{O}(\sqrt{\varepsilon/\theta})$) is significantly thinner than the normal scale ($\theta \gg 1$), then large gradients in time are observed initially and the magnitude of the approximation errors are adversely effected by the presence of this excessively thin pulse. In addition, we also consider the intermediate case of $C\varepsilon \ll \theta \ll C$, which lies between the case of no layer ($\theta \leq C\varepsilon$) and the case of a normal layer width ($\theta = C$) in the initial pulse. In this paper, we highlight how the error constants (in the theoretical error bounds) depend on this parameter θ. The asymptotic error bound given in Theorem 7, indicates a degradation in the error bound for the case of $\theta \neq \mathcal{O}(1)$. Note that, for any fixed value of the parameter θ, the numerical method is parameter-uniformly convergent with respect to the singular perturbation parameter ε, present in the differential equation.

In §2 we state the problem class examined in this paper and global parameter-explicit bounds on the solution are established. A transformation of the domain is introduced in §3, which is used to align the mesh with the trajectory of the interior layer. Sharper pointwise bounds on the partial derivatives of the solution are derived in §4, using a decomposition of the solution into a sum of regular, boundary layer and interior layer components. The numerical scheme is constructed in §5 and theoretical error bounds are established in §6. Some numerical results are presented and discussed in the final section.

Notation: In this paper C denotes a generic constant that is independent of the parameter ε and the mesh parameters N and M. For any function z, we set $\|z\|_{\bar{G}} := \max_{(s,t)\in\bar{G}} |z(s,t)|$.

2. Continuous problem

Consider the following singularly perturbed parabolic problem: Find $\hat{u}$ such that

$$\hat{\mathcal{L}}_\varepsilon \hat{u} = \hat{f}(s,t), \quad (s,t) \in Q := (-1,1) \times (0,T], \tag{1a}$$

$$\text{where} \quad \hat{\mathcal{L}}_\varepsilon \hat{u} := -\varepsilon \hat{u}_{ss} + \hat{a}(t)\hat{u}_s + \hat{b}(s,t)\hat{u} + \hat{c}(t)\hat{u}_t \,,$$

$$\hat{u}(s,0) = \phi(s;\varepsilon), -1 \leq s \leq 1, \tag{1b}$$

$$\hat{u}(-1,t) = \phi_L(t), \ \hat{u}(1,t) = \phi_R(t), \ 0 < t \leq T, \tag{1c}$$

$$\hat{a}(t) > \alpha > 0, \quad \hat{c}(t) \geq c_0 > 0, \tag{1d}$$

$$\hat{b}(s,t) \geq \beta \geq 0, \quad \hat{b}(s,t) + 2\hat{c}'(t) > 0, \quad (s,t) \in Q. \tag{1e}$$

Note that, by using the standard transformation of $\hat{u} = \hat{v}e^{\gamma t}$, we see that (1e) is a mild constraint on the data.

The initial condition ϕ is smooth, but has an ε-dependent Gaussian profile in the vicinity of $s = 0$. The initial condition is assumed to be of the form

$$\phi(s,\varepsilon) = g_1(s) + g_2(s)e^{-\theta\frac{s^2}{\varepsilon}}, \quad \theta > C\varepsilon; \tag{1f}$$

where $g_1(s), g_2(s)$ are smooth functions with the additional compatibility assumptions of

$$g_2^{(i)}(-1) = g_2^{(i)}(1) = 0, \qquad i = 0,1,2. \tag{1g}$$

These additional assumptions ensure that the pulse $e^{-\theta\frac{s^2}{\varepsilon}}$ has no influence on the smoothness of the solution at the end-points $(-1,0)$ and $(1,0)$.

The case where $0 < \theta \leq C\varepsilon$ is not of interest to us here, as in this case no interior layer will form in the solution. Observe that as $\theta/\varepsilon \to \infty$ the width of the pulse narrows and the pulse can be viewed as a regularized delta function. We limit our investigation of the effect of θ, by restricting the parameter to the case of $\theta = \mathcal{O}(1)$, so that we assume that

$$0 < C_* \leq \theta \qquad \text{and} \qquad \frac{\theta T}{c_0} \leq C^*. \tag{1h}$$

The error constants C in our final error bound do depend on the constant C^*. By assuming that $\frac{\theta T}{c_0} \leq C^*$, we can then utilize the transformation $\hat{u} = \hat{v}e^{\frac{2\theta t}{c_0}}$ so that there is no further loss in generality in assuming the constraint

$$\hat{b} \geq 2\theta > 0, \qquad (s,t) \in Q.$$

The characteristic curve associated with the reduced differential equation (formally set $\varepsilon = 0$ in (1a)) can be described by the set of points

$$\Gamma^* := \left\{ (d(t),t) \,|\, d'(t) = \frac{\hat{a}(t)}{\hat{c}(t)}, \ d(0) = 0 \right\}.$$

Note that $d'(t) > 0$, which implies that the center of the pulse moves rightwards with time.

We also define the two subdomains of Q either side of Γ^* by

$$Q^- := \{(s,t) \in Q \,|\, s < d(t) < 1\} \quad \text{and} \quad Q^+ := \{(s,t) \in Q \,|\, s > d(t) > -1\}.$$

In §4.2 we prove that the solution has an interior layer of width $\mathcal{O}(\sqrt{\varepsilon/\theta})$, which travels along Γ^*. In general a boundary layer of width $\mathcal{O}(\varepsilon)$ will also appear in the vicinity of the edge $x = 1$. We restrict the size of the final time T so that the interior layer does not interact with this boundary layer. Thus, we limit the final time T such that

$$d(T) = \int_{t=0}^{T} \frac{\hat{a}(t)}{\hat{c}(t)}\, dt < 1.$$

We define the parameter

$$\delta := 1 - d(T) > 0. \tag{1i}$$

In later sections, we construct a piecewise-uniform mesh, which is designed to be refined in the neighbourhood of the curve Γ^*. To analyse the parameter-uniform convergence of the resulting numerical approximations on such a mesh, it is more convenient to perform the analysis in a transformed domain where the location of the interior layer is fixed in time. As most of the paper deals with this transformed domain, we have adopted the notation $\hat{u}(s,t)$ for the solution in the original domain and we use the simpler notation of $u(x,t)$ for the solution in the transformed domain. In the next section, we define this transformation, which allows us refine the space mesh in the orthogonal direction to the curve Γ^*. To construct a parameter-uniform numerical method, this alignment of the layer-adapted mesh to the characteristic curve of the reduced problem is necessary [12] and is facilitated by the assumption that the coefficients $\hat{a}(t)$, $\hat{c}(t)$ are independent of the space variable.

Assume sufficient compatibility and regularity so that the solution of (1) is such that $\hat{u} \in \mathcal{C}^{n+\gamma}(\bar{Q}), n \geq 4$.[1] From the maximum principle, we have that

$$\|\hat{u}\|_{\bar{Q}} \leq \|\hat{f}\|_{\bar{Q}} \left(1 + T \max_{\bar{Q}} \left\{0, \frac{1-\hat{b}}{\hat{c}}\right\}\right) + \|\hat{u}\|_{\bar{Q}\setminus Q} \leq C.$$

We can write

$$\hat{y} = \hat{u} - \frac{(1-s)}{2}(\phi_L(t) - \phi_L(0)) - \frac{(1+s)}{2}(\phi_R(t) - \phi_R(0)).$$

[1]The space $\mathcal{C}^{0+\gamma}(D)$, where $D \subset \mathbf{R}^2$ is an open set, is the set of all functions that are Hölder continuous of degree γ with respect to the metric $\|\cdot\|$, where for all $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2) \in \mathbf{R}^2$, $\|\mathbf{u} - \mathbf{v}\|^2 = (u_1 - v_1)^2 + |u_2 - v_2|$. For f to be in $\mathcal{C}^{0+\gamma}(D)$ the following semi-norm needs to be finite

$$\lceil f \rceil_{0+\gamma,D} = \sup_{\mathbf{u}\neq\mathbf{v},\ \mathbf{u},\mathbf{v}\in D} \frac{|f(\mathbf{u}) - f(\mathbf{v})|}{\|\mathbf{u} - \mathbf{v}\|^{\gamma}}.$$

The space $\mathcal{C}^{n+\gamma}(D)$ is defined by

$$\mathcal{C}^{n+\gamma}(D) = \left\{ z : \frac{\partial^{i+j} z}{\partial x^i \partial t^j} \in \mathcal{C}^{0+\gamma}(D),\ 0 \leq i + 2j \leq n \right\},$$

and $\|\cdot\|_{n+\gamma}$, $\lceil\cdot\rceil_{n+\gamma}$ are the associated norms and semi-norms.

Then, $\hat{y}$ satisfies zero boundary conditions and the differential equation

$$\hat{\mathcal{L}}_\varepsilon \hat{y} = \hat{F} := \hat{f} - \hat{\mathcal{L}}_\varepsilon \left(\frac{(1-s)}{2}(\phi_L(t) - \phi_L(0)) + \frac{(1+s)}{2}(\phi_R(t) - \phi_R(0)) \right).$$

Introduce the stretched variables $\varsigma := s/\varepsilon$, $\tau := t/\varepsilon$, and we denote $\tilde{\omega}(\varsigma, \tau) := \hat{\omega}(s,t)$, then $\tilde{y}$ satisfies the problem

$$\begin{aligned} -\tilde{y}_{\varsigma\varsigma} + \tilde{a}\tilde{y}_\varsigma + \varepsilon\tilde{b}\tilde{y} + \tilde{c}\tilde{y}_\tau = \varepsilon\tilde{F}, \quad (\varsigma, \tau) \in (-1/\varepsilon, 1/\varepsilon) \times (0, T/\varepsilon], \\ \tilde{y}(-1/\varepsilon, \tau) = \tilde{y}(1/\varepsilon, \tau) = 0,\ 0 < \tau \le T/\varepsilon, \\ \tilde{y}(\varsigma, 0) = \tilde{g}_1(\varepsilon\varsigma) + \tilde{g}_2(\varepsilon\varsigma)e^{-\theta\varepsilon\varsigma^2}, \quad -1/\varepsilon < \varsigma < 1/\varepsilon. \end{aligned}$$

Note that the initial condition satisfies the bounds

$$\left| \frac{\partial^i}{\partial \varsigma^i} \tilde{y}(\varsigma, 0) \right| \le C(\varepsilon^i + (\sqrt{\varepsilon\theta})^i), \quad 1 \le i \le n.$$

Using [3, pg. 65] or [10, p. 320], we have the following bounds on the derivatives of y

$$\|\tilde{y}\|_{n+\gamma} \le C\|\tilde{y}\| + C\left(\varepsilon\|\tilde{F}\|_{n-2+\gamma} + \|\tilde{y}(\varsigma, 0)\|_{n+\gamma}\right), \quad n = i + 2j \ge 2.$$

Using the differential equation, we can deduce that

$$\|\tilde{y}\|_{1+\gamma} \le C\left(\|\tilde{y}\| + \varepsilon\|\tilde{F}\|_{0+\gamma} + \|\tilde{y}(\varsigma, 0)\|_{2+\gamma}\right).$$

Observe that $\|\tilde{y}\|_n \le \|\tilde{y}\|_{n+\gamma}$, $n = i + 2j$, and then in the original variables (s,t), we obtain the parameter-explicit bounds

$$\|\hat{u}\|_n \le C + C(1 + (\sqrt{\varepsilon\theta})^n)\varepsilon^{-n}.$$

These bounds do not suffice for the subsequent error analysis. Below we will obtain sharper bounds on the solution, via a suitable decomposition, which will illustrate that the large derivatives are confined to narrow layer regions of the domain.

3. Mapping to fix the location of interior layer

Consider the map $X : (s,t) \to (x,t)$ given by

$$x(s,t) = \begin{cases} \dfrac{s - d(t)}{1 + d(t)}, & s \le d(t), \\ \dfrac{s - d(t)}{1 - d(t)}, & s \ge d(t). \end{cases} \tag{2}$$

Note that $x = s$ at $t = 0$ and $x = 0$ for all t such that $s = d(t)$. This maps

$$\bar{Q}^- \to \bar{\Omega}^- := [-1, 0] \times [0, T], \quad \bar{Q}^+ \to \bar{\Omega}^+ := [0, 1] \times [0, T]. \tag{3}$$

In the transformed variables, the center of the interior layer is fixed for all time and is located at $x = 0$.

Remark 1. *We employ the following notation in subsequent sections:*

$$
\begin{aligned}
&u(x,t) := \hat{u}(X(s,t),t),\\
&\Omega^- := (-1,0)\times(0,T], \quad \Omega^+ := (0,1)\times(0,T],\\
&\bar{\Omega} := \bar{\Omega}^- \cup \bar{\Omega}^+, \quad \Omega := \Omega^- \cup \Omega^+.
\end{aligned}
$$

Noting that $\hat{c}(t)d'(t) = \hat{a}(t), d(0) = 0$, we have for $s < d(t)$ or $x < 0$ that

$$
\hat{c}(t)\frac{\partial \hat{u}}{\partial t} = -a(t)\frac{(1+x)}{1+d(t)}\frac{\partial u}{\partial x} + c(t)\frac{\partial u}{\partial t}, \quad \frac{\partial \hat{u}}{\partial s} = \frac{1}{1+d(t)}\frac{\partial u}{\partial x}.
$$

Using this map, the differential equation (1a) transforms into

$$
\begin{aligned}
\mathcal{L}_\varepsilon u &:= \left(-\varepsilon g^2(x,t)u_{xx} + \kappa(x,t)a(t)u_x\right) + b(x,t)u + c(t)u_t\\
&= f(x,t),\ x\in\Omega, \qquad (4a)\\
u(x,0) &= \phi(x;\varepsilon),\ -1\le x\le 1, \qquad (4b)\\
u(-1,t) &= \phi_L(t),\ u(1,t) = \phi_R(t),\ 0<t\le T, \qquad (4c)
\end{aligned}
$$

$$
\kappa(x,t) := x\begin{cases} -(1+d(t))^{-1}, & \text{if } x<0,\\ (1-d(t))^{-1}, & \text{if } x>0,\end{cases} \qquad (4d)
$$

$$
g(x,t) := \begin{cases} (1+d(t))^{-1}, & \text{if } x<0,\\ (1-d(t))^{-1}, & \text{if } x>0.\end{cases} \qquad (4e)
$$

Note that for all $t > 0$ such that $d(t) \neq 0$, then $g(0^-,t) \neq g(0^+,t)$. In this transformed problem, the coefficient of the first derivative in space is positive, except along the internal line $x = 0$ where it is zero. Since the map is only piecewise linear, the transformed partial differential equation has discontinuous coefficients. Recall that $\hat{u} \in \mathcal{C}^{4+\gamma}(\bar{Q})$ and hence we have the following transmission conditions at the interface

$$
[u](0,t) = 0, \quad \varepsilon[gu_x](0,t) = 0, \qquad (4f)
$$

where the jump across $x = 0$ is $[u](0,t) := u(0^+,t) - u(0^-,t)$.

As $\hat{u} \in \mathcal{C}^{4+\gamma}(\bar{Q})$ and $u \in \mathcal{C}^1(\bar{\Omega})$, then $u \in \left\{\mathcal{C}^{4+\gamma}(\bar{\Omega}^-) \cup \mathcal{C}^{4+\gamma}(\bar{\Omega}^+)\right\} \cap \mathcal{C}^1(\bar{\Omega})$.

4. Decomposition of the continuous solution

4.1. Regular and boundary layer components

In the next theorem we establish estimates, in the untransformed domain Q, for the derivatives of the regular and the boundary layer components of the solution $\hat{u}$ of problem (1). Note that these two components do not depend on the parameter θ.

Theorem 1. *There exists a function $r(t)$ such that the solutions $\hat{v}, \hat{w}$ of the problems*

$$\hat{\mathcal{L}}_\varepsilon \hat{v} = \hat{f}, \qquad (s,t) \in Q, \tag{5a}$$
$$\hat{v}(s,0) = g_1(s),\ \hat{v}(-1,t) = u(-1,t),\ \hat{v}(1,t) = r(t), \tag{5b}$$

$$\hat{\mathcal{L}}_\varepsilon \hat{w} = 0, \qquad (s,t) \in Q, \tag{5c}$$
$$\hat{w}(s,0) = 0,\ \hat{w}(-1,t) = 0,\ \hat{w}(1,t) = u(1,t) - r(t), \tag{5d}$$

satisfy $\hat{v}, \hat{w} \in \mathcal{C}^{4+\gamma}(\bar{Q})$, and the following bounds for $0 \le j + 2m \le 4$,

$$\Big\| \frac{\partial^{j+m} \hat{v}}{\partial s^j \partial t^m} \Big\|_Q \le C(1 + \varepsilon^{2-(j+m)}), \tag{6a}$$
$$\Big| \frac{\partial^{j+m} \hat{w}}{\partial s^j \partial t^m}(s,t) \Big| \le C\varepsilon^{-j}(1 + \varepsilon^{1-m}) e^{-\frac{\alpha(1-s)}{\varepsilon}}, \quad (s,t) \in Q. \tag{6b}$$

Proof. See the argument in [9, Appendix] or [5]. □

We next deduce the corresponding bounds in the transformed (computational) variables. Using the transformation in (2) we have that for $x < 0$

$$c(t)u_t = \hat{c}(t)\hat{u}_t + a(t)(1+x)\hat{u}_s\,,$$
$$u_{tt} = \hat{u}_{tt} + 2\frac{a(t)}{c(t)}(1+x)\hat{u}_{st} + \left(\frac{a(t)}{c(t)}\right)'(1+x)\hat{u}_s$$
$$+ \left(\frac{a(t)}{c(t)}(1+x)\right)^2 \hat{u}_{ss}; \qquad u_x = \hat{u}_s(1 + d(t)).$$

Hence, we can deduce the following bounds

$$\Big\| \frac{\partial^j v}{\partial x^j} \Big\|_{\Omega^- \cup \Omega^+} \le C(1 + \varepsilon^{2-j}),\ 0 \le j \le 3, \ \Big\| \frac{\partial^m v}{\partial t^m} \Big\|_{\Omega^- \cup \Omega^+} \le C,\ m = 1, 2, \tag{7}$$

$$|w(x,t)| \le Ce^{-\frac{\alpha\delta(1-x)}{\varepsilon}}, \quad (x,t) \in \Omega^- \cup \Omega^+; \tag{8a}$$

and for $(x,t) \in \Omega^+$,

$$\Big| \frac{\partial^j w}{\partial x^j}(x,t) \Big| \le C\varepsilon^{-j} e^{-\frac{\alpha\delta(1-x)}{\varepsilon}}, \quad 1 \le j \le 4, \tag{8b}$$
$$\Big| \frac{\partial^m w}{\partial t^m}(x,t) \Big| \le C(1 + \varepsilon^{1-m}) e^{-\frac{\alpha\delta(1-x)}{\varepsilon}}, \quad m = 1, 2. \tag{8c}$$

4.2. Interior layer component

Bounds on the derivatives of the interior layer component are established directly in the transformed variables. Consider the following decomposition of the solution

$$u = v + w + z.$$

The interior layer component z is the solution of the problem

$$\mathcal{L}_\varepsilon z = 0, \text{ on } \Omega^- \cup \Omega^+, \; z(x,0) = g_2(x)e^{-\theta\frac{x^2}{\varepsilon}}, \; z(-1,t) = 0, \; z(1,t) = 0, \tag{9}$$

and at $x = 0$ the following transmission conditions are satisfied

$$[z](0,t) = 0, \quad [gz_x](0,t) = -[g(v_x + w_x)](0,t),$$

where the function g is defined in (4e).

In the following theorem we state a comparison principle, whose proof follows from a standard argument.

Theorem 2. *Assume that a function* $\omega \in \mathcal{C}^0(\overline{\Omega}) \cap \mathcal{C}^2(\Omega^- \cup \Omega^+)$ *satisfies* $\mathcal{L}'_\varepsilon \omega(x,t) \geq 0$, *for all* $(x,t) \in \overline{\Omega}$ *with*

$$\mathcal{L}'_\varepsilon \omega(x,t) := \begin{cases} \omega(x,t), & (x,t) \in \bar{\Omega} \setminus \Omega, \\ -\varepsilon g^2(x,t)\omega_{xx} + \kappa(x,t)a(t)\omega_x + b(x,t)\omega + c(t)\omega_t, & (x,t) \in \Omega, \\ -\varepsilon[g\omega_x] & x = 0, \; t > 0. \end{cases}$$

Then, $\omega(x,t) \geq 0$ *for all* $(x,t) \in \overline{\Omega}$.

Based on this comparison principle, the next theorem establishes bounds on the derivatives of the interior layer component. In particular, the time derivatives of the interior layer component do not depend on ε in the transformed variables (x,t).

Theorem 3. *The solution of (9)* $z \in \mathcal{C}^{4+\gamma}(\bar{\Omega}^-) \cup \mathcal{C}^{4+\gamma}(\bar{\Omega}^+)$ *and we have*

$$|z(x,t)| \leq Ce^{\sqrt{\frac{\theta}{\varepsilon}}x} e^{\frac{-\theta}{\|c\|_{\bar{\Omega}}}t}, \quad (x,t) \in \Omega^-, \tag{10a}$$

$$|z(x,t)| \leq Ce^{\frac{-\sqrt{\theta}x(1-d(t))}{\sqrt{\varepsilon}}} e^{\frac{-\theta}{\|c\|_{\bar{\Omega}}}t}, \quad (x,t) \in \Omega^+. \tag{10b}$$

If $\frac{-1}{1+d(T)} < x \leq 0$ *and* $0 \leq t \leq T$, *then*

$$\left|\frac{\partial^j z}{\partial x^j}(x,t)\right| \leq C(1+\theta^{j/2})\varepsilon^{-j/2}e^{\sqrt{\frac{\theta}{\varepsilon}}x}, \; 1 \leq j \leq 4, \tag{10c}$$

$$\left|\frac{\partial^m z}{\partial t^m}(x,t)\right| \leq C(1+\theta^m), \; m = 1,2. \tag{10d}$$

If $0 \leq x < 1 - d(T)$ *and* $0 \leq t \leq T$, *then*

$$\left|\frac{\partial^j z}{\partial x^j}(x,t)\right| \leq C(1+\theta^{j/2})\varepsilon^{-j/2}e^{\frac{-\sqrt{\theta}x(1-d(T))}{\sqrt{\varepsilon}}}, \; 1 \leq j \leq 4, \tag{10e}$$

$$\left|\frac{\partial^m z}{\partial t^m}(x,t)\right| \leq C(1+\theta^m), \quad m = 1,2. \tag{10f}$$

Proof. Note first that for all s and any $\kappa > 0$

$$e^{-\kappa s^2} \leq e^{\frac{1}{4}} e^{-\sqrt{\kappa}|s|}. \tag{11}$$

Recall that $z = u - (v + w)$ and so $z \in \mathcal{C}^{4+\gamma}(\bar{\Omega}^-) \cup \mathcal{C}^{4+\gamma}(\bar{\Omega}^+)$. Since u, v, w are bounded, we have that $\|z\|_{\bar{\Omega}} \leq C$. Note that $|z(x,0)| \leq Ce^{\sqrt{\frac{\theta}{\varepsilon}}x}, -1 < x < 0$ and, using $b \geq 2\theta > 0$,

$$\mathcal{L}_\varepsilon \left(e^{\sqrt{\frac{\theta}{\varepsilon}}x} e^{\frac{-\theta}{\|c\|_{\bar{\Omega}}}t} \right) \geq 0, \ (x,t) \in \Omega^-.$$

Thus we have established the pointwise bound

$$|z(x,t)| \leq Ce^{\sqrt{\frac{\theta}{\varepsilon}}x} e^{\frac{-\theta}{\|c\|_{\bar{\Omega}}}t}, \quad (x,t) \in \bar{\Omega}^-. \tag{12a}$$

To the right of the interface $x = 0$ we introduce the barrier function

$$\Phi(x,t) := Ce^{-\frac{p(t)\sqrt{\theta}x}{\sqrt{\varepsilon}}} e^{-\frac{\theta}{\|c\|_{\bar{\Omega}}}t}, \quad \text{where} \quad p(t) := 1 - d(t).$$

Note that $|z(x,0)| \leq \Phi(x,0), 0 < x < 1$ and

$$\begin{aligned}\mathcal{L}_\varepsilon \Phi &= C \left\{ -\theta - xa(t)\sqrt{\frac{\theta}{\varepsilon}} + b(x,t) + c(t) \left(\frac{-\theta}{\|c\|_{\bar{\Omega}}} + \frac{a(t)}{c(t)}\sqrt{\frac{\theta}{\varepsilon}}\, x \right) \right\} \Phi \\ &= C \left(-\theta + b(x,t) - \frac{c}{\|c\|_{\bar{\Omega}}}\theta \right) \Phi \ \geq 0.\end{aligned}$$

Hence,

$$|z(x,t)| \leq Ce^{-\frac{(1-d(t))\sqrt{\theta}x}{\sqrt{\varepsilon}}} e^{-\frac{\theta}{\|c\|_{\bar{\Omega}}}t}, \ (x,t) \in \Omega^+. \tag{12b}$$

To obtain bounds on the derivatives of the interior layer, we follow the argument in [6, Theorem 4], which relies on the interior estimates from [10, p. 352]. Let us first determine bounds on the time derivatives of the solution u of (4) in the vicinity of the line $x = 0$.

Introduce the time dependent stretched variable

$$\eta = \frac{s - d(t)}{\sqrt{\varepsilon}},$$

and define $\check{u}(\eta, t) := \hat{u}(s,t)$. Then,

$$\frac{\partial \hat{u}}{\partial s} = \frac{1}{\sqrt{\varepsilon}} \frac{\partial \check{u}}{\partial \eta}, \quad \frac{\partial^2 \hat{u}}{\partial s^2} = \frac{1}{\varepsilon} \frac{\partial^2 \check{u}}{\partial \eta^2}, \quad \frac{\partial \hat{u}}{\partial t} = -\frac{1}{\sqrt{\varepsilon}} \frac{\check{a}(t)}{\check{c}(t)} \frac{\partial \check{u}}{\partial \eta} + \frac{\partial \check{u}}{\partial t}.$$

Hence the differential equation (1a) can be written in the form

$$-\check{u}_{\eta\eta} + \check{b}\,\check{u} + \check{c}\,\check{u}_t = \check{f}, \quad (\eta, t) \in \left(-\frac{1 + d(t)}{\sqrt{\varepsilon}}, \frac{1 - d(t)}{\sqrt{\varepsilon}} \right) \times (0, T].$$

On the ε-independent strip $R := (-\delta, \delta) \times (0, T]$, where δ is given in (1i), we use Ladyzhenskaya interior estimates [10, p. 352] to deduce that

$$\left| \frac{\partial^m \check{u}}{\partial t^m}(\eta, t) \right| \leq C(1 + \theta^m), \quad m \leq 2 + \gamma, \quad (\eta, t) \in R.$$

Observe that

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial \breve{u}}{\partial \eta}(\eta,t)\frac{x}{\sqrt{\varepsilon}}d'(t) + \frac{\partial \breve{u}}{\partial t}(\eta,t), \quad (x,t) \in \Omega^-,$$
$$\frac{\partial u}{\partial t}(x,t) = -\frac{\partial \breve{u}}{\partial \eta}(\eta,t)\frac{x}{\sqrt{\varepsilon}}d'(t) + \frac{\partial \breve{u}}{\partial t}(\eta,t), \quad (x,t) \in \Omega^+.$$

Therefore, along the line $x = 0$,

$$\frac{\partial u}{\partial t}(0,t) = \frac{\partial \breve{u}}{\partial t}(0,t).$$

Use the same argument to prove that $\frac{\partial^2 \breve{u}}{\partial t^2}(0,t) = \frac{\partial^2 u}{\partial t^2}(0,t)$. Thus, we have established that

$$\left|\frac{\partial^m u}{\partial t^m}(0,t)\right| \le C(1+\theta^m), \quad m = 1,2. \tag{13}$$

On the left domain Ω^-, we use the time dependent stretched variable

$$\zeta := \frac{x\sqrt{\alpha}}{\sqrt{\varepsilon}}(1+d(t)). \tag{14}$$

Note that with $\tilde{z}(\zeta,t) := z(x,t)$ then

$$\frac{\partial z}{\partial x} = \frac{\sqrt{\alpha}}{\sqrt{\varepsilon}}(1+d(t))\frac{\partial \tilde{z}}{\partial \zeta} \quad \text{and} \quad c(t)\frac{\partial z}{\partial t} = \frac{a(t)}{(1+d(t))}\zeta\frac{\partial \tilde{z}}{\partial \zeta} + c(t)\frac{\partial \tilde{z}}{\partial t}. \tag{15}$$

Hence the function $\tilde{z}(\zeta,t)$ satisfies the parabolic problem

$$-\alpha\tilde{z}_{\zeta\zeta} + (\tilde{b}\tilde{z} + \tilde{c}\tilde{z}_t) = 0; \quad (\zeta,t) \in \left(-\frac{\sqrt{\alpha}}{\sqrt{\varepsilon}}(1+d(t)), 0\right) \times (0,T].$$

Consider the ε-dependent rectangular region

$$(\zeta,t) \in S_\varepsilon := \left(-\frac{\sqrt{\alpha}}{\sqrt{\varepsilon}}, 0\right) \times (0,T].$$

Recall from (12a) that

$$|\tilde{z}(\zeta,t)| \le Ce^{\frac{\sqrt{\theta}\zeta}{\sqrt{\alpha}}}e^{\frac{-\theta}{\|c\|_{\bar{\Omega}}}t}, \quad (\zeta,t) \in S_\varepsilon.$$

From (7), (8), (13) and

$$\frac{\partial^m z}{\partial t^m}(0,t) = \frac{\partial^m \tilde{z}}{\partial t^m}(0,t), \quad m \le 2+\gamma,$$

where we used (15), we have

$$\left|\frac{\partial^m \tilde{z}}{\partial t^m}(0,t)\right| \le C(1+\theta^m), \quad 0 \le t \le T, \quad m \le 2+\gamma. \tag{16}$$

Note further that

$$\left|\frac{\partial^j \tilde{z}}{\partial \zeta^j}(\zeta,0)\right| \le C(\sqrt{\varepsilon}+\sqrt{\theta})^j e^{\frac{\sqrt{\theta}\zeta}{2\sqrt{\alpha}}}, \qquad j \le 4+\gamma, \quad -\frac{\sqrt{\alpha}}{\sqrt{\varepsilon}} < \zeta \le 0, \tag{17}$$

where we used (11).

From the bounds (16) and (17), [10, (10.5), p. 352] we deduce that for all $0 \le n := j+2m \le 4$,

$$\left|\frac{\partial^{j+m}\tilde{z}}{\partial \zeta^j \partial t^m}(\zeta,t)\right| \le C\left(1+\theta^{n/2}\right) e^{\frac{\sqrt{\theta}\zeta}{2\sqrt{\alpha}}}, \quad -\frac{\sqrt{\alpha}}{\sqrt{\varepsilon}} < \zeta \le 0, \quad t \ge 0,$$

by considering the cases $-\frac{\sqrt{\alpha}}{\sqrt{\varepsilon}} < \zeta < -1$ and $-1 \le \zeta \le 0$ separately.

Thus, in particular, the space derivatives satisfy for $0 \le j \le 4$

$$\left|\frac{\partial^j z}{\partial x^j}(x,t)\right| \le C\left(1+\theta^{j/2}\right)\varepsilon^{-j/2} e^{\frac{\sqrt{\theta}x}{2\sqrt{\varepsilon}}}, \ \frac{-1}{1+d(T)} < x \le 0, \ t \ge 0.$$

For the time derivatives, consider the problem (9) restricted to the region $\overline{\Omega}^-$. At the boundaries we have that

$$z_t(-1,t)=0, \ t \in (0,T], \quad |z_t(0,t)| \le C(1+\theta), \quad t \in (0,T].$$

At $t=0$, use a continuity argument to deduce that

$$\begin{aligned}|z_t(x,0)| &= \frac{1}{c(0)}\left|\varepsilon g^2(x,0) z_{xx}(x,0) - \kappa(x,0)a(0)z_x(x,0) - b(x,0)z(x,0)\right| \\ &\le C(1+\theta).\end{aligned}$$

Finally, at the points $(x,t) \in \Omega^-$, we have that

$$(\mathcal{L}+c'(t))z_t(x,t) = -\kappa_t(x,t)a(t)z_x(x,t) - \kappa(x,t)a'(t)z_x(x,t) - b_t(x,t)z(x,t),$$

and therefore

$$|(\mathcal{L}+c'(t))z_t(x,t)| \le C, \quad (x,t) \in \Omega^-.$$

Then by the maximum principle and (1e), one can establish that

$$|z_t(x,t)| \le C(1+\theta), \quad (x,t) \in \Omega^-.$$

Repeat this argument to deduce the bound on the second derivative in time.

An analogous argument applies on the region Ω^+, where we now use the time dependent stretched variable

$$\zeta_1 := \frac{x\sqrt{\alpha}}{\sqrt{\varepsilon}}(1-d(t)). \tag{18}$$

□

5. Numerical scheme

Let N and M be two positive integers. To approximate the solution of problem (4) we use a uniform mesh in time $\{t_j = j\Delta t, \mid \Delta t = T/M\}$ and a piecewise uniform mesh of Shishkin type in space $\{x_i\}_{i=0}^{N}$ (described below) in the transformed variables (x, t). The grid is given by

$$\overline{\Omega}^{N,M} = \{t_j\}_{j=0}^{M} \times \{x_i\}_{i=0}^{N}, \quad \overline{\Gamma}^{N,M} = \overline{\Omega}^{N,M} \cap (\overline{\Omega} \setminus \Omega), \quad \Omega^{N,M} = \overline{\Omega}^{N,M} \setminus \overline{\Gamma}^{N,M}.$$

The local spatial mesh sizes are denoted by $h_i = x_i - x_{i-1}$, $1 \le i \le N$.

To describe the numerical method we use the following notation for the finite difference approximations of the derivatives

$$D_t^- \Upsilon(x_i, t_j) := \frac{\Upsilon(x_i, t_j) - \Upsilon(x_i, t_{j-1})}{\Delta t}, \ D_x^- \Upsilon(x_i, t_j) := \frac{\Upsilon(x_i, t_j) - \Upsilon(x_{i-1}, t_j)}{h_i},$$

$$D_x^+ \Upsilon(x_i, t_j) := \frac{\Upsilon(x_{i+1}, t_j) - \Upsilon(x_i, t_j)}{h_{i+1}},$$

$$\delta_x^2 \Upsilon(x_i, t_j) := \frac{2}{h_i + h_{i+1}} (D_x^+ \Upsilon(x_i, t_j) - D_x^- \Upsilon(x_i, t_j)).$$

Discretize problem (4) using an Euler method to approximate the time variable and an upwind finite difference operator to approximate in space. The finite difference equation associated with each grid point is given by

$$L^{N,M} U(x_i, t_j) = f(x_i, t_j), \quad (x_i, t_j) \in \Omega^{N,M}, \quad \text{with} \tag{19a}$$

$$L^{N,M} U := -\varepsilon g^2 \delta_x^2 U + \kappa a D_x^- U + bU + cD_t^- U, \ \text{in } \Omega^{N,M}, \tag{19b}$$

$$\begin{aligned} L^{N,M} U(0, t_j) &:= -\varepsilon (g(0^-, t_j) D_x^- U - g(0^+, t_j) D_x^+ U)(0, t_j) \\ &= 0, \quad t_j > 0, \end{aligned} \tag{19c}$$

$$U(x_i, 0) = u(x_i, 0), \quad -1 < x_i < 1, \tag{19d}$$

$$U(-1, t_j) = u(-1, t_j), \quad U(1, t_j) = u(1, t_j), \ t_j \ge 0. \tag{19e}$$

The space domain is discretized using a piecewise uniform mesh which splits the space domain $[-1, 1]$ into four subintervals

$$[-1, -\tau_1] \cup [-\tau_1, \tau_2] \cup [\tau_2, 1 - \sigma] \cup [1 - \sigma, 1], \tag{19f}$$

where the transition parameters between the fine and coarse meshes are defined by

$$\tau_1 := \min \left\{ \frac{1}{1 + d(T)}, 2\sqrt{\frac{\varepsilon}{\theta}} \ln N \right\}, \ \tau_2 := \min \left\{ 1 - d(T), \frac{2}{\delta} \sqrt{\frac{\varepsilon}{\theta}} \ln N \right\},$$

$$\sigma := \min \left\{ \frac{1 - \tau_2}{2}, \frac{4\varepsilon}{\alpha \delta} \ln N \right\}.$$

Note that the parameter τ_1, τ_2 (associated with the internal layer) do not depend on the magnitude of the convective coefficient parameter α. The grid points are uniformly distributed within each subinterval such that

$$x_0 = -1, \ x_{N/4} = -\tau_1, \ x_{N/2} = 0, \ x_{3N/4} = \tau_2, \ x_{7N/8} = 1 - \sigma, \ x_N = 1.$$

Using standard arguments, one can establish the following discrete comparison principle:

Theorem 4. *For any mesh function Z, if $Z(x_i,t_j) \geq 0, (x_i,t_j) \in \overline{\Gamma}^{N,M}$ and $L^{N,M}Z(x_i,t_j) \geq 0,\ (x_i,t_j) \in \Omega^{N,M}$ then $Z(x_i,t_j) \geq 0$, for all $(x_i,t_j) \in \overline{\Omega}^{N,M}$.*

From this, we easily establish the discrete stability bound $\|U\|_{\overline{\Omega}^{N,M}} \leq C$.

In the next section, the error analysis will concentrate on the case when

$$\tau_1 = 2\sqrt{\frac{\varepsilon}{\theta}}\ln N, \quad \tau_2 = \frac{2}{\delta}\sqrt{\frac{\varepsilon}{\theta}}\ln N, \quad \sigma = \frac{4\varepsilon}{\alpha\delta}\ln N. \tag{20}$$

The other possibilities for the mesh parameters can be easily dealt with, using a classical argument.

6. Numerical Analysis

Let $U = V + W + Z$, where V, W, Z are the discrete counterparts to the continuous components v, w, z. The discrete regular component V is defined as the solution of

$$L^{N,M}V(x_i,t_j) = f(x_i,t_j), \quad (x_i,t_j) \in \Omega^{N,M}, \tag{21a}$$

$$L^{N,M}V(0,t_j) = \varepsilon\,[gv_x]\,(0,t_j), \quad t_j > 0, \tag{21b}$$

$$V(-1,t_j) = v(-1,t_j),\ V(1,t_j) = v(1,t_j), \quad t_j > 0, \tag{21c}$$

$$V(x_i,0) = v(x_i,0), \quad -1 < x_i < 1. \tag{21d}$$

The discrete boundary layer function W is defined by

$$W(x_i,t_j) \equiv 0, \quad (x_i,t_j) \in \bar{\Omega}^- \cap \bar{\Omega}^{N,M}, \tag{22a}$$

$$L^{N,M}W(x_i,t_j) = 0, \quad (x_i,t_j) \in \Omega^+ \cap \Omega^{N,M}, \tag{22b}$$

$$W(0,t_j) = 0,\ W(1,t_j) = w(1,t_j), \quad t_j > 0, \tag{22c}$$

$$W(x_i,0) = 0, \quad -1 < x_i < 1. \tag{22d}$$

The discrete interior layer function Z is finally defined by:

$$L^{N,M}Z(x_i,t_j) = 0, \quad (x_i,t_j) \in \Omega^{N,M}, \tag{23a}$$

$$Z(-1,t_j) = Z(1,t_j) = 0, \quad t_j > 0, \tag{23b}$$

$$Z(x_i,0) = g_2(x_i)e^{-\theta\frac{x_i^2}{\varepsilon}}, \quad -1 < x_i < 1, \tag{23c}$$

$$L^{N,M}Z(0,t_j) = (g(0^+,t_j)D_x^+(V+W) - g(0^-,t_j)D_x^-V)(0,t_j). \tag{23d}$$

Theorem 5. *Assume (20). For M sufficiently large,*

$$(a) \quad \|V - v\|_{\overline{\Omega}^{N,M}} \leq C(N^{-1} + M^{-1}),$$

$$(b) \quad \|W - w\|_{\overline{\Omega}^{N,M}} \leq C(N^{-1}\ln N + M^{-1})\ln N,$$

where the solutions V, W of problems (21) and (22) are, respectively, the discrete approximations to the regular component v (5b) and the boundary layer component w (5d).

Proof. (a) For $t_j > 0$ and using the bounds on the derivatives of the regular component,

$$\|L^{N,M}(V-v)\|_{\overline{\Omega}^{N,M}} \le C(N^{-1}+M^{-1}).$$

The truncation error at $x=0$ is

$$|(g(0^+,t_j)D_x^+(V-v) - g(0^-,t_j)D_x^-(V-v)(0,t_j))| \le CN^{-1}.$$

Use the barrier function

$$C(N^{-1}+M^{-1})(1+t_j) + CN^{-1}\psi(x_i), \quad \psi(x) := \begin{cases} 1-x, & \text{if } x \le 0, \\ 1+x, & \text{if } x > 0, \end{cases}$$

and a discrete comparison principle to bound $\|V-v\|_{\overline{\Omega}^{N,M}}$.

(b) As in [6], the solution of (22) satisfies the bound

$$\begin{aligned} |W(x_i,t_j)| &\le C\left(1-\frac{\alpha T}{c_0\delta M}\right)^{-j} \frac{\prod_{k=N/2}^{i}\left(1+\frac{\alpha\delta x_k h_k}{2\varepsilon}\right)}{\prod_{k=N/2}^{N}\left(1+\frac{\alpha\delta x_k h_k}{2\varepsilon}\right)} \\ &\le C\frac{\prod_{k=N/2}^{i}\left(1+\frac{\alpha\delta x_k h_k}{2\varepsilon}\right)}{\prod_{k=N/2}^{N}\left(1+\frac{\alpha\delta x_k h_k}{2\varepsilon}\right)}, \quad N/2 \le i \le N, \end{aligned}$$

if M is sufficiently large so that

$$0 < 1 - \frac{\alpha T}{c_0\delta M}.$$

Hence, for $0 \le x_i \le 1-\sigma$,

$$|W(x_i,t_j)| \le |W(1-\sigma,t_j)| \le C\prod_{k=7N/8+1}^{N}\left(1+\frac{\alpha\delta h_k}{4\varepsilon}\right)^{-1} \le CN^{-1},$$

where we used that $x_i \ge 1/2$ if $x_i \ge 1-\sigma$.

Use the exponential character of the continuous layer component w to get that $|(W-w)(x_i,t_j)| \le |W(x_i,t_j)| + |w(x_i,t_j)| \le CN^{-1}$, $x_i \le 1-\sigma$. In the fine region $[1-\sigma,1]\times[0,T]$ use a truncation error argument to deduce error estimates of the singular component W. The discrete maximum principle with the barrier function

$$(N^{-1}\ln N + M^{-1})\ln N(x_i-(1-\sigma))\sigma^{-1}, \quad 1-\sigma < x_i < 1;$$

establishes the desired bound. □

Theorem 6. *Assume (20). For sufficiently large $M \ge \mathcal{O}(\ln(N))$, the solution of (23) satisfies the bounds*

$$\begin{aligned} (a) \qquad |Z(x_i,t_j)| &\le C\frac{\prod_{k=1}^{i}\left(1+\frac{\sqrt{\theta}h_k}{2\sqrt{\varepsilon}}\right)}{\prod_{k=1}^{N/2}\left(1+\frac{\sqrt{\theta}h_k}{2\sqrt{\varepsilon}}\right)}, \quad x_i \le 0, \\ (b) \qquad |Z(x_i,t_j)| &\le C\prod_{n=N/2}^{i}\left(1+\frac{\delta\sqrt{\theta}h_n}{2\sqrt{\varepsilon}}\right)^{-1} + CN^{-1}\ln N, \quad x_i \ge 0. \end{aligned}$$

Proof. First, we note that $|Z(0,t_j)| \leq C$, $t_j \geq 0$, since U, W and V are all bounded.

(a) For $x_i \leq 0$, consider the following barrier function

$$\Phi(x_i,t_j) := C\left(1-\frac{\theta T}{c_0 M}\right)^{-j} \frac{\prod_{k=1}^{i}\left(1+\frac{\sqrt{\theta}h_k}{2\sqrt{\varepsilon}}\right)}{\prod_{k=1}^{N/2}\left(1+\frac{\sqrt{\theta}h_k}{2\sqrt{\varepsilon}}\right)},$$

where M is sufficiently large so that

$$0 < 1-\frac{\theta T}{c_0 M}. \tag{24}$$

Note that

$$\begin{aligned} 2\sqrt{\varepsilon}D_x^+\Phi(x_i,t_j) = \sqrt{\theta}\Phi(x_i,t_j), \quad cD_t^-\Phi(x_i,t_j) &\geq 0.5\theta\Phi(x_i,t_j), \\ -\varepsilon g^2\delta_x^2\Phi(x_i,t_j) &\geq -0.5\theta\Phi(x_i,t_j), \end{aligned}$$

and so, it follows that, for $t_j > 0$ and $-1 < x_i < 0$, we have that

$$(-\varepsilon g^2\delta_x^2 + \kappa a D_x^- + b\,I + cD_t^-)\Phi(x_i,t_j) \geq 0,$$

where I is the identity operator. Note also that $\Phi(-1,t_j) \geq 0, \Phi(0,t_j) \geq C > 0$ and

$$\Phi(x_i,0) \geq Ce^{\sqrt{\frac{\theta}{\varepsilon}}\frac{x_i}{2}} \geq Ce^{-\frac{\theta x_i^2}{\varepsilon}} \geq z(x_i,0), \quad x_i < 0,$$

where we used (11). Finish using a discrete comparison principle.

(b) For $x_i \geq 0$, consider the following barrier function

$$B(x_i,t_j) := C\Phi_1(x_i)\Psi_1(t_j) + C(N^{-1}\ln N)t_j,$$

where

$$\Phi_1(x_i) := \prod_{n=N/2+1}^{i}\left(1+\frac{\sqrt{\theta}\delta h_n}{2\sqrt{\varepsilon}}\right)^{-1}, \ i > N/2; \quad \Phi_1(0) := 1;$$

$$\Psi_1(t_j) := \left(1-\frac{\delta(\theta+1)+\|a\|_{\bar{\Omega}^+}\ln N}{c_0\delta}\frac{T}{M}\right)^{-j}, \ j > 0; \quad \Psi_1(0) := 1;$$

and $M(N)$ is chosen sufficiently large so that

$$0 < c < 1-\left(\frac{\delta(\theta+1)+\|a\|_{\bar{\Omega}^+}\ln N}{c_0\delta}\right)\frac{T}{M}. \tag{25}$$

Note first that $B(1,t_j) \geq 0,\ B(0,t_j) \geq C\Psi_1(t_j) > 0$ and we can choose C such that $B(x_i,0) \geq Z(x_i,0)$. In addition we have that, for $x_i > 0$,

$$2\sqrt{\varepsilon}D_x^-\Phi_1(x_i) = -\sqrt{\theta}\delta\Phi_1(x_i), \quad -\varepsilon\delta_x^2\Phi_1(x_i) \geq -0.5\theta\delta^2\Phi_1(x_i),$$

and for all $t_j > 0$,

$$c(t_j)D_t^-\Psi_1(t_j) \geq \left(\theta + 1 + \frac{\|a\|_{\bar{\Omega}^+}\ln N}{\delta}\right)\Psi_1(t_j).$$

For $0 < x_i \leq \tau_2$, and for N sufficiently large (independently of ε) we have that

$$\begin{aligned}
&\left(-\varepsilon g^2\delta_x^2 + \kappa a D_x^- + b\,I + cD_t^-\right)\Phi_1(x_i)\Psi_1(t_j) \\
&\qquad \geq \left(1 - \frac{\sqrt{\theta}\delta\|a\|_{\bar{\Omega}^+}x_i}{2\sqrt{\varepsilon}(1-d(t_j))} + \frac{\|a\|_{\bar{\Omega}^+}\ln N}{\delta}\right)\Phi_1(x_i)\Psi_1(t_j) \\
&\qquad \geq \Phi_1(x_i)\Psi_1(t_j), \quad 0 < x_i \leq \tau_2.
\end{aligned}$$

If $\tau_2 < x_i \leq 1-\sigma$, then using the inequality $nt \leq (1+t)^n, t \geq 0$,

$$\begin{aligned}
\frac{\sqrt{\theta}x_i}{2\sqrt{\varepsilon}}\Phi_1(x_i) &= \frac{\sqrt{\theta}\tau_2}{2\sqrt{\varepsilon}}\Phi_1(x_i) + \Phi_1(\tau_2)\sqrt{\theta}\frac{(x_i-\tau_2)}{2\sqrt{\varepsilon}}\left(1 + \frac{\sqrt{\theta}\delta H_\star}{2\sqrt{\varepsilon}}\right)^{-(i-3N/4)} \\
&\leq CN^{-1}\ln N,
\end{aligned}$$

where $NH_\star := 8(1-\sigma-\tau_2)$. If $1-\sigma < x_i < 1$, using the fact that $\Phi_1(x_i) \leq \Phi_1(1-\sigma)$ and $nt \leq (1+t)^n, t \geq 0$ in the fine mesh, we have

$$\frac{\sqrt{\theta}x_i}{2\sqrt{\varepsilon}}\Phi_1(x_i) \leq \sqrt{\theta}\frac{1-\sigma}{2\sqrt{\varepsilon}}\Phi_1(1-\sigma) + \sqrt{\theta}\frac{x_i-(1-\sigma)}{2\sqrt{\varepsilon}}\Phi_1(x_i) \leq CN^{-1}\ln N.$$

Then for sufficiently large N and $\tau_2 < x_i < 1$,

$$\begin{aligned}
&\left(-\varepsilon g^2\delta_x^2 + \kappa a D_x^- + b\,I + cD_t^-\right)\Phi_1(x_i)\Psi_1(t_j) \\
&\qquad \geq \left(1 + \frac{\theta}{2} - \frac{\|a\|_{\bar{\Omega}^+}\sqrt{\theta}x_i}{2\sqrt{\varepsilon}} + \frac{\|a\|_{\bar{\Omega}^+}}{\delta}\ln N\right)\Phi_1(x_i)\Psi_1(t_j) \\
&\qquad \geq \Phi_1(x_i)\Psi_1(t_j) - C\sqrt{\theta}N^{-1}\ln N.
\end{aligned}$$

Thus $C\Phi_1(x_i)\Psi_1(t_j) + C(N^{-1}\ln N)t_j$ is a suitable barrier function for Z. □

We are now ready to state the main result. We impose an additional constraint of $\alpha^2 \geq 16\theta\varepsilon$, which is satisfied if ε is sufficiently small.

Theorem 7. *Assume (20), M is sufficiently large so that (25) is satisfied and ε is sufficiently small so that $\alpha^2 \geq 16\theta\varepsilon$. Then, outside the interior layer,*

$$\|\bar{U} - u\|_{\bar{\Omega}\setminus\{(\tau_1,\tau_2)\times[0,T]\}} \leq C(N^{-1}\ln N + M^{-1})\ln N,$$

and within the interior layer region

$$\|\bar{U} - u\|_{(\tau_1,\tau_2)\times[0,T]} \leq C(\theta^{-1}+\theta)(N^{-1}\ln N + M^{-1})\ln N;$$

where $\bar{U}$ is the piecewise bilinear interpolant of the solution U of the discrete problem (19) and u is the solution of the continuous problem (1). The error constant C is independent of N, M and ε.

Proof. By virtue of Theorem 5 we first focus on the nodal error in approximating the interior layer component z. From Theorems 3 and 6 and the triangular inequality, we have that

$$|(Z-z)(x_i,t_j)| \leq CN^{-1}\ln N, \quad \text{for } x_i \in \ [-1,-\tau_1] \cup \ [\tau_2, 1]. \tag{26}$$

We now examine the truncation error $L^{N,M}(Z-z)$ in the layer region $(-\tau_1,\tau_2)\times[0,T]$. Note first that

$$\left|g(0^+,t_j)\left(z_x(0^+,t_j)-D_x^+z(0,t_j)\right)\right| \leq C\,(1+\theta)\,\frac{N^{-1}\tau_2}{\varepsilon},$$

with a similar bound to the left of 0. Hence, using (21b), (22a,c), (23d), we get that

$$\begin{aligned}
&L^{N,M}(Z-z)(0,t_j)\\
&\quad= (g(0^+,t_j)D_x^+ - g(0^-,t_j)D_x^-)V(0,t_j) + g(0^+,t_j)D_x^+W(0,t_j)\\
&\quad+ (g(0^+,t_j)D_x^+ - g(0^-,t_j)D_x^-)z(0,t_j)\\
&\quad= \ [gv_x](0,t_j) + \ [gw_x](0,t_j) + g(0^+,t_j)D_x^+w(0,t_j) - \ [gw_x](0,t_j)\\
&\quad+ (g(0^+,t_j)D_x^+ - g(0^-,t_j)D_x^-)z(0,t_j)\\
&\quad= g(0^+,t_j)D_x^+W(0,t_j) - \ [gw_x](0,t_j)\\
&\quad+ (g(0^+,t_j)D_x^+ - g(0^-,t_j)D_x^-)z(0,t_j) - [gz_x](0,t_j).
\end{aligned}$$

Recall the barrier function used in Theorem 5 to bound the discrete boundary layer W. Note that $x_k \geq \tau_2$ for $3N/4 \leq k \leq 7N/8$ and so, for sufficiently large N and ε sufficiently small so that $\alpha^2/(16\theta) \geq \varepsilon$, we have the following bound:

$$\begin{aligned}
|W(x_{N/2+1},t_j)| \ &\leq \ CN^{-1}\left(1+\frac{\alpha\delta\tau_2 H}{2\varepsilon}\right)^{-N/8}, \ H := \frac{8(1-\sigma-\tau_2)}{N}\\
&= \ CN^{-1}\left(1+\frac{\alpha H\ln N}{\sqrt{\theta}\sqrt{\varepsilon}}\right)^{-N/8},\\
&\leq \ CN^{-1}\left(1+\frac{8\ln N}{N}\right)^{-N/8}, \quad \text{as} \quad \frac{\alpha(1-\sigma-\tau_2)}{\sqrt{\theta}\sqrt{\varepsilon}} \geq 1\\
&\leq \ CN^{-2}.
\end{aligned}$$

Thus $|D_x^+W(0,t_j)| \leq CN^{-1}\tau_2^{-1}$. Hence, we deduce that

$$\begin{aligned}
|L^{N,M}(Z-z)(0,t_j)| \quad &\leq \quad C\,(1+\theta)\,\frac{N^{-1}(\tau_1+\tau_2)}{\varepsilon} + C\frac{N^{-1}}{\tau_2}\\
&\leq \quad C\,(1+\theta)\,\frac{N^{-1}\ln N}{\sqrt{\theta\varepsilon}}.
\end{aligned}$$

In the fine mesh region around the points $x=0$ the truncation error will be of the form

$$\varepsilon N^{-2}\tau^2|z_{xxxx}| + \kappa(x_i,t_j)N^{-1}\tau|z_{xx}| + M^{-1}|z_{tt}|, \quad \tau := \max\{\tau_1,\tau_2\}.$$

Hence the truncation error in the fine mesh (for $x_i \neq 0$) is bounded by

$$\begin{aligned}|L^{N,M}(Z-z)| &\leq C\left(1+\theta^2\right)N^{-2}\tau^2\varepsilon^{-1} + C\left(1+\theta\right)N^{-1}\tau^2\varepsilon^{-1} + C\left(1+\theta^2\right)M^{-1} \\ &\leq C\left(1+\theta^2\right)\left(\frac{N^{-1}(\ln N)^2}{\theta} + M^{-1}\right).\end{aligned}$$

Consider the barrier function

$$C(1+\theta^2)\left[\left(\frac{1}{\theta}N^{-1}(\ln N)^2 + M^{-1}\right)\frac{t_j}{c_0}\right] + C(1+\theta)\frac{N^{-1}(\ln N)^2}{\theta}\Phi^*(x_i,t_j),$$

where Φ^* is a piecewise linear function defined by

$$\Phi^*(-\tau_1,t_j) = 0 = \Phi^*(\tau_2,t_j), \qquad \Phi^*(0,t_j) = 1.$$

Note that

$$-(gD_x^+ - gD_x^-)\Phi^*(0,t_j) = \frac{g(0^-,t_j)\tau_2 + g(0^+,t_j)\tau_1}{\tau_2\tau_1} = \frac{C\sqrt{\theta}}{\sqrt{\varepsilon}\ln N}. \tag{27}$$

Hence, recalling that $\theta T/c_0$ is bounded, we have that

$$|Z-z| \leq C(\theta^{-1}+\theta)(N^{-1}(\ln N)^2 + M^{-1}).$$

The desired error bound on the nodal error $|U-u|$ at the nodes is obtained from Theorem 5, estimates (26), (27) and the triangular inequality. To extend this to a global error bound, follow the argument in [2, p. 56]. □

7. Numerical experiments

Consider the following test problem

$$\begin{aligned}&-\varepsilon\hat{u}_{ss} + \hat{u}_s + \hat{u}_t = 0, \quad (s,t) \in Q := (-1,1)\times(0,0.5], \\ &\hat{u}(s,0) = (1-s^2)(1+2s)e^{-\frac{\theta s^2}{\varepsilon}}, \ -1 \leq s \leq 1, \\ &\hat{u}(0,t) = \hat{u}(1,t) = 0, \ 0 < t \leq 0.5,\end{aligned} \tag{28}$$

with different values for the parameter θ.

In Figures 1-3 we display the computed solutions generated by the numerical scheme (19) for the parameter settings $\varepsilon = 2^{-5}, 2^{-10}$ and $N = M = 32$. The values of the parameter θ in the initial condition are $\theta = 0.01, 1, 100$ and we observe the influence of this parameter in the profile of the solution. For fixed ε, as the parameter θ increases in value, the width of the pulse narrows and the initial rate of decrease in time of the amplitude of the pulse increases. This effect of θ on the behaviour of the solution is reflected in the bounds, established in (10a), on the interior layer component.

We use the double mesh principle [2] to estimate the orders of convergence by first computing the two–mesh differences

$$F_\varepsilon^{N,M} := \max\left\{\|U^{N,M} - \bar{U}^{2N,2M}\|_{\bar{\Omega}^{N,M}}, \|\bar{U}^{N,M} - U^{2N,2M}\|_{\bar{\Omega}^{2N,2M}}\right\},$$

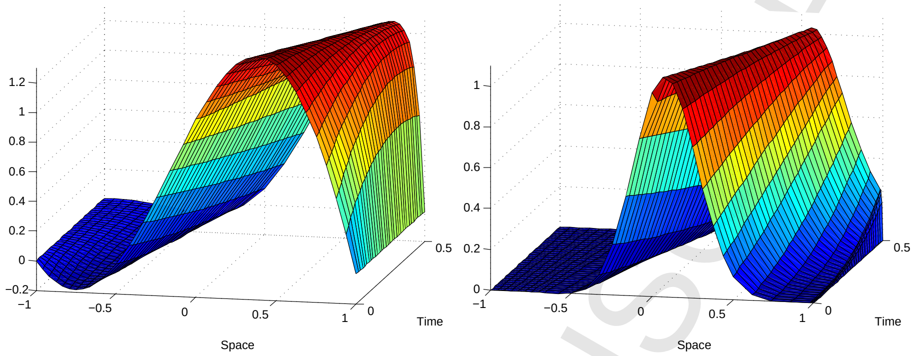

Figure 1: Test problem (28): Computed solution U^N generated by the numerical scheme (19) for $N = M = 32$, $\theta = 0.01$ and $\varepsilon = 2^{-5}$ (left figure) and $\varepsilon = 2^{-10}$ (right figure)

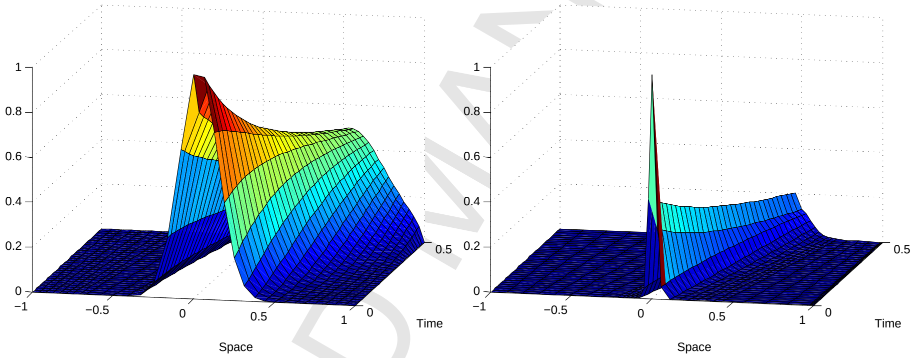

Figure 2: Test problem (28): Computed solution U^N generated by the numerical scheme (19) for $N = M = 32$, $\theta = 1$ and $\varepsilon = 2^{-5}$ (left figure) and $\varepsilon = 2^{-10}$ (right figure)

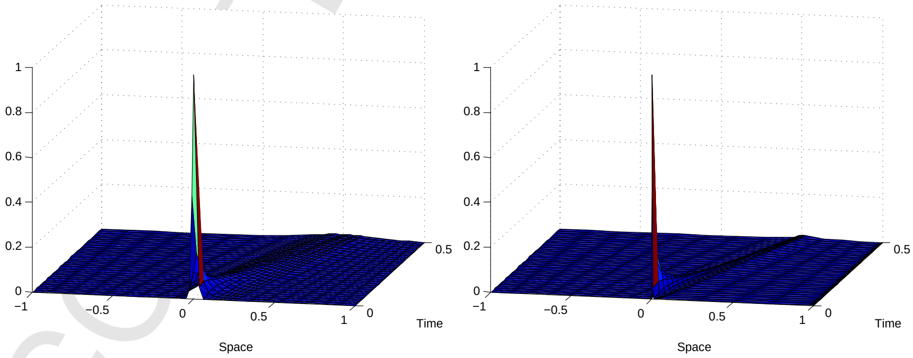

Figure 3: Test problem (28): Computed solution U^N generated by the numerical scheme (19) for $N = M = 32$, $\theta = 100$ and $\varepsilon = 2^{-5}$ (left figure) and $\varepsilon = 2^{-10}$ (right figure)

where $\bar{U}^{N,M}$ denotes the bilinear interpolant of the solution. These values are used to compute the approximate orders of global convergence using

$$Q_\varepsilon^{N,M} := \log_2(F_\varepsilon^{N,M}/F_\varepsilon^{2N,2M}).$$

The uniform global orders of convergence are estimated by computing

$$F^{N,M} := \max_{\varepsilon\in S} F_\varepsilon^{N,M}, \quad Q^{N,M} := \log_2(F^{N,M}/F^{2N,2M}),$$

with $S = \{2^0, 2^{-1}, 2^{-2}, \ldots, 2^{-30}\}$.

In Tables 1-3 we give some numerical results where we have taken $N = M = 2^j, j = 6, 7, \ldots 12$, $\varepsilon \in S$ and $\theta = 0.01, 1, 100$, respectively. We give for each value of ε the maximum two mesh-differences in the first row and the orders of convergence in the second row. In Table 1, where $\theta = 0.01$, the two mesh-differences stop changing only for ε smaller than 2^{-20}; while in Table 3, where $\theta = 100$, the two-mesh differences and the associated rates settle once $\varepsilon \leq 2^{-5}$. The maximum two-mesh differences and their corresponding orders of convergence are in the last row of the table. These numerical results are in agreement with the error bounds given in Theorem 7.

Recall that there are three potential layers appearing in the solution of problem (28): an interior layer near the line $t = s$, a boundary layer near $s = 1$ and an initial layer near $t = 0$. The location (in space and time) of the point where $F_\varepsilon^{N,M}$ is evaluated, can vary depending on the parameters N, M, θ and ε. If this location moves from one layer to another layer, then this location shift can result in an abrupt pattern change in the computed rates of convergence. This effect is most noticeable in Table 1 (see the entries highlighted), where $F_\varepsilon^{N,M}$ moves from the interior layer (for $N \leq 128$) to the boundary layer (for $N \geq 512$).

We consider the values of $\varepsilon = 2^{-16}$, $\theta = 2^{2j}, j = -10, -9, \ldots, 4, 5$ in problem (28) and the values of the discretisation parameter are $N = 2^j, j = 6, 7, \ldots, 12$ in the numerical scheme (19). Table 4 shows the maximum two-mesh differences and the orders of convergence for these particular values of the parameters and we observe that the orders of convergence degenerate as θ increases. These observations are in line with the error bound established in Theorem 7. We also observe that the two-mesh differences increase as θ decreases for the values of $\theta = 2^{-12}, 2^{-14}$ and 2^{-16}; in fact the maximum two-mesh differences are approximately doubled as θ is divided by 2. The special case of $C\varepsilon \leq \theta < 1$ is examined in further detail in [7] in the case of the corresponding reaction-diffusion problem.

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Table 1: Finite difference scheme (19) on the Shishkin mesh: Computed two-mesh differences $F_\varepsilon^{N,M}$ and uniform differences $F^{N,M}$ with their corresponding computed orders of convergence $Q_\varepsilon^{N,M}$, $Q^{N,M}$ for the test problem (28) with $\theta = 0.01$.

	N=M=64	N=M=128	N=M=256	N=M=512	N=M=1024	N=M=2048	N=M=4096
$\varepsilon = 2^{0}$	0.321E-01 0.943	0.167E-01 0.975	0.850E-02 0.993	0.427E-02 0.943	0.222E-02 0.904	0.119E-02 0.928	0.624E-03
$\varepsilon = 2^{-1}$	0.180E-01 1.013	0.891E-02 1.007	0.443E-02 0.875	0.242E-02 0.886	0.131E-02 0.914	0.694E-03 0.935	0.363E-03
$\varepsilon = 2^{-2}$	0.151E-01 1.013	0.748E-02 1.007	0.372E-02 1.004	0.186E-02 1.002	0.927E-03 1.001	0.463E-03 1.000	0.232E-03
$\varepsilon = 2^{-3}$	0.224E-01 0.997	0.112E-01 0.998	0.563E-02 0.999	0.282E-02 0.999	0.141E-02 1.000	0.705E-03 1.000	0.352E-03
$\varepsilon = 2^{-4}$	0.384E-01 0.981	0.195E-01 0.986	0.983E-02 0.993	0.494E-02 0.996	0.248E-02 0.998	0.124E-02 0.999	0.620E-03
$\varepsilon = 2^{-5}$	0.677E-01 0.969	0.346E-01 0.980	0.175E-01 0.985	0.886E-02 0.993	0.445E-02 0.996	0.223E-02 0.998	0.112E-02
$\varepsilon = 2^{-6}$	0.116E+00 0.963	0.596E-01 0.973	0.303E-01 0.982	0.154E-01 0.988	0.775E-02 0.993	0.389E-02 0.997	0.195E-02
$\varepsilon = 2^{-7}$	0.183E+00 0.967	0.934E-01 0.951	0.483E-01 0.968	**0.247E-01** 0.981	0.125E-01 0.990	0.630E-02 0.995	0.316E-02
$\varepsilon = 2^{-8}$	0.125E+00 0.689	0.773E-01 0.897	0.415E-01 0.816	0.236E-01 0.761	**0.139E-01** 0.780	**0.810E-02** 0.850	**0.449E-02**
$\varepsilon = 2^{-9}$	0.630E-01 0.676	0.394E-01 0.929	0.207E-01 0.827	0.117E-01 0.787	0.676E-02 0.829	0.381E-02 0.844	0.212E-02
$\varepsilon = 2^{-10}$	0.358E-01 0.856	0.198E-01 0.921	0.105E-01 0.959	0.537E-02 0.984	0.272E-02 0.998	0.136E-02 1.006	0.678E-03
$\varepsilon = 2^{-11}$	0.357E-01 0.876	0.195E-01 0.939	0.102E-01 0.970	0.518E-02 0.987	0.261E-02 0.995	0.131E-02 0.999	0.656E-03
$\varepsilon = 2^{-12}$	0.420E-01 1.153	0.189E-01 0.930	0.990E-02 0.960	0.509E-02 0.980	0.258E-02 0.990	0.130E-02 0.995	0.652E-03
$\varepsilon = 2^{-13}$	0.796E-01 1.603	0.262E-01 1.007	0.130E-01 1.010	0.648E-02 1.001	0.324E-02 0.987	0.163E-02 0.993	0.821E-03
$\varepsilon = 2^{-14}$	0.137E+00 1.481	0.491E-01 1.416	0.184E-01 1.004	0.918E-02 1.004	0.458E-02 1.001	0.229E-02 1.000	0.114E-02
$\varepsilon = 2^{-15}$	0.199E+00 1.601	0.656E-01 1.297	0.267E-01 1.044	0.129E-01 0.994	0.650E-02 1.002	0.325E-02 0.999	0.162E-02
$\varepsilon = 2^{-16}$	0.289E+00 1.508	0.102E+00 1.600	0.335E-01 1.235	0.142E-01 0.937	0.744E-02 0.968	0.380E-02 0.797	0.219E-02
$\varepsilon = 2^{-17}$	**0.325E+00** 1.008	**0.162E+00** 1.721	0.490E-01 1.370	0.190E-01 0.914	0.101E-01 0.957	0.518E-02 0.978	0.263E-02
$\varepsilon = 2^{-18}$	0.300E+00 0.912	0.159E+00 1.378	0.613E-01 1.357	0.239E-01 0.815	0.136E-01 0.939	0.710E-02 0.970	0.363E-02
$\varepsilon = 2^{-19}$	0.299E+00 0.905	0.160E+00 1.371	0.617E-01 1.501	0.218E-01 0.733	0.131E-01 0.793	0.757E-02 0.834	0.425E-02
$\varepsilon = 2^{-20}$	0.298E+00 0.899	0.160E+00 1.365	0.620E-01 1.512	0.217E-01 0.733	0.131E-01 0.793	0.755E-02 0.834	0.424E-02
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\varepsilon = 2^{-29}$	0.296E+00 0.887	0.160E+00 1.353	**0.627E-01** 1.529	0.217E-01 0.739	0.130E-01 0.794	0.750E-02 0.834	0.421E-02
$\varepsilon = 2^{-30}$	0.296E+00 0.887	0.160E+00 1.352	0.627E-01 1.529	0.217E-01 0.739	0.130E-01 0.794	0.750E-02 0.834	0.421E-02
$F^{N,M}$ $Q^{N,M}$	0.325E+00 1.008	0.162E+00 1.367	0.627E-01 1.343	0.247E-01 0.829	0.139E-01 0.780	0.810E-02 0.850	0.449E-02

Table 2: Finite difference scheme (19) on the Shishkin mesh: Computed two-mesh differences $F_\varepsilon^{N,M}$ and uniform differences $F^{N,M}$ with their corresponding computed orders of convergence $Q_\varepsilon^{N,M}$, $Q^{N,M}$ for the test problem (28) with $\theta = 1$.

	N=M=64	N=M=128	N=M=256	N=M=512	N=M=1024	N=M=2048	N=M=4096
$\varepsilon = 2^{0}$	0.275E-01 0.898	0.147E-01 0.940	0.769E-02 0.967	0.393E-02 0.983	0.199E-02 0.991	0.100E-02 0.996	0.502E-03
$\varepsilon = 2^{-1}$	0.187E-01 0.959	0.961E-02 0.969	0.491E-02 0.981	0.249E-02 0.990	0.125E-02 0.995	0.629E-03 0.997	0.315E-03
$\varepsilon = 2^{-2}$	0.162E-01 1.123	0.746E-02 1.031	0.365E-02 1.005	0.182E-02 1.002	0.908E-03 1.001	0.453E-03 1.001	0.227E-03
$\varepsilon = 2^{-3}$	0.195E-01 1.117	0.898E-02 1.047	0.435E-02 1.021	0.214E-02 1.010	0.106E-02 1.005	0.530E-03 1.003	0.265E-03
$\varepsilon = 2^{-4}$	0.264E-01 1.121	0.122E-01 1.057	0.584E-02 1.028	0.286E-02 1.014	0.142E-02 1.007	0.706E-03 1.004	0.352E-03
$\varepsilon = 2^{-5}$	0.403E-01 1.221	0.173E-01 1.081	0.818E-02 1.040	0.398E-02 1.020	0.196E-02 1.010	0.973E-03 1.005	0.485E-03
$\varepsilon = 2^{-6}$	0.681E-01 1.423	0.254E-01 1.118	0.117E-01 1.058	0.562E-02 1.029	0.275E-02 1.015	0.136E-02 1.007	0.678E-03
$\varepsilon = 2^{-7}$	0.118E+00 1.620	0.385E-01 1.177	0.170E-01 1.083	0.803E-02 1.042	0.390E-02 1.021	0.192E-02 1.011	0.954E-03
$\varepsilon = 2^{-8}$	0.168E+00 1.579	0.564E-01 1.161	0.252E-01 1.119	0.116E-01 1.060	0.556E-02 1.030	0.272E-02 1.015	0.135E-02
$\varepsilon = 2^{-9}$	0.239E+00 1.527	0.829E-01 1.529	0.287E-01 1.022	0.141E-01 0.968	0.723E-02 0.900	0.388E-02 1.022	0.191E-02
$\varepsilon = 2^{-10}$	**0.297E+00** 1.257	0.124E+00 1.686	0.387E-01 1.153	0.174E-01 1.011	0.863E-02 0.970	0.441E-02 0.957	0.227E-02
$\varepsilon = 2^{-11}$	0.249E+00 0.853	**0.138E+00** 1.239	**0.584E-01** 1.361	0.227E-01 1.067	0.109E-01 1.001	0.542E-02 0.974	0.276E-02
$\varepsilon = 2^{-12}$	0.245E+00 0.845	0.137E+00 1.326	0.545E-01 1.251	**0.229E-01** 0.959	**0.118E-01** 0.869	**0.645E-02** 0.895	**0.347E-02**
$\varepsilon = 2^{-13}$	0.243E+00 0.839	0.136E+00 1.323	0.543E-01 1.246	0.229E-01 0.958	0.118E-01 0.912	0.626E-02 0.897	0.337E-02
$\varepsilon = 2^{-14}$	0.241E+00 0.834	0.135E+00 1.320	0.542E-01 1.242	0.229E-01 0.957	0.118E-01 0.912	0.627E-02 0.896	0.337E-02
$\varepsilon = 2^{-15}$	0.240E+00 0.831	0.135E+00 1.318	0.541E-01 1.240	0.229E-01 0.957	0.118E-01 0.911	0.627E-02 0.896	0.337E-02
$\varepsilon = 2^{-16}$	0.239E+00 0.829	0.135E+00 1.317	0.540E-01 1.238	0.229E-01 0.957	0.118E-01 0.911	0.627E-02 0.896	0.337E-02
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$\varepsilon = 2^{-29}$	0.237E+00 0.824	0.134E+00 1.314	0.539E-01 1.235	0.229E-01 0.956	0.118E-01 0.911	0.628E-02 0.896	0.337E-02
$\varepsilon = 2^{-30}$	0.237E+00 0.824	0.134E+00 1.314	0.539E-01 1.235	0.229E-01 0.956	0.118E-01 0.911	0.628E-02 0.896	0.337E-02
$F^{N,M}$ $Q^{N,M}$	0.297E+00 1.111	0.138E+00 1.239	0.584E-01 1.350	0.229E-01 0.956	0.118E-01 0.871	0.645E-02 0.895	0.347E-02

Table 3: Finite difference scheme (19) on the Shishkin mesh: Computed two-mesh differences $F_\varepsilon^{N,M}$ and uniform differences $F^{N,M}$ with their corresponding computed orders of convergence $Q_\varepsilon^{N,M}$, $Q^{N,M}$ for the test problem (28) with $\theta = 100$.

	N=M=32	N=M=64	N=M=128	N=M=256	N=M=512	N=M=1024	N=M=4096
$\varepsilon = 2^{0}$	0.812E-01 0.238	0.688E-01 0.243	0.581E-01 0.369	0.450E-01 0.535	0.311E-01 0.691	0.193E-01 0.815	0.109E-01
$\varepsilon = 2^{-1}$	0.938E-01 0.229	0.800E-01 0.330	0.637E-01 0.421	0.476E-01 0.568	0.321E-01 0.716	0.195E-01 0.832	0.110E-01
$\varepsilon = 2^{-2}$	0.111E+00 0.356	0.866E-01 0.345	0.682E-01 0.432	0.506E-01 0.585	0.337E-01 0.741	0.202E-01 0.848	0.112E-01
$\varepsilon = 2^{-3}$	0.139E+00 0.465	0.101E+00 0.430	0.746E-01 0.490	0.531E-01 0.614	0.347E-01 0.747	0.207E-01 0.854	0.114E-01
$\varepsilon = 2^{-4}$	**0.140E+00** 0.235	**0.119E+00** 0.484	0.848E-01 0.561	0.575E-01 0.657	0.365E-01 0.772	0.214E-01 0.870	0.117E-01
$\varepsilon = 2^{-5}$	0.139E+00 0.313	0.112E+00 0.397	**0.849E-01** 0.503	**0.599E-01** 0.621	**0.389E-01** 0.798	**0.224E-01** 0.893	0.121E-01
$\varepsilon = 2^{-6}$	0.138E+00 0.311	0.111E+00 0.396	0.847E-01 0.503	0.598E-01 0.640	0.384E-01 0.774	0.224E-01 0.878	**0.122E-01**
$\varepsilon = 2^{-7}$	0.138E+00 0.309	0.111E+00 0.395	0.846E-01 0.502	0.597E-01 0.640	0.383E-01 0.774	0.224E-01 0.878	0.122E-01
$\varepsilon = 2^{-8}$	0.138E+00 0.308	0.111E+00 0.395	0.845E-01 0.502	0.597E-01 0.639	0.383E-01 0.774	0.224E-01 0.878	0.122E-01
$\varepsilon = 2^{-9}$	0.137E+00 0.308	0.111E+00 0.394	0.844E-01 0.501	0.596E-01 0.639	0.383E-01 0.773	0.224E-01 0.878	0.122E-01
$\varepsilon = 2^{-10}$	0.137E+00 0.307	0.111E+00 0.394	0.844E-01 0.501	0.596E-01 0.639	0.383E-01 0.773	0.224E-01 0.878	0.122E-01
$\varepsilon = 2^{-11}$	0.137E+00 0.307	0.111E+00 0.394	0.844E-01 0.501	0.596E-01 0.639	0.383E-01 0.773	0.224E-01 0.878	0.122E-01
$\varepsilon = 2^{-12}$	0.137E+00 0.306	0.111E+00 0.394	0.843E-01 0.501	0.596E-01 0.639	0.383E-01 0.773	0.224E-01 0.878	0.122E-01
$\varepsilon = 2^{-13}$	0.137E+00 0.306	0.111E+00 0.394	0.843E-01 0.501	0.596E-01 0.639	0.383E-01 0.773	0.224E-01 0.878	0.122E-01
$\varepsilon = 2^{-14}$	0.137E+00 0.306	0.111E+00 0.394	0.843E-01 0.501	0.596E-01 0.639	0.383E-01 0.773	0.224E-01 0.878	0.122E-01
$\varepsilon = 2^{-15}$	0.137E+00 0.306	0.111E+00 0.394	0.843E-01 0.501	0.596E-01 0.639	0.383E-01 0.773	0.224E-01 0.877	0.122E-01
$\varepsilon = 2^{-16}$	0.137E+00 0.306	0.111E+00 0.394	0.843E-01 0.501	0.596E-01 0.639	0.383E-01 0.773	0.224E-01 0.877	0.122E-01
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$\varepsilon = 2^{-29}$	0.137E+00 0.306	0.111E+00 0.394	0.843E-01 0.501	0.596E-01 0.639	0.383E-01 0.773	0.224E-01 0.877	0.122E-01
$\varepsilon = 2^{-30}$	0.137E+00 0.306	0.111E+00 0.394	0.843E-01 0.501	0.596E-01 0.639	0.383E-01 0.773	0.224E-01 0.877	0.122E-01
$F^{N,M}$ $Q^{N,M}$	0.140E+00 0.235	0.119E+00 0.483	0.849E-01 0.503	0.599E-01 0.621	0.389E-01 0.796	0.224E-01 0.878	0.122E-01

Table 4: Finite difference scheme (19) on the Shishkin mesh: Computed two-mesh differences $F_\varepsilon^{N,M}$ and their corresponding computed orders of convergence $Q_\varepsilon^{N,M}$ for the test problem (28) with $\varepsilon = 2^{-16}$.

	N=M=64	N=M=128	N=M=256	N=M=512	N=M=1024	N=M=2048	N=M=4096
$\theta = 2^{-20}$	0.251E+00 0.607	0.165E+00 0.834	0.925E-01 0.789	0.535E-01 0.829	0.301E-01 0.807	0.172E-01 0.860	0.949E-02
$\theta = 2^{-18}$	0.234E+00 0.618	0.153E+00 0.846	0.850E-01 0.784	0.493E-01 0.817	0.280E-01 0.815	0.159E-01 0.859	0.877E-02
$\theta = 2^{-16}$	0.186E+00 0.665	0.117E+00 0.866	0.643E-01 0.790	0.372E-01 0.774	0.217E-01 0.840	0.121E-01 0.856	0.671E-02
$\theta = 2^{-14}$	0.833E-01 0.680	0.520E-01 0.934	0.272E-01 0.789	0.158E-01 0.767	0.926E-02 0.836	0.519E-02 0.849	0.288E-02
$\theta = 2^{-12}$	0.370E-01 0.859	0.204E-01 0.922	0.108E-01 0.962	0.552E-02 0.981	0.280E-02 0.991	0.141E-02 0.995	0.706E-03
$\theta = 2^{-10}$	0.639E-01 1.420	0.239E-01 1.019	0.118E-01 0.985	0.596E-02 0.976	0.303E-02 0.988	0.153E-02 0.994	0.767E-03
$\theta = 2^{-8}$	0.172E+00 1.493	0.612E-01 1.344	0.241E-01 1.054	0.116E-01 0.999	0.581E-02 1.001	0.291E-02 1.000	0.145E-02
$\theta = 2^{-6}$	0.341E+00 1.325	0.136E+00 1.748	0.405E-01 1.255	0.170E-01 0.926	0.893E-02 0.961	0.459E-02 0.981	0.232E-02
$\theta = 2^{-4}$	0.279E+00 0.948	0.145E+00 1.354	0.566E-01 1.313	0.228E-01 0.931	0.120E-01 0.741	0.715E-02 0.830	0.402E-02
$\theta = 2^{-2}$	0.249E+00 0.889	0.134E+00 1.334	0.532E-01 1.170	0.237E-01 0.976	0.120E-01 0.930	0.631E-02 0.911	0.336E-02
$\theta = 2^{0}$	0.239E+00 0.829	0.135E+00 1.317	0.540E-01 1.238	0.229E-01 0.957	0.118E-01 0.911	0.627E-02 0.896	0.337E-02
$\theta = 2^{2}$	0.235E+00 0.779	0.137E+00 1.214	0.592E-01 1.312	0.238E-01 0.987	0.120E-01 0.922	0.634E-02 0.902	0.339E-02
$\theta = 2^{4}$	0.216E+00 0.643	0.138E+00 0.955	0.713E-01 1.103	0.332E-01 1.140	0.151E-01 0.995	0.756E-02 0.947	0.392E-02
$\theta = 2^{6}$	0.160E+00 0.391	0.122E+00 0.534	0.842E-01 0.659	0.533E-01 0.785	0.310E-01 0.889	0.167E-01 0.959	0.861E-02
$\theta = 2^{8}$	0.914E-01 0.168	0.813E-01 0.144	0.736E-01 0.185	0.647E-01 0.312	0.522E-01 0.483	0.373E-01 0.653	0.237E-01
$\theta = 2^{10}$	0.453E-01 0.139	0.411E-01 -.043	0.424E-01 -.153	0.471E-01 -.132	0.516E-01 0.004	0.515E-01 0.202	0.448E-01