The CMV Bispectral Problem

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A classical result due to Bochner classifies the orthogonal polynomials on the real line
which are common eigenfunctions of a second order linear differential operator. We
settle a natural version of the Bochner problem on the unit circle which answers a
similar question concerning orthogonal Laurent polynomials and can be formulated as
a bispectral problem involving CMV matrices. We solve this CMV bispectral problem
in great generality proving that, except the Lebesgue measure, no other one on the unit
circle yields a sequence of orthogonal Laurent polynomials which are eigenfunctions
of a linear differential operator of arbitrary order. Actually, we prove that this is the
case even if such an eigenfunction condition is imposed up to finitely many orthogonal
Laurent polynomials.

1 Introduction

The motivation for the problem we address here can be traced to work in signal pro-
cessing started by Shannon [39]. Addressing his problem required finding and exploiting
some remarkable mathematical miracles and was accomplished in a series of papers by
three workers at Bell Labs in the 1960s: David Slepian, Henry Landau, and Henry Pollak,
see [32, 33, 42–46]. The most important of these miracles is the existence of a second order

differential operator that commutes with Shannon's time-and-band limiting *integral* operator.

In an effort to understand and extend the range of applicability of these miracles one of us introduced the so called "bispectral problem," see [5]. For connections of this notion with the "time-and-band limiting problem" of Shannon, see for instance [4, 9–12, 24]. The basic idea is that bispectral instances should lead to situations featuring the remarkable algebraic properties exploited by D. Slepian, H. Landau, and H. Pollak. A strict connection between these two properties has not yet been established.

These algebraic properties have important numerical/practical consequences. For a very recent account of several computational issues, see [2, 29, 36]. For new areas of applications involving (sometimes) vector-valued quantities on the sphere, see [28, 37, 40, 41]. From a different numerical point of view see [7].

The study of the bispectral problem has moved in several fronts and led many unsuspected areas of mathematics, for a sample, see [13–16, 18–23, 51].

While the initial problem of C. Shannon was formulated in a continuous–continuous setup, the case of Fourier series (a discrete–continuous version) was handled by Slepian in [44], and the case of the DFT (a discrete–discrete version) was discussed in [8].

If one replaces the unit circle by the real line, these bispectral problems have a precedent in a continuous–discrete setup in the work of Bochner [1] and previous workers such as Routh [38]. They classified all families of orthogonal polynomials on the real line that admit a common second order differential operator having all of them as eigenfunctions. This constitutes a bispectral situation since orthogonal polynomials are (formal) eigenvectors of Jacobi matrices. The first step in going beyond second order differential operators was taken by [31]. This issue was later addressed in other situations, such as second order q-difference equations, where the Askey–Wilson polynomials were found to be the most general case. If one gets away from polynomials the class of solutions is much larger, see [16].

With all this as background we can state the contents of the paper: the natural extension of [44] by replacing the Lebesgue measure (the case of the Fourier series) by an arbitrary measure on the unit circle leads to a new bispectral problem, which we consider here. This takes us back to [50] who talked about orthogonal polynomials with respect to arbitrary measures on the unit circle. A better approach is taken in [3, 52] (see also [47, 48]), where one applies the Gram–Schmidt process to all the integer powers—and not only the positive ones as in [50]—and obtains an orthonormal basis for the corresponding L^2 space. We study the bispectral problem for this basis of Laurent polynomials.

This bispectral problem constitutes the natural analogue on the unit circle of the Bochner problem on the real line. The role of the Jacobi matrices in the ad-conditions is played now by its unitary counterpart, the CMV matrices [3, 47, 48, 52], which encode the recurrence relation for the orthonormal basis of Laurent polynomials. In other words, our aim is to find all the orthonormal Laurent polynomials (OLP) on the unit circle which are common eigenfunctions of a linear differential operator.

We will refer to this as the CMV bispectral problem since it can be formulated as a bispectral problem involving CMV matrices: to find all the CMV matrices whose (formal) eigenvectors, given by the corresponding OLP, are simultaneously eigenvectors of a linear differential operator.

The ad-conditions introduced in [5] have been the main workhorse to study different bispectral situations [14, 16, 18], and is the approach we are going to follow here, see Section 2. However, the standard ad-conditions become too messy to solve the CMV bispectral problem by applying them directly. Instead of this, we will exploit the unitarity and factorization properties of CMV matrices to transform the related ad-conditions into what we call the Hermitian ad-conditions because they come from the calculation of a Hermitian matrix. The result is a reduction of the ad-conditions in number and complexity, which allows us to solve them for second order linear differential operators, see Section 3.

Nevertheless, going beyond second order differential operators calls for more effective tools than solving ad-conditions by brute force. This is the aim of Sections 4 and 5, which develop the ad-integration and ad-factorization of ad-conditions. The ad-integration refers to a reduction in the order of the difference equations involved in the ad-conditions. The idea of solving ad-conditions by means of ad-integration was first advanced in [14] and then fully developed in [25] for the case of Jacobi matrices. The adaptation of this technique to CMV matrices is the objective of Section 4.

On the other hand, while the standard ad-conditions are defined in terms of the power of the ad-operator, given by a single commutator, the more useful Hermitian ad-conditions are not given by the power of any operator. Despite of this, Section 5 proves that the Hermitian ad-conditions factorize into lower order ones.

These are the main tools to tackle the general CMV bispectral problem in Section 6. We not only solve the CMV bispectral problem for linear differential operators of arbitrary order, but also assuming the corresponding eigenfunction condition up to finitely many OLP. Furthermore, the solution to this problem follows from the answer to a more general "bispectral" question in which a tridiagonal matrix takes the place of the diagonal matrix of eigenvalues for the differential operator. In all these

cases we find that the only solution to the CMV bispectral problem is given by the integer powers of a complex variable, which are the OLP related to the Lebesgue measure on the unit circle.

This is in contrast with the very rich structure of the solutions to the analogous problem on the real line. As it is pointed out in the conclusions of Section 7, this negative result should not be viewed as the end of the story, but could help us to focus our attention on those situations on the unit circle which could end in bispectral problems with non-trivial solutions. Besides, the triviality of the CMV bispectral problem can be used to test on the unit circle the not fully understood connections of bispectrality with the miracles behind the time-and-band limiting problem and its unexpected links with integrable systems.

2 Bispectral CMV matrices and ad-conditions

CMV matrices naturally arise in the study of orthogonality on the unit circle [3, 47, 48, 52]. For each probability measure μ with an infinite support lying on the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ we can consider the sequences $(\chi_n)_{n \geq 0}$ and $(x_n)_{n \geq 0}$ of OLP coming from the orthonormalization in L^2_μ of $(1, z, z^{-1}, z^2, z^{-2}, \dots)$ and $(1, z^{-1}, z, z^{-2}, z^2, \dots)$, respectively. Both sequences are related by the substar operation in the vector space $\mathbb{C}[z, z^{-1}]$ of Laurent polynomials,

$$\chi_n(z) = x_{n*}(z), \qquad f_*(z) = \overline{f(1/\bar{z})} \quad \forall f \in \mathbb{C}[z, z^{-1}].$$

The probability measures on $\mathbb{T}$ with infinite support are parameterized by the Verblunsky coefficients, a sequence $(\alpha_n)_{n \geq 0}$ in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ which generates the OLP via the five term recurrence relations

$$\mathcal{C}^t \chi(z) = z\chi(z), \quad \chi = \begin{pmatrix} \chi_0 \\ \chi_1 \\ \vdots \end{pmatrix}, \qquad \mathcal{C}x(z) = zx(z), \quad x = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \end{pmatrix}, \tag{1}$$

$$\mathcal{C} = \begin{pmatrix} \bar{\alpha}_0 & \rho_0\bar{\alpha}_1 & \rho_0\rho_1 & 0 & 0 & 0 & 0 & \dots \\ \rho_0 & -\alpha_0\bar{\alpha}_1 & -\alpha_0\rho_1 & 0 & 0 & 0 & 0 & \dots \\ 0 & \rho_1\bar{\alpha}_2 & -\alpha_1\bar{\alpha}_2 & \rho_2\bar{\alpha}_3 & \rho_2\rho_3 & 0 & 0 & \dots \\ 0 & \rho_1\rho_2 & -\alpha_1\rho_2 & -\alpha_2\bar{\alpha}_3 & -\alpha_2\rho_3 & 0 & 0 & \dots \\ 0 & 0 & 0 & \rho_3\bar{\alpha}_4 & -\alpha_3\bar{\alpha}_4 & \rho_4\bar{\alpha}_5 & \rho_4\rho_5 & \dots \\ 0 & 0 & 0 & \rho_3\rho_4 & -\alpha_3\rho_4 & -\alpha_4\bar{\alpha}_5 & -\alpha_4\rho_5 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \tag{2}$$

where $\rho_n = \sqrt{1 - |\alpha_n|^2}$. Both the five-diagonal unitary matrix $\mathcal{C}$ and its transpose $\mathcal{C}^t$ are named CMV matrices. The identities in (1) follow from the basic ones

$$z\boldsymbol{x}(z) = \mathcal{L}\boldsymbol{\chi}(z), \qquad \mathcal{L} = \begin{pmatrix} \Theta_0 & & & \\ & \Theta_2 & & \\ & & \Theta_4 & \\ & & & \ddots \end{pmatrix},$$

$$\boldsymbol{\chi}(z) = \mathcal{M}\boldsymbol{x}(z), \qquad \mathcal{M} = \begin{pmatrix} 1 & & & \\ & \Theta_1 & & \\ & & \Theta_3 & \\ & & & \ddots \end{pmatrix}, \qquad \Theta_n = \begin{pmatrix} \overline{\alpha}_n & \rho_n \\ \rho_n & -\alpha_n \end{pmatrix}, \tag{3}$$

hence $\mathcal{C} = \mathcal{L}\mathcal{M}$ and $\mathcal{C}^t = \mathcal{M}\mathcal{L}$ factorize as a product of a couple of 2×2-block diagonal symmetric unitary matrices. Using the shift matrix

$$S = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & \ddots & \ddots \end{pmatrix} \tag{4}$$

and its adjoint $S^\dagger$, these factors can be expressed as

$$\mathcal{L} = \mathcal{A}_e + \mathcal{B}_e S + S^\dagger \mathcal{B}_e, \qquad\qquad \mathcal{M} = \mathcal{A}_o + \mathcal{B}_o S + S^\dagger \mathcal{B}_o,$$

$$\mathcal{A}_e = \begin{pmatrix} \overline{\alpha}_0 & & & & \\ & -\alpha_0 & & & \\ & & \overline{\alpha}_2 & & \\ & & & -\alpha_2 & \\ & & & & \ddots \end{pmatrix}, \qquad \mathcal{A}_o = \begin{pmatrix} 1 & & & & & \\ & \overline{\alpha}_1 & & & & \\ & & -\alpha_1 & & & \\ & & & \overline{\alpha}_2 & & \\ & & & & -\alpha_2 & \\ & & & & & \ddots \end{pmatrix},$$

$$\mathcal{B}_e = \begin{pmatrix} \rho_0 & & & \\ & 0 & & \\ & & \rho_2 & \\ & & & 0 \\ & & & & \ddots \end{pmatrix}, \qquad \mathcal{B}_o = \begin{pmatrix} 0 & & & \\ & \rho_1 & & \\ & & 0 & \\ & & & \rho_3 \\ & & & & \ddots \end{pmatrix}. \tag{5}$$

For every $z \in \mathbb{C} \setminus \{0\}$, the relations in (1) identify $\boldsymbol{x}(z)$ and $\boldsymbol{\chi}(z)$ as formal eigenvectors with eigenvalue z for the matrices $\mathcal{C}$ and $\mathcal{C}^t$, respectively. Actually, Proposition A.1 in the Appendix implies that $\boldsymbol{x}(z)$ and $\boldsymbol{\chi}(z)$ span the set of such formal eigenvectors. Therefore, the search for OLP which are also eigenfunctions of a linear differential operator can be understood as a bispectral problem, which we will call the CMV bispectral problem. Every linear differential operator

$$D = \sum_{k=0}^{r} D_k(z) \frac{\mathrm{d}^k}{\mathrm{d}z^k} \tag{6}$$

arising from the CMV bispectral problem maps $\mathbb{C}[z, z^{-1}]$ on to itself because any Laurent polynomial is a finite linear combination of OLP. Applying D given by (6) to the powers z^k we see by induction on k that the linear differential operators $D \colon \mathbb{C}[z, z^{-1}] \to \mathbb{C}[z, z^{-1}]$ are those with Laurent polynomial coefficients $D_k(z)$.

The two CMV bispectral problems related to the OLP x_n or χ_n are essentially identical because both OLP are simultaneously eigenfunctions of a (different) linear differential operator. This is due to the equivalence

$$Lx_n = \lambda_n x_n \iff L_* \chi_n = \bar{\lambda}_n \chi_n, \qquad \lambda_n \in \mathbb{C},$$

for any linear operator L in $\mathbb{C}[z, z^{-1}]$, where L_* is the linear operator in $\mathbb{C}[z, z^{-1}]$ defined by

$$L_* f = (L f_*)_* \quad \forall f \in \mathbb{C}[z, z^{-1}].$$

In the case of a linear differential operator D, the substar operation yields also a linear differential operator D_* of the same order as D, a fact that follows from the general composition law $(\widetilde{LL})_* = \widetilde{L}_* L_*$ together with the substar of a single derivative,

$$\left(\frac{\mathrm{d}}{\mathrm{d}z} \right)_* = -z^2 \frac{\mathrm{d}}{\mathrm{d}z}.$$

Therefore, the relation

$$Dx_n = \lambda_n x_n \iff D_* \chi_n = \bar{\lambda}_n \chi_n,$$

shows that the CMV bispectral problem can be equivalently studied using $\mathcal{C}$ or $\mathcal{C}^t$.

For concreteness, in what follows we will use the CMV matrix $\mathcal{C}$. In other words, we will search for OLP x_n which are eigenfunctions of some linear differential operator D of arbitrary order $r \geq 1$, that is,

$$Dx_n = \lambda_n x_n, \qquad \lambda_n \in \mathbb{C}, \tag{7}$$

or equivalently

$$D\boldsymbol{x} = \Lambda \boldsymbol{x}, \qquad \Lambda = \begin{pmatrix} \lambda_0 & & & \\ & \lambda_1 & & \\ & & \lambda_2 & \\ & & & \ddots \end{pmatrix}.$$

In this case the CMV matrix $\mathcal{C}$ related to x_n will be called bispectral.

Actually, we will study a more general problem.

First, we will ask for the linear differential operator D in $\mathbb{C}[z, z^{-1}]$ to satisfy (7) up to finitely many OLP. This is equivalent to stating that $D\boldsymbol{x} = \Omega \boldsymbol{x}$ with Ω diagonal up to a finite submatrix, that is, $\Omega = \Omega_N \oplus \Lambda$ for some N, where Ω_N is the $N \times N$ principal submatrix of Ω and Λ is diagonal.

Second, we will search for solutions of more general relations than (7), namely,

$$Dx_n \in \mathrm{span}\{x_{n-1}, x_n, x_{n+1}\}. \tag{8}$$

This three-term difference-differential equation reads as $Dx = \Omega x$ with Ω tridiagonal. Indeed, we will assume (8) only from some index onwards, which is equivalent to stating that Ω is tridiagonal up to a finite submatrix.

These comments are the origin of the following definition.

Definition 2.1. We say that an infinite matrix Ω is almost (tri)diagonal if it is (tri)diagonal up to a finite submatrix, that is, $\Omega - \widetilde{\Omega}$ has finitely many non-zero entries for some infinite (tri)diagonal matrix $\widetilde{\Omega}$. $\qquad\square$

With this terminology, the existence of a linear differential operator D in $\mathbb{C}[z, z^{-1}]$ satisfying (7) or (8) for large enough n is equivalent to stating that $Dx = \Omega x$ with Ω almost diagonal and almost tridiagonal, respectively.

All these cases are covered by the relation

$$Dx = \Omega x, \qquad \Omega \text{ banded}, \tag{9}$$

a general situation characterized by "ad-conditions" (see [5]) involving the CMV matrix $\mathcal{C}$ related to x. Such CMV ad-conditions, made explicit in Theorem 2.2 below, are given in terms of a linear operator $(\mathrm{ad}\,\mathcal{C})$ in the vector space of band matrices, defined by the commutator

$$(\mathrm{ad}\,\mathcal{C})\Omega = [\mathcal{C}, \Omega] = \mathcal{C}\Omega - \Omega\mathcal{C}.$$

By induction, its powers can be seen to have the explicit form

$$(\mathrm{ad}\,\mathcal{C})^n \Omega = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \mathcal{C}^{n-k} \Omega \mathcal{C}^k. \tag{10}$$

This operator is essential in the following result, key for this paper, which is a translation of the general ideas in [5] to the case of CMV matrices. It characterizes the relation (9) in terms of CMV ad-conditions. Proposition A.1 in the Appendix will be crucial for the proof.

In what follows, I stands for the infinite identity matrix.

Theorem 2.2.   Given a sequence x_n of OLP on the unit circle with CMV matrix $\mathcal{C}$, the following conditions are equivalent for any band matrix Ω:

  (i)   There is a linear differential operator D of order at most r such that $Dx = \Omega x$.
  (ii) $(\operatorname{ad} \mathcal{C})^{r+1} \Omega = 0$.   □

Proof.   Condition *(i)* reads as $\Omega x(z) \in \operatorname{span}\{x(z), x'(z), \ldots, x^{(r)}(z)\}$, $z \in \mathbb{C} \setminus \{0\}$. In view of Proposition A.1 in the Appendix, this is equivalent to $\Omega x(z) \in \ker(\mathcal{C} - zI)^{r+1}$, that is, $(\mathcal{C} - zI)^{r+1} \Omega x(z) = 0$. Using the expansion

$$(\mathcal{C} - zI)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} z^k \, \mathcal{C}^{n-k}, \tag{11}$$

together with (1) and (10), we find that

$$(\mathcal{C} - zI)^{r+1} \Omega x(z) = 0 \;\Leftrightarrow\; \sum_{k=0}^{r+1} (-1)^k \binom{r+1}{k} \mathcal{C}^{r+1-k} \Omega \, \mathcal{C}^k x(z) = 0$$

$$\Leftrightarrow\; (\operatorname{ad} \mathcal{C})^{r+1} \Omega x(z) = 0.$$

Due to the linear independence of the OLP, the last condition is equivalent to *(ii)*.   ■

The fact that the ad-conditions involve no information about the explicit form of the operator D, apart from its order, makes the previous characterization particularly useful for discovering bispectral situations.

The linear differential operator D involved in Theorem 2.2 has the freedom of a constant factor and an additive constant, which corresponds to the freedom of the band matrix Ω in a numerical factor and the addition of a multiple of the identity. Among the solutions Ω of the ad-conditions we must discard the multiples of the identity as trivial solutions corresponding to differential operators of order zero. In what follows, we will use the expression "linear differential operators" to refer only to those of order greater than zero, that is, with the form (6) and $D_r(z) \neq 0$ for $r \geq 1$.

The operator $(\operatorname{ad} \mathcal{C})$ has a symmetry, inherited from the unitarity of the CMV matrix $\mathcal{C}$, which will be further exploited in the next section.

Proposition 2.3.   For any CMV matrix $\mathcal{C}$ and any band matrix Ω,

$$(\operatorname{ad} \mathcal{C})^n \Omega^\dagger = \mathcal{C}^n ((\operatorname{ad} \mathcal{C})^n \Omega)^\dagger \, \mathcal{C}^n. \qquad\qquad □$$

Proof.   For $n = 1$,

$$\mathcal{C}((\operatorname{ad}\mathcal{C})\Omega)^{\dagger}\mathcal{C} = \mathcal{C}(\mathcal{C}\Omega - \Omega\mathcal{C})^{\dagger}\mathcal{C} = \mathcal{C}\Omega^{\dagger} - \Omega^{\dagger}\mathcal{C} = (\operatorname{ad}\mathcal{C})\Omega^{\dagger}.$$

Assuming the identity for an index n,

$$\begin{aligned}
\mathcal{C}^{n+1}((\operatorname{ad}\mathcal{C})^{n+1}\Omega)^{\dagger}\mathcal{C}^{n+1} &= \mathcal{C}(\operatorname{ad}\mathcal{C})^{n}(((\operatorname{ad}\mathcal{C})\Omega)^{\dagger})\mathcal{C} \\
&= (\operatorname{ad}\mathcal{C})^{n}(\mathcal{C}((\operatorname{ad}\mathcal{C})\Omega)^{\dagger}\mathcal{C}) = (\operatorname{ad}\mathcal{C})^{n+1}\Omega^{\dagger}.
\end{aligned}$$

∎

As a consequence of the previous result,

$$(\operatorname{ad}\mathcal{C})^{n}\Omega = 0 \ \Leftrightarrow \ (\operatorname{ad}\mathcal{C})^{n}\Omega^{\dagger} = 0 \ \Leftrightarrow \ \begin{cases} (\operatorname{ad}\mathcal{C})^{n}\operatorname{Re}\Omega = 0, & \operatorname{Re}\Omega = \frac{1}{2}(\Omega + \Omega^{\dagger}), \\ (\operatorname{ad}\mathcal{C})^{n}\operatorname{Im}\Omega = 0, & \operatorname{Im}\Omega = \frac{1}{2i}(\Omega - \Omega^{\dagger}). \end{cases} \tag{12}$$

This means that the solutions of the CMV ad-conditions can be chosen Hermitian without loss, since any solution Ω splits into Hermitian ones, $\operatorname{Re}\Omega$, and $\operatorname{Im}\Omega$. Therefore, in the CMV bispectral problem we can assume that the eigenvalues of the linear differential operator are real.

In view of Theorem 2.2 and the previous comments, the CMV bispectral problem can be reduced to the search for CMV matrices $\mathcal{C}$ with non-trivial real diagonal solutions Λ of the CMV ad-conditions $(\operatorname{ad}\mathcal{C})^{n}\Lambda = 0$ for some $n \geq 2$.

3   The Hermitian ad-conditions

The CMV ad-conditions $(\operatorname{ad}\mathcal{C})^{n}\Lambda = 0$ are too difficult to solve the problem directly in this way, but they can be rewritten in a more manageable form.

The matrix $(\operatorname{ad}\mathcal{C})^{n}\Lambda$ is $(4n + 1)$-diagonal for any diagonal Λ, thus $(\operatorname{ad}\mathcal{C})^{n}\Lambda = 0$ gives $4n + 1$ difference equations, one for each diagonal. These difference equations are not all independent, so it should be possible to reorganize these ad-conditions in a smarter way. For this purpose we will introduce a narrower CMV ad-operator $(\operatorname{ad}_{n}\mathcal{C})$ which preserves hermiticity and such that $(\operatorname{ad}\mathcal{C})^{n}\Lambda = 0$ iff $(\operatorname{ad}_{n}\mathcal{C})\Lambda = 0$. This will reduce the number of difference equations. The key result is Proposition 2.3 which shows that, when Λ is real, $(\operatorname{ad}\mathcal{C})^{n}\Lambda = \mathcal{C}^{n}((\operatorname{ad}\mathcal{C})^{n}\Lambda)^{\dagger}\mathcal{C}^{n}$. Hence, we can get an Hermitian matrix by multiplying $(\operatorname{ad}\mathcal{C})^{n}\Lambda$ on the left and the right by "half" of the matrix factors in $(\mathcal{C}^{\dagger})^{n}$. This suggests the following definition.

Definition 3.1.   For any CMV matrix $\mathcal{C} = \mathcal{LM}$ and any band matrix Ω we define

$$(\mathrm{ad}_n\,\mathcal{C})\Omega := \begin{cases} (\mathcal{C}^\dagger)^m((\mathrm{ad}\,\mathcal{C})^n\Omega)(\mathcal{C}^\dagger)^m, & n = 2m, \\ \mathcal{L}^\dagger(\mathcal{C}^\dagger)^m((\mathrm{ad}\,\mathcal{C})^n\Omega)(\mathcal{C}^\dagger)^m\mathcal{M}^\dagger, & n = 2m+1. \end{cases} \qquad \square$$

Due to the unitarity of $\mathcal{L}$ and $\mathcal{M}$,

$$(\mathrm{ad}\,\mathcal{C})^n\Omega = 0 \ \Leftrightarrow\ (\mathrm{ad}_n\,\mathcal{C})\Omega = 0. \tag{13}$$

Therefore, Theorem 2.2 can be restated in the following way: given a sequence of OLP x_n on the unit circle with CMV matrix $\mathcal{C}$, for any band matrix Ω the Hermitian ad-conditions $(\mathrm{ad}_{r+1}\,\mathcal{C})\Omega = 0$ characterize the existence of a linear differential operator D of order at most r such that $D\boldsymbol{x} = \Omega\boldsymbol{x}$.

Also, Proposition 2.3 implies that

$$(\mathrm{ad}_n\,\mathcal{C})\Omega^\dagger = ((\mathrm{ad}_n\,\mathcal{C})\Omega)^\dagger, \tag{14}$$

so that $(\mathrm{ad}_n\,\mathcal{C})\Omega$ is Hermitian whenever $\Omega^\dagger = \Omega$, a requirement that we can assume without loss.

From the definition of $(\mathrm{ad}_n\,\mathcal{C})\Omega$ we obtain directly the recursion

$$(\mathrm{ad}_{n+1}\,\mathcal{C})\Omega = \begin{cases} \mathcal{M}((\mathrm{ad}_n\,\mathcal{C})\Omega)\mathcal{M}^\dagger - \mathcal{L}^\dagger((\mathrm{ad}_n\,\mathcal{C})\Omega)\mathcal{L}, & \text{even } n, \\ \mathcal{L}((\mathrm{ad}_n\,\mathcal{C})\Omega)\mathcal{L}^\dagger - \mathcal{M}^\dagger((\mathrm{ad}_n\,\mathcal{C})\Omega)\mathcal{M}, & \text{odd } n, \end{cases} \tag{15}$$

$$(\mathrm{ad}_0\,\mathcal{C})\Omega = \Omega,$$

which allows us to find easily the explicit form of $(\mathrm{ad}_n\,\mathcal{C})\Omega$ for small values of n,

$(\mathrm{ad}_1\,\mathcal{C})\Omega = \mathcal{M}\Omega\mathcal{M}^\dagger - \mathcal{L}^\dagger\Omega\mathcal{L},$

$(\mathrm{ad}_2\,\mathcal{C})\Omega = \mathcal{LM}\Omega\mathcal{M}^\dagger\mathcal{L}^\dagger - 2\Omega + \mathcal{M}^\dagger\mathcal{L}^\dagger\Omega\mathcal{LM},$

$(\mathrm{ad}_3\,\mathcal{C})\Omega = \mathcal{MLM}\Omega\mathcal{M}^\dagger\mathcal{L}^\dagger\mathcal{M}^\dagger - 3\mathcal{M}\Omega\mathcal{M}^\dagger + 3\mathcal{L}^\dagger\Omega\mathcal{L} - \mathcal{L}^\dagger\mathcal{M}^\dagger\mathcal{L}^\dagger\Omega\mathcal{LML},$

$(\mathrm{ad}_4\,\mathcal{C})\Omega = \mathcal{LMLM}\Omega\mathcal{M}^\dagger\mathcal{L}^\dagger\mathcal{M}^\dagger\mathcal{L}^\dagger - 4\mathcal{LM}\Omega\mathcal{M}^\dagger\mathcal{L}^\dagger + 6\Omega - 4\mathcal{M}^\dagger\mathcal{L}^\dagger\Omega\mathcal{LM} + \mathcal{M}^\dagger\mathcal{L}^\dagger\mathcal{M}^\dagger\mathcal{L}^\dagger\Omega\mathcal{LMLM}.$

For an arbitrary value of n, using the expansion (10) we obtain

$$(\mathrm{ad}_n\,\mathcal{C})\Omega = \big(\overset{r)}{\cdots}\mathcal{MLM}\big)\Omega\big(\mathcal{M}^\dagger\mathcal{L}^\dagger\mathcal{M}^\dagger\overset{r)}{\cdots}\big) + (-1)^r\big(\overset{r)}{\cdots}\mathcal{L}^\dagger\mathcal{M}^\dagger\mathcal{L}^\dagger\big)\Omega\big(\mathcal{LML}\overset{r)}{\cdots}\big)$$

$$+ \text{ narrower band matrices.}$$

Therefore, when Λ is real diagonal, bearing in mind that $\mathcal{M}\Lambda\mathcal{M}^\dagger$ and $\mathcal{L}^\dagger\Lambda\mathcal{L}$ are tridiagonal, we find that $(\mathrm{ad}_n\,\mathcal{C})\Lambda$ is a $(4n-1)$-diagonal Hermitian matrix, so the ad-conditions $(\mathrm{ad}_n\,\mathcal{C})\Lambda = 0$ only lead to $2n$ difference equations, corresponding to the main

and upper diagonals. We can write explicitly the equations of $(\mathrm{ad}_n\,\mathcal{C})\Lambda = 0$ for the first values of n. We will order the equations running from the top upper diagonal ($2n$th diagonal) to the main one (first diagonal), using the previous equations to simplify the new ones and omitting them when they yield no independent equation. Proceeding in this way we obtain the following results for $n = 2, 3$:

$(\mathrm{ad}_2\,\mathcal{C})\Lambda = 0$	
Fourth diagonal	$(\lambda_{k+1} - \lambda_k)\alpha_k = 0, \quad k \geq 1$
Third diagonal	$(\lambda_1 - \lambda_0)\alpha_0 = 0$
Second diagonal	$(\lambda_2 - \lambda_0)\alpha_0 = 0$
	$(\lambda_{k+2} - \lambda_{k-1})\alpha_k = 0, \quad k \geq 1$
First diagonal	$\lambda_2 - 2\lambda_0 + \lambda_1 = 0$
	$\lambda_3 - 2\lambda_1 + \lambda_0 = 0$
	$\lambda_{k+4} - 2\lambda_{k+2} + \lambda_k = 0, \quad k \geq 0$

$(\mathrm{ad}_3\,\mathcal{C})\Lambda = 0$	
Sixth diagonal	$(\lambda_{k+1} - \lambda_k)\alpha_k = 0, \quad k \geq 2$
Fifth diagonal	$(\lambda_2 - \lambda_1)\alpha_1 = 0$
Fourth diagonal	$(\lambda_3 - \lambda_0)\alpha_1 = (\lambda_1 - \lambda_0)\alpha_0(\overline{\alpha}_0\alpha_1 - \alpha_0)$
	$(\lambda_{k+2} - \lambda_{k-1})\alpha_k = 0, \quad k \geq 2$
Third diagonal	$(\lambda_2 - \lambda_0)\alpha_0 = 0, \quad (\lambda_1 - \lambda_0)\alpha_0 = 0$
Second diagonal	$(\lambda_3 - \lambda_0)\alpha_0 = 0, \quad (\lambda_4 - \lambda_0)\alpha_1 = 0$
	$(\lambda_{k+3} - \lambda_{k-2})\alpha_k = 0, \quad k \geq 2$
First diagonal	$\lambda_3 - 3\lambda_1 + 3\lambda_0 - \lambda_2 = 0$
	$\lambda_4 - 3\lambda_2 + 3\lambda_0 - \lambda_1 = 0$
	$\lambda_5 - 3\lambda_3 + 3\lambda_1 - \lambda_0 = 0$
	$\lambda_{k+6} - 3\lambda_{k+4} + 3\lambda_{k+2} - \lambda_k = 0, \quad k \geq 0$

We can reorganize the above equations in a more natural way. In the following tables the equations with the same shape are grouped in the same column, except for the equations in asterisk (*) at the top of some columns, which are slightly different.

$(\mathrm{ad}_2\,\mathcal{C})\Lambda = 0$		
Eq_1	Eq_2	RR
$(\lambda_1 - \lambda_0)\alpha_0 = 0$	$(\lambda_2 - \lambda_0)\alpha_0 = 0^*$	$\lambda_2 - 2\lambda_0 + \lambda_1 = 0^*$
$(\lambda_2 - \lambda_1)\alpha_1 = 0$	$(\lambda_3 - \lambda_0)\alpha_1 = 0$	$\lambda_3 - 2\lambda_1 + \lambda_0 = 0^*$
$(\lambda_3 - \lambda_2)\alpha_2 = 0$	$(\lambda_4 - \lambda_1)\alpha_2 = 0$	$\lambda_4 - 2\lambda_2 + \lambda_0 = 0$
$(\lambda_4 - \lambda_3)\alpha_3 = 0$	$(\lambda_5 - \lambda_2)\alpha_3 = 0$	$\lambda_5 - 2\lambda_3 + \lambda_1 = 0$
$\vdots$	$\vdots$	$\vdots$

$(\mathrm{ad}_3\,\mathcal{C})\Lambda = 0$			
Eq_1	Eq_2	Eq_3	RR
$(\lambda_1 - \lambda_0)\alpha_0 = 0$	$(\lambda_2 - \lambda_0)\alpha_0 = 0^*$	$(\lambda_3 - \lambda_0)\alpha_0 = 0^*$	$\lambda_3 - 3\lambda_1 + 3\lambda_0 - \lambda_2 = 0^*$
$(\lambda_2 - \lambda_1)\alpha_1 = 0$	$(\lambda_3 - \lambda_0)\alpha_1 = 0$	$(\lambda_4 - \lambda_0)\alpha_1 = 0^*$	$\lambda_4 - 3\lambda_2 + 3\lambda_0 - \lambda_1 = 0^*$
$(\lambda_3 - \lambda_2)\alpha_2 = 0$	$(\lambda_4 - \lambda_1)\alpha_2 = 0$	$(\lambda_5 - \lambda_0)\alpha_2 = 0$	$\lambda_5 - 3\lambda_3 + 3\lambda_1 - \lambda_0 = 0^*$
$(\lambda_4 - \lambda_3)\alpha_3 = 0$	$(\lambda_5 - \lambda_2)\alpha_3 = 0$	$(\lambda_6 - \lambda_1)\alpha_3 = 0$	$\lambda_6 - 3\lambda_4 + 3\lambda_2 - \lambda_0 = 0$
$(\lambda_5 - \lambda_4)\alpha_4 = 0$	$(\lambda_6 - \lambda_3)\alpha_4 = 0$	$(\lambda_7 - \lambda_2)\alpha_4 = 0$	$\lambda_7 - 3\lambda_5 + 3\lambda_3 - \lambda_1 = 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Let us use the previous results to find, for instance, the CMV matrices $\mathcal{C}$ with non-trivial real diagonal solutions Λ for $(\mathrm{ad}_2\,\mathcal{C})\Lambda = 0$. All but the first two entries in the column RR of the corresponding table yield the recurrence relation

$$\lambda_{k+4} - 2\lambda_{k+2} + \lambda_k = 0, \qquad k \geq 0,$$

whose general solution is

$$\lambda_k = a_0 + a_1 k + (-1)^k (b_0 + b_1 k), \qquad a_i, b_i \in \mathbb{R}.$$

Imposing the remaining two conditions of RR,

$$\lambda_2 - 2\lambda_0 + \lambda_1 = 0, \qquad \lambda_3 - 2\lambda_1 + \lambda_0 = 0,$$

yields $a_1 = 0$ and $b_1 = 2b_0$, that is,

$$\lambda_k = a_0 + b_0(-1)^k(1 + 2k). \tag{16}$$

Then, if $\alpha_j \neq 0$ for some index j, the equation $(\lambda_{j+1} - \lambda_j)\alpha_j = 0$ of the column Eq_1 implies that

$$0 = \lambda_{j+1} - \lambda_j = 4b_0(-1)^{j+1}(1 + j),$$

so that $b_0 = 0$ and Λ is a multiple of the identity.

Therefore, the only non-trivial solutions may appear when $\alpha_k = 0$ for all k. In this case the equations of the columns Eq_1 and Eq_2 are automatically satisfied and the general solution of $(\mathrm{ad}_2\,\mathcal{C})\Lambda = 0$ is given by (16), that is,

$$\lambda_k = \lambda_0 + (\lambda_0 - \lambda_1)\frac{(-1)^k(1 + 2k) - 1}{4}, \qquad \lambda_0, \lambda_1 \in \mathbb{R}.$$

In other words,

$$\Lambda = \begin{pmatrix} 0 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & -2 & \\ & & & & 2 \\ & & & & & \ddots \end{pmatrix} \tag{17}$$

up to numerical factors and addition of multiples of the identity. This solution corresponds to the OLP

$$x_{2m-1}(z) = z^{-m}, \qquad x_{2m}(z) = z^m,$$

associated with the Lebesgue measure on the unit circle, which satisfy $D\boldsymbol{x} = \Lambda\boldsymbol{x}$ for the first order linear differential operator

$$D = z\frac{\mathrm{d}}{\mathrm{d}z}.$$

Using the results of the table for $(\mathrm{ad}_3\,\mathcal{C})\Lambda = 0$ we find that a similar analysis works for these ad-conditions. Concerning the column RR, the first three equations impose on the general solution of the remaining equations

$$\lambda_{k+6} - 3\lambda_{k+4} + 3\lambda_{k+2} - \lambda_k = 0, \qquad k \geq 0,$$

given by

$$\lambda_k = a_0 + a_1 k + a_2 k^2 + (-1)^k (b_0 + b_1 k + b_2 k^2), \qquad a_i, b_i \in \mathbb{R},$$

the constraints $a_2 = a_1$, $b_1 = 2b_0$, and $b_2 = 0$. If $\alpha_j \neq 0$ for some j, using Eq_1 and Eq_2 we find again that Λ is a multiple of the identity. This leaves as the only non-trivial solution that one related to the Lebesgue measure as in the previous case.

As a consequence we have the following version of Bochner theorem for OLP on the unit circle.

Theorem 3.2. The only OLP on the unit circle which are eigenfunctions of a linear differential operator of order not greater than two are those orthonormal with respect to the Lebesgue measure. $\qquad\square$

We have seen that the simplicity of the Hermitian ad-conditions is enough to deal with the CMV bispectral problem for linear differential operators of lower degree just by brute force. However, to go beyond this we need to further develop the machinery of the CMV ad-conditions.

4 The CMV ad-conditions: ad-integration

The CMV ad-conditions involve commutators with a CMV matrix. Hence, the study of the centralizer of a CMV matrix can help us to find a short-cut to the solution of such ad-conditions.

Definition 4.1. We denote by $\mathcal{Z}(\mathcal{C})$ the centralizer of the CMV matrix $\mathcal{C}$ in the multiplicative group of infinite band matrices, that is,

$$\mathcal{Z}(\mathcal{C}) := \{\Omega \text{ band matrix} : [\mathcal{C}, \Omega] = 0\}.$$

We can write $\mathcal{Z}(\mathcal{C}) = \cup_{n\geq 0}\mathcal{Z}_n(\mathcal{C})$, where

$$\mathcal{Z}_n(\mathcal{C}) := \{\Omega \ (2n+1)\text{-diagonal matrix} : [\mathcal{C}, \Omega] = 0\}. \qquad\square$$

The centralizer of a CMV matrix among banded matrices can be explicitly determined. Concerning the result below, keep in mind that, due to the unitarity of a CMV matrix $\mathcal{C}$, its inverse $\mathcal{C}^{-1} = \mathcal{C}^{\dagger}$ is also banded.

Proposition 4.2.   For any CMV matrix $\mathcal{C}$,

$$\mathcal{Z}(\mathcal{C}) = \{f(\mathcal{C}) : f \in \mathbb{C}[z, z^{-1}]\},$$

$$\mathcal{Z}_{2m}(\mathcal{C}) = \mathcal{Z}_{2m+1}(\mathcal{C}) = \{f(\mathcal{C}) : f \in \operatorname{span}\{1, z^{-1}, z, z^{-2}, z^2, \ldots, z^{-m}, z^m\}\}. \qquad \square$$

Proof.   If x_n are the OLP related to $\mathcal{C}$ we know from Proposition A.1 that, for every $z \in \mathbb{C} \setminus \{0\}$, $\boldsymbol{x}(z)$ spans the set of formal eigenvectors of $\mathcal{C}$ with eigenvalue z. Thus, due to the linear independence of the OLP, given a band matrix Ω,

$$[\mathcal{C}, \Omega] = 0 \;\Leftrightarrow\; [\mathcal{C}, \Omega]\boldsymbol{x}(z) = 0 \;\Leftrightarrow\; (\mathcal{C} - zI)\Omega\boldsymbol{x}(z) = 0 \;\Leftrightarrow\; \Omega\boldsymbol{x}(z) = f(z)\boldsymbol{x}(z),$$

for some function $f \colon \mathbb{C} \to \mathbb{C}$. From the last equality,

$$f(z) = f(z)x_0(z) = \sum_k \Omega_{0,k}\, x_k(z),$$

hence $f \in \mathbb{C}[z, z^{-1}]$ because Ω is banded.

Also, if Ω is $(2n+1)$-diagonal,

$$f \in \operatorname{span}\{x_k\}_{k=0}^n = \begin{cases} \operatorname{span}\{1, z^{-1}, z, z^{-2}, z^2, \ldots, z^{-m}, z^m\}, & n = 2m, \\ \operatorname{span}\{1, z^{-1}, z, z^{-2}, z^2, \ldots, z^{-m}, z^m, z^{-m-1}\}, & n = 2m+1, \end{cases}$$

so that, for some $a_j \in \mathbb{C}$, we have that $\Omega = \sum_{j=-m}^m a_j \mathcal{C}^j$ if Ω is $(4m+1)$-diagonal, while $\Omega = \sum_{j=-m-1}^m a_j \mathcal{C}^j$ when Ω is $(4m+3)$-diagonal. However, since $\mathcal{C}^j$ and $\mathcal{C}^{-j} = (\mathcal{C}^{\dagger})^j$ are both strictly $(4j+1)$-diagonal, $a_{-m-1} = 0$ in the last case because otherwise Ω would be $(2n+3)$-diagonal but not $(2n+1)$-diagonal.

This proves that

$$\mathcal{Z}(\mathcal{C}) \subset \{f(\mathcal{C}) : f \in \mathbb{C}[z, z^{-1}]\},$$

$$\mathcal{Z}_{2m}(\mathcal{C}), \mathcal{Z}_{2m+1}(\mathcal{C}) \subset \{f(\mathcal{C}) : f \in \operatorname{span}\{1, z^{-1}, z, z^{-2}, z^2, \ldots, z^{-m}, z^m\}\}.$$

The reverse inclusions are obvious.   ∎

The above result permits the integration of the CMV ad-conditions: if $\mathcal{C}$ is a CMV matrix, for any band matrix Ω,

$$(\operatorname{ad}\mathcal{C})^{n+1}\Omega = 0 \;\Leftrightarrow\; (\operatorname{ad}\mathcal{C})^n\Omega = f(\mathcal{C}), \quad f \in \mathbb{C}[z, z^{-1}].$$

In the case of Ω diagonal we can say much more. Indeed, we will state the result for Ω tridiagonal because it needs no more effort due to the equality $\mathcal{Z}_{2m}(\mathcal{C}) = \mathcal{Z}_{2m+1}(\mathcal{C})$.

Proposition 4.3. If $\mathcal{C}$ is a CMV matrix and Ω is a tridiagonal matrix, then

$$(\operatorname{ad}\mathcal{C})^{n+1}\Omega = 0 \;\Leftrightarrow\; (\operatorname{ad}\mathcal{C})^n\Omega = a\mathcal{C}^n \;\Leftrightarrow\; (\operatorname{ad}_n\mathcal{C})\Omega = aI, \qquad a \in \mathbb{C}. \qquad \square$$

Proof. Let Ω be tridiagonal. Then, $(\operatorname{ad}\mathcal{C})^n\Omega$ is $(4n+3)$-diagonal and

$$(\operatorname{ad}\mathcal{C})^{n+1}\Omega = 0 \;\Leftrightarrow\; (\operatorname{ad}\mathcal{C})^n\Omega \in \mathcal{Z}_{2n+1}(\mathcal{C}) \;\Leftrightarrow\; (\operatorname{ad}\mathcal{C})^n\Omega = \sum_{j=-n}^{n} a_j\mathcal{C}^j, \; a_j \in \mathbb{C}.$$

If Ω is Hermitian, Proposition 2.3 states that $(\operatorname{ad}\mathcal{C})^n\Omega = \mathcal{C}^n((\operatorname{ad}\mathcal{C})^n\Omega)^\dagger\mathcal{C}^n$, hence

$$\sum_{j=-r}^{r} a_j\mathcal{C}^j = \mathcal{C}^r\left(\sum_{j=-r}^{r} \overline{a}_j\mathcal{C}^{-j}\right)\mathcal{C}^r = \sum_{j=-r}^{r} \overline{a}_j\mathcal{C}^{2r-j} = \sum_{j=r}^{3r} \overline{a}_{2r-j}\mathcal{C}^j.$$

On the other hand, the set $\{\mathcal{C}^j\}_{j\in\mathbb{Z}}$ is linearly independent: if $\sum_j b_j\mathcal{C}^j = 0$, $b_j \in \mathbb{C}$, then $0 = \sum_j b_j\mathcal{C}^j\boldsymbol{x}(z) = (\sum_j b_j z^j)\boldsymbol{x}(z)$, thus $\sum_j b_j z^j = 0$ and $b_j = 0$ for all j. From these results we conclude that $a_j = 0$ for $j \neq r$ and $a_r \in \mathbb{R}$, which proves the proposition for a Hermitian Ω.

The result for a non-Hermitian Ω follows from (12). $\blacksquare$

Similar ad-integration techniques to those in Proposition 4.3 have been considered previously in [25] for Jacobi bispectral problems.

As an illustration of the ad-integration techniques, we will use them to present a simplified resolution of the ad-conditions $(\operatorname{ad}\mathcal{C})^2\Lambda = 0$ for a diagonal matrix Λ. According to Proposition 4.3, this is equivalent to solve $(\operatorname{ad}_1\mathcal{C})\Lambda \propto I$. Using the notation λ_k, $k \geq 0$, for the diagonal coefficients of Λ, a simple calculation yields

$$(\operatorname{ad}_1\mathcal{C})\Lambda = \begin{pmatrix} b_0 & -a_0 & & & & \\ -\overline{a}_0 & -b_1 & \overline{a}_1 & & & \\ & a_1 & b_2 & -a_2 & & \\ & & -\overline{a}_2 & -b_3 & \overline{a}_3 & \\ & & & a_3 & b_4 & -a_4 \\ & & & & \ddots & \ddots & \ddots \end{pmatrix},$$

$$a_k = (\lambda_k - \lambda_{k+1})\rho_k\alpha_k, \qquad b_k = \begin{cases} (\lambda_0 - \lambda_1)\rho_0^2, & k = 0, \\ (\lambda_{k-1} - \lambda_k)\rho_{k-1}^2 + (\lambda_k - \lambda_{k+1})\rho_k^2, & k \geq 1. \end{cases}$$

Therefore,

$$(\mathrm{ad}_1\, \mathcal{C})\Lambda \propto I \;\Leftrightarrow\; \begin{cases} a_k = 0, \\ b_{k+1} = -b_k, \end{cases} \;\Leftrightarrow\; \begin{cases} (\lambda_k - \lambda_{k+1})\alpha_k = 0, \\ (\lambda_k - \lambda_{k+1})\rho_k^2 = (-1)^k(k+1)(\lambda_0 - \lambda_1)\rho_0^2. \end{cases}$$

If $\alpha_j \neq 0$ for some j, the above relations imply that $\lambda_k = \lambda_{k+1}$ for all k, that is, $\Lambda \propto I$. On the other hand, when $\alpha_k = 0$ for all k the condition $a_k = 0$ is automatically satisfied, while $b_{k+1} = -b_k$ determines Λ as in (17) up to numerical factors and addition of multiples of the identity.

5   The Hermitian ad-conditions: ad-factorization

Another useful tool to deal with the CMV bispectral problem is the ad-factorization of the Hermitian ad-operator. The original ad-operator $(\mathrm{ad}\,\mathcal{C})^n$ is by definition a power of the simple ad-operator $(\mathrm{ad}\,\mathcal{C})$, but this is no longer true for the Hermitian ad-operator $(\mathrm{ad}_n\, \mathcal{C})$. To understand the ad-factorization of $(\mathrm{ad}_n\, \mathcal{C})$ let us exploit again the possibility of approaching the CMV bispectral problem using two kinds of OLP, x_n and χ_n.

Section 2 shows that the previous results about the bispectral problem for x_n can be translated to the bispectral problem for χ_n just by performing the following transformations:

$$\begin{aligned} x_n &\longrightarrow \chi_n \\ \mathcal{C} &\longrightarrow \mathcal{C}^t \\ \mathcal{L} &\longrightarrow \mathcal{M} \\ \mathcal{M} &\longrightarrow \mathcal{L}. \end{aligned}$$

For instance, the bispectral problem $D\chi = \Lambda\chi$ can be solved by using the ad-conditions $(\mathrm{ad}\,\mathcal{C}^t)^n\Lambda = 0$. These ad-conditions are equivalent to the Hermitian ones $(\mathrm{ad}_n\, \mathcal{C}^t)\Lambda = 0$, where the definition and properties of $(\mathrm{ad}_n\, \mathcal{C}^t)$ can be obtained from those of $(\mathrm{ad}_n\, \mathcal{C})$ by simply making the exchanges $\mathcal{C} \leftrightarrow \mathcal{C}^t$ and $\mathcal{L} \leftrightarrow \mathcal{M}$.

A number of properties relate the ad-operators $(\mathrm{ad}_n\, \mathcal{C})$ and $(\mathrm{ad}_n\, \mathcal{C}^t)$, among them the ad-factorization that we are interested in. The following proposition summarizes these properties.

Proposition 5.1.   Given a CMV matrix $\mathcal{C}$, the following relations hold for any band matrix Ω:

(i) $((\mathrm{ad}_n\, \mathcal{C})\Omega)^t = (-1)^n(\mathrm{ad}_n\, \mathcal{C}^t)\Omega^t.$

(ii) $(\mathrm{ad}_n \, \mathcal{C})(\mathcal{L}\Omega\mathcal{M}) = \begin{cases} \mathcal{L}((\mathrm{ad}_n \, \mathcal{C}^t)\Omega)\mathcal{M}, & \text{even } n, \\ \mathcal{M}((\mathrm{ad}_n \, \mathcal{C}^t)\Omega)\mathcal{L}, & \text{odd } n. \end{cases}$

(iii) $(\mathrm{ad}_n \, \mathcal{C})\Omega = (\mathrm{ad}_{n-k} \, \mathcal{C}(k))((\mathrm{ad}_k \, \mathcal{C})\Omega), \quad \mathcal{C}(k) := \begin{cases} \mathcal{C}, & \text{even } k, \\ \mathcal{C}^t, & \text{odd } k. \end{cases}$ $\square$

Proof.   Property *(i)* follows from Definition 3.1 of $(\mathrm{ad}_n \, \mathcal{C})\Omega$ and the corresponding one for $(\mathrm{ad}_n \, \mathcal{C}^t)\Omega$, together with the relation $((\mathrm{ad}\,\mathcal{C})^n\Omega)^t = (-1)^n (\mathrm{ad}\,\mathcal{C}^t)^n\Omega^t$, obtained iterating $((\mathrm{ad}\,\mathcal{C})\Omega)^t = -(\mathrm{ad}\,\mathcal{C}^t)\Omega^t$.

Analogously, the iteration of $(\mathrm{ad}\,\mathcal{C})(\mathcal{L}\Omega\mathcal{M}) = \mathcal{L}((\mathrm{ad}\,\mathcal{C}^t)\Omega)\mathcal{M}$ gives rise to the identity $(\mathrm{ad}\,\mathcal{C})^n(\mathcal{L}\Omega\mathcal{M}) = \mathcal{L}((\mathrm{ad}\,\mathcal{C}^t)^n\Omega)\mathcal{M}$. Introducing in this equality Definition 3.1 and its counterpart for $(\mathrm{ad}_n \, \mathcal{C}^t)\Omega$, when $n = 2m + 1$ leads to

$$(\mathrm{ad}_n \, \mathcal{C})(\mathcal{L}\Omega\mathcal{M}) = \mathcal{L}^\dagger(\mathcal{C}^\dagger)^m \mathcal{L}(\mathcal{C}^t)^m \mathcal{M}((\mathrm{ad}_n \, \mathcal{C}^t)\Omega)\mathcal{L}(\mathcal{C}^t)^m \mathcal{M}(\mathcal{C}^\dagger)^m \mathcal{M}^\dagger$$

$$= \mathcal{M}((\mathrm{ad}_n \, \mathcal{C}^t)\Omega)\mathcal{L}.$$

Here we have used that $\mathcal{L}(\mathcal{C}^t)^m = \mathcal{C}^m\mathcal{L}$ and $(\mathcal{C}^t)^m\mathcal{M} = \mathcal{M}\mathcal{C}^m$ due to the factorizations $\mathcal{C} = \mathcal{L}\mathcal{M}$ and $\mathcal{C}^t = \mathcal{M}\mathcal{L}$. This proves Property *(ii)* for odd n. The proof for even n is similar.

Property *(iii)* is a direct consequence of Property *(ii)*. There are four cases to discuss depending on the parity of n and k. We will show the proof for one of the cases, the others having a very similar proof. Consider an even $n = 2m$ and an odd $k = 2j + 1$. Then, $n - k = 2(m - j - 1) + 1$ is odd and

$$(\mathrm{ad}\,\mathcal{C})^n\Omega = \mathcal{C}^m((\mathrm{ad}_n \, \mathcal{C})\Omega)\mathcal{C}^m, \qquad (\mathrm{ad}\,\mathcal{C})^k\Omega = \mathcal{C}^j\mathcal{L}((\mathrm{ad}_k \, \mathcal{C})\Omega)\mathcal{M}\mathcal{C}^j,$$

$$(\mathrm{ad}\,\mathcal{C})^{n-k}\Omega = \mathcal{C}^{m-j-1}\mathcal{L}((\mathrm{ad}_{n-k} \, \mathcal{C})\Omega)\mathcal{M}\mathcal{C}^{m-j-1}.$$

Using these relations and the factorization $(\mathrm{ad}\,\mathcal{C})^n\Omega = (\mathrm{ad}\,\mathcal{C})^{n-k}((\mathrm{ad}\,\mathcal{C})^k\Omega)$ we get

$$(\mathrm{ad}_n \, \mathcal{C})\Omega = \mathcal{C}^{-m}((\mathrm{ad}\,\mathcal{C})^{n-k}(\mathcal{C}^j\mathcal{L}((\mathrm{ad}_k \, \mathcal{C})\Omega)\mathcal{M}\mathcal{C}^j))\mathcal{C}^{-m}$$

$$= \mathcal{C}^{j-m}((\mathrm{ad}\,\mathcal{C})^{n-k}(\mathcal{L}((\mathrm{ad}_k \, \mathcal{C})\Omega)\mathcal{M}))\mathcal{C}^{j-m}$$

$$= \mathcal{C}^{-1}\mathcal{L}((\mathrm{ad}_{n-k} \, \mathcal{C})(\mathcal{L}((\mathrm{ad}_k \, \mathcal{C})\Omega)\mathcal{M}))\mathcal{M}\mathcal{C}^{-1}.$$

Finally, Property *(ii)* gives

$$(\mathrm{ad}_n \, \mathcal{C})\Omega = \mathcal{C}^{-1}\mathcal{L}\mathcal{M}((\mathrm{ad}_{n-k} \, \mathcal{C}^t)((\mathrm{ad}_k \, \mathcal{C})\Omega))\mathcal{L}\mathcal{M}\mathcal{C}^{-1} = (\mathrm{ad}_{n-k} \, \mathcal{C}^t)((\mathrm{ad}_k \, \mathcal{C})\Omega).$$ $\blacksquare$

Proposition 5.1 *(iii)* is the ad-factorization of $(\mathrm{ad}_n\,\mathcal{C})$. A special case is the recursive algorithm (15) for $(\mathrm{ad}_n\,\mathcal{C})\Omega$ since it can be written as

$$(\mathrm{ad}_{n+1}\,\mathcal{C})\Omega = (\mathrm{ad}_1\,\mathcal{C}(n))((\mathrm{ad}_n\,\mathcal{C})\Omega).$$

The opposite special case,

$$(\mathrm{ad}_{n+1}\,\mathcal{C})\Omega = (\mathrm{ad}_n\,\mathcal{C}^t)((\mathrm{ad}_1\,\mathcal{C})\Omega), \tag{18}$$

will be particularly useful in dealing with the CMV bispectral problem for linear differential operators of arbitrary order.

6 The general CMV bispectral problem

We have proved that the CMV bispectral problem for linear differential operators of order not greater than two is trivial, that is, the only bispectral CMV matrix is that one with null Verblunsky coefficients. The purpose of the present section is to generalize this result as much as possible by weakening the assumptions in different ways:

(A) Admitting linear differential operators of arbitrary order.

(B) Requiring the eigenfunction condition with respect to the linear differential operator up to finitely many OLP.

(C) Substituting the eigenfunction condition by the more general three-term difference–differential relation (8).

Remember that assuming (B) for a linear differential operator D in $\mathbb{C}[z, z^{-1}]$ can be restated by saying that $D\boldsymbol{x} = \Omega\boldsymbol{x}$ with Ω almost diagonal (or almost tridiagonal if combined with (C)), that is, $\Omega = \Omega_N \oplus \Lambda$ for some N, with Ω_N the $N \times N$ principal submatrix of Ω and Λ diagonal.

A first step in the direction pointed out in (B) is given by the following proposition.

Proposition 6.1. If $\mathcal{C}$ is a CMV matrix, then

$$\Omega \text{ almost diagonal}, \ (\mathrm{ad}_1\,\mathcal{C})\Omega \text{ diagonal} \ \Rightarrow \ \Omega \text{ diagonal}. \qquad \square$$

Proof. Obviously, Ω is (almost) diagonal iff $\mathrm{Re}\,\Omega$ and $\mathrm{Im}\,\Omega$ are simultaneously (almost) diagonal. Also, from (14), $\mathrm{Re}\,((\mathrm{ad}_1\,\mathcal{C})\Omega) = (\mathrm{ad}_1\,\mathcal{C})\mathrm{Re}\,\Omega$ and $\mathrm{Im}\,((\mathrm{ad}_1\,\mathcal{C})\Omega) = (\mathrm{ad}_1\,\mathcal{C})\mathrm{Im}\,\Omega$.

Therefore, by taking real and imaginary parts, it suffices to prove the proposition for an Hermitian Ω.

By induction on N, it is enough to see that $\Omega = \Omega_N \oplus \Lambda$ with Λ diagonal implies that $\Omega_N = \Omega_{N-1} \oplus \lambda_{N-1}$ with $\lambda_{N-1} \in \mathbb{R}$. If we write

$$\Omega_N = \begin{pmatrix} \Omega_{N-1} & u_{N-1} \\ u_{N-1}^\dagger & \lambda_{N-1} \end{pmatrix}, \qquad u_{N-1} \in \mathbb{C}^{N-1}, \qquad \lambda_{N-1} \in \mathbb{R},$$

all that we must prove is that $u_{N-1} = 0$ whenever $(\mathrm{ad}_1\, \mathcal{C})\Omega$ is diagonal.

Suppose that $\Omega = \Omega_N \oplus \Lambda$ for an even N, the proof for odd N follows similar arguments. Then, denoting in general by the subscript N the $N \times N$ principal submatrix and by the superscript (N) the submatrix obtained deleting the first N rows and columns, we have the splitting

$$A := \mathcal{L}^\dagger \Omega \mathcal{L} = A_N \oplus A^{(N)}, \qquad\qquad A_N = \mathcal{L}_N^\dagger \Omega_N \mathcal{L}_N,$$

$$B := \mathcal{M}\Omega\mathcal{M}^\dagger = B_{N+1} \oplus B^{(N+1)}, \qquad B_{N+1} = \mathcal{M}_{N+1}\Omega_{N+1}\mathcal{M}_{N+1}^\dagger,$$

because $\mathcal{L}_N$ and $\mathcal{M}_{N+1}$ are direct sums of complete blocks Θ_k. Since $\Omega_{N+1} = \Omega_N \oplus \lambda_N$ with $\lambda_N \in \mathbb{R}$ and

$$\mathcal{M}_{N+1} = \mathcal{M}_{N-1} \oplus \Theta_{N-1} = \begin{pmatrix} \mathcal{M}_N & \rho_{N-1}e_N \\ \rho_{N-1}e_N^\dagger & -\alpha_{N-1} \end{pmatrix}, \qquad e_N = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{C}^N,$$

we find that

$$B_{N+1} = \begin{pmatrix} B_N & v_N \\ v_N^\dagger & * \end{pmatrix}, \qquad v_N = \rho_{N-1}(\mathcal{M}_N\Omega_N - \lambda_N\overline{\alpha}_{N-1}I_N)e_N.$$

Therefore,

$$(\mathrm{ad}_1\, \mathcal{C})\Omega = B - A = \left(\begin{array}{c|c} B_N - A_N & \begin{matrix} v_N & 0 \end{matrix} \\ \hline \begin{matrix} v_N^\dagger \\ 0 \end{matrix} & B^{(N)} - A^{(N)} \end{array} \right),$$

where 0 stands for the null matrix of the appropriate size. Hence, $(\mathrm{ad}_1\, \mathcal{C})\Omega$ diagonal implies $v_N = 0$. Bearing in mind that $\mathcal{M}_N = \mathcal{M}_{N-1} \oplus \overline{\alpha}_{N-1}$ and $\mathcal{M}_{N-1}$ is unitary we conclude that

$$v_N = 0 \;\Rightarrow\; \lambda_N\overline{\alpha}_{N-1}e_N = \mathcal{M}_N\Omega_N e_N = \begin{pmatrix} \mathcal{M}_{N-1}u_{N-1} \\ \lambda_{N-1}\overline{\alpha}_{N-1} \end{pmatrix} \;\Rightarrow\; u_{N-1} = 0. \qquad \blacksquare$$

Combining the previous proposition and the results of the calculations at the end of Section 4 we get the following result.

Corollary 6.2.   If $\mathcal{C}$ is a CMV matrix whose Verblunsky coefficients are not all null, then

$$\Omega \text{ almost diagonal}, \ (\text{ad}_1 \mathcal{C})\Omega \propto I \ \Rightarrow \ \Omega \propto I. \qquad \square$$

We have seen that an almost diagonal matrix satisfying certain ad-conditions must be actually diagonal. A similar result states that, under some ad-conditions, an almost tridiagonal matrix becomes almost diagonal.

Proposition 6.3.   If $\mathcal{C}$ is a CMV matrix, for any $n \in \mathbb{N}$,

$$\Omega \text{ almost tridiagonal}, \ (\text{ad}_n \mathcal{C})\Omega = 0 \ \Rightarrow \ \Omega \text{ almost diagonal}. \qquad \square$$

Proof.   Supposing without loss that Ω is Hermitian, the fact that Ω is almost tridiagonal means that $\Omega = \widehat{\Omega} + \widetilde{\Omega}$, where $\widehat{\Omega}$ has finitely many non-null coefficients and $\widetilde{\Omega}$ is Hermitian tridiagonal. We can express

$$\widetilde{\Omega} = \begin{pmatrix} \widetilde{\lambda}_0 & \lambda_0 & & \\ \bar{\lambda}_0 & \widetilde{\lambda}_1 & \lambda_1 & \\ & \bar{\lambda}_1 & \widetilde{\lambda}_2 & \lambda_3 \\ & & \ddots & \ddots & \ddots \end{pmatrix} = \widetilde{\Lambda} + \Lambda S + S^\dagger \Lambda^\dagger, \qquad \lambda_k \in \mathbb{C}, \qquad \widetilde{\lambda}_k \in \mathbb{R},$$

in terms of the shift matrix S given in (4) and the two diagonal matrices

$$\Lambda = \begin{pmatrix} \lambda_0 & & & \\ & \lambda_1 & & \\ & & \lambda_2 & \\ & & & \ddots \end{pmatrix}, \qquad\qquad \widetilde{\Lambda} = \begin{pmatrix} \widetilde{\lambda}_0 & & & \\ & \widetilde{\lambda}_1 & & \\ & & \widetilde{\lambda}_2 & \\ & & & \ddots \end{pmatrix}.$$

We must prove that Ω is almost diagonal, that is, $\lambda_k = 0$ for big enough k, whenever $(\text{ad}_n \mathcal{C})\Omega = 0$. These ad-conditions imply that $(\text{ad}_n \mathcal{C})\widetilde{\Omega} = -(\text{ad}_n \mathcal{C})\widehat{\Omega}$ has only finitely many non-null coefficients. The conclusions of the proposition will follow from the analysis of the top upper diagonal of $(\text{ad}_n \mathcal{C})\widetilde{\Omega}$, whose coefficients must vanish up to finitely many ones.

To obtain the top upper diagonal in question it is useful to rewrite also $\mathcal{L}$ and $\mathcal{M}$ using the shift matrix, as in (5). The top upper diagonal of $(\text{ad}_n \mathcal{C})\widetilde{\Omega}$ is the term corresponding to the highest power of the shift. In the case of even $n = 2m$ such a term comes exclusively from the summands $\mathcal{C}^m \widetilde{\Omega}(\mathcal{C}^\dagger)^m + (\mathcal{C}^\dagger)^m \widetilde{\Omega}\mathcal{C}^m$ and is given by

$$(\mathcal{B}_e S \mathcal{B}_o S)^m \Lambda S (\mathcal{B}_o S \mathcal{B}_e S)^m + (\mathcal{B}_o S \mathcal{B}_e S)^m \Lambda S (\mathcal{B}_e S \mathcal{B}_o S)^m. \tag{19}$$

Using the identity (A.1) to permute any diagonal matrix with the powers of the shift, (19) reads as $\Delta(n)S^{2n}$ where

$$\Delta(n) = \mathcal{B}_e \mathcal{B}_o^{(1)} \mathcal{B}_e^{(2)} \mathcal{B}_o^{(3)} \cdots \mathcal{B}_e^{(n-2)} \mathcal{B}_o^{(n-1)} \Lambda^{(n)} \mathcal{B}_o^{(n+1)} \mathcal{B}_e^{(n+2)} \mathcal{B}_o^{(n+3)} \mathcal{B}_e^{(n+4)} \cdots \mathcal{B}_o^{(2n-1)} \mathcal{B}_e^{(2n)}$$
$$+ \mathcal{B}_o \mathcal{B}_e^{(1)} \mathcal{B}_o^{(2)} \mathcal{B}_e^{(3)} \cdots \mathcal{B}_o^{(n-2)} \mathcal{B}_e^{(n-1)} \Lambda^{(n)} \mathcal{B}_e^{(n+1)} \mathcal{B}_o^{(n+2)} \mathcal{B}_e^{(n+3)} \mathcal{B}_o^{(n+4)} \cdots \mathcal{B}_e^{(2n-1)} \mathcal{B}_o^{(2n)}$$
$$= \begin{pmatrix} \delta_0^{(n)} & & \\ & \delta_1^{(n)} & \\ & & \ddots \end{pmatrix}, \qquad \delta_k^{(n)} = \rho_k \rho_{k+1} \cdots \rho_{k+n-1} \lambda_{k+n} \rho_{k+n+1} \rho_{k+n+2} \cdots \rho_{k+2n}.$$

Since $\Delta(n)$ must have finitely many non-null coefficients, we conclude that $\lambda_k = 0$ for big enough k in the case of even n. A similar proof works for odd n.   ∎

Propositions 6.1 and 6.3, as well as Corollary 6.2, remain true when substituting $\mathcal{C}$ by $\mathcal{C}^t$, whose effect is simply exchanging $\mathcal{L} \leftrightarrow \mathcal{M}$. This can be used to obtain our main result.

Theorem 6.4.   If $\mathcal{C}$ is a CMV matrix whose Verblunsky coefficients are not all null, for any $n \in \mathbb{N}$,

$$\Omega \text{ almost tridiagonal, } (\mathrm{ad}_n \mathcal{C})\Omega = 0 \implies \Omega \propto I. \qquad \square$$

Proof.   Suppose Ω almost tridiagonal satisfying $(\mathrm{ad}_n \mathcal{C})\Omega = 0$. From Proposition 6.3 we know that Ω must be almost diagonal. Hence, $\Omega(1) := (\mathrm{ad}_1 \mathcal{C})\Omega$ is almost tridiagonal and, according to the ad-factorization property (18), $(\mathrm{ad}_{n-1} \mathcal{C}^t)\Omega(1) = (\mathrm{ad}_n \mathcal{C})\Omega = 0$. Then, the result analogous to Proposition 6.3 for $\mathcal{C}^t$ implies that $\Omega(1)$ is almost diagonal. Hence, $\Omega(2) := (\mathrm{ad}_1 \mathcal{C}^t)\Omega(1)$ is almost tridiagonal and satisfies $(\mathrm{ad}_{n-2} \mathcal{C})\Omega(2) = (\mathrm{ad}_{n-1} \mathcal{C}^t)\Omega(1) = 0$ due to the analogue of the ad-factorization (18) for $\mathcal{C}^t$. Thus, Proposition 6.3 implies that $\Omega(2)$ is almost diagonal. Proceeding by induction we finally find almost diagonal matrices $\Omega(0) = \Omega, \Omega(1), \Omega(2), \ldots, \Omega(n-1)$ such that

$$(\mathrm{ad}_{n-k} \mathcal{C}(k))\Omega(k) = 0, \qquad \Omega(k+1) = (\mathrm{ad}_1 \mathcal{C}(k))\,\Omega(k), \qquad \mathcal{C}(k) = \begin{cases} \mathcal{C}, & \text{even } k, \\ \mathcal{C}^t, & \text{odd } k. \end{cases}$$

In particular, $\Omega(n-1)$ is almost diagonal and $(\mathrm{ad}_1 \mathcal{C}(n-1))\Omega(n-1) = 0$. From Corollary 6.2 we find that $\Omega(n-1) \propto I$. Hence, $(\mathrm{ad}_1 \mathcal{C}(n-2))\,\Omega(n-2) \propto I$, so that $\Omega(n-2) \propto I$, again by Corollary 6.2. Proceeding in this way we obtain by induction that $\Omega = \Omega(0) \propto I$.   ∎

The hypothesis of the previous theorem are equivalent to the existence of a linear differential operator D such that $D\boldsymbol{x} = \Omega\boldsymbol{x}$ for an almost tridiagonal matrix Ω, where x_n are the OLP related to $\mathcal{C}$. This means that D preserves $\mathbb{C}[z, z^{-1}]$ and $Dx_n \in \mathrm{span}\{x_{n-1}, x_n, x_{n+1}\}$ for all but finitely many indices n. Therefore, bearing in mind the equivalence (13), Theorem 6.4 has the following translation in terms of linear differential operators and OLP on the unit circle.

Theorem 6.5.   The only OLP x_n on the unit circle satisfying

$$Dx_n \in \mathrm{span}\{x_{n-1}, x_n, x_{n+1}\}, \qquad \forall n \geq n_0, \qquad n_0 \in \mathbb{N},$$

for a linear differential operator $D\colon \mathbb{C}[z, z^{-1}] \to \mathbb{C}[z, z^{-1}]$ of arbitrary order, are those orthonormal with respect to the Lebesgue measure.   □

As a particular case of this theorem we get the triviality of the general CMV bispectral problem.

Corollary 6.6.   The only OLP on the unit circle which, up to finitely many ones, are eigenfunctions of a linear differential operator $D\colon \mathbb{C}[z, z^{-1}] \to \mathbb{C}[z, z^{-1}]$ of arbitrary order, are those orthonormal with respect to the Lebesgue measure.   □

7   Conclusions and outlook

We have shown that the CMV bispectral problem on the unit circle—at least in its traditional formulation, or even in some generalizations—admits only the trivial solution. The question is: does this result close the topic? Our intention, based in other experiences involving the bispectral problem and its connections with the problem of C. Shannon, is to keep looking in different directions in the context of the unit circle. Some hope is offered, for instance by results in [17, 26, 49, 53], where one sees how getting away from polynomials leads to interesting situations. These references also show that considering not necessarily positive definite measures can be fruitful. The richness of the matrix valued Bochner problem in the case of the real line (whose full solution is still unknown, as a small sample see [6, 19, 22, 23]), compared to the scalar case, suggests that the triviality of the CMV bispectral problem may disappear if one admits matrix valued measures. A different path in this direction could arise from the use of more exotic kind of orthogonality on the unit circle, such as the one related to Sobolev inner products. All of this remains as a challenge.

It is important to point out that the Bochner–Krall problem—see [25] for a very nice presentation—is intimately connected with the study of the Toda lattice and its Virasoro symmetries. In connection with CMV matrices very relevant references are [27, 30, 34, 35], where the Ablowitz–Ladik hierarchy is seen to be the integrable system that plays the role that the Toda lattice played for Jacobi matrices.

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Appendix

In this appendix we prove the following technical result, crucial for Theorem 2.2.

Proposition A.1. Let x_n and χ_n be the OLP related to the CMV matrices $\mathcal{C}$ and $\mathcal{C}^t$, respectively. Then, a basis of $\ker(\mathcal{C} - zI)^n$ is given by $\{\boldsymbol{x}(z), \boldsymbol{x}'(z), \ldots, \boldsymbol{x}^{(n-1)}(z)\}$ and a basis of $\ker(\mathcal{C}^t - zI)^n$ is given by $\{\boldsymbol{\chi}(z), \boldsymbol{\chi}'(z), \ldots, \boldsymbol{\chi}^{(n-1)}(z)\}$ for every $z \in \mathbb{C} \setminus \{0\}$. □

Proof. We will prove the result for $\mathcal{C}$ and x_n, the proof for $\mathcal{C}^t$ and χ_n begin similar.

From (1) we find that $(\mathcal{C} - zI)\boldsymbol{x}^{(k)}(z) = k\,\boldsymbol{x}^{(k-1)}(z)$ by induction on k. This leads to $(\mathcal{C} - zI)^k \boldsymbol{x}^{(k)} = k!\,\boldsymbol{x}(z)$ and $(\mathcal{C} - zI)^{k+1}\boldsymbol{x}^{(k)}(z) = 0$, which implies that $\text{span}\{\boldsymbol{x}(z), \boldsymbol{x}'(z), \ldots, \boldsymbol{x}^{(n-1)}(z)\} \subset \ker(\mathcal{C} - zI)^n$. Besides, $\{\boldsymbol{x}(z), \boldsymbol{x}'(z), \ldots, \boldsymbol{x}^{(n-1)}(z)\}$ is linearly independent for every $z \neq 0$ because applying $(\mathcal{C} - zI)^{n-1}$ to the equation $c_0(z)\boldsymbol{x}(z) + c_1(z)\boldsymbol{x}'(z) + \cdots + c_{n-1}(z)\boldsymbol{x}^{(n-1)}(z) = 0$ yields $(n-1)!\,c_{n-1}(z) = 0$, so an induction gives $c_k(z) = 0$ for all k. Hence, to prove that $\{\boldsymbol{x}(z), \boldsymbol{x}'(z), \ldots, \boldsymbol{x}^{(n-1)}(z)\}$ is a basis of $\ker(\mathcal{C} - zI)^n$ we only need to show that $\dim \ker(\mathcal{C} - zI)^n = n$.

To determine $\dim \ker(\mathcal{C} - zI)^n$ note that, due to the unitarity of $\mathcal{L}$ and $\mathcal{M}$, multiplying $(\mathcal{C} - zI)^n$ on the left by $\mathcal{L}^\dagger$ or $\mathcal{M}^\dagger$ does not change its kernel. In particular, $\ker(\mathcal{C} - zI)^n = \ker[K(n)(\mathcal{C} - zI)^n]$, where

$$K(2m) = (\mathcal{C}^\dagger)^m, \qquad K(2m + 1) = \mathcal{L}^\dagger (\mathcal{C}^\dagger)^m.$$

The advantage of $K(n)(\mathcal{C} - zI)^n$ over $(\mathcal{C} - zI)^n$ is its narrower band structure, which is shown by inserting the expansion (11), so that

$$
\begin{aligned}
K(2m)(\mathcal{C} - zI)^{2m} &= \sum_{j=-m}^{m} (-1)^{m-j} \binom{2m}{m-j} z^{m-j} \mathcal{C}^j \\
&= (-1)^m \binom{2m}{m} z^m + \sum_{j=1}^{m} (-1)^{m-j} \binom{2m}{m-j} \left(z^{m-j} \mathcal{C}^j + z^{m+j} (\mathcal{C}^\dagger)^j \right),
\end{aligned}
$$

$$
\begin{aligned}
K(2m+1)(\mathcal{C} - zI)^{2m+1} &= \sum_{j=-m-1}^{m} (-1)^{m-j} \binom{2m+1}{m-j} z^{m-j} \mathcal{M} \mathcal{C}^j \\
&= \sum_{j=0}^{m} (-1)^{m-j} \binom{2m+1}{m-j} \left(z^{m-j} \mathcal{M} \mathcal{C}^j - z^{m+j+1} \mathcal{L}^\dagger (\mathcal{C}^\dagger)^j \right).
\end{aligned}
$$

The above expressions prove that $K(n)(\mathcal{C} - zI)^n$ is $(2n+1)$-diagonal. Besides, the coefficients of its top upper diagonal are non-null for $z \neq 0$. To check this last statement note that this top upper diagonal comes exclusively from the terms

$$
\begin{aligned}
\mathcal{C}^m + z^n (\mathcal{C}^\dagger)^m, && n = 2m, \\
\mathcal{M} \mathcal{C}^m - z^n \mathcal{L}^\dagger (\mathcal{C}^\dagger)^m, && n = 2m + 1,
\end{aligned}
$$

and corresponds to the power S^n when expanding in powers of the shift S given in (4). Using (5) we find that such upper diagonal is

$$
\begin{aligned}
(\mathcal{B}_e S \mathcal{B}_o S)^m + z^n (\mathcal{B}_o S \mathcal{B}_e S)^m, && n = 2m, \\
\mathcal{B}_o S (\mathcal{B}_e S \mathcal{B}_o S)^m - z^n \mathcal{B}_e S (\mathcal{B}_o S \mathcal{B}_e S)^m, && n = 2m + 1.
\end{aligned}
$$

We can permute any diagonal matrix Λ with the powers of the shift S via the identity

$$
S^k \Lambda = \Lambda^{(k)} S^k, \tag{A.1}
$$

where $\Lambda^{(k)}$ is obtained by deleting the first k rows and columns of Λ. Therefore, the top upper diagonal in question reads as $\Gamma(n) S^n$ with

$$
\Gamma(n) = \begin{cases}
\mathcal{B}_e \mathcal{B}_o^{(1)} \cdots \mathcal{B}_e^{(n-2)} \mathcal{B}_o^{(n-1)} + z^n \mathcal{B}_o \mathcal{B}_e^{(1)} \cdots \mathcal{B}_o^{(n-2)} \mathcal{B}_e^{(n-1)}, & n = 2m, \\
\mathcal{B}_o \mathcal{B}_e^{(1)} \cdots \mathcal{B}_e^{(n-2)} \mathcal{B}_o^{(n-1)} - z^n \mathcal{B}_e \mathcal{B}_o^{(1)} \cdots \mathcal{B}_o^{(n-2)} \mathcal{B}_e^{(n-1)}, & n = 2m + 1.
\end{cases}
$$

More explicitly,

$$\Gamma(n) = \begin{pmatrix} \gamma_0^{(n)} & & \\ & \gamma_1^{(n)} & \\ & & \ddots \end{pmatrix}, \quad \gamma_k^{(n)} = \begin{cases} \rho_k\rho_{k+1}\cdots\rho_{k+n-1}, & k, n \text{ same parity}, \\ (-1)^n z^n \rho_k\rho_{k+1}\cdots\rho_{k+n-1}, & k, n \text{ different parity}, \end{cases}$$

which clearly has non-null diagonal coefficients for $z \neq 0$.

The fact that $K(n)(\mathcal{C}-zI)^n$ is $(2n+1)$-diagonal with non-null coefficients in the top upper diagonal implies that $\dim\ker[K(n)(\mathcal{C}-zI)^n] = n$. In other words, $\dim\ker(\mathcal{C}-zI)^n = n$, which ends the proof of the proposition. ∎

References

[1] Bochner S. "Über Sturm–Liouvillesche polynomsysteme." *Mathematische Zeitschrift* 29 (1929): 730–6.

[2] Bonami, A., and A. Karoui. "Uniform approximation and explicit estimates for the prolate spheroidal wave functions." *Constructive Approximation* 43, no. 1 (2016): 15–45.

[3] Cantero M. J., L. Moral, and L. Velázquez. "Five-diagonal matrices and zeros of orthogonal polynomials on the unit circle." *Linear Algebra and its Applications* 362 (2003): 29–56.

[4] Castro M., and F. A. Grünbaum. "The Darboux process and time-and-band limiting for matrix orthogonal polynomials." *Linear Algebra and its Applications* 487 (2015): 328–41.

[5] Duistermaat J. J., and F. A. Grünbaum. "Differential equations in the spectral parameter." *Communications in Mathematical Physics* 103, no. 2 (1986): 177–240.

[6] Durán A., and F. A. Grünbaum. "Orthogonal matrix polynomials satisfying second-order differential equations." *International Mathematics Research Notices* no. 10 (2004): 461–84.

[7] Edelman A., P. McCorquodale, and S. Toledo. "The future fast Fourier transform?." *SIAM Journal of Scientific Computing* 20, no. 3 (1999): 1094–114.

[8] Grünbaum, F. A. "Eigenvectors of a Toeplitz matrix: discrete version of the prolate spheroidal wave functions." *SIAM Journal on Algebraic and Discrete Methods* 2, no. 2 (1981): 136–41.

[9] Grünbaum F. A. "A new property of reproducing kernels of classical orthogonal polynomials." *Journal of Mathematical Analysis and Applications* 95, no. 2 (1983): 491–500.

[10] Grünbaum F. A. "Time-band limiting and the bispectral problem." *Communications on Pure and Applied Mathematics* 47, no. 3 (1994): 307–28.

[11] Grünbaum F. A. "Band-time-band limiting integral operators and commuting differential operators." *Algebra i Analiz* 8, no. 1 (1996): 122–26; *Translation in St. Petersburg Math. J.* 8, no. 1 (1997): 93–6.

[12] Grünbaum F. A. "The Bispectral Problem: An Overview." In *Special Functions 2000: Current Perspective and Future Directions (Tempe, AZ)*, edited by J. Bustoz, E. H. Ismail, and S. Suslov, 129–40. NATO Science Series II Mathematics, Physics and Chemistry, vol. 30. Dordrecht: Kluwer Academic Publishers, 2001.

[13]   Grünbaum, F. A., "The Darboux Process and a Noncommutative Bispectral Problem: Some Explorations and Challenges." In *Geometric Aspects of Analysis and Mechanics, Book in Honor of Hans Duistermaat*, edited by E. P. van den Ban and J. A. C. Kolk, 161–77. Progress in Mathematics, vol. 292. New York: Birkhäuser/Springer, 2011.

[14]   Grünbaum F. A., and L. Haine. "A Theorem of Bochner, Revisited." In *Algebraic Aspects of Integrable Systems: In Memory of Irene Dorfman*, edited by A. S. Fokas and I. M. Gelfand, 143–72. Progress in Nonlinear Differential Equations and their Applications, vol. 26. Boston, MA: Birkhuser, 1997.

[15]   Grünbaum F. A., and L. Haine. "Bispectral Darboux transformations: an extension of the Krall polynomials." *International Mathematics Research Notices* no. 8 (1997): 359–92.

[16]   Grünbaum F. A. and L. Haine. "Some functions that generalize the Askey–Wilson polynomials." *Communications in Mathematical Physics* 184, no. 1 (1997): 173–202.

[17]   Grünbaum F. A., and P. Iliev. "Heat kernel expansions on the integers." *Mathematical Physics Analysis and Geometry* 5, no. 2 (2002): 183–200.

[18]   Grünbaum F. A., and P. Iliev. "A noncommutative version of the bispectral problem." *Journal of Computational and Applied Mathematics* 161, no. 1 (2003): 99–118.

[19]   Grünbaum F. A., and J. A. Tirao. "The algebra of differential operators associated to a weight matrix." *Integral Equations and Operator Theory* 58, no. 4 (2007): 449–75.

[20]   Grünbaum F. A., and M. Yakimov. "The Prolate Spheroidal Phenomenon as a Consequence of Bispectrality." In *Superintegrability in Classical and Quantum Systems*, 301–12. CRM Proceedings and Lecture Notes, vol. 37. Providence, RI: American Mathematical Society, 2004.

[21]   Grünbaum F. A., L. Haine, and E. Horozov. "On the Krall–Hermite and the Krall–Bessel polynomials." *International Mathematics Research Notices* no. 19 (1997): 953–66.

[22]   Grünbaum F. A., I. Pacharoni, and J. A. Tirao. "Matrix valued spherical functions associated to the complex projective plane." *Journal of Functional Analysis* 188, no. 2 (2002): 350–441.

[23]   Grünbaum F. A., I. Pacharoni, and J. A. Tirao. "Matrix valued orthogonal polynomials of the Jacobi type." *Indagationes Mathematicae* 14, no. 3–4 (2003): 353–66.

[24]   Grünbaum F. A., I. Pacharoni, and I. Zurrian. "Time and band limiting for matrix valued functions, an example." *Symmetry, Integrability and Geometry: Methods and Applications* 11 (2015): 045 (14 pp).

[25]   Haine L. "The Bochner-Krall Problem: Some New Perspectives." In *Special Functions 2000: Current Perspective and Future Directions (Tempe, AZ)*, edited by J. Bustoz, E. H. Ismail, and S. Suslov, 141–78. NATO Science Series II Mathematics, Physics and Chemistry, vol. 30. Dordrecht: Kluwer Academic Publishers, 2001.

[26]   Haine L., and P. Iliev. "A rational analog of the Krall polynomials." Kowalevski Workshop on Mathematical Methods of Regular Dynamics (Leeds, 2000), *Journal of Physics A: Mathematical and General* 34, no. 11 (2001): 2445–57.

[27]   Haine L., and D. Vanderstichelen. "A centerless Virasoro algebra of master symmetries for the Ablowitz-Ladik hierarchy." *Symmetry, Integrability and Geometry: Methods and Applications* 9 (2013): 079 (42 pp).

[28]   Jahn K., and N. Bokor. "Revisiting the concentration problem of vector fields within a spherical cap: a commuting differential operator solution." *Journal of Fourier Analysis and Applications* 20, no. 2 (2014): 421–51.

[29]   Jamming P., A. Karoui, and S. Spektor. "The approximation of almost time and band limited functions by their expansion in some orthogonal polynomial bases." arXiv: 1501.03655.

[30]   Killip R., and I. Nenciu. "CMV: the unitary analogue of Jacobi matrices." *Communications on Pure and Applied Mathematics* 60, no. 8 (2007): 1148–88.

[31]   Krall H. L. "Orthogonal polynomials satisfying fourth order differential equations." *Proceedings of the Royal Society of Edinburgh: Section A* 87 (1981): 271–88.

[32]   Landau H. J., and H. O. Pollak. "Prolate spheroidal wave functions, Fourier analysis and uncertainty, II." *Bell System Technical Journal* 40, no. 1 (1961): 65–84.

[33]   Landau H. J., and H. O. Pollak. "Prolate spheroidal wave functions, Fourier analysis and uncertainty, III *Bell System Technical Journal* 41, no. 4 (1962): 1295–336.

[34]   Nenciu I. "Lax pairs for the Ablowitz-Ladik system via orthogonal polynomials on the unit circle." *International Mathematics Research Notices* no. 11 (2005): 647–86.

[35]   Nenciu I. "CMV matrices in random matrix theory and integrable systems: a survey." *Journal of Physics A: Mathematical and General* 39, no. 28 (2006): 8811–22.

[36]   Osipov A., V. Rokhlin, and H. Xiao. "Prolate Spheroidal Wave Functions of Order Zero." In *Mathematical Tools for Bandlimited Approximation*. Applied Mathematical Sciences, vol. 187. New York, Springer, 2013.

[37]   Plattner A., and F. Simons. "Spatiospectral concentration of vector fields on a sphere." *Applied and Computational Harmonic Analysis* 36, no. 1 (2014): 1–22.

[38]   Routh E. "On some properties of certain solutions of a differential equation of the second order." *Proceedings of the London Mathematical Society* S1–16, no. 1 (1884): 245–62.

[39]   Shannon C. E. "A mathematical theory of communication." *Bell System Technical Journal* 27, no. 3 (1948): 379–423 (no. 4, 623–56).

[40]   Simons F. J., and F. A. Dahlen. "Spherical Slepian functions and the polar gap in geodesy." *Geophysical Journal International* 166, no. 3 (2006): 1039–61.

[41]   Simons F. J., F. A. Dahlen, and M. A. Wieczorek. "Spatiospectral concentration on a sphere." *SIAM Review* 48, no. 3 (2006): 504–36.

[42]   Slepian D. "Prolate spheroidal wave functions, Fourier analysis and uncertainty, IV." *Bell System Technical Journal* 43, no. 6 (1964): 3009–57.

[43]   Slepian D. "On bandwidth." *Proceedings of the IEEE* 64, no. 3 (1976): 292–300.

[44]   Slepian D. "Prolate spheroidal wave functions, Fourier analysis and uncertainty, V." *Bell System Technical Journal* 57, no. 5 (1978): 1371–430.

[45]   Slepian D. "Some comments on Fourier analysis, uncertainty and modeling." *SIAM Review* 25, no. 3 (1983): 379–93.

[46]   Slepian D., and H. O. Pollak. "Prolate spheroidal wave functions, Fourier analysis and uncertainty, I." *Bell System Technical Journal* 40, no. 1 (1961): 43–64.

[47]   Simon B. *Orthogonal Polynomials on the Unit Circle*. AMS Colloquium Publications 54, vol. I and II. Providence, RI: American Mathematical Society, 2005.

[48]   Simon B. "CMV matrices: Five years after." *Journal of Computational and Applied Mathematics* 208, no. 1 (2007): 120–54.

[49]   Spiridonov V. P., and A. S. Zhedanov. "*Generalized Eigenvalue Problem and a New Family of Rational Functions Biorthogonal on Elliptic Grids*." In *Special Functions 2000: Current Perspective and Future Directions (Tempe, AZ)*, edited by J. Bustoz, E. H. Ismail, and S. Suslov, 365–8. NATO Science Series II Mathematics, Physics and Chemistry, vol. 30, Dordrecht: Kluwer Academic Publishers, 2001.

[50]   Szegő G. *Orthogonal Polynomials.* AMS Colloquium Publications, vol. 23. New York: American Mathematical Society, 1939.

[51]   Tirao J. A. "The matrix valued hypergeometric equation." *Proceedings of the National Academy of Sciences of the United States of America* 100, no. 14 (2003): 8138–41.

[52]   Watkins D. S. "Some perspectives on the eigenvalue problem." *SIAM Review* 35, no. 3 (1993): 430–71.

[53]   Zhedanov A. S. "Biorthogonal rational functions and generalized eigenvalue problem." *Journal of Approximation Theory* 101, no. 2 (1999): 303–29.