

ON NON-KÄHLER COMPACT COMPLEX MANIFOLDS WITH BALANCED AND ASTHENO-KÄHLER METRICS

SUR LES VARIÉTÉS COMPACTES COMPLEXES NON-KÄHLER AVEC DES MÉTRIQUES
EQUILIBRÉES ET ASTHENO-KÄHLER

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ABSTRACT. In this note we construct, for every $n \geq 4$, a non-Kähler compact complex manifold X of complex dimension n admitting a balanced metric and an astheno-Kähler metric which is in addition k -th Gauduchon for any $1 \leq k \leq n - 1$.

RÉSUMÉ. Dans cette note nous construisons, pour chaque $n \geq 4$, une variété compacte complexe non-Kähler X de dimension complexe n admettant une métrique équilibrée et une métrique astheno-Kähler; de plus, cette métrique est k -ième Gauduchon pour $1 \leq k \leq n - 1$.

1. INTRODUCTION

Let X be a compact complex manifold of complex dimension n , and let F be a Hermitian metric on X . It is well-known that the metric F is called *balanced* if the Lee form vanishes, equivalently the form F^{n-1} is closed. If $\partial\bar{\partial}F^{n-2} = 0$, then the Hermitian metric F is said to be *astheno-Kähler*. Balanced metrics are studied by Michelsohn in [15], and the class of astheno-Kähler metrics is considered by Jost and Yau in [13] to extend Siu's rigidity theorem to non-Kähler manifolds. This note is motivated by a question in the paper [16] by Székelyhidi, Tosatti and Weinkove about the existence of examples of non-Kähler compact complex manifolds admitting both balanced and astheno-Kähler metrics. Recently, two examples, in dimensions 4 and 11, are constructed by Fino, Grantcharov and Vezzoni in [4]. Our goal is to present examples in any complex dimension $n \geq 4$. Moreover, we show that our astheno-Kähler metrics satisfy the stronger condition of being k -th Gauduchon for every $1 \leq k \leq n - 1$.

When the Lee form is co-closed, equivalently F^{n-1} is $\partial\bar{\partial}$ -closed, the Hermitian metric F is called *standard* or *Gauduchon*. By [10] there is a Gauduchon metric in the conformal class of every Hermitian metric on X . Fu, Wang and Wu introduce and study in [9] the following generalization of Gauduchon metrics. Let k be an integer such that $1 \leq k \leq n - 1$, a Hermitian metric F on X is called *k -th Gauduchon* if $\partial\bar{\partial}F^k \wedge F^{n-k-1} = 0$.

By definition, $(n - 1)$ -th Gauduchon metrics are the usual Gauduchon metrics. Astheno-Kähler metrics are particular examples of $(n - 2)$ -th Gauduchon metrics, and any *pluriclosed* (*SKT*) metric, i.e. a metric satisfying $\partial\bar{\partial}F = 0$, is in particular 1-st Gauduchon.

In [9] a unique constant $\gamma_k(F)$ is associated to any Hermitian metric F on X . This constant is invariant by biholomorphisms and depends smoothly on F . Moreover, it is proved that $\gamma_k(F) = 0$ if and only if there exists a k -th Gauduchon metric in the conformal class of F .

On a compact complex surface any Hermitian metric is automatically astheno-Kähler, and the balanced condition is the same as the Kähler one. In complex dimension $n = 3$ the notion of astheno-Kähler metric coincides with that of SKT metric.

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SKT or astheno-Kähler metrics on a compact complex manifold X of complex dimension $n \geq 3$ cannot be balanced unless they are Kähler (see [1, 14]). If the Lee form is exact, then the Hermitian structure is conformally balanced. By [5, 11] a conformally balanced SKT or astheno-Kähler metric whose Bismut connection has (restricted) holonomy contained in $SU(n)$ is necessarily Kähler. Similar results for 1-st Gauduchon metrics are proved in [6]. Ivanov and Papadopoulos [12] have extended these results to any generalized k -th Gauduchon metric, for $k \neq n - 1$.

A recent conjecture in [7] asserts that if X has an SKT metric and another metric which is balanced, then X is Kähler. By a result of Chiose [3] a manifold in the Fujiki class \mathcal{C} has no SKT metrics unless it is Kähler. In [8] the conjecture is studied on the class of *complex nilmanifolds* $X = (\Gamma \backslash G, J)$, i.e. on compact quotients of simply-connected nilpotent Lie groups G by uniform discrete subgroups Γ endowed with an invariant complex structure J . In this note we construct, for every $n \geq 4$, a non-SKT complex nilmanifold X of complex dimension n admitting a balanced metric and an astheno-Kähler metric which additionally satisfies the stronger condition of being k -th Gauduchon for every $1 \leq k \leq n - 1$.

2. GENERALIZED GAUDUCHON METRICS ON COMPLEX NILMANIFOLDS

We first prove the following general result.

Proposition 2.1. *Let X be a compact complex manifold of complex dimension $n \geq 3$, and F any Hermitian metric on X . For any integer k such that $1 \leq k \leq n - 1$, we have*

$$(1) \quad \int_X \partial \bar{\partial} F^k \wedge F^{n-k-1} = \frac{k(n-k-1)}{n-2} \int_X \partial \bar{\partial} F \wedge F^{n-2}.$$

Proof. The equality (1) is trivial for $k = 1$ and for $k = n - 1$. Let us then suppose that $2 \leq k \leq n - 2$. By induction one has $\partial F^k = k \partial F \wedge F^{k-1}$ and $\bar{\partial} F^k = k \bar{\partial} F \wedge F^{k-1}$. Therefore,

$$(2) \quad \partial \bar{\partial} F^k \wedge F^{n-k-1} = k \partial \bar{\partial} F \wedge F^{n-2} + k(k-1) \partial F \wedge \bar{\partial} F \wedge F^{n-3}.$$

On the other hand,

$$\begin{aligned} \partial \bar{\partial} F^k \wedge F^{n-k-1} &= d \left(\bar{\partial} F^k \wedge F^{n-k-1} \right) + \bar{\partial} F^k \wedge \partial F^{n-k-1} \\ &= d \left(\bar{\partial} F^k \wedge F^{n-k-1} \right) - k(n-k-1) \partial F \wedge \bar{\partial} F \wedge F^{n-3}, \end{aligned}$$

so we get

$$\partial F \wedge \bar{\partial} F \wedge F^{n-3} = \frac{-1}{k(n-k-1)} \left[\partial \bar{\partial} F^k \wedge F^{n-k-1} - d \left(\bar{\partial} F^k \wedge F^{n-k-1} \right) \right].$$

Now, if we substitute this expression in (2) we have

$$\partial \bar{\partial} F^k \wedge F^{n-k-1} = k \partial \bar{\partial} F \wedge F^{n-2} - \frac{k-1}{n-k-1} \partial \bar{\partial} F^k \wedge F^{n-k-1} + \frac{k-1}{n-k-1} d \left(\bar{\partial} F^k \wedge F^{n-k-1} \right),$$

which leads to

$$(n-2) \partial \bar{\partial} F^k \wedge F^{n-k-1} = k(n-k-1) \partial \bar{\partial} F \wedge F^{n-2} + (k-1) d \left(\bar{\partial} F^k \wedge F^{n-k-1} \right).$$

By Stokes theorem we arrive at (1). \square

Next we apply the previous proposition to homogeneous compact complex manifolds X , of complex dimension n , endowed with an *invariant* Hermitian metric F . We recall that in [6, Lemma 4.7] the following duality result is proved: for each $k = 1, \dots, \lfloor \frac{n}{2} \rfloor - 1$, the Hermitian metric F is k -th Gauduchon if and only if it is $(n-k-1)$ -th Gauduchon. As a consequence of Proposition 2.1, the relation among these metrics turns out to be stronger:

Proposition 2.2. *Let F be an invariant Hermitian metric on a homogeneous compact complex manifold X of complex dimension $n \geq 3$, and let k be an integer such that $1 \leq k \leq n - 2$. Then,*

- (i) F is always Gauduchon, and
- (ii) if F is k -th Gauduchon for some k , then it is k -th Gauduchon for any other k .

Proof. For any invariant Hermitian metric F and any $1 \leq k \leq n - 2$, the real (n, n) -form $\frac{i}{2} \partial \bar{\partial} F^k \wedge F^{n-k-1}$ is proportional to the volume form F^n , hence

$$(3) \quad \frac{i}{2} \partial \bar{\partial} F^k \wedge F^{n-k-1} = C_{F,k} F^n,$$

for some constant $C_{F,k} \in \mathbb{R}$ (notice that $C_{F,k}$ is a multiple of the constant $\gamma_k(F)$ in [9]).

If $k = n - 1$ then $C_{F,n-1} = 0$, i.e. F is Gauduchon, because otherwise the form F^n would be exact. Now, let k be such that $1 \leq k \leq n - 2$. From (1) and (3) we get

$$C_{F,k} \int_X F^n = \frac{i}{2} \int_X \partial \bar{\partial} F^k \wedge F^{n-k-1} = \frac{k(n-k-1)}{n-2} \frac{i}{2} \int_X \partial \bar{\partial} F \wedge F^{n-2} = \frac{k(n-k-1)}{n-2} C_{F,1} \int_X F^n,$$

that is, $\left(C_{F,k} - \frac{k(n-k-1)}{n-2} C_{F,1}\right) \int_X F^n = 0$. Therefore,

$$(4) \quad C_{F,k} = \frac{k(n-k-1)}{n-2} C_{F,1},$$

for any k such that $1 \leq k \leq n - 2$. Hence, if F is k -th Gauduchon for some k , then $C_{F,k} = 0$ and by (4) we get $C_{F,1} = 0$. Using again (4) we conclude that $C_{F,k} = 0$ for any other k , i.e. F is k -th Gauduchon for any $1 \leq k \leq n - 2$. \square

Corollary 2.3. *Let X be a homogeneous compact complex manifold of complex dimension $n \geq 3$ and let F be an invariant Hermitian metric on X . If F is SKT or astheno-Kähler, then F is k -th Gauduchon for any $1 \leq k \leq n - 1$.*

Theorem 2.4. *For each $n \geq 4$, there is a non-Kähler compact complex manifold X of complex dimension n admitting a balanced metric \tilde{F} and an astheno-Kähler metric F which is additionally k -th Gauduchon for any $1 \leq k \leq n - 1$.*

Proof. We will construct such an X using the class of complex nilmanifolds. Let $(a_1, \dots, a_{n-1}) \in (\mathbb{R} \setminus \{0\})^{n-1}$, and let $\{\omega^j\}_{j=1}^n$ be a basis of forms of type $(1,0)$ satisfying

$$(5) \quad d\omega^1 = \dots = d\omega^{n-1} = 0, \quad d\omega^n = \sum_{j=1}^{n-1} a_j \omega^{j\bar{j}}.$$

(See Remark 2.5 below for more details.) We impose the ‘‘canonical’’ metric $\tilde{F} = \frac{i}{2}(\omega^{1\bar{1}} + \dots + \omega^{n\bar{n}})$ to be balanced, i.e. $d\tilde{F}^{n-1} = 0$. This condition is equivalent to

$$(6) \quad a_1 + \dots + a_{n-1} = 0.$$

Let us now consider a generic ‘‘diagonal’’ metric

$$(7) \quad F = \frac{i}{2}(b_1 \omega^{1\bar{1}} + \dots + b_{n-1} \omega^{(n-1)\bar{(n-1)}}) + \frac{i}{2} \omega^{n\bar{n}},$$

where $b_1, \dots, b_{n-1} \in \mathbb{R}^+$.

Let $r \leq n - 1$. We denote $A_r = a_1 \omega^{1\bar{1}} + \dots + a_r \omega^{r\bar{r}}$ and $B_r = b_1 \omega^{1\bar{1}} + \dots + b_r \omega^{r\bar{r}}$. Hence, in (5) and (7) we can write $d\omega^n = A_{n-1}$ and $F = \frac{i}{2} B_{n-1} + \frac{i}{2} \omega^{n\bar{n}}$.

Let us calculate $\partial \bar{\partial} F^{n-2}$. Using that the form B_{n-1} is closed, we get

$$\begin{aligned} (-2i)^{n-2} \partial \bar{\partial} F^{n-2} &= \partial \bar{\partial} (B_{n-1} + \omega^{n\bar{n}})^{n-2} = \partial \bar{\partial} (B_{n-1})^{n-2} + (n-2) \partial \bar{\partial} ((B_{n-1})^{n-3} \wedge \omega^{n\bar{n}}) \\ &= (n-2) (B_{n-1})^{n-3} \wedge \partial \bar{\partial} (\omega^{n\bar{n}}) = -(n-2) (A_{n-1})^2 \wedge (B_{n-1})^{n-3}, \end{aligned}$$

where in the last equality we have used that $\partial\bar{\partial}(\omega^{n\bar{n}}) = \bar{\partial}\omega^n \wedge \partial\omega^{\bar{n}} = -A_{n-1} \wedge A_{n-1}$.

Therefore, F is astheno-Kähler if and only if $(A_{n-1})^2 \wedge (B_{n-1})^{n-3} = 0$.

We now use the balanced condition (6), i.e. $a_{n-1} = -a_1 - \dots - a_{n-2}$. Writing $A_{n-1} = A_{n-2} + a_{n-1} \omega^{n-1\bar{n-1}}$ and $B_{n-1} = B_{n-2} + b_{n-1} \omega^{n-1\bar{n-1}}$, and noting that $(A_{n-2})^2 \wedge (B_{n-2})^{n-3} = 0$, one has that the astheno-Kähler condition is equivalent to

$$\begin{aligned} 0 &= (A_{n-1})^2 \wedge (B_{n-1})^{n-3} = (A_{n-2} + a_{n-1} \omega^{n-1\bar{n-1}})^2 \wedge (B_{n-2} + b_{n-1} \omega^{n-1\bar{n-1}})^{n-3} \\ &= \left[(A_{n-2})^2 + 2a_{n-1} A_{n-2} \wedge \omega^{n-1\bar{n-1}} \right] \wedge \left[(B_{n-2})^{n-3} + (n-3)b_{n-1} (B_{n-2})^{n-4} \wedge \omega^{n-1\bar{n-1}} \right] \\ &= [(n-3)b_{n-1} A_{n-2} - 2(a_1 + \dots + a_{n-2}) B_{n-2}] \wedge A_{n-2} \wedge (B_{n-2})^{n-4} \wedge \omega^{n-1\bar{n-1}}. \end{aligned}$$

Let us observe that in order to simplify this equation one can take $a_1, \dots, a_{n-2} > 0$ and $b_j = a_j$ for $1 \leq j \leq n-2$. Indeed, in this case we have that $B_{n-2} = A_{n-2}$, so it is enough to choose $b_{n-1} = \frac{2}{n-3}(a_1 + \dots + a_{n-2})$ to get an astheno-Kähler metric F given by (7).

Finally, by Corollary 2.3 the metric F is in addition k -th Gauduchon for any $1 \leq k \leq n-1$. We notice that it can be directly proved that these complex nilmanifolds do not admit any SKT metric. Let us also note that the canonical bundle is holomorphically trivial, since the $(n, 0)$ -form $\Omega = \omega^{1 \dots n}$ is closed. \square

Remark 2.5. The (real) nilmanifolds in (5) correspond to the Lie algebras $\mathfrak{g} = \mathfrak{h}_{2n+1} \times \mathbb{R}$, where \mathfrak{h}_{2n+1} is the $(2n+1)$ -dimensional Heisenberg algebra. Andrada, Barberis and Dotti proved in [2, Proposition 2.2] that every invariant complex structure J on these nilmanifolds is *abelian*, i.e. $[Jx, Jy] = [x, y]$ for any $x, y \in \mathfrak{g}$. Moreover, there are exactly $\lfloor \frac{n}{2} \rfloor + 1$ complex structures up to isomorphism. Let J_0 be the complex structure defined by taking all the coefficients a_j positive numbers, i.e. $a_1, \dots, a_{n-1} > 0$ in (5). One can prove the following result: *for any J not isomorphic to J_0 , the complex nilmanifold admits a balanced metric and an astheno-Kähler metric which is k -th Gauduchon for any k .*

Remark 2.6. The complex structure in the 4-dimensional example given in [4] as well as those given in (5) are all abelian. Here we present a more general family of 4-dimensional complex nilmanifolds where the complex structure is not of that special type. Let us consider the complex structure equations

$$(8) \quad d\omega^1 = d\omega^2 = d\omega^3 = 0, \quad d\omega^4 = A\omega^{12} + B\omega^{13} + C\omega^{23} + \omega^{1\bar{1}} + \omega^{2\bar{2}} - 2\omega^{3\bar{3}},$$

where we require the coefficients A, B, C to belong to $\mathbb{Q}(i)$ in order to ensure the existence of a lattice, so that equations (8) define a complex nilmanifold. Consider a metric $F_{\alpha, \beta, \gamma}$ of the form

$$F_{\alpha, \beta, \gamma} = \frac{i}{2}(\alpha\omega^{1\bar{1}} + \beta\omega^{2\bar{2}} + \gamma\omega^{3\bar{3}} + \omega^{4\bar{4}}),$$

with $\alpha, \beta, \gamma \in \mathbb{R}^+$. On the one hand, it is easy to see that $\alpha = \beta = \gamma = 1$ provides a balanced metric. On the other hand, the astheno-Kähler condition is satisfied if and only if $\gamma = \frac{\alpha(|C|^2+4)+\beta(|B|^2+4)}{2-|A|^2} > 0$, so it suffices to take any complex structure in (8) with $|A| < \sqrt{2}$. This provides a family of 4-dimensional complex nilmanifolds $X_{A, B, C}$ with balanced and astheno-Kähler metrics which are k -th Gauduchon for any k . Notice that if $(A, B, C) \neq (0, 0, 0)$, then the Lie algebra underlying $X_{A, B, C}$ is not isomorphic to $\mathfrak{h}_7 \times \mathbb{R}$.

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