YANO'S CONJECTURE FOR 2-PUISEUX PAIRS IRREDUCIBLE PLANE CURVE SINGULARITIES

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Abstract. In 1982, Tamaki Yano proposed a conjecture predicting the b-exponents of an irreducible plane curve singularity germ which is generic in its equisingularity class. In this article we prove the conjecture for the case in which the irreducible germ has two Puiseux pairs and its algebraic monodromy has distinct eigenvalues. This hypothesis on the monodromy implies that the b-exponents coincide with the opposite of the roots of the Bernstein polynomial, and we compute the roots of the Bernstein polynomial.

Introduction

The Bernstein polynomial of a singularity germ is a powerful analytic invariant, but it is, in general, extremely hard to compute, even in the case of irreducible plane curve singularities. It is well-known that the Bernstein polynomial vary in the μ-constant stratum of such germs. Since this stratum is irreducible, it is conceivable that a *generic* Bernstein polynomial exists, i.e., there exists a dense Zariski-open set in the stratum where the Bernstein polynomial remains constant. From the computational point of view it is even harder to effectively compute this generic polynomial. In 1982, Tamaki Yano conjectured a closed formula for the Bernstein polynomial of an irreducible plane curve which is generic in its equisingularity class, [22, Conjecture 2.6]. This conjecture is still open. The aim of this paper is to provide a significant progress by proving it for a *big* family of 2-Puiseux-pairs singularities.

Let $\mathcal{O}$ be the ring of germs of holomorphic functions on $(\mathbb{C}^n, 0)$, $\mathcal{D}$ the ring of germs of holomorphic differential operators of finite order with coefficients in $\mathcal{O}$. Let s be an indeterminate commuting with the elements of $\mathcal{D}$ and set $\mathcal{D}[s] = \mathcal{D} \otimes_{\mathbb{C}} \mathbb{C}[s]$.

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Given a holomorphic germ $f \in \mathcal{O}$, one considers $\mathcal{O}\left[\frac{1}{f}, s\right] f^s$ as a free $\mathcal{O}\left[\frac{1}{f}, s\right]$-module of rank 1 with the natural $\mathcal{D}[s]$-module structure. Then, there exits a non-zero polynomial $B(s) \in \mathbb{C}[s]$ and some differential operator $P = P(x, \frac{\partial}{\partial x}, s) \in \mathcal{D}[s]$, holomorphic in $x_1, \ldots, x_n$ and polynomial in $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$, which satisfy in $\mathcal{O}\left[\frac{1}{f}, s\right] f^s$ the following functional equation

$$P(s, x, D) \cdot f(x)^{s+1} = B(s) \cdot f(x)^s. \tag{1}$$

The monic generator $b_{f,0}(s)$ of the ideal of such polynomials $B(s)$ is called the Bernstein polynomial (or b-function or Bernstein-Sato polynomial) of f at 0. The same result holds if we replace $\mathcal{O}$ by the ring of polynomials in a field $\mathbb{K}$ of zero characteristic with the obvious corrections, see e.g. [9, Section 10, Theorem 3.3].

This result was first obtained for f polynomial by Bernstein in [3] and in general by Björk [4]. One can prove that $b_{f,0}(s)$ is divisible by $s+1$, and we also consider the reduced Bernstein polynomial $\tilde{b}_{f,0}(s) := \frac{b_{f,0}(s)}{s+1}$.

In the case where f defines an isolated singularity, one can consider the Brieskorn lattice $H_0'' := \Omega^n / df \wedge d\Omega^{n-2}$ and its saturated $\tilde{H}_0'' = \sum_{k\geq 0} (\partial_t t)^k H_0''$. Malgrange [15] showed that the reduced Bernstein polynomial $\tilde{b}_{f,0}(s)$ is the minimal polynomial of the endomorphism $-\partial_t t$ on the vector space $F := \tilde{H}_0'' / \partial_t^{-1} \tilde{H}_0''$, whose dimension equals the Minor number $\mu(f, 0)$ of f at 0. Following Malgrange [15], the set of *b-exponents* are the μ roots $\{\alpha_1, \ldots, \alpha_\mu\}$ of the characteristic polynomial of the endomorphism $-\partial_t t$. Recall also that $\exp(-2i\pi\partial_t t)$ can be identified with the (complex) *algebraic monodromy* of the corresponding Milnor fibre F_f of the singularity at the origin.

Kashiwara [12] expressed these ideas using differential operators and considered $\mathcal{M} := \mathcal{D}[s]f^s / \mathcal{D}[s]f^{s+1}$, where s defines an endomorphism of $P(s)f^s$ by multiplication. This morphism keeps invariant $\tilde{\mathcal{M}} := (s+1)\mathcal{M}$ and defines a linear endomorphism of $(\Omega^n \otimes_{\mathcal{D}} \tilde{\mathcal{M}})_0$ which is naturally identified with F and under this identification $-\partial_t t$ becomes the endomorphism defined by the multiplication by s.

In [15], Malgrange proved that the set $R_{f,0}$ of roots of the Bernstein polynomial is contained in $\mathbb{Q}_{<0}$, see also Kashiwara [12], who also restricts the set of candidate roots. The number $-\alpha_{f,0} := \max R_{f,0}$ is the opposite of the log canonical threshold of the singularity and Saito [18, Theorem 0.4] proved that

$$R_{f,0} \subset [\alpha_{f,0} - n, -\alpha_{f,0}]. \tag{2}$$

Now let f be an irreducible germ of plane curve. In 1982, Tamaki Yano [22] made a conjecture concerning the b-exponents of such germs. Let $(n, \beta_1, \beta_2, \ldots, \beta_g)$ be the characteristic sequence of f, see e.g. [21, Section 3.1]. Recall that this means that $f(x,y)=0$ has as root (say over x) a Puiseux expansion

$$x = \dots + a_1 y^{\frac{\beta_1}{n}} + \dots + a_g y^{\frac{\beta_g}{n}} + \dots$$

with exactly g characteristic monomials. Denote $\beta_0 := n$ and define recursively

$$e^{(k)} := \begin{cases} n & \text{if } k = 0, \\ \gcd(e^{(k-1)}, \beta_k) & \text{if } 1 \leq k \leq g. \end{cases}$$

We define the following numbers for $1 \leq k \leq g$:

$$R_k := \frac{1}{e^{(k)}} \left(\beta_k e^{(k-1)} + \sum_{j=0}^{k-2} \beta_{j+1} \left(e^{(j)} - e^{(j+1)} \right) \right), \qquad r_k := \frac{\beta_k + n}{e^{(k)}}.$$

Note that R_k admits the following recursive formula:

$$R_k := \begin{cases} n & \text{if } k = 0, \\ \frac{e^{(k-1)}}{e^{(k)}} \left(R_{k-1} + \beta_k - \beta_{k-1} \right) & \text{if } 1 \leq k \leq g. \end{cases}$$

We end with the following definitions $R'_0 := n$, $r'_0 := 2$ and for $1 \leq k \leq g$:

$$R'_k := \frac{R_k e^{(k)}}{e^{(k-1)}}, \quad r'_k := \left\lfloor r_k e^{(k)} / e^{(k-1)} \right\rfloor + 1.$$

Yano defined the following polynomial with fractional powers in t

$$R(n, \beta_1, \dots, \beta_g; t) := t + \sum_{k=1}^{g} t^{\frac{r_k}{R_k}} \frac{1-t}{1-t^{\frac{1}{R_k}}} - \sum_{k=0}^{g} t^{\frac{r'_k}{R'_k}} \frac{1-t}{1-t^{\frac{1}{R'_k}}}, \tag{3}$$

and he proved that $R(n, \beta_1, \dots, \beta_g; t)$ has non-negative coefficients.

The number of monomials in $R(n, \beta_1, \dots, \beta_g; t)$ is equal to $1 + \sum_{k=1}^{g} R_k - \sum_{k=0}^{g} R'_k$ and one can prove that this number is the Milnor number μ. The numbers R_k (resp. R'_k) are the multiplicities of the irreducible exceptional divisors of the minimal embedded resolution of the singularity whose smooth part has Euler characteristic -1 (resp. 1), see e.g. Lemma 3.6.1, Fig 3.5 and Theorem 8.5.2 in [21]. Using A'Campo formula [1] for the Euler characteristic of the Milnor fibre F_f of f at 0, that is $1 - \mu = \chi(F_f)$, one gets $\chi(F_f) = -\sum_{k=1}^{g} R_k + \sum_{k=0}^{g} R'_k$, that is that number equals to μ.

Yano's Conjecture ([22]). *For almost all irreducible plane curve singularity germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ with characteristic sequence $(n, \beta_1, \beta_2, \dots, \beta_g)$, the b-exponents $\{\alpha_1, \dots, \alpha_\mu\}$ are given by the generating series*

$$\sum_{i=1}^{\mu} t^{\alpha_i} = R(n, \beta_1, \dots, \beta_g; t).$$

For almost all means for an open dense subset in the μ-constant strata in a deformation space.

In 1989, B. Lichtin [13] proved that for $i = 1, \cdots, g$, the number $-\frac{r_i}{R_i}$ is a root of the Bernstein polynomial of f with characteristic sequence $(n, \beta_1, \beta_2, \ldots, \beta_g)$. These result has been extended to the general curve case (not necessarily irreducible) by F. Loeser in [14].

Yano's conjecture holds for $g = 1$ as it was proved by the second named author in [8].

In [16, Section 4.2] M. Saito described how can vary the Bernstein polynomial in μ-constant deformations. Let $\{f_t\}_{t\in T}$ be a μ-constant analytic deformation of an irreducible germ of an isolated curve singularity f_0. Then there exists an analytic stratification of T (by restricting T if necessary) such that the Bernstein polynomial is constant on each strata. Since the μ-constant strata is irreducible and smooth, the Bernstein polynomial of its open stratum, denoted by $b_{\mu,\mathrm{gen}}(s)$, is called the Bernstein polynomial of the generic μ-constant deformation of $f_0(x)$.

In this article we are interested in the case $g = 2$. Yano [22] claimed the case $(4, 6, 2n-3)$, with $n \geq 5$, but referred to a non published article. For $g = 2$, the characteristic sequence (n, β_1, β_2) can be written as $(n_1 n_2, m n_2, m n_2 + q)$ where $n_1, m, n_2, q \in \mathbb{Z}_{>0}$ satisfying

$$\gcd(n_1, m) = \gcd(n_2, q) = 1.$$

In this work we solve Yano's conjecture for the case

$$\gcd(q, n_1) = 1 \text{ or } \gcd(q, m) = 1. \tag{4}$$

The above condition is equivalent to ask for the algebraic monodromy to have distinct eigenvalues. In that case, the μ b-exponents are all distinct and they coincide with the opposite of roots of the reduced Bernstein polynomial (which turns out to be of degree μ).

Our goal is to compute the roots of the Bernstein polynomial for a generic function having characteristic sequence $(n_1 n_2, m n_2, m n_2 + q)$. To do this we follow the same method than in [8]. To prove that a rational number is a root of the Bernstein polynomial of some function f, we prove that this number is a pole of some integral with a transcendental residue.

For some exponents of the generating series we prove this property for families of functions which should contain *generic* elements in the μ-constant stratum. For the rest of exponents, the computations are very tricky, and we apply them only to particular functions. In order to ensure that the opposite of these exponents are roots of the Bernstein polynomial for a generic f, we use the following result.

Proposition 1 ([20, Corollary 21]). *Let $f_t(x)$ be a μ-constant analytic deformation of an isolated hypersurface singularity $f_0(x)$. If all eigenvalues of the monodromy are pairwise different, then all roots of the reduced Bernstein-Sato polynomial $\tilde{b}_{f_t}(s)$ depend lower semi-continously upon the parameter t.*

Then if α is root of the local Bernstein-Sato polynomial $b_{f_0}(s)$ for some f_0, and $\alpha+1$ is not root of $b_f(s)$ for any f with the same characteristic sequence, then by Proposition 1, α is root of the local Bernstein-Sato polynomial $b_f(s)$ for f generic with the same characteristic sequence.

In the first section we collect some results on integrals that will be crucial in the following. Some of the proofs are in the appendix of the paper. In the second section we express Yano's conjecture in our setting. In the third and fourth sections we compute poles of integrals that we shall need later, and in the fifth part we show how we can use these integrals to compute roots of the Bernstein polynomial and we prove Yano's conjecture in the sixth section.

We are very grateful to Driss Essouabry for providing us with Proposition 1.4.

1. Meromorphic integrals

1.1. **One-variable integrals.** Let $f \in \mathbb{R}[t]$ be a real polynomial such that $f(t) > 0$ for all $t \in [0,1]$ and let $a, b \in \mathbb{Z},\ a \geq 0,\ b \geq 1$ fixed. Consider the (complex) integral depending on a complex variable $s \in \mathbb{C}$

$$\mathcal{Y}_{f,a,b}(s) := \int_0^1 f(t)^s t^{as+b} \frac{dt}{t}. \tag{1.1}$$

Using classical techniques we can see that this integral defines a holomorphic function on a half-plane in $\mathbb{C}$ admitting a meromorphic continuation to the whole complex line, having only simple poles at some rational numbers (with bounded denominator), where the residues can be controlled.

Proposition 1.1. *The function $s \mapsto \mathcal{Y}_{f,a,b}(s)$ satisfies the following properties:*

(1) *It is absolutely convergent for $\Re(s) > \alpha_0 := -\frac{b}{a}$ (the whole $\mathbb{C}$ if $a = 0$).*

(2) *It has a meromorphic continuation on $\mathbb{C}$ with simple poles, which are contained in $S = \left\{-\frac{b+k}{a} \mid k \in \mathbb{Z}_{\geq 0}\right\}$.*

(3) $\operatorname{Res}_{s=-\frac{b+k}{a}} \mathcal{Y}_{f,a,b}(s)$ *is algebraic over the field of coefficients of f.*

Proof. For the first statement, there exists $M_s > 0$ such that $|f(t)^s| \leq M_s$ for $t \in [0,1]$. Hence,

$$\left|\int_0^1 t^{as+b-1} f^s(t) dt\right| \leq M_s \int_0^1 t^{a\Re(s)+b-1} dt = M_s \left.\frac{t^{a\Re(s)+b}}{a\Re(s)+b}\right|_0^1 = \frac{M_s}{a\Re(s)+b}.$$

For the second statement, we consider the Taylor expansion of $f(t)^s$ at $t=0$ of order k:

$$f^s(t)=\sum_{i=0}^{k}\frac{(f^s)^{(i)}(0)}{i!}t^i+t^{k+1}R_{s,k}(t),\quad R_{s,k}(t)=\frac{1}{k!}\int_0^1(1-u)^k(f^s)^{(k+1)}(ut)du.$$

Hence,

$$\mathcal{Y}_{f,a,b}(s)=\sum_{i=0}^{k}\frac{(f^s)^{(i)}(0)}{(as+b+i)i!}+H(s)$$

where

$$H(s):=\int_0^1 t^{as+b+k}R_{s,k}(t)dt.$$

Note that $H(s)$ is holomorphic for $\Re(s)>-\frac{b+k+1}{a}$, and the first terms are rational functions. Hence, the second statement is true.

For the third one, note that

$$\operatorname*{Res}_{s=-\frac{b+k}{a}}\mathcal{Y}_{f,a,b}(s)=\frac{(f^{-\frac{b+k}{a}})^{(k)}(0)}{ak!}$$

which satisfies the conditions. □

In general, we will deal with more general integrals which a priori, are not so well-defined. For example, let $f(t),g(t)$ be two real analytic functions in $t^{\frac{1}{N}}$ in $[0,T]$, for some $N\in\mathbb{Z}_{>0}$ and $T>0$. Let K be the field of coefficients of the power series of f,g at 0. Let r_f,r_g be the orders of f,g at 0, respectively, and assume that $f(t)>0$ for $t\in(0,T]$. Let $a,b\in\mathbb{Q}$, $a\geq 0$, $b>0$ fixed. Consider the improper integral

$$\mathcal{Y}_{f,g,a,b}(s):=\int_0^T f(t)^s g(t)t^{as+b}\frac{dt}{t}. \tag{1.2}$$

Let us denote $a_1=a+r_f$ and $b_1=b+r_g$. The following result is a direct consequence of the Proposition 1.1, using a simple change of variables.

Corollary 1.2. *The function $s\mapsto\mathcal{Y}_{f,g,a,b}(s)$ satisfies the following properties:*

(1) *It is absolutely convergent for $\Re(s)>\alpha_0:=-\frac{b_1}{a_1}$ (the whole $\mathbb{C}$ if $a_1=0$).*
(2) *It has a meromorphic continuation on $\mathbb{C}$ with simple poles, which are contained in $S=\left\{-\frac{Nb_1+k}{Na_1}\mid k\in\mathbb{Z}_{\geq 0}\right\}$.*
(3) $\operatorname*{Res}_{s=-\frac{Nb_1+k}{Na_1}}\mathcal{Y}_{f,g,a,b}(s)$ *is algebraic over K.*

1.2. **Two-variables integrals.**

Definition 1.3. We say that a real polynomial $f \in \mathbb{R}[x, y]$ is *positive* if $f(x, y) > 0$ for all $(x, y) \in [0, 1]^2$.

Let us state the two-variables counterpart of Proposition 1.1. Let $f \in \mathbb{R}[x, y]$ positive. Let $a_1, a_2, b_1, b_2 \in \mathbb{Z}$ such that $a_1, a_2 \geq 0, b_1, b_2 \geq 1$. We denote

$$\mathcal{Y}(s) = \mathcal{Y}_{f,a_1,b_1,a_2,b_2}(s) := \int_0^1 \int_0^1 f(x, y)^s x^{a_1 s+b_1} y^{a_2 s+b_2} \frac{dx}{x} \frac{dy}{y}. \tag{1.3}$$

Proposition 1.4 (Essouabri). *The function $\mathcal{Y}(s)$ satisfies the following porperties:*

(1) *It is absolutely convergent for $\Re(s) > \alpha_0$, where $\alpha_0 = \sup\left(-\frac{b_1}{a_1}, -\frac{b_2}{a_2}\right)$*
(2) *It has a meromorphic continuation on $\mathbb{C}$ with poles of order at most 2 contained in $S = \left\{-\frac{b_1+\nu_1}{a_1},\ \nu_1 \in \mathbb{Z}_{\geq 0}\right\} \cup \left\{-\frac{b_2+\nu_2}{a_2},\ \nu_2 \in \mathbb{Z}_{\geq 0}\right\}$*

In order to do not break the line of the exposition, the proof of this Proposition is given in the A. Note that no information is given in the above Proposition for the residues. Let us introduce some notation.

Notation 1.5. Let $f : [0, 1] \to \mathbb{R}$ be a continous function. We will denote by $G_f(s)$ the meromorphic continuation of

$$\int_0^1 f(t) t^s \frac{dt}{t}.$$

Proposition 1.6. *With the hypotheses of Proposition 1.4, let $\nu_1 \in \mathbb{Z}_{\geq 0}$ such that $\alpha = -\frac{b_1+\nu_1}{a_1} \neq -\frac{b_2+\nu_2}{a_2}$ for all $\nu_2 \in \mathbb{Z}_{\geq 0}$, then the pole of $\mathcal{Y}(s)$ at α is simple and*

$$\operatorname*{Res}_{s=\alpha} \mathcal{Y}(s) = \frac{1}{\nu_1! a_1} G_{h_{\nu_1,\alpha,x}}(a_2\alpha + b_2), \quad h_{\nu_1,\alpha,x}(y) := \frac{\partial^{\nu_1} f^\alpha}{\partial x^{\nu_1}}(0, y). \tag{1.4}$$

The proof of this Proposition is also given in the A. Note that, under the hypotheses of the Proposition, the function $G_{h_{\nu_1,\alpha,x}}$ admits an integral expression which is absolutely convergent and holomorphic for $\Re(s) > -N_2 - 1$, with N_2 such that $\alpha > -\frac{b_2+N_2+1}{a_2}$, see the proof of Proposition 1.4 in page 24. The following result is also a straightforward consequence of the proof of Proposition 1.4.

Proposition 1.7. *Let $(\nu_1, \nu_2) \in \mathbb{Z}^2_{\geq 0}$ such that $\alpha = -\frac{b_1+\nu_1}{a_1} = -\frac{b_2+\nu_2}{a_2}$, then the pole at α is of order at most 2 and*

$$\lim_{s \to \alpha} \mathcal{Y}(s)(s - \alpha)^2 = \frac{1}{\nu_1! \nu_2! a_1 a_2} \frac{\partial^{\nu_1+\nu_2} f^\alpha}{\partial x^{\nu_1} \partial y^{\nu_2}}(0, 0).$$

We finish this section with a result that relates these integrals with the beta function.

Lemma 1.8. *Let $p \in \mathbb{N}$ and $c \in \mathbb{R}_{>0}$. Given $s_1, s_2 \in \mathbb{C}$ such that $-\alpha = s_1 + s_2 > 0$ then*

$$G_{(y^p+c)^\alpha}(ps_1) + G_{(1+cx^p)^\alpha}(ps_2) = \frac{c^{-s_2}}{p}\boldsymbol{B}\left(s_1, s_2\right) \tag{1.5}$$

where $\boldsymbol{B}$ is the beta function.

The proof appears in the A.

2. Candidate roots

Since we are going to use mostly Bernstein polynomial instead of b-exponents, it will be more convenient to work with the opposite exponents. If we study closely the Yano's set of candidates for the b-exponents given by the exponents of the generating series (3), we can check that for a branch with g characteristic pairs, this set can be decomposed in a union of g subsets, each one associated to a characteristic pair. For example, in the case $g = 1$ and characteristic sequence (n_1, m), with $\gcd(n_1, m) = 1$, the set of opposite b-exponents is decomposed into only one set

$$A := \left\{-\frac{m+n_1+k}{mn_1} : 0 \leq k < mn_1,\ \frac{m+n_1+k}{m}, \frac{m+n_1+k}{n_1} \notin \mathbb{Z}\right\}. \tag{2.1}$$

Note that $\max\ A = -\frac{m+n_1}{mn_1}$, which is the opposite of the *log canonical threshold* of the singularity and we have

$$\max A - 1 < \rho \leq \max A, \qquad \forall \rho \in A$$

agreeing with (2). Recall that the conductor of the semigroup generated by (n_1, m) is $mn_1 - m - n_1$.

Let us consider the case $g = 2$. Let us fix some notations. We work with curve singularities with characteristic sequence $(n_1n_2, mn_2, mn_2 + q)$, where

- $1 < n_1 < m$, $\gcd(m, n_1) = 1$;
- $q > 0$, $n_2 > 1$, $\gcd(q, n_2) = 1$.

In order to use the integrals of §1, we will restrict to *real* singularities with Puiseux expansion

$$x = \cdots + a_1 y^{\frac{m}{n_1}} + \cdots + a_2 y^{\frac{mn_2+q}{n_1n_2}} + \ldots,$$

where $a_1, a_2 \in \mathbb{R}^*$ (only characteristic terms are shown, the other coefficients are also real). The semigroup Γ of these singularities is generated by n_1n_2, mn_2 and $mn_1n_2 + q$. Its conductor equals

$$n_2(mn_1n_2 + q) - (m + n_1)n_2 - q + 1.$$

We are going to deal with *most* local irreducible curve singularities with two Puiseux pairs, where *most* stands for *non-multiple eigenvalues for the algebraic monodromy.* The condition on the eigenvalues is equivalent to (4).

Example 2.1. Let us consider $(a, b) \in \mathbb{Z}^2_{\geq 1}$ such that $mn_1n_2 + q = am + bn_1$. Since the conductor of the semigroup generated by n_1, m equals $(m-1)(n_1-1)$, we deduce that such coefficients exist with the condition $a, b \geq 0$. We can prove that $a, b \geq 1$ using (4). Then the functions

$$F_{\pm}(x, y) = (x^{n_1} \pm y^m)^{n_2} + x^a y^b$$

define singularities of this type.

Let us express Yano's set of opposite candidates as the union of two subsets A_1, A_2. The first one looks like A:

$$A_1 := \left\{ \alpha = -\frac{m + n_1 + k}{mn_1n_2} : 0 \leq k < mn_1n_2, \text{ and } n_2 m\alpha, n_2 n_1 \alpha \notin \mathbb{Z} \right\}; \tag{2.2}$$

the last condition is equivalent to neither m nor n_1 are divisors of $m+n_1+k$. The second one corresponds to the second Puiseux pair:

$$A_2 := \left\{ \alpha = -\frac{\overbrace{(m + n_1)n_2 + q + k}^{N_k}}{n_2 \underbrace{(mn_1n_2 + q)}_{D}} \middle| 0 \leq k < n_2 D \text{ and } n_2\alpha, D\alpha \notin \mathbb{Z} \right\}; \tag{2.3}$$

the last condition is equivalent to neither n_2 nor D are divisors of N_k. They satisfy the following conditions:

(A1) These two subsets are disjoint under the condition (4).
(A2) $\max A_i - \min A_i < 1$ for $i = 1, 2$
(A3) $-\max A_1$ is the *log canonical threshold* of those singularities.
(A4) $0 < \max A_1 - \max A_2 < 1$.

These subsets are decomposed as disjoint unions $A_1 = A_{11} \sqcup A_{12}$ and $A_2 = A_{21} \sqcup A_{22}$ using the semigroups associated to the singularity. The set A_{11} is formed by the elements of A_1 whose numerator is in the semigroup generated by (m, n_1), i.e.,

$$A_{11} := \left\{ -\frac{m\beta_1 + n_1\beta_2}{mn_1n_2} \in A_1 \middle| \beta_1, \beta_2 \in \mathbb{Z}_{\geq 1} \right\}. \tag{2.4}$$

The set A_{21} is formed by the elements of A_2 whose numerator (minus q) is in Γ, i.e.,

$$A_{21} := \left\{ -\frac{N_k}{n_2 D} \middle| N_k - q \in \Gamma \right\}. \tag{2.5}$$

The following lemma means that A_{12} and A_{22} are somewhat *small.*

Lemma 2.2. *If* $\alpha \in A_{i2}$, $i = 1, 2$, *then* $\max A_1 - \alpha < 1$. *In an equivalent way*

(1) *if* $-\frac{m+n_1+k}{mn_1n_2} \in A_{11}$, *then* $k \leq mn_1 - m - n_1$;
(2) *if* $-\frac{N_k}{n_2D} \in A_{21}$, *then* $\frac{N_k}{n_2D} < \frac{m+n_1}{mn_1n_2} + 1$.

Proof. The first statement follows from the fact that $(m-1)(n_1-1)$ is the conductor of the semigroup generated by m, n_1.

For the second one, we use the conductor and Γ to obtain

$$N_k < n_2D - (m+n_1)n_2 + 1.$$

Then,

$$\frac{N_k}{n_2D} < 1 - \frac{(m+n_1)n_2 - 1}{n_2D} < 1 + \frac{m+n_1}{mn_1n_2}.$$

□

Remark 2.3. The connection between the set Spec(f) of spectral numbers and roots of the Bernstein polynomial has been investigated by many authors. The spectral numbers are such that $0 < \tilde{\alpha}_1 \leq \tilde{\alpha}_2 \leq \ldots \leq \tilde{\alpha}_\mu$, where μ is the Milnor number. We know that $\tilde{\alpha}_1 = -\max A_1$ and the set Spec(f) is constant under μ-constant deformation of the germ. The main results in [17, 11, 10], imply that the set $\tilde{\alpha} \in \operatorname{Spec}(f)$, such that $\tilde{\alpha} < \tilde{\alpha}_1 + 1$ are roots of the Bernstein polynomial $b_{f_t}(s)$ of every μ-constant deformation $\{f_t\}$ of f. In fact, it can be proved that those spectral numbers are contained in the set $A_{11} \cup A_{21}$ so a good chunk of the candidate roots are already known to be roots of the Bernstein polynomial. In a forthcoming paper [2] the authors will describe the set of all common roots of the Bernstein polynomial $b_{f_t}(s)$ of any μ-constant deformation $\{f_t\}$ of f with characteristic sequence (n_1n_2, mn_2, mn_2+q) such that $\gcd(q, n_1) = 1$ or $\gcd(q, m) = 1$.

3. Residues of integrals at poles in A_1

Definition 3.1. A polynomial $f \in \mathbb{R}[x, y]$ is called to be of *type* $(n_1n_2, mn_2, mn_2+q)^+$ if it satisfies:

$$f(x, y) = (x^{n_1} + y^m + h_1(x, y))^{n_2} + x^a y^b + h_2(x, y) \tag{3.1}$$

where

(G$^+$1) $h_1(x, y) = \sum_{(i,j)\in\mathcal{P}_{n_1,m}} a_{ij}x^iy^j \in \mathbb{R}[x, y]$, where
$$\mathcal{P}_{n_1,m} := \{(i, j) \in \mathbb{Z}^2_{\geq 0} \mid mi + n_1j > mn_1\};$$
(G$^+$2) $a, b \geq 0$ such that $am + bn_1 = mn_1n_2 + q$;
(G$^+$3) the polynomial $h_2 \in \mathbb{R}[x, y]$, whose support is disjoint from the first term, satisfies that the characteristic sequence of f is $(n_1n_2, mn_2, mn_2 + q)$;

(G$^+$4) $f > 0$ in $[0,1]^2 \setminus \{(0,0)\}$.

For $\beta_1, \beta_2 \in \mathbb{Z}_{\geq 1}$, and f of type $(n_1 n_2, mn_2, mn_2 + q)^+$ we set:

$$I(f, \beta_1, \beta_2)(s) = \int_0^1 \int_0^1 f(x,y)^s \, x^{\beta_1} y^{\beta_2} \frac{dx}{x} \frac{dy}{y}. \tag{3.2}$$

Note that f does not satisfy the conditions stated in §1 and we cannot ensure that $I(f, \beta_1, \beta_2)(s)$ is well-defined, because $f(0,0) = 0$. The purpose of the following Proposition is to prove that, after a suitable change of variables, $I(f, \beta_1, \beta_2)(s)$ is expressed as a linear combination of integrals as in Proposition 1.4. In order to simplify the notation, we denote $\tilde{h}_2(x,y) := x^a y^b + h_2(x,y)$. We will use the following properties:

(G$^+$5) The minimum degree of $h_1(x^m, y^{n_1})$ is greater than mn_1.
(G$^+$6) The minimum degree of $\tilde{h}_2(x^m, y^{n_1})$ is greater than $mn_1 n_2$.

Proposition 3.2. *Let f be of type $(n_1 n_2, mn_2, mn_2 + q)^+$ and $\beta_1, \beta_2 \in \mathbb{Z}_{\geq 1}$. The integral $I(f, \beta_1, \beta_2)(s)$ is absolutely convergent for $\Re(s) > -\frac{\beta_1 m + \beta_2 n_1}{mn_1 n_2}$ and may have simple poles only for $s = -\frac{\beta_1 m + \beta_2 n_1 + \nu}{mn_1 n_2}, \nu \in \mathbb{Z}_{\geq 0}$.*

Proof. In this proof we are going to transform $I(f, \beta_1, \beta_2)(s)$ in a sum of integrals of type $\mathcal{Y}(s)$, for which we may apply Proposition 1.4. For the first step, we apply the change of variables

$$x = x_1^m, \quad y = y_1^{n_1}.$$

Let us denote

$$\tilde{f}(x_1, y_1) := f(x_1^m, y_1^{n_1}) = (x_1^{mn_1} + y_1^{mn_1} + h_1(x_1^m, y_1^{n_1}))^{n_2} + \tilde{h}_2(x_1^m, y_1^{n_1}).$$

We obtain (after renaming back the coordinates to x, y):

$$I(f, \beta_1, \beta_2)(s) = mn_1 \int_0^1 \int_0^1 \tilde{f}(x,y)^s \, x^{m\beta_1} y^{n_1 \beta_2} \frac{dx}{x} \frac{dy}{y}.$$

Let us decompose the square $[0,1]^2$ into two triangles

$$D_1 := \{(x,y) \in [0,1]^2 \mid x \geq y\}, \quad D_2 := \{(x,y) \in [0,1]^2 \mid x \leq y\}.$$

We express

$$I(f, \beta_1, \beta_2)(s) = mn_1 (I_1(f, \beta_1, \beta_2)(s) + I_2(f, \beta_1, \beta_2)(s)) \tag{3.3}$$

where each integral I_j has as integration domain D_j:

$$I_1(f, \beta_1, \beta_2)(s) = \int_0^1 \left(\int_0^x \tilde{f}(x,y)^s \, y^{n_1 \beta_2} \frac{dy}{y} \right) x^{m\beta_1} \frac{dx}{x}$$

and

$$I_2(f,\beta_1,\beta_2)(s)=\int_0^1\left(\int_0^y \tilde f(x,y)^s\, x^{m\beta_1}\,\frac{dx}{x}\right) y^{n_1\beta_2}\,\frac{dy}{y}.$$

Let us study first $I_1(f,\beta_1,\beta_2)(s)$. We consider the change of variables

$$x=x_1,\quad y=x_1y_1.$$

There is a polynomial $f_1(x_1,y_1)$ determined by $\tilde f(x_1,x_1y_1)=x_1^{mn_1n_2}f_1(x_1,y_1)$. Renaming the variables,

$$f_1(x,y)=(1+y^{mn_1}+xh_{11}(x,y))^{n_2}+x\tilde h_{21}(x,y),\quad h_{11},\tilde h_{21}\in\mathbb{R}[x,y].$$

The integral becomes

$$I_1(f,\beta_1,\beta_2)(s)=\int_0^1\int_0^1 f_1(x,y)^s\, x^{m\beta_1+n_1\beta_2+mn_1n_2s}y^{n_1\beta_2}\,\frac{dx}{x}\,\frac{dy}{y}. \tag{3.4}$$

We study now $I_2(f,\beta_1,\beta_2)(s)$ with the change of variables

$$x=x_1y_1,\quad y=y_1.$$

As above, there is a polynomial $f_2(x_1,y_1)$ such that $\tilde f(x_1y_1,y_1)=y_1^{mn_1n_2}f_2(x_1,y_1)$. Renaming the variables,

$$f_2(x,y)=(x^{mn_1}+1+yh_{12}(x,y))^{n_2}+y\tilde h_{22}(x,y),\quad h_{12},\tilde h_{22}\in\mathbb{R}[x,y].$$

The integral becomes

$$I_2(f,\beta_1,\beta_2)(s)=\int_0^1\int_0^1 f_2(x,y)^s\, x^{m\beta_1}y^{m\beta_1+n_1\beta_2+mn_1n_2s}\,\frac{dx}{x}\,\frac{dy}{y}. \tag{3.5}$$

The key point is that the functions $f_1(x,y)$ and $f_2(x,y)$ are *positive*, i.e., they do not vanish at $(0,0)$ and we can apply Proposition 1.4. Therefore $I_1(f,\beta_1,\beta_2)(s)$ and $I_2(f,\beta_1,\beta_2)(s)$ are absolutely convergent for $\Re(s)>-\frac{m\beta_1+n_1\beta_2}{mn_1n_2}$, have meromorphic continuation to the whole plane $\mathbb{C}$ with possible simple poles at $\alpha=-\frac{m\beta_1+n_1\beta_2+\nu}{mn_1n_2}$ with $\nu\in\mathbb{Z}_{\geq 0}$. $\square$

We study the possible poles $\alpha\in A_1$, defined in (2.2).

3.1. Residues at poles in A_{11}.

In this subsection, let $\alpha\in A_{11}$, i.e. there exist $\beta_1,\beta_2\in\mathbb{Z}_{\geq 1}$ for which

$$\alpha=-\frac{m\beta_1+n_1\beta_2}{mn_1n_2}, \tag{3.6}$$

see (2.4).

Proposition 3.3. *Let f be of type $(n_1n_2,mn_2,mn_2+q)^+$. Then, the integral $I(f,\beta_1,\beta_2)(s)$ has a pole for $s=\alpha$ and its residue is* $\frac{1}{mn_1n_2}\boldsymbol{B}\left(\frac{\beta_1}{n_1},\frac{\beta_2}{m}\right)$.

Proof. With the notation in the proof of Proposition 3.2, one has

$$f_1^\alpha(0,y) = (1+y^{mn_1})^{n_2\alpha}, \quad f_2^\alpha(x,0) = (x^{mn_1}+1)^{n_2\alpha}.$$

The residues of the integrals I_1, I_2 are computed using Proposition 1.6. For I_1, we have $(a_1,b_1) = (mn_1n_2, m\beta_1 + n_1\beta_2)$ and $(a_2,b_2) = (0, n_2\beta_2)$:

$$\operatorname*{Res}_{s=\alpha} I_1(f,\beta_1,\beta_2)(s) = \frac{1}{mn_1n_2} G_{f_1^\alpha(0,\cdot)}(n_1\beta_2).$$

With the same ideas,

$$\operatorname*{Res}_{s=\alpha} I_2(f,\beta_1,\beta_2)(s) = \frac{1}{mn_1n_2} G_{f_2^\alpha(\cdot,0)}(m\beta_1).$$

Recall that $I = mn_1(I_1 + I_2)$. We apply Lemma 1.8 where $c = 1$, $p = mn_1$, $\alpha = n_2\alpha$, $s_1 = \frac{\beta_1}{n_1}$ and $s_2 = \frac{\beta_2}{m}$, and we obtain

$$\operatorname*{Res}_{s=\alpha} I(f,\beta_1,\beta_2)(s) = \frac{1}{mn_1n_2} \boldsymbol{B}\left(\frac{\beta_1}{n_1}, \frac{\beta_2}{m}\right).$$

□

Remark 3.4. Let $\alpha \in A_{11}$. Since $A_{11} \subset A_1$, the rational number $-n_2\alpha$ is not an integer by (2.2). From the definition of α in (3.6), it is clear that if $\frac{\beta_1}{n_1} \in \mathbb{Z}$, then $mn_2\alpha \in \mathbb{Z}$ also in contradiction with (2.2). Hence $\frac{\beta_1}{n_1}, \frac{\beta_2}{m}$ are not integers. Then, using a Theorem of Schneider in [19], we know that $\boldsymbol{B}\left(\frac{\beta_1}{n_1}, \frac{\beta_2}{m}\right)$ is transcendental.

3.2. Residues at poles in A_{12}.

In the above subsection, we have succeeded to compute the exact residue because in the application of Proposition 1.6, no derivation was needed. For elements in A_{12} the situation is much more complicated and we will restrict our computation to some particular examples. Let us fix $\alpha = -\frac{m+n_1+k}{mn_1n_2} \in A_{12}$. We can express

$$mi_0 + n_1j_0 = mn_1 + k \text{ for some } (i_0, j_0) \in \mathbb{Z}^2_{\geq 0}, \tag{3.7}$$

since mn_1 is greater than the conductor of the semigroup generated by m, n_1. Let

$$f_{+t}(x,y) := (x^{n_1} + y^m + tx^{i_0}y^{j_0})^{n_2} + x^a y^b, \quad t \in \mathbb{R}_{>0},$$

with a and b as in (3.1).

Proposition 3.5. *The function $I(f_{+t},1,1)(s)$ has a pole for $s = -\alpha$ and its residue is a polynomial of degree 1 in t whose coefficient of t equals*

$$\frac{\alpha}{n_2n_1m} \boldsymbol{B}\left(\frac{1+i_0}{n_1}, \frac{1+j_0}{m}\right).$$

Proof. From Lemma 2.2, $1 \leq k \leq mn_1 - m - n_1$. The computation of the residue of $I_1(f,1,1)(s)$ is quite involved for a general polynomial and this is why we restrict our attention to f_{+t}. In the notation of Proposition 3.2, we have

$$\tilde{f}_{+t}(x,y) = (x^{mn_1} + y^{mn_1} + tx^{i_0 m} y^{j_0 n_1})^{n_2} + x^{am} y^{bn_1}.$$

Then

$$f_1(x,y) = (1 + y^{mn_1} + tx^k y^{j_0 n_1})^{n_2} + x^q y^{bn_1}, \quad f_2(x,y) = (x^{mn_1} + 1 + tx^{i_0 m} y^k)^{n_2} + x^{am} y^q.$$

By Proposition 1.6, we have:

$$\operatorname*{Res}_{s=\alpha} I_1(f_{+t},1,1)(s) = \frac{1}{mn_1 n_2 k!} G_{h_{k,\alpha,x}}(n_1), \quad h_{k,\alpha,x}(y) = \frac{\partial^k f_1^\alpha}{\partial x^k}(0,y). \tag{3.8}$$

It is well-known that

$$\frac{\partial^k f_1^\alpha}{\partial x^k} = \alpha f_1^{\alpha-1} \frac{\partial^k f_1}{\partial x^k} + \text{terms involving } f_1^{\alpha-m} \text{ and } \frac{\partial^r f_1}{\partial x^r} \text{ with } r < k. \tag{3.9}$$

In the sequel $\dots$ will mean in this proof *independent of the variable* t. It is easy to obtain the coefficient of t (e.g., derivating with respect to t and replacing t by 0):

$$\frac{\partial^r f_1}{\partial x^r}(0,y) = \begin{cases} \dots & \text{if } r < k, \\ tk! y^{n_1 j_0} (1 + y^{n_1 m})^{n_2 - 1} + \dots & \text{if } r = k. \end{cases}$$

Thus

$$\frac{\partial^k f_1^\alpha}{\partial x^k}(0,y) = tk! \alpha y^{n_1 j_0} (1 + y^{n_1 m})^{n_2 \alpha - 1} + \dots$$

The same arguments yield

$$\operatorname*{Res}_{s=\alpha} I_2(f_{+t},1,1)(s) = \frac{1}{mn_1 n_2 k!} G_{h_{k,\alpha,y}}(n_1), \quad h_{k,\alpha,y}(x) = \frac{\partial^k f_2^\alpha}{\partial y^k}(x,0)$$

and

$$\frac{\partial^k f_2^\alpha}{\partial y^k}(x,0) = tk! \alpha x^{m i_0} (x^{n_1 m} + c)^{n_2 \alpha - 1} + \dots.$$

Hence

$$\operatorname*{Res}_{s=\alpha} I_1(f_{+t},1,1)(s) = t \frac{\alpha}{mn_1 n_2} G_{(1+y^{n_1 m})^{-n_2 \alpha - 1}}(n_1(j_0+1)) + \dots$$

and

$$\operatorname*{Res}_{s=\alpha} I_2(f_{+t},1,1)(s) = t \frac{\alpha}{mn_1 n_2} G_{(x^{n_1 m}+1)^{-n_2 \alpha - 1}}(m(i_0+1)) + \dots$$

If we apply Lemma 1.8 to $\alpha = -n_2 \alpha - 1$, $s_1 = \frac{i_0+1}{n_1}$, $s_2 = \frac{j_0+1}{m}$, $p = n_1 m$, we obtain

$$\operatorname*{Res}_{s=\alpha} I(f_{+t},1,1)(s) = t \frac{\alpha}{n_2 n_1 m} \boldsymbol{B}\left(\frac{1+i_0}{n_1}, \frac{1+j_0}{m} \right) + \dots$$

□

Remark 3.6. Since $\alpha \in A_{12} \subset A_1$, by (2.2), it is clear that $-n_2\alpha - 1 \notin \mathbb{Z}$, and this number is the sum of the arguments of $\boldsymbol{B}$. If $\frac{i_0+1}{n_1} \in \mathbb{Z}$, then n_1 divides $m+k$ and this is forbidden by (2.2). Hence $\frac{i_0+1}{n_1}, \frac{j_0+1}{m} \notin \mathbb{Z}$. Since these three rational numbers are non-integers, we deduce from [19] that $\boldsymbol{B}\left(\frac{1+i_0}{n_1}, \frac{1+j_0}{m}\right)$ is transcendental.

4. Residues of integrals at poles in A_2

Definition 4.1. A polynomial $f \in \mathbb{R}[x,y]$ is said of type $(n_1n_2, mn_2, mn_2+q)^-$ if it satisfies:

$$f(x,y) = g(x,y)^{n_2} + x^a y^b + h_2(x,y) \tag{4.1}$$

where $g(x,y) := x^{n_1} - y^m + h_1(x,y)$

(H1) $h_1(x,y)$ is as in (G$^+$1).
(H2) $a, b \geq 0$ such that $am + bn_1 = mn_1n_2 + q$.
(H3) There exists $a_1, \dots, a_k \in \mathbb{R}$ such that for

$$Y(x^{\frac{1}{m}}) := \left(x^{\frac{1}{m}} + a_1 x^{\frac{2}{m}} + \cdots + a_k x^{\frac{k+1}{m}}\right)^{n_1}$$

we have $\operatorname{ord}_x g(x, Y(x^{\frac{1}{m}})) > \frac{mn_1n_2+q}{mn_2}$ and $Y(x^{\frac{1}{m}}) > 0$ if $0 < x \leq 1$. Let

$$g_Y(x,y) := \prod_{\zeta_m^m=1} \left(y - Y(\zeta_m x^{\frac{1}{m}})\right) \in \mathbb{R}[x,y].$$

(H4) The polynomial $h_2 \in \mathbb{R}[x,y]$, whose support is disjoint from the first terms, satisfies that the characteristic sequence of f is (n_1n_2, mn_2, mn_2+q).
(H5) Let $\mathcal{D}_Y := \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, 0 \leq y \leq Y(x^{\frac{1}{m}})\}$. Then $f > 0$ on $\mathcal{D}_Y \setminus \{(0,0)\}$.

For $\beta_1, \beta_2 \in \mathbb{Z}_{\geq 1}$, $\beta_3 \in \mathbb{Z}_{\geq 0}$ and f of type $(n_1n_2, mn_2, mn_2+q)^-$ (with g, Y as above) we set:

$$\mathcal{I}(f, \beta_1, \beta_2, \beta_3)(s) = \iint_{\mathcal{D}_Y} f(x,y)^s \, x^{\beta_1} y^{\beta_2} g_Y(x,y)^{\beta_3} \, \frac{dx}{x} \frac{dy}{y}. \tag{4.2}$$

Proposition 4.2. *Let $f \in \mathbb{R}[x,y]$ be a polynomial of type $(n_1n_2, mn_2, mn_2+q)^-$, $\beta_1, \beta_2 \in \mathbb{Z}_{\geq 1}$ and $\beta_3 \in \mathbb{Z}_{\geq 0}$. Then the integral $\mathcal{I}(f, \beta_1, \beta_2, \beta_3)(s)$ is convergent for $\Re(s) > -\frac{\beta_1 m + \beta_2 n_1 + \beta_3 mn_1}{mn_1n_2}$ and its set of poles is contained in the set*

$$P_1 \cup \bigcup_{i \in \mathbb{Z}_{\geq 1}, j \in \mathbb{Z}_{\geq 0}} P_{2,i,j}$$

where

$$P_1 := \left\{ -\frac{m\beta_1 + n_1\beta_2 + mn_1\beta_3 + \nu}{mn_1n_2} \,\middle|\, \nu \in \mathbb{Z}_{\geq 0} \right\}$$

and

$$P_{2,i,j} := \left\{ -\frac{n_2(m\beta_1 + n_1\beta_2 + mn_1\beta_3 + j) + q(\beta_3 + i) + \nu}{n_2(mn_1n_2 + q)} \middle| \nu \in \mathbb{Z}_{\geq 0} \right\}$$

The poles have at most order two. The poles may have order two at the values contained in P_1 and $P_{2,i,j}$ for some i, j.

Proof. We proceed as in the proof of Proposition 3.2. We start with the change: $x = x_1^m$, $y = y_1^{n_1}$. Note that after this change, the integration domain is exactly

$$\mathcal{D}_1 := \{(x, y) \in \mathbb{R}^2, 0 \leq x \leq 1, 0 \leq y \leq Y_1(x)\},$$

where $Y_1(x) = Y(x)^{\frac{1}{n_1}} = x + a_1x^2 + \cdots + a_kx^{k+1}$. We rename the coordinates and we obtain,

$$\mathcal{I}(f, \beta_1, \beta_2, \beta_3)(s) = mn_1 \iint_{\mathcal{D}_1} f(x^m, y^{n_1})^s\, x^{m\beta_1} y^{n_1\beta_2} g_{0,Y}(x, y)^{\beta_3}\, \frac{dx}{x}\frac{dy}{y}.$$

where $g_0(x, y) := g(x^m, y^{n_1})$ and $\operatorname{ord} g_0(x, Y_1(x)) > \frac{mn_1n_2+q}{n_2}$ and $g_{0,Y}(x, y)$ is defined in the same way and satisfies $g_{0,Y}(x, Y_1(x)) \equiv 0$.

The following change is $x = x_1$, $y = x_1y_1$. Let $\tilde{g}(x, y)$ be defined such that $g_0(x_1, x_1y_1) = x_1^{n_1m}\tilde{g}(x_1, y_1)$. Let $Y_2(x) = \frac{Y_1(x)}{x}$, note that $\operatorname{ord}\tilde{g}(x, y) > \frac{q}{n_2}$. In the same way, we define $\tilde{f}(x, y)$ such that $f(x_1^m, x_1^{n_1}y_1^{n_1}) = x_1^{n_1mn_2}\tilde{f}(x_1, y_1)$. It is easily seen that

$$\tilde{g}(x, y) = 1 - y^{n_1m} + x^{-n_1m}h_1(x^m, x^{n_1}y^{n_1}),\ \tilde{f}(x, y) = \tilde{g}(x, y)^{n_2} + x^qy^{n_1b} + \tilde{h}_2(x, y+1),$$

where the Newton polygon of $\tilde{h}_2(x, y)$ is above the one of $y^{n_2} + x^q$ (from the condition of f having the chosen characteristic sequence). We define $g_{0,Y}(x_1, x_1y_1) := x_1^{n_1m}\tilde{g}_Y(x, y)$ in the same way and $\tilde{g}_Y(x, Y_2(x)) \equiv 0$.

Let

$$\mathcal{D}_2 = \{(x, y) \in \mathbb{R}^2, 0 \leq x \leq 1, 0 \leq y \leq Y_2(x)\}.$$

With the renaming of coordinates, we have

$$\mathcal{I}(f, \beta_1, \beta_2, \beta_3)(s) = mn_1 \iint_{\mathcal{D}_2} \tilde{f}(x, y)^s\, x^{M+mn_1n_2s} y^{n_1\beta_2} \tilde{g}_Y(x, y)^{\beta_3}\, \frac{dx}{x}\frac{dy}{y}, \tag{4.3}$$

where $M := m\beta_1 + n_1\beta_2 + mn_1\beta_3$.

Note that $\tilde{f}$ is strictly positive on $\mathcal{D}_2 \setminus \{x = 0\}$ and $\tilde{f}(0, y) = 1 - y^{mn_1}$. Then $\tilde{f} > 0$ on $\mathcal{D}_2 \setminus \{(0, 1)\}$. This is why we perform the change of variables $x = x_1, y = (1 - y_1)Y_2(x_1)$. From the above properties if $\hat{g}(x, y) = \tilde{g}(x, (1 - y)Y_2(x))$, its Newton polygon is *more horizontal* than the one of $y^{n_2} + x^q$ and the coefficient of y equals mn_1. In particular, if $\hat{f}(x, y) = \tilde{f}(x, (1 - y)Y_2(x))$, then

$$\hat{f}(x, y) = (mn_1y)^{n_2} + x^q + \hat{h}(x, y)$$

where the Newton polygon of $\hat{h}(x,y)$ is above the one of the first two monomials.

Since $\tilde{g}_Y(x, Y_2(x)) \equiv 0$ then

$$\tilde{g}(x,(1-y)Y_2(x)) = yq_Y(x,y), \quad q_Y(0,0) = -n_1.$$

Let us define $\hat{g}_Y(x,y)$ by

$$y^{\beta_3}\hat{g}_Y(x,y) = \tilde{g}_Y(x,(1-y)Y_2(x))^{\beta_3}((1-y)Y_2(x))^{n_1\beta_2-1},\ \hat{g}_Y(x,y) = \sum b_{ij}x^j y^{i-1}.$$

This change of variables transforms the integration domain $\mathcal{D}_2$ into the square $[0,1]^2$. Then,

$$\mathcal{I}(f,\beta_1,\beta_2,\beta_3)(s) = mn_1 \int_0^1\!\!\int_0^1 \hat{f}(x,y)^s\, x^{M+mn_1n_2s} y^{\beta_3+1}\hat{g}_Y(x,y)\,\frac{dx}{x}\,\frac{dy}{y}, \tag{4.4}$$

where $\hat{g}_Y(x,y) \in \mathbb{R}[x,y]$.

We break this integral as

$$\mathcal{I}(f,\beta_1,\beta_2,\beta_3)(s) = mn_1 \sum_{i\geq 1, j\geq 0} b_{i,j}J_{i,j}(s), \quad b_{1,0} = 1, \tag{4.5}$$

where

$$J_{i,j}(s) := \int_0^1\!\!\int_0^1 \hat{f}(x,y)^s x^{M+j+mn_1n_2s} y^{\beta_3+i}\frac{dx}{x}\,\frac{dy}{y}.$$

Each of these integrals looks like the ones in Proposition 3.2 and we apply the same procedure where (n_1,m) is replaced by (q,n_2). Hence, we get $J_{i,j}(s) = J_{i,j,1}(s) + J_{i,j,2}(s)$. Replacing β_1 by $M+j+mn_1n_2s$ and β_2 by β_3+i in the statement of Proposition 3.2, we obtain

$$J_{i,j,1}(s) = n_2q\int_0^1\!\!\int_0^1 F_1(x,y)^s\, x^{sn_2D+B_{i,j}} y^{q(\beta_3+i)}\,\frac{dx}{x}\,\frac{dy}{y}, \tag{4.6}$$

where $B_{i,j} = n_2(M+j) + q(\beta_3+i)$ and $D = mn_1n_2 + q$ as in (2.3), and

$$J_{i,j,2}(s) = n_2q\int_0^1\!\!\int_0^1 F_2(x,y)^s\, x^{n_2(M+mn_1n_2s+j)} y^{sn_2D+B_{i,j}}\,\frac{dx}{x}\,\frac{dy}{y}, \tag{4.7}$$

where F_1, F_2 are strictly positive in the square. The poles of $J_{i,j,1}(s)$ are simple and given by

$$\alpha = -\frac{n_2(m\beta_1 + n_1\beta_2 + mn_1\beta_3 + j) + q(\beta_3+i) + \nu}{n_2(mn_1n_2+q)}, \quad \nu \in \mathbb{Z}_{\geq 0}.$$

The poles of $J_{i,j,2}(s)$ are the above ones and

$$\alpha = -\frac{m\beta_1 + n_1\beta_2 + mn_1\beta_3 + j + \nu}{mn_1n_2}, \quad \nu \in \mathbb{Z}_{\geq 0};$$

they may be double if one element is of both types (for fixed $i, j, \beta_1, \beta_2, \beta_3$). $\square$

4.1. **Residues at poles in A_{21}.**

Let $\alpha \in A_{21}$. Because of the definition (2.5) of A_{21} and the structure of the semigroup Γ, there exist $\beta_1, \beta_2 \in \mathbb{Z}_{\geq 1}$ and $\beta_3 \in \mathbb{Z}_{\geq 0}$ such that

$$\alpha = -\frac{n_2(\beta_1 m + \beta_2 n_1) + \beta_3(mn_1n_2 + q) + q}{n_2(mn_1n_2 + q)}. \tag{4.8}$$

Proposition 4.3. *For any f of type $(n_1n_2, mn_2, mn_2 + q)^-$, α is a pole of the integral $\mathcal{I}(f, \beta_1, \beta_2, \beta_3)(s)$ with residue*

$$\frac{1}{n_2(mn_1n_2 + q)}\boldsymbol{B}\left(\frac{\beta_3 + 1}{n_2}, -\alpha - \frac{\beta_3 + 1}{n_2}\right)$$

Proof. We keep the notations of Proposition 4.2. If $i > 1$ or $j > 0$ then

$$\operatorname*{Res}_{s=\alpha} J_{i,j,1}(s) = \operatorname*{Res}_{s=\alpha} J_{i,j,2}(s) = 0,$$

since the starting point of the poles is shifted by 1 to the left and α is in the semiplane of holomorphy.

We compute the residues for $J_{1,0,1}(s)$ and $J_{1,0,2}(s)$ using Proposition 1.6. Using (4.6), we have $\nu_1 = 0$, $a_1 = n_2(mn_1n_2+q)$, $b_1 = n_2(\beta_1 m+\beta_2 n_1)+\beta_3(n_2m_1n_1+q)+q$, $a_2 = 0$, $b_2 = q(\beta_3 + 1)$; hence

$$\operatorname*{Res}_{s=\alpha} J_{1,0,1}(s) = \frac{q}{mn_1n_2 + q} G_{((mn_1)^{n_2} y^{n_2 q}+1)^{\alpha}}(q(\beta_3 + 1)).$$

We apply the same computations (the roles of x and y exchange), where now $a_2 = mn_1n_2^2$, $b_2 = n_2(\beta_1 m + \beta_2 n_1 + \beta_3 m_1 n_1)$. Hence,

$$\operatorname*{Res}_{s=\alpha} J_{1,0,2}(s) = \frac{q}{mn_1n_2 + q} G_{((mn_1)^{n_2}+x^{n_2 q})^{\alpha}}(n_2(m\beta_1 + n_1\beta_2 + mn_1\beta_3 + mn_1n_2\alpha)).$$

Let us apply Lemma 1.8 (x, y are exchanged). We have $\alpha = \alpha$, $s_2 = \frac{\beta_3+1}{n_2}$, $s_1 = \frac{m\beta_1+n_1\beta_2+mn_1\beta_3+mn_1n_2\alpha}{q}$, $p = n_2 q$ and $c = (mn_1)^{n_2}$. The condition is fullfilled:

$$\begin{aligned} s_2+s_1 &= \frac{\beta_3 + 1}{n_2} + \frac{m\beta_1 + n_1\beta_2 + mn_1\beta_3}{q} - mn_1 \frac{n_2(\beta_1 m+\beta_2 n_1)+\beta_3(n_2m_1n_1+q)+q}{q(mn_1n_2 + q)} \\ &= \frac{\beta_3 + 1}{n_2} + \frac{m\beta_1 + n_1\beta_2}{q}\left(1 - \frac{mn_1n_2}{mn_1n_2 + q}\right) - \frac{mn_1}{mn_1n_2 + q} = \\ &\frac{\beta_3 + 1}{n_2} + \frac{m\beta_1 + n_1\beta_2}{mn_1n_2 + q} - \frac{mn_1}{mn_1n_2 + q} = -\alpha. \end{aligned}$$

Hence,

$$\operatorname*{Res}_{s=\alpha}(J_{1,0,1}(s) + J_{1,0,2}(s)) = \frac{1}{(mn_1)^{\beta_3+1} n_2(mn_1n_2 + q)}\boldsymbol{B}\left(\frac{\beta_3 + 1}{n_2}, -\alpha - \frac{\beta_3 + 1}{n_2}\right)$$

and the result follows from (4.5). $\square$

Remark 4.4. It is obvious that $-\alpha \notin \mathbb{Z}$. Assume that $\frac{\beta_3+1}{n_2} \in \mathbb{Z}$. From (4.8) and (2.3), we get a contradiction, hence $\frac{\beta_3+1}{n_2} \notin \mathbb{Z}$. On the other side, if $-\alpha - \frac{\beta_3+1}{n_2} \in \mathbb{Z}$, we obtain that $n_2\alpha \in \mathbb{Z}$ which is in contradiction with (2.3). Hence, $\boldsymbol{B}\left(\frac{\beta_3+1}{n_2}, -\alpha - \frac{\beta_3+1}{n_2}\right)$ is transcendental.

4.2. **Residues at poles in A_{22}.**

As in §3.2, we perform now a partial computation of the residue for $\alpha \in A_{22}$,

$$\alpha = -\frac{n_2(m+n_1)+q+k}{n_2(mn_1n_2+q)}.$$

From the definition of A_{22} and the properties of the semigroup Γ, we can find non-negative integers a', b', ℓ are such that

$$(a'm + b'n_1)n_2 + \ell(mn_1n_2+q) = (mn_1n_2+q)n_2 + k$$

Let

$$f_{-t}(x,y) := (x^{n_1} - y^m)^{n_2} + x^a y^b + t(x^{n_1} - y^m)^{\ell} x^{a'} y^{b'}, \quad t \in \mathbb{R}_{>0}.$$

Proposition 4.5. *The function $I(f_{-t},1,1,0)(s)$ has a pole for $s = -\alpha$ and its residue is a polynomial of degree* 1 *in t whose coefficient of t equals*

$$\frac{\alpha(mn_1)^{1-\frac{\ell(\ell+1)}{n_2}}}{n_2(mn_1n_2+q)} \boldsymbol{B}\left(\frac{\ell+1}{n_2}, -\alpha+1-\frac{\ell+1}{n_2}\right).$$

Proof. The poles we are interested in for $J_{i,j,1}, J_{i,j,2}$ start, for each i,j, at

$$-\frac{n_2(m+n_1+j)+qi}{n_2(mn_1n_2+q)}.$$

For (i,j) such that $n_2 j + qi \leq k$ the integrals $J_{i,j,1}, J_{i,j,2}$ may have poles at α. We follow the strategy of the proof of Proposition 3.5. The residues are computed using a derivative of order $k - (n_2 j + qi)$ (the steps from the first pole). It is not hard to see that if $j \neq 0$ or $i \neq 1$, then the residues are independent of t.

Let us study the behavior of $J_{1,0,1}(s)$ and $J_{1,0,2}(s)$. As in the proof of Proposition 3.5, we have

$$\frac{\partial^k F_1^{\alpha}}{\partial x^k}(0,y) = \alpha k! t (mn_1)^{\ell} y^{q\ell} {F_1}^{\alpha-1}(0,y) + \dots$$

and

$$\operatorname*{Res}_{s=\alpha} J_{1,0,1}(f)(s) = \frac{q}{(mn_1n_2+q)k!} G_{\left(\partial^{(k,0)}(F_1)^{\alpha}(0,\cdot)\right)}(q) =$$
$$t\frac{\alpha q (mn_1)^{\ell}}{mn_1n_2+q} G_{((mn_1)^{n_2} y^{n_2 q}+1)}(q(\ell+1)) + \dots$$

With the same arguments,

$$\frac{\partial^k F_2^\alpha}{\partial y^k}(x,0) = \alpha k! t (mn_1)^\ell x^{(n_2-\ell)q+k}(F_2)^{\alpha-1}(x,0) + \dots$$

and

$$\operatorname*{Res}_{s=\alpha} J_{1,0,2}(f)(s) = \frac{q}{(mn_1n_2+q)k!} G_{\left(\partial^{(0,k)}(F_2)^\alpha(\cdot,0)\right)}(n_2(mn_1n_2\alpha+n_1+m)) =$$
$$t\frac{\alpha q (mn_1)^\ell}{mn_1n_2+q} G_{((mn_1)^{n_2}+x^{n_2q})}(n_2(mn_1n_2\alpha+n_1+m)+(n_2-\ell)q+k) + \dots$$

Let us denote

$$s_1 = \frac{mn_1n_2\alpha+n_1+m}{q} - \frac{\ell}{n_2} + \frac{k}{qn_2} + 1, \quad s_2 = \frac{\ell+1}{n_2}, \quad p = qn_2, \quad c = (mn_1)^{n_2}.$$

Since $s_1+s_2 = -\alpha+1$, applying Lemma 1.8, we have

$$\operatorname*{Res}_{s=\alpha} J_1(s) = \frac{\alpha (mn_1)^{-\frac{\ell(\ell+1)}{n_2}} t}{n_2(mn_1n_2+q)} \boldsymbol{B}\left(\frac{\ell+1}{n_2}, -\alpha+1-\frac{\ell+1}{n_2}\right).$$

□

Remark 4.6. Note again that $\boldsymbol{B}\left(\frac{\ell+1}{n_2}, -\alpha+1-\frac{\ell+1}{n_2}\right)$ is transcendental.

5. Relation of integrals with Bernstein polynomial

We are using ideas from [5, 6, 7]. Let us fix notations that may cover all the cases. We fix $f, g, Y, g_Y, \mathcal{D}_Y$ with the following properties:

(B1) The characteristic sequence of $f \in \mathbb{R}[x,y]$ is (n_1n_2, mn_2, mn_2+q).

(B2) The characteristic sequence of $g \in \mathbb{R}[x,y]$ is (n_1, m) and it has maximal contact with f among all the singularities with the same characteristic sequence.

(B3) The polynomial $Y(x^{\frac{1}{m}}) \in \mathbb{R}[x^{\frac{1}{m}}]$ (where one of its n_1-roots is still in $\mathbb{R}[x^{\frac{1}{m}}]$) satisfies one of the following conditions:
- $\operatorname{ord}_x(g(x, Y(x^{\frac{1}{m}}))) > \frac{mn_1n_2+q}{mn_2}$ and it is monotonically increasing in $\mathbb{R}_{\geq 0}$.
- $Y \equiv 1$.

(B4) g_Y is as in (H3) in §4.

(B5) $\mathcal{D}_Y := \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, 0 \leq y \leq Y(x^{\frac{1}{m}})\}$.

(B6) $f(x,y) > 0\ \forall (x,y) \in \mathcal{D}_Y \setminus \{(0,0)\}$.

Let $\beta_1, \beta_2 \in \mathbb{Z}_{\geq 1}$ and $\beta_3 \in \mathbb{Z}_{\geq 0}$. Let us consider the integral

$$\mathcal{I}(f,\beta_1,\beta_2,\beta_3)(s) = \iint_{\mathcal{D}_Y} f(x,y)^s\, x^{\beta_1} y^{\beta_2}\, g_Y(x,y)^{\beta_3} \frac{dx}{x}\frac{dy}{y}. \tag{5.1}$$

These integrals cover those studied in Sections 3 and 4. For those of §3, we take $Y \equiv 1$ and $\beta_3 = 0$ (hence g_Y is not longer used). If we need to distinguish them, we will denote by $\mathcal{I}_+$ those coming from §3 and by $\mathcal{I}_-$ those coming from §4. For $\mathcal{I}_+$ we may drop the argument β_3.

Let us recall the definition of Bernstein-Sato polynomial $b_f(s)$, see the Introduction. It is the lowest-degree non-zero polynomial satisfying the existence of an s-differential operator

$$\mathbf{D} = \sum_{j=0}^{N} D_j s^j, \qquad D_j = \sum_{i_1+i_2<M} a_{j,i_1,i_2}(x,y)\frac{\partial^{i_1}}{\partial x^{i_1}}\frac{\partial^{i_2}}{\partial y^{i_2}}, a_{j,i_1,i_2} \in \mathbb{C}[x,y]$$

such that

$$\mathbf{D}\cdot f^{s+1} = b_f(s) f^s. \tag{5.2}$$

Moreover, see e.g. [9], if $f \in \mathbb{K}[x,y]$, $\mathbb{K} \subset \mathbb{R}$, the polynomials a_{j,i_1,i_2} have coefficients over $\mathbb{K}$. Applying (5.2), we have

(5.3)
$$\mathcal{I}(f,\beta_1,\beta_2,\beta_3)(s) = \frac{1}{b_f(s)}\mathcal{J}, \quad \mathcal{J} := \iint_{\mathcal{D}_Y} \mathbf{D}[f(x,y)^{s+1}]x^{\beta_1}y^{\beta_2}g_Y(x,y)^{\beta_3}\frac{dx}{x}\frac{dy}{y}.$$

Following the definition of $\mathbf{D}$, $\mathcal{J}$ is a linear combination (with coefficients in $\mathbb{K}[s]$) of integrals

$$\mathcal{I}_{i_1,i_2}(\beta_1',\beta_2',\beta_3)(s) = \iint_{\mathcal{D}_Y} \frac{\partial^{i_1+i_2} f^{s+1}(x,y)}{\partial x^{i_1}\partial y^{i_2}} x^{\beta_1'-1}y^{\beta_2'-1}g_Y(x,y)^{\beta_3}dxdy,$$

with $\beta_i' \geq \beta_i$.

Using (3.9), we could express these integrals using derivatives of f and powers of the type f^{s+1-m} (for some non-negative integer m). But, following the ideas in [6], we will use integration by parts in order to do not decrease the exponent $s+1$.

Let us define $X(y^{\frac{1}{n_1}})$ the inverse of the function $Y(x^{\frac{1}{m}})$, when Y is not constant; we set $X \equiv 0$ if Y is constant. Note that $X(y^{\frac{1}{n_1}})$ is an analytic function in $y^{\frac{1}{n_1}}$ with coefficients in $\mathbb{K}$. The integration by parts with respect to x (if $i_1 > 0$) yields

$$\mathcal{I}_{i_1,i_2}(\beta_1',\beta_2',\beta_3)(s) = U - W$$

where

$$U = \int_0^{Y(1)} \left[\frac{\partial^{i_1+i_2-1} f^{s+1}(x,y)}{\partial x^{i_1-1}\partial y^{i_2}} x^{\beta_1'-1}(g_Y(x,y))^{\beta_3}\right]_{X(y^{\frac{1}{n_1}})}^{1} y^{\beta_2'}\frac{dy}{y},$$

$$W = \iint_{\mathcal{D}_Y} \frac{\partial^{i_1+i_2-1} f^{s+1}}{\partial x^{i_1-1}\partial y^{i_2}}(x,y)\,\frac{\partial(x^{\beta_1'-1}(g_Y(x,y))^{\beta_3})}{\partial x} y^{\beta_2'}dx\frac{dy}{y}.$$

A similar formula is obtained with respect to y.

Using again (3.9), we can see that U is a linear combination with coefficients in $\mathbb{K}$ of integrals as in Corollary 1.2 (where the exponents may decrease). The term W is again a linear combination with coefficients in $\mathbb{K}$ of integrals $\mathcal{I}_{i_1-1,i_2}(\beta_1'',\beta_2'',\beta_3')(s)$.

Since the index i_1 decreases (and the same happens with i_2 integrating with respect to y) we can summarize these arguments in the following Proposition.

Proposition 5.1. *Let $f \in \mathbb{K}[x, y]$ be a polynomial whose local complex singularity at the origin has two Puiseux pairs and such that $\mathbb{K}$ is an algebraic extension of $\mathbb{Q}$. If $\beta_1, \beta_2 \geq 1$, and $\beta_3 \geq 0$ then $\mathcal{I}_{i_1,i_2}(\beta_1, \beta_2, \beta_3)(s)$ is a linear combination over $\mathbb{K}[s]$ of:*

(1) *meromorphic functions having only simple poles whose residues are algebraic over $\mathbb{K}$;*
(2) *and integrals $\mathcal{I}(f, \beta'_1, \beta'_2, \beta'_3)(s+1)$ for some triples $(\beta'_1, \beta'_2, \beta'_3)$ with $\beta'_i \geq \beta_i$ for $1 \leq i \leq 3$.*

Corollary 5.2. *Let $f \in \mathbb{K}[x, y]$ be a polynomial whose local complex singularity at the origin has two Puiseux pairs and such that $\mathbb{K}$ is an algebraic extension of $\mathbb{Q}$. Then the integral $\mathcal{I}(f, \beta_1, \beta_2, \beta_3)(s)$ is the product of $b_f(s)^{-1}$ and a linear combination over $\mathbb{K}[s]$ of meromorphic functions whose residues are algebraic over $\mathbb{K}$ and integrals $\mathcal{I}(f, \beta'_1, \beta'_2, \beta'_3)(s + 1)$.*

These results allow to detect roots of Bernstein polynomials in some cases.

Theorem 5.3. *Let $f \in \mathbb{K}[x, y]$ be a polynomial whose local complex singularity at the origin has two Puiseux pairs and its algebraic monodromy has distinct eigenvalues and such that $\mathbb{K}$ is an algebraic extension of $\mathbb{Q}$. Let α be a pole of $\mathcal{I}(f, \beta_1, \beta_2, \beta_3)(s)$ with transcendental residue, and such that $\alpha + 1$ is not a pole of $\mathcal{I}(f, \beta'_1, \beta'_2, \beta'_3)(s)$ for any $(\beta'_1, \beta'_2, \beta'_3)$. Then α is a root of the Bernstein-Sato polynomial $b_f(s)$ of f.*

Proof. Let us consider the equality (5.3). On the left-hand side of the integral, α is a pole with transcendental residue. Let us study the situation on the right-hand side. It can be either a pole of $\mathcal{J}$ or a root of $b_f(s)$ (only simple roots!). Note that by Corollary 5.2, if α is a pole of $\mathcal{J}$ then its residue must be algebraic. Then, α must be a root of $b_f(s)$. $\square$

6. Yano's conjecture for two-Puiseux-pair singularities

Let $(n_1n_2, mn_1, mn_2 + q)$ be a characteristic sequence such that $\gcd(q, m) = \gcd(q, n_1) = 1$, i.e., the monodromy has distinct eigenvalues. The Bernstein-Sato polynomial of a germ f with this characteristic sequence, depends on f, but there is a *generic* Bernstein polynomial $b_{\mu,\text{gen}}(s)$: for any versal deformation of such an f, there exists a Zariski dense open set $\mathcal{U}$ on which the Bernstein-Sato polynomial of any germ in $\mathcal{U}$ equals $b_{\mu,gen}(s)$.

Recall that the hypothesis on the eigenvalues of the monodromy implies that the set of b-exponents consists in a set of μ distinct values, which are opposite to the roots of the Bernstein polynomial, being μ the Milnor number of any irreducible germ with (n_1n_2, mn_1, mn_2+q) as characteristic sequence. Hence, in order to prove that Yano's Conjecture holds for those characteristic sequences, we need to prove that the set of roots of the Bernstein polynomial $b_{\mu,\mathrm{gen}}(s)$ is $A_1 \cup A_2$.

Theorem 6.1. *Let $f(x,y) \in \mathbb{C}\{x,y\}$ be an irreducible germ of plane curve which has two Puiseux pairs and its algebraic monodromy has distinct eigenvalues. Then Yano's Conjecture holds for generic polynomials having as characteristic sequence (n_1n_2, mn_1, mn_2+q) such that $\gcd(q,m) = \gcd(q,n_1) = 1$, that is the set of opposite b-exponents is $A_1 \cup A_2$.*

Proof. Let us fix an element $\alpha \in A_1 \cup A_2$.

Let us start with $\alpha \in A_1$. Note that $\alpha+1 \geq -\frac{m+n}{mn_1n_2}$, which is the greater abscissa of convergence of $\mathcal{I}(f,\beta_1',\beta_2')(s)$ for all β_1',β_2'. As a consequence, α satisfies the second hypothesis of Theorem 5.3 for any f of type $(n_1n_2, mn_2, mn_2+q)^+$.

Assume that $\alpha \in A_{11}$. Let us pick-up f of type $(n_1n_2, mn_2, mn_2+q)^+$ and let $\mathcal{V}$ be the set of such polynomials. We have proved in Proposition 3.3 that there exist $\beta_1,\beta_2 \in \mathbb{Z}_{\geq 1}$ such that $\mathcal{I}(f,\beta_1,\beta_2)(s)$ has a simple pole for $s=\alpha$ and its residue equals (up to a rational number) $\boldsymbol{B}\left(\frac{\beta_1}{n_1},\frac{\beta_2}{m}\right)$, and neither $\frac{\beta_1}{n_1},\frac{\beta_2}{m}$ nor its sum (which equals $-n_2\alpha$) are integers. As a consequence, this residue is a transcendental number, see Remark 3.4. Then, if we choose f with algebraic coefficients, all the hypotheses of Theorem 5.3 are fulfilled and α is a root of the Bernstein polynomial of f.

Since $\mathcal{V}$ determines a non-empty open set in the real part of a versal deformation, there is a non-empty real open set $\mathcal{V}_1$ of real polynomials whose Bernstein polynomial is $b_{\mu,gen}(s)$. Since polynomials with algebraic coefficients are dense, we conclude that α is a root of $b_{\mu,\mathrm{gen}}(s)$, $\forall \alpha \in A_{11}$.

Now let us assume $\alpha \in A_{12}$. By Proposition 3.5, we know that there is an f_{+t} of type $(n_1n_2, mn_1, mn_2+q)^+$ (and algebraic coefficients) such that $\mathcal{I}(f_{+t},1,1)(s)$ has a simple pole for $s=\alpha$ with a transcendental residue. As above, Theorem 5.3 ensures that α is a root of the Bernstein polynomial of this particular f_{+t}. Recall, from Lemma 2.2, that $\forall \alpha \in A_{12}$, $\alpha+1 > -\frac{m+n_1}{n_1n_2m}$, in particular $\alpha+1$ cannot be a root of the Bernstein polynomial for any f with characteristic sequence (n_1n_2, mn_1, mn_2+q). We are in the hypothesis of Proposition 1; The lower semicontinuity implies that either α or $\alpha+1$ are roots of $b_{\mu,\mathrm{gen}}(s)$, hence, α is a root of $b_{\mu,gen}(s)$, $\forall \alpha \in A_{12}$.

Once the statement is done for the set A_1 we can use the same kind of arguments for the set A_2. If $\alpha \in A_2$, by (2.3), $\alpha+1 > \frac{(m+n_1)n_2+q}{n_2(mn_1n_2+q)}$ which is the maximum pole that can be congruent with $\alpha \bmod \mathbb{Z}$. This ensures the fulfillment of the second hypothesis of Theorem 5.3 for any f of type $(n_1n_2, mn_2, mn_2+q)^-$. The rest of the arguments follow the same ideas as above using instead Propositions 4.3 and 4.5. $\square$

Appendix A. Technical proofs

Proof of Proposition 1.4. The proof follows the same ideas as in Proposition 1.1. Let us consider first the Taylor expansion of f^s with respect to x:

$$f^s(x,y) = \sum_{\nu_1=0}^{N_1} \frac{1}{\nu_1!}\frac{\partial^{\nu_1} f^s}{\partial x^{\nu_1}}(0,y)x^{\nu_1} + \frac{1}{N_1!}\int_0^1 x^{N_1+1}(1-t_1)^{N_1}\frac{\partial^{N_1+1} f^s}{\partial x^{N_1+1}}(t_1x,y)dt_1.$$

We apply to each function above its Taylor expansion with respect to y:

$$f^s(x,y) = \sum_{\nu_1=0}^{N_1}\sum_{\nu_2=0}^{N_2} \frac{1}{\nu_1!\nu_2!}\frac{\partial^{\nu_1+\nu_2} f^s}{\partial x^{\nu_1}\partial y^{\nu_2}}(0,0)x^{\nu_1}y^{\nu_2}+$$

$$\sum_{\nu_1=0}^{N_1}\frac{x^{\nu_1}}{\nu_1!N_2!}\int_0^1 y^{N_2+1}(1-t_2)^{N_2}\frac{\partial^{\nu_1+N_2+1} f^s}{\partial x^{\nu_1}\partial y^{N_2+1}}(0,t_2y)dt_2+$$

$$\sum_{\nu_2=0}^{N_2}\frac{y^{\nu_2}}{N_1!\nu_2!}\int_0^1 x^{N_1+1}(1-t_1)^{N_1}\frac{\partial^{N_1+\nu_2+1} f^s}{\partial x^{N_1+1}\partial y^{\nu_2}}(t_1x,0)dt_1+$$

$$\frac{1}{N_1!N_2!}\int_0^1\int_0^1 x^{N_1+1}y^{N_2+1}(1-t_1)^{N_1}(1-t_2)^{N_2}\frac{\partial^{N_1+N_2+2} f^s}{\partial x^{N_1+1}\partial y^{N_2+1}}(t_1x,t_2y)dt_1dt_2.$$

Consider the following notation:

$$\begin{aligned}
\psi^1_{N_1,\nu_2}(x,s) :=& \frac{1}{N_1!\nu_2!}\int_0^1 (1-t_1)^{N_1}\frac{\partial^{N_1+\nu_2+1} f^s}{\partial x^{N_1+1}\partial y^{\nu_2}}(t_1x,0)dt_1\\
\psi^2_{\nu_1,N_2}(y,s) :=& \frac{1}{\nu_1!N_2!}\int_0^1 (1-t_2)^{N_2}\frac{\partial^{\nu_1+N_2+1} f^s}{\partial x^{\nu_1}\partial y^{N_2+1}}(0,t_2y)dt_2\\
\mathcal{S}_{N_1,N_2}(x,y,s) :=& \frac{1}{N_1!N_2!}\int_0^1\int_0^1 (1-t_1)^{N_1}(1-t_2)^{N_2}\frac{\partial^{N_1+N_2+2} f^s}{\partial x^{N_1+1}\partial y^{N_2+1}}(t_1x,t_2y)dt_1dt_2.
\end{aligned}$$

These functions are holomorphic for $s \in \mathbb{C}$. Hence, one can write

$$
\begin{aligned}
\mathcal{Y}(s) = \sum_{\nu_1=0}^{N_1}\sum_{\nu_2=0}^{N_2} \frac{1}{\nu_1!\nu_2!}\frac{\partial^{\nu_1+\nu_2} f^s}{\partial x^{\nu_1}\partial y^{\nu_2}}(0,0)\frac{1}{(a_1 s+b_1+\nu_1)(a_2 s+b_2+\nu_2)}+ \\
\sum_{\nu_1=0}^{N_1}\frac{1}{a_1 s+b_1+\nu_1}\int_0^1 y^{a_2 s+b_2+N_2}\psi^2_{\nu_1,N_2}(y,s)dy+ \\
\sum_{\nu_2=0}^{N_2}\frac{1}{a_2 s+b_2+\nu_2}\int_0^1 x^{a_1 s+b_1+N_1}\psi^1_{N_1,\nu_2}(x,s)dx+ \\
\int_0^1\int_0^1 x^{a_1 s+b_1+N_1}y^{a_2 s+b_2+N_2}\mathcal{S}_{N_1,N_2}(x,y,s)dxdy.
\end{aligned}
\tag{A.1}
$$

Let us denote

$$
\begin{aligned}
\varphi^1_{a_1,b_1,\nu_2}(s) &:= \int_0^1 x^{a_1 s+b_1+N_1}\psi^1_{N_1,\nu_2}(x,s)dx \\
\varphi^2_{a_2,b_2,\nu_1}(s) &:= \int_0^1 y^{a_2 s+b_2+N_2}\psi^2_{\nu_1,N_2}(y,s)dy \\
\mathcal{R}_{a_1,b_1,a_2,b_2}(s) &:= \int_0^1\int_0^1 x^{a_1 s+b_1+N_1}y^{a_2 s+b_2+N_2}\mathcal{S}_{N_1,N_2}(x,y,s)dxdy.
\end{aligned}
$$

The integral function $\varphi^1_{a_1,b_1,\nu_2}$ is absolutely convergent and holomorphic for $\Re(s) > -\frac{b_1+N_1+1}{a_1}$, while $\varphi^2_{a_2,b_2,\nu_1}$ is holomorphic for $\Re(s) > -\frac{b_2+N_2+1}{a_2}$.

The function $\mathcal{R}_{a_1,b_1,a_2,b_2}$ is absolutely convergent and holomorphic for $\Re(s) > \max\{-\frac{b_1+N_1+1}{a_1}, -\frac{b_2+N_2+1}{a_2}\}$. The result follows. □

Proof of Proposition 1.6. The hypothesis ensures that the pole is simple. Choose $N_1 \geq \nu_1$ and N_2 such that $\alpha > -\frac{b_2+N_2+1}{a_2}$. We use the functions and equalities introduced in the proof of Proposition 1.4. The residue is obtained by evaluating $\frac{a_1 s+b_1+\nu_1}{a_1}\mathcal{Y}(s)$ at α. Using (A.1), we have

$$
\sum_{\nu_2=0}^{N_2}\frac{1}{(a_2\alpha+b_2+\nu_2)a_1\nu_1!\nu_2!}\frac{\partial^{\nu_1+\nu_2} f^\alpha}{\partial x^{\nu_1}\partial y^{\nu_2}}(0,0)+\frac{1}{a_1}\int_0^1 y^{a_2\alpha+b_2+N_2}\psi^2_{\nu_1,N_2}(y,\alpha)dy
$$

Then,

$$
\operatorname*{Res}_{s=\alpha}\mathcal{Y}(s) = \sum_{\nu_2=0}^{N_2}\frac{1}{(a_2\alpha+b_2+\nu_2)a_1\nu_1!\nu_2!}\frac{\partial^{\nu_1+\nu_2} f^\alpha}{\partial x^{\nu_1}\partial y^{\nu_2}}(0,0)+\frac{1}{a_1}\varphi^2_{\nu_1,N_2}(\alpha).
$$

Consider the integral

$$
\int_0^1 \partial^{(\nu_1,0)}(f^\alpha)(0,y)y^s\frac{dy}{y}
$$

The Taylor formula yields

$$\partial^{(\nu_1,0)}(f^\alpha)(0,y)=\frac{\partial^{\nu_1}f^\alpha}{\partial x^{\nu_1}}(0,y)=$$

$$\sum_{\nu_2=0}^{N_2}\frac{1}{\nu_2!}\frac{\partial^{\nu_1+\nu_2}f^\alpha}{\partial x^{\nu_1}\partial y^{\nu_2}}(0,0)y^{\nu_2}+\frac{1}{N_2!}\int_0^1 y^{N_2+1}(1-t_2)^{N_2}(f^\alpha)^{(N_2+1)}(0,t_2y)dt_2=$$

$$\sum_{\nu_2=0}^{N_2}\frac{1}{\nu_2!}\frac{\partial^{\nu_1+\nu_2}f^\alpha}{\partial x^{\nu_1}\partial y^{\nu_2}}(0,0)y^{\nu_2}+\nu_1!y^{N_2+1}\psi^2_{\nu_1,N_2}(y,\alpha).$$

We integrate that function (multiplied by y^{s-1}) to get

$$\sum_{\nu_2=0}^{N_2}\frac{1}{(\nu_2+s)\nu_2!}\frac{\partial^{\nu_1+\nu_2}f^\alpha}{\partial x^{\nu_1}\partial y^{\nu_2}}(0,0)+\frac{1}{a_1}\int_0^1 y^{s+N_2}\psi^2_{\nu_1,N_2}(y,\alpha)dy$$

and the equality holds. $\square$

Proof of Lemma 1.8. Let $G_1:=G_{(\,y^p+c)^\alpha}(ps_1)$,

$$G_1=\int_0^1(y^p+c)^\alpha y^{ps_1}\frac{dy}{y}=\frac{c^\alpha}{p}\int_0^1\left(\frac{y}{c}+1\right)^\alpha y^{s_1}\frac{dy}{y}=\frac{c^{-s_2}}{p}\int_0^{c^{-1}}(\,y+1)^\alpha\,y^{s_1}\frac{dy}{y}.$$

Let $G_2:=G_{(\,1+cx^p)^\alpha}(ps_2)$,

$$G_2=\int_0^1(\,1+cx^p)^\alpha\,x^{ps_2}\frac{dx}{x}=\frac{1}{p}\int_0^1(\,1+cx)^\alpha\,x^{s_2}\frac{dx}{x}=$$

$$\frac{1}{p}\int_1^\infty(\,x+c)^\alpha\,x^{s_1}\frac{dx}{x}=\frac{c^{-s_2}}{p}\int_{c^{-1}}^\infty(\,x+1)^\alpha\,x^{s_1}\frac{dx}{x}.$$

Thus:

$$G_1+G_2=\frac{c^{-s_2}}{p}\int_0^\infty(\,x+1)^\alpha\,x^{s_1}\frac{dx}{x}=\frac{c^{-s_2}}{p}\boldsymbol{B}\left(s_1,s_2\right).$$

$\square$

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