Projective Geometry (a brief introduction including projective duality and plane curves)

Trabajo de fin de grado en Matemáticas
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por *(by)*

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To SOPHIE GERMAIN,
because only algebraic objects should change its name to be considered in the dual space.

"All geometry is projective geometry"

Arthur Cayley

"In the house of mathematics there are many mansions and of these the most elegant is projective geometry. The beauty of the concepts, the logical perfection of its structure, and its fundamental role in geometry recommend the subject to every student of mathematics"

Morris Kline

Prologue

In the same way, when we are kids, we are told first about the natural numbers, then about the integers, then we discover the real numbers and finally as a completion of them, the complex numbers, in the beginning there was the linear geometry. Then, some steps later affine geometry arises, and as a completion for affine geometry, projective geometry arises.

The historical origins of projective geometry should be found in the 17th century, when Renaissance painters struggled to find a way to draw realistic representations of spatial scenes on a plane. There was some method already, discovered by Brunelleschi around 1420 and first published by Alberti in 1436. Nowadays known as *Alberti's Veil*, the artist marked on a glass screen a point where one of the light rays from the scene to the artist's eye intersected the screen, as shown in Figure 1. This method was fine for painting actual scenes, but to paint imaginary scenes in perspective some theory was required.

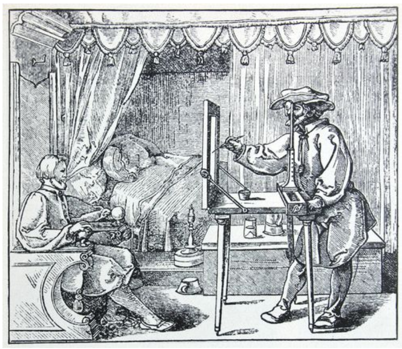

Figure 1: Dürer's depiction of Alberti's veil

The first man to supply such insight was Gérard Desargues whose motivation was to help the artists. He used the concept of "points at infinity" (vanishing points) which had already been also used by Kepler (1604). Projective geometry is born. Together with Pascal they stablish some first fundamental theorems in the subject. Nevertheless, the innovations of these men were not immediately appreciated by their fellow mathematicians. "Radical" new ideas that were contrary to Euclid such as the points at infinity where parallels meet or transformations that change lengths and angles (projections) made Desargues be called crazy and dismissed projective geometry. Every printed copy of Desargues' book *Brouillon projet d'une atteinte aux événemens des rencontres du cône avec un plan*, originally published in 1639, was lost. In the 19th century a copy of the book made by a pupil of Desargues was found by the geometer Michel Chasles and thereby the world learned the full extent of Desargues' major work.

Resumen

¿Qué es la geometría proyectiva? De pequeños, los primeros números que aprendemos son los números naturales. Conforme vamos creciendo, vamos descubriendo nuevas propiedades y nos topamos con los enteros, los reales, y por último los números complejos, como culminación de todos los anteriores. De igual modo, cuando nos sumergimos en la geometría, lo primero que aprendemos es la geometría lineal. A continuación descubrimos la geometría afín, y por último y como culminación que engloba todo lo anterior, aparece la geometría proyectiva. La geometría proyectiva nos proporciona un nuevo escenario con menos restricciones y ataduras. Todas las propiedades de la geometría lineal y la geometría afín tienen cabida en la geometría proyectiva. Es además el hábitat natural de las curvas algebraicas y una de las áreas donde la teoría de la dualidad tiene más aplicaciones.

Nuestro objetivo es introducirnos en la geometría proyectiva y usar diferentes herramientas para descubrir nuevas propiedades y conceptos proyectivos. En el primer capítulo, entenderemos qué son los espacios proyectivos, cómo se construyen y cuáles son sus propiedades fundamentales. Trataremos asimismo de familiarizarnos a trabajar en ellos. En el segundo capítulo veremos cómo se aplica la teoría de la dualidad en los espacios proyectivos y usaremos las herramientas que ésta nos proporciona para resolver problemas de geometría clásica (Teoremas de Pappus y Desargues). El último capítulo está dedicado a las curvas proyectivas, y en especial a las curvas proyectivas planas. Estudiaremos algunas propiedades de estas curvas y trataremos de entender qué es el dual de una curva plana y qué significado tiene. Realizaremos asimismo algunos ejercicios usando toda la teoría desarrollada a lo largo de los tres capítulos.

Las asignaturas de *Algebra lineal* (27000) y *Geometría lineal* (27010) me han proporcionado los conocimientos básicos que he necesitado para dar los primeros pasos en geometría proyectiva. El curso de *Topología general* (27008) me ha ayudado en todo lo referente a aplicaciones proyectivas y teoría de la dualidad. Para el estudio de las curvas proyectivas y el significado de la curva dual he empleado conceptos y teoría de las asignaturas de *Análisis Matemático II* (27006) y *Geometría de curvas y superficies* (27013). En mi estudio, me he apoyado fundamentalmente en dos textos, Audin [1] y Traves [10]. El primero me ha servido para aprender cómo funcionan los espacios proyectivos y las aplicaciones entre ellos, así como teoría básica de dualidad y cónicas. El segundo lo he empleado para la parte de curvas planas y resultados clásicos de la geometría proyectiva y las curvas. Newman [8] y Stillwell [9] son dos textos de fácil lectura que explican resultados generales de la geometría proyectiva de un modo práctico, claro y sencillo (si bien no muy riguroso). Los he utilizado para el contexto histórico y para dar puntos de vista prácticos y entender el significado de conceptos definidos de forma analítica. Debo citar por último el texto de Coxeter ([2]), un espléndido resumen de toda la teoría fundamental de los espacios proyectivos, con una gran cantidad de resultados demostrados. He recurrido a él a menudo para contrastar definiciones y para ver demostraciones de resultados que son frecuentemente enunciados en otros textos pero no probados.

A partir de un espacio vectorial E sobre un cuerpo $\mathbb{K}$ se define el *espacio proyectivo* $\mathcal{P}(E)$ como el conjunto de las clases de equivalencia sobre $E\backslash\{0\}$ bajo relación $u \sim u \iff \exists\lambda \in \mathbb{K}^\star$ tal que $u = \lambda v \; \forall u, v \in E\backslash\{0\}$. Se define de igual manera $\dim \mathcal{P}(E) = \dim E - 1$. Así, un espacio proyectivo $\mathcal{P}(E)$ de dimensión n proviene de un espacio vectorial E de dimensión $n+1$. Desde un punto de

vista afín, $\mathcal{P}(E)$ se puede entender como la unión de un hiperplano afín en E (que tendrá dimensión n) y un hiperplano proyectivo (proyección de un hiperplano vectorial de E) llamado hiperplano del infinito. Esto provoca importantes consecuencias, como por ejemplo el hecho de que dos rectas en un plano proyectivo siempre se corten (si son paralelas en el plano afín entonces se cortan en algún punto de la recta del infinito).

Al igual que sobre un espacio vectorial E, con dim $E = n$ podemos identificar cada vector con una $n-$tupla de escalares mediante la elección de una base, en el espacio proyectivo $\mathcal{P}(E)$ podemos hacer algo análogo. Dado un punto $m \in \mathcal{P}(E)$ proveniente de la recta $\tilde{m} \subseteq E$, éste se puede ver como la $n-$tupla de coordenadas de un vector $u \in E$ que genera la recta $\tilde{m}$. Obviamente, la recta $\tilde{m}$ puede ser generada por más de un vector en E, todos los múltiplos escalares de u de hecho, por lo que dos $n-$tuplas $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, $(\beta_1, \beta_2, \ldots, \beta_n)$ representan el mismo punto en $\mathcal{P}(E)$ si y solo si $\exists \lambda \in \mathbb{K}^\star$ tal que $\alpha_i = \lambda \beta_i \, \forall i$. De esta forma, el punto m puede ser representado por la clase de equivalencia de la $n-$tupla $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, que se denomina *cordenadas homogéneas* de m y se suele denotar por $[\alpha_1 : \alpha_2 : \ldots : \alpha_n]$.

Sean E y E' dos espacios vectoriales, y $p : E \backslash \{0\} \longrightarrow \mathcal{P}(E)$, $p' : E' \backslash \{0\} \longrightarrow \mathcal{P}(E')$ dos proyecciones que asocian a cada vector de E, E' el punto del espacio proyectivo correspondiente a la recta vectorial que generan en E y E' respectivamente. Una aplicación $g : \mathcal{P}(E) \longrightarrow \mathcal{P}(E')$ tal que existe un isomorfismo $f : E \longrightarrow E'$ que cumple $p' \circ f = g \circ p$ se llama *transformación proyectiva*. Las transformaciones proyectivas son el análogo a los homomorfismos en los espacios vectoriales. Sin embargo, no preservan ni los ángulos ni las distancias. Lo que sí es invariante bajo las transformaciones proyectivas es la razón doble de cuatro puntos, que se define así: dados cuatro puntos a, b, c, d alineados sobre una recta proyectiva, siendo los tres primeros distintos, la *razón doble* $[a, b, c, d]$ es $\frac{d-b}{d-a} / \frac{c-b}{c-a}$.

El concepto de dualidad en espacios proyectivos abre la puerta a importantes propiedades y herramientas en este campo. Si F es un subespacio vectorial del espacio vectorial E, definimos el *dual* de F como $F' = \{\varphi \in E^* \mid \varphi_{|F} = 0\}$, que es un subespacio de E^* de dimension dim $E-$ dim F (E^* denota el espacio dual de E). El dual de un espacio proyectivo $\mathcal{P}(E)$ se define como $\mathcal{P}(E^*)$. Consideremos un plano proyectivo $\mathcal{P}(E)$ (esto significa que dim $E = 3$). Su dual será otro plano proyectivo ya que $E \cong E^*$ y por lo tanto dim $E^* =$ dim $E = 3$. Ahora bien, el dual de una recta en $\mathcal{P}(E)$ es un punto en $\mathcal{P}(E^*)$. Asimismo, el dual de un punto en $\mathcal{P}(E)$ es una recta en $\mathcal{P}(E^*)$. Relacionamos así puntos con rectas. Notar que dos puntos generan una recta de la misma forma que dos rectas se cortan en un punto (recordar lo dicho antes de que dos rectas en el plano proyectivo siempre se cortan en un punto). Análogamente, tenemos que el dual de dos puntos $a, b \in \mathcal{P}(E)$ son dos rectas $a^*, b^* \in \mathcal{P}(E^*)$ cuyo punto de intersección $a^* \cap b^*$ es el dual de la recta en $\mathcal{P}(E)$ generada por los puntos a y b. El dual de n rectas concurrentes en $\mathcal{P}(E)$ serán n puntos alineados en $\mathcal{P}(E^*)$.

Con esta nueva teoría podemos enfrentarnos a la resolución de problemas clásicos de geometría como el Teorema de Pappus y el Teorema de Desargues. En ambos casos podemos reducir los teoremas a resultados más débiles que pueden ser probados más fácilmente.

Consideremos ahora el plano proyectivo complejo $\mathcal{P}(\mathbb{C}^3) = \mathbb{P}_2$. Definimos en él una *curva de grado* d como el conjunto $\mathcal{C} = \{[x : y : z] \mid F(x, y, z) = 0\}$ donde F es in polinomio homogéneo de grado d. Podemos ver una curva como el conjunto de puntos que la componen, pero también podemos obtenerla mediante el conjunto de rectas tangentes a dicha curva (mediante el método de la evolvente). Para cada punto de la curva existe una única recta tangente a la curva en ese punto. El dual de este conjunto de rectas será un conjunto de puntos que también forman una curva en $\mathbb{P}_2^*$. El dual de una curva proyectiva es otra curva proyectiva.

Sea $\mathcal{H}$ una curva de grado d dada por $F(x, y, z) = 0$, donde F es un polinomio homogéneo de grado d. Entonces con los ceros de cualquier polinomio de la forma $\lambda F(x, y, z)$ con $\lambda \in \mathbb{C}$ también obtenemos la misma curva $\mathcal{H}$. Hay $D = \binom{d+2}{2}$ monomios de grado d en tres variables por lo que si $(a_1, a_2, \dots, a_D)$ son los coeficientes de los monomios en tres variables de F, entonces $\mathcal{H}$ se puede identificar con el punto $[a_1 : a_2 : \dots : a_D] \in \mathbb{P}_{D-1}$. Si imponemos que la curva $\mathcal{H}$ pase por m puntos de $\mathbb{P}_2$, podemos obtener un sistema de m ecuaciones donde las incógnitas son los D coeficientes $(a_1, a_2, \dots, a_D)$. Si las ecuaciones son independientes decimos que los m puntos están en posición general. Si imponemos que $m = D - 1$ puntos en posición general pertenezcan a la curva $\mathcal{H}$, obtendremos un sistema con $D - 1$ ecuaciones y D incógnitas. La solución tendrá dimensión 1, luego contando la multiplicación por escalares, habrá una sola curva de grado d pasando por $D - 1$ puntos en posición general. De esto también podemos deducir que no hay ninguna curva de grado d que pasando D puntos en posición general.

Uno de los resultados más importantes de curvas complejas proyectivas planas es el teorema de Bézout, que sirve de base para probar otros resultados como el teorema de Chasles y el teorema de Pascal. El teorema de Pascal es una generalización del teorema de Pappus antes mencionado. También se demuestra el resultado inverso del teorema de Pascal, llamado teorema de Braikenridge-MacLaurin.

Las aplicaciones de estos resultados y la geometría proyectiva también se pueden emplear para resolver problemas que no son puramente geométricos. En álgebra, los puntos de una curva elíptica (una curva proyectiva plana de grado 3 y no singular) forman un grupo, donde tres puntos distintos suman la unidad si y solo si están alineados. Podemos demostrar que la operación de este grupo es asociativa (se trata de hecho de un grupo abeliano).

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Chapter 1

Projective Geometry

Definition. Let E be a vector space over the field $\mathbb{K}$. Let $\sim$ denote the relation

$$\mathbf{u} \sim \mathbf{v} \Longleftrightarrow \exists \lambda \in \mathbb{K}^* \text{ such that}, \mathbf{u} = \lambda \mathbf{v}, \text{ for} \mathbf{u}, \mathbf{v} \in E \backslash \{\mathbf{0}\}$$

where $\mathbb{K}^\star$ denotes $\mathbb{K}\backslash\{0\}$. Then, the *projective space* $\mathcal{P}(E)$ deduced from the vector space E is the set of all equivalence classes of $E\backslash\{\mathbf{0}\}$ under the relation $\sim$. We will say that the *dimension* of $\mathcal{P}(E)$ is $\dim E - 1$.

One may check that $\sim$ is in fact an equivalence relation over the vector space E. We will move from the vector space E to the projective space $\mathcal{P}(E)$ using the projection mappings.

Definition. Let E be a vector space over a field $\mathbb{K}$. The map $p : E\backslash\{\mathbf{0}\} \longrightarrow \mathcal{P}(E)$ given by $p(\mathbf{u}) = u$, where u is the point in $\mathcal{P}(E)$ that comes from the vector line in E generated by $\mathbf{u}$, is called a *projection*.

In case the field $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$, the space $\mathcal{P}(E)$ has also a topological structure, as it can be seen as the quotient $\mathcal{P}(E) = {}^{E\backslash\{\mathbf{0}\}}/_{\sim}$ defined so that the projection $p : E\backslash\{\mathbf{0}\} \longrightarrow \mathcal{P}(E)$ is a continuous mapping.

1.1 Affine point of view

In order to further understand how a projective space is built and what it looks like, let us analyse the construction of a projective space from an affine point of view. We may consider now $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

Consider a vector plane E over the field $\mathbb{K}$. Let (e_1, e_2) be a basis of E. Then, every element $\mathbf{u}$ in E can be written as $\mathbf{u} = \alpha e_1 + \beta e_2$, for some $(\alpha, \beta) \in \mathbb{K} \times \mathbb{K}$ called coordinates. Lines in E can be defined as sets $\{(x,y) \in \mathbb{K} \times \mathbb{K} : ax + by = 0\}$, where (x,y) are the coordinates of the points on the line and $a, b \in \mathbb{K}$ are two arbitrary elements, one of which is at least different from 0. This set can also be written as $\{\lambda(-b,a) : \lambda \in \mathbb{K}\}$. For instance, the line passing through $\mathbf{u} = (\alpha, \beta)$ would be $l_{\mathbf{u}} \equiv \{\lambda(\alpha, \beta) : \lambda \in \mathbb{K}\}$. The set of all lines in E can be expressed as

$$\begin{aligned} \{\lambda(a,b) : a, b \in \mathbb{K}\} &= \{\lambda(a/b, 1) : a, b \in \mathbb{K}, b \neq 0\} \cup \{\lambda(a,0) : a \in \mathbb{K}\} \\ &= \underbrace{\{\lambda(a,1) : a \in \mathbb{K}\}}_{\text{points in the line } y=1} \cup \underbrace{\{\lambda(1,0)\}}_{\text{point at } \infty}. \end{aligned}$$

Recall that the set of all lines in E is the set of points in $\mathcal{P}(E)$. Hence, we have split the set of points in $\mathcal{P}(E)$ into two different sets. The points in the first set, correspond to the lines that intersect the affine line $y = 1$ (coloured in green in Figure 1.1). We take this intersection point as the representative for the equivalence class of the line. On the other hand, the line $x = 0$ does not cut off the affine line $y = 1$ (red line in Figure 1.1). Notice that this is the only line that does not intersect with $y = 1$. We take as a representative of this line the point with coordinates $(1,0)$. Then, $\mathcal{P}(E)$ can be seen as the affine

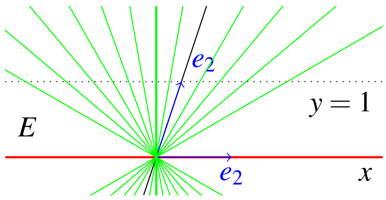


Figure 1.1: All lines cross $y=1$ but the x-axis one.

line $y=1$ plus the point $(1,0)$. This extra point is usually referred to as *point at infinity*. Hence, a one-dimensional projective space can be seen as an affine line plus a point (at infinity).

Let us consider now a 3-dimensional vector space H over the field $\mathbb{K}$. We choose again a basis of H and denote $(x,y,z)\in\mathbb{K}^3$ the coordinates of the vectors in this basis. As in the previous case, all the lines in H will intersect the affine plane $F: z=1$, except for the lines contained in the plane $F': z=0$. We may then split again the set of lines in H as

$$\{\lambda(a,b,c): a,b,c\in\mathbb{K}\} = \underbrace{\{\lambda(a,b,1): a,b\in\mathbb{K}\}}_{\text{points in the plane } z=1} \cup \underbrace{\{\lambda(a,b,0): a,b\in\mathbb{K}\}}_{\text{points in the line at } \infty}.$$

Therefore, a two-dimensional projective space can be seen as the union of an affine plane, and the set of lines in a plane, which is in fact a projective line (as we have just already seen). This projective line is called *line at infinity*.

For the general case, where G is an $(n+1)$-dimensional vector space, an analogous procedure can be followed to derive that the projective space $\mathcal{P}(G)$ can be seen as the union of an affine hyperplane and a projective hyperplane called the *hyperplane at infinity*.

We go back to the previous case and study how and where lines in the projective plane intersect. Recall that the projective plane $\mathcal{P}(H)$ is obtained from a vector space H, with $\dim H=3$. Lines in the projective space come from vector planes in H. We see the projective plane $\mathcal{P}(H)$ as the union of the affine plane $z=1$ and a line at infinity. Given a line $r\subseteq\mathcal{P}(H)$, there are two possibilities: 1) r intersects the affine plane $z=1$ (so r is not the line at infinity). In this case, r is obtained from the vector plane in E containing r (this plane is well defined since there is just one plane going through the origin and the line r). So, if we call this plane $\tilde{r}$, we denote $r=p(\tilde{r})$, where p is the projection map. 2) r does not intersect the affine plane $z=1$. Then, r is the line at infinity and it comes from the vector plane $z=0$.

Let $r,s\in\mathcal{P}(H)$ be two projective lines. There are two options:

- Both r,s intersect the affine plane $z=1$. If these lines meet also in the affine plane, then the intersection point would be that one. This is showed in Figure 1.2, the intersection between the two planes $\tilde{r},\tilde{s}$ that generate the lines r,s respectively is a line (coloured in green) passing through this intersection point. In case they do not meet in the affine plane (they are parallel), we consider again the vector plane that generate each line, $\tilde{r},\tilde{s}$. Since the lines are parallel, these planes will intersect over a line $\tilde{t}$ contained in the plane $z=0$ (coloured in green). This line produces a projective point $t=p(\tilde{t})$ which lies in the line at infinity. So r and s meet at some point in the line at infinity.

- The line r intersects the affine plane $z=1$ and s is the line at infinity. Then, consider again the vector planes the lines come from, called $\tilde{r}$ and $\tilde{s}$. The plane $\tilde{r}$ would be the vector plane going through r, and $\tilde{s}$ the plane $z=0$. Clearly $\tilde{r}\cap\tilde{s}$ is a line $\tilde{t}$ which contained in the $z=0$ plane itself (coloured in green). Hence, the intersection point $t=p(\tilde{t})$ is a point lying on the line at infinity.

Remark. In a projective plane, the choice of the line at infinity depends on the choice of coordinates, which is not a canonical definition. For a particular choice of coordinates any line can become the line

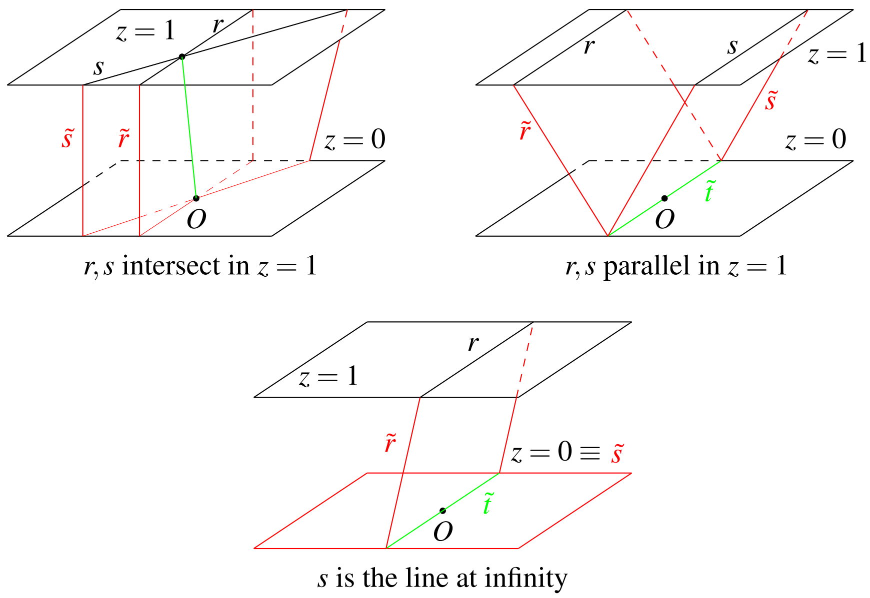


Figure 1.2: Intersection of two lines in the projective plane

at infinity. This is extended to projective spaces in general. We can always find a basis for which any given hyperplane is the hyperplane at infinity.

1.2 Projective transformations and frames

Definition. Let E_1 and E_2 be two vector spaces over a field $\mathbb{K}$, and consider the two projections $p : E_1 \backslash \{\mathbf{0}\} \longrightarrow \mathcal{P}(E_1)$, $p' : E_2 \backslash \{\mathbf{0}\} \longrightarrow \mathcal{P}(E_2)$. A *projective transformation* $g : \mathcal{P}(E_1) \longrightarrow \mathcal{P}(E_2)$ is a mapping such that there exists a linear isomorphism $f : E_1 \longrightarrow E_2$ with $p' \circ f = g \circ p$. Sometimes it is also called a *projectivity*. A projective transformation of a projective line is called a *homography*.

An equivalent definition is that the diagram

$$\begin{array}{ccc} E_1\backslash\{\mathbf{0}\} & \xrightarrow{f} & E_2\backslash\{\mathbf{0}\} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ \mathcal{P}(E_1) & \xrightarrow{g} & \mathcal{P}(E_2) \end{array}$$

commutes.

Remark. What we are doing with this new concept of projection is a general mathematical description for the change of the fixed view point. In the case $E_1 = E_2$, recall that a projective subspace in $\mathcal{P}(E_1)$ is just the result of seeing a vector subspace of a greater dimension (since it 'lives' in E_1) with the eyes in a one lower dimension. It is clear that while looking at the same object, what we see changes if we change the position of the eyes all over the space. Therefore, it should not be strange to relate all of these different 'pictures' of the same object. This is in fact what projective transformations do. Two objects in $\mathcal{P}(E_1)$ for which there is a projective transformation going from one to the other are actually two different points of view in a lower dimension space of the same object in E_1.

Let E be a vector space over a field $\mathbb{K}$ with $\dim E = n+1$. The same way we represent a vector in E by its coordinates in a given basis of E, we can do it for the projective space $\mathcal{P}(E)$. Given a point $m \in \mathcal{P}(E)$, it can be described as the $(n+1)$-tuple of coordinates of a vector $\mathbf{u} \in E$ which generates the line $\tilde{m}$, where $m = p(\tilde{m})$. Obviously, the same line $\tilde{m}$ can be generated by more than one vector in E, and hence two different $(n+1)$-tuples, $(\alpha_1, \alpha_2, \ldots, \alpha_{n+1}), (\beta_1, \beta_2, \ldots, \beta_{n+1})$ say, will be representing the same point in $\mathcal{P}(E)$ if and only if there exists a non-zero scalar $\lambda \in \mathbb{K}^\star$ such that $\alpha_i = \lambda \beta_i \ \forall i$. The equivalent class of $(\alpha_1, \alpha_2, \ldots, \alpha_{n+1})$ is called a set of *homogeneous coordinates* for m. It is often denoted $[\alpha_1 : \alpha_2 : \ldots : \alpha_{n+1}]$. They were first invented by Möbius and Plücker (see [9, Chapter 8]).

Definition. If E is a vector space of dimension $n+1$, a *projective frame* of $\mathcal{P}(E)$ is a system of $n+2$ points $(m_0, m_1, \ldots, m_{n+1})$ of $\mathcal{P}(E)$ such that $m_1, \ldots, m_{n+1}$ are the images of the vectors $e_1, \ldots, e_{n+1}$, which are a basis of E and m_0 is the image of $e_1 + e_2 + \ldots + e_{n+1}$.

There is an important relation between projective transformations and projective frames which will support later definitions on this area.

Proposition 1.1. *Let $\mathcal{P}(E_1)$ and $\mathcal{P}(E_2)$ be two projective spaces of dimension n. Any projective mapping from $\mathcal{P}(E_1)$ to $\mathcal{P}(E_2)$ maps a projective frame of $\mathcal{P}(E_1)$ onto a projective frame of $\mathcal{P}(E_2)$. Moreover, if $(m_0, \ldots, m_{n+1})$ and $(m'_0, \ldots, m'_{n+1})$ are projective frames of $\mathcal{P}(E_1)$ and $\mathcal{P}(E_2)$ respectively, then there exists a unique projective transformation $g : \mathcal{P}(E_1) \longrightarrow \mathcal{P}(E_2)$ such that $m'_i = g(m_i)$ for all i.*

It is not hard to prove using isomorphisms. A short and simple proof can be found in Audin ([1, Proposition 5.6]). Notice that over a projective line, three distinct points form a projective frame. This fact, together with Proposition 1.1, motivates the following definition.

Definition. Let a, b, c be three different points on a projective line D. Then, there exists a unique projective mapping, g, from the line to $\mathbb{K} \cup \{\infty\}$ such that $g(a) = \infty$, $g(b) = 0$, $g(c) = 1$. If d is another point of D, then $g(d)$ is called the *cross-ratio* of (a, b, c, d) and is denoted by $[a, b, c, d]$.

The cross-ratio answers a natural question first raised by the Renaissance man Alberti ([9]): since length and angle are not preserved by projection, what is? The cross-ratio is in fact a projective invariant of four ordered points. We formalize this in the next proposition.

Proposition 1.2. *Let a_1, a_2, a_3, a_4 be four points on a projective line D (the first three being distinct) and a'_1, a'_2, a'_3, a'_4 be four points of a line D' (the first three being distinct). Then, there exists a homography $f : D \longrightarrow D'$ such that $f(a_i) = a'_i$ if and only if $[a_1, a_2, a_3, a_4] = [a'_1, a'_2, a'_3, a'_4]$.*

Proof. First, let $f : D \longrightarrow D'$ be a homography such that $f(a_i) = a'_i$. We define now the projective homography $g : D' \longrightarrow \mathbb{K} \cup \{\infty\}$ such that $g(a'_1) = \infty$, $g(a'_2) = 0$, $g(a'_3) = 1$. Then, by the definition of cross-ratio, we have that $[a'_1, a'_2, a'_3, a'_4] = g(a'_4)$. Consider now the map $h = g \circ f : D \longrightarrow \mathbb{K} \cup \{\infty\}$. Then,

$$\left.\begin{array}{r} h(a_1) = g(f(a_1)) = g(a'_1) = \infty \\ h(a_2) = g(f(a_2)) = g(a'_2) = 0 \\ h(a_3) = g(f(a_3)) = g(a'_3) = 1 \end{array}\right\} \overset{def}{\Longrightarrow} \quad [a_1, a_2, a_3, a_4] = h(a_4).$$

Summing up, we obtain the equality

$$[a_1, a_2, a_3, a_4] = h(a_4) = g(f(a_4)) = g(a'_4) = [a'_1, a'_2, a'_3, a'_4].$$

For the converse, notice that the three distinct points a_1, a_2, a_3 are a projective frame of D, and a'_1, a'_2, a'_3 are also a projective frame of D'. Then, we apply Proposition 1.1 to state that there is a unique homography $f : D \longrightarrow D'$ such that $f(a_i) = a'_1$, $i = 1, 2, 3$. The only thing we have left is to check that $f(a_4) = a'_4$. Let $t : D' \longrightarrow \mathbb{K} \cup \{\infty\}$ be such that

$$\left.\begin{array}{r} t(a'_1) = \infty \\ t(a'_2) = 0 \\ t(a'_3) = 1 \end{array}\right\} \overset{def}{\Longrightarrow} \quad t(a'_4) = [a'_1, a'_2, a'_3, a'_4].$$

On the other hand, we have

$$\left.\begin{array}{r} t\circ f(a_1)=t(f(a_1))=t(a_1')=\infty \\ t\circ f(a_2)=t(f(a_2))=t(a_2')=0 \\ t\circ f(a_3)=t(f(a_3))=t(a_3')=1 \end{array}\right\} \overset{def}{\Longrightarrow} \quad t\circ f(a_4)=[a_1,a_2,a_3,a_4].$$

Our hypothesis now is $[a_1',a_2',a_3',a_4']=[a_1,a_2,a_3,a_4]$, so $t(a_4')=t(f(a_4))$. But we also have that $t(a_i')=t(f(a_i))\ \forall i=1,2,3$. Hence, we can apply the direct part of the proposition to the homography t and conclude that $[a_1',a_2',a_3',a_4']=[f(a_1),f(a_2),f(a_3),f(a_4)]$, and hence $[a_1',a_2',a_3',a_4']=[a_1',a_2',a_3',f(a_4)]$. Therefore $f(a_4)=a_4'$. □

Our definition of cross-ratio may seem a bit strange compared with the definition given in most of books and papers, which is the following.

Definition. The *cross-ratio* of four collinear points a,b,c,d, the first three being distinct is

$$[a,b,c,d]=\frac{d-b}{d-a}\bigg/\frac{c-b}{c-a}$$

Proposition 1.3. *The two definitions given for the cross-ratio of four collinear points are equivalent.*

Proof. Assuming the general convention that identifies $\frac{k}{0}$ with ∞ $\forall k\neq 0$, then the homography given by

$$z\longmapsto \frac{z-b}{z-a}\bigg/\frac{c-b}{c-a}$$

is the only one that maps a to ∞, b to 0 and c to 1. Therefore, its image for d would be the cross-ratio of the points a,b,c,d, which is exactly the formula given for the cross-ratio in the second definition. □

1.3 Topology of the real projective plane

Let us take a look into the topology of the real projective plane. For now on, we will write $\mathbb{RP}_2$ to denote $\mathcal{P}(\mathbb{R}^3)$.

Our vector space is now $\mathbb{R}^3$ over the field $\mathbb{R}$. We consider the canonical basis over $\mathbb{R}^3$. Notice that all (vector) lines in $\mathbb{R}^3$ intersect the sphere S_2=$\{(x,y,z)\in\mathbb{R}^3\,|\,x^2+y^2+z^2=1\}$, in two points. For instance, if a line intersects S_2 at the point (a,b,c), then the line can be written as the set of point of the form $\lambda(a,b,c)$, for any $\lambda\in\mathbb{R}$. Hence, the other intersection point at S_2 would be $(-a,-b,-c)$. So for any line in $\mathbb{R}^3$, there are two intersection points with S_2. We now identify these points to get a complete set of representative for any line in $\mathbb{R}^3$. Hence, we get

$$\mathbb{RP}_2=\{(x,y,z)\in\mathbb{R}^3\,|\,x^2+y^2+z^2=1\}/(x,y,z)\sim(-x,-y,-z).$$

The relation $\sim$ is an equivalence relation and therefore $\mathbb{RP}_2$ can be seen as a hemisphere where the antipodal points at the boundary are identified. Moreover, this hemisphere is homeomorphic with a disk where points at the boundary are identified with its antipodes. Consider the mapping

$$\begin{array}{cccc} \Phi: & \mathbb{RP}_2 & \longrightarrow & \{(a,b,0)\mid a^2+b^2\le 1\} \\ & (x,y,z) & \longmapsto & (x,y,0) \end{array}$$

which is well-defined, is continuous and has inverse

$$\begin{array}{cccc} \Phi^{-1}: & \{(a,b,0)\mid a^2+b^2\le 1\} & \longrightarrow & \mathbb{RP}_2 \\ & (x,y,0) & \longmapsto & (x,y,+\sqrt{1-x^2-y^2}) \end{array}$$

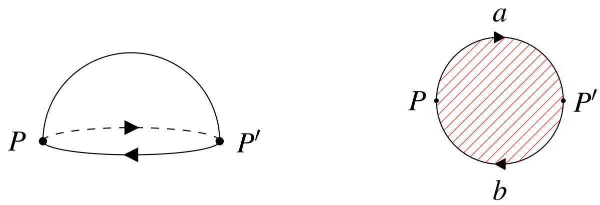


Figure 1.3: The points P and P' are identified, as well as the semi-circumferences a and b following the direction of the arrows. We may then say $P \equiv P'$, $b \equiv a$.

which is also well-defined since $x^2 + y^2 \leq 1 \Longrightarrow 1 - x^2 - y^2 \geq 0$, and it is continuous. Therefore, $\mathbb{RP}_2$ can be seen as a disk with the antipodal identification at the boundary circumference (see Figure 1.3). We cut off a disk centred at the point P. Since the points at the boundary are identified, the disk will look like the green part in the left object of Figure 1.4. If we get rid of the disk, we are left with the central object in Figure 1.4. This is in fact a strip, were points in the two edges denoted by $\hat{a}$ (coming from the former semi-circumference a) are identified following the arrows. The last object is just another way to draw the same thing, where it is easier to see that this object is a Möbius strip.

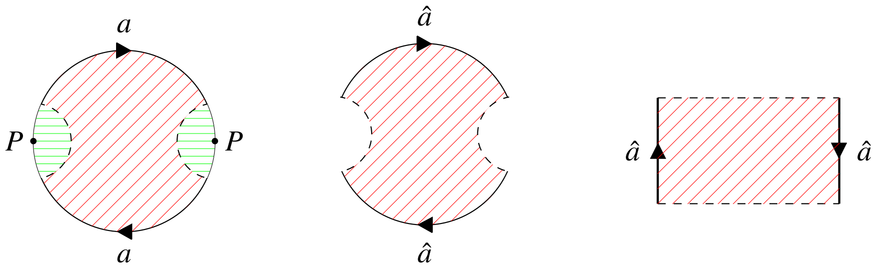


Figure 1.4: If we remove a disk (centred at P for instance) from the original disk we obtain a Möbius strip.
t

Hence, it follows that a Möbius strip together with a disk (the one we removed which was centred in P) is homeomorphic to $\mathbb{RP}_2$. The disk should be place around the dashed lines in the right picture of Figure 1.4, since it was the place where it was firstly removed. This was first noticed and proved by Klein in 1874 ([6]), and we state it formally in the following corollary:

Corollary. $\mathbb{RP}_2$ is obtained by gluing a disk and a Möbius strip along their boundaries.

Chapter 2

Projective Duality

The french mathematicians J. D. Gergonne and J. V. Poncelet may be called the founders of Projective Duality ([4, page 141]). They both developed this subject spreading through two different paths. Gergonne focused over the duality of points and lines whereas Poncelet was more interested in the dual of conics and curves.

Definition. Let E be a vector space over a field $\mathbb{K}$. The *dual* space of E is defined as

$$E^* = \{\varphi : E \longrightarrow \mathbb{K} \mid \varphi \text{ linear}\}.$$

The dual space E^* is in fact another vector space over the field $\mathbb{K}$. Taking basis for E and E^* as will be explained in the proof of Proposition 2.2, it is easy to see that both vector spaces are isomorphic. Therefore, it makes sense to define the dual of the projective space $\mathcal{P}(E)$ as the projective space $\mathcal{P}(E^*)$. We take a further step to show the following result.

Lemma 2.1. $E = (E^*)^*$

Proof. We prove the statement for fields $\mathbb{K}$ with 0-characteristic. Let us consider the following mapping,

$$\begin{array}{rccc} h: & E & \longrightarrow & (E^*)^* \\ & \mathbf{u} & \longmapsto & h_{\mathbf{u}} \end{array} \quad \text{where} \quad \begin{array}{rccc} h_{\mathbf{u}}: & E^* & \longrightarrow & \mathbb{K} \\ & \varphi & \longmapsto & \varphi(\mathbf{u}) \end{array}.$$

We will prove that h is a natural isomorphism. First, note that h is a homomorphism: let $\mathbf{u},\mathbf{v} \in E$, then $h_{\mathbf{u}+\mathbf{v}}(\varphi) = \varphi(\mathbf{u}+\mathbf{v}) = \varphi(\mathbf{u})+\varphi(\mathbf{v}) = h_{\mathbf{u}}(\varphi)+h_{\mathbf{v}}(\varphi) = (h_{\mathbf{u}}+h_{\mathbf{v}})(\varphi)$, and $h_{\lambda\mathbf{u}}(\varphi) = \varphi(\lambda\mathbf{u}) = \lambda\varphi(\mathbf{u}) = \lambda h_{\mathbf{u}}(\varphi) = (\lambda h_{\mathbf{u}})(\varphi)\ \forall\lambda \in \mathbb{K}$. Since $E \cong E^*$, we deduce dim E = dim E^*, and this also implies dim E = dim $(E^*)^*$. Therefore, it is enough to show that h is injective. Over a basis of E^*, we can define the dot product $\langle,\rangle$ of two vectors by the multiplication of their coordinates. Then, the mapping $\theta_{\mathbf{v}} : E \longrightarrow \mathbb{K}$, given by $\theta(\mathbf{w}) = \langle\mathbf{v},\mathbf{w}\rangle\,\forall\mathbf{w} \in E$, is a linear mapping for all $\mathbf{v} \in E$. Let $\mathbf{u} \in E$ be such that $h_{\mathbf{u}} = \widehat{0}$, where $\widehat{0}$ is the null map in $(E^*)^*$ (this means $\widehat{0}(\varphi) = 0\forall\varphi \in E^*$). Therefore, $\widehat{0}(\theta_{\mathbf{v}}) = \langle\mathbf{u},\mathbf{v}\rangle = 0\forall\mathbf{v} \in E$, and this can only happen if $\mathbf{u} = 0$. Hence, $ker\,\varphi = 0$ and h is injective. □

Proposition 2.2. *Let F be a vector subspace in a vector space E (over the field $\mathbb{K}$). Then, the subset $F' = \{\varphi \in E^* \mid \varphi_{|_F} = 0\}$ is a vector subspace of E^* of dimension dim $E-$ dim F (i.e. the codimension is dim F). We will call F' the dual subspace of F.*

Proof. Again this proof is for fields with 0-characteristic. It is clear that $F' \subseteq E^*$ since it is formed from elements in E^*. The null map, φ_0 is in F' since $\varphi_0(\mathbf{u}) = 0\forall\mathbf{u} \in E$, so in particular this holds for $\mathbf{v} \in F \subseteq E$. Let φ, $\psi \in F'$, then $(\varphi+\psi)(\mathbf{v}) = \varphi(\mathbf{v})+\psi(\mathbf{v}) = 0+0 =\ 0\forall\mathbf{v} \in F$. If $\lambda \in \mathbb{K}$, then $(\lambda\varphi)(\mathbf{x}) = \lambda\varphi(\mathbf{x}) = \lambda 0 = 0\forall\mathbf{x} \in F$. Hence, F' is a subspace of E^*. Now, let dim $F = m$, dim $E = n$. Consider the dot product $\langle\, ,\rangle$ in the vector space E given by the multiplication of the coordinates in a given basis. Let us take an orthogonal basis of F and complete it so that we end up with an orthogonal basis of E. In other words, we have $(e_1,e_2,\dots,e_m,e_{m+1},\dots,e_n)$ a basis of E where $e_i \in F$ for $i = 1,\dots,m$, $e_i \in E\backslash F$ for

$i = m+1,\ldots,n$ and $\langle e_i, e_j\rangle = 0$ for $i \neq j$. For each e_i, consider $\varphi_{e_i} : E \longrightarrow \mathbb{K}$, given by the dot product in E, i.e. $\varphi_{e_i}(\mathbf{u}) = \langle e_i, \mathbf{u}\rangle\, \forall \mathbf{u} \in E$. We claim $(\varphi_{e_1},\ldots,\varphi_{e_n})$ is a basis of E^*. Indeed, for if $\lambda_1\varphi_{e_1} + \lambda_2\varphi_{e_2} + \ldots + \lambda_n\varphi_{e_n} = \varphi_0$ for some $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{K}$, then $\lambda_1\langle e_1, \mathbf{u}\rangle + \lambda_2\langle e_2, \mathbf{u}\rangle + \ldots + \lambda_n\langle e_n, \mathbf{u}\rangle = 0\,\forall \mathbf{u} \in E$, and therefore by the linearity of the dot product this implies, $\langle \lambda_1 e_1 + \lambda_2 e_2 + \ldots + \lambda_n e_n, \mathbf{u}\rangle = 0\,\forall \mathbf{u} \in E$, which means, $\lambda_1 e_1 + \lambda_2 e_2 + \ldots + \lambda_n e_n = 0$, and since $(e_1, e_2, \ldots, e_n)$ is a basis of E, this can only happen if $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$. This proves that $(\varphi_{e_1},\ldots,\varphi_{e_n})$ is a basis of E^*. For $i = 1,\ldots,m$, $\varphi_{e_i} \notin F'$ since $\varphi_{e_i}(e_i) = \langle e_i, e_i\rangle \neq 0$, and $e_i \in F\, \forall i = 1,\ldots,m$. On the other hand, let $\mathbf{v} \in F$. Then $\mathbf{v} = \alpha_1 e_1 + \ldots + \alpha_m e_m$, for some $\alpha_1,\ldots,\alpha_m \in \mathbb{K}$. So, for $i = m+1,\ldots,n$, $\varphi_{e_i}(\mathbf{v}) = \langle e_i, \alpha_1 e_1 + \ldots + \alpha_m e_m\rangle = \alpha_1\langle e_1, e_i\rangle + \ldots + \alpha_m\langle e_m, e_i\rangle = 0$. Hence, $(\varphi_{e_{m+1}},\ldots,\varphi_{e_n})$ are all in F' and moreover, they are a basis of F'. This implies that dim $F' = n - m =$ dim $E -$ dim F. □

The dual space gives us a new perspective to approach classic geometry results. All the things that at this point we know that work on a vector (or projective) space can be translated to the dual space (and its projective space) and we may find new properties and relations. This connexion between vector and projective spaces and its dual space where things in the first one are translated into its dual space is sometimes called *metamorphosis*. Let us give some examples to illustrate how this works.

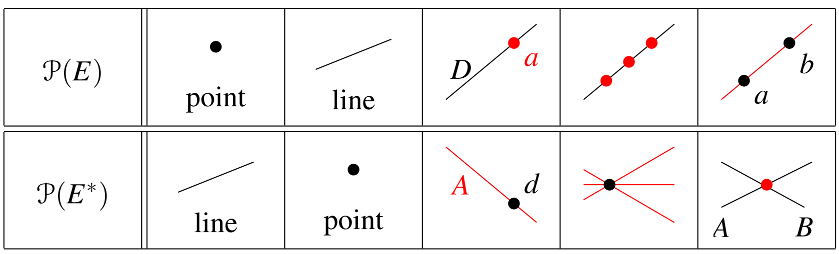
Take for instance a 3-dimensional vector space E, over a field $\mathbb{K}$. A line $\tilde{d} \in E$ defines a plane $\tilde{d}^* \in E^*$ (since dim $\tilde{d} = 1$ and dim $\tilde{d}^* = 3 - 1 = 2$). But remember that a line $\tilde{d}$ in E produces a point d in $\mathcal{P}(E)$, and also a plane $\tilde{d}^*$ in E^* produces a line d^* in $\mathcal{P}(E^*)$. Hence, a point d in $\mathcal{P}(E)$ defines a line d^* in $\mathcal{P}(E^*)$. On the other hand, a plane $\tilde{P} \in E$ defines a line $\tilde{P}^* \in E^*$, since dim $\tilde{P} = 2$, and dim $\tilde{P}^* = 3 - 2 = 1$. Again, since a plane $\tilde{P}$ in E generates a line P in $\mathcal{P}(E)$ and a line $\tilde{P}^*$ in E^* generates a point P^* in $\mathcal{P}(E^*)$, we conclude that a line P in $\mathcal{P}(E)$ produces a point P^* in $\mathcal{P}(E^*)$.

Proposition 2.3. *Let F, G be two vector subspaces in a vector space E. Then, $F \subseteq G \Longleftrightarrow G' \subseteq F'$.*

The proof is a direct consequence from the last two results. Let us compute a shortcut to go from $\mathcal{P}(E)$ to $\mathcal{P}(E^*)$ and vice versa. Let $V \subseteq \mathcal{P}(E)$ be a projective subspace where dim $\mathcal{P}(E) = n$, and dim $V = k$. Then, codim $V = n - k$. When moving to the vector space E, V becomes a vector subspace $\tilde{V} \subseteq E$, with dim $\tilde{V} = k+1$. Therefore, since dim $E = n+1$, codim $\tilde{V} = n - k$, which was the same as the codimension for V. Hence, the codimension is preserved when moving from the projective space to the original vector space. Now, we move forward, to the dual space E^*. We obtain a vector subspace $\tilde{V}^* \subseteq E^*$, dim $\tilde{V}^* = n - (k+1)$. Since dim $E^* = n$, we follow codim $\tilde{V}^* = k+1$. Therefore, in this step, we kind of interchange the codimension with the dimension to move from one place to the other. Finally, we move to $\mathcal{P}(E^*)$, and we get the subspace V^*, with dim $V^* = n - (k+1)$. Since dim $\mathcal{P}(E^*) = n$, we conclude that codim $V^* = k+1$. Reading from the beginning, the dual of a vector subspace $V \subseteq E$, dim $V = k$, turns out to become a subspace $V^* \subseteq E^*$ with codim $V^* = k+1$.

$\mathcal{P}(E)$	E	E^*	$\mathcal{P}(E^*)$
dim V = k codim $V = n-k$	dim $\tilde{V} = k+1$ codim $\tilde{V} = n-k$	dim $\tilde{V}^* = n-k$ codim $\tilde{V}^* = k+1$	dim $V^* = n-(k+1)$ **codim $\mathbf{V^*} = \mathbf{k+1}$**

Corollary [*Metamorphosis*]. Over a 3-dimensional vector space E, the following objects are equivalent:



Working over $\mathbb{C}^3$ now, let $\mathcal{P}(\mathbb{C}^3) = \mathbb{P}_2$. There is a really quick way to compute the intersection point between two lines in $\mathbb{P}_2$. Considering coordinates in the canonical basis, we take two different lines $a_1x+b_1y+c_1z=0$, and $a_2x+b_2y+c_2z=0$. They both come from two different vector planes in $\mathbb{C}^3$. Hence, they meet at a line. If $c_1=c_2=0$, then $a_1b_2-a_2b_1 \neq 0$ since the planes are different and they meet at the line $\lambda(0,0,1)$. For $c_1=0$, $c_2 \neq 0$, then they meet at the line $\lambda(b_1c_2, -c_2a_1, a_2b_1-b_2a_1)$ for $a_1 \neq 0$, or $\lambda(c_2,0,-a_2)$ for $a_1=0$. Finally, if $c_1 \neq 0 \neq c_2$, then they meet at the line $\lambda(c_2b_1-c_1b_2\,, c_1a_2-a_1c_2\,, a_1b_2-a_2b_1)$. In all the cases the following result is satisfied.

Lemma 2.4. *The projective lines $a_1x+b_1y+c_1z=0$, $a_2x+b_2y+c_2z=0$ in $\mathbb{P}_2$ meet at the point $[a_3:b_3:c_3]$ given by $(a_3,b_3,c_3)=(a_1,b_1,c_1)\times(a_2,b_2,c_2)$.*

If $ax+by+cz=0\,, a,b,c\in\mathbb{C}$ is a line in $\mathbb{P}_2$, then $ax+by+cz=0$ is a vector plane in $\mathbb{C}^3$. We know that in $(\mathbb{C}^3)^*$ this plane will become a line, but how can we write equations for this line? On the one hand, we can refer to it as $\lambda\varphi$, where φ is the linear map given by

$$\begin{array}{rccc} \varphi: & \mathbb{C}^3 & \longrightarrow & \mathbb{C} \\ & (x,y,z) & \longmapsto & ax+by+cz \end{array}.$$

By abuse of notation, we may just write it as $\lambda(ax+by+cz)$, being aware that the variables x,y,z refer to the coordinates in the canonical basis over $\mathbb{C}^3$ (and NOT over $(\mathbb{C}^3)^*$, since we have not properly defined a canonical basis here!). Then, the point in $\mathbb{P}_2^*$ can be written as $ax+by+cz$ $(\equiv\varphi)$ noting the same as before. But, on the other hand, we can rewrite this expressions in a better way considering a basis in $(\mathbb{C}^3)^*$ and taking coordinates in this basis.

Following the construction in the proof of Proposition 2.2 over the canonical basis on $\mathbb{C}^3$, we define the canonical basis over $(\mathbb{C}^3)^*$ to be the mappings $\{\varepsilon_1,\varepsilon_2,\varepsilon_2\}$ given by

$$\begin{array}{rccc} \varepsilon_i: & \mathbb{C}^3 & \longrightarrow & \mathbb{C} \\ & (u_1,u_2,u_3) & \longmapsto & u_i, \end{array}$$

for $i=1,2,3$. This way, $\varphi = a\varepsilon_1+b\varepsilon_2+c\varepsilon_3$, and hence the line $\lambda\varphi$ can be expressed as $\lambda(a,b,c)$, where (a,b,c) are now coordinates with respect to the canonical basis in $(\mathbb{C}^3)^*$.

We have a nice way to find the intersection point between two projective lines in $\mathbb{P}_2$, but what about an easy way to compute the equation of the projective line through two points in $\mathbb{P}_2$? We just need to change into the dual space and apply what we already know.

Let $[a_1:b_1:c_1], [a_2:b_2:c_2]$ be two points in $\mathbb{P}_2$. We can see them as lines in $\mathbb{P}_2^*$. The point $a=[a_1:b_1:c_1]$ comes from the line $\tilde{a}=\lambda(a_1,b_1,c_1)\in\mathbb{C}^3$, which can also be written as

$$\tilde{a}:\left\{\begin{array}{l} a_1y-b_1x=0 \\ c_1y-b_1z=0 \end{array}\right. \quad\Longrightarrow\quad \tilde{a}^*:\ \alpha(-b_1,a_1,0)+\beta(0,c_1,-b_1),\ \alpha,\beta\in\mathbb{C}$$

and lead to a plane $\tilde{a}^*$ in $(\mathbb{C}^3)^*$ which can be written as $a_1X+b_1Y+c_1Z=0$, where X,Y,Z denotes coordinates in the canonical basis of $(\mathbb{C}^3)^*$. Following this path with $[a_2:b_2:c_2]$ we are able to write the correspondent lines in $\mathbb{P}_2^*$ as

$$a_1X+b_1Y+c_1Z=0, \qquad a_2X+b_2Y+c_2Z=0.$$

We use now the formula we already know for the intersection point of two lines (Lemma 2.4) to obtain the meeting point of these two lines.

$$(a_1,b_1,c_1)\times(a_2,b_2,c_2)=(\eta_1,\eta_2,\eta_3), \qquad \begin{array}{l} \eta_1=b_1c_2-b_2c_1 \\ \eta_2=c_1a_2-a_1c_2 \\ \eta_3=b_2a_2-b_1a_2 \end{array}$$

The point $[\eta_1 : \eta_2 : \eta_3] \in \mathbb{P}_2^*$ comes from the line $\lambda(\eta_1, \eta_2, \eta_3)$ in $(\mathbb{C}^3)^*$, and this from the plane $\eta_1 x + \eta_2 y + \eta_3 z = 0$ in $\mathbb{C}^3$, whose projective line can be written as $\eta_1 x + \eta_2 y + \eta_3 z = 0$ in $\mathbb{P}_2$ which is in fact the line through the points $[a_1 : b_1 : c_1]$, $[a_2 : b_2 : c_2]$.

Lemma 2.5. *The projective line through the points* $[a_1 : b_1 : c_1]$, $[a_2 : b_2 : c_2]$ *in* $\mathbb{P}_2$ *is given by* $\eta_1 x + \eta_2 y + \eta_3 z = 0$ *where* $(\eta_1, \eta_2, \eta_3) = (a_1, b_1, c_1) \times (a_2, b_2, c_2)$.

Using coordinates, we have seen a nice and quick way to move from the projective space to its dual. For instance, the dual of the line $ax + by + cz = 0$ in $\mathbb{P}_2$, is the point $(a, b, c) \in \mathbb{P}_2^*$. On the other hand, the dual of the point $(u_1, u_2, u_3) \in \mathbb{P}^2$ is the line $u_1 X + u_2 Y + u_3 Z = 0$.

Exercise. A *pencil of lines* in a projective plane $\mathcal{P}(E)$ is the family, denoted by m', of all the lines through a point m. Prove that a pencil of lines of $\mathcal{P}(E)$ is a line of $\mathcal{P}(E^*)$.

Solution. Let $\tilde{m}$ denote the vector line in E such that $m = \mathcal{P}(\tilde{m})$. Let us consider a basis over E, in which the vector $\mathbf{u} \in E$ that generates the line $\tilde{m}$ has coordinates $(\beta_1, \beta_2, \beta_3)$. In $\mathcal{P}(E)$, let m' be the set of lines through the point m, i.e., $m' = \{L : ax + by + cz = 0 \,|\, a, b, c, \in \mathbb{K}, a\beta_1 + b\beta_2 + c\beta_3 = 0\}$. In E, $\tilde{m}'$ is the set of planes $\{\tilde{L} : ax + by + cz = 0 \,|\, a, b, c, \in \mathbb{K}\}$ containing the line $\tilde{m}$. For each different plane $\tilde{L} : ax + by + cz = 0$ in E we get the line $\tilde{L}^* = \lambda\varphi$ where $\varphi \in E^*$ is given by $\varphi(x, y, z) = ax + by + cz \,\forall (x, y, z) \in \mathbb{K}^3$ (coordinates in the basis of E). Since the line $\tilde{m}$ is contained in every plane $\tilde{L} \in \tilde{m}'$, by Proposition 2.3, all the correspondent lines $\tilde{L}^*$ in E^* will be contained in the plane $\tilde{m}^*$. Finally, all lines $\tilde{L}^*$ turn into points in $\mathcal{P}(E^*)$, which are all contained in the line m^*. Moreover, for every point in this line, we can construct a line in E^* contained in the plane $\tilde{m}^*$, which will be a plane in E containing the line $\tilde{m}$, and hence, it will be a plane in $\tilde{m}'$. Therefore, it also comes from a line contained in m'. Moreover, it follows that m' is not just a line, it is equal to m^*.

Exercise. Let H and H' be two hyperplanes over the projective space $\mathcal{P}(E)$, m be a point which is neither in H nor in H'. Let x be a point in H. Prove that the line mx intersects H' at a unique point, that we denote by $g(x)$. Prove that g is a projective transformation. The mapping g is called the *perspectivity* of center m from H to H'.

Solution. Since $x \notin H', m \notin H'$, the line $mx \notin H'$. This means that the vector plane $\tilde{P} \in E$ such that $\mathcal{P}(\tilde{P}) = mx$ is not contained in the hyperplane $\tilde{H}' \in E$. Therefore, $\tilde{P} \cap \tilde{H}' \neq \tilde{P}$. They are both vector spaces so by the dimension formula they should meet along a line in E. The projective point in $\mathcal{P}(E)$ for this line would be the intersection point between the line mx and the hyperplane H' in $\mathcal{P}(E)$.
$H \neq H'$ in $\mathcal{P}(E) \implies \tilde{H} \neq \tilde{H}'$ in E. $m \notin H, H'$ means that the line $\tilde{m} \notin \tilde{H}, \tilde{H}'$ in E. Therefore, writing $\tilde{m} = \langle \mathbf{u} \rangle = \{\lambda\mathbf{u} \mid \lambda \in \mathbb{K}\}$ for some vector $\mathbf{u} \in E$ belonging to $\tilde{m}$, the vector space E can be seen as $E = \tilde{H}' \oplus \langle \mathbf{u} \rangle$. Using this decomposition, the following mapping is well defined:

$$f : \begin{array}{ccc} E & \longrightarrow & \tilde{H}' \\ \mathbf{v} + \lambda\mathbf{u} & \longmapsto & \mathbf{v} \end{array}$$

where $\mathbf{v} \in \tilde{H}'$, $\lambda \in \mathbb{K}$. Clearly it is also surjective. Moreover, $E = \tilde{H} \oplus \langle \mathbf{u} \rangle$, and hence the restriction of f to $\tilde{H}$ is a linear isomorphism between $\tilde{H}$ and $\tilde{H}'$. Let us finally check that f is the linear isomorphism satisfying $p^{-1} \circ g \circ p = f$. Let $\tilde{d} \in \tilde{H}$ be a line. Let's prove, firstly, that $f(\tilde{d})$ is just the intersection line of the plane $\tilde{T}$ defined by the lines $\tilde{d}$ and $\tilde{m}$ with the hyperplane $\tilde{H}'$. We write $\tilde{d} = \langle \mathbf{w} \rangle$ for some $\mathbf{w} \in \tilde{H}$. Since, $\mathbf{w} \in \tilde{H} \subseteq E$, we can write it as $\mathbf{w} = \mathbf{v} + \lambda\mathbf{u}$, for some $\mathbf{v} \in \tilde{H}'$, $\lambda \in \mathbb{K}$. Since $\tilde{d} \in \tilde{H}'$, $\tilde{m} \notin \tilde{H}'$, then $\tilde{T} = \langle \mathbf{w}, \mathbf{u} \rangle$ is a vector plane in E. Moreover, $\tilde{m} \notin \tilde{H}'$ implies that the intersection between $\tilde{T}$ and the vector hyperplane $\tilde{H}'$ is just a line. What line? We can rewrite the expression for $\tilde{T}$ in the following terms:

$$\tilde{T} = k_1\mathbf{w} + k_2\mathbf{u} = k_1(\mathbf{v} + \lambda\mathbf{u}) + k_2\mathbf{u} = k_1\mathbf{v} + (k_1\lambda + k_2)\mathbf{u}$$

so $\tilde{T}$ can also be seen as the plane through the lines $\langle \mathbf{v} \rangle$ and $\langle \mathbf{u} \rangle = \tilde{m}$. But the line $\langle \mathbf{v} \rangle$ also belongs to $\tilde{H}'$ since the vector $\mathbf{v} \in \tilde{H}'$. Hence, $\tilde{T} \cap \tilde{H}' = \langle \mathbf{v} \rangle$. Recall now that $f(\tilde{d}) = f(\beta(\mathbf{v} + \lambda\mathbf{u})) = \beta f(\mathbf{v} + \lambda\mathbf{u}) =$

$\beta\mathbf{v} = \langle\mathbf{v}\rangle$, for $\beta \in \mathbb{K}$. So $f(\tilde{d})$ is just the intersection line of the plane $\tilde{P}$ with the hyperplane $\tilde{H}'$. On the other hand, $d = p(\tilde{d})$ is a point in H. $g(d)$ is just the point in H' given by the intersection of the line dm and the hyperplane H'. Going now back to $\tilde{H}'$, $p^{-1}(dm \cap H')$ is the intersection line between the plane defined by the lines $\tilde{d}$ and $\tilde{m}$, which is precisely $\tilde{T}$ and the hyperplane $\tilde{H}'$. So $p^{-1}(g(p(\tilde{d}))) = \langle\mathbf{v}\rangle = f(\tilde{d})$.

Exercise. Let P be a projective plane. Let D be a line and m be a point in P not in D. Let $m^* \subseteq P^*$ be the dual line, that is, the set of lines through m. One defines the *incidence mapping* $i : m^* \longrightarrow D$ associating with any line through m its intersection point with D. Prove that i is a projective transformation.
Prove that, if D and D' are two lines in the projective plane P and if m is a point of P (neither in D nor in D'), the perspectivity of center m from D to D' is the composition of two incidences.

Solution. Let E be the 3-dimensional vector space such that $P = \mathcal{P}(E)$. Let D' be a line in P, $D \neq D'$ and $m \notin D'$. Then, the map

$$\begin{array}{rccc} j: & m^* & \longrightarrow & D' \\ & p^* & \longmapsto & p \cap D' \end{array}$$

where p is the line in P whose dual point in P^* is $p^*(\in m^*)$, is a bijection. For every $d \in D'$, there is a unique line $l = dm$ passing through d and m. Therefore, l belongs to the pencil of lines through m, and d is the image of the point $l^* \in m^*$. We can therefore identify $D' \equiv m^*$.Taking P as the projective space $\mathcal{P}(E)$ from the last exercise, the two lines D, D' are actually hyperplanes in P. We can therefore use it to claim that the mapping $g : D' \longrightarrow D$ associating any point p in D' the intersection point of the lines pm and D, is a projective transformation. Using again the identification $m^* \equiv D'$, we get that the mapping $i : m^* \longrightarrow D$ is also a projective transformation.
The way we have proceeded also shows the last part of the exercise. Take for instance the perspectivity $g : D' \longrightarrow D$. Notice that j is precisely an incidence mapping, and therefore

$$g: \quad D' \xrightarrow{j^{-1}} m^* \xrightarrow{i} D$$

is the composition of two incidences.

2.1 Projectivities of the dual space

For this section, let dim $\mathcal{P}(E_2) = n$. Consider a projectivity $g : \mathcal{P}(E_2^*) \longrightarrow \mathcal{P}(E_1^*)$. We know from Proposition 1.1 that the image of $(n+2)$ points in general position fixes g (since dim $\mathcal{P}(E_2^*) = n$ and therefore a collection of $n+2$ points in general position establish a frame). What we understand as $n+2$ points in general position is that there is no hyperplane in $\mathcal{P}(E_2^*)$ containing all points. If we translate this in terms of $\mathcal{P}(E_2)$, the $n+2$ points in $\mathcal{P}(E_2^*)$ become $n+2$ hyperplanes in $\mathcal{P}(E_2)$. The condition that no hyperplane contains all points means that no point in $\mathcal{P}(E_2)$ is contained in all the hyperplanes, i.e., the $n+2$ hyperplanes do not intersect at one point. If this happens, we say that the hyperplanes are in general position. Therefore, it follows that the image of $n+2$ hyperplanes in general position also fixes a projectivity.

Proposition 2.6. *Any projective mapping from a projective space of dimension n can be described by the image of $n+2$ projective hyperplanes in general position.*

Given a projective mapping $f : \mathcal{P}(E_1) \longrightarrow \mathcal{P}(E_2)$, one may wonder whether the dual of f makes sense as a mapping

$$f^* : \mathcal{P}(E_2^*) \longrightarrow \mathcal{P}(E_1^*)$$

such that the following diagram

$$\begin{array}{ccc} \mathcal{P}(E_1) & \xrightarrow{f} & \mathcal{P}(E_2) \\ \downarrow \pi & & \downarrow \pi \\ \mathcal{P}(E_1^*) & \xleftarrow[f^*]{} & \mathcal{P}(E_2^*) \end{array}$$

commutes, where $\pi(V_i) = V_i^*$ for all $V_i \subseteq \mathcal{P}(E_i)$. In fact, it makes sense and we will try to describe its matrix in terms of the matrix of f. Let us work it out for $\mathbb{P}_2$. We consider coordinates over the canonical basis in $\mathbb{C}^3$. Let $f : \mathbb{P}_2 \xrightarrow{A} \mathbb{P}_2$, denote the projective mapping given by the (3×3)-matrix A. This means that

$$f(p) = A \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{where } p \equiv (x,y,z) \in \mathbb{P}_2.$$

Our aim is to find the matrix that describes $f^* : \mathbb{P}_2^* \longrightarrow \mathbb{P}_2^*$. Let $(a,b,c) \in \mathbb{P}_2^*$, what is $(f^*)^{-1}(a,b,c)$? The point (a,b,c) in the dual space comes from the line $r : \{ax+by+cz = 0\} = \{(a\,b\,c)\ {}^t(x\,y\,z) = 0\} \subseteq \mathbb{P}_2$, where ${}^t(x\,y\,z)$ denotes the transposed of $(x\,y\,z)$. Therefore,

$$f^{-1}(r) = \{p \in \mathbb{P}_2 \mid A\ {}^t(x\,y\,z) \in r\} = \{p \in \mathbb{P}_2 \mid (a\,b\,c)\,A\,{}^t(x\,y\,z) = 0\}.$$

If we do the multiplication $(a\,b\,c)\,A$ we obtain a row vector $(a'\,b'\,c') \in \mathbb{C}^3$, so $f^{-1}(r)$ is in fact another line. What is the dual of this line? It would be the point ${}^tA\,{}^t(a\,b\,c) \in \mathbb{P}_2^*$, where ${}^tA$ denotes the transposed of A. Hence,

$$(f^*)(a,b,c) = {}^tA \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

and therefore ${}^tA$ is the matrix of f^* with respect to the dual basis.

2.2 Pappus' Theorem

Nowadays known as Pappus' Theorem, it was first stated and proved by Pappus of Alexandria in the fourth century A.D. (Proposition 139 of his Book VII in *Mathematical Collection*, see Coxeter [2]). The original statement requires a basic knowledge of projective geometry which was completely unknown until Desargues time in the 17th century. Nevertheless, excluding some arrangements for the case when the lines are parallel, the theorem makes full sense in spite of the lack of projective geometry.

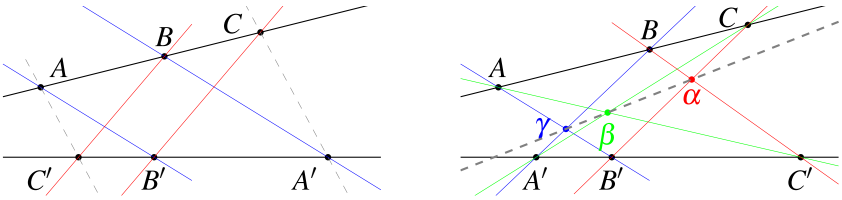
Theorem 2.7 (Pappus' Theorem)**.** *Let D and D' be two lines; let A, B and C be three points of D, and A', B' and C' three points of D'. Let α, β and γ be the interseciton points of $B'C$ and $C'B$, $C'A$ and $A'C$, and $A'B$ and $B'A$ respectively. Then α, β and γ are collinear.*

A generalisation of this theorem will be proved in Chapter 3, but now, in an attempt to show the nice tools projective geometry supply, let us show how the proof can be reduced to the proof of a weaker result.

Proposition 2.8. *Let A, B, C be three points of a line D, and A', B', C' be three points of a line D' distinct from D. If AB' is parallel to BA', and BC' is parallel to CB', then AC' is parallel to CA'.*

Since we are working on a projective plane, we can see it as an affine plane by choosing any line to take the role of line at infinity. We consider, for instance the line $\alpha\beta$. Translating the statement to this affine plane, the lines $B'C$ and $C'B$ meeting at α, which is at the line at infinity, means that they are parallel in the affine plane. The same happens to $A'B$ and $B'A$. Then, we apply Proposition 2.8 to conclude that $C'A$ and $A'C$ are also parallel and hence, as lines in the projective plane they meet at a point β which is in the line at infinity. Therefore α, β, γ belong to the line at infinity and are in fact collinear. Just by a quick translation to the dual space, we can formulate what is sometimes called the dual of Pappus' Theorem.

Proposition 2.9. *Let A^*, B^*, C^*, and $(A')^*$, $(B')^*$, $(C')^*$ be two sets of concurrent lines in two different points. Let α^* be the line through the points $B^* \cap (C')^*$ and $(B')^* \cap C^*$, β^* be the line through the points $(A')^* \cap C^*$ and $A^* \cap (C')^*$, and γ^* the line through $A^* \cap (B')^*$ and $(A')^* \cap B^*$. Then, the lines α^*, β^* and γ^* are concurrent.*



Proposition 2.8 Pappus' Theorem

Figure 2.1: Comparation between both statements

Nevertheless, it is not very hard to show that this proposition is actually equivalent to the original Pappus' Theorem. This is why Pappus' Theorem is also its own dual. To see this equivalence it is enough to prove that Proposition 2.9 implies that Pappus' Theorem holds. Notice that if Pappus' Theorem holds, then it also must hold for the dual space, which is what Proposition 2.9 states.

- Let us call D^*, X and Y the dual of the lines D, AB' and AC' respectively. Since the three lines are collinear in P (over the point A), then the three points in the dual space will be collinear, over the line A^* (which is the dual line for the point A).
- Denote the dual points for the lines D, $A'C$ and $A'B$ by $(D')^*$, X' and Y' respectively. Again, since the lines in P meet all at the point A', then the three dual point will belong to the line $(A')^*$ and therefore, be collinear.
- Let C^* and $(B')^*$ denote the dual lines for the points $C, B' \in P$. Then, $D^*, X' \in C^*$, $(D')^*, X \in (B')^*$ and both lines meet at the point $\pi = (B'C)^*$, dual of the line $B'C \in P$.
- Let B^* and $(C')^*$ denote the dual lines for the points $B, C' \in P$. Then, $D, Y' \in B^*$, $(D')^*, Y \in (C')^*$ and both lines meet at the point $\rho = (BC')^*$, dual of the line $BC' \in P$.
- Let us call α^*, β^*, γ^* the dual lines for α, β, γ. Notice that $\gamma = AB' \cap A'B$ in P imply $X, Y' \in \gamma^*$ in P^*. Similarly, $\beta = AC' \cap C'A$ imply $X', Y \in \beta^*$. Let $\upsilon = \beta^* \cap \gamma^*$.

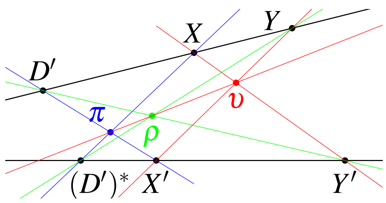


Figure 2.2: Dual to Pappus' Theorem

Proposition 2.9 tells us that the three lines α^*, β^* and γ^* are concurrent. The point of concurrence should be υ, so $\upsilon \in \alpha^*$. Since $\alpha = BC' \cap B'C$, we also have that $\pi, \rho \in \alpha^*$. And therefore π, ρ and υ are concurrent. This is what original Pappus' Theorem stated, as showed in Figure 2.2.

2.3 Desargues' Theorem

Definition. A set of three lines in general position that intersect in three different points A, B, C is called a *triangle*. We may denote it by $\widehat{ABC}$, or just ABC.

Girard Desargues main result, still known as Desargues' Theorem, plays a fundamental role in the whole projective geometry. It states a significant property common to two sections of the same projection of a triangle.

Theorem 2.10 (Desargues' Theorem)**.** *Let ABC and $A'B'C'$ be two triangles. Let α, β and γ be the intersection points of BC and $B'C'$, CA and $C'A'$, AB and $A'B'$. Then the points α, β and γ are collinear if and only if the lines AA', BB' and CC' are concurrent.*

What means that the lines AA', BB', CC' are concurrent? Looking at Figure 2.3, we realize that both triangles are actually two sections of a 3-dimensional pyramid with vertix O. In fact, what this means is that both triangles are just the result of two different points of view of the 3-dimensional pyramid over a 2-dimensional space. This point of view is explained in more detail in [8, Chapter 7, IV]. Recalling the beginning of this Chapter when we defined the projective transformations we can say that there is a projective transformation between the two triangles. Therefore, the statement of the Theorem can actually be replace by saying that α, β, γ are collinear if and only if there is a projective transformation between the two triangles.

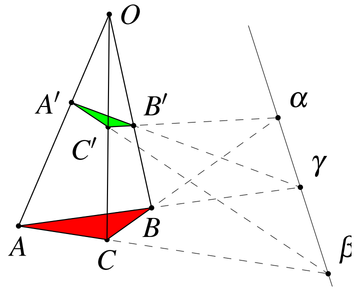


Figure 2.3: Desargues' Theorem

Desargues' Theorem can be derived from Pappus' Theorem as showed by Coxeter ([2]). Although the Theorem states an equivalence, thanks to duality we can show that it is enough to prove just one direction of the theorem since the other implication is just the dual of the first one. Let us see how.

- The dual of the triangles $\widehat{ABC}$ and $\widehat{A'B'C'}$ in P are the triangles given by the lines A^*, B^*, C^* and $(A')^*$, $(B')^*$, $(C')^*$ in P^*.
- If we denote by $\widehat{abc}$, and $\widehat{a'b'c'}$ these new triangles in P^*, where $a = B^* \cap C^*$, $a' = (B')^* \cap (C')^*$ and so on, then a is the dual point to the line BC, a' the dual to the line $B'C'$, and so on.
- The point $\alpha = BC \cap B'C'$ becomes the line α^* passing through the points a and a'. Denoting this line as $\alpha^* = aa'$ we also have that $\beta^* = bb'$ and $\gamma^* = cc'$.
- The lines AA', BB' and CC' become points in P^*. Let us call them δ, μ, σ respectively. Then, by Proposition 2.3, $A, A' \in AA' \implies \delta \in A^*, \delta \in (A')^* \implies \delta = A^* \cap (A')^*$. Similarly we can say $\mu = B^* \cap (B')^*$ and $\sigma = C^* \cap (C')^*$.

We can apply now the "only if" part of Desargues' Theorem (given two triangles ABC and $A'B'C'$, if the three points α, β, γ are collinear, then the three lines AA', BB', CC' are concurrent) to the two triangles $\widehat{abc}$ and $\widehat{a'b'c'}$ in P^*. This means that if the three points

$$bc \cap b'c' = A^* \cap (A')^* = \delta, \qquad ca \cap c'a' = B^* \cap (B')^* = \mu, \qquad ab \cap a'b' = C^* \cap (C')^* = \sigma,$$

are collinear, the the lines $aa' = \alpha^*$, $bb' = \beta^*$ and $cc' = \gamma^*$ are concurrent. What does this mean back in P?

- Using the fourth case in the Metamorphosis Corollary, the points δ, μ, σ being collinear in P^* means that the lines AA', BB' and CC' where they come from in P are concurrent.
- To say that the three lines α^*, β^* and γ^* are concurrent in P^* is to say that the points α, β and γ are collinear in P.

This shows how the converse Desargues' Theorem is just the dual of the direct statement and thanks to duality theory it holds as a consequence of it.

Chapter 3

Projective Curves

Unless otherwise specified, along this chapter we will be considering the complex projective plane $\mathcal{P}(\mathbb{C}^3) = \mathbb{P}_2$. We will use the same notation as in former chapters and denote by $[x:y:z]$ the equivalence class of the line $\lambda(x,y,z) \in \mathbb{C}^3$, taking coordinates over the canonical basis of $\mathbb{C}^3$. Let us introduce the concept of a curve in the projective plane relying on homogeneous polynomials (all the terms in the polynomial have the same degree).

Definition. Let $F(x,y,z)$ be a degree-d homogeneous polynomial (sometimes called degree-d form). Then, $\mathcal{C} = \{[x:y:z] \mid F(x,y,z) = 0\}$ is called a degree-d curve in the projective plane. We say that $\mathcal{C}$ is an irreducible curve when $F(x,y,z,)$ is an irreducible polynomial.

We can see $\mathbb{P}_2$ as an extension of the complex affine plane $\mathbb{C}^2$ where a projective line has been added. Curves in $\mathbb{C}^2$ are then easily transformed into projective curves in $\mathbb{P}_2$. Let $\tilde{f}(x,y)$ be a degree-k polynomial defining a curve in $\mathbb{C}^2$. Then, we can split $\tilde{f}$ such that $\tilde{f}(x,y) = f_k(x,y) + f_{k-1}(x,y) + \ldots + f_1(x,y) + f_0$, where f_i is a homogeneous degree-i polynomial. Then, we can *homogenize* the equation of the affine curve considering the projective curve defined by the set of zeroes of the homogeneous polynomial $f(x,y,z) = f_k(x,y) + zf_{k-1}(x,y) + \ldots + z^{k-1}f_1(x,y) + z^k f_0$, which is a degree-$k$ curve in the projective plane.

Example 1 (*The dual of a curve*)**.** Some fair questions arise right after our introduction of curves in the projective plane. What is the dual of a curve? What is its meaning? What does it look like?

We know that a curve is just a collection of points satisfying some algebraic condition. The dual of a point in $\mathbb{P}_2$ is a line in $\mathbb{P}_2^*$. Hence, for each point in the curve, we will get a line in the dual space. These lines will determine the dual curve in a very special manner. This is not a surprise if the notice that for each point in a curve, there is also a unique line which is tangent to the curve at that point. The collection of lines we obtain in the dual space will be in fact the set of all tangent lines to the dual curve. The dual of a curve will hence be another curve.

Let $\mathcal{K}$ be a curve in $\mathbb{P}_2$ given by the equation $g(x,y,z) = 0$. We are going to check that the dual of the set of tangent lines to $\mathcal{K}$ is a curve in $\mathbb{P}_2^*$. Let $P \in \mathcal{K}$, the tangent line to $\mathcal{K}$ through P has equation

$$t_P \,:\, x\,\frac{\partial g}{\partial x}\bigg|_P + y\,\frac{\partial g}{\partial y}\bigg|_P + z\,\frac{\partial g}{\partial z}\bigg|_P = 0$$

where $\dfrac{\partial g}{\partial x}$ denotes the partial derivative of g with respect to its first variable and so on. So, the set of all tangent lines to $\mathcal{K}$ can be written as:

$$\left\{ x\,\frac{\partial g}{\partial x}\bigg|_P + y\,\frac{\partial g}{\partial y}\bigg|_P + z\,\frac{\partial g}{\partial z}\bigg|_P = 0 \;\Big|\; g(P) = 0 \right\}.$$

The set of dual points for these lines can be expressed as

$$\left\{ \left(\left.\frac{\partial g}{\partial x}\right|_P , \left.\frac{\partial g}{\partial y}\right|_P , \left.\frac{\partial g}{\partial z}\right|_P \right) \,\Big|\, g(P) = 0 \right\}.$$

Notice that the points are expressed as a 3-tuple of coordinates in the canonical basis of $\mathbb{P}_2^*$, so we may call:

$$X = \left.\frac{\partial g}{\partial x}\right|_P, \qquad Y = \left.\frac{\partial g}{\partial y}\right|_P, \qquad Z = \left.\frac{\partial g}{\partial z}\right|_P.$$

Writing $P = [p_1 : p_2 : p_3]$, we can solve this system of three equations for p_1, p_2 and p_3 and replace them in the equation $g(p_1, p_2, p_3) = 0$, to get another equation now depending only on the coordinates of the canonical basis in $\mathbb{P}_2^*$, i.e. $g(p_1, p_2, p_3) = 0 \iff g^*(X,Y,Z) = 0$. This is the equation for the dual curve $\mathcal{K}^*$ of $\mathcal{K}$. Take for instance the conic $\mathcal{C} : 4xy - 2xz + y^2 - yz = 0$ in $\mathbb{P}_2$. The set of tangent lines to the conic can be written as

$$\{(4b-2c)x + (4a+2b-c)y + (-2a-b)z = 0 \mid 4ab - 2ac + b^2 - bc = 0\}$$

whose dual is the set of points

$$\{(4b-2c, 4a+2b-c, -2a-b) \mid 4ab - 2ac + b^2 - bc = 0\}.$$

Calling $X = 4b - 2c, Y = 4a + 2b - c, Z = -2a - b$, we have that

$$a = -\frac{1}{8}X + \frac{1}{4}Y, \qquad b = \frac{1}{4}X - \frac{1}{2}Y - Z, \qquad c = -Y - 2Z.$$

Substituting a, b, c in the equation $4ab - ac + b^2 - bc = 0$, we get the final equation for the dual curve

$$\mathcal{C}^* : -\frac{1}{16}X^2 + \frac{1}{4}XY - \frac{1}{4}Y^2 - YZ - Z^2 = 0.$$

Notice that not only have we obtained a curve for $\mathbb{P}_2^*$, but also a conic curve. The dual of a conic will also be a conic. We generalise what happens with a generic degree-d curve in the following proposition.

Proposition 3.1. *The dual of a smooth degree-d curve is another curve of degree $d(d-1)$.*

Proof. The idea of the proof is to choose the line at infinity such that it intersects the curve $\mathcal{C}$ transversally and choose a point on it $P = [0:1:0]$ such that the pencil of lines through P contains no tangencies to inflection points of the curve, that is, only transversal lines or tangent lines. The degree of the dual curve will be the number of tangencies of $\mathcal{C}$ through P. Projecting from P this corresponds with points of $\mathcal{C}$ whose gradient is horizontal, that is, the solution to the system

$$\begin{cases} f &= 0 \\ \frac{\partial f}{\partial y} &= 0 \end{cases} \qquad \text{where } f \text{ is the degree-}d \text{ form defining } \mathcal{C}.$$

Since the degree of f is d and the degree of $\frac{\partial f}{\partial y}$ is $d-1$, the number of solutions by Bézout – see Theorem 3.2 – is $d(d-1)$. □

3.1 Geometry of projective plane curves

Let us consider a degree-2 curve given by $a_{200}\, x^2 + a_{110}\, xy + a_{101}\, xz + a_{020}\, y^2 + a_{011}\, yz + a_{002}\, z^2 = 0$. Notice that the same curve can be defined by any non-0 scalar multiple of the given polynomial. So, the curve can be identified with the point $[a_{200} : a_{110} : a_{101} : a_{020} : a_{011} : a_{002}] \in \mathbb{P}_5$. We can do this for every degree-d curve.

The space of curves of degree d in the projective plane can be expressed as the quotient

$$\mathcal{P}_d = \left\{ \sum_{i+j+k=d} a_{ijk}\, x^i y^j z^k = 0 \right\} / \sim \tag{3.1}$$

where $\sim$ relates two polynomials if there if they are non-0 scalar multiples of each other. This way the degree-d curves in $\mathbb{P}_2$ are identified with points in the projective space $\mathcal{P}(S_d)$ (recall that $S = \mathbb{C}[x,y,z] = \oplus_{d \geq 0} S_d$). There are $D = \binom{d+2}{2}$ monomials of degree d in three variables, and together they form a basis for the vector space S_d, so we may identify

$$\mathcal{P}_d \equiv \mathcal{P}([a_{ijk}]) \equiv \mathcal{P}(S_d) \cong \mathbb{P}_{D-1}.$$

In our initial conic, $D = \binom{2+2}{2} = 6$, so the curve is in fact a point in $\mathcal{P}(S_2) \cong \mathbb{P}_5$. If we require now the curve to pass through 5 points in $\mathbb{P}_2$, then the coefficients $a_{200}, \dots, a_{002}$ need to satisfy 5 linear equations. Assuming that the points are in general position, the solution for the homogeneous system will have dimension 1, and hence, there is just 1 degree-2 curve passing through 5 points. In general, for a degree-d curve we have a linear system with D unknowns and as many equations as points we want to pass through the curve. In this sense, k points will be said to be in general position if they impose k independent linear equations over this set of unknowns. For $D-1$ independent conditions the solution for the system will have dimension 1. This means that there will be just one degree-d curve passing through $D-1$ points in general position, and no degree-d will go over D points in general position. This concept of independent conditions for the coefficients a_{ijk} (unknowns of the system) will be used again in next definitions.

Note. The description we have given for the space of curves of degree d in (3.1) does not hold for $\mathbb{R}$. It is due to Hilbert's Nullstellensatz (theorem of zeroes). In $\mathbb{R}$ we may find two different degree-d polynomials whose set of zeroes are equal (and therefore they define the same curve) but they are not scalar multiples of each other. For instance we may consider the conics given by $\{x^2 + y^2 + z^2 = 0\}$ and $\{x^2 + 2y^2 + 3z^2 = 0\}$. They are both representing the same conic (the empty set), but the equations are not scalar multiples.

3.2 Basic results on plane curves

Theorem 3.2 (Bézout)**.** *If $\mathcal{C}_1$ and $\mathcal{C}_2$ are curves of degrees d_1 and d_2 in the complex projective plane $\mathbb{P}_2$ sharing no common components, then they meet in $d_1 d_2$ points, counted appropriately.*

Michael Chasles proved a simple and powerful statement that we will be used to solve many upcoming results.

Theorem 3.3 (Chasles)**.** *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two plane cubic curves meeting in 9 distinct points. Then any other cubic passing through any 8 of the nine points must pass through the ninth point too.*

Two different proofs for this theorem can be found in [3, Theorem CB3] and [10, Theorem 7]. We will not go over it now. The French mathematician and philosopher Blaise Pascal, stated (and most likely showed), at the age of sixteen, a generalization of Pappus' Theorem (in his now lost work *Essai sur les Coniques* -see [9]-).

Theorem 3.4 (Pascal)**.** *If 6 points A, B, C, a, b, c lie on a conic section, then the lines Aa, Bb, Cc meet the lines aB, bC, cA in three new points and these new points are collinear.*

Proof. Let $\mathcal{C}_1$ be the cubic containing the three lines of three non-consecutive edges in the hexagon with vertices A, B, C, a, b, c. And let $\mathcal{C}_2$ be the cubic containing the lines of the other three edges. $\mathcal{C}_1$ and $\mathcal{C}_2$ meet then in 6 points on the conic $\mathcal{K}$, and in 3 other points outside $\mathcal{K}$. Let L be the line through 2 of these 3 points. Then, $\mathcal{K} \cup L$ is a cubic going through 8 of the 9 points of intersection of cubics $\mathcal{C}_1 \cap \mathcal{C}_2$. Hence, by Theorem 3.3, $\mathcal{K} \cup L$ must pass through the ninth point too. It cannot lie in $\mathcal{K}$ since $\mathcal{C}_1$ only intersects $\mathcal{C}_2$ in 6 points on $\mathcal{K}$ and so, it must lie in L. Therefore, the three points of intersection not in $\mathcal{K}$ are aligned. □

It is not hard to realize that Pascal's Theorem is a generalisation of Pappus' Theorem, and therefore the latter holds as a particular case of the first one. We can take the conic formed by two lines, then take three points, A, B, C, at one of the lines and three points, C', A', B' in the other one. Then the lines AC', BA' and BA' meet the lines $C'B$, $A'C$, and $B'A$ in three collinear points.

Example 2 (*Brianchon's Theorem*)**.** Let us pause for a second to look into the dual statement of Pascal's Theorem. We know that the dual to a conic is another conic. The points A, B, C, a, b, c of the conic will become tangent lines $A^*, B^*, C^*, a^*, b^*, c^*$ to the dual conic. Calling $\alpha = Aa \cap bC$, $\beta = Bb \cap cA$ and $\gamma = Cc \cap aB$, the dual lines $\alpha^*, \beta^*, \gamma^*$ would be lines through the points

$$\alpha^* : \begin{cases} A^* \cap a^* \\ b^* \cap C^* \end{cases} \quad \beta^* : \begin{cases} B^* \cap b^* \\ c^* \cap A^* \end{cases} \quad \gamma^* : \begin{cases} C^* \cap c^* \\ a^* \cap B^* \end{cases} .$$

If Pascal's Theorem holds, and α, β, γ are collinear, then the lines $\alpha^*, \beta^*, \gamma^*$ will be concurrent. We can check easily what does this mean by taking a look at Figure 3.1. Our conclusion for the dual of Pascal's Theorem is the following statement, which is generally named after Monge's student, Charles Brianchon, French mathematician in the 19th century, who discovered it by applying the principle of duality to Pascal's Theorem similarly to what we have done (see [8, page 637]).

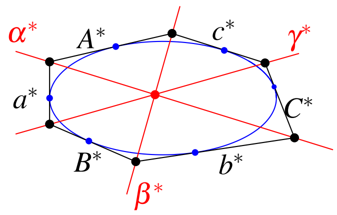


Figure 3.1: Dual of Pascal's Theorem

Corollary [*Brianchon's Theorem*]. Let A, B, C, D, E, F be the vertices of an hexagon circumscribed around a conic (which means the lines AB, BC, CD, DE, EF, FA are tangent to the conic). Then the diagonals AD, BE and CF are concurrent.

The converse to Pascal's Theorem was showed independently by William Braikenridge and Colin MacLaurin and is usually called after them.

Theorem 3.5 (Braikenridge-MacLaurin)**.** *If three lines meet three other lines in nine points and if three of these points lie in a line, then the remaining six points lie in a conic.*

Proof. Let $\mathcal{C}_1$ be the cubic containing the first three lines, and $\mathcal{C}_2$ the cubic containing the other three lines. Then, $\mathcal{C}_1$ and $\mathcal{C}_2$ are two cubics meeting in 9 points. Let L be the line containing the 3 aligned points. Recall that 5 points always lie in a conic, so let $\mathcal{K}$ be a conic passing through 5 of the remaining 6 points. Then, $L \cup \mathcal{K}$ is a cubic going through 8 of the 9 points in $\mathcal{C}_1 \cap \mathcal{C}_2$, and hence by Theorem 3.3 it must go through the ninth point too. If it were in L, the we would have 4 aligned intersection points, and the only way for this to happen is that $\mathcal{C}_1$ and $\mathcal{C}_2$ share a common line. But $|\mathcal{C}_1 \cap \mathcal{C}_2| = 9$, and this cannot be the case. Therefore, the point must lie on the conic $\mathcal{K}$ and the six points lie on the conic. □

Definition. If a set Γ of γ points imposes only λ independent linear conditions on the coefficients of a curve of degree d, then we say that Γ *fails to impose* $\gamma - \lambda$ *independent linear conditions on forms of degree* d.

Recall our previous notation –(3.1)– to express a degree-d curve as $\{\sum_{i+j+k=d} a_{ijk}\, x^i y^j z^k\}$. Then, in order for a point to be on the curve we impose a condition on the D coefficients a_{ijk} of the curve. We obtain a linear system with an equation for every different point we want it to be in the curve. The independent linear conditions stated in the definition refers to independent linear equations for this system where a_{ijk} are the unknowns.

Theorem 3.6 (Cayley-Bacharach)**.** *Suppose that two curves of degrees d_1 and d_2 meet in a finite collection of points $\Gamma \subseteq \mathbb{P}^2$. Partition Γ into two disjoint subsets $\Gamma = \Gamma' \cup \Gamma''$ and set $s = d_1 + d_2 - 3$. If $d \leq s$ is a non-negative integer, then the dimension of the space of forms of degree d vanishing on Γ' modulo those vanishing on Γ is equal to the failure of Γ'' to impose independent conditions on forms of degree $s-d$.*

We introduce some notation from Katz [5] in order to help the understanding and simplify upcoming statements.

Definition. Let $\mathcal{K}_1, \mathcal{K}_2$ be two plane curves given by the union of d and e lines respectively. Then, the union $\mathcal{K}_1 \cup \mathcal{K}_2$ is called a $(d \times e)$*-cage*. In order to simplify the terminology, we colour the lines from $\mathcal{K}_1$ red, and the lines from $\mathcal{K}_2$ blue. The points given by the intersection of a red and a blue line are called the *nodes* of the cage.

Theorem 3.7. *Suppose that k nodes of a $(k \times k)$- cage are collinear through a different line. Then, there is a unique curve of degree $k-1$ passing through the remaining $k^2 - k$ nodes.*

Proof. The result is trivial for $k = 1$. For $k = 2$, there is always a unique line passing through two given points. Let us assume now that $k \geq 3$. Suppose that the red lines $\mathcal{L}_1, \dots, \mathcal{L}_k$ in the cage are given by the zeroes of the homogeneous forms $L_1, \dots, L_k$, the blue lines $\mathcal{M}_1, \dots, \mathcal{M}_k$ by the forms $M_1, \dots, M_k$, and the line G containing the k nodes by g. Let Γ be the set of k^2 nodes. Notice now that there is no degree-$(k-1)$ curve going through all the points in Γ. For if there is such a degree-$(k-1)$ curve $\mathcal{K}$ (given by the degree-$(k-1)$ form K), then $|\mathcal{K} \cap \mathcal{M}_i| = k$, for $i_1, \dots k$. Hence, since $k > (k-1) \cdot 1$, by Bézout's Theorem (Theorem 3.2), $\mathcal{K}$ shares a component with $\mathcal{M}_i$, which means that M_i divides $K\ \forall i = 1, \dots k$, which is a contradiction for the degree of K. Therefore such a curve cannot exist. Define $\Gamma'' = \{x \in \Gamma \mid g(x) = 0\}$, and $\Gamma' = \Gamma \backslash \Gamma''$. Since there are no degree-$(k-1)$ curves going through all the points of Γ, Theorem 3.6 says that the dimension of the space of forms of degree $k-1$ vanishing on Γ' is equal to the failure of Γ'' to impose independent conditions on forms of degree $k+k-3-(k-1) = k-2$. Since the k points in Γ'' are collinear (they all lie in G), and $k-1 > (k-2) \cdot 1$, again by Theorem 3.6, if a degree-$(k-2)$ form contains $k-1$ aligned points, then it will contain the line they lie on. Therefore, the failure is equal to $k-(k-1) = 1$. Hence, up to scaling there is a unique equation of degree-$(k-1)$ passing through the $k^2 - k$ nodes of Γ off the line G. □

Theorem 3.8. *Suppose that a $(k \times k)$-cage shapes a polygon with sides belonging alternatively to blue and red lines. Assume that the polygon is inscribed in an irreducible conic. If $k-1$ of the $k^2 - 2k$ nodes not in the polygon lie on a green line, then another of these points lies on the green line as well.*

A nice proof for this theorem can be found in [10, Theorem 9]. We step forward and present a couple of results which are consequences of the last two theorems.

Corollary 1. Let Λ be a set of k red points, and Ω a set of k blue points. Consider the k^2 lines joining a point in Λ with a point in Ω. If k of these lines are concurrent over another point (not in $\Lambda \cup \Omega$), then there is a unique curve of degree $(k-1)(k-2)$ which is tangent to the remaining $k^2 - k$ lines.

Proof. We just need to read Theorem 3.7 in terms of the dual space. The k^2 nodes of the $(k \times k)$-cage become lines in the dual space. The $2k$ lines become points. Each red line of the cage contains k points given by the intersection with the other k blue lines. So, in the dual space, k lines go through every red point, and each of this k lines contains also a blue point. Therefore, the dual of a $(k \times k)$-cage is a collection of k red points, another of k blue points, and the k^2 lines obtained by joining a blue and a red point. Each of the lines contains therefore just one red point, and just one blue point. k nodes of the cage being collinear through a different line means in the dual space that k lines are concurrent to a different point (of the original $2k$ points). Theorem 3.7 says that if this happen, then there is a unique curve of degree $k-1$ passing through the remaining k^2-k points, which means in the dual space, using what we have seen in Examples 1 and 2 and Proposition 3.1, that there is a unique curve of degree $(k-1)(k-2)$ which is tangent to the remaining k^2-k lines. Hence, the statement holds. □

Example 3 (*The dual of an irreducible conic*)**.** We know from Example 1 that the dual of a particular conic is another conic. In order to state the dual of Theorem 3.8 we need to find out now what the dual of an irreducible conic looks like. We now that a curve is an irreducible conic if it is given as the set of zeroes of an irreducible degree-2 polynomial, but how can we check this easily? Let $a_1x^2+2a_2xy+2a_3xz+a_4y^2+2a_5yz+a_6z^2=0$ describe a conic. We can rewrite the expression as

$$\begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} a_1 & a_2 & a_3 \\ a_2 & a_4 & a_5 \\ a_3 & a_5 & a_6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=0.$$

If A is a (3×3) regular matrix, it can be seen as the matrix of a change of coordinates in $\mathbb{C}^3$. Over this new basis, the conic will be expressed as

$$\begin{pmatrix} x & y & z \end{pmatrix}\underbrace{({}^tA)\begin{pmatrix} a_1 & a_2 & a_3 \\ a_2 & a_4 & a_5 \\ a_3 & a_5 & a_6 \end{pmatrix}A}_{\tilde{C}}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=0$$

where $\tilde{C}$ is the new matrix describing the conic. Notice that a change of coordinates does not change the nature of the conic. This means that the conic will remain irreducible under a change of coordinates, if the conic is a product of two lines it will remain a product of two (different) lines and if the conic is a double line it will remain so under a change of coordinates. We note that both matrices that describe the conic before and after the change of coordinates are congruent, so the congruent class of the conic matrix is preserved under a change of coordinates. There are three different congruence classes for (3×3) matrices in $\mathbb{C}$, which we express by the following representatives:

$$G_1=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\quad G_2=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\quad G_3=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

G_3 will contain the irreducible conics, G_2 the product of two lines and G_1 the double lines. Therefore, given a conic, we just need to check the congruent class of the matrix to see if it is irreducible or not. The quickest way to do so is to compute the rank of the matrix, in our case $\tilde{C}$ for instance (our the initial matrix).

$$\begin{array}{lllll} \text{rank } (\tilde{C})=3 & \Rightarrow & \tilde{C}\in G_3 & \Rightarrow & \text{Irreducible conic} \\ \text{rank } (\tilde{C})=2 & \Rightarrow & \tilde{C}\in G_2 & \Rightarrow & \text{Two lines} \\ \text{rank } (\tilde{C})=1 & \Rightarrow & \tilde{C}\in G_1 & \Rightarrow & \text{Double line} \end{array}$$

Let's go no back to our initial question. We take the four points $[0:0:1]$, $[0:1:0]$, $[1:1:1]$, $[1:0:0]\in\mathbb{P}^2$. We consider now all conics in $\mathbb{C}^3$ that go through these four points, namely $\mathcal{H}: b_1x^2+b_2xy+b_3xz+b_4y^2+b_5yz+b_6z^2=0$. Then,

$$\begin{cases} [0:0:1]\in\mathcal{H} & \Rightarrow & b_6=0 \\ [0:1:0]\in\mathcal{H} & \Rightarrow & b_4=0 \\ [1:0:0]\in\mathcal{H} & \Rightarrow & b_1=0 \\ [1:1:1]\in\mathcal{H} & \Rightarrow & b_2+b_3+b_5=0 \end{cases}$$

So $\mathcal{H} : 2b_2xy + 2b_3xz - 2(b_2+b_3)yz = 0$ where $b_2, b_3 \in \mathbb{C}$ includes all the conic passing through the four points. We want also it to be an irreducible conic, so the matrix must have full rank (its determinant should be different from 0).

$$\det \begin{pmatrix} 0 & b_2 & b_3 \\ b_2 & 0 & -(b_2+b_3) \\ b_3 & -(b_2+b_3) & 0 \end{pmatrix} = -2b_2b_3(b_2+b_3) = 0 \quad \Longleftrightarrow \quad \begin{cases} b_2 = 0 \\ b_3 = 0 \\ b_2 = -b_3 \end{cases}$$

Assuming $b_3 = 1$ (this does not mean a loss of generality), we end up with the generic irreducible conic $\mathcal{H} : 2b_2xy + 2xz - 2(b_2+1)yz = 0$, $b_2 \neq 0, -1$. What is its dual conic? We follow the procedure as in Example 1 to obtain: $\mathcal{H}^* : (b_2+1)^2X^2 - 2(b_2+1)XY + 2(b_2+1)b_2XZ - 3Y^2 + 2b_2YZ - b_2Z^2 = 0$. Then,

$$\det \begin{pmatrix} (b_2+1)^2 & -(b_2+1) & b_2(b_2+1) \\ -(b_2+1) & -3 & b_2 \\ b_2(b_2+1) & b_2 & -b_2^2 \end{pmatrix} = 4b_2^2(b_2+1)^2 \neq 0 \text{ since } b_2 \neq 0, -1.$$

Hence, the dual conic is also irreducible. We have then proved that the dual of all irreducible conics passing through $[0:0:1]$, $[0:1:0]$, $[1:0:0]$, $[1:1:1]$ are irreducible. Notice that then, the dual of all irreducible conics are irreducible since there is always a projectivity that maps any four different points (in general position so that they form a frame) to these four points, and all the conics can be reduced to one of the considered ones.

Corollary 2. Assume a $2k$-polygon is formed by the intersection of $2k$ lines is circumscribed along an irreducible conic. Colour the $2k$ vertices of the polygon alternatively red and blue. Consider the remaining $k^2 - 2k$ lines obtained by joining a red an a blue point. If $k-1$ of these lines concur in another point, then in fact another of those line should meet at that point too.

Proof. This is just the dual of Theorem 3.8. The sides of the polygon forms another polygon in the dual space with vertices alternatively red and blue. As the vertices of the original polygon were in an irreducible conic, the $2k$ sides of the dual polygon will be tangent to an irreducible conic. Therefore, the polygon is circumscribed around the irreducible conic. The rest of the points in the original $(k \times k)$-cage will become lines meeting a red and a blue point. If $k-1$ of these $k^2 - 2k$ lines not in the dual polygon are concurrent, then by Theorem 3.8, another of these line will be concurrent through this point too. □

3.3 Further applications and exercises

We call a smooth plane curve of degree 3 an elliptic curve. The points of an elliptic curve form an abelian group, where three distinct points add up to the identity element in the elliptic group if and only if they are collinear. We denote the group law of an elliptic curve χ by '+', and the identity element by 0_χ. Hence, if P_1, P_2 and P_3 are three collinear points in χ, then $P_1 + P_2 + P_3 = 0_\chi$. We have enough background now to show a nice result on the group law on an elliptic curve.

Proposition 3.9. *The group law on an elliptic curve χ is associative.*

Proof. Let $P_1, P_2, P_3 \in \chi$. We need to show that $(P_1+P_2)+P_3 = P_1+(P_2+P_3)$, or analogously, that $-((P_1+P_2)+P_3) = -(P_1+(P_2+P_3))$. Notice that χ is a smooth plane degree 3 curve, and hence it doesn't have a line as a component in any case. Therefore, by Bézout's Theorem (Theorem 3.2), any line will meet χ in three points (counting multiplicity). Let L_1 be the line through P_1 and P_2. Then, $-(P_1+P_2)$ is also a point in χ and it is collinear to P_1 and P_2 since $P_1+P_2+(-(P_1+P_2)) = 0_\chi$. Hence, L_1 also meets χ at the point $-(P_1+P_2)$. We consider now the line L_2 through the points P_1+P_2 and P_3, which will also meet χ at $-((P_1+P_2)+P_3)$, and the line L_3 that joins P_2+P_3 and 0_χ, which will meet χ also at the point $-(P_2+P_3)$. On the other hand, let N_1 be the line joining P_1+P_2 and 0_χ, which meets

χ also at $-(P_1+P_2)$, N_2 the line joining P_2 and P_3, which meets χ also at $-(P_1+P_2)$ and let N_3 be the line joining P_1 and P_2+P_3, meeting the curve χ also at $-(P_1+(P_2+P_3))$.

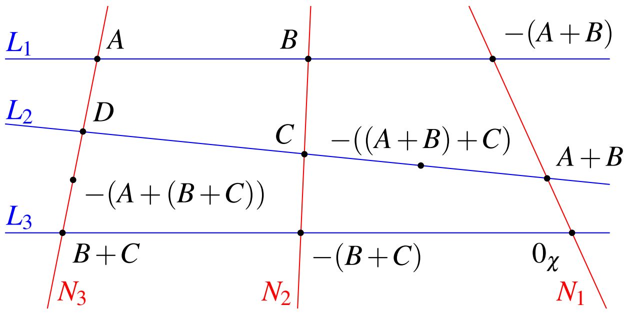


Figure 3.2: The only point which is not in χ initially is D

Let D be the intersection point of the lines L_2 and N_3. We will see that this point is actually equal to $-(P_1+(P_2+P_3))$ and to $-((P_1+P_2)+P_3)$, and therefore these points are actually the same one. Let C_1 be the cubic formed by the three lines L_1, L_2 and L_3, and let C_2 be the cubic formed by the lines N_1, N_2 and N_3. Then, C_1 and C_2 are two cubics meeting at nine distinct points ($0_\chi, P_1, P_2, P_3, P_1+P_2, -(P_1+P_2), P_2+P_3, -(P_2+P_3), D$). Hence, by Chasles' Theorem (Theorem 3.3), any other cubic meeting at 8 of these nine points must also pass through the ninth point too. Recall that χ is a cubic curve going through 8 of the points (all of them but D), and therefore it must also pass through D. Finally, D is different from P_1 and P_2+P_3 by definition, and if $D \neq -(P_1+(P_2+P_3))$ then the line N_3 would meet χ in four different points, which cannot happen. Therefore $D = -(P_1+(P_2+P_3))$. Also, $D \neq P_3, P_1+P_2$ by definition, and if $D \neq -((P_1+P_2)+P_3)$ then L_2 would meet the cubic χ in four different points. Therefore $D = -((P_1+P_2)+P_3)$ and both points are equal. □

Exercise. Consider two polygons P_1 and P_2, each with m edges ($m > 3$) and different vertices, inscribed in an irreducible conic, and associate one edge from P_1 with one edge from P_2. Working counter-clockwise in each polygon, associate the other edges of P_1 with the edges of P_2. Extending the edges to lines, prove that if $m-1$ of the intersections of pairs of corresponding edges lie on a line, then the last pair of corresponding edges also meets in a point on this line.

Solution. We split the proof in three cases. First we assume m is even, then we prove the case $m = 5$ and finally we reduce the remaining odd cases to the previous ones.

We start assuming m is even. Let us build a $(m \times m)$-cage with $2m$ nodes inside the conic $\mathcal{H}$, and $m-1$ of the remaining nodes aligned through a line G. We associate an edge from P_1, B_1, with one from P_2, R_1 (marked with the symbol $*$ in Figure 3.3) and we colour them in blue and red respectively. Working counter-clockwise in P_1 we colour the edges alternatively red and blue and we call them $R_{\frac{m}{2}+1}, B_2, R_{\frac{m}{2}+2}, B_3, \ldots$. We do the same in P_2 with blue and red and we call them $B_{\frac{m}{2}+1}, R_2, B_{\frac{m}{2}+2}, R_3, \ldots$ as in Figure 3.3. Since m is even, both polygons will have $\frac{m}{2}$ edges alternatively red and blue, so the $2m$ points of both polygons will be nodes of the cage. Moreover, the associated lines are the couples of lines $\{R_i, B_i\}$ corresponding to a red and a blue line. Therefore all the intersections of pairs of corresponding edges are also nodes of the cage.

Now, let us compute the dimension of the degree $m-2$ forms through all the nodes off the conic. We set

$$\Gamma' \equiv m^2 - 2m \text{ nodes off the conic}$$

$$\Gamma'' \equiv 2m \text{ nodes on the conic.}$$

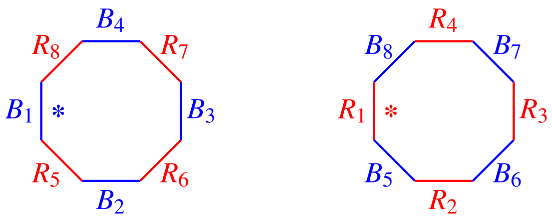


Figure 3.3: Colouring for the case $m = 8$

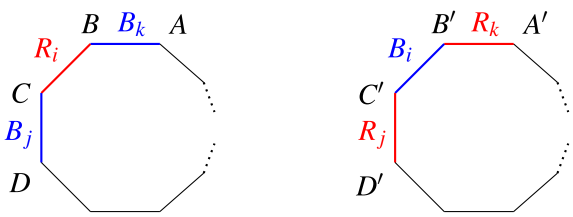
Theorem 3.6 tells us that this dimension will be equal to the failure of Γ'' to impose independent conditions on forms of degree $2m-3-(m-2) = m-1$. Since the points in Γ'' lie on an irreducible conic, it should be a component of the $(m-2)$-form. Therefore, since $2(m-1) = 2m-2 > 2m-1$, by Bézout's Theorem 3.2, $2m-1$ points are needed to satisfy this condition. So the failure equals

$$|\Gamma''| - (2m-1) = 2m - (2m-1) = 1.$$

Hence, up to scaling there is just one curve $\mathcal{C}$ of degree $m-2$ through Γ'. It will meet the line G in $m-1$ points, so it will contain this line as a component. Therefore we may consider the curve $(\mathcal{C}/G) \cup \mathcal{H}$ (curve given by the set of zeroes of the polynomial $(f/g) \cdot h$, where f, g, h are the forms defining the curves $\mathcal{C}, G$ and $\mathcal{H}$ respectively) which has degree $m-3+2 = m-1$ and goes through all the nodes off the line G. What is the dimension of the forms of degree $m-1$ going through all the points off G? It will be equal to the failure of the points in G to impose independent conditions on forms of degree $2m-3-(m-1) = m-2$ (by Cayley-Bacharach -Theorem 3.6). The line should be a component of the form, so the failure equals

$$\#(\text{points in } G) - (m-1) \geq 1$$

and it should be at least 1 since we have already obtained a degree $m-1$ curve going through all the points off G. Therefore, $\#(\text{points in } G) \geq m$. But if $\#(\text{points in } G) > m$, then the curve will meet the m red lines in more than m points, which cannot happen. Then, $\#(\text{points in } G) = m$. The only thing that is still to be proved is that the extra mth node is the one corresponding to the intersection of the remaining edges in the polygons, $R_m \cap B_m$. But this should also be the case, for if this node lied in any other red line, $R_i \neq R_m$ say, then R_i would meet G in two points, so both lines would be equal. Let us show that this cannot happen. R_i contains m nodes given by the intersections with the m blue lines. Let's call B, C the vertices of the edge R_i, and A, D the next two adjacent nodes. We call B_k, B_j the two blue lines that form the two adjacent edges to R_i. We do the same for the polygon P_2 and its associate edge B_i as described in the figure.



Recall now that a line meets an irreducible conic in two points. Since the vertices A', B', C', D' are all in $\mathcal{H}$, it follows

$$\begin{aligned} R_k \cap \mathcal{H} &= \{A', B'\} \\ R_j \cap \mathcal{H} &= \{C', D'\} \end{aligned} .$$

Recall that we are assuming $R_i \neq R_m$. Note that R_m should be different from either R_j or R_k, hence we may assume $R_m \neq R_j$. Then

$$\left. \begin{array}{c} B_j \cap R_j \in G = R_i \\ B_j \cap R_i = \{C\} \end{array} \right\} \implies B_j \cap R_j = \{C\}.$$

Therefore, $C \in R_j$, but $C \in \mathcal{H}$ and $R_j \cap \mathcal{H} = \{C', D'\}$. Hence, it follows that C is equal to C' or D'. In any case, this cannot happen since all the vertices are different.

We prove it now for the case $m = 5$, where the two polygons P_1 and P_2 are two pentagons. Let us call the vertices A,B,C,D,E,a,b,c,d,e such that $\mathsf{P}_1 = \widehat{ABCDE}$, $\mathsf{P}_2 = \widehat{abcde}$. Recall that all the vertices lie on an irreducible conic $\mathcal{H}$. We associate the edge AB from P_1 with the edge ab from P_2. By our hypothesis, let's assume that the intersection points $AB \cap ab$, $BC \cap bc$, $CD \cap cd$ and $DE \cap de$ are aligned along a projective line G. What we want to prove is that the intersection point coming from the remaining two edges, $AE \cap ae$ is also in G. Recall from Chapter 1 that we can always find a projectivity that sends the line G to the "line at infinity" in the projective plane. Then,

$$\left.\begin{array}{c} AB \cap ab \\ BC \cap bc \\ CD \cap cd \\ DE \cap de \end{array}\right\} \begin{array}{c}\text{aligned} \\ \text{through } G\end{array} \quad \Longleftrightarrow \quad \begin{array}{c} AB \parallel ab \\ BC \parallel bc \\ CD \parallel cd \\ DE \parallel de \end{array} \tag{3.2}$$

where $AB \parallel ab$ means that the lines AB and ab are parallel (they meet at the line at infinity). Our aim is now to prove that the lines AE and ae are also parallel. The points B,C,D,b,c,d lie on the conic $\mathcal{H}$. Therefore, we can apply Pascal's Theorem over them to conclude that the points $BC \cap bc$, $CD \cap cd$, $Bd \cap Db$ are collinear along a line G. Since $BC \parallel bc$, $CD \parallel cd$ this means that the line G is the line at infinity and therefore

$$Bd \parallel Db. \tag{3.3}$$

A quick way see how Pascal's Theorem 3.4 applies to a set of points A,B,C,a,b,c lying on a conic is by giving them an order. This means that we consider an hexagon, say $\widehat{ABCabc}$. Then, for every point, we consider the two lines joining this point with its two neighbouring vertices of the hexagon. We will end up with the six lines AB, BC, Ca, ab, bc, cA. Then, Pascal's Theorem says that the points $AB \cap ab$, $BC \cap bc$ and $Ca \cap Ac$ are collinear. Hence, if we know that the two pairs of lines $AB \parallel ab$ and $BC \parallel bc$ are parallel, we can assert $Ac \parallel Ca$.

Since all the points A,B,C,D,E,a,b,c,d,e lie on the same conic, we can go on choosing ordered sets of 6 points as mentioned and apply Pascal's Theorem.

$$\text{For } \widehat{ABCabc}, \quad \left.\begin{array}{cc} AB \parallel ab & (3.2) \\ BC \parallel bc & (3.2) \end{array}\right\} \quad \Longrightarrow \quad Ac \parallel Ca \tag{3.4}$$

$$\text{For } \widehat{BdebDE}, \quad \left.\begin{array}{cc} Bd \parallel bD & (3.3) \\ de \parallel DE & (3.2) \end{array}\right\} \quad \Longrightarrow \quad BE \parallel eb \tag{3.5}$$

$$\text{For } \widehat{DCadcA}, \quad \left.\begin{array}{cc} DC \parallel dc & (3.2) \\ Ca \parallel cA & (3.4) \end{array}\right\} \quad \Longrightarrow \quad DA \parallel ad \tag{3.6}$$

$$\text{For } \widehat{ADEade}, \quad \left.\begin{array}{cc} AD \parallel ad & (3.6) \\ DE \parallel de & (3.2) \end{array}\right\} \quad \Longrightarrow \quad Ae \parallel Ea \tag{3.7}$$

Let us now state the following extension of Pascal's Theorem for 8 points lying on a conic due to Möbius [7].

Proposition 3.10. *If 8 points A,B,C,D,a,b,c,d lie on a conic section, then the lines Aa, Ba, Dd, Cb meet the lines Cc, Dc, Bb, Ad in four new points, and these points are collinear.*

The same method works now, but with an octagon instead of an hexagon. Therefore, if we consider the octagon

$$\widehat{ebDAEBda}, \quad \left.\begin{array}{cc} eb \parallel EB & (3.5) \\ bD \parallel Bd & (3.3) \\ DA \parallel da & (3.6) \end{array}\right\} \quad \Longrightarrow \quad ea \parallel AE \tag{3.8}$$

So the lines EA and ea are also parallel, as we wanted to prove.

What we are left to show is the general case for m an odd number greater than 5. Let P_1 and P_2 be two polygons with $m = 2k+1$, $k \geq 3$ edges inscribed in an irreducible conic. We associate an edge x_1 from P_1 with vertices A, B with an edge y_1 from P_2 with vertices A', B'. Working counter-clockwise we call x_2, x_3 the following two edges of P_1 (corresponding to the vertices B, C, D) and y_2, y_3 the following edges of P_2 (corresponding to the vertices B', C', D'), as shown in Figure 3.4. Let y_m be the other adjacent edge to A in P_1 and y_m be the other adjacent edge to A' in P_2 (we call E, E' the remaining vertices in X_m and Y_m respectively). By hypothesis, the intersection point of $m-1$ pairs of associated edges lie on a common line G. Assume these $m-1$ pairs are all x_i, y_i but x_1, y_1. Let us show that the point $x_1 \cap y_1$ also lies in G.

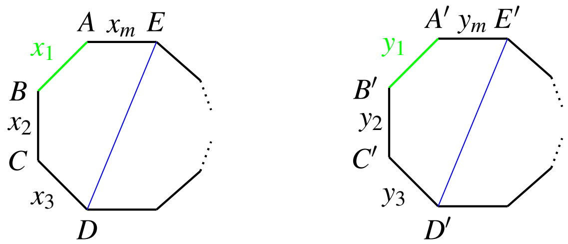


Figure 3.4: The case m odd holds from the cases m even and $m = 5$.

Let DE denote line joining the vertices D, E. We construct a new polygon based on P_1 by taking the new edge DE and removing x_m, x_1, x_2, x_3, and another one from P_2 by adding the edge $D'E'$ and removing y_m, y_1, y_2, y_3. These new polygons have $2k+1-4+1 = 2k-2$ edges. Since $k \geq 3$ these new polygons are well defined and have an even number of edges. If we associate DE with $D'E'$, we get two polygons whose vertices are all on a conic and where the intersection points of the $2k-3$ pairs of associated lines (all of them but DE and $D'E'$) lie on the line G. Then, we know from the even case, that also the remaining intersection point $DE \cap D'E'$ lies in G.

Consider now the pentagon Q_1 formed by x_m, x_1, x_2, x_3 and DE, and the pentagon Q_2 formed by y_m, y_1, y_2, y_3 and $E'D'$. We know that the intersection points of the associated edges $x_m \cap y_m$, $x_2 \cap y_2$, $x_3 \cap y_3$, $DE \cap D'E'$ lie in G. Therefore, we apply the result for $m = 5$ and conclude that the remaining intersection point $x_1 \cap y_1$ also lies in G.

Exercise. If d red lines and d blue lines intersect in d^2 points and if $2d$ of these points lie on an irreducible conic, then there is a unique curve of degree $d-2$ through the other $d^2 - 2d$ intersection points.

Solution. We have a $(d \times d)$-cage, where $2d$ nodes lie on an irreducible conic. Using Cayley-Bacharach (Theorem 3.6), let Γ' be the nodes of the cage off the conic and Γ'' be the $2d$ nodes in the conic. The dimension of the degree $d-2$ curves going through Γ' will be equal to the failure of Γ'' to impose independent conditions on forms of degree $2d-3-(d-2) = d-1$. Since the points in Γ'' lie on an irreducible conic, it will be a component of the form. By Bézout's Theorem, since $2(d-1) = 2d-2 < 2d-1$, then $2d-1$ points in Γ'' will be enough to impose the condition. Therefore the failure equals:

$$|\Gamma''| - (2d-1) = 2d - (2d-1) = 1.$$

Hence, up to scaling there is a unique degree-$(d-2)$ curve passing through the remaining $d^2 - 2d$ nodes of the cage off the conic.

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