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Embedded $\mathbb{Q}$-Resolutions and Yomdin-Lê Surface Singularities

DOCTORAL THESIS

Supervised by Professors Enrique Artal Bartolo
and José Ignacio Cogolludo Agustín
from University of Zaragoza

by

Jorge Martín-Morales

&

Zaragoza, October 10, 2011

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Jorge Martín-Morales Zaragoza, May 31, 2011

RESUMEN (Spanish)

Primera Parte (Capítulos I–V)

Uno de los principales invariantes de una singularidad de hipersuperficie es la estructura de Hodge mixta (EHM) sobre la cohomología de la fibra de Milnor. En el caso aislado, Steenbrink dio un método para calcular esta estructura usando una sucesión espectral construida a partir de los divisores asociados a la normalización semiestable de una resolución encajada [**Ste77**].

Sin embargo, en la práctica la combinatoria del divisor excepcional de la resolución es tan compleja que el estudio de la sucesión espectral resulta muy complicado, ver por ejemplo [**Art94b**] donde se calcula una resolución encajada y la correspondiente normalización semiestable de singularidades superaisladas de superficie usando explosiones de puntos y curvas racionales.

Después de la normalización semiestable el nuevo espacio ambiente contiene singularidades normales que se obtienen como cociente de una bola en $\mathbb{C}^n$ por la acción de un grupo finito. Los espacios que tienen solamente este tipo de singularidades se llaman V*-variedad*. Fueron introducidas en [**Sat56**] y tienen las mismas propiedades sobre $\mathbb{Q}$ que las variedades diferenciables, por ejemplo, admiten dualidad de Poincaré si son compactas y tienen estructura de Hodge pura si son compactas y Kähler [**Bai56**]. Además, se puede definir la noción de divisor con cruces normales [**Ste77**].

Motivado por esto y para tratar de simplificar la combinatoria del divisor excepcional, introducimos la noción de **Q**-resolución encajada. La idea es la siguiente. Clásicamente una resolución encajada de $\{f = 0\} \subset \mathbb{C}^{n+1}$ es una aplicación propia $\pi : X \to (\mathbb{C}^{n+1}, 0)$ de una variedad lisa X verificando, entre otras condiciones, que $\pi^*(\{f = 0\})$ es un divisor con cruces normales. Para debilitar la condición sobre la preimagen de la singularidad, permitimos que el nuevo espacio ambiente X tenga singularidades cocientes abelianas y el divisor $\pi^*(\{f = 0\})$ cruces normales en X.

Más concretamente, aquí presentamos la definición de uno de los objetos más importantes de nuestro estudio.

Definición. Sea M un espacio cociente abeliano. Consideramos $H \subset M$ una subvariedad analítica de codimensión 1. Una **Q**-*resolución encajada* de $(H,0) \subset (M,0)$ es una aplicación analítica propia $\pi : X \to (M,0)$ tal que:

(1) X es una V-variedad con singularidades cocientes abelianas.
(2) π es un isomorfismo sobre $X \setminus \pi^{-1}(\mathrm{Sing}(H))$.
(3) $\pi^*(H)$ es una hipersuperficie con $\mathbb{Q}$-cruces normales en X.

El presente trabajo está dedicado al estudio de invariantes de hipersuperficies singulares $(H,0) \subset (\mathbb{C}^{n+1},0)$ a través de una **Q**-resolución encajada o la normalización semiestable asociada. Nos centraremos en dos invariantes importantes de H: el polinomio característico de la monodromía compleja (Capítulo IV) y la estructura de Hodge mixta sobre la cohomología de la fibra de Milnor (Capítulo V).

Como hemos dicho anteriormente, la motivación de usar **Q**-resoluciones encajadas en lugar de las clásicas es doble. Por un lado, son generalizaciones de las resoluciones encajadas usuales, para las que se espera que los invariantes anteriores se puedan calcular de manera efectiva. Por otro lado, la complejidad combinatoria y computacional de las **Q**-resoluciones encajadas es mucho más sencilla, pero conservan la misma información necesaria para la comprensión de la topología de la singularidad.

Notación. Para tratar estas resoluciones, necesitamos introducir algo de notación. Sea $G := \mu_{d_0} \times \cdots \times \mu_{d_r}$ un grupo finito abeliano arbitrario escrito como productor de grupos finitos cíclicos, esto es, μ_{d_i} es el grupo cíclico de las raíces d_i-ésimas de la unidad. Consideramos una matriz de pesos

$$A := (a_{ij})_{i,j} = [\mathbf{a}_0 \,|\, \cdots \,|\, \mathbf{a}_n] \in Mat((r+1)\times(n+1), \mathbb{Z})$$

y la acción

$$(1) \qquad \begin{gathered} (\mu_{d_0} \times \cdots \times \mu_{d_r}) \times \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^{n+1}, \\ \big(\boldsymbol{\xi}_{\mathbf{d}}, \mathbf{x}\big) \mapsto (\xi_{d_0}^{a_{00}} \cdots \xi_{d_r}^{a_{r0}}\, x_0, \, \ldots \, , \xi_{d_0}^{a_{0n}} \cdots \xi_{d_r}^{a_{rn}}\, x_n). \end{gathered}$$

El conjunto de todas las órbitas $\mathbb{C}^{n+1}/G$ se llama *espacio cociente* (*cíclico*) *de tipo* $(\mathbf{d};A)$ y se denota por

$$X(\mathbf{d};A) := X \left(\begin{array}{c|ccc} d_0 & a_{00} & \cdots & a_{0n} \\ \vdots & \vdots & \ddots & \vdots \\ d_r & a_{r0} & \cdots & a_{rn} \end{array} \right).$$

La órbita de un elemento $(x_0,\ldots,x_n)$ bajo esta acción se denota por $[(x_0,\ldots,x_n)]_{(\mathbf{d};A)}$ y el subíndice se omite si no hay lugar a confusión.

Nota. La condición (3) de la definición anterior significa que si $f : (M, 0) \to (\mathbb{C}, 0)$ es el germen de una función analítica y $(H, 0)$ es la hipersuperficie definida por f, entonces la transformada total $\pi^{-1}(H) = (f \circ \pi)^{-1}(0)$ está localmente dada por una función de la forma $x_0^{m_0} \cdots x_k^{m_k} : X(\mathbf{d}; A) \to \mathbb{C}$, donde $X(\mathbf{d}; A) := \mathbb{C}^{n+1}/\mu_{\mathbf{d}}$ y $\mu_{\mathbf{d}}$ actúan diagonalmente como en (1).

Los números anteriores m_i no tienen ningún significado cuando $\mu_{\mathbf{d}}$ no induce una acción "small" sobre $GL(n+1, \mathbb{C})$. Esto motiva lo siguiente.

Definición. El tipo $(\mathbf{d}; A)$ se dice *normalizado* si la acción es libre en $(\mathbb{C}^*)^{n+1}$ y $\mu_{\mathbf{d}}$ se identifica con un subgrupo small de $GL(n+1, \mathbb{C})$.

Un problema clásico en Teoría de Singularidades es describir o dar un método para calcular invariantes una vez conocida una resolución encajada. La existencia de tal resolución está garantizada por los trabajos de Hironaka. Con el mismo espíritu, uno de los propósitos de este trabajo es proporcionar información sobre la singularidad a través de una $\mathbf{Q}$-resolución encajada.

Como herramienta para encontrar $\mathbf{Q}$-resoluciones encajadas, usaremos explosiones ponderadas con centro liso. Prestaremos especial atención a los casos de dimensión 2 y 3 y explosiones de puntos. Tales explosiones pueden entenderse desde la geometría tórica pero en este trabajo las presentamos más geométricamente, generalizando las usuales.

Ejemplo. Supongamos $(d; a, b)$ normalizado y $\gcd(\omega) = 1$, $\omega := (p, q)$. Entonces, el espacio total de la explosión ω-ponderada del origen de $X(d; a, b)$,

$$\pi_{(d;a,b),\omega} : \widehat{X(d; a, b)}_{\omega} \longrightarrow X(d; a, b), \tag{2}$$

se puede escribir como

$$\widehat{U}_1 \cup \widehat{U}_2 = X\left(\frac{pd}{e}; 1, \frac{-q+\beta pb}{e}\right) \cup X\left(\frac{qd}{e}; \frac{-p+\mu qa}{e}, 1\right)$$

y las cartas están dadas por

$$\text{Primera carta} \quad \left| \begin{array}{c} X\left(\dfrac{pd}{e}; 1, \dfrac{-q+\beta pb}{e}\right) \longrightarrow \widehat{U}_1, \\[2ex] \big[(x^e, y)\big] \mapsto \big[((x^p, x^q y), [1:y]_\omega)\big]_{(d;a,b)}. \end{array} \right.$$

$$\text{Segunda carta} \quad \left| \begin{array}{c} X\left(\dfrac{qd}{e}; \dfrac{-p+\mu qa}{e}, 1\right) \longrightarrow \widehat{U}_2, \\[2ex] \big[(x, y^e)\big] \mapsto \big[((x y^p, y^q), [x:1]_\omega)\big]_{(d;a,b)}. \end{array} \right.$$

En lo anterior, $e = \gcd(d, pb - qa)$ y $\beta a \equiv \mu b \equiv 1 \pmod{d}$, ver I.3–1 para los detalles. Nótese que el origen de las dos cartas son singularidades cocientes cíclicas. Están situadas en el divisor excepcional E que es isomorfo a $\mathbb{P}^1$.

Para estudiar las **Q**-resoluciones encajadas necesitamos una teoría de intersección. Para ello tenemos que trabajar con divisores en V-variedades. Dos clases de divisores aparecen en la literatura: divisores de Weil y de Cartier. Los divisores de Weil son combinaciones lineales localmente finitas con coeficientes enteros de subvariedades de codimensión 1 y los divisores de Cartier son secciones globales del haz cociente de funciones meromorfas módulo funciones holomorfas que no se anulan nunca. La relación entre divisores de Cartier y fibrados en línea proporciona una buena manera de definir la multiplicidad de intersección de dos divisores.

En la categoría lisa, ambos conceptos coinciden pero no es el caso para variedades singulares, ni siquiera para las normales. También podemos considerar $\mathbb{Q}$-divisores de Weil y de Cartier (tensorizando los correspondientes grupos por $\mathbb{Q}$). El primer resultado importante de este trabajo es que estos dos conceptos coinciden para V-variedades.

Teorema 1. *Sea X una V-variedad. Entonces, la aplicación definida a través de la noción de divisor de Weil asociado,*

$$T_X \otimes 1 : \mathrm{CaDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathrm{WeDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q},$$

es un isomorfismo de $\mathbb{Q}$-espacios vectoriales. En particular, dado un divisor de Weil D en X, siempre existe $k \in \mathbb{Z}$ tal que $kD \in \mathrm{CaDiv}(X)$.

Probablemente este resultado es conocido por los especialistas pero no hemos encontrado una demostración en la literatura. Existen algunos resultados parciales para variedades tóricas. Además, en este trabajo damos un algoritmo para presentar explícitamente un $\mathbb{Q}$-divisor de Weil como un $\mathbb{Q}$-divisor de Cartier, ver (II.2.14). Ilustramos el uso de este algoritmo con un ejemplo de un espacio obtenido después de una explosión ponderada.

Ejemplo. Sea $\pi_{(d;a,b),\omega}$ el morfismo propio definido en (2). Entonces, su divisor excepcional E es un divisor de Weil que no se corresponde con ningún divisor de Cartier. Sin embargo, siguiendo la discusión anterior, se puede escribir como $\mathbb{Q}$-divisor de Cartier del modo $\frac{e}{dpq}\big\{(\widehat{U}_1, x^{dq}), (\widehat{U}_2, y^{dp})\big\}$.

El teorema 1 anterior nos permite desarrollar una teoría de intersección racional sobre V-variedades con las propiedades usuales esperadas, que están recogidas en la proposición (III.1.3).

Definición. Sea X una V-variedad y consideremos $D_1, D_2 \in \mathbb{Q}\text{-Div}(X)$. El *número de intersección* está definido como $D_1 \cdot D_2 := \frac{1}{k_1 k_2} (k_1 D_1 \cdot k_2 D_2) \in \mathbb{Q}$, donde $k_1, k_2 \in \mathbb{Z}$ se eligen para que $k_1 D_1 \in \mathrm{WeDiv}(X)$ y $k_2 D_2 \in \mathrm{CaDiv}(X)$. Análogamente, se define el *número de intersección local* en $P \in D_1 \cap D_2$, si $D_1 \not\subseteq D_2$. Idem el *pull-back* está definido por $F^*(D_2) := \frac{1}{k_2} F^*(k_2 D_2)$ si $F : Y \to X$ es un morfismo propio entre dos V-variedades irreducibles.

Este número de intersección racional fue primero introducido por Mumford para superficies normales, ver [**Mum61**, Pag. 17]. Nuestra definición coincide con la de Mumford gracias al buen comportamiento con respecto al pull-back, ver Theorem (III.1.5). La principal ventaja es que la nuestra no involucra una resolución del espacio ambiente y, por ejemplo, esto nos permite encontrar fácilmente fórmulas para la auto-intersección de los divisores excepcionales de explosiones ponderadas sin calcular ninguna resolución.

De hecho, esto es el segundo resultado importante de este trabajo.

Proposición 2. *Sea $\pi := \pi_{(d;a,b),\omega}$ el morfismo definido en (2). Consideramos dos $\mathbb{Q}$-divisores C y D en $X(d;a,b)$. Entonces,*

(1) $E \cdot \pi^*(C) = 0$, (4) $E^2 = -\dfrac{e^2}{dpq}$,

(2) $\pi^*(C) = \widehat{C} + \dfrac{\nu}{e}E$, (5) $\widehat{C} \cdot \widehat{D} = C \cdot D - \dfrac{\nu\mu}{dpq}$,

(3) $E \cdot \widehat{C} = \dfrac{e\nu}{dpq}$, (6) $\widehat{D}^2 = D^2 - \dfrac{\mu^2}{dpq}$ *(D compacto)*,

donde ν y μ denotan la (p,q)-multiplicidad de C y D en P, es decir, x (resp. y) tienen (p,q)-multiplicidad p (resp. q).

Nuestro tercer resultado importante es una versión del teorema de Bézout para cocientes de planos proyectivos ponderados.

Proposición 3. *Sean m_1, m_2, m_3 los determinantes de los tres menores de orden 2 de la matriz $\left(\begin{smallmatrix} p & q & r \\ a & b & c \end{smallmatrix}\right)$. Supongamos que $\gcd(p,q,r) = 1$ y escribamos $e = \gcd(d, m_1, m_2, m_3)$. Si $\omega = (p,q,r)$, entonces el número de intersección de dos $\mathbb{Q}$-divisores en $\mathbb{P}^2_\omega(d;a,b,c) := \mathbb{P}^2_\omega/\mu_d$ es*

$$D_1 \cdot D_2 = \frac{e}{dpqr} \deg_\omega(D_1) \deg_\omega(D_2) \in \mathbb{Q}.$$

Nótese que el divisor excepcional de la explosión (p,q,r)-ponderada de un punto de tipo $(d;a,b,c)$ es isomorfo a $\mathbb{P}^2_\omega(d;a,b,c)$, ver §I.3–2. Así este resultado nos ayudará a describir **Q**-resoluciones encajadas de superficies en $\mathbb{C}^3$, ver el capítulo VI donde se trata con detalle el caso superaislado.

Ahora ya tenemos todos los ingredientes necesarios para estudiar los dos invariantes mencionados en términos de una **Q**-resolución encajada de la singularidad y la normalización semiestable asociada. Ambos resultados dependen de una estratificación de un $\mathbb{Q}$-divisor con cruces normales. Así, necesitamos introducir algo de notación.

Notación. Sea $f : (M,0) \to (\mathbb{C},0)$ el germen de una función analítica y sea $(H,0)$ la hipersuperficie definida por f. Dada una **Q**-resolución encajada de $(H,0)$, $\pi : X \to (M,0)$, consideramos $E_1, \ldots, E_s$ las componentes irreducibles del divisor excepcional y $\widehat{H}$ la transformada estricta.

Normalmente se escribe $E_0 = \widehat{H}$ y $S = \{0, 1, \ldots, s\}$ para que la estratificación de X asociada al $\mathbb{Q}$-divisor con cruces normales $\pi^{-1}(H) = \bigcup_{i \in S} E_i$ esté definida por

$$E_I^\circ := \Big(\cap_{i \in I} E_i \Big) \setminus \Big(\cup_{i \notin I} E_i \Big),$$

para $I \subseteq S$ posiblemente vacío.

Sea también $X = \bigsqcup_{j \in J} Q_j$ una estratificación de X dada por los puntos singulares cocientes de manera que la ecuación local de $g := f \circ \pi$ en $P \in E_I^\circ \cap Q_j$ sea de la forma

$$x_0^{m_0} \cdot \ldots \cdot x_k^{m_k} : X(\mathbf{d}; A) \longrightarrow \mathbb{C}, \quad (0 \leq k \leq n)$$

y las multiplicidades m_i y la acción $\mu_{\mathbf{d}}$ son la misma a lo largo de cada estrato $E_I^\circ \cap Q_j$. En este contexto $m(E_I^\circ \cap Q_j)$ está definido por

$$m(E_I^\circ \cap Q_j) := \gcd \Big(m_0, \ldots, m_k, \frac{\sum_{j=0}^k a_{0j} m_j}{d_0}, \ldots, \frac{\sum_{j=0}^k a_{rj} m_j}{d_r} \Big).$$

A veces también lo denotamos por $m(E, P)$ o incluso $m(P)$, $P \in E_I^\circ \cap Q_j$, si no hay lugar a confusión, ver (IV.3.12) y (V.1.4).

El cuarto resultado importante de este trabajo es la generalización de la fórmula de A'Campo para **Q**-resoluciones encajadas, ver (IV.3.14) para un enunciado más completo. Su demostración está basada en [**Dim04**, Th. 6.1.14.] y así necesitamos trabajar con complejos constructibles de haces con respecto a una estratificación y el "nearby cycles" de f.

Teorema 4. $Z(f;t) \ = \displaystyle\prod_{i=1,\ldots,s,\ j \in J} \Big(1 - t^{m(E_{\{i\}}^\circ \cap Q_j)} \Big)^{\chi(E_{\{i\}}^\circ \cap Q_j)}.$

Nótese que solo los estratos $E_{\{i\}}^\circ \cap Q_j$ que provienen del divisor excepcional contribuyen a $Z(f;t)$. Esto refleja el buen comportamiento de las singularidades cocientes abelianas con respecto a los cruces normales. Por el contrario, las no abelianas parecen funcionar de otra manera, ver §IV.5 donde se muestra que los "puntos dobles" pueden contribuir a $Z(f;t)$.

Nota. Si la ecuación de g en $P \in E_{\{i\}}^\circ \cap Q_j$ es de la forma $x^m : X(d; a, b) \to \mathbb{C}$ y el tipo $(d; a, b)$ está normalizado, entonces $m(P) = \frac{m}{d}$. Así esta fórmula ya ha sido estudiada en [**Vey97**] para singularidades de curvas planas.

Vamos a describir la normalización semiestable de $g : X \to D_\eta^2$. Sea e el mínimo común múltiplo de todas las multiplicidades que aparecen en el divisor $E := g^{-1}(0) = E_0 \cup \cdots \cup E_s$ y consideremos $\sigma : D_{\eta^{1/e}}^2 \to D_\eta^2$ la cubierta ramificada definida por $\sigma(t) = t^e$. Denotamos por (X_1, g_1, σ_1) el pull-back de g y σ. Finalmente, sea $\nu : \widetilde{X} \to X_1$ la normalización de X_1 y denotemos por $\widetilde{g} := g_1 \circ \nu$ y $\varrho := \sigma_1 \circ \nu$ los morfismos naturales. También pongamos $D_i = \varrho^{-1}(E_i)$ para $i = 0, \ldots, s$ y $D = \bigcup_{i=0}^s D_i$.

Este diagrama conmutativo representa el proceso completo de la normalización semiestable.

$$\begin{array}{ccccccc} D_i & \hookrightarrow & \widetilde{X} & \xrightarrow{\nu} & X_1 & \xrightarrow{g_1} & D^2_{\eta^{1/e}} \\ \varrho\big\downarrow & & \varrho\big\downarrow & & \sigma_1\big\downarrow & & \big\downarrow\sigma \\ E_i & \hookrightarrow & X & = & X & \xrightarrow[g]{} & D^2_{\eta} \end{array}$$

En esta situación, $m(g^*(0), P)$ con $P \in g^{-1}(0)$ se puede interpretar como el cardinal de la fibra sobre P de la cubierta $\varrho : \widetilde{X} \to X$. Nuestro quinto resultado importante es una descripción detallada de esta cubierta. Su demostración está basada en el cálculo explícito de la normalización de $t^e - x_0^{m_0} \cdots x_k^{m_k}$ visto como elemento de $\mathbb{C}[x_0, \ldots, x_n]^{\mu_{\mathbf{d}}} \otimes_{\mathbb{C}} \mathbb{C}[t]$, ver (V.1.7).

Proposición 5. *La variedad $\widetilde{X}$ solo tiene singularidades cocientes abelianas situadas en $\widetilde{g}^{-1}(0) = D$, el cual es un divisor reducido con cruces normales en $\widetilde{X}$. Además, $\varrho : \widetilde{X} \to X$ es una cubierta cíclica de e hoja no ramificada sobre $X \setminus g^{-1}(0)$. Para $\emptyset \neq I \subseteq S := \{0, 1, \ldots, s\}$ y $j \in J$, se tiene:*

1. *La restricción $\varrho| : \varrho^{-1}(\overline{E_I^\circ \cap Q_j}) \to \overline{E_I^\circ \cap Q_j}$ es una cubierta cíclica ramificada de $m(E_I^\circ \cap Q_j)$ hojas no ramificada sobre $E_I^\circ \cap Q_j$.*
2. *El espacio $\varrho^{-1}(\overline{E_I^\circ \cap Q_j})$ es una V-variedad con singularidades cocientes abelianas con $\gcd(\{m(P) \mid P \in \overline{E_I^\circ \cap Q_j}\})$ componentes conexas.*
3. *Sea $\varphi : \widetilde{X} \to \widetilde{X}$ el generador canónico de la monodromía de la cubierta ϱ. Entonces, su restricción a $\varrho^{-1}(\overline{E_I^\circ \cap Q_j})$ es un generador de la monodromía de $\varrho| : \varrho^{-1}(\overline{E_I^\circ \cap Q_j}) \to \overline{E_I^\circ \cap Q_j}$.*

La idea principal que hay detrás de esta construcción es que en el caso clásico después de considerar la normalización semiestable, el espacio ambiente contiene singularidades cocientes. La proposición anterior prueba que lo mismo es cierto para **Q**-resoluciones encajadas y así la construcción de Steenbrink con la sucesión espectral se puede adaptar para proporcionar una EHM sobre los grupos de cohomología (V.3.4). El propósitos del capítulo V es la descripción explícita de una sucesión espectral que converge a la cohomología de la fibra de Milnor a partir de una **Q**-resolución encajada §V.3.

Puesto que la **Q**-resolución encajada se puede elegir para que “casi todo” divisor excepcional contribuya a la monodromía, nuestra sucesión espectral es mejor en el sentido de que menos divisores aparecerán en la normalización semiestable y por tanto la combinatoria será más sencilla. Vamos a ver con un ejemplo cómo se aplican todos los resultados anteriormente presentados.

Ejemplo. Supongamos $\gcd(p,q) = \gcd(r,s) = 1$ y $\frac{p}{q} < \frac{r}{s}$. Sea $f = (x^p + y^q)(x^r + y^s)$ y consideremos $\mathbf{C}_1 = \{x^p + y^q = 0\}$ y $\mathbf{C}_2 = \{x^r + y^s = 0\}$. Una $\mathbf{Q}$-resolución encajada de $\{f = 0\} \subset \mathbb{C}^2$ se puede calcular con la (q,p)-explosión del origen de $\mathbb{C}^2$, seguida de la $(s, qr - ps)$-explosión de un punto de tipo $(q; -1, p)$, comparar con (2), ver figura 1.

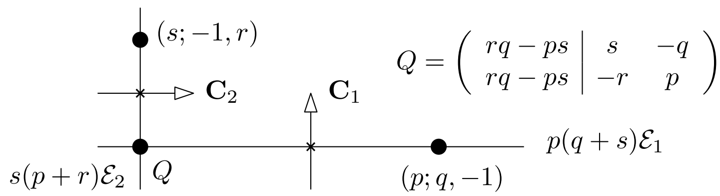


FIGURA 1. $\mathbf{Q}$-resolución encajada de $f = (x^p + y^q)(x^r + y^s)$.

La auto-intersecciones se calculan con la proposición 2 y la matriz de intersección es $A = \frac{1}{rq-ps}\begin{pmatrix} -r/p & 1 \\ 1 & -q/s \end{pmatrix}$. Por el teorema 4, el polinomio característico es

$$\Delta(t) = \frac{\left(t-1\right)\left(t^{p(q+s)}-1\right)\left(t^{s(p+r)}-1\right)}{\left(t^{q+s}-1\right)\left(t^{p+r}-1\right)}.$$

Estudiamos la normalización semiestable con la proposición 5. Su grafo dual ponderado se muestra en la figura 2.

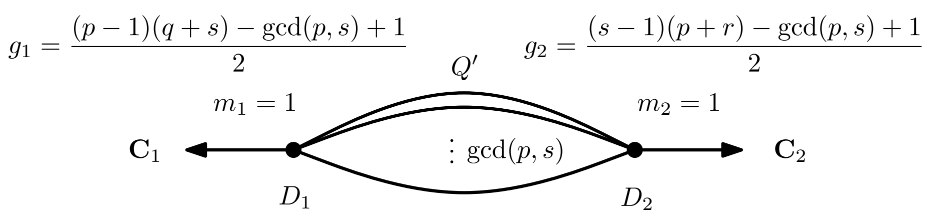


FIGURA 2. Grafo dual de la normalización semiestable de f.

La estructura de Hodge mixta de la cohomología de la fibra de Milnor $H^1(F,\mathbb{C})$ se obtiene de la sucesión espectral de Steenbrink:

$$H^1(F,\mathbb{C}) = \underbrace{H^{0,0}}_{\mathrm{Gr}_0^W H^1(F,\mathbb{C})} \oplus \underbrace{H^{0,1} \oplus H^{1,0}}_{\mathrm{Gr}_1^W H^1(F,\mathbb{C})} \oplus \underbrace{H^{1,1}}_{\mathrm{Gr}_2^W H^1(F,\mathbb{C})},$$

donde

$$H^{0,0} = \mathbb{C}^{\gcd(p,s)-1}, \qquad H^{0,1} = \mathbb{C}^{g_1} \oplus \mathbb{C}^{g_2}, \qquad H^{1,1} = \mathbb{C}^{\gcd(p,s)}.$$

Los géneros g_1 y g_2 se han calculado en la figura 2. La acción de la monodromía sobre $\mathrm{Gr}_0^W H^1(F,\mathbb{C})$ está dada por el polinomio $\frac{t^{\gcd(p,s)}-1}{t-1}$. Nótese que esto proporciona los autovalores de la monodromía con bloques de Jordan de tamaño 2, ver (V.4.3) para más detalles.

Segunda Parte (Capítulos VI y VII)

En estos dos capítulos aplicamos parcialmente las nuevas técnicas desarrolladas anteriormente para el estudio de singularidades superaisladas de hipersuperficies y singularidades de Yomdin-Lê (ponderadas) de superficies.

Estas singularidades han sido estudiadas ampliamente por muchos autores, ver el "survey" [**ALM06**] donde se repasa parte de la teoría de estas singularidades y sus aplicaciones incluyendo algunos desarrollos recientes y novedosos. Fueron introducidas por Luengo y también aparecen en un artículo de Stevens, donde se considera el estrado μ-constante, ver [**Lue87**] y [**Ste89**]. Después, Artal describió en su tesis doctoral [**Art94b**] una resolución encajada de tales singularidades usando explosiones de puntos y curvas racionales.

Aquí, en el capítulo VI, presentamos una descripción de una $\mathbf{Q}$-resolución encajada de singularidades superaisladas de superficies en términos de una $\mathbf{Q}$-resolución encajada (global) de su cono tangente. Probamos que solamente se necesitan explosiones ponderadas de puntos. Por el contrario, el espacio total que aparece tiene singularidades cocientes abelianas.

Más concretamente, sea $f = f_m + f_{m+1} + \cdots$ la descomposición de f en componentes homogéneas. Denotamos por $\mathbf{C} := V(f_m) \subset \mathbb{P}^2$ el cono tangente y supongamos que $V := V(f)$ es superaislada, es decir, $\operatorname{Sing}(\mathbf{C}) \cap V(f_{m+1}) = \emptyset$. El principal resultado de esta parte es una colección de resultados que se pueden resumir como sigue, ver (VI.2.2), (VI.2.10), (VI.2.13).

Teorema 6. *Sea $\varrho^P : Y^P \to (\mathbf{C}, P)$ una $\mathbf{Q}$-resolución encajada del cono tangente para $P \in \operatorname{Sing}(\mathbf{C})$. Supongamos que*

$$(\varrho^P)^*(\mathbf{C}, P) = \widehat{\mathbf{C}} + \sum_{a \in S(\Gamma_+^P)} m_a^P \mathcal{E}_a^P$$

es la transformada total de $(\mathbf{C}, P)$, donde $\mathcal{E}_a^P$ es el divisor excepcional de la (p_a^P, q_a^P)-explosión de un punto P_a que pertenece al lugar de no transversalidad. Denotemos por ν_a^P la (p_a^P, q_a^P)-multiplicidad de $\mathbf{C}$ en P_a.

Entonces, se puede construir una $\mathbf{Q}$-resolución encajada $\rho : X \to (V, 0)$ de la singularidad de superficie tal que la transformada total es

$$\rho^*(V, 0) = \widehat{V} + mE_0 + \sum_{\substack{P \in \operatorname{Sing}(\mathbf{C}) \\ a \in S(\Gamma_+^P)}} (m+1) m_a^P E_a^P,$$

y E_a^P aparece después de la (p_a^P, q_a^P, ν_a^P)-explosión del punto P_a (nótese que el lugar de no transversalidad en dimensión 2 y 3 se identifican).

La principal ventaja comparada con la de Artal [**Art94b**] es que en ésta última se necesitan en cada paso ν_a^P (en lugar de solo una) explosión de puntos y curvas racionales para llegar a una situación parecida.

En el capítulo VI, aplicamos el teorema 4 (fórmula de A'Campo's generalizada) para calcular el polinomio característico y el número de Milnor, ver teorema (VI.3.5) y corolario (VI.3.7). En particular, las fórmulas de [**Sie90**] y [**Ste89**] se pueden obtener de esta manera. En el futuro estudiaremos otros invariantes más sofisticados como la estructura de Hodge mixta de la cohomología de la fibra de Milnor.

Como consecuencia probamos que un divisor excepcional de la **Q**-resolución encontrada para $(V, 0)$ contribuye a la monodromía compleja si y solo si lo hace el correspondiente divisor en el cono tangente, ver (VI.3.3). Así los pesos se pueden elegir para que todo divisor excepcional de la **Q**-resolución encajada de $(V, 0)$ contribuya a la monodromía.

Estas técnicas se pueden aplicar para estudiar singularidades superaisladas en dimensión superior, ver §VI.4, y lo mismo ocurre para singularidades de Yomdin-Lê (ponderadas) de superficies, ver capítulo VII.

Tercera Parte (Capítulo VIII)

El último capítulo trata sobre el algoritmo `checkRoot` y sus aplicaciones para calcular el polinomio de Bernstein-Sato con bases de Gröbner. Para dar una descripción más detallada de los problemas que estamos interesados y los resultados que hemos obtenido, pasamos a recordar algunas definiciones básicas en campo de los D-módulos.

Dado un polinomio $f \in \mathbb{C}[\mathbf{x}]$ en varias variable el *polinomio de Bernstein-Sato* (también llamado *b-función global*) de f se define como el polinomio mónico no nulo $b_f(s)$ de menor grado que verifica

$$P(s) f^{s+1} = b_f(s) f^s \in \mathbb{C}[\mathbf{x}, s, 1/f] \cdot f^s$$

para $P(s) \in D_n[s] := D_n \otimes_{\mathbb{C}} \mathbb{C}[s]$, donde D_n denota la n-ésima álgebra de Weyl. La existencia de tal polinomio no nulo está garantizada por [**Ber72**]. Análogamente se define el *polinomio de Bernstein-Sato local* (también llamado *b-función local*) de f en $p \in \mathbb{C}^n$ y se denota por $b_{f,p}(s)$.

Se conocen varios algoritmos para calcular la b-función de un polinomio, ver por ejemplo [**Oak97c**], [**SST00**], [**BM02**], [**Nor02**], [**Sch04a**], [**LM08**]. Sin embargo, desde el punto de vista computacional, es muy complejo obtener este polinomio en general. A pesar de recientes progresos, en la práctica solo se pueden tratar un número limitado de ejemplos.

Motivados por esto y para mejorar el cálculo del polinomio de Bernstein-Sato con bases de Gröbner, estudiamos los siguientes problemas computacionales:

(1) Encontrar $B(s) = \prod_{i=1}^{d}(s-\alpha_i)^{m_i} \in \mathbb{C}[s]$ tal que $b_f(s)|B(s)$.
(2) Comprobar si α_i es raíz de la b-función.
(3) Calcular la multiplicidad de α_i como raíz de $b_f(s)$.

Existen algunos métodos conocidos para obtener una cota superior para el polinomio de Bernstein-Sato de una hipersuperficie singular, una vez conocida, por ejemplo, una resolución encajada de tal singularidad [**Kas77**], ver sección VIII.3. Sin embargo, no conocemos ningún algoritmo para calcular la b-función a partir de esta cota superior.

El resultado más importante de esta parte final es el teorema (VIII.2.1), que tiene varias consecuencias, ver (VIII.2.6) y (VIII.5.1). En particular, se tiene lo siguiente que resuelve los problemas (2) y (3) anteriores.

Corolario 7. *Sea m_α (resp. $m_\alpha(p)$) la multiplicidad de α como raíz de $b_f(-s)$ (resp. $b_{f,p}(-s)$). Sean los ideales $I = \operatorname{Ann}_{D_n[s]}(f^s) + D_n[s]\langle f\rangle$ e $I_{\alpha,i} = \big(I : (s+\alpha)^i\big) + D[s]\langle s+\alpha\rangle$. Entonces,*

(1) $m_\alpha > i \Longleftrightarrow I_{\alpha,i} \neq D_n[s]$,
(2) $m_\alpha(p) > i \Longleftrightarrow p \in V(I_{\alpha,i} \cap \mathbb{C}[\mathbf{x}])$.

El correspondiente algoritmo se llama `checkRoot` y en general es mucho más rápido que el cálculo de todo el polinomio de Bernstein, debido a que en (1) no hace falta usar órdenes de eliminación para calcular una base de Gröbner de $I_{\alpha,i}$. Además, el elemento $(s+\alpha)^i$, añadido como generador, parece simplificar tremendamente los cálculos, comparar con [**Nak09**].

Como primera aplicación, después de calcular una resolución encajada, hemos hallado $b_f(s)$ de la singularidad no aislada $f = (xz+y)(x^4+y^5+xy^4)$ en unos 30 segundos, ver (VIII.3.3). Este ejemplo (que apareció primero en [**CU05**]) era intratable con cualquier sistema de álgebra computacional.

Este algoritmo tiene varias aplicaciones como el cálculo de la b-función cuando se puede encontrar una cota superior (mediante resolución encajada, para singularidades topológicamente equivalentes o usando la fórmula de A'Campo y los número espectrales), las raíces enteras de $b_f(s)$ (importantes por ejemplo en el problema de comparación logarítmico) y una estratificación de $\mathbb{C}^n$ con la b-función local constante en cada estrato (el algoritmo propuesto no emplea descomposición primaria, comparar con [**NN10**]).

Los métodos de este capítulo han sido implementados en SINGULAR en las librerías `dmod.lib` y `bfun.lib`. Todos los ejemplos que se presentan aquí han sido calculados con esta implementación.

INTRODUCTION

First Part (Chapters I–V)

One of the main invariant of a given hypersurface singularity is the mixed Hodge structure (MHS) on the cohomology of the Milnor fiber. In the isolated case, Steenbrink gave a method for computing this Hodge structure using a spectral sequence that is constructed from the divisors associated with the semistable reduction of an embedded resolution, see [**Ste77**].

However, in practice the combinatorics of the exceptional divisor of the resolution is often so complicated that the study of the spectral sequence becomes very hard, see e.g. [**Art94b**] where an embedded resolution and its associated semistable reduction for superisolated surface singularities is computed using blow-ups at points and rational curves.

After the semistable reduction process the new ambient space contains normal singularities which are obtained as the quotient of a ball in $\mathbb{C}^n$ by the linear action of a finite group. Spaces admitting only such singularities are called *V-manifolds*. They were introduced in [**Sat56**] and have the same homological properties over $\mathbb{Q}$ as manifolds, e.g. they admit a Poincaré duality if they are compact and carry a pure Hodge structure if they are compact and Kähler, see [**Bai56**]. Moreover, a natural notion of normal crossing divisor can be defined on V-manifolds, see [**Ste77**].

Motivated by this fact and in order to try to simplify the combinatorics of the exceptional divisor mentioned above, we introduce the notion of embedded **Q**-resolution. The idea is as follows. Classically an embedded resolution of $\{f = 0\} \subset \mathbb{C}^{n+1}$ is a proper map $\pi : X \to (\mathbb{C}^{n+1}, 0)$ from a smooth variety X satisfying, among other conditions, that $\pi^*(\{f = 0\})$ is a normal crossing divisor. To weaken the condition on the preimage of the singularity we allow the new ambient space X to contain abelian quotient singularities and the divisor $\pi^*(\{f = 0\})$ to have "normal crossings" on X.

More precisely, here is the formal definition of one of the main objects of our study.

Definition. Let M be an abelian quotient space. Consider $H \subset M$ an analytic subvariety of codimension one. An *embedded* **Q**-*resolution* of $(H, 0) \subset (M, 0)$ is a proper analytic map $\pi : X \to (M, 0)$ such that:

(1) X is a V-manifold with abelian quotient singularities.
(2) π is an isomorphism over $X \setminus \pi^{-1}(\mathrm{Sing}(H))$.
(3) $\pi^*(H)$ is a hypersurface with $\mathbb{Q}$-normal crossings on X.

The present work is devoted to the study of invariants of a hypersurface singularity $(H, 0) \subset (\mathbb{C}^{n+1}, 0)$ by looking at either an embedded **Q**-resolution or its associated semistable reduction. We will focus on two important invariants of H, namely the characteristic polynomial of the complex monodromy (Chapter IV) and the mixed Hodge structure on the cohomology of the Milnor fiber (Chapter V).

As mentioned above, the motivation for using embedded **Q**-resolutions rather than standard ones is twofold. On the one hand, they are natural generalization of the usual embedded resolutions, for which the invariant above are expected to be calculated effectively. On the other hand, the combinatorial and computational complexity of embedded **Q**-resolutions is much simpler, but they keep as much information as needed for the comprehension of the topology of the singularity.

Notation. To deal with these resolutions, some notations need to be introduced. Let $G := \mu_{d_0} \times \cdots \times \mu_{d_r}$ be an arbitrary finite abelian group written as a product of finite cyclic groups, that is, μ_{d_i} is the cyclic group of d_i-th roots of unity. Consider a matrix of weight vectors

$$A := (a_{ij})_{i,j} = [\mathbf{a}_0 \,|\, \cdots \,|\, \mathbf{a}_n] \in Mat((r+1) \times (n+1), \mathbb{Z})$$

and the action

$$\begin{gathered} (\mu_{d_0} \times \cdots \times \mu_{d_r}) \times \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^{n+1}, \\ (\boldsymbol{\xi}_{\mathbf{d}}, \mathbf{x}) \mapsto (\xi_{d_0}^{a_{00}} \cdots \xi_{d_r}^{a_{r0}}\, x_0, \,\ldots\, , \xi_{d_0}^{a_{0n}} \cdots \xi_{d_r}^{a_{rn}}\, x_n). \end{gathered} \tag{3}$$

The set of all orbits $\mathbb{C}^{n+1}/G$ is called (*cyclic*) *quotient space of type* $(\mathbf{d}; A)$ and it is denoted by

$$X(\mathbf{d}; A) := X \left(\begin{array}{c|ccc} d_0 & a_{00} & \cdots & a_{0n} \\ \vdots & \vdots & \ddots & \vdots \\ d_r & a_{r0} & \cdots & a_{rn} \end{array} \right).$$

The orbit of an element $(x_0, \ldots, x_n)$ under this action is denoted by $[(x_0, \ldots, x_n)]_{(\mathbf{d};A)}$ and the subindex is omitted if no ambiguity seems likely to arise.

Remark. Condition (3) of the previous definition means that if $f : (M,0) \to (\mathbb{C},0)$ is a non-constant analytic function germ and $(H,0)$ is the hypersurface defined by f on $(M,0)$, then the total transform $\pi^{-1}(H) = (f \circ \pi)^{-1}(0)$ is locally given by a function of the form $x_0^{m_0} \cdots x_k^{m_k} : X(\mathbf{d}; A) \to \mathbb{C}$, where $X(\mathbf{d}; A) := \mathbb{C}^{n+1}/\mu_{\mathbf{d}}$ and $\mu_{\mathbf{d}}$ acts diagonally as in (3).

The previous numbers m_i's have no intrinsic meaning when $\mu_{\mathbf{d}}$ does not induce a small action on $GL(n+1,\mathbb{C})$. This motivates the following.

Definition. The type $(\mathbf{d}; A)$ is said to be *normalized* if the action is free on $(\mathbb{C}^*)^{n+1}$ and $\mu_{\mathbf{d}}$ is identified with a small subgroup of $GL(n+1,\mathbb{C})$.

A classic problem in Singularity Theory is to describe or give a method for calculating invariants once an embedded resolution is known. The existence of such a resolution is guaranteed by the works of Hironaka. In the same spirit, one of the main aims of this work is to provide information about the singularity by looking at an embedded **Q**-resolution of it.

As a tool for finding embedded **Q**-resolutions we will use weighted blow-ups with smooth center. Special attention is paid to the case of dimension 2 and 3 and blow-ups at points. Such blow-ups can be understood from toric geometry but in this work they are presented more geometrically, generalizing the standard ones.

Example. Assume $(d; a, b)$ is normalized and $\gcd(\omega) = 1$, $\omega := (p, q)$. Then, the total space of the ω-weighted blow-up at the origin of $X(d; a, b)$,

$$\pi_{(d;a,b),\omega} : \widehat{X(d; a, b)}_{\omega} \longrightarrow X(d; a, b), \tag{4}$$

can be written as

$$\widehat{U}_1 \cup \widehat{U}_2 = X\left(\frac{pd}{e}; 1, \frac{-q+\beta pb}{e}\right) \cup X\left(\frac{qd}{e}; \frac{-p+\mu qa}{e}, 1\right)$$

and the charts are given by

$$\text{First chart} \quad \left| \begin{array}{c} X\left(\frac{pd}{e}; 1, \frac{-q+\beta pb}{e}\right) \longrightarrow \widehat{U}_1, \\ \big[(x^e, y)\big] \mapsto \big[((x^p, x^q y), [1 : y]_{\omega})\big]_{(d;a,b)}. \end{array} \right.$$

$$\text{Second chart} \quad \left| \begin{array}{c} X\left(\frac{qd}{e}; \frac{-p+\mu qa}{e}, 1\right) \longrightarrow \widehat{U}_2, \\ \big[(x, y^e)\big] \mapsto \big[((x y^p, y^q), [x : 1]_{\omega})\big]_{(d;a,b)}. \end{array} \right.$$

Above, $e = \gcd(d, pb - qa)$ and $\beta a \equiv \mu b \equiv 1 \pmod{d}$, see I.3–1 for details. Observe that the origins of the two charts are cyclic quotient singularities; they are located at the exceptional divisor E which is isomorphic to $\mathbb{P}^1$.

To manage to study embedded **Q**-resolutions an intersection theory is needed. It is also required to deal with divisors on V-manifolds. Two kinds of divisors appear in the literature: Weil and Cartier divisors. Weil divisors are locally finite linear combination with integral coefficients of irreducible subvarieties of codimension 1 and Cartier divisors are global sections of the quotient sheaf of meromorphic functions modulo non-vanishing holomorphic functions. The relationship between Cartier divisors and line bundles provides a nice way to define the intersection multiplicity of two divisors.

In the smooth category, both notions coincide but this is not the case for singular varieties, not even for normal ones. One can also consider Weil and Cartier $\mathbb{Q}$-divisors (tensoring the corresponding groups by $\mathbb{Q}$). The first main result of this work is that these two notions coincide for V-manifolds.

Theorem 1. *Let X be a V-manifold. Then, the linear map defined using the notion of associated Weil divisor,*

$$T_X \otimes 1 : \mathrm{CaDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathrm{WeDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q},$$

is an isomorphism of $\mathbb{Q}$-vector spaces. In particular, for a given Weil divisor D on X, there always exists $k \in \mathbb{Z}$ such that $kD \in \mathrm{CaDiv}(X)$.

This result is probably known for specialists but we have not found a proof in the literature. There are some partial results for toric varieties (defined with simplicial cones). Moreover, in this work we give an algorithm to explicitly represent a Weil $\mathbb{Q}$-divisor as a Cartier $\mathbb{Q}$-divisor, see (II.2.14). We illustrate the use of this algorithm with an example living in the space obtained after a weighted blow-up.

Example. Let $\pi_{(d;a,b),\omega}$ be the proper morphism defined in (4). Then, its exceptional divisor E is a Weil divisor which does not correspond to a Cartier divisor. However, following the preceding discussion, it can be written as Cartier $\mathbb{Q}$-divisor like $\frac{e}{dpq}\big\{(\widehat{U}_1, x^{dq}), (\widehat{U}_2, y^{dp})\big\}$.

Theorem 1 above allows one to develop a rational intersection theory on V-manifolds with the usual expected properties collected in Proposition (III.1.3).

Definition. Let X be a V-surface and consider $D_1, D_2 \in \mathbb{Q}\text{-Div}(X)$. The *intersection number* is defined as $D_1 \cdot D_2 := \frac{1}{k_1 k_2}(k_1 D_1 \cdot k_2 D_2) \in \mathbb{Q}$, where $k_1, k_2 \in \mathbb{Z}$ are chosen so that $k_1 D_1 \in \mathrm{WeDiv}(X)$ and $k_2 D_2 \in \mathrm{CaDiv}(X)$. Analogously, it is defined the *local intersection number* at $P \in D_1 \cap D_2$, if the condition $D_1 \not\subseteq D_2$ is satisfied. Idem the *pull-back* is defined by $F^*(D_2) := \frac{1}{k_2} F^*(k_2 D_2)$ if $F : Y \to X$ is a proper morphism between two irreducible V-surfaces.

This rational intersection number was first introduced by Mumford for normal surfaces, see [**Mum61**, Pag. 17]. Our definition coincides with Mumford's because it has good behavior with respect to the pull-back, see Theorem (III.1.5). The main advantage is that ours does not involve a resolution of the ambient space and, for instance, this allows us to easily find formulas for the self-intersection numbers of the exceptional divisors of weighted blow-ups, without computing any resolution.

In fact, this is essentially the second main result of this work.

Proposition 2. *Let $\pi := \pi_{(d;a,b),\omega}$ be the morphism defined in* (4)*. Consider two $\mathbb{Q}$-divisors C and D on $X(d;a,b)$. Then,*

(1) $E \cdot \pi^*(C) = 0$,

(2) $\pi^*(C) = \widehat{C} + \frac{\nu}{e} E$,

(3) $E \cdot \widehat{C} = \frac{e\nu}{dpq}$,

(4) $E^2 = -\frac{e^2}{dpq}$,

(5) $\widehat{C} \cdot \widehat{D} = C \cdot D - \frac{\nu\mu}{dpq}$,

(6) $\widehat{D}^2 = D^2 - \frac{\mu^2}{dpq}$ *(D compact)*,

where ν and μ denote the (p,q)-multiplicities of C and D at P, i.e. x (resp. y) has (p,q)-multiplicity p (resp. q).

Our third main result is a version of Bézout's Theorem for quotients of weighted projective planes.

Proposition 3. *Let us denote by m_1, m_2, m_3 the determinants of the three minors of order* 2 *of the matrix $\left(\begin{smallmatrix} p & q & r \\ a & b & c \end{smallmatrix}\right)$. Assume that* $\gcd(p,q,r) = 1$ *and write $e = \gcd(d, m_1, m_2, m_3)$. If $\omega = (p,q,r)$, then the intersection number of two $\mathbb{Q}$-divisors on $\mathbb{P}^2_\omega(d;a,b,c) := \mathbb{P}^2_\omega/\mu_d$ is*

$$D_1 \cdot D_2 = \frac{e}{dpqr} \deg_\omega(D_1) \deg_\omega(D_2) \in \mathbb{Q}.$$

Note that the exceptional divisor of the (p,q,r)-weighted blow-up at a point of type $(d;a,b,c)$ is naturally isomorphic to $\mathbb{P}^2_\omega(d;a,b,c)$, see §I.3–2. Hence this result will help us describe embedded **Q**-resolutions of surfaces in $\mathbb{C}^3$, see e.g. Chapter VI where the superisolated case is treated in detail.

Now we have all the necessary ingredients to study the two commented invariants in terms of an embedded **Q**-resolution of the singularity and its associated semistable reduction. Both results depend on the stratification of a $\mathbb{Q}$-normal crossing divisor. Hence some notation need to be introduced.

Notation. Let $f : (M,0) \to (\mathbb{C},0)$ be a non-constant analytic function germ and let $(H,0)$ be the hypersurface defined by f. Given an embedded **Q**-resolution of $(H,0)$, $\pi : X \to (M,0)$, consider $E_1, \ldots, E_s$ the irreducible components of the exceptional divisor and $\widehat{H}$ the strict transform.

One writes $E_0 = \widehat{H}$ and $S = \{0, 1, \ldots, s\}$ so that the stratification of X associated with the $\mathbb{Q}$-normal crossing divisor $\pi^{-1}(H) = \bigcup_{i\in S} E_i$ is defined by setting

$$E_I^\circ := \Big(\cap_{i\in I} E_i \Big) \setminus \Big(\cup_{i\notin I} E_i \Big),$$

for a given possibly empty subset $I \subseteq S$.

Also, let $X = \bigsqcup_{j\in J} Q_j$ be a finite stratification of X given by its quotient singularities so that the local equation of $g := f \circ \pi$ at $P \in E_I^\circ \cap Q_j$ is of the form

$$x_0^{m_0} \cdot \ldots \cdot x_k^{m_k} : X(\mathbf{d}; A) \longrightarrow \mathbb{C}, \quad (0 \le k \le n)$$

and the multiplicities m_i's and the action $\mu_{\mathbf{d}}$ are the same along each stratum $E_I^\circ \cap Q_j$. In this context $m(E_I^\circ \cap Q_j)$ is defined as

$$m(E_I^\circ \cap Q_j) := \gcd \left(m_0, \ldots, m_k, \frac{\sum_{j=0}^k a_{0j} m_j}{d_0}, \ldots, \frac{\sum_{j=0}^k a_{rj} m_j}{d_r} \right).$$

Sometimes it is denoted by $m(E, P)$ or even $m(P)$, $P \in E_I^\circ \cap Q_j$, if no ambiguity seems likely to arise, cf. (IV.3.12) and (V.1.4).

The fourth main result of this work is the generalized A'Campo's formula for embedded **Q**-resolutions, see Theorem (IV.3.14) for a more complete statement. Its proof is based on the result [**Dim04**, Th. 6.1.14.] and hence one needs to deal with constructible complexes of sheaves with respect to a stratification and the nearby cycles associated with an analytic function.

Theorem 4. $Z(f; t) = \prod_{i=1,\ldots,s,\ j\in J} \left(1 - t^{m(E_{\{i\}}^\circ \cap Q_j)}\right)^{\chi(E_{\{i\}}^\circ \cap Q_j)}.$

Note that only the strata $E_{\{i\}}^\circ \cap Q_j$ coming from the exceptional divisor contribute to $Z(f; t)$. This reflects the good behavior of abelian quotient singularities with respect to normal crossing divisors. By contrast, non-abelian groups seem to work differently, see §IV.5 where it is shown that "double points" may contribute to $Z(f; t)$.

Remark. If the equation of g at $P \in E_{\{i\}}^\circ \cap Q_j$ is of the form $x^m : X(d; a, b) \to \mathbb{C}$ and the latter quotient space is normalized, then $m(P) = \frac{m}{d}$. Hence this formula has already been studied in [**Vey97**] for plane curve singularities.

Let us briefly describe the semistable reduction of $g : X \to D_\eta^2$. Let e be the least common multiple of all possible multiplicities appearing in the divisor $E := g^{-1}(0) = E_0 \cup \cdots \cup E_s$ and consider $\sigma : D_{\eta^{1/e}}^2 \to D_\eta^2$ the branched covering defined by $\sigma(t) = t^e$. Denote by (X_1, g_1, σ_1) the pull-back of g and σ. Finally, let $\nu : \widetilde{X} \to X_1$ be the normalization of X_1 and denote by $\widetilde{g} := g_1 \circ \nu$ and $\varrho := \sigma_1 \circ \nu$ the natural maps. Write $D_i = \varrho^{-1}(E_i)$ for $i = 0, \ldots, s$ and $D = \bigcup_{i=0}^s D_i$.

This commutative diagram illustrates the whole process of the semistable reduction.

$$
\begin{array}{ccccccc}
D_i & \hookrightarrow & \widetilde{X} & \xrightarrow{\nu} & X_1 & \xrightarrow{g_1} & D^2_{\eta^{1/e}} \\
\Big\downarrow \varrho & & \Big\downarrow \varrho & & \Big\downarrow \sigma_1 & & \Big\downarrow \sigma \\
E_i & \hookrightarrow & X & = & X & \xrightarrow[g]{} & D^2_{\eta}
\end{array}
$$

In this situation, $m(g^*(0), P)$ with $P \in g^{-1}(0)$ can be interpreted geometrically as the cardinality of the fiber over P of the covering $\varrho : \widetilde{X} \to X$. Our fifth main result is a detailed description of this covering. Its proof is based on the explicit computation of the normalization of $t^e - x_0^{m_0} \cdots x_k^{m_k}$ regarded as an element in $\mathbb{C}[x_0, \ldots, x_n]^{\mu_{\mathbf{d}}} \otimes_{\mathbb{C}} \mathbb{C}[t]$, cf. (V.1.7).

Proposition 5. *The variety $\widetilde{X}$ only has abelian quotient singularities located at $\widetilde{g}^{-1}(0) = D$ which is a reduced divisor with normal crossings on $\widetilde{X}$. Also, $\varrho : \widetilde{X} \to X$ is a cyclic branched covering of e sheets unramified over $X \setminus g^{-1}(0)$. Moreover, for $\emptyset \neq I \subseteq S := \{0, 1, \ldots, s\}$ and $j \in J$, one has:*

(1) *The restriction $\varrho| : \varrho^{-1}(\overline{E_I^\circ \cap Q_j}) \to \overline{E_I^\circ \cap Q_j}$ is a cyclic branched covering of $m(E_I^\circ \cap Q_j)$ sheets unramified over $E_I^\circ \cap Q_j$.*

(2) *The variety $\varrho^{-1}(\overline{E_I^\circ \cap Q_j})$ is a V-manifold with abelian quotient singularities with $\gcd(\{m(P) \mid P \in \overline{E_I^\circ \cap Q_j}\})$ connected components.*

(3) *Let $\varphi : \widetilde{X} \to \widetilde{X}$ be the canonical generator of the monodromy of the covering ϱ. Then, its restriction to $\varrho^{-1}(\overline{E_I^\circ \cap Q_j})$ is a generator of the monodromy of $\varrho| : \varrho^{-1}(\overline{E_I^\circ \cap Q_j}) \to \overline{E_I^\circ \cap Q_j}$.*

The main idea behind this construction is that in the classical case after considering the semistable reduction the ambient space already contains quotient singularities. Proposition 5 says that the same is true for embedded **Q**-resolutions and thus the construction by Steenbrink with the spectral sequence can be adapted to provide a mixed Hodge structure on the cohomology groups, see Theorem (V.3.4). In fact, one of the aims of Chapter V is to describe explicitly a similar spectral sequence converging to the cohomology of the Milnor fiber starting with an embedded **Q**-resolution, §V.3.

Since the embedded **Q**-resolution can be chosen so that "almost every" exceptional divisor contributes to the complex monodromy, our spectral sequence is better in the sense that less divisors will appear in the semistable reduction and thus the combinatorial of the spectral sequence will be simpler. We illustrate the use of all the preceding results presented in this part with an example.

Example. Assume $\gcd(p,q)=\gcd(r,s)=1$ and $\frac{p}{q}<\frac{r}{s}$. Let $f=(x^p+y^q)(x^r+y^s)$ and consider $\mathbf{C}_1=\{x^p+y^q=0\}$ and $\mathbf{C}_2=\{x^r+y^s=0\}$. An embedded $\mathbf{Q}$-resolution of $\{f=0\}\subset\mathbb{C}^2$ can be computed with the (q,p)-blow-up at the origin of $\mathbb{C}^2$, followed by the $(s,qr-ps)$-blow-up at a point of type $(q;-1,p)$, cf. (4), see Figure 3.

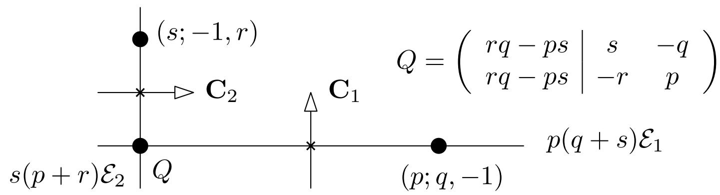


FIGURE 3. Embedded $\mathbf{Q}$-resolution of $f=(x^p+y^q)(x^r+y^s)$.

The self-intersection numbers are calculated using Proposition 2 and the intersection matrix is $A=\frac{1}{rq-ps}\begin{pmatrix}-r/p & 1\\ 1 & -q/s\end{pmatrix}$. By Theorem 4, the characteristic polynomial is

$$\Delta(t)=\frac{\big(t-1\big)\big(t^{p(q+s)}-1\big)\big(t^{s(p+r)}-1\big)}{\big(t^{q+s}-1\big)\big(t^{p+r}-1\big)}.$$

The semistable reduction is studied using Proposition 5. Its weighted dual graph is shown in Figure 4.

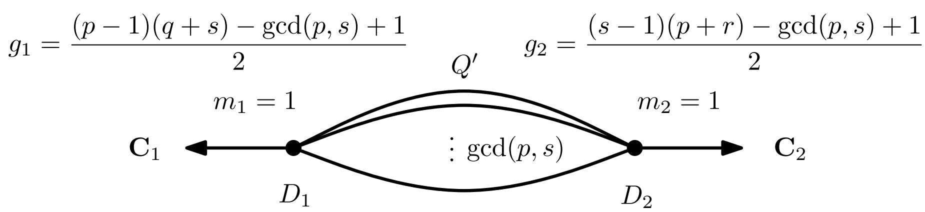


FIGURE 4. Dual graph of the semistable reduction of f.

The mixed Hodge structure (MHS) on the cohomology of the Milnor fiber $H^1(F,\mathbb{C})$ is obtained from Steenbrink's spectral sequence:

$$H^1(F,\mathbb{C})=\underbrace{H^{0,0}}_{\operatorname{Gr}_0^W H^1(F,\mathbb{C})}\oplus\underbrace{H^{0,1}\oplus H^{1,0}}_{\operatorname{Gr}_1^W H^1(F,\mathbb{C})}\oplus\underbrace{H^{1,1}}_{\operatorname{Gr}_2^W H^1(F,\mathbb{C})},$$

where

$$H^{0,0}=\mathbb{C}^{\gcd(p,s)-1},\qquad H^{0,1}=\mathbb{C}^{g_1}\oplus\mathbb{C}^{g_2},\qquad H^{1,1}=\mathbb{C}^{\gcd(p,s)}.$$

The genera g_1 and g_2 are calculated in Figure 4. The action of the monodromy on $\operatorname{Gr}_0^W H^1(F,\mathbb{C})$ is given by the polynomial $\frac{t^{\gcd(p,s)}-1}{t-1}$. Note that this provides the eigenvalues of the monodromy with Jordan blocks of size 2, see Example (V.4.3) for details.

Second Part (Chapters VI and VII)

In these two chapters the new techniques developed above are partially applied to the study of superisolated hypersurface singularities and (weighted) Yomdin-Lê surface singularities.

These singularities have been extensively studied by many authors, see the survey [**ALM06**] where part of the theory of these singularities and their applications including some new and recent developments are reviewed. They were introduced by Luengo and also appear in a paper by Stevens, where the μ-constant stratum is considered, see [**Lue87**] and [**Ste89**]. Afterward Artal described in his PhD thesis [**Art94b**] an embedded resolution of such singularities using blow-ups at points and rational curves.

Here, in chapter VI, we present a detailed description of an embedded $\mathbf{Q}$-resolution for superisolated surface singularities in terms of a (global) embedded $\mathbf{Q}$-resolution of its tangent cone. It is proven that only weighted blow-ups at points are needed. By contrast, the final total space produced has abelian quotient singularities.

More precisely, let $f = f_m + f_{m+1} + \cdots$ be the decomposition of f into its homogeneous parts. Denote by $\mathbf{C} := V(f_m) \subset \mathbb{P}^2$ its tangent cone and assume that $V := V(f)$ is superisolated, i.e. $\operatorname{Sing}(\mathbf{C}) \cap V(f_{m+1}) = \emptyset$. The main result of this part is a collection of several results that can be summarized as follows, cf. (VI.2.2), (VI.2.10), (VI.2.13).

Theorem 6. *Let $\varrho^P : Y^P \to (\mathbf{C}, P)$ be an embedded $\mathbf{Q}$-resolution of the tangent cone for $P \in \operatorname{Sing}(\mathbf{C})$. Suppose that*

$$(\varrho^P)^*(\mathbf{C}, P) = \widehat{\mathbf{C}} + \sum_{a \in S(\Gamma_+^P)} m_a^P \mathcal{E}_a^P$$

is the total transform of $(\mathbf{C}, P)$, where $\mathcal{E}_a^P$ is the exceptional divisor of the (p_a^P, q_a^P)-blow-up at a point P_a belonging to the locus of non-transversality. Denote by ν_a^P the (p_a^P, q_a^P)-multiplicity of $\mathbf{C}$ at P_a.

Then, one can construct an embedded $\mathbf{Q}$-resolution $\rho : X \to (V, 0)$ of the superisolated singularity such that the total transform is

$$\rho^*(V, 0) = \widehat{V} + mE_0 + \sum_{\substack{P \in \operatorname{Sing}(\mathbf{C}) \\ a \in S(\Gamma_+^P)}} (m+1) m_a^P E_a^P,$$

and E_a^P appears after the (p_a^P, q_a^P, ν_a^P)-blow-up at the point P_a (note that the locus of non-transversality in dimension 2 and 3 are identified).

The main advantage compared with Artal's resolution [**Art94b**] is that in the latter ν_a^P (rather than just one) blow-ups at points and rational curves at each step are needed to achieve a similar situation.

The generalized A'Campo's formula, Theorem 4, is applied and the characteristic polynomial and the Milnor number are calculated as an application, see Theorem (VI.3.5) and Corollary (VI.3.7). In particular, the formulas in [**Sie90**] and [**Ste89**] can be obtained in this way. Other more sophisticated invariants, including mixed Hodge structure of the cohomology of the Milnor fiber, are the subjects of our study for the future.

As a consequence, we prove that an exceptional divisor in the $\mathbf{Q}$-resolution obtained for $(V, 0)$ contributes to the complex monodromy if and only if so does the corresponding divisor in the tangent cone, see (VI.3.3). Thus the weights can be chosen so that every exceptional divisor in the embedded $\mathbf{Q}$-resolution of $(V, 0)$ contributes to its monodromy.

This techniques can be applied to study superisolated singularities in higher dimension, see §VI.4, and the same applies to (weighted) Yomdin-Lê surface singularities, see Chapter VII.

Third Part (Chapter VIII)

The last chapter is about the `checkRoot` algorithm and its applications for the computation of the Bernstein-Sato polynomial by means of non-commutative Gröbner bases. In order to give a more precise description of the problems we are interested in and the results we obtain, let us recall some basic definitions from the realm of D-modules.

Given a polynomial $f \in \mathbb{C}[\mathbf{x}]$ in several variables, the *Bernstein-Sato polynomial* (also called *global b-function*) of f is defined as the (non-zero) monic polynomial $b_f(s) \in \mathbb{C}[s]$ of minimal degree satisfying

$$P(s) f^{s+1} = b_f(s) f^s \in \mathbb{C}[\mathbf{x}, s, 1/f] \cdot f^s$$

for some $P(s) \in D_n[s] := D_n \otimes_{\mathbb{C}} \mathbb{C}[s]$, where D_n denotes the n-th Weyl algebra. The existence of such a non-zero polynomial is guaranteed by [**Ber72**]. Analogously, it is defined the *local Bernstein-Sato polynomial* (also called *local b-function*) of f at $p \in \mathbb{C}^n$, and it is denoted by $b_{f,p}(s)$.

Several algorithms for computing the b-function associated with a polynomial are known, see for instance [**Oak97c**], [**SST00**], [**BM02**], [**Nor02**], [**Sch04a**], [**LM08**]. However, from the computational point of view it is very hard to obtain this polynomial in general. Despite significant recent progress, only restricted number of examples can be actually treated.

Motived by this fact and in order to enhance the computation of the Bernstein-Sato polynomial via Gröbner bases, we study the following computational problems:

(1) Find $B(s) = \prod_{i=1}^{d}(s-\alpha_i)^{m_i} \in \mathbb{C}[s]$ such that $b_f(s)$ divides $B(s)$.
(2) Check whether α_i is a root of the b-function.
(3) Compute the multiplicity of α_i as a root of $b_f(s)$.

There exist some well-known methods to obtain an upper bound for the Bernstein-Sato polynomial of a hypersurface singularity once we know, for instance, an embedded resolution of such singularity [**Kas77**], see Section VIII.3. However, as far as we know, there is no algorithm for computing the b-function from this upper bound.

The main result of this final part is Theorem (VIII.2.1), which has several consequences, see e.g. (VIII.2.6) and (VIII.5.1). In particular, one obtains the following result solving problems (2) and (3) above.

Corollary 7. *Let m_α (resp. $m_\alpha(p)$) be the multiplicity of α as a root of $b_f(-s)$ (resp. $b_{f,p}(-s)$). Consider the ideals $I = \mathrm{Ann}_{D_n[s]}(f^s) + D_n[s]\langle f\rangle$ and $I_{\alpha,i} = \big(I : (s+\alpha)^i\big) + D[s]\langle s+\alpha\rangle$. Then,*

(1) $m_\alpha > i \Longleftrightarrow I_{\alpha,i} \neq D_n[s]$,
(2) $m_\alpha(p) > i \Longleftrightarrow p \in V(I_{\alpha,i} \cap \mathbb{C}[\mathbf{x}])$.

The corresponding algorithm is called `checkRoot` and in general is much faster than the computation of the whole Bernstein polynomial because no elimination ordering is needed in (1) for computing a Gröbner basis of $I_{\alpha,i}$. Also, the element $(s+\alpha)^i$, added as a generator, seems to simplify tremendously such a computation, cf. [**Nak09**].

As a first application, after computing an embedded resolution, we could obtain $b_f(s)$ for the non-isolated singularities $f = (xz+y)(x^4+y^5+xy^4)$ in about 30 seconds, see Example (VIII.3.3). This example (first appeared in [**CU05**]) was intractable by any computer algebra system.

Applications of this algorithm includes the computation of the b-function where there is a possibility to compute an upper (it can be achieved by means of embedded resolution, for topologically equivalent singularities or using the formula by A'Campo and spectral numbers), the integral roots of $b_f(s)$ (important e.g. for the logarithmic comparison problem), and a stratification of $\mathbb{C}^n$ with the local b-function being constant on each stratum (the algorithm we propose does not employ primary decomposition, cf. [**NN10**]).

The methods from this chapter have been implemented in Singular as libraries `dmod.lib` and `bfun.lib`. All the examples presented here have been computed with this implementation.

I

Quotient Singularities and Embedded $\mathbb{Q}$-Resolutions

The purpose of this chapter is to fix the notation and provide several tools to calculate a special kind of embedded resolutions allowing the ambient space to contain abelian quotient singularities. These resolutions are called *embedded* **Q**-*resolutions*, see Definition (I.3.2) below. To do this, we study weighted blow-ups with smooth center. Special attention is paid to the case of dimension 2 and 3 and blow-ups at points.

In Chapter III, we develop an intersection theory on this natural context of varieties with abelian quotient singularities. This theory was first introduced by Mumford over normal surfaces, see [**Mum61**]. The tools presented in this chapter will permit computing the self-intersection numbers of the exceptional divisors of weighted blow-ups in dimension two, see Proposition (III.3.2).

All these techniques are applied in Chapters VI and VII, and they are essential for our study of (weighted) Yomdin-Lê singularities. We do not pretend to be exhaustive and though objects presented here have many interesting properties, we focus on those that are used later.

As for notation through this work we often use $(i_1, \ldots, i_k)$ instead of $\gcd(i_1, \ldots, i_k)$ in case of complicated and long formulas if no ambiguity seems likely to arise.

Section § I.1
V-manifolds and Quotient Singularities

Definition (I.1.1). A V-manifold of dimension n is a complex analytic space which admits an open covering $\{U_i\}$ such that U_i is analytically isomorphic to B_i/G_i where $B_i \subset \mathbb{C}^n$ is an open ball and G_i is a finite subgroup of $GL(n, \mathbb{C})$.

V-manifolds were introduced in [**Sat56**] and have the same homological properties over $\mathbb{Q}$ as manifolds. For instance, they admit a Poincaré duality if they are compact and carry a pure Hodge structure if they are compact and Kähler, see [**Bai56**]. They have been classified locally by Prill [**Pri67**]. To state this local result we need the following.

Definition (I.1.2). A finite subgroup G of $GL(n,\mathbb{C})$ is called *small* if no element of G has 1 as an eigenvalue of multiplicity precisely $n-1$, that is, G does not contain rotations around hyperplanes other than the identity.

(I.1.3). For every finite subgroup G of $GL(n,\mathbb{C})$ denote by G_{big} the normal subgroup of G generated by all rotations around hyperplanes. Then, the G_{big}-invariant polynomials form a polynomial algebra and hence $\mathbb{C}^n/G_{\text{big}}$ is isomorphic to $\mathbb{C}^n$.

The group G/G_{big} maps isomorphically to a small subgroup of $GL(n,\mathbb{C})$, once a basis of invariant polynomials has been chosen. Hence the local classification of V-manifolds reduces to the classification of actions of small subgroups of $GL(n,\mathbb{C})$.

Theorem (I.1.4) ([**Pri67**]). *Let G_1 and G_2 be small subgroups of $GL(n,\mathbb{C})$. Then $\mathbb{C}^n/G_1$ is isomorphic to $\mathbb{C}^n/G_2$ if and only if G_1 and G_2 are conjugate subgroups.* □

I.1–1. The abelian case: normalized types

We are interested in V-manifolds where the quotient spaces B_i/G_i are given by (finite) abelian groups. In this case the following notation is used.

(I.1.5). Let $G := \mu_{d_1}\times\cdots\times\mu_{d_r}$ be an arbitrary finite abelian group written as a product of finite cyclic groups, that is, μ_{d_i} is the cyclic group of d_i-th roots of unity. Consider a matrix of weight vectors

$$A := (a_{ij})_{i,j} = [\mathbf{a}_1\,|\,\cdots\,|\,\mathbf{a}_n] \in Mat(r\times n,\mathbb{Z})$$

and the action

$$(\mu_{d_1}\times\cdots\times\mu_{d_r})\times\mathbb{C}^n \longrightarrow \mathbb{C}^n, \qquad (5)$$
$$(\boldsymbol{\xi}_{\mathbf{d}},\mathbf{x}) \mapsto (\xi_{d_1}^{a_{11}}\cdots\xi_{d_r}^{a_{r1}}\,x_1,\,\ldots\,,\xi_{d_1}^{a_{1n}}\cdots\xi_{d_r}^{a_{rn}}\,x_n).$$

Note that the i-th row of the matrix A can be considered modulo d_i. The set of all orbits $\mathbb{C}^n/G$ is called (*cyclic*) *quotient space of type* $(\mathbf{d};A)$ and it is denoted by

$$X(\mathbf{d};A) := X\left(\begin{array}{c|ccc} d_1 & a_{11} & \cdots & a_{1n}\\ \vdots & \vdots & \ddots & \vdots\\ d_r & a_{r1} & \cdots & a_{rn}\end{array}\right).$$

The orbit of an element $(x_1,\ldots,x_n)$ under this action is denoted by $[(x_1,\ldots,x_n)]_{(\mathbf{d};A)}$ and the subindex is omitted if no ambiguity seems likely to arise. Sometimes we use multi-index notation

$$\mathbf{d}=(d_1,\ldots,d_r), \qquad a_j=(a_{1j},\ldots,a_{rj}),$$
$$\boldsymbol{\xi}_{\mathbf{d}}=(\xi_{d_1},\ldots,\xi_{d_r}), \quad \mathbf{x}=(x_1,\ldots,x_n), \quad \mu_{\mathbf{d}}=\mu_{d_1}\times\cdots\times\mu_{d_r},$$

so that the action takes the simple form

$$\mu_{\mathbf{d}}\times\mathbb{C}^n \longrightarrow \mathbb{C}^n, \quad (\boldsymbol{\xi}_{\mathbf{d}},\mathbf{x})\mapsto(\boldsymbol{\xi}_{\mathbf{d}}^{\mathbf{a}_1}\,x_1,\ldots,\boldsymbol{\xi}_{\mathbf{d}}^{\mathbf{a}_n}\,x_n).$$

The following result shows that the family of varieties which can locally be written like $X(\mathbf{d};A)$ is exactly the same as the family of V-manifolds with abelian quotient singularities.

Lemma (I.1.6). *Let G be a finite abelian subgroup of $GL(n,\mathbb{C})$. Then, $\mathbb{C}^n/G$ is isomorphic to some quotient space of type $(\mathbf{d};A)$.*

Proof. Let us write $G=C_{d_1}\times\cdots\times C_{d_r}$ as a product of cyclic groups. Let $M_1,\ldots,M_r$ be generators of these cyclic groups so that

$$G=\{M_1^{i_1}\cdots M_r^{i_r}\mid i_k=0,\ldots,d_k-1\}.$$

Each of these matrices M_i, $i=1,\ldots,r$, is conjugated to a diagonal matrix of the form

$$M_i\sim\begin{pmatrix}\zeta_{d_i}^{a_{i1}} & & \\ & \ddots & \\ & & \zeta_{d_i}^{a_{in}}\end{pmatrix},$$

where ζ_{d_i} is a primitive d_i-th root of unity. Moreover, they are simultaneously diagonalizable because they commute.

This proves that $\mathbb{C}^n/G\simeq X((d_1,\ldots,d_r);(a_{ij})_{i,j})$. □

Different types $(\mathbf{d};A)$ can give rise to isomorphic quotient spaces, see Remark (I.1.7). We shall prove that they can always be represented by an upper triangular matrix of dimension $(n-1)\times n$, see Lemma (I.1.8). Finding a simpler type $(\mathbf{d};A)$ to represent a quotient space will lead us to the notion of normalized type, see Definition (I.1.10).

Remark **(I.1.7).** Assume just for a while that $n=3$. The simple group automorphism on $\mu_d\times\mu_d$ given by $(\xi,\eta)\mapsto(\xi\eta^{-1},\eta)$ shows that the following two spaces are isomorphic under the identity map.

$$X\left(\begin{array}{c|ccc} d & a_{11} & a_{12} & a_{13}\\ d & a_{21} & a_{22} & a_{23}\end{array}\right)=X\left(\begin{array}{c|ccc} d & a_{11} & a_{12} & a_{13}\\ d & a_{21}-a_{11} & a_{22}-a_{12} & a_{23}-a_{13}\end{array}\right)$$

Note that the determinants of the minors of order 2 are the same in both side of the previous equation. Analogous considerations hold for higher dimension.

Lemma (I.1.8). *The space* $X(\mathbf{d};A) = \mathbb{C}^n/\mu_{\mathbf{d}}$ *can always be represented by an upper triangular matrix of dimension* $(n-1)\times n$*. More precisely, there exist a vector* $\mathbf{e} = (e_1,\ldots,e_{n-1})$*, a matrix* $B = (b_{i,j})_{i,j}$*, and an isomorphism* $[(x_1,\ldots,x_n)] \mapsto [(x_1,\ldots,x_n^k)]$ *for some* $k \in \mathbb{N}$ *such that*

$$X(\mathbf{d};A) \cong \left(\begin{array}{c|cccc} e_1 & b_{1,1} & \cdots & b_{1,n-1} & b_{1,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ e_{n-1} & 0 & \cdots & b_{n-1,n-1} & b_{n-1,n} \end{array}\right) = X(\mathbf{e};B).$$

Proof. To keep the proof as simple as possible, consider only the case $n=3$. The general case is analogous. Let $(d_1;a_{11},a_{12},a_{13})$ and $(d_2;a_{21},a_{22},a_{23})$ be the first two rows of the matrix defining the quotient space. Multiplying conveniently, one can assume $d_1 = d_2$. Choose α,β satisfying Bézout's identity $\alpha a_{11} + \beta a_{21} = \gcd(a_{11},a_{21})$. Using repeatedly Remark (I.1.7), one finds an isomorphism induced by the identity map between our space $X\left(\begin{smallmatrix} d; & a_{11} & a_{12} & a_{13} \\ d; & a_{21} & a_{22} & a_{23}\end{smallmatrix}\right)$ and

$$X\left(\begin{array}{c|ccc} d & \gcd(a_{11},a_{21}) & \alpha a_{12}+\beta a_{22} & \alpha a_{13}+\beta a_{23} \\ d & 0 & \frac{a_{11}a_{22}-a_{21}a_{12}}{\gcd(a_{11},a_{21})} & \frac{a_{11}a_{23}-a_{21}a_{13}}{\gcd(a_{11},a_{21})} \end{array}\right).$$

This process allows one to reduce the claim to the case $n=1$. The proof is complete after Example (I.1.12). □

(I.1.9). The action shown in (5) is free on $(\mathbb{C}^*)^n$, that is,

$$\big[\mathbf{x}\in(\mathbb{C}^*)^n,\ \boldsymbol{\xi}_{\mathbf{d}}\cdot\mathbf{x} = \mathbf{x}\big] \Longrightarrow \boldsymbol{\xi}_{\mathbf{d}} = \mathbf{1},$$

if and only if the group homomorphism $\mu_{\mathbf{d}} \to GL(n,\mathbb{C})$ given by

$$\boldsymbol{\xi}_{\mathbf{d}} = (\xi_{d_1},\ldots,\xi_{d_r}) \longmapsto \begin{pmatrix} \boldsymbol{\xi}_{\mathbf{d}}^{\mathbf{a}_1} & & \\ & \ddots & \\ & & \boldsymbol{\xi}_{\mathbf{d}}^{\mathbf{a}_r}\end{pmatrix} \tag{6}$$

is injective. If this is not the case, let H be the kernel of this group homomorphism. Then $\mathbb{C}^n/H \equiv \mathbb{C}^n$ and the group $\mu_{\mathbf{d}}/H$ acts freely on $(\mathbb{C}^*)^n$ under the previous identification.

Thus one can always assume that the free (as well as the small) condition is satisfied. This motivates the following definition.

Definition (I.1.10). The type $(\mathbf{d};A)$ is said to be *normalized* if the following two conditions hold.

(1) The action is free on $(\mathbb{C}^*)^n$.
(2) The group $\mu_{\mathbf{d}}$ is identified with a small subgroup of $GL(n,\mathbb{C})$ under the group homomorphism given in (6).

By abuse of language we often say the space $X(\mathbf{d};A)$ is written in a normalized form when we actually mean the type $(\mathbf{d};A)$ is normalized.

Proposition (I.1.11). *The space $X(\mathbf{d};A)$ is written in a normalized form if and only if the stabilizer subgroup of P is trivial for all $P \in \mathbb{C}^n$ with exactly $n-1$ coordinates different from zero.*

In the cyclic case the stabilizer of a point as above (with exactly $n-1$ coordinates different from zero) has order $\gcd(d, a_1, \ldots, \widehat{a}_i, \ldots, a_n)$. □

The procedures described in (I.1.9) and (I.1.3) can be used to convert general types $(\mathbf{d};A)$ into their normalized form. Theorem (I.1.4) allows one to decide whether two quotient spaces are isomorphic. In particular, one can use this result to compute the singular points of the space $X(\mathbf{d};A)$. This method is specially simple in the cyclic case, see (I.1.15) below.

I.1–2. Dimension 1, 2, 3 and the cyclic case

Now, in the following examples, we discuss the previous normalization process in dimension one, two, and three separately. Also a paragraph is devoted to the cyclic case.

Example (I.1.12). (Dimension 1). When $n=1$ all spaces $X(\mathbf{d};A)$ are isomorphic to $\mathbb{C}$. Note that $X((d_1,\ldots,d_r);(a_{11},\ldots,a_{r1})^t)$ is the same space as $X((d'_1,\ldots,d'_r);(a'_{11},\ldots,a'_{r1})^t)$ where $d'_i = \frac{d_i}{\gcd(d_i,a_{i1})}$ and $a'_{i1} = \frac{a_{i1}}{\gcd(d_i,a_{i1})}$. Therefore we can assume that $\gcd(d_i,a_{i1})=1$.

The map $[x] \mapsto x^{d_1}$ gives an isomorphism between $X(d_1;a_{11})$ and $\mathbb{C}$. For $r=2$ one has that (we write the symbol "$=$" when the isomorphism is induced by the identity map)

$$\begin{array}{ccccccc} \dfrac{\mathbb{C}}{\mu_{d_1}\times\mu_{d_2}} = \dfrac{\mathbb{C}/\mu_{d_1}}{\mu_{d_2}} & \stackrel{\cong}{\longrightarrow} & \mathbb{C}/\mu_{d_2} \stackrel{(*)}{=} X(d_2;a_{21}d_1) & \stackrel{\cong}{\longrightarrow} & \mathbb{C}, \\ [x] & \mapsto & x^{d_1}, \qquad\qquad [x] & \mapsto & x^{\frac{d_2}{\gcd(d_1,d_2)}}. \end{array}$$

To see the equality $(*)$ observe that

$$\xi_{d_2}\cdot x^{d_1} \equiv \xi_{d_2}\cdot[x] = [\xi_{d_2}^{a_{21}}x] \equiv \xi_{d_2}^{a_{21}d_1}x^{d_1}.$$

It follows that the corresponding quotient space is isomorphic to $\mathbb{C}$ under the map $[x]\mapsto x^{\operatorname{lcm}(d_1,d_2)}$.

In higher dimension (without assuming $\gcd(d_i,a_{i1})=1$) the isomorphism takes the form

$$X((d_1,\ldots,d_r);(a_{11},\ldots,a_{r1})^t) \longrightarrow \mathbb{C}: \quad [x]\mapsto x^{\ell},$$
$$\ell = \operatorname{lcm}\left(\frac{d_1}{\gcd(d_1,a_{11})},\ldots,\frac{d_r}{\gcd(d_r,a_{r1})}\right).$$

This integer ℓ is closely related to our notion of multiplicity (at a point) of a normal crossing divisor, see (IV.3.12) and (V.1.4).

Example (I.1.13). (Dimension 2). Following Lemma (I.1.8), all quotient spaces for $n = 2$ are cyclic. The space $X(d; a, b)$ is written in a normalized form if and only if $\gcd(d, a) = \gcd(d, b) = 1$. If this is not the case, one uses the isomorphism[1] (assuming $\gcd(d, a, b) = 1$)

$$\begin{array}{rcl} X(d; a, b) & \longrightarrow & X\left(\frac{d}{(d,a)(d,b)}; \frac{a}{(d,a)}, \frac{b}{(d,b)}\right), \\ \big[(x, y)\big] & \mapsto & \big[(x^{(d,b)}, y^{(d,a)})\big] \end{array}$$

to convert it into a normalized one.

On the other hand, one can have spaces like $X\left(\begin{smallmatrix} d; & a & b \\ e; & r & s \end{smallmatrix}\right)$ also written in a normalized form. In fact, the previous quotient space is written in a normalized form if and only if so are both rows and $\gcd(d, e) = 1$.

Example (I.1.14). (Dimension 3). The space $X(d; a, b, c)$ is written in a normalized form if and only if $\gcd(d, a, b) = \gcd(d, a, c) = \gcd(d, b, c) = 1$. As above, isomorphisms of the form $[(x, y, z)] \mapsto [(x, y, z^k)]$ can be used to convert types $(d; a, b, c)$ into their normalized form.

For $n = 3$ there exists non-cyclic quotient spaces written in a normalized form. As an example we give $X\left(\begin{smallmatrix} 2; & 1 & 1 & 0 \\ 2; & 1 & 0 & 1 \end{smallmatrix}\right)$. In fact, the general space $X\left(\begin{smallmatrix} d; & a & b & c \\ e; & r & s & t \end{smallmatrix}\right)$ is written a in normalized form if and only if so are both rows and $(d, e, m_1) = (d, e, m_2) = (d, e, m_3) = 1$, where m_1, m_2, m_3 are the determinants of the three minors of order 2.

(I.1.15). (Cyclic case). In the cyclic case the order of the stabilizer subgroup is specially easy to compute and hence the normalized form can be described explicitly. In fact, $X(d; a_1, \ldots, a_n)$ is written in a normalized form if and only if $\gcd(d, a_1, \ldots, \widehat{a}_i, \ldots, a_n) = 1$, $\forall i = 1, \ldots, n$. Here we summarize how to convert types $(d; a_1, \ldots, a_n)$ into their normalized form.

(1) $X(d; a_1, \ldots, a_n) \simeq X(d;\, a_{\sigma(1)}, \ldots, a_{\sigma(n)})$, $\forall \sigma \in \Sigma_n$.
(2) $X(d; 0, a_2, \ldots, a_n) = \mathbb{C} \times X(d;\, a_2, \ldots, a_n)$.
(3) $X(d; a_1, \ldots, a_n) = X(\frac{d}{k};\, \frac{a_1}{k}, \ldots, \frac{a_n}{k})$ if k divides d and all a_i's.
(4) $X(d; a_1, \ldots, a_n) = X(d; ka_1, \ldots, ka_n)$ if $\gcd(d, k) = 1$.
(5) $X(d; a_1, \ldots, a_n) \simeq X(\frac{d}{k}; a_1, \frac{a_2}{k}, \ldots, \frac{a_n}{k})$, the isomorphism is given by $[(x_1, x_2, \ldots, x_n)] \mapsto [(x_1^k, x_2, \ldots, x_n)]$.

In [**Fuj75**], the author computes resolutions of these cyclic quotient singularities and also studies, among others, the properties shown above.

[1] Recall the notation $(i_1, \ldots, i_k) = \gcd(i_1, \ldots, i_k)$ in case of complicated or long formulas.

I.1–3. Working with local equations

Let $X(\mathbf{d}; A) = \mathbb{C}^n/\mu_{\mathbf{d}}$ be a quotient singularity not necessarily cyclic or written in a normalized form. Let $f : X(\mathbf{d}; A) \to \mathbb{C}$ be a global function, that is, f is a holomorphic function preserving the action. One is interested in finding a local equation for the divisor defined by $f : (X(\mathbf{d}; A), [P]) \to (\mathbb{C}, 0)$ as a germ of functions at $P = (p_1, \ldots, p_n) \in \mathbb{C}^n \setminus \{0\}$.

Note that the usual change of coordinates $\varphi : \mathbb{C}^n \to \mathbb{C}^n$ given by $x_i \mapsto x_i + p_i$ induces an isomorphism on $X(\mathbf{d}; A)$ if and only if the condition $\big[p_j \neq 0 \Longrightarrow d_i | a_{ij},\ \forall i\big]$ is satisfied. Equivalently, the j-th column of A is zero (modulo $\mathbf{d}$) whenever $p_j \neq 0$.

Now the trick is to find an isomorphism induced by the identity map $(X(\mathbf{d}; A), [P]) \simeq (X(\mathbf{d}'; A'), [P])$ such that $(\mathbf{d}'; A')$ verifies the condition above. The following result illustrates this idea.

Lemma (I.1.16). *Let $P = (p_1, \ldots, p_k, 0, \ldots, 0) \in \mathbb{C}^n$, $1 \leq k \leq n$, $p_i \neq 0$. Let $(\mu_{\mathbf{d}})_P$ be the stabilizer subgroup of P. There exist $\mathbf{d}'$ and A' such that $X(\mathbf{d}'; A') = \mathbb{C}^n/(\mu_{\mathbf{d}})_P$.*

The natural projection $X(\mathbf{d}'; A') \to X(\mathbf{d}; A)$ defines a branched covering unramified over a small neighborhood of $[P]$. In particular, as germs, one has $\big(X(\mathbf{d}; A),\, [P]\big) = \big(X(\mathbf{d}'; A'),\, [P]\big)$.

In the cyclic case the order of $(\mu_d)_P$ is $\gcd(d, a_1, \ldots, a_k)$.

Proof. Note that $\mu_{\mathbf{d}}/(\mu_{\mathbf{d}})_P$ acts freely on $\big((\mathbb{C}^*)^k \times \mathbb{C}^{n-k}\big)/(\mu_{\mathbf{d}})_P$. Thus the natural projection

$$\frac{(\mathbb{C}^*)^k \times \mathbb{C}^{n-k}}{(\mu_{\mathbf{d}})_P} \longrightarrow \frac{\Big((\mathbb{C}^*)^k \times \mathbb{C}^{n-k}\Big)\Big/(\mu_{\mathbf{d}})_P}{\mu_{\mathbf{d}}\,/(\mu_{\mathbf{d}})_P} = \frac{(\mathbb{C}^*)^k \times \mathbb{C}^{n-k}}{\mu_{\mathbf{d}}} \ni [P]$$

is an unramified covering and the claim follows. The order of the stabilizer subgroup in the cyclic case can be computed directly. □

Observe that the new data $(\mathbf{d}'; A')$ obtained in the previous lemma satisfies the required condition. In fact,

$$X(\mathbf{d}'; A') = X(\mathbf{d}'; 0, \ldots, 0, \mathbf{a}'_{k+1}, \ldots, \mathbf{a}'_n) = \mathbb{C}^k \times (\mathbb{C}^{n-k}/\mu_{\mathbf{d}'}).$$

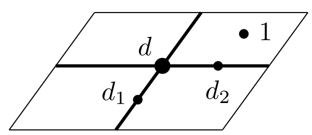
Now the usual change of coordinates can be used to compute the local equation of f at $[P]$.

(I.1.17). Let $\mathcal{O}_{X(\mathbf{d};A)}$ be the sheaf of analytic functions on $X(\mathbf{d}; A)$ and let $\mathcal{O}_{X(\mathbf{d};A),[P]}$ be the corresponding local ring at $[P] \in X(\mathbf{d}; A)$. Then one has

$$\begin{array}{rcl} \mathcal{O}_{X(\mathbf{d};A),[P]} = (\mathcal{O}_{\mathbb{C}^n,P})^{(\mu_{\mathbf{d}})_P} & \stackrel{\cong}{\longrightarrow} & (\mathcal{O}_{\mathbb{C}^n,0})^{(\mu_{\mathbf{d}})_P}, \\ x_i & \mapsto & x_i + p_i. \end{array}$$

Define the equivalence relation on $X(\mathbf{d};A)$ given by $[(p_1,\ldots,p_n)] \sim [(q_1,\ldots,q_n)]$ if and only if $\forall i = 1,\ldots,n, [p_i \neq 0 \Leftrightarrow q_i \neq 0]$. Then the local ring $\mathcal{O}_{X(\mathbf{d};A),[P]}$ only depends on the equivalence class of $[P]$. In particular, for a given $\lambda \in \mathbb{C}^*$, $\mathcal{O}_{X(\mathbf{d};A),[P]} \cong \mathcal{O}_{X(\mathbf{d};A),[\lambda P]}$ holds. This has to do with the notion of cone in topology.

Example (I.1.18). Let us examine a special case that will be used later. Assume that $X = X(d;a,b,c)$ is written in a normalized form, see Example (I.1.14). Let $f : X \to \mathbb{C}$ be the polynomial map given by $f = x_3^d$. The support of the divisor defined by f can be decomposed into several strata depending of their quotient singularities, or equivalently, depending on the order of the stabilizer subgroup, as the following picture shows. However, the local equation of the divisor is always the same, since $(d;a,b,c)$ is normalized.



$d_1 = \gcd(d,a)$
$d_2 = \gcd(d,b)$

FIGURE I.1. Stratification of $\{x_3^d = 0\} \subset X(d;a,b,c)$.

We finish this section with a general result about quotient spaces and V-manifolds. The proof is a consequence of all the properties that have been studied.

Proposition (I.1.19). *The spaces $X(\mathbf{d};A) = \mathbb{C}^n/\mu_{\mathbf{d}}$ are normal irreducible algebraic varieties of dimension n. Their singular locus has codimension greater than or equal to 2 and it is located on the coordinate axes.*

The Euler characteristic is $\chi(X(\mathbf{d};A)) = 1$ because in fact they are contractible. Therefore V-manifolds are normal varieties and their singular locus forms a subvariety of codimension at least 2. □

SECTION § **I.2**

Weighted Projective Spaces

The main reference that has been used in this section is [**Dol82**]. Here we concentrate our attention on the analytic structure.

Let $\omega = (q_0,\ldots,q_n)$ be a weight vector, i.e. a finite set of positive integers. There is a natural action of the multiplicative group $\mathbb{C}^*$ on $\mathbb{C}^{n+1} \setminus \{0\}$ given by

$$(x_0,\ldots,x_n) \longmapsto (t^{q_0}x_0,\ldots,t^{q_n}x_n).$$

The set of orbits $\frac{\mathbb{C}^{n+1}\setminus\{0\}}{\mathbb{C}^*}$ under this action is denoted by $\mathbb{P}^n_\omega$ (or $\mathbb{P}^n(\omega)$ in case of complicated weight vectors) and it is called the *weighted projective space* of type ω. The class of a nonzero element $(x_0,\ldots,x_n) \in \mathbb{C}^{n+1}$ is denoted by $[x_0 : \ldots : x_n]_\omega$ and the weight vector is omitted depending on the context. When $(q_0,\ldots,q_n) = (1,\ldots,1)$ one obtains the usual projective space and the weight vector is always omitted. For $\mathbf{x} \in \mathbb{C}^{n+1} \setminus \{\mathbf{0}\}$, the closure of $[\mathbf{x}]_\omega$ in $\mathbb{C}^{n+1}$ is obtained by adding the origin; it is an algebraic curve.

(I.2.1). (Another way to present $\mathbb{P}^n_\omega$). Let $\mathbb{P}^n$ be the classical projective space and $\mu_\omega = \mu_{q_0} \times \cdots \times \mu_{q_n}$ the product of cyclic groups. Consider the group action

$$\begin{array}{ccc} \mu_\omega \times \mathbb{P}^n & \longrightarrow & \mathbb{P}^n, \\ \big((\xi_{q_0},\ldots,\xi_{q_n}),[x_0 : \ldots : x_n]\big) & \mapsto & [\xi_{q_0}x_0 : \ldots : \xi_{q_n}x_n]. \end{array}$$

Then the set of all orbits $\mathbb{P}^n/\mu_\omega$ is isomorphic to the weighted projective space of type ω and the isomorphism is induced by the branched covering

$$\mathbb{P}^n \ni [x_0 : \ldots : x_n] \longmapsto [x_0^{q_0} : \ldots : x_n^{q_n}]_\omega \in \mathbb{P}^n_\omega.$$

Note that this branched covering is unramified over

$$\mathbb{P}^n_\omega \setminus \{[x_0,\ldots,x_n]_\omega \mid x_0 \cdots x_n = 0\}$$

and has $\frac{q_0\cdots q_n}{\gcd(q_0,\ldots,q_n)}$ sheets. Moreover, the covering respects the coordinate axes.

Example (I.2.2). Let $\mathbb{P}^2 \to \mathbb{P}^2_\omega$ be the branched covering defined above with weights $\omega = (1,2,3)$. For instance, the preimage of $[1:1:1]_\omega$ consists of 6 points, namely the set $\{[1:\xi_2:\xi_3] \in \mathbb{P}^2 \mid \xi_2 \in \mu_2,\ \xi_3 \in \mu_3\}$.

More generally, the degree (the number of sheets) of a covering of the form $\mathbb{P}^2 \to \mathbb{P}^2_\omega/\mu_d$, where μ_d defines an action of type $(d;a,b,c)$, is calculated in Lemma (III.4.2) and Proposition (III.4.3). This degree will be essential to state Bézout's Theorem on $\mathbb{P}^2_\omega/\mu_d$.

(I.2.3). (Analytic structure). As in the classical case, the weighted projective spaces can be endowed with an analytic structure. However, in general they contain cyclic quotient singularities.

Consider the decomposition $\mathbb{P}^n_\omega = U_0 \cup \cdots \cup U_n$, where U_i is the open set consisting of all elements $[x_0 : \ldots : x_n]_\omega$ with $x_i \neq 0$. The map

$$\widetilde{\psi}_0 : \mathbb{C}^n \longrightarrow U_0, \quad \widetilde{\psi}_0(x_1,\cdots,x_n) := [1 : x_1 : \ldots : x_n]_\omega$$

is clearly a surjective analytic map but it is not a chart since injectivity fails.

In fact, $[1 : x_1 : \ldots : x_n]_\omega = [1 : x'_1 : \ldots, x'_n]_\omega$ if and only if there exists $\xi \in \mu_{q_0}$ such that $x'_i = \xi^{q_i} x_i$ for all $i = 1, \ldots, n$. Hence the map above induces the isomorphism

$$\begin{array}{rcl} \psi_0 \ : \ X(q_0; q_1, \ldots, q_n) & \longrightarrow & U_0, \\ {[(x_1, \ldots, x_n)]} & \mapsto & [1 : x_1 : \ldots : x_n]_\omega. \end{array}$$

Analogously, $X(q_i; q_0, \ldots, \widehat{q_i}, \ldots, q_n) \cong U_i$ under the obvious analytic map. Therefore $\mathbb{P}^n_\omega$ is an analytic space with cyclic quotient singularities as claimed.

(I.2.4). (Simplifying the weights). For different weight vectors ω and ω' the corresponding spaces $\mathbb{P}^n_\omega$ and $\mathbb{P}^n_{\omega'}$ can be isomorphic. Consider

$$\begin{aligned} d &= \gcd(q_0, \ldots, q_n), \\ d_i &= \gcd(q_0, \ldots, \widehat{q_i}, \ldots, q_n), \\ e_i &= \operatorname{lcm}(d_0, \ldots, \widehat{d_i}, \ldots, d_n). \end{aligned}$$

Note that $e_i | q_i$, $\gcd(d_i, d_j) = d$ for $i \neq j$, and $\gcd(e_i, d_i) = d$.

Proposition (I.2.5). *Using the notation above, the following map is an isomorphism:*

$$\begin{array}{rcl} \mathbb{P}^n(q_0, \ldots, q_n) & \longrightarrow & \mathbb{P}^n\big(\frac{q_0}{e_0}, \ldots, \frac{q_n}{e_n}\big), \\ {[x_0 : \ldots : x_n]} & \mapsto & \Big[x_0^{\frac{d_0}{d}} : \ldots : x_n^{\frac{d_n}{d}} \Big]. \end{array}$$

Proof. Assume first that $d = 1$. Then $\gcd(q_i, d_i) = 1$ and $e_i = d_0 \cdots \widehat{d_i} \cdots d_n$. Now from (I.1.15), one has the following sequence of isomorphisms of analytic spaces:

$$\begin{aligned} & X(q_0; q_1, \ldots, q_n) \overset{\text{id}}{=} X(q_0; \tfrac{q_1}{d_0}, \tfrac{q_2}{d_0}, \ldots, \tfrac{q_n}{d_0}) \overset{\text{1st}}{\cong} X(\tfrac{q_0}{d_1}; \tfrac{q_1}{d_0}, \tfrac{q_2}{d_0 d_1}, \ldots, \tfrac{q_n}{d_0 d_1}) \\ & \overset{\text{2nd}}{\cong} X(\tfrac{q_0}{d_1 d_2}; \tfrac{q_1}{d_0 d_2}, \tfrac{q_2}{d_0 d_1}, \tfrac{q_3}{d_0 d_1 d_2}, \ldots, \tfrac{q_n}{d_0 d_1 d_2}) \overset{\text{3rd}}{\cong} \cdots \overset{n\text{th}}{\cong} X(\tfrac{q_0}{e_0}; \tfrac{q_1}{e_1}, \ldots, \tfrac{q_n}{e_n}). \end{aligned}$$

Observe that in the i-th step, we divide the corresponding weight vector by d_i except the i-th coordinate and hence the associated map in each step is $[(x_1, \ldots, x_i, \ldots, x_n)] \mapsto [(x_1, \ldots, x_i^{d_i}, \ldots, x_n)]$. Therefore

$$[1 : x_1 : \ldots : x_n]_\omega \longmapsto [1 : x_1^{d_1} : \ldots : x_n^{d_n}]_{\omega'}$$

is an isomorphism by composition. Analogously one proceeds with the other charts.

The general case $\gcd(q_0, \ldots, q_n) = d$ can easily be deduced from the previous one. $\square$

Remark **(I.2.6).** Note that, due to the preceding proposition, one can always assume the weight vector satisfies $\gcd(q_0,\ldots,\widehat{q}_i,\ldots,q_n)=1$, for $i=0,\ldots,n$. In particular, $\mathbb{P}^1(q_0,q_1)\cong\mathbb{P}^1$ and for $n=2$ we can take (q_0,q_1,q_2) pairwise relatively prime numbers. In higher dimension the situation is a bit more complicated.

We conclude with a general result.

Theorem (I.2.7). *The space $\mathbb{P}^n_\omega$ is a normal irreducible projective algebraic variety of dimension n. All singularities are cyclic quotient and form a subanalytic space of codimension greater than or equal to 2. The Euler characteristic is $\chi(\mathbb{P}^n_\omega)=\chi(\mathbb{P}^n)=n+1$.* □

Remark **(I.2.8).** In what follows we need to work over $(\mathbb{P}^k_\omega\times\mathbb{C}^{n-k})/\mu_{\mathbf{d}}$ where the action is as in (5). These spaces are also normal irreducible algebraic varieties of dimension n with singular locus of codimension at least 2. The Euler characteristic is $k+1$.

Section § I.3
Weighted Blow-ups and Embedded Q-Resolutions

Classically an embedded resolution of $\{f=0\}\subset\mathbb{C}^n$ is a proper map $\pi: X\to(\mathbb{C}^n,0)$ from a smooth variety X satisfying, among other conditions, that $\pi^{-1}(\{f=0\})$ is a normal crossing divisor. To weaken the condition on the preimage of the singularity we allow the new ambient space X to contain abelian quotient singularities and the divisor $\pi^{-1}(\{f=0\})$ to have "normal crossings" over this kind of varieties. This notion of normal crossing divisor on V-manifolds was first introduced by Steenbrink in [**Ste77**].

Definition (I.3.1). Let X be a V-manifold with abelian quotient singularities. A hypersurface D on X is said to be with $\mathbb{Q}$-*normal crossings* if it is locally isomorphic to the quotient of a normal crossing divisor under a group action of type $(\mathbf{d};A)$.

That is, given $x\in X$, there is an isomorphism of germs $(X,x)\simeq(X(\mathbf{d};A),[0])$ such that $(D,x)\subset(X,x)$ is identified under this morphism with a germ of the form

$$\big(\big\{[\mathbf{x}]\in X(\mathbf{d};A)\mid x_1^{m_1}\cdots x_k^{m_k}=0\big\},[(0,\ldots,0)]\big).$$

Let $M=\mathbb{C}^{n+1}/\mu_{\mathbf{d}}$ be an abelian quotient space not necessarily cyclic or written in normalized form. Consider $H\subset M$ an analytic subvariety of codimension one.

Definition (I.3.2). An *embedded* **Q**-*resolution* of $(H,0) \subset (M,0)$ is a proper analytic map $\pi : X \to (M,0)$ such that:

(1) X is a V-manifold with abelian quotient singularities.
(2) π is an isomorphism over $X \setminus \pi^{-1}(\mathrm{Sing}(H))$.
(3) $\pi^{-1}(H)$ is a hypersurface with $\mathbb{Q}$-normal crossings on X.

Remark **(I.3.3).** Let $f : (M,0) \to (\mathbb{C},0)$ be a non-constant analytic function germ. Consider $(H,0)$ the hypersurface defined by f. Let $\pi : X \to (M,0)$ be an embedded **Q**-resolution of $(H,0) \subset (M,0)$. Then $\pi^{-1}(H) = (f \circ \pi)^{-1}(0)$ is locally given by a function of the form $x_1^{m_1} \cdots x_k^{m_k} : X(\mathbf{d}; A) \to \mathbb{C}$.

In what follows we will use weighted blow-ups with smooth center as a tool for finding embedded **Q**-resolutions.

(I.3.4). (Classical blow-up of $\mathbb{C}^{n+1}$ with smooth center). Assume the center is $L : \{x_0 = \cdots = x_k = 0\}$. Let us use multi-index notation

$$\mathbf{x} = (x_0, \ldots, x_n) \in \mathbb{C}^{n+1}, \quad [\mathbf{u}] = [u_0 : \ldots : u_k] \in \mathbb{P}^k,$$

and consider

$$\widehat{\mathbb{C}}^{n+1}_L := \big\{ (\mathbf{x}, [\mathbf{u}]) \in \mathbb{C}^{n+1} \times \mathbb{P}^k \mid (x_0, \ldots, x_k) \in \overline{[u_0 : \ldots : u_k]} \big\}.$$

Then the natural projection $\pi : \widehat{\mathbb{C}}^{n+1}_L \to \mathbb{C}^{n+1}$ is an isomorphism over the complement $\widehat{\mathbb{C}}^{n+1}_L \setminus \pi^{-1}(L)$. The *exceptional divisor* $E := \pi^{-1}(L)$ is identified with $\mathbb{P}^k \times \mathbb{C}^{n-k}$. The space

$$\widehat{\mathbb{C}}^{n+1}_L = U_0 \cup \cdots \cup U_k$$

can be covered by $k+1$ charts each of them isomorphic to $\mathbb{C}^{n+1}$. For instance, the following map defines an isomorphism:

$$\begin{array}{ccl} \mathbb{C}^{n+1} & \longrightarrow & U_0 = \{u_0 \neq 0\} \ \subset \ \widehat{\mathbb{C}}^{n+1}_L, \\ \mathbf{x} & \mapsto & \big((x_0, x_0 x_1, \ldots, x_0 x_k, \, x_{k+1}, \ldots, x_n), [1 : x_1 : \ldots : x_k] \big). \end{array}$$

(I.3.5). (Weighted $(p_0, \ldots, p_k)$-blow-up of $\mathbb{C}^{n+1}$ with smooth center). Assume the center is $L : \{x_0 = \cdots = x_k = 0\}$. Let $\omega = (p_0, \ldots, p_k)$ be a weight vector. As above, consider the space

$$\widehat{\mathbb{C}}^{n+1}_L(\omega) := \big\{ (\mathbf{x}, [\mathbf{u}]_\omega) \in \mathbb{C}^{n+1} \times \mathbb{P}^k_\omega \mid (x_0, \ldots, x_k) \in \overline{[u_0 : \ldots : u_k]_\omega} \big\}.$$

Here the condition about the closure means that

$$\exists t \in \mathbb{C}, \quad x_i = t^{p_i} u_i, \quad i = 0, \ldots, k.$$

Then the natural projection $\pi : \widehat{\mathbb{C}}^{n+1}_L(\omega) \to \mathbb{C}^{n+1}$ is an isomorphism over $\widehat{\mathbb{C}}^{n+1}_L(\omega) \setminus \pi^{-1}(L)$ and the exceptional divisor $E := \pi^{-1}(L)$ is identified with the V-manifold $\mathbb{P}^k_\omega \times \mathbb{C}^{n-k}$.

Again $\widehat{\mathbb{C}}^{n+1}_L(\omega) = U_0 \cup \cdots \cup U_k$ can be covered by $k+1$ charts. However, the map $\varphi_0 : \mathbb{C}^{n+1} \to U_0$ given by

$$\begin{array}{ccl} \mathbb{C}^{n+1} & \stackrel{\varphi_0}{\longrightarrow} & U_0 = \{u_0 \neq 0\} \ \subset \ \widehat{\mathbb{C}}^{n+1}_L(\omega), \\ \mathbf{x} & \mapsto & \big((x_0^{p_0}, x_0^{p_1} x_1, \ldots, x_0^{p_k} x_k,\, x_{k+1}, \ldots, x_n), [1 : x_1 : \ldots : x_k]_\omega \big), \end{array}$$

is surjective but not injective. In fact, $\varphi_0(\mathbf{x}) = \varphi_0(\mathbf{y})$ if and only if

$$\exists \xi \in \mu_{p_0} : \begin{cases} y_0 = \xi^{-1} x_0, & \\ y_i = \xi^{p_i} x_i, & i = 1, \ldots, k, \\ y_i = x_i, & i = k+1, \ldots, n. \end{cases}$$

Hence the previous map φ_0 induces the isomorphism

$$X(p_0; -1, p_1, \ldots, p_k) \times \mathbb{C}^{n-k} \longrightarrow U_0.$$

Note that these charts are compatible with the ones described in (I.2.3) for the weighted projective space. In U_0 the exceptional divisor is $\{x_0 = 0\}$ and the first chart of $\mathbb{P}^k_\omega$ is the quotient space $X(p_0;\, p_1, \ldots, p_k)$.

(I.3.6). ($(p_0, \ldots, p_k)$-blow-up of $X(\mathbf{d}; A)$ with smooth center). Assume the center is $L : \{x_0 = \cdots = x_k = 0\}$. Let $\omega = (p_0, \ldots, p_k)$ be a weight vector. The action $\mu_{\mathbf{d}}$ on $\mathbb{C}^{n+1}$ extends naturally to an action on $\widehat{\mathbb{C}}^{n+1}_L(\omega)$ as follows,

$$\boldsymbol{\xi}_{\mathbf{d}} \cdot \big(\mathbf{x}, [\mathbf{u}]_\omega \big) \stackrel{\mu_{\mathbf{d}}}{\longmapsto} \big((\boldsymbol{\xi}_{\mathbf{d}}^{\mathbf{a}_0} x_0, \ldots, \boldsymbol{\xi}_{\mathbf{d}}^{\mathbf{a}_n} x_n), [\boldsymbol{\xi}_{\mathbf{d}}^{\mathbf{a}_0} u_0 : \ldots : \boldsymbol{\xi}_{\mathbf{d}}^{\mathbf{a}_k} u_k]_\omega \big).$$

Let $\widehat{X(\mathbf{d}; A)}_L(\omega) := \widehat{\mathbb{C}}^{n+1}_L(\omega)/\mu_{\mathbf{d}}$ denote the quotient space under this action. Then the induced projection

$$\pi : \widehat{X(\mathbf{d}; A)}_L(\omega) \longrightarrow X(\mathbf{d}; A), \quad \big[(\mathbf{x}, [\mathbf{u}]_\omega) \big]_{(\mathbf{d};A)} \mapsto [\mathbf{x}]_{(\mathbf{d};A)}$$

is an isomorphism over $\widehat{X(\mathbf{d}; A)}_L(\omega) \setminus \pi^{-1}(L)$ and the exceptional divisor $E := \pi^{-1}(L)$ is identified with the variety $(\mathbb{P}^k_\omega \times \mathbb{C}^{n-k})/\mu_{\mathbf{d}}$.

The action $\mu_{\mathbf{d}}$ above respects the charts of $\widehat{\mathbb{C}}^{n+1}_L(\omega)$ so that the new ambient space can be covered as

$$\widehat{X(\mathbf{d}; A)}_L(\omega) = \widehat{U}_0 \cup \cdots \cup \widehat{U}_k,$$

where $\widehat{U}_i := U_i/\mu_{\mathbf{d}} = \{u_i \neq 0\}$.

Let us study, for instance, the first chart. By using φ_0 one identifies U_0 with

$$X(p_0; -1, p_1, \ldots, p_k) \times \mathbb{C}^{n-k}$$

and $\mu_{\mathbf{d}} = \mu_{d_1} \times \cdots \times \mu_{d_r}$ with the group

$$\frac{\mu_{p_0 \mathbf{d}}}{\mu_{p_0} \times \overset{(r)}{\cdots} \times \mu_{p_0}}.$$

Finally, one has the following action

$$\left(\mu_{p_0\mathbf{d}}/(\mu_{p_0}\times \overset{(k)}{\cdots} \times\mu_{p_0})\right)\times\left(X(p_0;-1,p_1,\ldots,p_k)\times\mathbb{C}^{n-k}\right)$$

defined by

$$\left([(\boldsymbol{\xi}^{\mathbf{a}_0}x_0,\boldsymbol{\xi}^{p_0\mathbf{a}_1-p_1\mathbf{a}_0}x_1,\ldots,\boldsymbol{\xi}^{p_0\mathbf{a}_k-p_k\mathbf{a}_0}x_k)],\ (\boldsymbol{\xi}^{p_0\mathbf{a}_{k+1}}x_{k+1},\ldots,\boldsymbol{\xi}^{p_0\mathbf{a}_n}x_n)\right).$$

This shows that

$$X\left(\begin{array}{c|ccccccc} p_0 & -1 & p_1 & \cdots & p_k & 0 & \cdots & 0\\ p_0\mathbf{d} & \mathbf{a}_0 & p_0\mathbf{a}_1-p_1\mathbf{a}_0 & \cdots & p_0\mathbf{a}_k-p_k\mathbf{a}_0 & p_0\mathbf{a}_{k+1} & \cdots & p_0\mathbf{a}_n\end{array}\right)$$

is isomorphic to $\widehat{U}_0$ and the isomorphism is defined by

$$[\mathbf{x}] \overset{\widehat{\varphi}_0}{\longmapsto} \left((x_0^{p_0},x_0^{p_1}x_1,\ldots,x_0^{p_k}x_k,\,x_{k+1},\ldots,x_n),[1:x_1:\ldots:x_k]_\omega\right).$$

For $i=1,\ldots,k$, one proceeds analogously.

As for the the exceptional divisor $E=\pi^{-1}(L)=(\mathbb{P}^k_\omega\times\mathbb{C}^{n-k})/\mu_{\mathbf{d}}$, it is usually written as

$$E=\widehat{V}_0\cup\cdots\cup\widehat{V}_k$$

so that these charts are compatible with the ones of $\widehat{X(\mathbf{d};A)}_L(\omega)$ in the sense that $\widehat{V}_i=\widehat{U}_i|_{\{x_i=0\}}$, $i=0,\ldots,k$. Hence, for example,

$$\widehat{V}_0\cong X\left(\begin{array}{c|cccccc} p_0 & p_1 & \cdots & p_k & 0 & \cdots & 0\\ p_0\mathbf{d} & p_0\mathbf{a}_1-p_1\mathbf{a}_0 & \cdots & p_0\mathbf{a}_k-p_k\mathbf{a}_0 & p_0\mathbf{a}_{k+1} & \cdots & p_0\mathbf{a}_n\end{array}\right).$$

Remark **(I.3.7).** Let $\omega=(p_0,\ldots,p_k)$ be a weight vector and write $e=\gcd(p_0,\ldots,p_k)$. Denote $p_i'=p_i/e$ for $i=0,\ldots,k$ and $\omega'=(p_0',\ldots,p_k')$. Using the previous notation there is an isomorphism

$$F:\widehat{X(\mathbf{d};A)}_L(\omega)\longrightarrow\widehat{X(\mathbf{d};A)}_L(\omega')$$

of blowing-ups (i.e. $F\circ\pi_{\omega'}=\pi_\omega$) induced by the identity map. Hence one can always assume that $\gcd(p_0,\ldots,p_k)=1$.

For instance, in the first chart $F:\widehat{U}_{\omega,0}\to\widehat{U}_{\omega',0}$ takes the form

$$F_0:\quad [(x_0,x_1,\ldots,x_n)]\quad\longmapsto\quad [(x_0^e,x_1,\ldots,x_n)],$$

$$\begin{array}{ccc}
\left(\begin{array}{c|ccccccc} p_0 & -1 & p_1 & \cdots & p_k & 0 & \cdots & 0\\ p_0\mathbf{d} & \mathbf{a}_0 & p_0\mathbf{a}_1-p_1\mathbf{a}_0 & \cdots & p_0\mathbf{a}_k-p_k\mathbf{a}_0 & p_0\mathbf{a}_{k+1} & \cdots & p_0\mathbf{a}_n\end{array}\right) & \xrightarrow{\widehat{\varphi}_{\omega,0}} & \widehat{U}_{\omega,0}\\
\Big\downarrow F_0 & \# & \Big\downarrow F\\
\left(\begin{array}{c|ccccccc} p_0' & -1 & p_1' & \cdots & p_k' & 0 & \cdots & 0\\ p_0'\mathbf{d} & \mathbf{a}_0 & p_0'\mathbf{a}_1-p_1'\mathbf{a}_0 & \cdots & p_0'\mathbf{a}_k-p_k'\mathbf{a}_0 & p_0'\mathbf{a}_{k+1} & \cdots & p_0'\mathbf{a}_n\end{array}\right) & \xrightarrow[\widehat{\varphi}_{\omega',0}]{} & \widehat{U}_{\omega',0}\,.
\end{array}$$

Definition (I.3.8). Let $\pi : \widehat{X(\mathbf{d}; A)}_L(\omega) \to X(\mathbf{d}; A)$ be the ω-blow-up with smooth center $L : \{x_0 = \cdots = x_k = 0\}$. Then the *total transform* $\pi^*(H)$ decomposes as

$$\pi^*(H) = \widehat{H} + mE,$$

where $E := \pi^{-1}(L)$ is the *exceptional divisor* of π, $\widehat{H} := \overline{\pi^{-1}(H \setminus L)}$ is the *strict transform* of H, and m is the *multiplicity* of E at a smooth point.

I.3–1. Dimension 2

Let X be an analytic surface with abelian quotient singularities. Consider $\pi : \widehat{X} \to X$ the weighted blow-up at a point $P \in X$ with respect to $\omega = (p, q)$. We distinguish three different situations.

(i) The point P is smooth. Without lost of generality one can assume that $X = \mathbb{C}^2$ and $\pi = \pi_\omega : \widehat{\mathbb{C}}^2_\omega \to \mathbb{C}^2$ is the weighted blow-up at the origin with respect to $\omega = (p, q)$. The new ambient space is covered as

$$\widehat{\mathbb{C}}^2_\omega = U_1 \cup U_2 = X(p; -1, q) \cup X(q;\, p, -1)$$

and the charts are given by

$$\text{First chart} \left| \begin{array}{rcl} X(p; -1, q) & \longrightarrow & U_1, \\ {[(x, y)]} & \mapsto & ((x^p, x^q y), [1 : y]_\omega). \end{array} \right.$$

$$\text{Second chart} \left| \begin{array}{rcl} X(q;\, p, -1) & \longrightarrow & U_2, \\ {[(x, y)]} & \mapsto & ((xy^p, y^q), [x : 1]_\omega). \end{array} \right.$$

The exceptional divisor $E = \pi_\omega^{-1}(0)$ is isomorphic to $\mathbb{P}^1_\omega$ which is in turn isomorphic to $\mathbb{P}^1$ under the map

$$[x : y]_\omega \longmapsto [x^{q_1} : y^{p_1}], \quad p_1 = \frac{p}{\gcd(p, q)}, \quad q_1 = \frac{q}{\gcd(p, q)}.$$

The singular points of $\widehat{\mathbb{C}}^2_\omega$ are cyclic quotient singularities located at the exceptional divisor. They actually coincide with the origins of the two charts; in the case $\gcd(p, q) = 1$ they are written in a normalized form.

Example (I.3.9). Let $f : \mathbb{C}^2 \to \mathbb{C}$ be the function given by $f = x^p + y^q$ with $\gcd(p, q) = 1$. Consider $\pi_{(q,p)} : \widehat{\mathbb{C}}^2_{(q,p)} \to \mathbb{C}^2$ the (p, q)-weighted blow-up at the origin. In U_1 the total transform is given by the function

$$x^{pq}(1 + y^q) \,:\, X(q; -1, p) \longrightarrow \mathbb{C}.$$

The equation $y^q = -1$ has just one solution in U_1 and the local equation of the total transform at this point is of the form $x^{pq} y = 0$.

Hence the proper map $\pi_{(q,p)}$ is an embedded $\mathbf{Q}$-resolution of the plane curve $\mathbf{C} = \{f = 0\} \subset \mathbb{C}^2$ where all spaces are written in a normalized form.

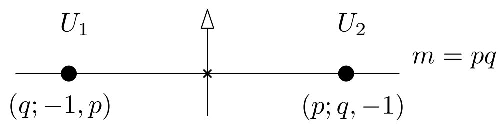


FIGURE I.2. Embedded $\mathbf{Q}$-resolution of $\{x^p + y^q = 0\} \subset \mathbb{C}^2$.

(ii) The point P **is of type** $(d; p, q)$**.** Assume $X = X(d; p, q)$ and it is written in a normalized form, i.e. $\gcd(d, p) = \gcd(d, q) = 1$. Also assume $\pi = \pi_{\omega,d} : \widehat{\mathbb{C}}^2_{\omega,d} \to X(d; p, q)$ is the weighted blow-up at the origin with respect to $\omega = (p, q)$. The new ambient space is cover as

$$\widehat{\mathbb{C}}^2_{\omega,d} = U_1 \cup U_2 = X(p; -d, q) \cup X(q; p, -d)$$

and the charts are given by

$$\text{First chart} \left| \begin{array}{ccc} X(p; -d, q) & \longrightarrow & U_1, \\ \big[(x^d, y)\big] & \mapsto & \big([(x^p, x^q y)]_d, [1 : y]_\omega\big). \end{array} \right.$$

$$\text{Second chart} \left| \begin{array}{ccc} X(q;\, p, -d) & \longrightarrow & U_2, \\ \big[(x, y^d)\big] & \mapsto & \big([(x y^p, y^q)]_d, [x : 1]_\omega\big). \end{array} \right.$$

As above, the exceptional divisor $E = \pi_\omega^{-1}(0)$ is identified with $\mathbb{P}^1_\omega$ which is isomorphic to $\mathbb{P}^1$ under the map

$$[x : y]_\omega \longmapsto [x^{q_1} : y^{p_1}], \quad p_1 = \frac{p}{\gcd(p, q)}, \quad q_1 = \frac{q}{\gcd(p, q)}.$$

The singular points of $\widehat{\mathbb{C}}^2_{\omega,d}$ are cyclic quotient singularities and coincide with the origins of the two charts. They are written in a normalized form if $\gcd(p, q) = 1$.

Example (I.3.10). Assume $\gcd(p, q) = 1$ and $p < q$. Let $f = (x^p + y^q)(x^q + y^p)$ and consider $\mathbf{C}_1 = \{x^p + y^q = 0\}$ and $\mathbf{C}_2 = \{x^q + y^p = 0\}$ the two irreducible components of $\{f = 0\}$.

Let $\pi_{(q,p)} : \widehat{\mathbb{C}}^2_{(q,p)} \to \mathbb{C}^2$ be the (q, p)-weighted blow-up at the origin. The new space has two singular points of type $(q; -1, p)$ and $(p; q, -1)$ located at the exceptional divisor $\mathcal{E}_1$. The local equation of the total transform in the first chart is given by the function

$$x^{p(p+q)}(1 + y^q)(x^{q^2-p^2} + y^p) : \; X(q; -1, p) \longrightarrow \mathbb{C}.$$

Here $x = 0$ is the equation of the exceptional divisor and the other factors correspond to the strict transform of $\mathbf{C}_1$ and $\mathbf{C}_2$ (denoted again by the same symbol).

Hence $\mathcal{E}_1$ has multiplicity $p(p+q)$; it intersects transversely $\mathbf{C}_1$ at a smooth point while it intersects $\mathbf{C}_2$ at a singular point (the origin of the first chart) without $\mathbb{Q}$-normal crossings.

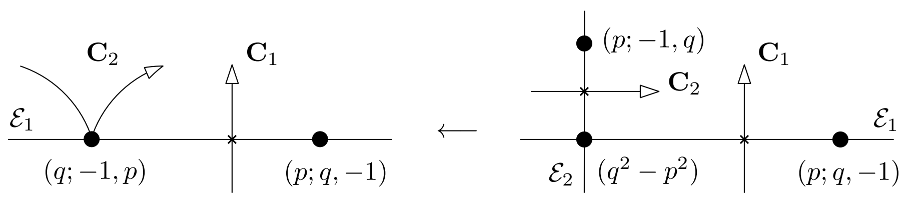


FIGURE I.3. Embedded $\mathbf{Q}$-resolution of $f = (x^p + y^q)(x^q + y^p)$.

Let us consider $\pi_{(p,q^2-p^2),q}$ the (p, q^2-p^2)-weighted blow-up at the origin of $X(q;-1,p)$,

$$\pi_{(p,q^2-p^2),q} : \widehat{\mathbb{C}}^2_{(p,q^2-p^2),q} \longrightarrow X(q;p,q^2-p^2) = X(q;-1,p).$$

The new space has two singular points of type $(p;-q,q^2-p^2) = (p;-1,q)$ and $(q^2-p^2;p,-q)$. In the first chart, the local equation of the total transform of $x^{p(p+q)}(x^{q^2-p^2} + y^p)$ is given by the function

$$x^{p(p+q)}(1+y^p) : X(p;-1,q) \longrightarrow \mathbb{C}.$$

Thus the new exceptional divisor $\mathcal{E}_2$ has multiplicity $p(p+q)$ and intersects transversely the strict transform of $\mathbf{C}_2$ at a smooth point. Hence the composition $\pi_{(p,q^2-p^2),q} \circ \pi_{(q,p)}$ is an embedded $\mathbf{Q}$-resolution of $\{f=0\} \subset \mathbb{C}^2$ where all quotient spaces are written in a normalized form. Figure I.3 illustrates the whole process.

(iii) The point P is of type $(d;a,b)$. As above, assume that $X = X(d;a,b)$ and the map

$$\pi = \pi_{(d;a,b),\omega} : \widehat{X(d;a,b)}_\omega \longrightarrow X(d;a,b)$$

is the weighted blow-up at the origin of $X(d;a,b)$ with respect to $\omega = (p,q)$. The new space is covered as

$$\widehat{U}_1 \cup \widehat{U}_2 = X\left(\begin{array}{c|cc} p & -1 & q \\ pd & a & pb-qa \end{array}\right) \cup X\left(\begin{array}{c|cc} q & p & -1 \\ qd & qa-pb & b \end{array}\right).$$

The charts are given by

$$\text{First chart}\ \left|\ \begin{array}{l} X\left(\begin{array}{c|cc} p & -1 & q \\ pd & a & pb-qa \end{array}\right) \longrightarrow \widehat{U}_1, \\ \big[(x,y)\big] \mapsto \big[((x^p, x^q y), [1:y]_\omega)\big]_{(d;a,b)}. \end{array}\right.$$

$$\text{Second chart}\ \left|\ \begin{array}{l} X\left(\begin{array}{c|cc} q & p & -1 \\ qd & qa-pb & b \end{array}\right) \longrightarrow \widehat{U}_2, \\ \big[(x,y)\big] \mapsto \big[((xy^p, y^q), [x:1]_\omega)\big]_{(d;a,b)}. \end{array}\right.$$

The exceptional divisor $E = \pi^{-1}_{(d;a,b),\omega}(0)$ is identified with the quotient space $\mathbb{P}^1_\omega(d;a,b) := \mathbb{P}^1_\omega/\mu_d$ which is isomorphic to $\mathbb{P}^1$ under the map

$$\begin{array}{ccc} \mathbb{P}^1_\omega(d;a,b) & \longrightarrow & \mathbb{P}^1 \\ [x:y]_\omega & \mapsto & [x^{dq/e} : y^{dp/e}], \end{array}$$

where $e = \gcd(dp, dq, pb-qa)$. Again the singular points are cyclic and correspond to the origins. They may be not written in normalized form even if $\gcd(p,q)=1$ and $(d;a,b)$ is normalized.

(I.3.11). Let us give another expression for the previous charts. We follow the proof of Lemma (I.1.8) and Remark (I.1.7). Let α and β satisfying $\alpha d + \beta a = \gcd(d,a)$. One has the following isomorphisms induced by the identity map[2].

$$(7)\qquad \begin{aligned} \left(\begin{array}{c|cc} p & -1 & q \\ pd & a & pb-qa \end{array}\right) &= \left(\begin{array}{c|cc} pd & d & -qd \\ pd & a & pb-qa \end{array}\right) \\ &= \left(\begin{array}{c|cc} pd & (d,a) & -q(d,a)+\beta pb \\ pd & 0 & \frac{dpb}{(d,a)} \end{array}\right) \end{aligned}$$

For the last equality note that $\alpha(-qd) + \beta(pb-qa) = -q\gcd(d,a) + \beta pb$ and the determinant of the minor of the matrix representing the second quotient space is dpb. From (7), assuming $\boxed{\gcd(d,a,b)=1}$, one also has the isomorphism

$$\begin{array}{ccc} X\left(\begin{array}{c|cc} pd & (d,a) & -q(d,a)+\beta pb \\ (d,a) & 0 & b \end{array}\right) & \stackrel{\cong}{\longrightarrow} & X\left(\frac{pd}{(d,a)}; 1, -q(d,a)+\beta pb\right), \\ {}[(x,y)] & \mapsto & [(x, y^{(d,a)})]. \end{array}$$

Analogously one can proceed with the second chart. Choose λ, μ satisfying Bézout's identity $\lambda d + \mu b = \gcd(d,b)$.

[2]Recall once again the notation $(i_1,\ldots,i_k) = \gcd(i_1,\ldots,i_k)$ for long formulas.

Then the equations of the two charts in these new coordinates are given by the isomorphism

$$\text{First chart} \left| \begin{array}{l} X\left(\dfrac{pd}{(d,a)}; 1, -q(d,a)+\beta pb\right) \longrightarrow \widehat{U}_1, \\ \big[(x, y^{(d,a)})\big] \mapsto \big[((x^p, x^q y), [1:y]_\omega)\big]. \end{array} \right.$$

$$\text{Second chart} \left| \begin{array}{l} X\left(\dfrac{qd}{(d,b)}; -p(d,b)+\mu qa, 1\right) \longrightarrow \widehat{U}_2, \\ \big[(x^{(d,b)}, y)\big] \mapsto \big[((xy^p, y^q), [x:1]_\omega)\big]. \end{array} \right.$$

These spaces are written in a normalized form if and only if the following greatest common divisor equals one:

$$\left(\frac{dp}{(d,a)}, -q(d,a)+\beta pb\right) = \Big(dp, dq, pb-qa\Big) = \left(\frac{dq}{(d,b)}, -p(d,b)+\mu qa\right).$$

Although elementary, the proof of the preceding equalities are not intuitive. That is why the first one is commented separately in the result below.

Lemma (I.3.12). *With the assumption above,*

$$\left(\frac{dp}{(d,a)}, -q(d,a)+\beta pb\right) = \Big(dp, dq, pb-qa\Big).$$

Moreover, $\left(\frac{dp}{(d,a)}, \frac{dq}{(d,b)}, pb-qa\right)$ *also equals the previous number.*

Proof. Note that $\frac{pd}{\gcd(d,a)} = \gcd\left(pd, \frac{dpb}{\gcd(d,a)}\right)$, since $\gcd(d,a,b)=1$, and consequently

$$\left(\frac{dp}{(d,a)}, -q(d,a)+\beta pb\right) = \left(dp, \frac{dpb}{(d,a)}, -q(d,a)+\beta pb\right).$$

The following two couples of equalities complete the first part of the proof.

- $\frac{a}{(d,a)} \cdot \big[-q(d,a)+\beta pb \,\big] + \alpha \cdot \frac{dpb}{(d,a)} = pb-qa.$
- $\frac{-d}{(d,a)} \cdot \big[-q(d,a)+\beta pb \,\big] + \beta \frac{dpb}{(d,a)} = dq.$

- $-q(d,a)+\beta pb = \alpha(-qd)+\beta(pb-qa).$
- $\frac{dpb}{(d,a)} = \frac{d}{(d,a)}(pb-qa) + \frac{a}{(d,a)}(dq).$

The second part of the statement is again rather artificial but elementary; the details are left to the reader. □

Remark **(I.3.13).** Assume that $\gcd(d,a,b)=1$. The (p,q)-weighted blow-up at the origin of $X(d;a,b)$ is isomorphic to the ω'-weighted blow-up at the origin of $X\big(d';a',b'\big)$, where the new vectors are

$$\begin{aligned} \omega' &= \big(p\cdot\gcd(d,b),\ q\cdot\gcd(d,a)\big),\\ (d',a',b') &= \left(\frac{d}{\gcd(d,a)\gcd(d,b)}, \frac{a}{\gcd(d,a)}, \frac{b}{\gcd(d,b)}\right). \end{aligned}$$

In fact, there is a commutative diagram of blowing-ups

$$\begin{array}{ccc} \widehat{X(d;a,b)}_\omega & \xrightarrow{H} & \widehat{X(d';a',b')}_{\omega'} \\ {\scriptstyle \pi_{(d;a,b),\omega}}\downarrow & \# & \downarrow{\scriptstyle \pi_{(d';a',b'),\omega'}} \\ X(d;a,b) & \xrightarrow[h]{} & X(d';a',b') \end{array}$$

where H and h are isomorphisms of analytic spaces defined by

$$\begin{array}{ccc} [((x,y),[u:v])_\omega]_{(d;a,b)} & \stackrel{H}{\longmapsto} & [((x^{(d,b)},y^{(d,a)}),[u^{(d,b)}:v^{(d,a)}])_{\omega'}]_{(d';a',b')};\\ [(x,y)]_{(d;a,b)} & \stackrel{h}{\longmapsto} & [(x^{(d,b)},y^{(d,a)})]_{(d';a',b')}, \end{array}$$

and H gives rise to the identity map on each chart.

Note also that if

$$\mathbf{C}=\{f=0\}\subseteq X(d;a,b), \quad \mathbf{C}'=\{f'=0\}\subseteq X(d';a',b')$$

such that $h^*(\mathbf{C}')=\mathbf{C}$, then $\operatorname{ord}_\omega(f)=\operatorname{ord}_{\omega'}(f')=\operatorname{ord}(f(x^p,y^q))$. Hence the order is preserved under this construction.

Remark **(I.3.14).** Using the notation in (I.3.11), assume $\gcd(p,q)=1$ and $X(d;a,b)$ is written in a normalized form. To normalize the last cyclic quotient spaces obtained in that paragraph, let

$$e=\gcd(pd,-q+\beta pb)=\gcd(d,pb-qa).$$

Then one has the isomorphism

$$\begin{array}{ccc} X(pd;1,-q+\beta pb) & \stackrel{\cong}{\longrightarrow} & X\left(\frac{pd}{e};1,\frac{-q+\beta pb}{e}\right),\\ [(x,y)] & \mapsto & [(x^e,y)]. \end{array}$$

One proceeds analogously with the second chart. Finally the equations of the two charts in these new coordinates are given by

$$
(8)\qquad
\begin{array}{l|c}
\text{First chart} & X\left(\dfrac{pd}{e}; 1, \dfrac{-q+\beta p b}{e}\right) \longrightarrow \widehat{U}_1, \\
 & \big[(x^e, y)\big] \mapsto \big[((x^p, x^q y), [1:y]_\omega)\big]_{(d;a,b)}. \\
\text{Second chart} & X\left(\dfrac{qd}{e}; \dfrac{-p+\mu q a}{e}, 1\right) \longrightarrow \widehat{U}_2, \\
 & \big[(x, y^e)\big] \mapsto \big[((x y^p, y^q), [x:1]_\omega)\big]_{(d;a,b)}.
\end{array}
$$

Recall that β and μ are the inverse of a and b modulo d, respectively. Note that both quotient spaces are now written in their normalized form.

Example (I.3.15). Assume $\gcd(p,q) = \gcd(r,s) = 1$ and $\frac{p}{q} < \frac{r}{s}$. Let $f = (x^p + y^q)(x^r + y^s)$ and consider

$$\mathbf{C}_1 = \{x^p + y^q = 0\}, \quad \mathbf{C}_2 = \{x^r + y^s = 0\}$$

the two irreducible components of f.

Working as in Example (I.3.10), one obtains the following picture representing an embedded $\mathbf{Q}$-resolution of $\{f = 0\} \subset \mathbb{C}^2$.

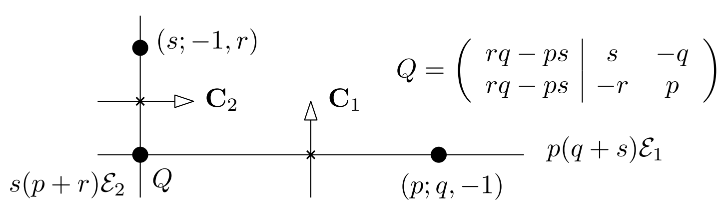


FIGURE I.4. Embedded $\mathbf{Q}$-resolution of $f = (x^p + y^q)(x^r + y^s)$.

After writing the quotient spaces in their normalized form one checks that this resolution coincides with the one given in Example (I.3.10) assuming $r = q$ and $s = p$.

(I.3.16). (Puiseux expansion). Let us study the behavior of Puiseux pairs under weighted blow-ups. Let $\mathbf{C} = \{f = 0\} \subset \mathbb{C}^2$ be the irreducible plane curve given by

$$\prod_{j=1}^{d} \Big[-y + \big(a_{1j} x^{\frac{p_1}{q}} + \cdots + a_{kj} x^{\frac{p_k}{q}}\big) + \big(b_{1j} x^{\frac{r_1}{s}} + \cdots + b_{lj} x^{\frac{r_l}{s}}\big) + \cdots \Big],$$

where $p_1 < \cdots < p_k$, $r_1 < \cdots < r_l$, $\frac{p_1}{q} < \frac{r_i}{s}$ and each fraction is irreducible.

Let $\pi_{(q,p_1)} : \widehat{\mathbb{C}}^2_{(q,p_1)} \to \mathbb{C}^2$ be the (q,p_1)-blow-up at the origin. In the first chart, that is, after performing the substitution

$$(x,y) \longmapsto (x^q, x^{p_1} y),$$

one obtains the following equation for the total transform

$$x^{p_1 d} \cdot \prod_{j=1}^{d} \big[-y + (a_{1j} + a_{2j} x^{p_2 - p_1} + \cdots + a_{kj} x^{p_k - p_1}) + (b_{1j} x^{\frac{r_1 q - p_1 s}{s}} + \cdots + b_{lj} x^{\frac{r_l q - p_1 s}{s}}) + \cdots \big] = 0.$$

At first sight the exceptional divisor and the strict transform intersect at d different smooth points. However, since a_{1j}^q does not depend on j by conjugation, all of them are the same. After the following change of coordinates

$$y \longmapsto y + (a_{2j} x^{p_2 - p_1} + \cdots + a_{kj} x^{p_k - p_1}),$$

the local equation of the total transform $\pi^{-1}_{(q,p_1)}(\mathbf{C})$ at this point is

$$x^{p_1 d} \cdot \prod_{j=1}^{d/q} \big[-y + (b_{1j} x^{\frac{r_1 q - p_1 s}{s}} + \cdots + b_{lj} x^{\frac{r_l q - p_1 s}{s}}) + \cdots \big] = 0.$$

This proves that in the irreducible case, only a weighted blow-up is needed for each Puiseux pair in order to compute an embedded **Q**-resolution. Moreover, the embedded **Q**-resolution obtained is as in Figure I.5.

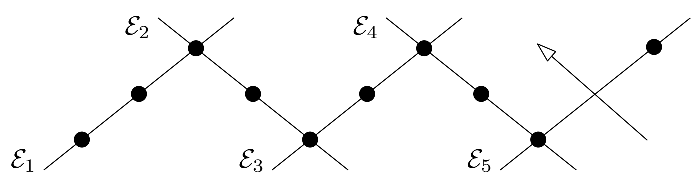


FIGURE I.5. Embedded **Q**-resolution of an irreducible plane curve.

In the non-irreducible case, the situation is a bit more complicated but can still be described in terms of the Puiseux pairs of each irreducible component and their intersection multiplicities.

I.3–2. Dimension 3

Let X be a 3-dimensional variety with abelian quotient singularities and consider $\pi : \widehat{X} \to X$ the weighted blow-up at a point $P \in X$ with respect to $\omega = (p,q,r)$. Two special situations are considered.

(i) The point P is smooth. As usual one assumes that $X = \mathbb{C}^2$ and the map $\pi = \pi_\omega : \widehat{\mathbb{C}}^3_\omega \to \mathbb{C}^3$ is the weighted blow-up at the origin with respect to $\omega = (p,q,r)$. Also assume $\gcd(p,q,r) = 1$. The new space is covered as

$$\widehat{\mathbb{C}}^3_\omega = U_1 \cup U_2 \cup U_3 = X(p;-1,q,r) \cup X(q;p,-1,r) \cup X(r;p,q,-1),$$

and the charts are given by

$$\begin{aligned} &X(p;-1,q,r) \longrightarrow U_1 : \quad [(x,y,z)] \mapsto ((x^p, x^q y, x^r z), [1:y:z]_\omega),\\ (9)\qquad &X(q;p,-1,r) \longrightarrow U_2 : \quad [(x,y,z)] \mapsto ((xy^p, y^q, y^r z), [x:1:z]_\omega),\\ &X(r;p,q,-1) \longrightarrow U_3 : \quad [(x,y,z)] \mapsto ((xz^p, yz^q, z^r), [x:y:1]_\omega). \end{aligned}$$

In general $\widehat{\mathbb{C}}^3_\omega$ has three lines of (cyclic quotient) singular points located at the exceptional divisor $\pi_\omega^{-1}(0) \simeq \mathbb{P}^2_\omega$. They correspond to the three lines at infinity of the previous weighted projective plane. The stratification of the exceptional divisor is shown below in terms of its quotient singularities, or equivalently, in terms of the order of the stabilizer subgroups. For example, the stratum labeled as (p,q) is isomorphic to $\mathbb{C}^*$ and the order of the stabilizer subgroup is $\gcd(p,q)$.

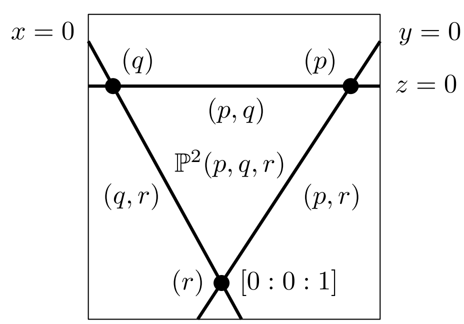


FIGURE I.6. Stratification of the exceptional divisor of the (p,q,r)-weighted blow-up at a smooth point.

Note that although the quotient spaces are written in their normalized form, there is an isomorphism of weighted projective spaces that simplifies the expression of the exceptional divisor:

$$\begin{array}{ccc} \mathbb{P}^2(p,q,r) & \longrightarrow & \mathbb{P}^2\left(\dfrac{p}{(p,r)\cdot(p,q)}, \dfrac{q}{(q,p)\cdot(q,r)}, \dfrac{r}{(r,p)\cdot(r,q)}\right), \\ [x:y:z] & \mapsto & [x^{\gcd(q,r)} : y^{\gcd(p,r)} : z^{\gcd(p,q)}]. \end{array}$$

However, this simplification may be not useful when working with the whole ambient space because its charts are not compatible with $\widehat{\mathbb{C}}^3_\omega$.

Thus the natural covering of the exceptional divisor is

$$\mathbb{P}^2_\omega = V_1 \cup V_2 \cup V_3 = X(p; q, r) \cup X(q; p, r) \cup X(r; p, q),$$

and the charts are given by

$$\begin{array}{rclcl} X(p;q,r) & \longrightarrow & V_1: & [(y,z)] & \mapsto & [1:y:z]_\omega, \\ X(q;p,r) & \longrightarrow & V_2: & [(x,z)] & \mapsto & [x:1:z]_\omega, \\ X(r;p,q) & \longrightarrow & V_3: & [(x,y)] & \mapsto & [x:y:1]_\omega. \end{array}$$

Now one sees that

$$V_1 = U_1|_{\{x=0\}}, \quad V_2 = U_2|_{\{y=0\}}, \quad V_3 = U_3|_{\{z=0\}}.$$

In other words, the restriction of the charts of $\widehat{\mathbb{C}}^3_\omega$ gives rise to the charts of the projective plane $\mathbb{P}^2_\omega$.

Remark **(I.3.17).** Using just a weighted blow-up of this kind, one can find an embedded **Q**-resolution for Brieskorn-Pham surfaces singularities, i.e. $x^p + y^q + z^r = 0$, see Example (IV.2.6). This can be generalized to higher dimension obtaining an embedded **Q**-resolution for $x_1^{p_1} + \cdots + x_n^{p_n}$ by blowing-up the origin with suitable weights.

(ii) The point P is of type $(d; a, b, c)$. Assume $X = X(d; a, b, c)$ and the map

$$\pi = \pi_{(d;a,b,c),\omega} : \widehat{X(d;a,b,c)}_\omega \longrightarrow X(d;a,b,c)$$

is the weighted blow-up at the origin of $X(d; a, b, c)$ with respect to $\omega = (p, q, r)$. The new space is covered as

$$\widehat{X(d;a,b,c)}_\omega = \frac{\widehat{\mathbb{C}}^3_\omega}{\mu_d} = \frac{U_1 \cup U_2 \cup U_3}{\mu_d} = \widehat{U}_1 \cup \widehat{U}_2 \cup \widehat{U}_3,$$

where

$$\widehat{U}_1 = \frac{U_1}{\mu_d} = \frac{X(p; -1, q, r)}{\mu_d} = X\left(\begin{array}{c|ccc} p & -1 & q & r \\ pd & a & pb - qa & pc - ra \end{array}\right),$$

$$\widehat{U}_2 = \frac{U_2}{\mu_d} = \frac{X(q; p, -1, r)}{\mu_d} = X\left(\begin{array}{c|ccc} q & p & -1 & r \\ qd & qa - pb & b & qc - rb \end{array}\right),$$

$$\widehat{U}_3 = \frac{U_3}{\mu_d} = \frac{X(r; p, q, -1)}{\mu_d} = X\left(\begin{array}{c|ccc} r & p & q & -1 \\ rd & ra - pc & rb - qc & c \end{array}\right).$$

The charts are given by the induced maps on the corresponding quotient spaces, see (9). For instance, the first map is

$$[(x, y, z)] \longmapsto [((x^p, x^q y, x^r z), [1 : y : z]_\omega)].$$

The exceptional divisor $E = \pi^{-1}_{(d;a,b,c),\omega}(0)$ is identified with

$$\mathbb{P}^2_\omega(d; a, b, c) := \frac{\mathbb{P}^2_\omega}{\mu_d}.$$

There are three lines of quotient singular points in E and outside E the map $\pi_{(d;a,b,c),\omega}$ is an isomorphism.

The expression of the quotient spaces can be modified as in dimension 2, see (I.3.11). Let α and β be such that $\alpha d + \beta a = \gcd(d, a)$, then one has that the space $X\left(\begin{smallmatrix} p; & -1 & q & r \\ pd; & a & pb-qa & pc-ar \end{smallmatrix}\right)$ equals

$$X\left(\begin{array}{c|ccc} pd & (d,a) & -q(d,a)+\beta pb & -r(d,a)+\beta pc \\ (d,a) & 0 & b & c \end{array}\right).$$

Note that in general the previous space is not written in a normalized form, even if $(d; a, b, c)$ is already normalized and $\gcd(p, q, r) = 1$.

To obtain the normalized one, follow the processes described in (I.1.9) and (I.1.3). For instance, the previous space Q is cyclic if either $\gcd(d, a) = 1$ or $\gcd(p, a) = 1$.

- $\gcd(d, a) = 1 \implies Q = X(pd; 1, -q + \beta pb, -r + \beta pc)$.
- $\gcd(p, a) = 1 \implies Q = X(pd; a, pb - qa, pc - ar)$.

As the following example shows this is not always the case.

Example (I.3.18). Blowing-up the origin of $X(2; 2, 1, 1)$ with respect to $(2, 1, 2)$, one obtains the following decomposition of the new space into normalized quotient spaces,

$$X(\widehat{2;2,1},1)_{(2,1,2)} = X\left(\begin{array}{c|ccc} 2 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right) \cup X(2; 0, 1, 1) \cup X(4; 2, 1, 1).$$

In particular, the origin of the first chart is a quotient singular point which is not isomorphic to a cyclic singularity.

(I.3.19). Turning to the exceptional divisor $E = \mathbb{P}^2_\omega(d; a, b, c)$, it can be written as

$$\mathbb{P}^2_\omega(d; a, b, c) = \frac{\mathbb{P}^2_\omega}{\mu_d} = \frac{V_1 \cup V_2 \cup V_3}{\mu_d} = \widehat{V}_1 \cup \widehat{V}_2 \cup \widehat{V}_3,$$

where

$$\widehat{V}_1 = \frac{V_1}{\mu_d} = \frac{X(p;q,r)}{\mu_d} = X\left(\begin{array}{c|cc} p & q & r \\ pd & pb-qa & pc-ra \end{array} \right),$$

$$\widehat{V}_2 = \frac{V_2}{\mu_d} = \frac{X(q;p,r)}{\mu_d} = X\left(\begin{array}{c|cc} q & p & r \\ qd & qa-pb & qc-rb \end{array} \right),$$

$$\widehat{V}_3 = \frac{V_3}{\mu_d} = \frac{X(r;p,q)}{\mu_d} = X\left(\begin{array}{c|cc} r & p & q \\ rd & ra-pc & rb-qc \end{array} \right).$$

Hence these charts are compatible with the ones of $X\widehat{(d;a,b,c)}_\omega$ in the sense that $\widehat{V}_1 = \widehat{U}_1|_{\{x=0\}}$, $\widehat{V}_2 = \widehat{U}_2|_{\{y=0\}}$, and $\widehat{V}_3 = \widehat{U}_3|_{\{z=0\}}$. Note that all these singular quotient spaces can be rewritten as cyclic singularities.

I.3–3. Higher dimension

Here we only consider a special kind of weighted blow-ups with smooth center where the ambient space is smooth too, as in (I.3.4) and (I.3.5). These blow-ups will be used later to compute an embedded **Q**-resolution for superisolated singularities in higher dimension, see Section VI.4.

Let $\pi_\omega : \widehat{\mathbb{C}}^{n+1}_L(\omega) \to \mathbb{C}^{n+1}$ be the ω-weighted blow-up of $\mathbb{C}^{n+1}$ with smooth center $L = \{x_0 = \cdots = x_k = 0\}$ where $\omega = (p_0, 1, \overset{(k)}{\ldots}, 1)$. The new space is covered as

$$\widehat{\mathbb{C}}^{n+1} = U_0 \cup \cdots \cup U_k,$$

where

$$U_0 = X(p_0; -1, 1, \overset{(k)}{\ldots}, 1) \times \mathbb{C}^{n-k},$$

and $U_i = \mathbb{C}^{n+1}$ for all $i \neq 0$.

The charts are given by

$$0 \left\{ \begin{array}{l} X(p_0; -1, 1, \overset{(k)}{\ldots}, 1) \times \mathbb{C}^{n-k} \xrightarrow{\varphi_0} U_0 = \{u_0 \neq 0\} \subset \widehat{\mathbb{C}}^{n+1}_L(\omega), \\ [\mathbf{x}] \mapsto \big((x_0^{p_0}, x_0x_1, \ldots, x_0x_k, x_{k+1}, \ldots, x_n), [1 : x_1 : \ldots : x_k]_\omega \big); \end{array} \right.$$

$$1 \left\{ \begin{array}{l} \mathbb{C}^{n+1} \xrightarrow{\varphi_1} U_1 = \{u_1 \neq 0\} \subset \widehat{\mathbb{C}}^{n+1}_L(\omega), \\ \mathbf{x} \mapsto \big((x_0x_1^{p_0}, x_1, \ldots, x_1x_k, x_{k+1}, \ldots, x_n), [x_0 : 1 : \ldots : x_k]_\omega \big); \end{array} \right.$$

$$\vdots$$

$$k \left\{ \begin{array}{l} \mathbb{C}^{n+1} \xrightarrow{\varphi_k} U_k = \{u_k \neq 0\} \subset \widehat{\mathbb{C}}^{n+1}_L(\omega), \\ \mathbf{x} \mapsto \big((x_0x_k^{p_0}, x_1x_k, \ldots, x_k, x_{k+1}, \ldots, x_n), [x_0 : x_1 : \ldots : 1]_\omega \big). \end{array} \right.$$

The exceptional divisor is isomorphic to $\mathbb{P}^k_\omega \times \mathbb{C}^{n-k}$. The singular locus of $\widehat{\mathbb{C}}^{n+1}_L(\omega)$ is the subset $[(0,\ldots,0)] \times \mathbb{C}^{n-k}$. These quotient singular points are all cyclic and the space

$$U_0 = X(p_0; -1, 1, \overset{(k)}{\ldots}, 1) \times \mathbb{C}^{n-k}$$

is always written in a normalized form.

Remark **(I.3.20).** If $n = k$ the previous singular locus is reduced to a point. This is the case, for instance, of the blowing-up at the origin in dimension 3 (i.e. $n = k = 2$).

II

Cartier and Weil Divisors on V-Manifolds: Pull-Back of a $\mathbb{Q}$-Divisor

This chapter is based on [**AMO11a**] and its aim is to show that when X is a V-manifold there is an isomorphism of $\mathbb{Q}$-vector spaces between Cartier and Weil divisors, see Theorem (II.2.6) below. It is explained in (II.2.14) how to write explicitly a $\mathbb{Q}$-Weil divisor as a $\mathbb{Q}$-Cartier divisor. Also, the case of the exceptional divisor of a weighted blow-up in dimension 2 (which is in general just a Weil divisor) is treated in Example (II.2.15).

Following the theory of holomorphic line bundles, the pull-back of a $\mathbb{Q}$-divisor can be defined using this approach, see Section II.4. This provides all the necessary ingredients to develop a rational intersection theory on variety with quotient singularities. Although Chapter III is devoted to the details, an illustrative example is shown at the end, see (II.4.5).

Section § II.1
Divisors on Complex Analytic Varieties

Let X be an irreducible complex analytic variety. As usual, consider $\mathcal{O}_X$ the structure sheaf of X and $\mathcal{K}_X$ the sheaf of total quotient rings of $\mathcal{O}_X$. Denote by $\mathcal{K}_X^*$ the (multiplicative) sheaf of invertible elements in $\mathcal{K}_X$. Similarly $\mathcal{O}_X^*$ is the sheaf of invertible elements in $\mathcal{O}_X$.

Remark **(II.1.1).** By a *complex analytic variety* we mean a reduced complex space. A *subvariety* V of X is a reduced closed complex subspace of X, or equivalently, an analytic set in X, cf. [**GR84**]. An irreducible subvariety V corresponds to a prime ideal in the ring of sections of any local complex model space meeting V.

Definition (II.1.2). A *Cartier divisor* on X is a global section of the sheaf $\mathcal{K}_X^*/\mathcal{O}_X^*$, that is, an element in $\Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*) = H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$. Any Cartier divisor can be represented by giving an open covering $\{U_i\}_{i\in I}$ of X and, for all $i \in I$, an element $f_i \in \Gamma(U_i, \mathcal{K}_X^*)$ such that

$$\frac{f_i}{f_j} \in \Gamma(U_i \cap U_j, \mathcal{O}_X^*), \quad \forall i, j \in I.$$

Two systems $\{(U_i, f_i)\}_{i\in I}$, $\{(V_j, g_j)\}_{j\in J}$ represent the same Cartier divisor if and only if on $U_i \cap V_j$, f_i and g_j differ by a multiplicative factor in $\mathcal{O}_X(U_i \cap V_j)^*$. The abelian group of Cartier divisors on X is denoted by $\operatorname{CaDiv}(X)$. If $D := \{(U_i, f_i)\}_{i\in I}$ and $E := \{(V_j, g_j)\}_{j\in J}$, then

$$D + E = \{(U_i \cap V_j, f_i g_j)\}_{i\in I, j\in J}.$$

The functions f_i above are called *local equations* of the divisor on U_i. A Cartier divisor on X is *effective* if it can be represented by $\{(U_i, f_i)\}_i$ with all local equations $f_i \in \Gamma(U_i, \mathcal{O}_X)$.

Any global section $f \in \Gamma(X, \mathcal{K}_X^*)$ determines a *principal* Cartier divisor $(f)_X := \{(X, f)\}$ by taking all local equations equal to f. That is, a Cartier divisor is principal if it is in the image of the natural map

$$\Gamma(X, \mathcal{K}_X^*) \longrightarrow \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*).$$

Two Cartier divisors D and E are *linearly equivalent*, denoted by $D \sim E$, if they differ by a principal divisor. The *Picard group* $\operatorname{Pic}(X)$ denotes the group of linear equivalence classes of Cartier divisors.

The *support* of a Cartier divisor D, denoted by $\operatorname{Supp}(D)$ or $|D|$, is the subset of X consisting of all points x such that a local equation for D is not in $\mathcal{O}_{X,x}^*$. The support of D is a closed subset of X.

Definition (II.1.3). A *Weil divisor* on X is a locally finite linear combination with integral coefficients of irreducible subvarieties of codimension one. The abelian group of Weil divisors on X is denoted by $\operatorname{WeDiv}(X)$. If all coefficients appearing in the sum are non-negative, the Weil divisor is called *effective*.

Remark **(II.1.4).** In the algebraic category meromorphic functions are assumed to be regular functions and hence the locally finite sum of Definition (II.1.3) is automatically finite. Therefore $\operatorname{WeDiv}(X)$ is the free abelian group on the codimension one irreducible algebraic subvarieties of X. Similar considerations hold if X is a compact analytic variety.

Given a Cartier divisor there is a Weil divisor associated with it. To see this, the notion of order of a divisor along an irreducible subvariety of codimension one is needed.

(II.1.5). (Order function). Let $V \subset X$ be an irreducible subvariety of codimension one. It corresponds to a prime ideal in the ring of sections of any local complex model space meeting V. The *local ring of* X *along* V, denoted by $\mathcal{O}_{X,V}$, is the localization of such ring of sections at the corresponding prime ideal; it is a one-dimensional local domain.

For a given $f \in \mathcal{O}_{X,V}$ define $\mathrm{ord}_V(f)$ to be

$$\mathrm{ord}_V(f) := \mathrm{length}_{\mathcal{O}_{X,V}} \left(\frac{\mathcal{O}_{X,V}}{\langle f \rangle} \right),$$

where $\mathrm{length}_{\mathcal{O}_{X,V}}$ denotes the length as an $\mathcal{O}_{X,V}$-module. This determines a well-defined group homomorphism

$$\mathrm{ord}_V : \Gamma(X, \mathcal{K}_X^*) \longrightarrow \mathbb{Z}$$

that satisfies, for a given $f \in \Gamma(X, \mathcal{K}_X^*)$, the following local finiteness property: (U_x is assumed to be an open neighborhood of x)

$$\forall x \in X,\ \exists U_x \subset X \ \mid \ \#\{\mathrm{ord}_V(f) \neq 0 \mid V \cap U_x \neq \emptyset\} < +\infty.$$

The previous length, X being a complex analytic variety of dimension $n \geq 2$, can be computed as follows. Choose $x \in V$ such that x is smooth in X and (V, x) defines an irreducible germ. Thus, this germ is the zero set of an irreducible $g \in \mathcal{O}_{X,x}$. Then

$$\mathrm{ord}_V(f) = \mathrm{ord}_{V,x}(f),$$

where $\mathrm{ord}_{V,x}(f)$ is the classical order of a meromorphic function at a smooth point with respect to an irreducible subvariety of codimension one; it is known to be given by the equality

$$f = g^{\mathrm{ord}} \cdot h \in \mathcal{O}_{X,x}, \quad h \nmid g.$$

The same applies if X is 1-dimensional and smooth.

Remark **(II.1.6).** The order $\mathrm{ord}_{V,x}(f)$ does not depend on the defining equation g, as long as we choose g irreducible. In fact, two irreducible $g, g' \in \mathcal{O}_{X,x}$ with $V(g) = V(g')$ only differ by a unit in $\mathcal{O}_{X,x}$. Moreover, $\mathrm{ord}_{V,x}(f)$ does not depend on x, since the set of regular points V_{red} is connected if V is irreducible.

Now if D is a Cartier divisor on X, one writes $\operatorname{ord}_V(D) = \operatorname{ord}_V(f_i)$ where f_i is a local equation of D on any open set U_i with $U_i \cap V \neq \emptyset$. This is well defined since f_i is uniquely determined up to multiplication by units and the order function is a homomorphism. Define the *associated Weil divisor* of a Cartier divisor D by setting

$$\begin{array}{rccc} T_X: & \operatorname{CaDiv}(X) & \longrightarrow & \operatorname{WeDiv}(X) \\ & D & \mapsto & \displaystyle\sum_{V \subset X} \operatorname{ord}_V(D) \cdot [V], \end{array}$$

where the sum is taken over all codimension one irreducible subvarieties V of X. The previous sum is locally finite, i.e. for any $x \in X$ there exists an open neighborhood U such that the set

$$\{\operatorname{ord}_V(D) \neq 0 \mid V \cap U \neq \emptyset\}$$

is finite. By the additivity of the order function, the mapping T_X is a homomorphism of abelian groups.

A Weil divisor is *principal* if it is the image of a principal Cartier divisor under T_X; they form a subgroup of $\operatorname{WeDiv}(X)$. If $\operatorname{Cl}(X)$ denotes the quotient group of their equivalence classes, then T_X induces a morphism

$$\operatorname{Pic}(X) \longrightarrow \operatorname{Cl}(X).$$

These two homomorphisms (T_X and the induced one) are in general neither injective nor surjective. In this sense one has the following result.

Theorem (II.1.7). (cf. [**GD67**, 21.6]). *If X is normal (resp. locally factorial) then the previous maps* $\operatorname{CaDiv}(X) \to \operatorname{WeDiv}(X)$ *and* $\operatorname{Pic}(X) \to \operatorname{Cl}(X)$ *are injective (resp. bijective). The image of the first map is the subgroup of locally principal*[1] *Weil divisors.* □

Remark **(II.1.8).** Locally factorial essentially means that every local ring $\mathcal{O}_{X,x}$ is a unique factorization domain. In particular, every smooth analytic variety is locally factorial. In such a case, Cartier and Weil divisors are identified and denoted by

$$\operatorname{Div}(X) := \operatorname{CaDiv}(X) = \operatorname{WeDiv}(X).$$

Their equivalence classes coincide under this identification and we often write $\operatorname{Pic}(X) = \operatorname{Cl}(X)$.

[1] A Weil divisor D on X is said to be *locally principal* if X can be covered by open sets U such that $D|_U$ is principal for each U.

Example (II.1.9). Let X be the surface in $\mathbb{C}^3$ defined by the equation $z^2 = xy$. The line $V = \{x = z = 0\}$ defines a Weil divisor which is not a Cartier divisor. In this case $\operatorname{Pic}(X) = 0$ and $\operatorname{Cl}(X) = \mathbb{Z}/(2)$. Note that X is normal but not locally factorial. However, the associated Weil divisor of $\{(X, x)\}$ is

$$T_X\big(\{(X,x)\}\big) = \sum_{\substack{Z \subset X,\ \text{irred} \\ \operatorname{codim}(Z)=1}} \operatorname{ord}_Z(x) \cdot [Z] = 2[V].$$

Thus $[V]$ is principal as an element in $\operatorname{WeDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ and corresponds to the $\mathbb{Q}$-Cartier divisor $\frac{1}{2}\{(X, x)\}$.

Using the notation of Chapter I, this fact can be interpreted as follows. First note that identifying our surface X with $X(2; 1, 1)$ under

$$[(x, y)] \longmapsto (x^2, y^2, xy),$$

the previous Weil divisor corresponds to $D = \{x = 0\}$. Although $f = x$ defines a zero set on $X(2; 1, 1)$, it does not induce a function on the abelian quotient space. However, $x^2 : X(2; 1, 1) \to \mathbb{C}$ is a well-defined function and gives rise to the same zero set as f. Hence as $\mathbb{Q}$-Cartier divisors

$$D = \frac{1}{2}\{(X(2; 1, 1), x^2)\}.$$

Section § II.2
Divisors on V-Manifolds: ℚ-Divisor

Example (II.1.9) above illustrates the general behavior of Cartier and Weil divisors on V-manifolds, namely Weil divisors are all locally principal over $\mathbb{Q}$. To prove it we need some preliminaries.

(II.2.1). If X is smooth, contractible, and Stein, then $H^i(X, \mathcal{O}_X^*) = 0$, $\forall i \geq 1$. Indeed, there is a short exact sequence of sheaves of abelian groups

$$0 \longrightarrow \underline{\mathbb{Z}}_X \longrightarrow (\mathcal{O}_X, +) \stackrel{\exp}{\longrightarrow} (\mathcal{O}_X^*, \ \cdot\ ) \longrightarrow 0$$

that gives rise to the following long exact sequence in cohomology

$$\begin{aligned} 0 \longrightarrow H^0(X, \underline{\mathbb{Z}}_X) &\longrightarrow H^0(X, \mathcal{O}_X) \longrightarrow H^0(X, \mathcal{O}_X^*) \longrightarrow \\ H^1(X, \underline{\mathbb{Z}}_X) &\longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X^*) \longrightarrow \\ H^2(X, \underline{\mathbb{Z}}_X) &\longrightarrow H^2(X, \mathcal{O}_X) \longrightarrow H^2(X, \mathcal{O}_X^*) \longrightarrow \ \cdots \end{aligned}$$

Let $i \geq 1$. Since X is contractible, $H^i(X, \underline{\mathbb{Z}}_X) = 0$. The cohomology $H^i(X, \mathcal{O}_X)$ vanishes too because X is Stein and $\mathcal{O}_X$ is a coherent sheaf. Hence $H^i(X, \mathcal{O}_X^*) = 0$ as claimed and the previous long exact sequence is nothing but $0 \longrightarrow \underline{\mathbb{Z}}_X(X) \longrightarrow \mathcal{O}_X(X) \longrightarrow \mathcal{O}_X^*(X) \longrightarrow 0$.

(II.2.2). The short exact sequence of sheaves of multiplicative groups

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathcal{K}_X^* \longrightarrow \mathcal{K}_X^*/\mathcal{O}_X^* \longrightarrow 0$$

gives the long exact sequence in cohomology

$$\begin{aligned} 0 \longrightarrow H^0(X,\mathcal{O}_X^*) \longrightarrow H^0(X,\mathcal{K}_X^*) \longrightarrow H^0(X,\mathcal{K}_X^*/\mathcal{O}_X^*) \longrightarrow \\ H^1(X,\mathcal{O}_X^*) \longrightarrow H^1(X,\mathcal{K}_X^*) \longrightarrow H^1(X,\mathcal{K}_X^*/\mathcal{O}_X^*) \longrightarrow \\ H^2(X,\mathcal{O}_X^*) \longrightarrow H^2(X,\mathcal{K}_X^*) \longrightarrow H^2(X,\mathcal{K}_X^*/\mathcal{O}_X^*) \longrightarrow \cdots \end{aligned}$$

If, as above, $H^i(X,\mathcal{O}_X^*) = 0$, $\forall i \geq 1$, then the previous long exact sequence gives rise to the short exact sequence

$$0 \longrightarrow \mathcal{O}_X^*(X) \longrightarrow \mathcal{K}_X^*(X) \longrightarrow \mathrm{CaDiv}(X) \longrightarrow 0$$

together with an isomorphism $H^i(X,\mathcal{K}_X^*) \to H^i(X,\mathcal{K}_X^*/\mathcal{O}_X^*)$, $\forall i \geq 1$. In particular, every Cartier divisor on X is principal, that is, it is of the form $\{(X,f)\}$ where $f \in \Gamma(X,\mathcal{K}_X^*)$.

Remark **(II.2.3).** As an easy consequence of (II.2.1) and (II.2.2), one has that every effective Weil divisor on an open ball $B \subset \mathbb{C}^n$ is given by the zero set of a holomorphic function $f : B \to \mathbb{C}$. The Weil divisor is irreducible on B if and only if f defines a prime ideal in $\mathcal{O}_{\mathbb{C}^n}(B)$. In the algebraic category the corresponding holomorphic function is a polynomial.

Lemma (II.2.4). *Let $B \subset \mathbb{C}^n$ be an open ball and let G be a finite group acting on B. Then one has* $\mathrm{Cl}(B/G) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$.

Proof. Let $V \subset B/G =: U$ be an irreducible subvariety of codimension one. We shall prove that there exists $k \geq 1$ such that $k[V] \in \mathrm{WeDiv}(U)$ is principal.

Consider the natural projection $\pi : B \to U$. Then $W := \pi^{-1}(V)$ gives rise to an effective Weil divisor on the open ball B. By Remark (II.2.3), there exists $f : B \to \mathbb{C}$ a holomorphic function such that $W = \{f = 0\} \subset B$. Thus,

$$V = \pi(W) = \{[\mathbf{x}] \mid \mathbf{x} \in B,\ f(\mathbf{x}) = 0\} = \{f = 0\} \subset U.$$

Moreover, by construction the holomorphic function f satisfies the following property

$$\forall P \in U,\ \big[f(P) = 0 \implies f(\sigma \cdot P) = 0, \forall \sigma \in G\big]. \tag{10}$$

Note that f does not necessarily defines an analytic function on U. This reflects the fact that, although V is given by just one equation, $[V] \in \mathrm{WeDiv}(U)$ is not principal, see Example (II.1.9). Now the main idea is to change f by another holomorphic function F such that $V = \{F = 0\}$ but now with $F \in \Gamma(U,\mathcal{O}_U)$.

Let us consider $F = \prod_{\sigma \in G} f^\sigma$ where $f^\sigma(\mathbf{x}) = f(\sigma \cdot \mathbf{x})$; clearly it verifies the previous conditions. Then $\{(U, F)\}$ is a principal Cartier divisor and its associated Weil divisor is

$$T_U\big(\{(U,F)\}\big) = \sum_{\substack{Z \subset U,\ \text{irred} \\ \text{codim}(Z)=1}} \text{ord}_Z(F) \cdot [Z] = \text{ord}_V(F) \cdot [V].$$

Note that $\text{ord}_Z(F) \neq 0$ implies $Z = V$, since V is irreducible. □

Remark **(II.2.5).** The proof of this result is based on an idea extracted from [**Ful98**, Ex. 1.7.6].

Theorem (II.2.6). *Let X be a V-manifold. The notion of Cartier and Weil divisor coincide over $\mathbb{Q}$. More precisely, the linear map*

$$T_X \otimes 1 : \text{CaDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \text{WeDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism of $\mathbb{Q}$-vector spaces. In particular, for a given Weil divisor D on X, there always exists $k \in \mathbb{Z}$ such that $kD \in \text{CaDiv}(X)$.

Proof. By Proposition (I.1.19), the variety X is normal and then Theorem (II.1.7) applies. Therefore the linear map $T_X \otimes 1$ is injective and its image is the $\mathbb{Q}$-vector space generated by the locally principal Weil divisors on X.

Let $V \subset X$ be an irreducible subvariety of codimension one. Consider $\{U_i\}_i$ an open covering of X such that U_i is analytically isomorphic to B_i/G_i where $B_i \subset \mathbb{C}^n$ is an open ball and G_i is a finite subgroup of $GL(n, \mathbb{C})$. By Lemma (II.2.4), $\text{Cl}(U_i) \otimes \mathbb{Q} = 0$ for all i.

Thus $[V|_{U_i}]$ is principal as an element in $\text{WeDiv}(U_i) \otimes_{\mathbb{Z}} \mathbb{Q}$ which implies that V is locally principal over $\mathbb{Q}$; hence it belongs to the image of $T_X \otimes 1$. □

Definition (II.2.7). Let X be a V-manifold. The vector space of $\mathbb{Q}$-Cartier divisors is identified under T_X with the vector space of $\mathbb{Q}$-Weil divisors. A $\mathbb{Q}$-*divisor* on X is an element in $\text{CaDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q} = \text{WeDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. The set of all $\mathbb{Q}$-divisors on X is denoted by $\mathbb{Q}$-$\text{Div}(X)$.

II.2–1. Writing a Weil divisor as a $\mathbb{Q}$-Cartier divisor

Following the proofs of Lemma (II.2.4) and Theorem (II.2.6), every Weil divisor on X can locally be written as $\mathbb{Q}$-Cartier divisor like

$$[V|_U] = \frac{1}{\text{ord}_V(F)}\{(U,F)\}$$

where $F = \prod_{\sigma \in G} f^\sigma$ and $V \cap U = \{f = 0\}$ with $f : B \to \mathbb{C}$ being holomorphic on an open ball and satisfying (10).

The rest of this section is devoted to explicitly calculating $\operatorname{ord}_V(F)$. First, in Proposition (II.2.8), it is shown that F is essentially a power of f, if the latter is chosen properly. Then, $\operatorname{ord}_V(F)$ is computed in Proposition (II.2.12).

Proposition (II.2.8). *Let $f : B \to \mathbb{C}$ be a non-zero holomorphic function on an open ball $B \subset \mathbb{C}^n$ such that the germ $f_x \in \mathcal{O}_{B,x}$ is reduced for all $x \in B$. Let G be a finite subgroup of $GL(n, \mathbb{C})$ acting on B. As above, consider*

$$F = \prod_{\sigma \in G} f^\sigma$$

where $f^\sigma(\mathbf{x}) = f(\sigma \cdot \mathbf{x})$ for $\sigma \in G$.

The following conditions are equivalent:

(1) $\forall P \in B$, $\big[f(P) = 0 \implies f(\sigma \cdot P) = 0, \forall \sigma \in G\big]$.

(2) $\forall \sigma \in G$, $\exists h_\sigma \in \Gamma(B, \mathcal{O}_B^*)$ *such that* $f^\sigma = h_\sigma f$.

(3) $\exists h \in \Gamma(B, \mathcal{O}_B^*)$ *such that* $F = hf^{|G|}$.

(4) $\exists k \geq 1$, $\exists h \in \Gamma(B, \mathcal{O}_B^*)$ *such that* $hf^k \in \Gamma(B/G, \mathcal{O}_{B/G})$.

Proof. For (1) $\Rightarrow$ (2), first note that $f^\sigma \in IV(f)$. Now fix $x \in B$. Since f_x is reduced, there exists a holomorphic function h on a small enough open neighborhood of x such that as germs $(f^\sigma)_x = h_x f_x$. The order of the converging power series $(f^\sigma)_x$ and f_x are equal because the action is linear. Thus h_x is a unit in $\mathcal{O}_{B,x}$. In particular, $\frac{f^\sigma}{f}$ is holomorphic and does not vanish at $x \in B$.

For (2) $\Rightarrow$ (3), consider $h = \prod_{\sigma \in G} h_\sigma$. Then one has

$$F = \prod_{\sigma \in G} f^\sigma = \prod_{\sigma \in G} (h_\sigma f) = \left(\prod_{\sigma \in G} h_\sigma \right) \cdot f^{|G|} = hf^{|G|}.$$

For (3) $\Rightarrow$ (4), since $F : B/G \to \mathbb{C}$ is analytic, take $k = |G|$. Finally, note that $\forall P \in B$,

$$f(P) = 0 \Longleftrightarrow (hf^k)(P) = (hf^k)(\sigma \cdot P) = 0 \Longleftrightarrow f(\sigma \cdot P) = 0.$$

Hence (4) $\Rightarrow$ (1) follows and the proof is complete. □

The following example shows that the reduceness condition in the statement of the previous result is necessary.

Example (II.2.9). Let $f = (x^2 + y)(x^2 - y)^3 \in \mathbb{C}[x, y]$ and consider the cyclic quotient space $M = X(2; 1, 1)$. Then $\{f = 0\} \subset M$ defines a zero set, i.e. condition (1) holds, but there are no $k \geq 1$ and $h \in \Gamma(B, \mathcal{O}_B^*)$ such that hf^k is a well-defined function over M.

Remark **(II.2.10).** If the holomorphic function $f : B \to \mathbb{C}$ in Proposition (II.2.8) is given by a polynomial, then the condition $[f_x \in \mathcal{O}_{B,x}$ reduced $\forall x \in B]$ holds if and only if f is reduced as a polynomial. In such a case, the holomorphic nowhere-vanishing function h_σ (and hence h) above is a non-zero constant. Therefore $f^{|G|}$ itself (without multiplying by a unit) is a well-defined analytic function on B/G, cf. (IV.4.1).

The situation of Remark (II.2.10) above is specific for polynomials and, in general, it does not apply in the holomorphic case, as the following example indicates.

Example (II.2.11). Let $f : \mathbb{C}^2 \to \mathbb{C}$ be the holomorphic function given by $f = e^x xy$ and consider the quotient space $M = X(2;1,1)$. Then f defines a zero set on M and it verifies the four equivalent conditions of Proposition (II.2.8). However, there is no $k \geq 1$ such that f^k induces a function over M. As it is said, this happens because f is not a polynomial.

Proposition (II.2.12). *Let $B \subset \mathbb{C}^n$ be an open ball and G a finite subgroup of $GL(n,\mathbb{C})$ acting on B. Let $V \subset B/G =: U$ be an irreducible subvariety of codimension one and consider*

$$F = \prod_{\sigma \in G} f^\sigma$$

where $f : B \to \mathbb{C}$ is a holomorphic function defining V.

If G is small and f is chosen so that $f_x \in \mathcal{O}_{B,x}$ is reduced $\forall x \in B$, then $\operatorname{ord}_V(F : U \to \mathbb{C}) = |G|$.

Proof. Choose $[P] \in V$ such that $[P]$ is smooth in U and $(V,[P])$ defines an irreducible germ, then $\operatorname{ord}_V(F) = \operatorname{ord}_{V,[P]}(F)$, see (II.1.5).

By Theorem (I.1.4), since G is small and $[P] \in U$ is smooth, using the covering $\pi : B \to U$, one finds an isomorphism of germs

$$(U,[P]) \cong (B/G_P,[P]) = (B,P)$$

induced by the identity map[2]. The germ $(V,[P])$ is converted under this isomorphism into (W,P) where W is the zero set of $f_P \in \mathcal{O}_{B,P}$.

On the other hand, by Proposition (II.2.8), there exists $h \in \Gamma(B,\mathcal{O}_B^*)$ such that $F = hf^{|G|}$. Putting all together the wanted order is

$$\operatorname{ord}_{V,[P]}(F : U \to \mathbb{C}) = \operatorname{ord}_{V(f_P),P}(hf^{|G|} : B \to \mathbb{C}) = |G|$$

as claimed. □

[2]See also Lemma (I.1.16) where the abelian case in treated in detail.

Remark **(II.2.13).** Recall that $\pi : B \to B/G =: U$ denotes the projection. Without any condition on G and f (i.e. neither G small nor $f_x \in \mathcal{O}_{B,x}$ reduced $\forall x \in B$ are required), the order can still be computed as follows

$$\operatorname{ord}_V(F) = \sum_i \deg(W_i/V) \cdot \operatorname{ord}_{W_i}(f),$$

where the W_i's are the irreducible components of $\pi^{-1}(V)$ (assumed to be a finite number) and $\deg(W_i/V)$ is the degree of the restriction mapping $\pi|_{W_i} : W_i \to V$.

Note that under the assumption of Proposition (II.2.12), $\operatorname{ord}_{W_i}(f) = 1$ and $\sum_i \deg(W_i/V) = |G|$.

(II.2.14). Here we summarize how to write a Weil divisor as an element in $\operatorname{CaDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ where X is an algebraic V-manifold.

(1) Write $D = \sum_{i \in I} a_i[V_i] \in \operatorname{WeDiv}(X)$, where $a_i \in \mathbb{Z}$ and $V_i \subset X$ irreducible. Also choose $\{U_j\}_{j \in J}$ an open covering of X such that $U_j = B_j/G_j$ where $B_j \subset \mathbb{C}^n$ is an open ball and G_j is a **small** finite subgroup of $GL(n, \mathbb{C})$.

(2) For each $(i,j) \in I \times J$ choose a polynomial $f_{i,j} : U_j \to \mathbb{C}$ satisfying the condition $[(f_{i,j})_x \in \mathcal{O}_{B_j,x}$ **reduced** $\forall x \in B_j]$ and such that $V_i \cap U_j = \{f_{i,j} = 0\}$. Then,

$$[V_i|_{U_j}] = \frac{1}{|G_j|}\{(U_j, f_{i,j}^{|G_j|})\}.$$

(3) Identifying $\{(U_j, f_{i,j}^{|G_j|})\}$ with its image under the natural inclusion $\operatorname{CaDiv}(U_j) \hookrightarrow \operatorname{CaDiv}(X)$, one finally writes D as a sum of locally principal Cartier divisors over $\mathbb{Q}$,

$$D = \sum_{(i,j) \in I \times J} \frac{a_i}{|G_j|}\{(U_j, f_{i,j}^{|G_j|})\}.$$

We finish this section with an example where the exceptional divisor of a weighted blow-up (which is in general just a Weil divisor) is explicitly written as a $\mathbb{Q}$-Cartier divisor.

Example (II.2.15). Let X be a surface with abelian quotient singularities. Let $\pi : \widehat{X} \to X$ be the weighted blow-up at a point of type $(d; a, b)$ with respect to $\omega = (p, q)$. In general, the exceptional divisor $E := \pi^{-1}(0) \cong \mathbb{P}^1_\omega(d; a, b)$ is a Weil divisor on $\widehat{X}$ which does not correspond to a Cartier divisor. Let us write E as an element in $\operatorname{CaDiv}(\widehat{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

As in I.3–1(iii), assume $\pi = \pi_{(d;a,b),\omega} : \widehat{X(d;a,b)}_\omega \to X(d;a,b)$. Assume also that $\gcd(p,q) = 1$ and $(d;a,b)$ is normalized, see Remark (I.3.14). Using the notation in that remark, the space $\widehat{X}$ is covered by $\widehat{U}_1 \cup \widehat{U}_2$ and the first chart is given by

$$(11)\qquad \begin{array}{rcl} Q_1 := X\left(\frac{pd}{e}; 1, \frac{-q+\beta pb}{e}\right) & \longrightarrow & \widehat{U}_1, \\ \big[(x^e, y)\big] & \mapsto & \big[((x^p, x^q y), [1:y]_\omega)\big]_{(d;a,b)}, \end{array}$$

where $e = \gcd(d, pb - qa)$. See (8) for details.

In the first chart, E is the Weil divisor $\{x = 0\} \subset Q_1$. Note that the type representing the space Q_1 is in a normalized form and hence the corresponding subgroup of $GL(2,\mathbb{C})$ is small.

Following the discussion (II.2.14), the divisor $\{x = 0\} \subset Q_1$ is written as an element in $\mathrm{CaDiv}(Q_1) \otimes_{\mathbb{Z}} \mathbb{Q}$ like $\frac{e}{pd}\{(Q_1, x^{\frac{pd}{e}})\}$, which is mapped to

$$\frac{e}{pd}\{(\widehat{U}_1, x^d)\} \in \mathrm{CaDiv}(\widehat{U}_1) \otimes_{\mathbb{Z}} \mathbb{Q}$$

under the isomorphism (11).

Analogously, E in the second chart is $\frac{e}{qd}\{(\widehat{U}_2, y^d)\}$. Finally, one writes the exceptional divisor of π as claimed,

$$\begin{aligned} E &= \frac{e}{dp}\big\{(\widehat{U}_1, x^d), (\widehat{U}_2, 1)\big\} + \frac{e}{dq}\big\{(\widehat{U}_1, 1), (\widehat{U}_2, y^d)\big\} \\ &= \frac{e}{dpq}\big\{(\widehat{U}_1, x^{dq}), (\widehat{U}_2, y^{dp})\big\}. \end{aligned}$$

Example (II.2.16). Now consider the ambient space to be the quotient weighted projective line $\mathbb{P}^1_\omega(d;a,b) = \widehat{V}_1 \cup \widehat{V}_2$ which is isomorphic to $\mathbb{P}^1$ under the map

$$\begin{array}{rcl} \mathbb{P}^1_\omega(d;a,b) & \longrightarrow & \mathbb{P}^1 \\ {[x:y]_\omega} & \mapsto & [x^{dq/e} : y^{dp/e}], \end{array}$$

where $e = \gcd(dp, dq, pb - qa)$. Here we are not assuming $(d;a,b)$ is normalized or $\gcd(p,q) = 1$, cf. I.3–1(iii).

Then, for instance, the Weil divisor $e \cdot [\{v = 0\}]$ on $\mathbb{P}^1_\omega(d;a,b)$ corresponds to the Cartier divisor

$$\left\{\left(\widehat{V}_1, \frac{v^{dp}}{u^{dq}}\right), \left(\widehat{V}_2, 1\right)\right\}.$$

Note that in this example Cartier and Weil divisors coincide over $\mathbb{Z}$, since the quotient weighted projective line $\mathbb{P}^1_\omega(d;a,b)$ is smooth.

Section § **II.3**

Holomorphic Line Bundles and their Sections

Let $F : Y \to X$ be a morphism between two irreducible complex analytic varieties. The *pull-back* of a Cartier divisor $D = \{(U_i, f_i)\}_{i \in I}$ on X can be defined by pulling back the local equations of D as

$$F^*(D) = \left\{(F^{-1}(U_i), f_i \circ F|_{F^{-1}(U_i)}\right\}_{i \in I}$$

and it is a Cartier divisor on Y provided $F(Y) \not\subseteq \mathrm{Supp}(D)$. Moreover, F^* respects sums of divisors and preserves linear equivalence.

The main purpose of this section is to define $F^*(D)$ without any restriction on the support of D, so that F^* gives rise to a group homomorphism between $\mathrm{Pic}(X)$ and $\mathrm{Pic}(Y)$, cf. (II.4.1) and Definition (II.4.2). To do so, the relationship between Cartier divisors and holomorphic line bundles is needed, see Theorem (II.3.12) below. Recall that the pull-back of a divisor is the main object to develop an intersection theory.

II.3–1. Line bundle associated with a Cartier divisor

Definition (II.3.1). A surjective holomorphic map $\pi : E \to X$ is called *complex* (*or holomorphic*) *line bundle* on X if it is a complex vector bundle of rank one, that is, there exists an open covering $\{U_i\}_{i \in I}$ of X satisfying:

- For every $i \in I$ there is a biholomorphic map
$$\Phi_i : \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{C}$$
such that $(\mathrm{pr}_1 \circ \Phi_i)(e) = \pi(e)$ for $e \in \pi^{-1}(U_i)$, where pr_1 is the projection $U_i \times \mathbb{C} \to U_i$ and,
- the restriction $\Phi_i| : \pi^{-1}(x) \to \{x\} \times \mathbb{C}$ is an isomorphism of vector spaces.

The pair (U_i, Φ_i) is called a *local trivialization*. For two local trivializations (U_i, Φ_i) and (U_j, Φ_j) the map

$$\Phi_i \circ \Phi_j^{-1} : (U_i \cap U_j) \times \mathbb{C} \longrightarrow (U_i \cap U_j) \times \mathbb{C}$$

induces a holomorphic function (called *transition function*)

$$\phi_{ij} : U_i \cap U_j \longrightarrow \mathbb{C}^*$$

such that $(\Phi_i \circ \Phi_j^{-1})(x, t) = (x, \phi_{ij}(x)t)$ for $x \in U_i \cap U_j$ and $t \in \mathbb{C}$. The transition functions satisfy the following compatibility conditions. Let $i, j, k \in I$ and $x \in U_i \cap U_j \cap U_k \neq \emptyset$, then one has

$$\phi_{ij}(x)\phi_{jk}(x) = \phi_{ik}(x) \quad \left[\Longrightarrow \phi_{ii}(x) = 1,\ \phi_{ji}(x)^{-1} = \phi_{ij}(x)\right]. \tag{12}$$

(II.3.2). Conversely, suppose now that we are given an open covering $\{U_i\}_{i\in I}$ of X and a family of holomorphic functions $\{\phi_{i,j} : U_i \cap U_j \to \mathbb{C}^*\}_{i,j\in I}$ verifying the previous compatibility conditions (12). Then one can construct a line bundle having $\{\phi_{ij}\}_{i,j}$ as transition functions. An outline of the construction is as follows. Consider $E = \coprod_{i\in I}(U_i \times \mathbb{C})$ and define an equivalence relation by setting

$$(x,t)_i \sim (y,s)_j \iff x = y \quad \text{and} \quad t = \phi_{ij}(x)s.$$

The fact that this is a well-defined equivalence relation is a consequence of the compatibility conditions (12). One shows that the quotient $E^\phi = E/\sim$ under this relation is a complex analytic variety and the projection $\pi : E^\phi \to X$, $[(x,t)_i] \mapsto x$ is a line bundle with transition functions $\{\phi_{ij}\}_{i,j}$. In fact, the map

$$\begin{array}{ccc} \pi^{-1}(U_i) & \longrightarrow & U_i \times \mathbb{C} \\ [(x,t)_j] & \mapsto & (x, \phi_{ij}(x)t), \end{array}$$

defines a local trivialization on U_i; its inverse is given by the obvious map $(x,t) \mapsto [(x,t)_i]$.

Remark **(II.3.3).** Both constructions are inverse to each other. Let $\pi : E \to X$ be a line bundle with local trivializations $\{(U_i, \Phi_i)\}_{i\in I}$ and transition functions $\{\phi_{ij}\}_{i,j\in I}$. Then the map $E^\phi \to E$, $[(x,t)_i] \mapsto \Phi_i^{-1}(x,t)$ is an analytic isomorphism of line bundles.

Definition (II.3.4). Two families of holomorphic functions associated with an open covering $\{U_i\}_{i\in I}$ of X,

$$\{\phi_{ij} : U_i \cap U_j \to \mathbb{C}^*\}_{i,j\in I}, \quad \{\psi_{ij} : U_i \cap U_j \to \mathbb{C}^*\}_{i,j\in I},$$

are said to satisfied the *coboundary condition* if there exists another family of holomorphic functions $\{\alpha_i : U_i \to \mathbb{C}^*\}_i$ such that $\psi_{ij} = \frac{\alpha_i}{\alpha_j}\phi_{ij}$ on $U_i \cap U_j$, $\forall i,j \in I$ with $U_i \cap U_j \neq \emptyset$.

(II.3.5). In the bijection $(\pi : E \to X) \longleftrightarrow (\{\phi_{ij} : U_i \cap U_j \to \mathbb{C}^*\}_{i,j})$, the notion of isomorphic line bundles corresponds to the notion of families satisfying the coboundary condition.

More precisely, let $F : E \to E'$ be an isomorphism of line bundles. Consider $\{(U_i, \Phi_i)\}_{i\in I}$ and $\{\phi_{ij}\}_{i,j}$ (resp. $\{(U_i, \Psi_i)\}_{i\in I}$ and $\{\psi_{ij}\}_{i,j}$) a local trivializing cover and the transition functions of E (resp. E'). Then one has that

$$\psi_{ij} = \frac{\alpha_i}{\alpha_j} \cdot \phi_{ij}, \quad \text{on} \quad U_i \cap U_j \neq \emptyset, \quad \forall i,j \in I,$$

where the holomorphic nowhere-vanishing functions $\alpha_i : U_i \to \mathbb{C}^*$ are induced by $(\Psi_i \circ F \circ \Phi_i^{-1})(x,t) = (x, \alpha_i(x)t)$.

Conversely, let $\{\phi_{ij} : U_i \cap U_j \to \mathbb{C}^*\}_{i,j}$ and $\{\psi_{ij} : U_i \cap U_j \to \mathbb{C}^*\}_{i,j}$ be two families satisfying the coboundary condition, say for $\{\alpha_i : U_i \to \mathbb{C}^*\}_i$. Consider E^ϕ and E^ψ the respective line bundles obtained as in (II.3.2). The map $F : E^\phi \to E^\psi$ given by $[(x,t)_i] \mapsto [(x, \alpha_i(x)t)_i]$ is well defined and it gives an isomorphism of line bundles.

Definition (II.3.6). Let $D = \{(U_i, f_i)\}_{i\in I}$ be a Cartier divisor on X. The *line bundle associated with* D is the bundle with transition functions

$$\phi_{ij} = \frac{f_i}{f_j} : U_i \cap U_j \longrightarrow \mathbb{C}^*, \quad \forall i,j \in I.$$

It is denoted by $\mathcal{O}_X(D)$ and is well defined up to isomorphisms of line bundles by (II.3.5).

The set of line bundles (up to isomorphism) with the tensor product forms an abelian group. The trivial line bundle[3] $\mathcal{O}_X$ is the identity element and the inverse element is given by the dual line bundle E^*. This operation has a good behavior with respect to the sum of Cartier divisors.

Lemma (II.3.7). *Let D and E be two Cartier divisors on X and consider $h : X \to \mathbb{C}$ a non-zero meromorphic function. The following properties hold:*

(1) $\mathcal{O}_X(\{(X,h)\}) \simeq \mathcal{O}_X$.
(2) $\mathcal{O}_X(D+E) \simeq \mathcal{O}_X(D) \otimes \mathcal{O}_X(E)$.
(3) $\mathcal{O}_X(-D) \simeq \mathcal{O}_X(D)^*$.
(4) $D \sim E \Longleftrightarrow \mathcal{O}_X(D) \simeq \mathcal{O}_X(E)$. □

This shows that the map $\text{Pic}(X) \to \{\text{line bundles}\}/\simeq$, defined by $[D] \mapsto \mathcal{O}_X(D)$ is an injective group homomorphism. It is also surjective whenever a non-zero global meromorphic section exists for a given line bundle, see below (II.3.9).

II.3–2. Meromorphic sections of a line bundle

Definition (II.3.8). Given a line bundle $\pi : E \to X$ and an open set $U \subset X$, a holomorphic (resp. meromorphic) map $s : U \to E$ is said to be a *holomorphic* (resp. *meromorphic*) *section* if the composition $\pi \circ s$ is the inclusion $U \hookrightarrow X$. When $U = X$ the section is called *global.* The sheaf of meromorphic sections is a $\mathcal{K}_X$-module.

[3] Note the symbol $\mathcal{O}_X$ denotes both the structure sheaf of X as a complex analytic variety, and the trivial line bundle $X \times \mathbb{C} \to \mathbb{C}$. This is justified because the sheaf of sections of the trivial line bundle is identified with the structure sheaf.

(II.3.9). Let $\{\phi_{ij} : U_i \cap U_j \to \mathbb{C}^*\}_{i,j\in I}$ be the transition functions of a line bundle $\pi : E \to X$. Consider a family of holomorphic maps $f_i : U_i \to \mathbb{C}$, $i \in I$, satisfying the compatibility conditions

$$f_j(x)\phi_{ij}(x) = f_i(x), \quad x \in U_i \cap U_j \neq \emptyset.$$

Then $\{f_i\}_{i\in I}$ defines a global holomorphic section $s : X \to E$, since each f_i gives a section of $U_i \times \mathbb{C}$ and this pulls back by the trivialization to a section of $\pi^{-1}(U_i)$. The compatibility conditions imposed on $\{f_i\}_{i\in I}$ ensure that these sections of $\pi^{-1}(U_i)$ agree on $U_i \cap U_j$. Conversely, any global holomorphic section gives rise to a family as above. The functions f_i on U_i are determined by the equation $(\Phi_i \circ s)(x) = (x, f_i(x))$ for $x \in U_i$.

In the preceding paragraph the word "holomorphic" can be replaced by the word "meromorphic". In particular, any non-zero global meromorphic section $s : X \to E$ will give an open covering $\{U_i\}_{i\in I}$ of X and a family of non-zero meromorphic functions $\{f_i : U_i \to \mathbb{C}\}_{i\in I}$ such that the quotient

$$\phi_{ij} = \frac{f_i}{f_j} : U_i \cap U_j \longrightarrow \mathbb{C}^* \quad (i, j \in I)$$

is holomorphic and never vanishes. Therefore s determines a Cartier divisor on X, denoted by $(s) := \{(U_i, f_i)\}_{i\in I}$.

This implies that the map $(\mathrm{Pic}(X), +) \to (\{\text{line bundles}\}/\simeq, \otimes)$ above defined by $[D] \mapsto \mathcal{O}_X(D)$ is a group isomorphism if there exist non-zero global meromorphic sections for any given line bundle.

Lemma (II.3.10). *Let $s_1, s_2 : X \to E$ be two non-zero global meromorphic sections of a line bundle $\pi : E \to X$.*

(1) *There exists $h : X \to \mathbb{C}$ a non-zero global meromorphic function such that $s_2 = hs_1$ and hence $(s_2) = (s_1) + \{(X, h)\}$.*

(2) *$(s_1) = (s_2)$ if and only if $h \in \Gamma(\mathcal{O}_X^*, X)$. In such a case, the sections are called* equivalent*.*

(3) *The Cartier divisor is effective if and only if the section is holomorphic.* □

Definition (II.3.11). Let $D = \{(U_i, f_i)\}_{i\in I}$ be a Cartier divisor on X. The *canonical section associated with D* is the non-zero global meromorphic section of $\mathcal{O}_X(D)$ defined by the collection $\{f_i : U_i \to \mathbb{C}\}_{i\in I}$. It is denoted by $s_D : X \to \mathcal{O}_X(D)$.

Consider a family of holomorphic functions $\{\alpha_i : U_i \to \mathbb{C}^*\}_{i\in I}$. Then $\{(U_i, \alpha_i f_i)\}_{i\in I}$ defines the same Cartier divisor as D above. The associated line bundles are isomorphic by (II.3.5), and the isomorphism respects the sections. As a consequence, one has the following result.

Theorem (II.3.12). *Given two pairs (E,s) and (E',s') of line bundles with sections, let us write $(E,s)\sim(E',s')$ if there exists an isomorphism of line bundles $F: E\to E'$ such that $s'=F\circ s$.*

Then there is a natural bijection

$$\mathrm{CaDiv}(X) \longrightarrow \left\{ \left(\textit{line bundle}, \begin{array}{c} \textit{non-zero global} \\ \textit{mero. section} \end{array} \right) \right\} \Big/ \sim$$

given by $D\mapsto[(\mathcal{O}_X(D),s_D)]$; its inverse is $[(E,s)]\mapsto(s)$. □

Remark **(II.3.13).** A couple of comments about the theorem:

- Define another equivalence relation by setting $(E,s)\sim_2(E',s')$ if there exists an isomorphism of line bundles $F: E\to E'$ such that the sections $F\circ s$ and s' are equivalent. Then $(E,s)\sim(E',s')$ if and only if $(E,s)\sim_2(E',s')$.
- Since there is an isomorphism between $\mathrm{Pic}(X)$ and line bundles modulo isomorphisms, taking classes modulo being linearly equivalent on the left-hand side corresponds to forgetting the section on the right-hand side.

Section § II.4
Pull-Back of a $\mathbb{Q}$-Divisor

(II.4.1). (Pull-back of a line bundle). Let $F: Y\to X$ be a morphism between two irreducible complex analytic varieties. Let $\pi: E\to X$ be a complex line bundle with local trivialization cover $\{(U_i,\Phi_i)\}_{i\in I}$ and transition functions $\{\phi_{ij}\}_{i,j\in I}$.

Then its *pull-back*, denoted by $F^*\pi: F^*E\to Y$,

$$\left[\ F^*E := \{(y,e)\in Y\times E \mid F(y)=\pi(x)\} =: Y\times_X E\ \right]$$

is a complex line bundle with local trivialization $\{(F^{-1}(U_i),\Psi_i)\}_{i\in I}$, where

$$\begin{array}{rrcl} \Psi_i: & (F^*\pi)^{-1}(F^{-1}(U_i)) & \longrightarrow & F^{-1}(U_i)\times\mathbb{C}, \\ & (y,e) & \mapsto & (y, pr_2\Phi_i(e)), \end{array}$$

and transition functions

$$\left\{\phi_{ij}\circ F|_{F^{-1}(U_i\cap U_j)} : F^{-1}(U_i\cap U_j)\to\mathbb{C}^*\right\}_{i,j}.$$

The inverse of Ψ_i is given by $(y,t)\mapsto(y,\Phi_i^{-1}(F(y),t))$.

As for the behavior with respect to the sections, if $s : X \to E$ is a non-zero global meromorphic section of E defined by a collection of meromorphic functions $\{f_i : U_i \to \mathbb{C}\}_{i \in I}$, then its *pull-back*, denoted by $F^*s : Y \to F^*E$,

$$(F^*s)(y) := ((s \circ F)(y), y)$$

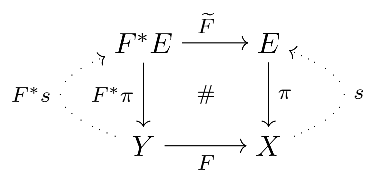
is the global meromorphic section of F^*E associated with

$$\left\{f_i \circ F|_{F^{-1}(U_i)} : F^{-1}(U_i) \to \mathbb{C}\right\}_{i \in I}.$$

Moreover, F^*s is the zero section of F^*E if and only if

$$F(Y) \subseteq \operatorname{Supp}(s) := \operatorname{Supp}((s)).$$

The following diagram represents the pull-back of a line bundle with a global meromorphic section. Note that locally $\widetilde{F}$ is $F \times 1_{\mathbb{C}}$.



Definition (II.4.2). Let $F : Y \to X$ be a morphism between irreducible complex analytic varieties. Let D be a Cartier divisor on X and consider $[D]$ its equivalence class in $\operatorname{Pic}(X)$. Define $F^*[D]$ to be the equivalence class in $\operatorname{Pic}(Y)$ of the divisor associated with any non-zero global meromorphic section of the bundle $F^*\mathcal{O}_X(D)$, i.e. $F^*[D] = [(t)]$ where t is a non-zero meromorphic section as above.

Remark **(II.4.3).** The pull-back is well defined and it has the following properties:

(1) In our setting, there always exist non-zero global meromorphic sections of a line bundle of the form $F^*\mathcal{O}_X(D)$.

(2) The pull-back $F^*[D] \in \operatorname{Pic}(Y)$ only depends on the equivalence class of D. Assume $D \sim D'$ and consider t and t' two non-zero global meromorphic sections of $F^*\mathcal{O}_X(D)$ and $F^*\mathcal{O}_X(D')$, respectively. Then, using Lemma (II.3.10)(1) and the functoriality of the pull-back, one sees that $[(t)] = [(t')] \in \operatorname{Pic}(Y)$.

(3) If $F(Y) \nsubseteq \operatorname{Supp}(D)$, then $F^*[D]$ coincides with the one given at the beginning of this section. This follows from (II.4.1) and the fact that $t = F^*s_D$ is a **non-zero** global meromorphic section of $F^*\mathcal{O}_X(D)$. Hence Definition (II.4.2) gives rise to a group homomorphism $F^* : \operatorname{Pic}(X) \to \operatorname{Pic}(Y)$ as claimed.

(4) The pull-back is a contravariant functor, that is, if Z is another irreducible complex analytic variety and $G : Z \to Y$ is a morphism, then $(G \circ F)^* = F^* \circ G^*$.

Although $F : Y \to X$ induces a group homomorphism between the Picard groups of X and Y, in practice it is convenient to treat the following two cases separately: (Here D is a Cartier divisor on X)

- If $F(Y) \nsubseteq \operatorname{Supp}(D)$, then $F^*(D) \in \operatorname{CaDiv}(Y)$.
- Otherwise $F^*(D)$ is only defined up to linear equivalence.

This approach is essentially the one presented by Fulton in [**Ful98**, Ch. 2] where the notion of pseudo-divisor is introduced. There, if $F(Y) \subseteq \operatorname{Supp}(D)$, then the pull-back $F^*[D]$ is defined as the equivalence class in $\operatorname{Pic}(Y)$ of any Cartier divisor E on Y whose line bundle $\mathcal{O}_Y(E)$ is isomorphic to $F^*\mathcal{O}_X(D)$.

Definition (II.4.4). Let $F : Y \to X$ be a morphism between two irreducible V-manifolds and consider $D \in \mathbb{Q}\text{-Div}(X)$. Then D can be written as a finite sum $\sum_{i=1}^r \alpha_i D_i$ where $D_i \in \operatorname{CaDiv}(X)$ and $\alpha_i \in \mathbb{Q}$. The *pull-back* of D is defined as

$$F^*(D) := \sum_{i=1}^r \alpha_i \cdot F^*(D_i),$$

where $F^*(D_i)$ is the pull-back of a Cartier divisor as in (II.4.2).

Hence $F^*(D)$ is an element in $\operatorname{CaDiv}(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$ if $F(Y) \nsubseteq |D_i|$, for all $i = 1, \ldots, r$, and it is only defined up to $\mathbb{Q}$-linear equivalence if $F(Y) \subseteq |D_{i_0}|$ for some $i_0 \in \{1, \ldots, r\}$. In any case,

$$[F^*(D)] \in \operatorname{Pic}(Y) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Now we have all the necessary ingredients to develop a rational intersection theory on varieties with quotient singularities. Chapter III is devoted to working out all the details, but first the following illustrative example will be given.

Example (II.4.5). Let $X = X(2; 1, 1)$ and consider the Weil divisors $D_1 = \{x = 0\}$ and $D_2 = \{y = 0\}$. Let us compute the Weil divisor associated with $j_{D_1}^* D_2$, where $j_{D_1} : D_1 \hookrightarrow X$ is the inclusion. Following (II.2.14), the divisor D_2 can be written as $\frac{1}{2}\{(X, y^2)\}$. By definition, since $D_1 \nsubseteq D_2$, the pull-back is

$$j_{D_1}^* D_2 = \frac{1}{2}\big\{(D_1, y^2|_{D_1})\big\}.$$

Thus its associated Weil divisor is

$$T_{D_1}(j_{D_1}^* D_2) = \frac{1}{2} \sum_{P \in D_1} \mathrm{ord}_P(y^2|_{D_1}) \cdot [P]$$
$$= \frac{1}{2} \mathrm{ord}_{[(0,0)]}(y^2|_{D_1}) \cdot [(0,0)] = \frac{1}{2} \cdot [(0,0)].$$

Note that there is an isomorphism $D_1 = X(2;1) \simeq \mathbb{C}$, $[y] \mapsto y^2$, and the function $y^2 : D_1 \to \mathbb{C}$ is converted into the identity map $\mathbb{C} \to \mathbb{C}$ under this isomorphism. Hence $\mathrm{ord}_{[(0,0)]}(y^2|_{D_1}) = 1$. It is natural to define the (global and local) intersection multiplicity as

$$D_1 \cdot D_2 = (D_1 \cdot D_2)_{[(0,0)]} = \frac{1}{2}.$$

III

Intersection Theory on Surfaces with Quotient Singularities

Previously in Chapter I we saw how useful weighted blow-ups can be to compute embedded **Q**-resolutions. In this chapter, to study this special kind of resolutions, we develop an intersection theory on varieties with quotient singularities.

Roughly speaking, given X a complex analytic variety, the intersection product $D \cdot E$ is well understood whenever D is a compact Weil divisor on X and E is a Cartier divisor on X. Over varieties with quotient singularities the notion of Cartier and Weil divisor coincide after tensoring with $\mathbb{Q}$, see Theorem (II.1.7), and hence a rational intersection theory can be defined on this kind of varieties.

This theory was first introduced by Mumford on normal surfaces, see [**Mum61**]. We give an alternative equivalent definition, without involving an embedded resolution of the ambient space, that allows us to compute the self-intersection numbers of the exceptional divisors of weighted blow-ups in dimension two. Also Bézout's theorem for quotients of weighted projective planes is studied.

See [**AMO11b**] for further applications including the computation of abstract resolutions of surfaces via Jung method. Also, see [**AMO11c**] for an overview on this chapter and [**Ort10**] for a more direct approach.

Section § III.1
Intersection Numbers: Generalities

Base on Example (II.4.5) the intersection number of two $\mathbb{Q}$-divisors is defined in terms of the degree map as follows.

(III.1.1). (Degree of a $\mathbb{Q}$-divisor). Let $\mathcal{C}$ be an irreducible analytic curve. Given a Weil divisor on $\mathcal{C}$ with finite support, $D = \sum_{i=1}^{r} n_i \cdot [P_i]$, its *degree* is defined as

$$\deg(D) = \sum_{i=1}^{r} n_i \in \mathbb{Z}.$$

It is a group homomorphism. Moreover, if $\mathcal{C}$ is compact, the degree of a principal divisor is zero and thus passes to the quotient yielding the map $\deg : \mathrm{Cl}(\mathcal{C}) \to \mathbb{Z}$, cf. [**Ful98**, Prop. 1.4].

The *degree of a Cartier divisor* is the degree of its associated Weil divisor, that is, by definition

$$\deg(D) := \deg(T_{\mathcal{C}} D).$$

Finally, extending to rational coefficients, one obtains a group homomorphism

$$\deg : \Big\{ D \in \mathbb{Q}\text{-}\mathrm{Div}(\mathcal{C}) \text{ with finite support} \Big\} \longrightarrow \mathbb{Q} \tag{13}$$

that passes to the quotient $\mathrm{Pic}(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{Q}$ when the curve is compact.

Definition (III.1.2). Let X be a V-manifold of dimension 2 and consider $D_1, D_2 \in \mathbb{Q}\text{-}\mathrm{Div}(X)$. If D_1 is irreducible, then the *intersection number*, denoted by $D_1 \cdot D_2$, is defined as

$$D_1 \cdot D_2 := \deg \left(j_{D_1}^* D_2 \right) \in \mathbb{Q},$$

where $j_{D_1} : |D_1| \hookrightarrow X$ denotes the inclusion and deg is the map in (13). The expression above extends linearly if D_1 is a finite sum of irreducible $\mathbb{Q}$-divisors.

Following (III.1.1) and Definition (II.4.4), this number is only well defined if either $|D_1| \nsubseteq |D_2|$ and $|D_1| \cap |D_2|$ is finite, or the divisor D_1 has a compact support.

Let us discuss these two cases separately. To simplify assume D_1 is an irreducible $\mathbb{Q}$-divisor.

- If D_1 has compact support, then extending the order function to rational coefficients $\mathrm{ord}_P : \mathrm{CaDiv}(|D_1|) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$, one writes the intersection number $D_1 \cdot D_2$ as

$$\deg(E) = \deg \left(\sum_{P \in D_1} \mathrm{ord}_P(E) \cdot [P] \right) = \sum_{P \in D_1} \mathrm{ord}_P(E),$$

  where E is any $\mathbb{Q}$-Cartier divisor on $|D_1|$ representing the rational class $[j_{D_1}^* D_2] \in \mathrm{Pic}(|D_1|) \otimes_{\mathbb{Z}} \mathbb{Q}$.

- If $|D_1| \not\subseteq |D_2|$, then $j_{D_1}^* D_2 \in \operatorname{CaDiv}(|D_1|) \otimes_{\mathbb{Z}} \mathbb{Q}$ and its support is the set $|D_1| \cap |D_2|$. In this situation the order at P its-self
$$(D_1 \cdot D_2)_P := \operatorname{ord}_P \left(j_{D_1}^* D_2 \right) \in \mathbb{Q}$$
is well defined and it is called the *local intersection number* at P. In addition, if $|D_1| \cap |D_2|$ is finite, then by definition
$$D_1 \cdot D_2 = \sum_{P \in |D_1| \cap |D_2|} (D_1 \cdot D_2)_P.$$

If D_1 is not irreducible, then the local intersection number $(D_1 \cdot D_2)_P$ is extended by linearity so that the previous formula still holds.

In the following result the main usual properties of intersection numbers are collected. Its proofs is omitted since it is well known for the classical case (i.e. without tensoring with $\mathbb{Q}$), cf. [**Ful98**], and our generalization is based on extending the classical definition to rational coefficients.

Proposition (III.1.3). *Let X be a V-manifold of dimension 2 and consider $D_1, D_2, D_3 \in \mathbb{Q}$-$\operatorname{Div}(X)$. Then the local and the global intersection numbers, provided the indicated operations make sense according to Definition (III.1.2), satisfy the following properties: ($\alpha \in \mathbb{Q}$, $P \in X$)*

(1) **Bilinear:**

Global
$$\begin{aligned} & D_1 \cdot (D_2 + D_3) = D_1 \cdot D_2 + D_1 \cdot D_3 \\ & (D_1 + D_2) \cdot D_3 = D_1 \cdot D_3 + D_2 \cdot D_3 \\ & (\alpha D_1) \cdot D_2 = D_1 \cdot (\alpha D_2) = \alpha (D_1 \cdot D_2) \end{aligned}$$

Local
$$\begin{aligned} & \big(D_1 \cdot (D_2 + D_3)\big)_P = (D_1 \cdot D_2)_P + (D_1 \cdot D_3)_P \\ & \big((D_1 + D_2) \cdot D_3\big)_P = (D_1 \cdot D_3)_P + (D_2 \cdot D_3)_P \\ & \big((\alpha D_1) \cdot D_2\big)_P = (D_1 \cdot (\alpha D_2))_P = \alpha (D_1 \cdot D_2)_P \end{aligned}$$

(2) **Commutative:** *If $D_1 \cdot D_2$ and $D_2 \cdot D_1$ are both defined, then $D_1 \cdot D_2 = D_2 \cdot D_1$. Analogously $(D_1 \cdot D_2)_P = (D_2 \cdot D_1)_P$ if both local numbers are defined.*

(3) **Non-negative:** *Assume D_1 and D_2 are effective, irreducible, and distinct. Then $D_1 \cdot D_2$ and $(D_1 \cdot D_2)_P$ are greater than or equal to zero if they are defined. Moreover, $(D_1 \cdot D_2)_P = 0$ if and only if $P \notin |D_1| \cap |D_2|$, and hence $D_1 \cdot D_2 = 0$ if and only if $|D_1| \cap |D_2| = \emptyset$.*

(4) **Non-rational:** *If $D_2 \in \operatorname{CaDiv}(X)$ and $D_1 \in \operatorname{WeDiv}(X)$, then $D_1 \cdot D_2$ and $(D_1 \cdot D_2)_P$ are integral numbers. By the commutative property, the same holds if D_1 is a Cartier divisor and D_2 is a Weil divisor.*

(5) **$\mathbb{Q}$-Linear equivalence:** *Assume D_1 has compact support. If D_2 and D_3 are $\mathbb{Q}$-linearly equivalent, i.e. $[D_2] = [D_3] \in \operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, then $D_1 \cdot D_2 = D_1 \cdot D_3$. Due to the commutativity, the roles of D_1 and D_2 can be exchanged. In particular, $D_1 \cdot D_2 = 0$ for every principal $\mathbb{Q}$-divisor D_2.*

(6) **Normalization:** *Let $\nu : \widetilde{|D_1|} \to |D_1|$ be the normalization of the support of D_1 and $j_{D_1} : |D_1| \hookrightarrow X$ the inclusion. Then $D_1 \cdot D_2 = \deg \left(j_{D_1} \circ \nu\right)^* D_2$. Observe that in this situation the normalization is a smooth complex analytic curve.* □

Remark **(III.1.4).** This rational intersection number was first introduced by Mumford for normal surfaces, see [**Mum61**, Pag. 17]. Our Definition (III.1.2) coincides with Mumford's because it has good behavior with respect to the pull-back, see Theorem (III.1.5). The main advantage is that ours does not involve a resolution of the ambient space and, for instance, this allows us to easily find formulas for the self-intersection numbers of the exceptional divisors of weighted blow-ups, without computing any resolution, see Proposition (III.3.2).

The following result (*the pull-back formula*) is essential for obtaining Bézout's Theorem on quotients of weighted projective planes as well as for studying the local intersection number on $X(\mathbf{d}; A)$. Again its proofs follows from the fact that our generalization is based on extending the classical definition to rational coefficients.

Theorem (III.1.5). *Let $F : Y \to X$ be a proper morphism between two irreducible V-manifolds of dimension 2, and $D_1, D_2 \in \mathbb{Q}\text{-}\operatorname{Div}(X)$.*

(1) *The cardinal of $F^{-1}(P)$, $P \in X$, is finite and generically constant. This generic number is denoted by $\deg(F)$.*

(2) *If $D_1 \cdot D_2$ is defined, then so is $F^*(D_1) \cdot F^*(D_2)$. In such a case, one has $F^*(D_1) \cdot F^*(D_2) = \deg(F)\,(D \cdot E)$.*

(3) *If $(D_1 \cdot D_2)_P$ is defined for some $P \in X$, then so is the local number $(F^*(D_1) \cdot F^*(D_2))_Q$, $\forall Q \in F^{-1}(P)$. In such a case, it is verified that $\sum_{Q \in F^{-1}(P)} (F^*(D_1) \cdot F^*(D_2))_Q = \deg(F)(D_1 \cdot D_2)_P$.* □

The rest of this section is devoted to reviewing some classical results concerning the intersection multiplicity, namely the computation of the local intersection number at a smooth point, the self-intersection numbers of the exceptional divisors of blow-ups at a smooth point, and the classical Bézout's Theorem on $\mathbb{P}^2$. Afterward, these results are generalized in the upcoming sections.

(III.1.6). (Local intersection number at a smooth point). Let X be a smooth analytic surface. Consider D_1, D_2 two effective (Cartier or Weil)[1] divisors on X and $P \in X$ a point. From Remark (II.2.3), the divisor D_i is locally given by a holomorphic function f_i, $i = 1, 2$, in a neighborhood of P. Then $(D_1 \cdot D_2)_P$ equals

$$\operatorname{ord}_P(f_2|D_1) = \operatorname{length}_{\mathcal{O}_{D_1,P}}\left(\frac{\mathcal{O}_{D_1,P}}{f_2|_{D_1}}\right) = \dim_{\mathbb{C}}\left(\frac{\mathcal{O}_{X,P}}{\langle f_1, f_2\rangle}\right).$$

Moreover, X being a smooth variety, $\mathcal{O}_{X,P}$ is isomorphic to $\mathbb{C}\{x, y\}$ and hence the previous dimension can be computed, for instance, by means of Gröbner bases with respect to local orderings.

(III.1.7). (Classical blow-up at a smooth point). Let X be an analytic surface. Let $\pi : \widehat{X} \to X$ be the classical blow-up at a smooth point P. Consider C and D two (Cartier or Weil) divisors on X with multiplicities m_C and m_D at P. Denote by E the exceptional divisor of π, and by $\widehat{C}$ (resp. $\widehat{D}$) the strict transform of C (resp. D). Then,

(1) $E \cdot \pi^*(C) = 0$,

(2) $\pi^*(C) = \widehat{C} + m_C E$,

(3) $E \cdot \widehat{C} = m_C$,

(4) $E^2 = -1$,

(5) $\widehat{C} \cdot \widehat{D} = C \cdot D - m_C m_D$.

In addition, if D has compact support, then $\widehat{D}^2 = D^2 - m_D^2$. Note that the exceptional divisor has multiplicity 1 at every point. This is why for the self-intersection numbers of the exceptional divisors every time we blow up a point on them, when computing an embedded resolution of a plane curve, one only has to subtract 1.

Example (III.1.8). The fourth property can easily be deduced assuming the first three. Let us prove it here by using directly Definition (III.1.2). Assume $X = \mathbb{C}^2$ and $\pi : \widehat{\mathbb{C}}^2 \to \mathbb{C}^2$ is the blow-up at the origin. By definition, $E^2 = \deg(j_E^* E) = \deg(t)$, where $t : E \to j_E^* \mathcal{O}_X(E)$ is any non-zero global meromorphic section of $j_E^* \mathcal{O}_X(E)$.

- Let us cover $\widehat{\mathbb{C}}^2$ by $U_1 \cup U_2$ and use coordinates $((x, y), [u : v])$ for $\mathbb{C}^2 \times \mathbb{P}^1$. As a Cartier divisor, the exceptional divisor of π is $E = \{(U_1, x), (U_2, y)\}$.

[1] Recall that on smooth analytic varieties, Cartier and Weil divisors are identified and their equivalence classes coincide under this identification, i.e. $\operatorname{Pic}(X) = \operatorname{Cl}(X)$, see Theorem (II.1.7).

- Then $\mathcal{O}_X(E)$ is the line bundle on $\widehat{\mathbb{C}}^2$ with transition function $\phi_{12}: U_1 \cap U_2 \to \mathbb{C}^*$, $\phi_{12}(((x,y),[u:v])) = \frac{x}{y}$. Thus $j_E^*\mathcal{O}_X(E)$ is the line bundle on $E = V_1 \cup V_2$ with transition function

$$\psi_{12} : V_1 \cap V_2 \longrightarrow \mathbb{C}^*, \quad \psi_{12}([u:v]) = \frac{u}{v}.$$

- The family $\{(V_1, \frac{u}{v}), (V_2, 1)\}$ gives rise to a non-zero global meromorphic section of $j_E^*\mathcal{O}_X(E)$. Its associated Weil divisor on $\mathbb{P}^1$ is $-\{v=0\} \in \operatorname{WeDiv}(\mathbb{P}^1)$ which has degree -1.

Another way to proceed is to show directly that the dual of $j_E^*\mathcal{O}_X(E)$ is isomorphic to the line bundle on E associated with the Weil divisor $\{v=0\}$.

(III.1.9). (Bézout's Theorem on $\mathbb{P}^2$). Every analytic (Cartier or Weil) divisor on $\mathbb{P}^2$ is algebraic and thus it can be written as a difference of two effective divisors. On the other hand, every effective divisor is defined by a homogeneous polynomial. The *degree of an effective divisor on* $\mathbb{P}^2$ is the degree, $\deg(F)$, of the corresponding homogeneous polynomial. This degree map is extended linearly yielding a group homomorphism $\deg : \operatorname{Div}(\mathbb{P}^2) \to \mathbb{Z}$ that characterizes the linear equivalence classes in the following sense: $\forall D_1, D_2 \in \operatorname{Div}(\mathbb{P}^2)$,

$$(14) \qquad [D_1] = [D_2] \in \operatorname{Pic}(\mathbb{P}^2) = \operatorname{Cl}(\mathbb{P}^2) \iff \deg(D_1) = \deg(D_2).$$

Let D_1, D_2 be two divisors on $\mathbb{P}^2$, then $D_1 \cdot D_2 = \deg(D_1)\deg(D_2)$. In particular, the self-intersection number of a divisor D on $\mathbb{P}^2$ is given by $D^2 = \deg(D)^2$. In addition, if $|D_1| \not\subseteq |D_2|$, then $|D_1| \cap |D_2|$ is a finite set of points and, by the discussion after Definition (III.1.2), one has

$$\deg(D_1)\deg(D_2) = D_1 \cdot D_2 = \sum_{P \in |D_1| \cap |D_2|} (D_1 \cdot D_2)_P.$$

The proof of this result is an easy consequence of (III.1.3), and the fact that D_i is linearly equivalent to $\deg(D_i)L_i$, where L_i is a linear form, $i = 1, 2$, by (14). The rest of this chapter is devoted to generalizing the classical results of (III.1.6), (III.1.7), and (III.1.9) to V-manifolds, weighted blow-ups, and quotients of weighted projective planes, respectively.

Section § III.2
Computing Local Intersection Numbers

Let X be an algebraic V-manifold of dimension 2. Consider D_1 and D_2 two effective $\mathbb{Q}$-divisors on X, and $P \in X$ a point. From (II.2.3), cf. proof of Lemma (II.2.4), the divisor D_i is locally given in a neighborhood of P by a reduced polynomial f_i, $i = 1, 2$.

On the other hand, the point P can be assumed to be a normalized type of the form $(d;a,b)$. Hence the computation of $(D_1 \cdot D_2)_P$ is reduced to the following particular case.

(III.2.1). (Local intersection number on $X(d;a,b)$). Denote by X the cyclic quotient space $X(d;a,b)$ and consider two divisors $D_1 = \{f_1 = 0\}$ and $D_2 = \{f_2 = 0\}$ given by reduced polynomials. Assume that,

- $(d;a,b)$ is normalized,
- D_1 is irreducible,
- f_1 induces a function on X,
- $|D_1| \nsubseteq |D_2|$.

Then as $\mathbb{Q}$-Cartier divisors $D_1 = \{(X, f_1)\}$, $D_2 = \frac{1}{d}\{(X, f_2^d)\}$, and the pull-back is $j_{D_1}^* D_2 = \frac{1}{d}\{(D_1, f_2^d|_{D_1})\}$. Following the definition, the local number $(D_1 \cdot D_2)_{[P]}$ equals

$$\frac{1}{d}\operatorname{ord}_{[P]}(f_2^d|_{D_1}) = \frac{1}{d}\operatorname{length}_{\mathcal{O}_{D_1,[P]}}\left(\frac{\mathcal{O}_{D_1,[P]}}{\langle f_2^d|_{D_1}\rangle}\right) = \frac{1}{d}\dim_{\mathbb{C}}\left(\frac{\mathcal{O}_{X,[P]}}{\langle f_1, f_2^d\rangle}\right).$$

The local ring $\mathcal{O}_{X,[P]}$ is described in detail in (I.1.17); there is an isomorphism of local rings if $P = (\alpha,\beta) \neq (0,0)$,

$$\begin{array}{ccc} \mathcal{O}_{X,[P]} & \stackrel{\cong}{\longrightarrow} & \mathcal{O}_{\mathbb{C}^2,(0,0)} \\ (x,y) & \mapsto & (x+\alpha, y+\beta), \end{array}$$

and for $P = (0,0)$ one has $\mathcal{O}_{X,[(0,0)]} \cong \mathbb{C}\{x,y\}^{\mu_d}$.

Also $\frac{1}{d}\dim_{\mathbb{C}}(\mathbb{C}\{x,y\}/\langle f_1, f_2^d\rangle)$ coincides with $\dim_{\mathbb{C}} \mathbb{C}\{x,y\}/\langle f_1, f_2\rangle$ when f_1 and f_2 are converging power series. So finally,

$$(D_1 \cdot D_2)_{[P]} = \begin{cases} \dfrac{1}{d}\dim_{\mathbb{C}}\left(\dfrac{\mathbb{C}\{x,y\}^{\mu_d}}{\langle f_1, f_2^d\rangle}\right), & P = (0,0)\,; \\ \dim_{\mathbb{C}}\left(\dfrac{\mathbb{C}\{x-\alpha, y-\beta\}}{\langle f_1, f_2\rangle}\right), & P = (\alpha,\beta) \neq (0,0)\,. \end{cases}$$

Analogously, if f_1 does not define a function on X, for computing the intersection number at $[(0,0)]$, one substitutes f_1 by f_1^d and divides the result by d.

Another way to calculate $(D_1 \cdot D_2)_{[(0,0)]}$ is to consider the projection $\operatorname{pr} : \mathbb{C}^2 \to X(d;a,b)$ and apply the local pull-back formula, see Theorem (III.1.5)(3). Indeed, let $\widetilde{D}_i$ be the pull-back divisor of D_i under the projection, $i = 1,2$. Then,

$$(D_1 \cdot D_2)_{[(0,0)]} = \frac{1}{d}(\widetilde{D}_1 \cdot \widetilde{D}_2)_{(0,0)} = \frac{1}{d}\dim_{\mathbb{C}}\left(\frac{\mathbb{C}\{x,y\}}{\langle f_1, f_2\rangle}\right).$$

In particular, combining these two expressions obtained for $(D_1 \cdot D_2)_{[(0,0)]}$, if two polynomials f and g define functions on X, then

$$\dim_{\mathbb{C}} \left(\frac{\mathbb{C}\{x,y\}^{\mu_d}}{\langle f,g \rangle} \right) = \frac{1}{d} \dim_{\mathbb{C}} \left(\frac{\mathbb{C}\{x,y\}}{\langle f,g \rangle} \right).$$

As in the smooth case, all the preceding dimensions can be computed by means of Gröbner bases with respect to local orderings.

Example (III.2.2). Let $X = X(2;1,1)$ and consider the Weil divisors $D_1 = \{x=0\}$ and $D_2 = \{y=0\}$. In Example (II.4.5), it is shown, by directly using the definition of the intersection product, that

$$(D_1 \cdot D_2)_{[(0,0)]} = \frac{1}{2}.$$

In this section two expressions have been obtained for computing this local number:

- $(D_1 \cdot D_2)_{[(0,0)]} = \frac{1}{2} \dim_{\mathbb{C}} \left(\frac{\mathbb{C}\{x,y\}}{\langle x,y \rangle} \right) = \frac{1}{2}.$
- $(D_1 \cdot D_2)_{[(0,0)]} = \frac{1}{4} \dim_{\mathbb{C}} \left(\frac{\mathbb{C}\{x,y\}^{\mu_2}}{\langle x^2,y^2 \rangle} \right) = \frac{1}{4} \cdot 2 = \frac{1}{2}.$

The isomorphism $\mathbb{C}\{z_1,z_2,z_3\}/\langle z_1z_2 - z_3^2 \rangle \to \mathbb{C}\{x^2,y^2,xy\} = \mathbb{C}\{x,y\}^{\mu_2}$ defined by $(z_1,z_2,z_3) \mapsto (x^2,y^2,xy)$ was used to prove that

$$\dim_{\mathbb{C}} \left(\frac{\mathbb{C}\{x,y\}^{\mu_2}}{\langle x^2,y^2 \rangle} \right) = \dim_{\mathbb{C}} \left(\frac{\mathbb{C}\{z_1,z_2,z_3\}}{\langle z_1,z_2,z_3^2 \rangle} \right) = 2.$$

Remark **(III.2.3).** If the point $P \in X$ is represented by a type of the form $(\mathbf{d};A)$, where $A \in Mat(r \times 2, \mathbb{Z})$, one considers the natural projection $\mathrm{pr} : \mathbb{C}^2 \to X(\mathbf{d};A)$ and applies the pull-back formula as above. Hence in general one has

$$(D_1 \cdot D_2)_{[(0,0)]} = \frac{1}{\deg[\ \mathrm{pr} : \mathbb{C}^2 \to X(\mathbf{d};A)]} (\widetilde{D}_1 \cdot \widetilde{D}_2)_{(0,0)},$$

where $\widetilde{D}_i$ is the pull-back divisor of D_i under the projection. A generalization of Lemma (III.4.2) could be useful in this sense.

Section § III.3
Intersection Numbers and Weighted Blow-ups

In chapter I weighted blow-ups were introduced as a tool for computing embedded **Q**-resolutions. To obtain information about the corresponding embedded singularity, an intersection theory on V-manifolds has been developed.

Here we calculate self-intersection numbers of exceptional divisors of weighted blow-ups on analytic varieties with abelian quotient singularities, see Proposition (III.3.2).

A preliminary lemma is presented separately so that the proof of the main result of this section becomes simpler.

Lemma (III.3.1). *Let $h : Y \to X$ be a morphism between two irreducible V-manifolds of dimension 2.*

Consider $\pi_X : \widehat{X} \to X$ (resp. $\pi_Y : \widehat{Y} \to Y$) a weighted blow-up at a point of X (resp. Y) and take C_X a $\mathbb{Q}$-divisor on X. Denote by E_X (resp. E_Y) the exceptional divisor of π_X (resp. π_Y), and $\widehat{C}_X$ the strict transform of C_X.

Let us suppose that there exist two rational numbers, e and ν, and a finite proper morphism $H : \widehat{Y} \to \widehat{X}$ completing the commutative diagram

$$\begin{array}{ccc} \widehat{Y} & \xrightarrow{H} & \widehat{X} \\ {\scriptstyle \pi_Y}\downarrow & \# & \downarrow{\scriptstyle \pi_X} \\ Y & \xrightarrow[h]{} & X \end{array}$$

such that:

(a) $H^*(E_X) = eE_Y$,
(b) $\pi_Y^*(h^*(C_X)) = H^*(\widehat{C}_X) + \nu E_Y$.

Then, the following hold:

(1) $\pi_X^*(C_X) = \widehat{C}_X + \frac{\nu}{e} E_X$,
(2) $E_X \cdot \widehat{C}_X = \frac{-e\,\nu}{\deg(h)} E_Y^2$,
(3) $E_X^2 = \frac{e^2}{\deg(h)} E_Y^2$.

Proof. For (1) note the total transform $\pi_X^*(C_X)$ can always be written as $\widehat{C}_X + mE_X$ for some $m \in \mathbb{Q}$. Considering its pull-back under H^*, one obtains two expressions for the same $\mathbb{Q}$-divisor on $\widehat{Y}$,

$$\begin{aligned} H^*(\pi_X^*(C_X)) &\overset{\text{diagram}}{=} \pi_Y^*(h^*(C_X)) \overset{\text{(b)}}{=} H^*(\widehat{C}_X) + \nu E_Y, \\ H^*(\widehat{C}_X + mE_X) &= H^*(\widehat{C}_X) + mH^*(E_X) \overset{\text{(a)}}{=} H^*(\widehat{C}_X) + meE_Y. \end{aligned}$$

It follows that $m = \frac{\nu}{e}$.

For (2) first note that $\deg(H) = \deg(h)$. From (III.3.2)(1), see below, one has that $E_Y \cdot \pi_Y^*(h^*(C_X)) = 0$. On the other hand, H being proper, Theorem (III.1.5)(2) can be applied thus obtaining

$$\begin{aligned} \deg(h)(E_X \cdot \widehat{C}_X) &= H^*(E_X) \cdot H^*(\widehat{C}_X) \overset{\text{(a)-(b)}}{=} \\ &= eE_Y \cdot \left[\pi_Y^*(h^*(C_X)) - \nu E_Y\right] \\ &= -e\nu E_Y^2. \end{aligned}$$

Analogously, $\deg(h)E_X^2 = H^*(E_X)^2 = e^2 E_Y^2$ and (3) follows. □

Now we are ready to present the main result of this section.

Proposition (III.3.2). *Let X be an analytic surface with abelian quotient singularities and let $\pi : \widehat{X} \to X$ be the (p,q)-weighted blow-up at a point $P \in X$ of type $(d; a, b)$. Assume* $\gcd(p,q) = 1$ *and $(d; a, b)$ is a normalized type, i.e.* $\gcd(d,a) = \gcd(d,b) = 1$*. Also write* $e = \gcd(d, pb - qa)$.

Consider two $\mathbb{Q}$-divisors C and D on X. As usual, denote by E the exceptional divisor of π, and by $\widehat{C}$ (resp. $\widehat{D}$) the strict transform of C (resp. D). Let ν and μ be the (p,q)-multiplicities of C and D at P, i.e. x (resp. y) has (p,q)-multiplicity p (resp. q).

Then, there are the following equalities:

(1) $E \cdot \pi^*(C) = 0$.

(2) $\pi^*(C) = \widehat{C} + \dfrac{\nu}{e} E$.

(3) $E \cdot \widehat{C} = \dfrac{e\nu}{dpq}$.

(4) $E^2 = -\dfrac{e^2}{dpq}$.

(5) $\widehat{C} \cdot \widehat{D} = C \cdot D - \dfrac{\nu\mu}{dpq}$.

In addition, if D has compact support then $\widehat{D}^2 = D^2 - \dfrac{\mu^2}{dpq}$.

Proof. Using Proposition (III.1.3)(5), the first item can be proved as in the smooth case since $\pi^*(C)$ is locally principal as $\mathbb{Q}$-divisor on $\widehat{X}$. The fifth item, and final conclusion, is an easy consequence of (2)-(4) and the fact that $\pi^*(C) \cdot \pi^*(D) = C \cdot D$.

For the rest of the proof, one assumes that

$$\pi = \pi_X : \widehat{X(d; a, b)}_\omega \longrightarrow X(d; a, b)$$

is the weighted blow-up at the origin of $X(d; a, b)$ with respect to $\omega = (p, q)$. Now the idea is to apply Lemma (III.3.1) to the commutative diagram

$$\begin{array}{ccc} \widehat{Y} := \widehat{\mathbb{C}}^2 & \xrightarrow{H} & \widehat{X(d; a, b)}_\omega =: \widehat{X} \\ \pi_Y \big\downarrow & \# & \big\downarrow \pi_X \\ Y := \mathbb{C}^2 & \xrightarrow[h]{} & X(d; a, b) =: X. \end{array}$$

Above, the morphisms H and h are defined by

$$\begin{array}{rcl} ((x,y),[u:v]) & \stackrel{H}{\longmapsto} & [((x^p,y^q),[u^p:v^q])_\omega]_{(d;a,b)}; \\ (x,y) & \stackrel{h}{\longmapsto} & [(x^p,y^q)]_{(d;a,b)}, \end{array}$$

and π_Y is the classical blowing-up at the origin. In this situation $E_Y^2 = -1$. The claim is reduced to the calculation of $\deg(h)$ and the verification of the conditions (a)-(b) in Lemma (III.3.1).

The degree is $\deg(h) = pq \cdot \deg\big[\mathrm{pr} : \mathbb{C}^2 \to X(d;a,b)\big] = dpq$. For (a), first recall the decompositions

$$\widehat{X(d;a,b)}_\omega = \widehat{U}_1 \cup \widehat{U}_2 \quad \text{and} \quad \widehat{\mathbb{C}}^2 = U_1 \cup U_2 \tag{15}$$

given by the non-cancellation of the variables u and v. By Example (II.2.15), one writes the exceptional divisor of π_X as

$$E_X = \frac{e}{dpq}\Big\{(\widehat{U}_1, x^{dq}), (\widehat{U}_2, y^{dp})\Big\}.$$

Hence its pull-back under H, computed by pulling back the local equations, is

$$H^*(E_X) = \frac{e}{dpq}\Big\{(U_1, x^{dpq}), (U_2, y^{dpq})\Big\} = e\Big\{(U_1, x), (U_2, y)\Big\} = eE_Y.$$

Finally, for (b), one uses local equations to check that $\pi_Y^*(h^*(C)) = H^*(\widehat{C}) + \nu E_Y$. Suppose the divisor C is locally given by a meromorphic function $f(x,y)$ defined on a neighborhood of the origin of $X(d;a,b)$; note that $\nu = \mathrm{ord}_{(p,q)}(f)$.

The charts associated with the decompositions (15) are described in detail in Section I.3–1(iii). As a summary, we recall here the first chart of each blowing-up:

$$\pi_X \left| \begin{array}{rcl} Q_1 := X\left(\begin{array}{c|cc} p & -1 & q \\ pd & a & pb-qa \end{array}\right) & \longrightarrow & \widehat{U}_1, \\ \big[(x,y)\big] & \mapsto & \big[((x^p, x^q y),[1:y]_\omega)\big]. \end{array}\right.$$

$$\pi_Y \left| \begin{array}{rcl} \mathbb{C}^2 & \longrightarrow & U_1, \\ (x,y) & \mapsto & ((x,xy),[1:y]). \end{array}\right.$$

Note that H respects the decompositions and takes the form $(x,y) \mapsto [(x,y^q)]$ in the first chart.

Then, one has the following local equations for the divisors involved:

Divisor	**Equation**	**Ambient space**
$h^*(C)$	$f(x^p, y^q) = 0$	$\mathbb{C}^2$
$\pi_Y^*(h^*(C))$	$f(x^p, x^q y^q) = 0$	$\mathbb{C}^2 \cong U_1$
$\widehat{C}$	$\dfrac{f(x^p, x^q y)}{x^\nu} = 0$	$Q_1 \cong \widehat{U}_1$
$H^*(\widehat{C})$	$\dfrac{f(x^p, x^q y^q)}{x^\nu} = 0$	$\mathbb{C}^2 \cong U_1$
E_Y	$x = 0$	$\mathbb{C}^2 \cong U_1$

From these local equations (b) is satisfied and the proof is complete. □

(III.3.3). Under the conditions of Proposition (III.3.2) and using the notation of its proof, let us compute the self-intersection number of the exceptional divisor of $\pi : \widehat{X} \to X$, following directly Definition (III.1.2) as in Example (III.1.8). Write $E = \frac{e}{dpq} E_1$, where

$$E_1 = \left\{ (\widehat{U}_1, x^{dq}), (\widehat{U}_2, y^{dp}) \right\}.$$

By definition, $E \cdot E_1 = \deg(j_E^* E_1) = \deg(t)$ where $t : E \to j_E^* \mathcal{O}_X(E_1)$ is any non-zero global meromorphic section of $j_E^* \mathcal{O}_X(E_1)$.

- The sheaf $\mathcal{O}_X(E_1)$ is the line bundle on $\widehat{X}$ with transition function $\phi_{12} : \widehat{U}_1 \cap \widehat{U}_2 \to \mathbb{C}^*$, $\phi_{12}(((x,y),[u:v]_\omega)) = \frac{x^{dq}}{y^{dp}}$. Thus $j_E^* \mathcal{O}_X(E_1)$ is the line bundle on $E = \widehat{V}_1 \cup \widehat{V}_2$ with transition function
$$\psi_{12} : \widehat{V}_1 \cap \widehat{V}_2 \to \mathbb{C}^*, \quad \psi_{12}([u:v]_\omega) = \frac{u^{dq}}{v^{dp}}.$$
- The family $\{(\widehat{V}_1, \frac{u^{dq}}{v^{dp}}), (\widehat{V}_2, 1)\}$ gives rise to a non-zero global meromorphic section of $j_E^* \mathcal{O}_X(E_1)$. Its associated Weil divisor on $\mathbb{P}^1_\omega / \mu_d$ is $-e \cdot [\{v = 0\}] \in \text{WeDiv}(\mathbb{P}^1_\omega(d; a, b))$ which has degree $-e$, compare with (II.2.16).

Consequently, $E^2 = \frac{e}{dpq}(E \cdot E_1) = -\frac{e^2}{dpq}$ as claimed. Another way to proceed is to show directly that the dual of $j_E^* \mathcal{O}_X(E_1)$ is isomorphic to the line bundle on E associated with the Weil divisor $e \cdot [\{v = 0\}]$.

(III.3.4). In the same spirit of the preceding example, let us calculate $E\cdot\widehat{C}$ using directly Definition (III.1.2) and the fact that

$$E\cdot\widehat{C}=\sum_{P\in E\ \cap\widehat{C}}(E\cdot\widehat{C})_P.$$

Suppose C is locally given by a meromorphic function $f(x,y)=0$ defined on a neighborhood of the origin of $X(d;a,b)$. Consider

$$f=f_\nu+f_{\nu+l}+\cdots$$

the decomposition of $f(x,y)$ into (p,q)-homogeneous parts. The global equation of $E\cap\widehat{C}=\{f_\nu=0\}\subset\mathbb{P}^1_\omega(d;a,b)$ can be written as

$$f_\nu(x,y)=x^\alpha y^\beta\prod_{i=1}^{k}(x^q-\varepsilon_i^q y^p)^{m_i}.$$

Note that $\nu=\operatorname{ord}_{(p,q)}(f)=p\alpha+q\beta+pq\sum_{i=1}^r m_i$. The intersection multiplicity of E and $\widehat{C}$ at the point $[\varepsilon_i:1]_\omega$ is m_i, while it is $\frac{\alpha e}{dq}$ (resp. $\frac{\beta e}{dp}$), not necessarily an integer, at the possibly singular point $[0:1]$ (resp. $[1:0]$). All these statements follows from §III.2, since by (I.3.14) the local equations of E and $\widehat{C}$ in the second chart are

$$X\left(\frac{dq}{e};\frac{-p+\delta qa}{e},1\right)\ \supseteq\ \begin{cases}E: & y=0;\\ \widehat{C}: & x^\alpha\prod_{i=1}^k(x^q-\varepsilon_i^q)^{m_i}=0,\end{cases}$$

where δ is the inverse of b modulo d. To compute the intersection multiplicity at $[1:0]$ the first chart is needed, but the details are omitted.

On the other hand, the isomorphism

$$\begin{array}{ccc}\mathbb{P}^1_\omega(d;a,b) & \longrightarrow & \mathbb{P}^1\\ [x:y]_\omega & \mapsto & [x^{dq/e}:y^{dp/e}],\end{array}$$

tells us that

$$[\varepsilon_i:1]_\omega=[\varepsilon_j:1]_\omega\in\mathbb{P}^1_\omega(d;a,b)\iff(\varepsilon_i^q)^{\frac{d}{e}}=(\varepsilon_j^q)^{\frac{d}{e}}.$$

Consequently, the cardinality of $E\cap\widehat{C}\setminus\{[0:1],[1:0]\}$ is $\frac{k}{d/e}$ and in fact one has

$$\sum_{i=1}^r m_i=\sum_{i=1}^r(E\cdot\widehat{C})_{[\varepsilon_i:1]_\omega}=\frac{d}{e}\sum_{P\neq[0:1],[1:0]}(E\cdot\widehat{C})_P.$$

Finally, collecting all the information above, it follows that

$$\sum_{E\cap\widehat{C}}(E\cdot\widehat{C})_P=(E\cdot\widehat{C})_{[0:1]}+(E\cdot\widehat{C})_{[1:0]}+\sum_{P\neq[0:1],[1:0]}(E\cdot\widehat{C})_P=$$

$$=\ \frac{\alpha e}{dq}\ +\ \frac{\beta e}{dp}\ +\ \frac{e}{d}\sum_{i=1}^r m_i\ =\ \frac{e}{dpq}\left(p\alpha+q\beta+pq\sum_{i=1}^r m_i\right)\ =\ \frac{e\nu}{dpq}.$$

Another way to proceed in order to calculate $E \cdot \widehat{C}$ is to realize that the required intersection product is the degree of the Weil divisor on $\mathbb{P}^1_\omega(d;a,b)$ given by $f_\nu(x,y) = x^\alpha y^\beta \prod_{i=1}^r (x^q - \varepsilon_i^q y^p)^{m_i}$. This expression is mapped to

$$x^{\frac{\alpha e}{dq}} y^{\frac{\beta e}{dp}} \prod_{i=1}^{k} (x^{\frac{e}{d}} - \varepsilon_i^q y^{\frac{e}{d}})^{m_i}$$

under the isomorphism $\mathbb{P}^1_\omega(d;a,b) \to \mathbb{P}^1$. The latter is clear to have degree $\frac{e\nu}{dpq}$ as a Weil divisor on $\mathbb{P}^1$.

Remark **(III.3.5).** Although elementary, the computation of the self-intersection numbers E^2 and $E \cdot \widehat{C}$ by using directly the definition is long and tedious. That is why Proposition (III.3.2) is proven with the pull-back formula (III.1.5) so that the proof becomes simpler and clearer.

Let us discuss two special cases of Prop. (III.3.2) according to I.3–1, namely the point $P \in X$ is smooth and the point P is of type $(d;p,q)$ with $\gcd(d,p) = \gcd(d,q) = 1$. Consider the ω-blow-up $\pi := \pi_\omega : \widehat{\mathbb{C}}^2_\omega \to \mathbb{C}^2$ (resp. $\pi := \pi_{\omega,d} : \widehat{\mathbb{C}}^2_{\omega,d} \to X(d;p,q)$). The following properties hold:

(1) $E \cdot \pi^*(C) = 0$ (in both cases).

(2) $\pi^*(C) = \widehat{C} + \nu E$ (resp. $\pi^*(C) = \widehat{C} + \frac{\nu}{d} E$).

(3) $E \cdot \widehat{C} = \frac{\nu}{pq}$ (in both cases).

(4) $E^2 = -\frac{1}{pq}$ (resp. $E^2 = -\frac{d}{pq}$).

(5) $\widehat{C} \cdot \widehat{D} = C \cdot D - \frac{\nu\mu}{pq}$ (resp. $\widehat{C} \cdot \widehat{D} = C \cdot D - \frac{\nu\mu}{dpq}$).

Remark **(III.3.6).** To state formulas when $P \in X$ is a point represented by a type of the form $(\mathbf{d};A)$, where $A \in Mat(r \times 2, \mathbb{Z})$, one proceeds as in the proof of Proposition (III.3.2). In particular, one has to compute $\deg(h)$, e, and ν.

For instance, to calculate e such that $H^*(E_X) = eE_Y$, one needs to write E_X as a $\mathbb{Q}$-Cartier divisor as in Example (II.2.15), or equivalently, to find the number e such that the map

$$\begin{array}{ccc} \big(X(\mathbf{d};A), [(0,1)]\big) & \longrightarrow & \big(\mathbb{C}^2, (0,1)\big) \\ [(x,y)] & \mapsto & (x^e, y) \end{array}$$

is an isomorphism of analytic germs. In fact, one can show that

$$e = \frac{\deg(\mathrm{pr})}{\deg(\mathrm{pr}\,|_{x=0})},$$

where $\mathrm{pr} : \mathbb{C}^2 \to Q_1$ is the projection on the first chart.

When $(\mathbf{d}; A) = (d; a, b)$ but the type is not necessarily normalized or $\gcd(p, q) \neq 1$, then

$$e = \frac{\gcd(dp, dq, pb - qa)}{\gcd(d, a, b)}.$$

For $\deg(h)$ a generalization of Lemma (III.4.2) is needed. The details are left to the reader.

Example (III.3.7). Let us consider the following divisors on $\mathbb{C}^2$,

$$C_1 = \{((x^3 - y^2)^2 - x^4 y^3) = 0\}, \quad C_2 = \{x^3 - y^2 = 0\},$$

$$C_3 = \{x^3 + y^2 = 0\}, \quad C_4 = \{x = 0\}, \quad C_5 = \{y = 0\}.$$

The local intersection numbers $(C_i \cdot C_j)_0$, $i, j \in \{1, \ldots, 5\}$, $i \neq j$, are encoded in the intersection matrix associated with any embedded $\mathbf{Q}$-resolution of $C = \bigcup_{i=1}^5 C_i$, see [**AMO11b**] for a proof of this result.

Let $\pi_1 : \mathbb{C}^2_{(2,3)} \to \mathbb{C}^2$ be the $(2,3)$-weighted blow-up at the origin. The new space has two cyclic quotient singular points of type $(2; 1, 1)$ and $(3; 1, 1)$ located at the exceptional divisor $\mathcal{E}_1$. The local equation of the total transform in the first chart is given by the function

$$x^{29}\ ((1 - y^2)^2 - x^5 y^3)\ (1 - y^2)\ (1 + y^2)\ y : \ X(2; 1, 1) \longrightarrow \mathbb{C},$$

where $x = 0$ is the equation of the exceptional divisor and the other factors correspond in the same order to the strict transform of C_1, C_2, C_3, C_5 (denoted again by the same symbol). To study the strict transform of C_4 one needs the second chart, the details are left to the reader.

Hence $\mathcal{E}_1$ has multiplicity 29 and self-intersection number $-\frac{1}{6}$; it intersects transversely C_3, C_4, and C_5 at three different points, while it intersects C_1 and C_2 at the same smooth point P, different from the other three. The local equation of the divisor $\mathcal{E}_1 \cup C_2 \cup C_1$ at this point P is

$$x^{29}\, y\, (x^5 - y^2) = 0,$$

see Figure III.1 below.

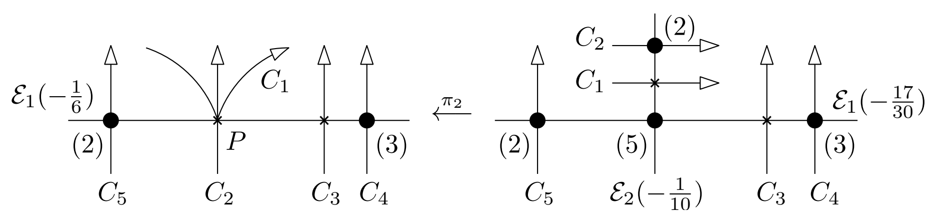


FIGURE III.1. Embedded $\mathbf{Q}$-resolution of $C = \bigcup_{i=1}^5 C_i \subset \mathbb{C}^2$.

Let π_2 be the $(2,5)$-weighted blow-up at the point P above. The new ambient space has two singular points of type $(2;1,1)$ and $(5;1,2)$. The local equations of the total transform of $\mathcal{E}_1 \cup C_2 \cup C_1$ are given by the following two functions:

1st chart
$\underbrace{x^{73}}_{\mathcal{E}_2} \cdot \underbrace{y}_{C_2} \cdot \underbrace{(1-y^2)}_{C_1} : X(2;1,1) \longrightarrow \mathbb{C}$

2nd chart
$\underbrace{x^{29}}_{\mathcal{E}_1} \cdot \underbrace{y^{73}}_{\mathcal{E}_2} \cdot \underbrace{(x^5-1)}_{C_1} : X(2;1,1) \longrightarrow \mathbb{C}$

Thus the new exceptional divisor $\mathcal{E}_2$ has multiplicity 73 and intersects transversely the strict transform of C_1, C_2, and $\mathcal{E}_1$. Hence the composition $\pi_2 \circ \pi_1$ is an embedded $\mathbf{Q}$-resolution of $C = \bigcup_{i=1}^5 C_i \subset \mathbb{C}^2$.

Figure III.1 above illustrates the whole process. As for the self-intersection numbers,

$$\mathcal{E}_2^2 = -\frac{1}{10}, \qquad \mathcal{E}_1^2 = -\frac{1}{6} - \frac{2^2}{1 \cdot 2 \cdot 5} = -\frac{17}{30}.$$

The intersection matrix associated with the embedded $\mathbf{Q}$-resolution obtained and its opposite inverse are

$$A = \begin{pmatrix} -17/30 & 1/5 \\ 1/5 & -1/10 \end{pmatrix}, \qquad B = -A^{-1} = \begin{pmatrix} 6 & 12 \\ 12 & 34 \end{pmatrix}.$$

Now one observes the intersection number is encoded in B as follows. For $i = 1, \ldots, 5$, set $k_i \in \{1, \ldots, 5\}$ such that $\emptyset \neq C_i \cap \mathcal{E}_{k_i} =: \{P_i\}$. Denote by $O(C_i)$ the order of the cyclic group acting on P_i. Then,

$$(C_i \cdot C_j)_0 = \frac{b_{k_i,k_j}}{O(C_i)\, O(C_j)}.$$

Looking at the figure one sees that

$$(k_1, \ldots, k_5) = (2,2,1,1,1),$$
$$(O(C_1), \ldots, O(C_5)) = (1,2,1,3,2).$$

Hence, for instance,

$$(C_1 \cdot C_2)_0 = \frac{b_{k_1,k_2}}{O(C_1)\, O(C_2)} = \frac{b_{22}}{1 \cdot 2} = \frac{34}{2} = 17,$$

which is indeed the intersection multiplicity at the origin of C_1 and C_2. Analogously for the other indices.

Remark **(III.3.8).** Consider the group action of type $(5;2,3)$ on $\mathbb{C}^2$. The previous plane curve C is invariant under this action and then it makes sense to compute an embedded $\mathbf{Q}$-resolution of $\overline{C} := C/\mu_5 \subset X(5;2,3)$. Similar calculations, as in the previous example, lead to a figure as the one obtained above with the following relevant differences:

- $\mathcal{E}_1 \cap \mathcal{E}_2$ is a smooth point.
- $\mathcal{E}_1$ (resp. $\mathcal{E}_2$) has self-intersection number $-\frac{17}{6}$ (resp. $-\frac{1}{2}$).
- The intersection matrix is $A' = \begin{pmatrix} -17/6 & 1 \\ 1 & -1/2 \end{pmatrix}$ and its opposite inverse is
$$B' = -(A')^{-1} = \begin{pmatrix} 6/5 & 12/5 \\ 12/5 & 34/5 \end{pmatrix}.$$

Hence, for instance,
$$(\overline{C}_1 \cdot \overline{C}_2)_0 = \frac{b'_{22}}{1 \cdot 2} = \frac{34/5}{2} = \frac{17}{5},$$
which is exactly the intersection number of these two curves, since that local number can also be computed as $(\overline{C}_1 \cdot \overline{C}_2)_0 = \frac{1}{5}(C_1 \cdot C_2)_0$. Analogous considerations hold for $(\overline{C}_i \cdot \overline{C}_j)_0$, $i,j = 1,\ldots,5$.

Section § III.4
Bézout's Theorem for Weighted Projective Planes

For a given weight vector $\omega = (p,q,r) \in \mathbb{N}^3$ and an action on $\mathbb{C}^3$ of type $(d;a,b,c)$, consider the quotient weighted projective plane
$$\mathbb{P}^2_\omega(d;a,b,c) := \mathbb{P}^2_\omega/\mu_d$$
and the corresponding morphism $\tau_{(d;a,b,c),\omega} : \mathbb{P}^2 \to \mathbb{P}^2_\omega(d;a,b,c)$ defined by

$$\tau_{(d;a,b,c),\omega}([x:y:z]) = [x^p : y^q : z^r]_\omega. \tag{16}$$

Recall that $\mathbb{P}^2_\omega(d;a,b,c)$ is a variety with abelian quotient singularities; its charts are described in (I.3.19). The *degree of a* $\mathbb{Q}$*-divisor on* $\mathbb{P}^2_\omega(d;a,b,c)$ is the degree of its pull-back under the map $\tau_{(d;a,b,c),\omega}$, that is, by definition,
$$D \in \mathbb{Q}\text{-Div}\left(\mathbb{P}^2_\omega(d;a,b,c)\right), \quad \deg_\omega(D) := \deg\left(\tau^*_{(d;a,b,c),\omega}(D)\right).$$

Thus if $D = \{F = 0\}$ is a $\mathbb{Q}$-divisor on $\mathbb{P}^2_\omega(d;a,b,c)$ given by a ω-homogeneous polynomial that indeed defines a zero set on the quotient projective space, then $\deg_\omega(D)$ is the classical degree, denoted by $\deg_\omega(F)$, of the quasi-homogeneous polynomial.

(III.4.1). The degree of a $\mathbb{Q}$-divisor on $\mathbb{P}^2_\omega(d;a,b,c)$ has the following behavior with respect to the normalization process of weighted projective planes.

- Let $\omega=(p,q,r)\in\mathbb{N}^3$ and $\omega'=\frac{1}{\gcd(p,q,r)}\omega$. Consider the morphism $\mathbb{P}^2_\omega\to\mathbb{P}^2_{\omega'}$ induced by the identity map. Let D' be a $\mathbb{Q}$-divisor on $\mathbb{P}^2_{\omega'}$ and D its pull-back under the previous map. Then,
$$\deg_{\omega'}(D')=\frac{1}{\gcd(p,q,r)}\deg_\omega(D).$$
- Let $\omega=(p,q,r)\in\mathbb{N}^3$ and $\omega'=\left(\frac{p}{(p,q)\cdot(p,r)},\frac{q}{(q,p)\cdot(q,r)},\frac{r}{(r,p)\cdot(r,q)}\right)$. Consider the morphism $\mathbb{P}^2_\omega\to\mathbb{P}^2_{\omega'}$ defined by
$$[x:y:z]_\omega\longmapsto[x^{\gcd(q,r)}:y^{\gcd(p,r)}:z^{\gcd(p,q)}]_{\omega'}.$$
Let D' be a $\mathbb{Q}$-divisor on $\mathbb{P}^2_{\omega'}$ and D its pull-back under the previous map which is a $\mathbb{Q}$-divisor on $\mathbb{P}^2_\omega$. Then,
$$\deg_{\omega'}(D')=\frac{\deg_\omega(D)}{\gcd(p,q)\cdot\gcd(p,r)\cdot\gcd(q,r)}.$$

The following result can be stated in a more general setting. However, it is presented in this way to keep the exposition as simple as possible.

Lemma (III.4.2). *The degree of the projection* $\mathrm{pr}:\mathbb{C}^2\to X\left(\begin{smallmatrix} d; & a & b\\ e; & r & s\end{smallmatrix}\right)$ *is given by the formula*
$$\frac{d\cdot e}{\gcd\big[d\cdot\gcd(e,r,s),\ e\cdot\gcd(d,a,b),\ as-br\big]}.$$

Proof. Assume $\gcd(d,a,b)=\gcd(e,r,s)=1$; the general formula is obtained easily from this one, since
$$\left(\begin{array}{c|cc} d & a & b\\ e & r & s\end{array}\right)=\left(\begin{array}{c|cc} \frac{d}{\gcd(d,a,b)} & \frac{a}{\gcd(d,a,b)} & \frac{b}{\gcd(d,a,b)}\\ \frac{e}{\gcd(e,r,s)} & \frac{r}{\gcd(e,r,s)} & \frac{s}{\gcd(e,r,s)}\end{array}\right).$$

The degree of the required projection $\mathbb{C}^2\to X\left(\begin{smallmatrix} d; & a & b\\ e; & r & s\end{smallmatrix}\right)$ is $\frac{de}{\ell}$, where ℓ is the order of the abelian group
$$H=\Big\{(\xi,\eta)\in\mu_d\times\mu_e\mid\xi^a\eta^r=1,\ \xi^b\eta^s=1\Big\}\lhd(\mu_d\times\mu_e).$$

To calculate ℓ, let us consider $(\xi,\eta)\in\mu_d\times\mu_e$ and solve the system
$$\begin{cases}\xi^a\eta^r=1,\\ \xi^b\eta^s=1.\end{cases}$$
Raising both equations to the e-th power, one obtains $\xi^{ae}=1$ and $\xi^{be}=1$. Hence,
$$\xi\in\mu_d\cap\mu_{ae}\cap\mu_{be}=\mu_{\gcd(d,ae,be)}=\mu_{\gcd(d,e)}.$$
Note that the assumption $\gcd(d,a,b)=1$ was used in the last equality. Analogously, it follows that $\eta\in\mu_{\gcd(d,e)}$, provided that $\gcd(e,r,s)=1$.

Thus there exist $i,j \in \{0,1,\ldots,\gcd(d,e)-1\}$ such that $\xi = \zeta^i$ and $\eta = \zeta^j$, where ζ is a fixed primitive (d,e)-th root of unity. Now the claim is reduced to finding the number of solutions of the system of congruences

$$\left\{ \begin{array}{rcl} ai + rj & \equiv & 0 \\ bi + sj & \equiv & 0 \end{array} \right., \qquad \big(\mathrm{mod} \;\; \gcd(d,e)\big).$$

This is known to be $\gcd(d,e,as-br)$ and now the proof is complete. □

Proposition (III.4.3). *Using the notation above, let us denote by m_1, m_2, m_3 the determinants of the three minors of order 2 of the matrix $\left(\begin{smallmatrix} p & q & r \\ a & b & c \end{smallmatrix}\right)$. Assume that $\gcd(p,q,r)=1$ and write $e = \gcd(d,m_1,m_2,m_3)$.*

Then, the intersection number of two $\mathbb{Q}$-divisors on $\mathbb{P}^2_\omega(d;a,b,c)$ is

$$D_1 \cdot D_2 = \frac{e}{dpqr} \deg_\omega(D_1) \deg_\omega(D_2) \in \mathbb{Q}.$$

In particular, the self-intersection number of a $\mathbb{Q}$-divisor is given by $D^2 = \frac{e}{dpqr}\deg_\omega(D)^2$. Moreover, if $|D_1| \not\subseteq |D_2|$, then $|D_1| \cap |D_2|$ is a finite set of points and

$$\frac{e}{dpqr} \deg_\omega(D_1) \deg_\omega(D_2) = \sum_{P \in |D_1| \cap |D_2|} (D_1 \cdot D_2)_P. \tag{17}$$

Proof. For simplicity, let us write just τ for the map in (16), omitting the subindex. Note that τ is a proper morphism between two irreducible V-manifolds of dimension 2. Thus by Theorem (III.1.5)(2) and the classical Bézout's theorem on $\mathbb{P}^2$ (III.1.9), one has the following sequence of equalities,

$$\begin{aligned} \deg(\tau)\,(D_1 \cdot D_2) &= \tau^*(D_1) \cdot \tau^*(D_2) \\ &= \deg\left(\tau^*(D_1)\right) \deg\left(\tau^*(D_2)\right) \\ &= \deg_\omega(D_1) \deg_\omega(D_2). \end{aligned}$$

The rest of the proof is the computation of $\deg(\tau)$; the final part is a consequence of discussion after Definition (III.1.2).

In the first chart τ takes the form $\mathbb{C}^2 \to X\left(\begin{smallmatrix} p & ; & q & r \\ pd & ; & m_1 & m_2 \end{smallmatrix}\right)$, $(y,z) \mapsto [(y^q, z^r)]$, see (I.3.19) for details. By decomposing this morphism into $\mathbb{C}^2 \to \mathbb{C}^2$, $(y,z) \mapsto (y^q,z^r)$ and the projection $\mathbb{C}^2 \to X\left(\begin{smallmatrix} p & ; & q & r \\ pd & ; & m_1 & m_2 \end{smallmatrix}\right)$, $(y,z) \mapsto [(y,z)]$, one obtains

$$\deg(\tau) = qr \cdot \deg\left[\mathbb{C}^2 \xrightarrow{\mathrm{pr}} X\left(\begin{smallmatrix} p & ; & q & r \\ pd & ; & m_1 & m_2 \end{smallmatrix}\right)\right].$$

The determinant of the corresponding matrix is $qm_2 - rm_1 = pm_3$. From Lemma (III.4.2), the latter degree is

$$\frac{p \cdot pd}{\gcd\left(p \cdot \gcd(pd,m_1,m_2), pd, pm_3\right)} = \frac{dp}{\gcd\left(d,m_1,m_2,m_3\right)},$$

and hence the proof is complete. □

Corollary (III.4.4). *Let X, Y, Z be the Weil divisors on the quotient space $\mathbb{P}^2_\omega(d;a,b,c)$ given by $\{x=0\}$, $\{y=0\}$, and $\{z=0\}$, respectively. Using the notation of Proposition (III.4.3), one has:*

(1) $X^2 = \dfrac{ep}{dqr}, \quad Y^2 = \dfrac{eq}{dpr}, \quad Z = \dfrac{er}{dpq}.$

(2) $X \cdot Y = \dfrac{e}{dr}, \quad X \cdot Z = \dfrac{e}{dq}, \quad Y \cdot Z = \dfrac{e}{dp}.$ □

Remark **(III.4.5).** Some comments about the previous results.

(1) The local intersection numbers $(D_1 \cdot D_2)_P$ in (17) are computed in (III.2.1) in terms of the dimension of a $\mathbb{C}$-vector space. This dimension can in turn be computed by means of Gröbner bases with respect to local orderings as usual.

(2) If $d=1$, then $e=1$ too and the formulas above become a bit simpler. In particular, one obtains Bézout's theorem on weighted projective planes, (the last equality if $|D_1| \nsubseteq |D_2|$ only)

$$D_1 \cdot D_2 = \frac{1}{pqr} \deg_\omega(D_1) \deg_\omega(D_2) = \sum_{P \in |D_1| \cap |D_2|} (D_1 \cdot D_2)_P.$$

(3) To state Bézout's theorem on $\mathbb{P}^2_\omega(\mathbf{d};A)$, where $A \in Mat(r \times 3, \mathbb{Z})$, one proceeds in the same way. First consider the natural morphism $\tau : \mathbb{P}^2 \to \mathbb{P}^2_\omega(\mathbf{d};A)$ defined by $[x:y:z] \mapsto [x^p : y^q : z^r]_\omega$, then apply the pull-back formula, and finally compute the degree of τ. That is, $\forall D_1, D_2 \in \mathbb{Q}\text{-}\operatorname{Div}\left(\mathbb{P}^2_\omega(\mathbf{d};A)\right)$, one has

$$D_1 \cdot D_2 = \frac{1}{\deg(\tau)} \deg_\omega(D_1) \deg_\omega(D_2).$$

The latter degree is reduced, as in the proof of Prop. (III.4.3), to the calculation of the degree of the projection $\mathbb{C}^2 \to X(\mathbf{e};B)$, $(y,z) \mapsto [(y,z)]$, where the type $(\mathbf{e};B)$ is obtained after taking charts on the corresponding projective planes. In this sense a generalization of Lemma (III.4.2) is welcome.

Example (III.4.6). Without assuming $\gcd(p,q,r)=1$ in (III.4.3), the degree of τ is $\frac{dpqr}{e}$ where $e = \gcd\left[d \cdot \gcd(p,q,r), m_1, m_2, m_3)\right]$. The general formula for the degree of $\tau : \mathbb{P}^2 \to \mathbb{P}^2_\omega(\mathbf{d};A)$ is left to the reader.

IV

Monodromy Zeta Function and Lefschetz Numbers

In this chapter the behavior of the Lefschetz numbers and the zeta function of the monodromy with respect to an embedded **Q**-resolution is investigated, cf. [**Mar11c**]. These two invariants have already been studied in different contexts by several authors. Hence before going into details, let us recall some of those approaches.

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of a non-constant analytic function and let $(H, 0)$ be the hypersurface singularity defined by f. Consider the Milnor fiber $F = \{x \in \mathbb{C}^{n+1} \ : \ ||x|| \leq \varepsilon,\ f(x) = \eta\}$ $(0 < \eta << \varepsilon$, where ε small enough) and $h : F \to F$ the corresponding geometric monodromy. The induced automorphisms on the complex cohomology groups are often denoted by $h := H^q(h) : H^q(F, \mathbb{C}) \to H^q(F, \mathbb{C})$.

In [**A'C75**], A'Campo gives a method for computing the Lefschetz number of the iterates $h^k := h \circ \cdots \circ h$ of the geometric monodromy, defined by

$$\Lambda(h^k) := \sum_{q \geq 0} (-1)^q \operatorname{tr} H^q(h^k),$$

in terms of an embedded resolution of the singularity $(H, 0) \subset (\mathbb{C}^{n+1}, 0)$. These Lefschetz numbers are related to the monodromy zeta function

$$Z(f) := \prod_{q \geq 0} \det(\mathrm{Id}^* - tH^q(h))^{(-1)^q}$$

by the following well-known formula

$$Z(f) = \exp\left(- \sum_{k \geq 1} \Lambda(h^k) \frac{t^k}{k} \right). \tag{18}$$

Using this relationship he derives a new expression for $Z(f)$. More precisely, let $\pi : X \to (\mathbb{C}^{n+1}, 0)$ be an embedded resolution of $(H, 0)$. Consider the total transform of H written as

$$\pi^*(H) = \widehat{H} + \sum_{i=1}^{r} m_i E_i,$$

where $\widehat{H}$ is the strict transform of H, and $E_1, \ldots, E_r$ are the irreducible components of the exceptional divisor $\pi^*(0)$. Now, define

$$\check{E}_i := E_i \setminus \left(E_i \cap \Big(\bigcup_{j \neq i} E_j \cup \widehat{H} \Big) \right).$$

Then, the Lefschetz numbers and the complex monodromy zeta function are given by

$$\Lambda(h^k) = \sum_{i=1,\ m_i | k}^{r} m_i \chi(\check{E}_i), \qquad Z(f) = \prod_{i=1}^{r} (1 - t^{m_i})^{\chi(\check{E}_i)}.$$

The Euler characteristic of the Milnor fiber is therefore

$$\chi(F) = \Lambda(h^0) = \sum_{i=1}^{r} m_i \chi(\check{E}_i). \tag{19}$$

When $(H, 0)$ defines an isolated singularity, both the characteristic polynomial of the monodromy $\Delta(t)$ and the Milnor number

$$\mu = \dim H^n(F, \mathbb{C}) = \deg \Delta(t)$$

can be obtained from the zeta function as follows,

$$\Delta(t) = \left[\frac{1}{t-1} \prod_{i=1}^{r} (t^{m_i} - 1)^{\chi(\check{E}_i)} \right]^{(-1)^n};$$

$$\mu = (-1)^n \Big[-1 + \sum_{i=1}^{r} m_i \chi(\check{E}_i) \Big],$$

and in particular by (19), $\mu = (-1)^n[-1 + \chi(F)]$ holds.

Another contribution in the same direction is found in [**GLM97**], where the authors give a generalization of A'Campo's formula for the zeta function via partial resolutions, that is, the map $\pi : X \to (\mathbb{C}^{n+1}, 0)$ is assumed to be just a modification (i.e. the condition about normal crossing divisor in the embedded resolution is removed). Also Dimca, using the machinery of constructible sheaves, proved the same result allowing X to be an arbitrary analytic space, see [**Dim04**, Th. 6.1.14.].

The aim of this chapter is to generalize all the previous results, giving the corresponding A'Campo's formula and the Lefschetz numbers in terms of an embedded **Q**-resolution, see Theorem (IV.3.14) below. Note that Veys has already considered this problem for plane curve singularities [**Vey97**].

From now on, and depending on the context, we shall denote the monodromy zeta function by $Z(f)$, $Z(f)(t)$, $Z(f;t)$, $Z_f(t)$ or $Z(t)$, interchangeably. The same applies for the Lefschetz numbers and the characteristic polynomial.

Section § IV.1
Toward A'Campo's Formula

Before giving the precise statement, let us see some examples to motivate A'Campo's formula in this setting. First, note that the zeta function and the Lefschetz numbers also exist in case of singular underlying spaces, such as $X(\mathbf{d};A)$. Moreover, if the function f is defined by a quasi-homogeneous polynomial, then $f : X(\mathbf{d};A) \setminus f^{-1}(0) \to \mathbb{C}^*$ is a locally trivial fibration and the global Minor fibration is equivalent to the local one.

(IV.1.1). Let $X(d;A) = \mathbb{C}^n/\mu_d$ be a cyclic quotient singularity, not necessarily written in a normalized form, i.e $(d;A) = (d;a_1,\ldots,a_n)$. Consider $f : X(d;A) \to \mathbb{C}$ a global algebraic function of the form $f = x_1^m$ (analogously one could proceed with x_i^m). Since it is a well-defined function, d must divide $a_1 m$. Let us write

$$e = \gcd(d,a_1), \qquad d = d'e, \qquad a_1 = a_1'e.$$

Then $d'|m$ and the equation $x_1^m = 1$ has m/d' different solutions over the space $X(d;A)$, as one can easily check by direct computations.

Consider $\zeta = \exp(2\pi i/m)$ a primitive m-th root of unity, it follows that the corresponding Milnor fiber $F = f^{-1}(1) = \{\mathbf{x} \in X(d;A) \mid x_1^m = 1\}$ is homeomorphic to the affine variety

$$\bigsqcup_{i=0}^{m/d'-1} \Big\{[(\zeta^i, x_2,\ldots,x_n)] \in X(d;A) \mid x_2,\ldots,x_n \in \mathbb{C}\Big\},$$

which has the same homotopy type as m/d' different points.

Let $\alpha : [0,1] \to \mathbb{C}^*$ be a generator of the fundamental group of $\mathbb{C}^*$, for instance $\alpha(t) = \exp(2\pi i t)$, and take $[(x_1,\ldots,x_n)] \in F$. Then the path $\widetilde{\alpha} : [0,1] \to X(d;A) \setminus f^{-1}(0)$ given by

$$\widetilde{\alpha}(t) = [(e^{\frac{2\pi i}{m}t}x_1, x_2,\ldots,x_n)]$$

defines a lifting of α with initial point $[(x_1,\ldots,x_n)]$.

Thus the geometric monodromy $h : F \to F$ corresponds to the map

$$\widetilde{\alpha}(0) = [(x_1, \dots, x_n)] \overset{h}{\longmapsto} [(e^{\frac{2\pi i}{m}} x_1, \dots, x_n)] = \widetilde{\alpha}(1).$$

From this expression, one deduces that the monodromy at the level zero $H^0(h) : H^0(F, \mathbb{C}) \to H^0(F, \mathbb{C})$ is the linear map $\mathbb{C}^{m/d'} \to \mathbb{C}^{m/d'}$ associated with the matrix $A = [e_2 | \dots | e_{m/d'} | e_1]$, where $\{e_1, \dots, e_{m/d'}\}$ is the canonical basis of $\mathbb{C}^{m/d'}$. Finally, one has that

$$\Lambda(h^k) = \begin{cases} \frac{m}{d} \gcd(d, a_1) & \text{if } \ \frac{m}{d'} | k, \\ 0 & \text{otherwise}; \end{cases} \qquad Z_{x_1^m}(t) = 1 - t^{\frac{m}{d} \gcd(d, a_1)}.$$

(IV.1.2). The next step would be to consider a global function defining a normal crossing divisor with more that one irreducible component, i.e. $f = x_1^{m_1} \cdots x_k^{m_k} : X(d; A) \to \mathbb{C}$, $k \geq 2$. To simplify the main ideas, assume also that $\gcd(m_1, \dots, m_k) = 1$. Let us use the notation,

$$e = \gcd(d, a_1, \dots, a_k), \quad d = d'e, \quad a_i = a_i' e, \quad (i = 1, \dots, k).$$

The Milnor fiber $F = f^{-1}(1)$ is homotopic to

$$\Big\{ [(x_1, \dots, x_k)] \in X(d; a_1, \dots, a_k) \mid x_1^{m_1} \cdots x_k^{m_k} = 1 \Big\},$$

which can be identified with

$$F' := \Big\{ [(x_1, \dots, x_k)] \in X(d'; a_1', \dots, a_k') \mid x_1^{m_1} \cdots x_k^{m_k} = 1 \Big\}.$$

As above, the path $\widetilde{\alpha} : [0,1] \to X(d; A) \setminus f^{-1}(0)$ given by

$$\widetilde{\alpha}(t) = [(e^{\frac{2\pi i}{m_1} t} x_1, x_2, \dots, x_n)]$$

defines a lifting of $\alpha : [0,1] \to \mathbb{C}^*$, $\alpha(t) = \exp(2\pi i t)$, with initial point $[(x_1, \dots, x_n)] \in F$. Thus the geometric monodromy $h : F \to F$ corresponds to the map

$$\widetilde{\alpha}(0) = [(x_1, \dots, x_n)] \overset{h}{\longmapsto} [(e^{\frac{2\pi i}{m_1}} x_1, \dots, x_n)] = \widetilde{\alpha}(1).$$

Hence $H^q(h)^{m_1} = \mathrm{Id}_{H^q(F, \mathbb{C})}$, $\forall q = 0, \dots, n$. Since $H^q(h)$ is a topological invariant of f, one can also prove that

$$H^q(h)^{m_2} = \mathrm{Id}^*, \dots, H^q(h)^{m_k} = \mathrm{Id}^* .$$

Consequently, all the induced automorphisms on the cohomology groups $H^q(h)$ are indeed the identity maps.

One finally has that

$$\Lambda(h^k) = \chi(F) = 0, \qquad Z(x_1^{m_1} \cdots x_k^{m_k}; t) = (1 - t)^{\chi(F)} = 1.$$

Note that $(\mathbb{C}^*)^k \supset \widetilde{F}' := \{x_1^{m_1} \cdots x_k^{m_k} = 1\} \to F'$ is an unramified covering of d' sheets and $\widetilde{F}'$ is homotopic to the $(k-1)$-dimensional real torus $\mathcal{T}_{k-1} := (S^1)^{k-1}$. This can be used to show that $\chi(F) = \chi(F') = 0$.

The general case $\gcd(m_1, \ldots, m_k) \neq 1$ is discussed later where the Lefschetz fixed point theorem is used to prove that all Lefschetz numbers are zero too, see Lemma (IV.3.11).

Section § **IV.2**

Partial Statement and Examples

Now, from the discussion above, the following result becomes very natural. See Theorem (IV.3.14) for a more general and complete result allowing abelian quotient singularities in the ambient spaces.

Theorem (IV.2.1). *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a non-constant analytic germ defining an isolated singularity and let $H = \{f = 0\}$. Assume that $\pi : X \to (\mathbb{C}^{n+1}, 0)$ is an embedded $\mathbf{Q}$-resolution of $(H, 0)$ having X cyclic quotient singularities. Let $X_0 = \pi^{-1}(H)$ be the total transform and denote by $S = \pi^{-1}(0)$ the exceptional divisor. Consider $S_{m,d'}$ to be the set*

$$\left\{ s \in S \;\middle|\; \begin{array}{l} \text{the equation of } X_0 \text{ in } s \text{ is given by the function} \\ x_i^m : X(d; A) \to \mathbb{C}, \text{ where } x_i \text{ is a local coordinate} \\ \text{of } X \text{ in } s \text{ and } d/\gcd(d, a_i) = d'. \end{array} \right\}.$$

Then, the characteristic polynomial of the complex monodromy of the germ $(H, 0)$ is

$$\Delta(t) = \left[\frac{1}{t-1} \prod_{m,d'} (t^{m/d'} - 1)^{\chi(S_{m,d'})} \right]^{(-1)^n}. \tag{20}$$

Remark **(IV.2.2).** If all cyclic quotient singularities appearing in X are written in their normalized form and $\gcd(d, a_i) \neq 1$, then $X \setminus X_0$ must contain singular points. This, however, contradicts that π is an embedded $\mathbf{Q}$-resolution. Therefore, after normalizing, one can always assume $d = d'$.

This theorem has already been proven by Veys in [**Vey97**] for plane curve singularities, that is, for $n = 1$. If all d's are equal to one, then $\pi : X \to (\mathbb{C}^{n+1}, 0)$ is an embedded resolution of $(H, 0)$ in the classical sense and one obtains exactly the formula by A'Campo [**A'C75**]. We postpone the complete proof of the theorem, devoting the rest of this section to showing several examples.

The tools developed in Chapters I are used without explicit mention. The rational self-intersection numbers of the exceptional divisors, when computing an embedded $\mathbf{Q}$-resolution of the singularity, see Chapter III, are omitted because they are not needed in Theorem (IV.2.1).

Example (IV.2.3). Let $f : \mathbb{C}^2 \to \mathbb{C}$ be the polynomial function given by $f = x^p + y^q$. Let us write

$$e = \gcd(p,q), \qquad p = p_1 e, \qquad q = q_1 e.$$

Consider $\pi : \widehat{\mathbb{C}}^2(q_1,p_1) \to \mathbb{C}^2$ the (q_1,p_1)-weighted blow-up at the origin. Recall that $\widehat{\mathbb{C}}^2(q_1,p_1) = U_0 \cup U_1$ has two singular points corresponding to the origin of each chart.

In $U_0 = X(q_1; -1, p_1)$, the total transform of f is given by the function $x^{p_1 q_1 e}(1 + y^q)$. The equation $y^q = -1$ only has $q/q_1 = e$ different solutions in U_0 and the local equation of the total transform at each of these points is of the form $x^{p_1 q_1 e}\, y$.

Hence π is an embedded **Q**-resolution of $\mathbf{C} = \{f = 0\}$ where all spaces are written in their normalized form.

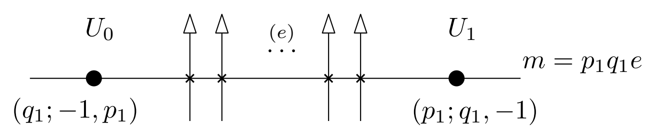


FIGURE IV.1. Embedded **Q**-resolution of $f = x^p + y^q$.

The set $S_{m,d}$ is not empty for the pairs $(m,d) = (p_1q_1e, 1)$, (p_1q_1e, q_1), and (p_1q_1e, p_1). Their Euler characteristics are

$$\chi(S_{p_1q_1e,1}) = 2 - (e+2) = -e,$$
$$\chi(S_{p_1q_1e,q_1}) = \chi(S_{p_1q_1e,p_1}) = 1.$$

Now, we apply Theorem (IV.2.1) and obtain

$$\Delta(t) = \frac{(t-1)(t^{\frac{pq}{e}} - 1)^e}{(t^p - 1)(t^q - 1)}.$$

Another interesting way to calculate the characteristic polynomial could be the following. Consider $\pi : \widehat{\mathbb{C}}^2(q,p) \to \mathbb{C}$ the (q,p)-weighted blow-up at the origin. Now $U_0 = X(q; -1, p)$ and the equation of the total transform in this chart is $x^{pq}(1 + y^q)$. As above, the map π is an embedded **Q**-resolution of **C** and our formula can be applied. However, the exceptional divisor, outside the two singular points, is not given by x^{pq} as one can expect at first sight. The reason is that $X(q; -1, p)$ is not written in its normalized form.

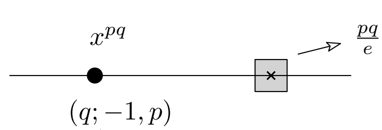


FIGURE IV.2. Non-normalized cyclic quotient singularity.

The isomorphism $X(q;-1,p) \cong X(q_1;-1,p_1)$ sends the well-defined function $x^{pq} : X(q;-1,p) \to \mathbb{C}$ to $x^{\frac{pq}{e}} : X(q_1;-1,p_1) \to \mathbb{C}$, and thus the required equation is $x^{\frac{pq}{e}} : \mathbb{C}^2 \to \mathbb{C}$. After applying the formula, one obtains the same characteristic polynomial.

This example shows that although one can blow up using non coprime weights, if possible, it is better to do it with the corresponding coprime weights to simplify calculations. However, the normalized condition is not necessary in the hypothesis of the statement.

Example (IV.2.4). Assume $p_1/q_1 < p_2/q_2$ are two irreducible fractions and $\gcd(q_1,q_2)=1$. Let $\mathbf{C}$ be the complex plane curve with Puiseux expansion

$$y = x^{\frac{p_1}{q_1}} + x^{\frac{p_2}{q_2}}.$$

Consider $\pi_1 : \widehat{\mathbb{C}}^2(q_1,p_1) \to \mathbb{C}^2$ the (q_1,p_1)-weighted blow-up at the origin. The exceptional divisor $\mathcal{E}_0$ has multiplicity $p_1q_1q_2$ and it contains two singular points of type $(q_1;-1,p_1)$ and $(p_1;q_1,-1)$. The strict transform $\widehat{\mathbf{C}}$ of the curve and $\mathcal{E}_0$ intersect at one smooth point, say P. The Puiseux expansion of $\widehat{\mathbf{C}}$ in a small neighborhood of this point is

$$y = x^{\frac{p_2q_1-p_1q_2}{q_2}},$$

and thus π_1 is not a embedded $\mathbf{Q}$-resolution.

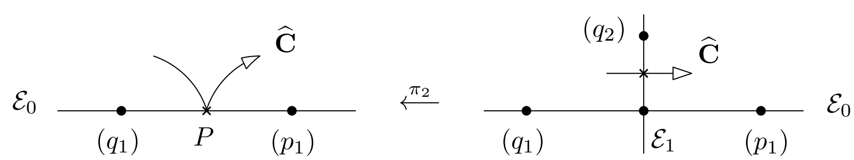


FIGURE IV.3. Embedded $\mathbf{Q}$-resolution of $\mathbf{C} = \{y = x^{\frac{p_1}{q_1}} + x^{\frac{p_2}{q_2}}\}$.

Now let π_2 be the $(q_2, p_2q_1 - p_1q_2)$-blow-up at P. The multiplicity of the new exceptional divisor $\mathcal{E}_1$ is $q_2(p_1q_1q_2 + p_2q_1 - p_1q_2)$. It intersects transversely $\mathcal{E}_0$ at a singular point of type

$$(p_2q_1 - p_1q_2;\, q_2, -1)$$

and also contains another singular point of type $(q_2;-1,p_2q_1)$. The strict transform of the curve is a smooth variety and it cuts transversely $\mathcal{E}_1$ at a smooth point.

Hence the composition $\pi_1 \circ \pi_2$ defines an embedded $\mathbf{Q}$-resolution of the curve $\mathbf{C} \subset \mathbb{C}^2$ where all cyclic quotient spaces are written in their normalized form. Figure IV.3 illustrates the whole process.

The corresponding Euler characteristics are $\chi = 1$, for the three singular points, and

$$\chi(\mathcal{E}_0 \setminus \{3 \text{ points}\}) = \chi(\mathcal{E}_1 \setminus \{3 \text{ points}\}) = -1.$$

Note that the singular point of type $(p_2q_1 - p_1q_2)$ does not contribute to the monodromy zeta function, since it belongs to more than one divisor. After applying formula (20), one obtains

$$\Delta(t) = \frac{\big(t-1\big)\big(t^{p_1q_1q_2}-1\big)\big(t^{q_2(p_1q_1q_2+p_2q_1-p_1q_2)}-1\big)}{\big(t^{p_1q_2}-1\big)\big(t^{q_1q_2}-1\big)\big(t^{p_1q_1q_2+p_2q_1-p_1q_2}-1\big)}, \quad \mu = \deg \Delta(t).$$

In case q_1 and q_2 are not coprime, the same arguments apply and one can find a formula for the characteristic polynomial of an irreducible plane curve with two (and then with arbitrary) Puiseux pairs. These formulas are quite involved and we omit them.

Example (IV.2.5). Let e_1, e_2, e_3 be three positive integers and denote by $e = \gcd(e_1, e_2, e_3)$. Assume that $\omega = (\frac{e_1}{e}, \frac{e_2}{e}, \frac{e_3}{e})$ is a weight vector of pairwise relatively prime numbers. Let $\mathbf{C}$ be the projective curve in $\mathbb{P}^2_\omega$ defined by the polynomial

$$F = x^{\frac{e_2e_3}{e}} + y^{\frac{e_1e_3}{e}} + z^{\frac{e_1e_2}{e}}.$$

Note that this polynomial is quasi-homogeneous of degree $e_1e_2e_3/e^2$. One is interested in computing the Euler characteristic of $\mathbf{C}$.

Consider $\pi : \widehat{\mathbb{C}}^3_\omega \to \mathbb{C}^3$ the weighted blow-up at the origin with respect to ω and take the affine variety $H = \{F = 0\} \subset \mathbb{C}^3$. The space $\widehat{\mathbb{C}}^3_\omega = U_0 \cup U_1 \cup U_2$ has just three singular points, corresponding to the origin of each chart and located at the exceptional divisor $E = \pi^*(0) \cong \mathbb{P}^2_\omega$. The order of the cyclic groups are $\frac{e_3}{e}$, $\frac{e_2}{e}$, and $\frac{e_1}{e}$, respectively.

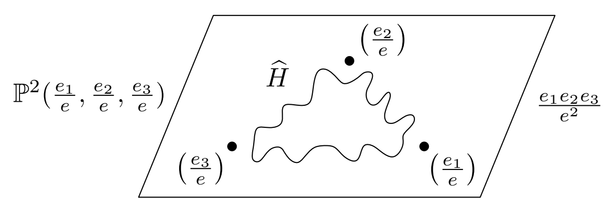


FIGURE IV.4. Embedded $\mathbf{Q}$-resolution of $x^{\frac{e_2e_3}{e}} + y^{\frac{e_1e_3}{e}} + z^{\frac{e_1e_2}{e}}$.

In the third chart $U_2 = X(\frac{e_3}{e}; \frac{e_1}{e}, \frac{e_2}{e}, -1)$, an equation of the total transform is

$$z^{\frac{e_1e_2e_3}{e^2}}(x^{\frac{e_2e_3}{e}} + y^{\frac{e_1e_3}{e}} + 1).$$

Using Lemma (I.1.16), one sees that the exceptional divisor and the strict transform are smooth varieties intersecting transversely. Thus π is an embedded $\mathbf{Q}$-resolution of H where all the quotient spaces are written in their normalized form.

The set $S_{m,d}$ is not empty for $m = e_1e_2e_3/e^2$ and $d \in \{1, \frac{e_1}{e}, \frac{e_2}{e}, \frac{e_3}{e}\}$. Since the intersection $E \cap \widehat{H}$ can be identified with $\mathbf{C}$, the Euler characteristics are

$$\begin{gathered}\chi(S_{m,1}) = -\chi(\mathbf{C}),\\ \chi(S_{m,\frac{e_1}{e}}) = \chi(S_{m,\frac{e_2}{e}}) = \chi(S_{m,\frac{e_3}{e}}) = 1.\end{gathered}$$

From Theorem (IV.2.1), the characteristic polynomial of H is

$$\Delta(t) = \frac{\left(t^{\frac{e_1e_2}{e}} - 1\right)\left(t^{\frac{e_1e_3}{e}} - 1\right)\left(t^{\frac{e_2e_3}{e}} - 1\right)}{(t-1)\left(t^{\frac{e_1e_2e_3}{e^2}} - 1\right)^{\chi(\mathbf{C})}}.$$

On the other hand, the Milnor number is known to be

$$\mu = \left(\frac{e_1e_2}{e} - 1\right)\left(\frac{e_1e_3}{e} - 1\right)\left(\frac{e_2e_3}{e} - 1\right).$$

Using that $\mu = \deg \Delta(t)$, one finally obtains

$$\chi(\mathbf{C}) = e_1 + e_2 + e_3 - \frac{e_1e_2e_3}{e}.$$

Example (IV.2.6). Let p, q, r be three positive integers and consider the polynomial function $f : \mathbb{C}^3 \to \mathbb{C}$ given by

$$f = x^p + y^q + z^r.$$

To simplify notation, we set $e_1 = \gcd(q,r)$, $e_2 = \gcd(p,r)$, $e_3 = \gcd(p,q)$, $e = \gcd(p,q,r)$, and $k = e_1e_2e_3$. The following information will be useful later:

$$\gcd(qr, pr, pq) = \frac{e_1e_2e_3}{e} = \frac{k}{e},$$

$$d_1 := \gcd\left(\frac{epr}{k}, \frac{epq}{k}\right) = \frac{ep}{e_2e_3}; \quad a_1 := \operatorname{lcm}(d_2, d_3) = \frac{e^2qr}{e_1k} = d_2d_3\,,$$

$$d_2 := \gcd\left(\frac{eqr}{k}, \frac{epq}{k}\right) = \frac{eq}{e_1e_3}; \quad a_2 := \operatorname{lcm}(d_1, d_3) = \frac{e^2pr}{e_2k}\,,$$

$$d_3 := \gcd\left(\frac{eqr}{k}, \frac{epr}{k}\right) = \frac{er}{e_1e_2}; \quad a_3 := \operatorname{lcm}(d_1, d_2) = \frac{e^2pq}{e_3k}\,.$$

Take the weight vector $\omega = \frac{e}{k}(qr, pr, pq)$ and let $\pi : \widehat{\mathbb{C}}^3_\omega \to \mathbb{C}^3$ be the weighted blow-up at the origin with respect to ω. The new space

$$\widehat{\mathbb{C}}^3_\omega = U_0 \cup U_1 \cup U_2$$

has three lines (each of them isomorphic to $\mathbb{P}^1$) of singular points located at the exceptional divisor $E = \pi^*(0) \cong \mathbb{P}^2_\omega$. They actually coincide with the three lines L_0, L_1, L_2 at infinity of $\mathbb{P}^2_\omega$.

In the third chart $U_2 = X(\frac{epq}{k}; \frac{eqr}{k}, \frac{epr}{k}, -1)$, an equation of the total transform is

$$z^{\frac{epqr}{k}}(x^p + y^q + 1),$$

where $z = 0$ is the exceptional divisor and the other equation corresponds to the strict transform.

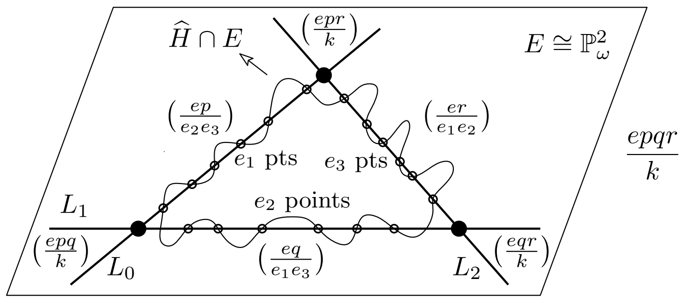


FIGURE IV.5. Embedded **Q**-resolution of $f = x^p + y^q + z^r$.

Working in this coordinate system, one sees that the line L_0 (resp. L_1) and $\widehat{H}$ intersect at exactly e_1 (resp. e_2) points. Analogously, $L_2 \cap \widehat{H}$ consists of e_3 points. Moreover, using Lemma (I.1.16), we have that $\widehat{H}$ and E are smooth varieties that intersect transversely. Hence the map π is an embedded **Q**-resolution of $\{f = 0\} \subset \mathbb{C}^3$ where all the cyclic quotient spaces are presented in their normalized form.

The Euler characteristics as well as the fractions m/d for the non-empty sets $S_{m,d}$ are calculated in the two tables below.

	$S_{\frac{epqr}{k},1}$	$S_{\frac{epqr}{k},\frac{ep}{e_2e_3}}$	$S_{\frac{epqr}{k},\frac{eq}{e_1e_3}}$	$S_{\frac{epqr}{k},\frac{er}{e_1e_2}}$
$\frac{m}{d}$	$\frac{epqr}{k}$	$\frac{qr}{e_1}$	$\frac{pr}{e_2}$	$\frac{pq}{e_3}$
χ	$e_1 + e_2 + e_3$ $-\chi(\mathbf{C})$	$-e_1$	$-e_2$	$-e_3$

	$S_{\frac{epqr}{k},\frac{eqr}{k}}$	$S_{\frac{epqr}{k},\frac{epr}{k}}$	$S_{\frac{epqr}{k},\frac{epq}{k}}$
m/d	p	q	r
χ	1	1	1

Here we denote by $\mathbf{C}$ the variety in $\mathbb{P}^2_\omega$ defined by the quasi-homogeneous polynomial $x^p + y^q + z^r$. Recall that from Proposition (I.2.5), the map $\mathbb{P}^2_\omega \to \mathbb{P}^2(\frac{e_1}{e}, \frac{e_2}{e}, \frac{e_3}{e})$ given by

$$[x : y : z]_\omega \longmapsto [x^{\frac{ep}{e_2 e_3}} : y^{\frac{eq}{e_1 e_3}} : z^{\frac{er}{e_1 e_2}}]_{(\frac{e_1}{e}, \frac{e_2}{e}, \frac{e_3}{e})}$$

is an isomorphism and it maps the hypersurface $\mathbf{C}$ to

$$\left\{x^{\frac{e_2 e_3}{e}} + y^{\frac{e_1 e_3}{e}} + z^{\frac{e_1 e_2}{e}} = 0\right\}.$$

By Example (IV.2.5), its Euler characteristic is

$$\chi(\mathbf{C}) = e_1 + e_2 + e_3 - \frac{e_1 e_2 e_3}{e},$$

and finally, from Theorem (IV.2.1), one obtains the characteristic polynomial of f,

$$\Delta(t) = \frac{\left(t^{\frac{epqr}{e_1 e_2 e_3}} - 1\right)^{\frac{e_1 e_2 e_3}{e}} \left(t^p - 1\right)\left(t^q - 1\right)\left(t^r - 1\right)}{\left(t-1\right)\left(t^{\frac{qr}{e_1}} - 1\right)^{e_1}\left(t^{\frac{pr}{e_2}} - 1\right)^{e_2}\left(t^{\frac{pq}{e_3}} - 1\right)^{e_3}}.$$

Note that the Euler characteristic of $\mathbf{C}$ could also be obtained using that the Milnor number is $\mu = (p-1)(q-1)(r-1) = \deg \Delta(t)$, as in the preceding example.

Example (IV.2.7). Let $f : \mathbb{C}^3 \to \mathbb{C}$ be the polynomial function defined by $f = z^{m+k} + h_m(x, y, z)$. Assume that $\mathbf{C} = \{h_m = 0\} \subseteq \mathbb{P}^2$ has only one singular point $P = [0 : 0 : 1]$, which is locally isomorphic to the cusp $x^q + y^p$, $\gcd(p, q) = 1$. Denote $k_1 = \gcd(k, p)$ and $k_2 = \gcd(k, q)$.

Consider the classical blow-up at the origin $\pi_1 : \widehat{\mathbb{C}}^3 \to \mathbb{C}^3$. In the third chart, the local equation of the total transform is

$$z^m(z^k + x^q + y^p) = 0.$$

The strict transform $\widehat{H}$ and the exceptional divisor E_0 intersect transversely at every point but in $P \in \mathbf{C} \equiv E_0 \cap \widehat{H}$. Also $\widehat{H} \setminus P$ is smooth.

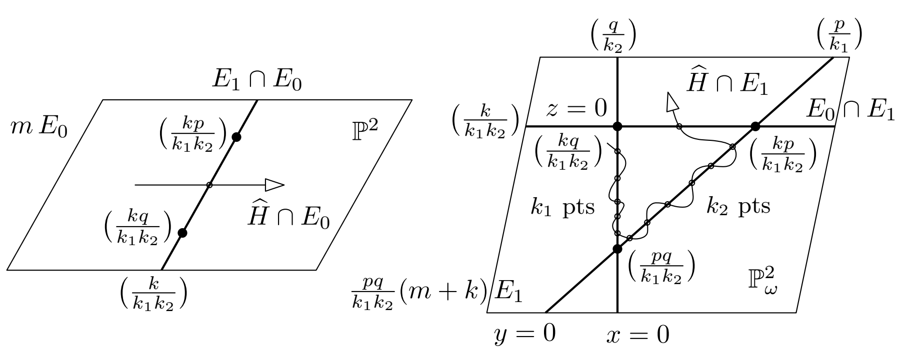


FIGURE IV.6. Intersection of E_0 (resp. E_1) with the rest of components.

One is therefore interested in the blowing-up at the point P with respect to (kp, kq, pq). However, in order to obtain cyclic quotient spaces in their normalized form, it is more suitable to choose $\omega = (\frac{kp}{k_1k_2}, \frac{kq}{k_1k_2}, \frac{pq}{k_1k_2})$ instead. Let π_2 be the weighted blow-up at P with respect to the vector ω. The local equation of the total transform in the second chart is given by the polynomial function

$$\left\{ y^{\frac{pq}{k_1k_2}(m+k)} z^m (z^k + x^q + 1) = 0 \right\} \subset X\left(\frac{kq}{k_1k_2}; \frac{kp}{k_1k_2}, -1, \frac{pq}{k_1k_2} \right),$$

where $y = 0$ represents the new exceptional divisor E_1.

The composition $\pi = \pi_1 \circ \pi_2$ is an embedded $\mathbf{Q}$-resolution. The final situation is illustrated in Figure IV.6, see Chapter VII for details.

The sets for which the Euler characteristic has to be computed are

$$S_{m,1}, \quad S_{\ell,1}, \quad S_{\ell,\frac{p}{k_1}}, \quad S_{\ell,\frac{q}{k_2}}, \quad S_{\ell,\frac{pq}{k_1k_2}}; \qquad \ell = \frac{pq}{k_1k_2}(m+k).$$

Clearly $\chi(S_{\ell,\,pq/k_1k_2}) = 1$, $\chi(S_{\ell,\,p/k_1}) = -k_2$, and $\chi(S_{\ell,\,q/k_2}) = -k_1$, since they are homeomorphic to a point, $\mathbb{P}^1 \setminus \{k_2{+}2 \text{ points}\}$ and $\mathbb{P}^1 \setminus \{k_1{+}2 \text{ points}\}$ respectively. The set $S_{m,1}$ is $\mathbb{P}^2 \setminus \mathbf{C}$. Finally, we use the additivity of the Euler characteristic to compute $\chi(S_{\ell,1})$.

Indeed, let $\mathbf{D} \subset \mathbb{P}^2(k_1, k_2, 1)$ be the variety defined by the equation $z^{k_1k_2} + x^{k_2} + y^{k_1} = 0$. Note that, by Proposition (I.2.5), $\mathbf{D}$ is isomorphism to the surface

$$\widehat{H} \cap E_1 = \{z^k + x^q + y^p = 0\} \subset \mathbb{P}^2_\omega$$

and, by Example (IV.2.5) (using $e_1 = k_1$, $e_2 = k_2$, $e_3 = 1$), its Euler characteristic is $k_1 + k_2 + 1 - k_1k_2$. Then,

$$\chi(S_{\ell,1}) = 3 - (2 + 2 + 2 + \chi(\mathbf{D})) + k_1 + k_2 + 4 = k_1k_2.$$

Every cyclic quotient singularity is written in their normalized form and thus the generalized A'Campo's formula can be applied with $d' = d$,

$$\begin{aligned} \Delta(t) &= \frac{\left(t^m - 1\right)^{\chi(\mathbb{P}^2\setminus\mathbf{C})}}{t-1} \cdot \frac{\left(t^{m+k} - 1\right)\left(t^{\frac{pq}{k_1k_2}(m+k)} - 1\right)^{k_1k_2}}{\left(t^{\frac{p}{k_1}(m+k)} - 1\right)^{k_1}\left(t^{\frac{q}{k_2}(m+k)} - 1\right)^{k_2}} \\ &= \frac{\left(t^m - 1\right)^{\chi(\mathbb{P}^2\setminus\mathbf{C})}}{t-1} \cdot \Delta_P^k(t^{m+k}). \end{aligned}$$

Let us explain the notation. The symbol $\Delta_P(t)$ denotes the characteristic polynomial of $\mathbf{C}$ at $P = [0:0:1]$, where the curve is locally isomorphic to $x^q + y^p$, and if $\Delta(t) = \prod_i (t^{m_i} - 1)^{a_i}$, then $\Delta^k(t)$ denotes

$$\Delta^k(t) = \prod_i \left(t^{\frac{m_i}{\gcd(m_i,k)}} - 1 \right)^{\gcd(m_i,k)a_i}.$$

The family of examples $z^{m+k} + h_m(x,y,z)$, where h_m defines a reduced projective plane curve such that $\mathrm{Sing}(h_m) \cap \{z=0\} = \emptyset$ as a subset in $\mathbb{P}^2$, i.e. Yomdin-Lê surface singularities, is studied in Chapter VII.

We conclude by emphasizing that in the classical A'Campo's formula one has to pay attention to compute the Euler characteristic while the multiplicities remain trivial. Using our formula we also have to take care of computing the multiplicities and the order of the corresponding cyclic groups, especially when the quotient singularity is not in its normalized form. Discussion (I.1.15) and Lemma (I.1.16) are very useful in this sense.

Section § IV.3
Proof of the Theorem

One way to proceed is to rebuild A'Campo's paper [**A'C75**], thus giving a model of the Milnor fibration in our setting. This method is very natural but perhaps a bit long and tedious. In [**GLM97**], the authors give a generalization of A'Campo's formula for the monodromy zeta function via partial resolution but the ambient space considered there is still smooth and the proof can not be generalized to an arbitrary analytic variety.

That is why a very general result by Dimca is used instead, see Theorem (IV.3.6) below. This leads us to talk about constructible complexes of sheaves with respect to a stratification and also about the nearby cycles associated with an analytic function. Using this theorem, only the monodromy zeta function of a monomial defining a function over a quotient space of type $X(\mathbf{d}; A)$ is needed.

IV.3–1. A result by Dimca

To state the result we need some notions about sheaves and constructibility. We refer, for instance, to [**Dim04**] and the references listed there for further details.

(IV.3.1). Consider $Sh(X, \mathrm{Vect}_{\mathbb{C}})$ the abelian category of sheaves of $\mathbb{C}$-vector spaces on a topological space X. To simplify notation its derived category is often denoted by $D^*(X)$. The constant sheaf corresponding to $\mathbb{C}$ is denoted by $\underline{\mathbb{C}}_X$; it is by definition the sheaf associated with the constant presheaf which sends every open subset of X to $\mathbb{C}$. If $U \subset X$ is connected open, then $\underline{\mathbb{C}}_X(U) = \mathbb{C}$.

Let $f : X \to Y$ be a continuous mapping between two topological spaces. The direct image functor $f_* : Sh(X, \mathrm{Vect}_{\mathbb{C}}) \to Sh(Y, \mathrm{Vect}_{\mathbb{C}})$ is defined on objects by $(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$, for any sheaf $\mathcal{F}$ on X and any open set $V \subset Y$. This functor is additive and left exact; its derived functor is denoted by $Rf_* : D^*(X) \to D^*(Y)$.

The inverse image functor $f^{-1} : Sh(Y, \mathrm{Vect}_{\mathbb{C}}) \to Sh(X, \mathrm{Vect}_{\mathbb{C}})$ is defined as $f^{-1}\mathcal{G}$ being the sheaf associated with the presheaf

$$U \longmapsto \varinjlim_{f(U) \subset V} \mathcal{G}(V).$$

Here $\mathcal{G}$ is a sheaf on Y and $U \subset X$ is open. This functor is exact and hence the corresponding derived functor $Rf^{-1} : D^*(Y) \to D^*(X)$ is usually denoted again by f^{-1}.

If $f(U) \subset Y$ is open, then $(f^{-1}\mathcal{G})(U) = \mathcal{G}(f(U))$. In particular, if the map $i_U : U \hookrightarrow X$ denotes the inclusion of an open set, then $i_U^{-1}\mathcal{F} = \mathcal{F}|_U$. The restriction to an arbitrary subspace $Z \subset X$ is defined by

$$\mathcal{F}|_Z := i_Z^{-1}\mathcal{F},$$

where $i_Z : Z \hookrightarrow X$ is the inclusion.

Using this notation one has $\underline{\mathbb{C}}_X|_Z := i_Z^{-1}\underline{\mathbb{C}}_X = \underline{\mathbb{C}}_Z$.

(IV.3.2). Let X be a complex analytic space and $\mathcal{S} = \{X_j\}_{j \in J}$ a locally finite partition of X into non-empty, connected, locally closed subsets called *strata* of $\mathcal{S}$. The partition $\mathcal{S}$ is called a *stratification* if it satisfies the following conditions:

(1) The boundary condition, i.e. each boundary $\partial X_j = \overline{X_j} \setminus X_j$ is a union of strata in $\mathcal{S}$.

(2) Constructibility, i.e. for all $j \in J$ the spaces $\overline{X_j}$ and ∂X_j are closed complex analytic subspaces in X.

(3) Stratification, i.e. all the strata are smooth constructible subvarieties of X.

Definition (IV.3.3). Let $\mathcal{S} = \{X_j\}_{j \in J}$ be a stratification on X.

(i) A sheaf complex $\mathcal{F}^\bullet \in D^*(X)$ is called *$\mathcal{S}$-constructible* if the restriction of each cohomology sheaf $\mathcal{H}^q(\mathcal{F}^\bullet)|_{X_j}$ is a $\underline{\mathbb{C}}_{X_j}$-local system of finite rank, that is, one has the isomorphisms of $\underline{\mathbb{C}}_{X_j}$-vector spaces

$$\mathcal{H}^q(\mathcal{F}^\bullet)|_{X_j} \simeq \underline{\mathbb{C}}_{X_j}^{r_{j,q}}.$$

(ii) Given $u : \mathcal{F}^\bullet \to \mathcal{F}^\bullet$ an automorphisms of $\underline{\mathbb{C}}_X$-vector spaces, the complex $\mathcal{F}^\bullet$ is called *equivariantly $\mathcal{S}$-constructible* with respect to u, if it is $\mathcal{S}$-constructible and the induced automorphisms on the cohomology groups $\mathcal{H}^q(u)_x : \mathcal{H}^m(\mathcal{F}^\bullet)_x \to \mathcal{H}^m(\mathcal{F}^\bullet)_x$ are all conjugate.

(IV.3.4). Let X be a complex analytic variety and $g : X \to \mathbb{C}$ a non-constant analytic function. Consider the diagram,

$$\begin{array}{ccccccc} g^{-1}(0) & \overset{i}{\hookrightarrow} & X & \overset{j}{\hookleftarrow} & X \setminus g^{-1}(0) & \overset{\hat{\pi}}{\longleftarrow} & E \\ & & & & \downarrow f & \# & \downarrow \hat{f} \\ & & & & \mathbb{C}^* & \underset{\exp}{\longleftarrow} & \widetilde{\mathbb{C}}^* \end{array}$$

where $i : g^{-1}(0) \hookrightarrow X$ and $j : X \setminus g^{-1}(0) \hookrightarrow X$ are inclusions, $\widetilde{\mathbb{C}}^*$ is the universal cover of $\mathbb{C}^*$, and E denotes the pull-back.

Definition (IV.3.5). Let $\mathcal{F}^\bullet \in D^*(X)$ be a complex. The *nearby cycles* of $\mathcal{F}^\bullet$ with respect to the function $g : X \to \mathbb{C}$ is defined to be the sheaf complex given by

$$\psi_g \mathcal{F}^\bullet := i^{-1} R(j \circ \hat{\pi})_* (j \circ \hat{\pi})^{-1} \mathcal{F}^\bullet \ \in \ D^*(g^{-1}(0)).$$

The nearby cycles is a local operation in the sense that if $U \subset X$ is an open set, then $(\psi_g \mathcal{F}^\bullet)|_W = \psi_{g|_W} \mathcal{F}^\bullet|_W$ holds. Also note that $\psi_g \mathcal{F}^\bullet$ only depends on g and $\mathcal{F}^\bullet|_{X \setminus g^{-1}(0)}$.

There is an associated monodromy deck transformation $h : E \to E$ coming from the action of the natural generator of $\pi_1(\mathbb{C}^*)$ which satisfies $\hat{\pi} \circ h = \hat{\pi}$. This homeomorphism induces an isomorphism of complexes

$$M_g : \psi_g \mathcal{F}^\bullet \longrightarrow \psi_g \mathcal{F}^\bullet.$$

For every point $x \in g^{-1}(0)$ there is a natural isomorphism from the stalk cohomology of $\psi_g \mathcal{F}^\bullet$ at x to the cohomology of the Milnor fiber at x with coefficients in $\mathcal{F}^\bullet$, that is, for all $\epsilon > 0$ small enough and all $t \in \mathbb{C}^*$ with $|t| << \epsilon$, one has

$$\begin{aligned} \mathcal{H}^q(\psi_g \mathcal{F}^\bullet)_x &\simeq \mathbb{H}^q(g^{-1}(t) \cap B_\epsilon(x), \mathcal{F}^\bullet_|) \\ &\simeq \mathbb{H}^q(g^{-1}(t) \cap \overline{B_\epsilon(x)}, \mathcal{F}^\bullet_|), \end{aligned} \tag{21}$$

where the open ball $B_\epsilon(x)$ is taken inside any local embedding of (X, x) in an affine space.

The monodromy morphism $M_{g,x}$ on the left-hand side corresponds to the morphism on the right-hand side induced by the monodromy homeomorphism of the local Milnor fibration associated with $g : (X, x) \to (\mathbb{C}, 0)$.

Now we are ready to state Dimca's theorem. To be precise, he only considered the case when the ambient space is smooth $M = \mathbb{C}^{n+1}$, see below. Repeating exactly the same arguments, one obtains the result for any analytic variety.

Theorem (IV.3.6) ([**Dim04**], Th. 6.1.14)**.** *Let $f:(M,p)\to(\mathbb{C},0)$ be the germ of a non-constant analytic function which is defined on a small neighborhood U of p. Let H be the hypersurface $\{x\in U \mid f(x)=0\}$. Assume $\pi: X\to U$ is a proper analytic map such that π induces an isomorphism between $X\setminus\pi^{-1}(H)$ and $U\setminus H$.*

Let $g=f\circ\pi$ denote the composition and $j: X\setminus\pi^{-1}(H)\hookrightarrow X$ the inclusion. Let $\mathcal{S}$ be a finite stratification of the exceptional divisor $\pi^{-1}(0)$ such that $\psi_g\big(Rj_*\underline{\mathbb{C}}_{X\setminus\pi^{-1}(H)}\big)$ is equivariantly $\mathcal{S}$-constructible with respect to the semisimple part of M_g. Then,

$$\Lambda(h)=\sum_{S\in\mathcal{S}}\chi(S)\Lambda(g,x_S)\,;\qquad Z(f)=\prod_{S\in\mathcal{S}}Z(g,x_S)^{\chi(S)},$$

where x_S is an arbitrary point in the stratum S and $Z(g,x_S)$, $\Lambda(g,x_s)$ are the zeta function and the Lefschetz number of the germ g at x_S.

Remark **(IV.3.7).** Let $\mathcal{F}^\bullet=Rj_*\underline{\mathbb{C}}_{X\setminus\pi^{-1}(H)}$. Using the notation of the previous theorem, the isomorphism of (21) tells us that

$$\mathcal{H}^q(\psi_g\mathcal{F}^\bullet)_x=H^q(F_x,\mathbb{C})$$

where F_x is the Milnor fiber at x.

This clarifies when the complex of sheaves $\psi_g\mathcal{F}^\bullet$ is equivariantly $\mathcal{S}$-constructible with respect to the semisimple part of M_g. In particular, this condition is satisfied, for instance, when the local equation of g along each stratum is the same.

IV.3–2. Zeta function of a normal crossing divisor

Let $M=\mathbb{C}^n/\mu_{\mathbf{d}}$ be a quotient space of type $X(\mathbf{d};A)$, not necessarily cyclic or written in a normalized form. Recall the multi-index notation:

$$X(\mathbf{d};A)=X\left(\begin{array}{c|ccc} d_1 & a_{11} & \dots & a_{1n}\\ \vdots & \vdots & \ddots & \vdots\\ d_r & a_{r1} & \dots & a_{rn}\end{array}\right),\qquad \begin{array}{c}\mathbf{d}=(d_1,\dots,d_r),\\ \mathbf{a}_j=(a_{1j},\dots,a_{rj}).\end{array}$$

In Section III, we have seen that for each $j=1,\dots,n$ there is an isomorphism

$$\begin{array}{rcl} X(\mathbf{d};\mathbf{a}_j) & \longrightarrow & \mathbb{C}\\ {[x_j]} & \mapsto & x_j^{\ell_j},\end{array}\tag{22}$$

where

$$\ell_j=\operatorname{lcm}\left(\frac{d_1}{\gcd(d_1,a_{1j})},\dots,\frac{d_r}{\gcd(d_r,a_{rj})}\right).$$

Given a homogeneous polynomial defined over M, the classical formula for the monodromy zeta function depending on the degree of the polynomial and the Euler characteristic of the Milnor fiber seems to be more complicated in this setting. Using the techniques developed in Chapter VI, one can provide formulas at least for plane curves and surfaces but the trick of applying the fixed point theorem does not work anymore. However, for our purpose, only the normal crossing case is needed.

(IV.3.8). We first proceed to compute the geometric monodromy of a homogeneous polynomial $f : M \to \mathbb{C}$ of degree $N := \deg(f)$. Let $\alpha : [0,1] \to \mathbb{C}^*$ be a generator of the fundamental group of $\mathbb{C}^*$, for example, $\alpha(t) = \exp(2\pi i t)$ and consider $[\mathbf{x}] \in F = f^{-1}(1)$. The path

$$\begin{array}{rccl} \widetilde{\alpha} : & [0,1] & \longrightarrow & M \setminus f^{-1}(0), \\ & t & \mapsto & \big[(e^{\frac{2\pi i}{N} t} x_1, \ldots, e^{\frac{2\pi i}{N} t} x_n)\big] \end{array}$$

defines a lifting of α with initial point $[(x_1, \ldots, x_n)]$. Thus the geometric monodromy $h : F \to F$ corresponds to the map

$$\widetilde{\alpha}(0) = \big[(x_1, \ldots, x_n)\big] \stackrel{h}{\longmapsto} \big[(e^{\frac{2\pi i}{N}} x_1, \ldots, e^{\frac{2\pi i}{N}} x_n)\big] = \widetilde{\alpha}(1).$$

As in the case $M = \mathbb{C}^n$, this also works for quasi-homogeneous polynomials, replacing the exponentials for suitable numbers according to the weights.

(IV.3.9). Let us study the monodromy zeta function in the simplest normal crossing case, i.e. $f = x_1^{m_1} : M \to \mathbb{C}$. The Milnor fiber

$$F := f^{-1}(1) = \{[\mathbf{x}] \in M \mid x_1^{m_1} = 1\}$$

has the same homotopy type as $F' := \{[(x_1, 0, \ldots, 0)] \in M \mid x_1^{m_1} = 1\}$ which can be identified with

$$\big\{[x_1] \in X(\mathbf{d}; \mathbf{a}_1) \mid x_1^{m_1} = 1\big\}.$$

In fact, $r : F \to F' : [\mathbf{x}] \mapsto [x_1]$ is a strong deformation retraction. Since $h(F') \subset F'$, the geometric monodromy $h : F \to F$ is homotopic to its restriction $h' := h|_{F'} : F' \to F'$. Using the isomorphism (22),

$$X(\mathbf{d}; \mathbf{a}_1) \simeq \mathbb{C} : \ [x] \mapsto x^{\ell_1},$$

the claim is reduced to the calculation of the zeta function of the polynomial $x_1^{m_1/\ell_1} : \mathbb{C} \to \mathbb{C}$. But this is known to be $1 - t^{m_1/\ell_1}$.

(IV.3.10). Assume now that $f = x_1^{m_1}\cdots x_k^{m_k} : M \to \mathbb{C}$, $k \geq 2$. The Milnor fiber $F := f^{-1}(1)$ has the same homotopic type as

$$F' := \Big\{ \big[(x_1,\dots x_k)\big] \in \frac{S^1\times \overset{(k)}{\cdots} \times S^1}{\mu_{\mathbf{d}}} \ \mid\ x_1^{m_1}\cdots x_k^{m_k} = 1 \Big\},$$

where $\mu_{\mathbf{d}}$ defines an action of type $(\mathbf{d}; \mathbf{a}_1,\dots,\mathbf{a}_k)$ on the space $(S^1)^k$. As above, there is a strong deformation retraction

$$\begin{array}{rcl} r : F & \longrightarrow & F', \\ {[\mathbf{x}]} & \mapsto & \Big[\Big(\dfrac{x_1}{|x_1|},\dots,\dfrac{x_k}{|x_k|},0,\dots,0\Big)\Big] \end{array}$$

satisfying that $h(F') \subset F'$.

We shall see that the Lefschetz numbers $\Lambda((h')^j) = \Lambda(h^j)$ equal zero for all $j \geq 1$. This would imply $Z_f(t) = 1$ by virtue of (18). Two cases arise:

- If $(h')^j$ does not have fixed points, then by the classical fixed point theorem $\Lambda((h')^j) = 0$.
- Otherwise, $(h')^j$ is the identity map and $\Lambda((h')^j) = \chi(F') = 0$.

Note that there is an unramified covering

$$(S^1)^k \supset \widetilde{F'} := \{x_1^{m_1}\cdots x_k^{m_k} = 1\} \xrightarrow{\ \pi\ } F'$$

with a finite number of sheets. The first of the preceding spaces $\widetilde{F'}$ has $e = \gcd(m_1,\dots,m_k)$ disjoint components, each of them homotopically equivalent to a real $(k-1)$-dimensional torus $\mathcal{T}_{k-1} = (S^1)^{k-1}$. It follows that

$$\chi(F') = \frac{1}{\deg \pi}\, e\, \chi(\mathcal{T}_{k-1}) = 0.$$

Note that the condition $k \geq 2$ has only been used at the end. In the case $k = 1$, one has

$$\deg \pi = \ell_1, \quad e = m_1, \quad \chi(\mathcal{T}_0) = 1, \quad \chi(F') = m_1/\ell_1.$$

We summarize the previous discussion in the following lemma.

Lemma (IV.3.11). *The monodromy zeta function of a normal crossing divisor given by* $x_1^{m_1}\cdots x_k^{m_k} : X(\mathbf{d}; A) \to \mathbb{C}$, $k \geq 1$, *is*

$$Z\big(x_1^{m_1}\cdots x_k^{m_k} : X(\mathbf{d}; A) \to \mathbb{C};\, t\big) = \begin{cases} 1 - t^{\frac{m_1}{\ell_1}} & k = 1; \\ 1 & k \geq 2, \end{cases}$$

where $\ell_1 = \operatorname{lcm}\left(\dfrac{d_1}{\gcd(d_1,a_{11})},\dots,\dfrac{d_r}{\gcd(d_r,a_{r1})}\right)$.

As we see, the strata belonging to more than one irreducible components do not contribute to the monodromy zeta function. This reflects the good behavior of abelian quotient singularities with respect to normal crossing divisors. By contrast, non-abelian groups seem to work differently, see §IV.5 where it is shown that "double points" may contribute to $Z(f;t)$.

IV.3–3. A'Campo's formula for embedded Q-resolutions

Let $f : (M,0) \to (\mathbb{C},0)$ be a non-constant analytic function germ and let $(H,0) \subset (M,0)$ be the hypersurface defined by f. Given an embedded **Q**-resolution of $(H,0)$, $\pi : X \to (M,0)$, consider as in the classical case,

$$\check{E}_i := E_i \setminus \left(E_i \cap \Big(\bigcup_{\substack{k=1,\ldots,s \\ k\neq i}} E_k \cup \widehat{H} \Big) \right),$$

where $E_1,\ldots,E_s$ are the irreducible components of the exceptional divisor of π, and $\widehat{H}$ is the strict transform of H.

Definition (IV.3.12). Let X be a complex analytic space having only abelian quotient singularities and consider D a $\mathbb{Q}$-divisor with normal crossings on X. Let $q \in D$ be a point living in exactly one irreducible component of D. Then, the equation of D at q is given by a function of the form $x_j^m : X(\mathbf{d};A) \to \mathbb{C}$, where x_j is a local coordinate of X in q.

The *multiplicity* of D at q, denoted by $m(D,q)$, is defined by

$$m(D,q) := \frac{m}{\ell_j}, \qquad \ell_j = \operatorname{lcm}\left(\frac{d_1}{\gcd(d_1,a_{1j})}, \ldots, \frac{d_r}{\gcd(d_r,a_{rj})} \right).$$

If there exists T contained in exactly one irreducible component of D and the function $q \in T \mapsto m(D,q)$ is constant, then we use the notation $m(T) := m(D,q_0)$, where q_0 is an arbitrary point in T.

Remark **(IV.3.13).** The integer $m(D,q)$ does not depend on the type $(\mathbf{d};A)$ representing the quotient space. A more general definition, including the case when $q \in D$ belongs to more than one irreducible component, will be given in (V.1.4).

To simplify the notation one writes $E_0 = \widehat{H}$ and $S = \{0,1,\ldots,s\}$ so that the stratification of X associated with the $\mathbb{Q}$-normal crossing divisor $\pi^{-1}(H) = \bigcup_{i\in S} E_i$ is defined by setting

$$E_I^\circ := \Big(\cap_{i\in I} E_i \Big) \setminus \Big(\cup_{i\notin I} E_i \Big), \tag{23}$$

for a given possibly empty set $I \subseteq S$. Note that, for $i = 1,\ldots,s$, one has that $E_{\{i\}}^\circ = \check{E}_i$.

Let $X = \bigsqcup_{j\in J} Q_j$ be a finite stratification on X given by its quotient singularities so that the local equation of $g = f\circ\pi$ at $q \in E_I^\circ \cap Q_j$ is of the form

$$x_1^{m_1}\cdots x_k^{m_k} : B/G \longrightarrow \mathbb{C},$$

where B is an open ball around q, and G is an abelian group acting diagonally as in $(\mathbf{d}; A)$. The multiplicities m_i's and the action G are the same along each stratum $E_I^\circ \cap Q_j$, i.e. they do not depend on the chosen point $q \in E_I^\circ \cap Q_j$. Let us denote

$$\check{E}_{i,j} := \check{E}_i \cap Q_j, \qquad m_{i,j} := m(\check{E}_{i,j}).$$

The following result is nothing but a generalization of (IV.2.1) written in the language of divisors. To use the classical convection on indices, $M = \mathbb{C}^{n+1}/\mu_{\mathbf{d}}$ (instead of $\mathbb{C}^n/\mu_{\mathbf{d}}$) in the theorem below.

Theorem (IV.3.14). *Let $f : (M,0) \to (\mathbb{C},0)$ be a non-constant analytic function germ and let $H = \{f = 0\}$. Consider F the Milnor fiber and $h : F \to F$ the geometric monodromy. Assume $\pi : X \to (M,0)$ is an embedded $\mathbf{Q}$-resolution of $(H,0)$. Then, using the notation above, one has: $(i = 1,\ldots,s,\ j\in J)$*

(1) *The Lefschetz number of $h^k = h \circ \overset{(k)}{\cdots} \circ h : F \to F$, $k \geq 0$, and the Euler characteristic of F are*

$$\Lambda(h^k) = \sum_{i,j,\ k|m_{i,j}} m_{i,j}\cdot \chi(\check{E}_{i,j}),$$

$$\chi(F) = \sum_{i,j} m_{i,j}\cdot\chi(\check{E}_{i,j}) = \Lambda(h^0).$$

(2) *The local monodromy zeta function of f at 0 is*

$$Z(t) = \prod_{i,j} (1 - t^{m_{i,j}})^{\chi(\check{E}_{i,j})}.$$

(3) *In the isolated case, the characteristic polynomial of the complex monodromy of $(H,0) \subset (M,0)$ is*

$$\Delta(t) = \left[\frac{1}{t-1}\prod_{i,j}(t^{m_{i,j}} - 1)^{\chi(\check{E}_{i,j})}\right]^{(-1)^n},$$

and the Milnor number is

$$\mu = (-1)^n\Big[-1 + \sum_{i,j} m_{i,j}\cdot\chi(\check{E}_{i,j})\Big] = \deg\Delta(t).$$

Proof. Only the proof of (2) is given; the other items follow from this one. Using that $E_0 = \widehat{H}$ and $S = \{0, 1, \ldots, s\}$, the support of the total transform can be written as

$$\pi^{-1}(H) = \widehat{H} \cup \pi^{-1}(0) = \bigcup_{i \in S} E_i.$$

Let $X = \bigsqcup_{I \subseteq S} E_I^\circ$ be the stratification of X given in (23) associated with this $\mathbb{Q}$-normal crossing divisor. This partition gives rise to a stratification on $\pi^{-1}(0) = \bigsqcup E_I^\circ$, where the intersection is taken over

$$I \in \mathcal{P}(S) \setminus \{\emptyset, \{0\}\}.$$

However, the equivariant property is not satisfied in general, since the strata may contain singular points of X. Instead, let $\mathcal{S}$ be the following finer stratification

$$\mathcal{S} = \Big\{ E_I^\circ \cap Q_j \Big\}_{\substack{I \subset S,\ j \in J \\ I \neq \emptyset, \{0\}}}.$$

Now the family $\mathcal{S}$ is a finite stratification of the exceptional divisor of π such that the complex $\psi_{f \circ \pi}(Rj_* \underline{\mathbb{C}}_{X \setminus \pi^{-1}(H)})$ is equivariantly $\mathcal{S}$-constructible, where

$$j : X \setminus \pi^{-1}(H) \hookrightarrow X$$

is the inclusion. Hence Theorem (IV.3.6) applies. Moreover, given $q \in \pi^{-1}(0)$, there exist $I = \{i_1, \ldots, i_k\} \subset S$, $k \geq 1$ ($k = 1 \Rightarrow i_1 \neq 0$), and $j \in J$ such that the local equation of $g := f \circ \pi$ at q is given by the function

$$x_{i_1}^{m_{i_1}} \cdots x_{i_k}^{m_{i_k}} : B_j / G_j \longrightarrow \mathbb{C}.$$

The numbers m_{i_j}'s and the action G_j are the same along each stratum of $\mathcal{S}$. By Lemma (IV.3.11), the strata with $k \geq 2$ do not contribute to the monodromy zeta function.

Take $x_T = x_{I,j}$ an arbitrary point in $E_I^\circ \cap Q_j$, then from the previous discussion one has

$$\begin{aligned} Z(f) = \prod_{T \in \mathcal{S}} Z(g, x_T) &= \prod_{\substack{I \subset S,\ j \in J \\ I \neq \emptyset, \{0\}}} Z(g, x_{I,j})^{\chi(E_I^\circ \cap Q_j)} \\ &= \prod_{\substack{i=1,\ldots,s \\ j \in J}} Z(g, x_{\{i\},j})^{\chi(E_{\{i\}}^\circ \cap Q_j)} = \prod_{\substack{i=1,\ldots,s \\ j \in J}} (1 - t^{m_{i,j}})^{\chi(\check{E}_{i,j})}. \end{aligned}$$

Above, Lemma (IV.3.11) is used for the computation of the monodromy zeta function at $x_{\{i\},j}$. Observe also that $E_{\{i\}}^\circ \cap Q_j = \check{E}_{i,j}$. Now the proof is complete. □

Remark **(IV.3.15).** Let $X = \bigsqcup_{j\in J} Q'_j$ be another finite stratification of X such that the function

$$q \in \check{E}_i \cap Q'_j \longmapsto m(E_i, q)$$

is constant. Then the previous theorem still holds replacing $\check{E}_{i,j} = \check{E}_i \cap Q_j$ by $\check{E}_i \cap Q'_j$.

Remark **(IV.3.16).** When $\operatorname{Sing}(M) \subset H$, then $M \setminus H$ is smooth and thus so is $X \setminus \pi^{-1}(H)$. Consequently, all singularities of X are contained in the total transform $\pi^{-1}(H)$, and the numbers $m_{i,j}$'s take the simple form

$$m_{i,j} = \frac{m}{\operatorname{lcm}(d_1, \ldots, d_r)},$$

after having normalized the types involved in the corresponding embedded **Q**-resolution of the singularity, cf. (IV.2.2).

Section § IV.4
Zeta Function of Not-Well-Defined Functions

In what follows, the monodromy zeta function associated with not well-defined functions over $M = X(\mathbf{d}; A)$ is needed. Assume $f \in \mathbb{C}[x_1, \ldots, x_n]$ is a polynomial such that the following condition holds for all $P \in \mathbb{C}^n$,

$$f(P) = 0 \implies f(\xi_{\mathbf{d}} \cdot P) = 0, \quad \forall \xi_{\mathbf{d}} \in \mu_{\mathbf{d}}.$$

Then the zero set $\{[\mathbf{x}] \in M \mid f(\mathbf{x}) = 0\} =: \{f = 0\} \subset M$ is well defined, although f may not induce a function over M.

Proposition (IV.4.1). *Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a reduced polynomial. The following conditions are equivalent:*

(1) $\forall P \in \mathbb{C}^n$, $\big[f(P) = 0 \implies f(\xi_{\mathbf{d}} \cdot P) = 0, \forall \xi_{\mathbf{d}} \in \mu_{\mathbf{d}}\big]$.

(2) $\exists \mathbf{v} \in \mathbb{N}^r$ *such that* $f(\xi_{\mathbf{d}} \cdot \mathbf{x}) = \xi_{\mathbf{d}}^{\mathbf{v}} f(\mathbf{x})$, $\forall \xi_{\mathbf{d}} \in \mu_{\mathbf{d}}$.

(3) $\exists k \geq 1$ *such that* $f^k := f \cdot \overset{(k)}{\cdots} \cdot f : M \to \mathbb{C}$ *is a function.*

Proof. The only non-trivial part is perhaps (1) ⇒ (2). Define $g_i(\mathbf{x})$ for each $i = 1, \ldots, r$ to be the polynomial

$$g_i(\mathbf{x}) := f((1, \ldots, \zeta_i, \ldots, 1) \cdot \mathbf{x}) = f(\zeta_i \cdot \mathbf{x}),$$

where ζ_i is a fixed primitive d_i-th root of unity. By (1), since f is reduced, one has

$$g_i \in IV(f) = \sqrt{f} = \langle f \rangle.$$

There exists $h_i \in \mathbb{C}[\mathbf{x}]$ such that $g_i = h_i f$. Taking degrees the polynomials h_i's must be constants. But,

$$\begin{aligned} f(\mathbf{x}) &= f(\zeta_i^{d_i} \cdot \mathbf{x}) = g_i(\zeta_i^{d_i-1} \cdot \mathbf{x}) \\ &= h_i \cdot f(\zeta_i^{d_i-1} \cdot \mathbf{x}) = \cdots = h_i^{d_i} \cdot f(\mathbf{x}). \end{aligned}$$

Hence $h_i = \zeta_i^{v_i}$ for some $v_i \in \mathbb{N}$. Now the vector $\mathbf{v} = (v_1, \ldots, v_r) \in \mathbb{N}^r$ satisfies (2) and the claim follows. $\square$

The following example shows that the reduceness condition in the statement of the previous result is necessary.

Example (IV.4.2). Let $f = (x^2+y)(x^2-y)^3 \in \mathbb{C}[x,y]$ and consider the cyclic quotient space $M = X(2;1,1)$. Then $\{f = 0\} \subset M$ defines a zero set but there is no k such that f^k is a function over M. This is basically Example (II.2.9).

(IV.4.3). If $f : X(\mathbf{d}; A) \to \mathbb{C}$ is a well-defined function, using A'Campo's formula, one easily sees that $Z(f^k;t) = Z(f;t^k)$. Therefore, when f is not a function but is f^k is, it is natural to define the monodromy zeta function of f as follows

$$Z(f;t) := Z(f^k; t^{\frac{1}{k}}).$$

One can prove that it is well defined, that is, it does not depend on k. Indeed, assume that f^l also induces a function over M, for some $l \geq 1$. Using Bézout's identity for k, l we have that $f^{\gcd(k,l)} : M \to \mathbb{C}$ is a function too. Denote $e := \gcd(k,l)$, $k = k_1 e$, and $l = l_1 e$. Then,

$$Z(f^k; t^{\frac{1}{k}}) = Z(f^{k_1 e}; t^{\frac{1}{k_1 e}}) = Z(f^e; t^{\frac{1}{e}}) = Z(f^{l_1 e}; t^{\frac{1}{l_1 e}}) = Z(f^l; t^{\frac{1}{l}}).$$

The zeta function defined is a rational function on $\mathbb{C}[t^{\frac{1}{k}}]$, where k is the minimum $l \geq 1$ such that f^l is a function over M. When f itself is a function, that is $k=1$, then it is a rational function on $\mathbb{C}[t]$ as usual.

The Euler characteristic of the Milnor fiber and the Milnor number are taken by definition as

$$\begin{aligned} \chi_f &:= \deg Z(f;t)\,; \\ \mu_f &:= (-1)^n(-1+\chi_f)\,, \end{aligned}$$

where the degree of $t^{i/k}$ is i/k. They are, in general, rational numbers and they verify

$$\begin{aligned} \chi_f &= \frac{\chi_{f^k}}{k}, \\ \mu_f &= \frac{(-1)^n(1-k)+\mu_{f^k}}{k}. \end{aligned}$$

In this situation, our generalized A'Campo's formula can be applied directly to f, that is, without going through f^k. See Section III for the notion of embedded $\mathbf{Q}$-resolution in this setting. Note that in this case, the numbers $m_{i,j}$'s of Theorem (IV.3.14) are rational numbers.

Let us see an example.

Example (IV.4.4). Let $f = x^a y^b (x^2 + y^3) \in \mathbb{C}[x, y]$ and consider the space $M = X(d; p, q)$ not necessarily written in a normalized form but assume $\gcd(d, p, q) = 1$ and $d|(2p - 3q)$ hold. Then f defines a zero set but does not induce a function over M.

Figure IV.7 represents an embedded $\mathbf{Q}$-resolution of $\{f = 0\} \subset M$ which has been obtained with the $\left(\frac{3}{\gcd(d,p)}, \frac{2}{\gcd(d,q)}\right)$-weighted blowing-up at the origin. The numbers in brackets are the order of the cyclic groups after normalizing and the others are the multiplicities of the corresponding divisors.

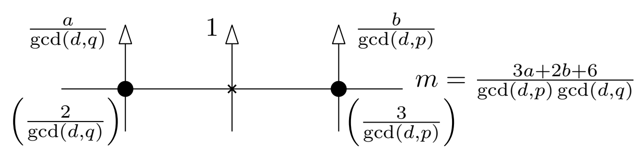


FIGURE IV.7. $\mathbf{Q}$-resolution of $\left\{x^a y^b (x^2 + y^3) = 0\right\} \subset X(d; p, q)$.

Hence the monodromy zeta function, the Euler characteristic of the Milnor fiber, and the Milnor number are:

$$\begin{aligned} Z(t) &= (1 - t^m)^{-1}, \\ \chi_f &= -m, \\ \mu_f &= m + 1. \end{aligned}$$

Here a, b are assumed to be non-zero, since otherwise the singular points of the final total space would also contribute to $Z(f; t)$. We show some special values for μ_f.

(d, p, q)	$(6, 3, 2)$	$(1, -, -)$	$(6, 3, 2)$
(a, b)	$(2, 3)$	$(1, 1)$	$(1, 1)$
μ_f	4	12	17/6

Observe that the first two values correspond to the polynomial functions $xy(x + y)$ and $xy(x^2 + y^3)$ defining over $\mathbb{C}^2$, respectively.

(IV.4.5). In the previous example, $X(d;p,q)$ can be normalized to

$$X\left(\frac{d}{(d,p)(d,q)};\frac{p}{(d,p)},\frac{q}{(d,q)}\right).$$

Under this isomorphism the polynomial $f=x^a y^b(x^2+y^3)$ is sent to

$$x^{\frac{a}{(d,q)}}\cdot y^{\frac{b}{(d,p)}}\left(x^{\frac{2}{(d,q)}}+y^{\frac{3}{(d,p)}}\right),$$

which is not a polynomial in general. This seems to force one to work with non-normalized spaces.

However, since $d|(2p-3q)$ and $\gcd(d,p,q)=1$, then $\gcd(d,q)|2$ and $\gcd(d,p)|3$. Thus the preceding expression is a polynomial times a monomial with rational exponents.

This fact is not a coincidence as the following result clarifies, see also Remark (VI.2.7). Although it can be stated in a more general setting, to simplify the ideas, we only consider polynomials in two variables over cyclic quotient singularities.

Proposition (IV.4.6). *Let d,p,q be three numbers, $\gcd(d,p,q)=1$. Let $f(x,y)\in\mathbb{C}[x,y]$ be a polynomial such that*

$$f(\xi_d^p x,\xi_d^q y)=\xi_d^v f(x,y).$$

If $x\nmid f(x,y)$ and $y\nmid f(x,y)$, then $f(x^{1/\gcd(d,q)},y^{1/\gcd(d,p)})$ is again a polynomial.

As a consequence, an arbitrary polynomial $g(x,y)$ satisfying

$$g(\xi_d^p x,\xi_d^q y)=\xi_d^v g(x,y),$$

is converted after normalizing $X(d;p,q)$ into a polynomial times a monomial with rational exponents, that is, it can be written in the form,

$$g\left(x^{\frac{1}{\gcd(d,q)}},y^{\frac{1}{\gcd(d,p)}}\right)=x^a y^b h(x,y),$$

where $h(x,y)\in\mathbb{C}[x,y]$ and $a,b\in\mathbb{Q}_{\geq 0}$.

Proof. Since $y\nmid f(x,y)$, there exists $k'\geq 0$ such that $x^{k'}$ is a monomial of f. The action is diagonal and does not change the form of the monomials. Hence $x^{k'}$ has the same behavior with respect to the action as f, that is, $\xi_d^{k'p}x^{k'}=\xi_d^v x^{k'}$. This implies that $d|(k'p-v)$. Take $k\geq 0$ such that $k\equiv -k'$ modulo d.

Now $x^k f(x,y):X(d;p,q)\to\mathbb{C}$ is a function with $x\nmid f(x,y)$. Then $\gcd(d,q)|k$ and $f(x^{1/\gcd(d,q)},y)$ is a polynomial, see Remark (VI.2.7). Analogously the expression $f(x,y^{1/\gcd(d,p)})$ is a polynomial too and the proof is complete. □

(IV.4.7). As for weighted projective planes, let $F \in \mathbb{C}[x,y,z]$ be a (p,q,r)-homogeneous polynomial with $\gcd(p,q,r)=1$. The monodromy zeta function of $F(x,y,z)$ at a point of the form $[a:b:1]$ is defined by

$$Z\big(F(x,y,z),[a:b:1];\, t\big) := Z\big(f(x,y,1),(a,b);\, t\big).$$

Note that $f(\xi_r^p x, \xi_r^q, 1) = \xi_r^{\deg(f)} f(x,y,1)$ and thus $f(x,y,1)$ satisfies the conditions of Proposition (IV.4.1)(2), where the quotient space is simply $M = X(r;p,q)$. Therefore the expression above equals

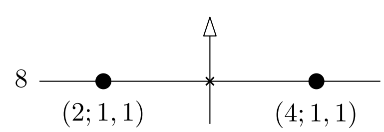
$$Z(f(x,y,1)^r,(a,b);\, t^{1/r}).$$

Analogously, the zeta function at every point of $\mathbb{P}^2(p,q,r)$ is defined and one sees that it is independent of the chosen chart.

This can be generalized to spaces like $\mathbb{P}^n_\omega/\mu_{\mathbf{d}}$, where $\mu_{\mathbf{d}}$ is an abelian finite group acting diagonally as usual.

Remark **(IV.4.8).** To define the monodromy zeta function for polynomials defining a zero set but there is no k such that f^k is a function over the quotient space, one could use A'Campo's formula and try to prove that the rational function obtained is independent of the chosen embedded **Q**-resolution. We do not insist on the veracity of this fact because it is not the purpose of this work.

Example (IV.4.9). We continue here with Example (IV.4.2). Blowing up the origin of $X(2;1,1)$ with respect to the weights $(1,2)$, an embedded **Q**-resolution of $\{f=0\} \subset X(2;1,1)$ is computed and thus it makes sense to define the zeta function using this resolution.



$$Z(t) = \frac{(1-t^4)(1-t^2)}{(1-t^8)}$$

FIGURE IV.8. Embedded **Q**-resolution of $\{(x^2+y)(x^2-y)^3=0\} \subset X(2;1,1)$ and its monodromy zeta function.

SECTION § IV.5
Why Abelian? D_4 as a Quotient Singularity

All over the chapter, the ambient space X is assumed to be $\mathbb{C}^n/G$, where G is an abelian finite subgroup of $GL(n,\mathbb{C})$. In this final part, using D_4 as a quotient singularity, it is exemplified the behavior for non-abelian groups.

As we shall see, double points in an embedded **Q**-resolution of a well-defined function $f : X \to \mathbb{C}$ contributes, in general, to its monodromy zeta function. In this sense abelian groups are the largest family for which Theorem (IV.2.1) applies.

Let $\mathbb{C}^2$ with coordinate (x, y) and consider the subgroup of $GL(2, \mathbb{C})$ generated by the matrices

$$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Thus $A^2 = B^2 = (AB)^2 = -Id_2$. This group of order 8, often denoted by BD_8, is called the *binary dihedral group*. The quotient singularity $\mathbb{C}^2/\mathrm{BD}_8$ is denoted by D_4.

Let us compute the zeta function of $f := (xy)^m : D_4 \to \mathbb{C}$, where m is an even positive integer so that the map is well defined. Consider $\pi : \widehat{\mathbb{C}}^2 \to \mathbb{C}^2$ the usual blow-up at the origin. The action BD_8 on $\mathbb{C}^2$ extends naturally to an action on $\widehat{\mathbb{C}}^2$ such that the induced map $\bar{\pi} : \widehat{\mathbb{C}}^2/\mathrm{BD}_8 \to \mathbb{C}^2/\mathrm{BD}_8 =: D_4$ defines an embedded **Q**-resolution of $\{f = 0\} \subset D_4$.

More precisely, there are three quotient singular points all of them of type $(2; 1, 1)$ located at the exceptional divisor. They correspond to the points $[0 : 1]$, $[1 : 1]$, $[i : 1] \in \mathbb{P}^1/\mathrm{BD}_8$. The strict transform intersects transversely the exceptional divisor at $P := ((0, 0), [0 : 1])$ and the equation of the total transform at this point is given by $x^m y^m : X(2; 1, 1) \to \mathbb{C}$, see Figure IV.9.

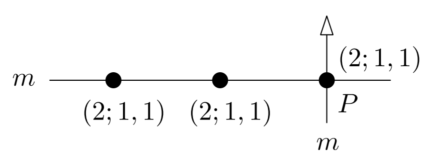


FIGURE IV.9. Embedded **Q**-resolution of $\{(xy)^m = 0\} \subset D_4$.

From Theorem (IV.2.1), the monodromy zeta function of f and the Euler characteristic of the Milnor fiber are

$$Z(t) = \frac{(1 - t^{m/2})^2}{1 - t^m} = \frac{1 - t^{m/2}}{1 + t^{m/2}}, \qquad \chi(F) = \deg Z(t) = 0.$$

In particular, $Z(t)$ is not trivial although f defines a "double point" on D_4, as claimed.

V

Mixed Hodge Structure on the Cohomology of the Milnor Fiber

Steenbrink in [**Ste77**] gave a mixed Hodge structure (MHS) on the cohomology of the Milnor fiber using a spectral sequence that is constructed from the divisors associated with the semistable reduction of an embedded resolution. The aim of this chapter is to describe explicitly a similar spectral sequence converging to the cohomology of the Milnor fiber starting with an embedded **Q**-resolution.

The main idea behind this construction is that in the classical case after considering the semistable reduction the ambient space already contains quotient singularities. We prove that the same is true for embedded **Q**-resolutions and thus the construction by Steenbrink with the spectral sequence can be adapted to provide a MHS on the cohomology $H^q(F,\mathbb{C})$.

Since the embedded **Q**-resolution can be chosen so that every exceptional divisor contributes to the complex monodromy, our spectral sequence is better in the sense that less divisors will appear in the semistable reduction and thus the combinatorial complexity of the spectral sequence will be simpler, cf. [**Mar11b**].

Section § V.1
The Semistable Reduction

This tool was introduced by Mumford in [**KKMS73**, pp. 53-108] and roughly speaking the mission of the semistable reduction is to get a reduced divisor that provides a model of the Milnor fibration. The spectral sequence converging to the cohomology of the Milnor fiber will be defined in terms of this reduced divisor, see Section V.3. Here we present a more general approach than the needed for the Milnor fibration.

(V.1.1). Let X be a complex analytic variety and let $g : X \to D^2_\eta$ be a non-constant analytic function. Assume X only has abelian quotient singularities and $g^{-1}(0)$ is a $\mathbb{Q}$-normal crossing divisor, that is, g is locally given by a function of the form $x_0^{m_0} \cdots x_k^{m_k} : X(\mathbf{d}; A) \to \mathbb{C}$. Let e be the least common multiple of all possible multiplicities appearing in the divisor $g^{-1}(0)$ and consider $\sigma : D^2_{\eta^{1/e}} \to D^2_\eta$ the branched covering defined by $\sigma(t) = t^e$.

Denote by (X_1, g_1, σ_1) the pull-back of g and σ.

$$\begin{array}{ccc} X_1 & \xrightarrow{g_1} & D^2_{\eta^{1/e}} \\ \sigma_1 \downarrow & & \downarrow \sigma \\ X & \xrightarrow[g]{} & D^2_\eta \end{array}$$

The map σ_1 is a cyclic covering of e sheets ramified over $g^{-1}(0)$. If F denotes the Milnor fiber of $g : X \to \mathbb{C}$, then $\sigma_1^{-1}(F)$ has e connected components which are projected diffeomorphically onto F.

We have not yet completed the construction of the semistable reduction because X_1 is not normal. Indeed, given $P \in g^{-1}(0)$ there exist integers $k \geq 0$ and $m_0, \ldots, m_k \geq 1$ such that

$$g(x_0, \ldots, x_n) = x_0^{m_0} \cdots x_k^{m_k} : B^{2n+2}/\mu_{\mathbf{d}} \longrightarrow \mathbb{C},$$

where B^{2n+2} is an open ball of $\mathbb{C}^{n+1}$ and the group $\mu_{\mathbf{d}}$ acts diagonally as in $(\mathbf{d}; A)$. Denote by P_1 the unique point in $\sigma_1^{-1}(P)$. Then, X_1 in a neighborhood of P_1 is of the form

$$\Big\{ \big(\big[(x_0, \ldots, x_n)\big], t \big) \in X(\mathbf{d}; A) \times \mathbb{C} \quad \Big| \quad t^e = x_0^{m_0} \cdots x_k^{m_k} \Big\}, \tag{24}$$

and hence the space X_1 is not necessarily normal.

Let $\nu : \widetilde{X} \to X_1$ be the normalization and denote by $\widetilde{g} := g_1 \circ \nu$ and $\varrho := \sigma_1 \circ \nu$ the natural maps. The normalization process has essentially two steps when the corresponding ring is a unique factorization domain (UFD). First, separate the irreducible components, and then find the normalization of each component. In the latter case, the ring in question is a domain and the following result applies.

Lemma (V.1.2). *Let $A \subset B$ be an integral extension of commutative rings. Suppose that B is an integrally closed domain such that $Q(B)|Q(A)$ is a Galois extension. Then, the normalization of the ring A is $\overline{A} = B^{\mathrm{Gal}(Q(B)|Q(A))}$.*

PROOF. Since B is normal and the extension $A \subset B$ is integral, then $\overline{A} = B \cap Q(A)$. Now the statement follows from the Galois condition. $\square$

Example (V.1.3). The ring of functions of $X(2;1,1)$ is isomorphic to $\mathbb{C}[x^2, xy, y^2]$ as an algebraic variety. In this ring the polynomial xy is irreducible but not prime. To compute the normalization of the quotient ring $\mathbb{C}[x^2, xy, y^2]/\langle xy\rangle$, one can not proceed in the same way as in a UFD. This happens because μ_2 does not define an action on the factors of the polynomial xy.

Although the ring of functions of the previous space (24) is not a UFD, see Example (V.1.3) above, to compute the normalization of X_1 one can proceed in the same spirit because of the special form of the polynomial $t^e - x_0^{m_0} \cdots x_k^{m_k}$, see proof of Proposition (V.1.7). Before that we need to introduce some notations.

Definition (V.1.4). Let X be a complex analytic space having only abelian quotient singularities and consider E a $\mathbb{Q}$-normal crossing divisor on X. Assume $P \in |E|$ is a point such that the local equation of E at P is given by the function

$$x_0^{m_0} \cdots x_k^{m_k} : X(\mathbf{d}; A) := \mathbb{C}^{n+1}/\mu_{\mathbf{d}} \longrightarrow \mathbb{C}, \quad (0 \le k \le n)$$

where $x_0, \ldots, x_n$ are local coordinates of X at P, $\mathbf{d} = (d_0, \ldots, d_r)$, and $A = (a_{ij})_{i,j} \in \mathrm{Mat}((r+1) \times (n+1), \mathbb{Z})$.

The *multiplicity* of E at P, denoted by $m(E,P)$ or simply $m(P)$ if the divisor is clear from de context, is defined by

$$m(E,P) := \gcd\left(m_0, \ldots, m_k, \frac{\sum_{j=0}^{k} a_{0j} m_j}{d_0}, \ldots, \frac{\sum_{j=0}^{k} a_{rj} m_j}{d_r}\right).$$

If there exists $T \subset |E|$ such that the function $P \in T \mapsto m(E,P)$ is constant, then we use the notation $m(T) := m(E, P_0)$, where P_0 is an arbitrary point in T.

Remark **(V.1.5).** Using the general fact $\mathrm{lcm}(\frac{m}{b_0}, \ldots, \frac{m}{b_r}) = \frac{m}{\gcd(b_0,\ldots,b_r)}$, one easily checks that this definition coincides with the one of (IV.3.12) for $k = 0$, cf. (25), that is,

$$m(E,P) := \frac{m}{L}, \qquad L = \mathrm{lcm}\left(\frac{d_0}{\gcd(d_0, a_{00})}, \ldots, \frac{d_r}{\gcd(d_r, a_{r0})}\right),$$

where E is a $\mathbb{Q}$-divisor on X locally given at the point P by the well-defined function $x_0^m : X(\mathbf{d}; A) \to \mathbb{C}$.

In the situation of (V.1.1), the multiplicity $m(g^*(0), P)$ with $P \in g^{-1}(0)$ can be interpreted geometrically as follows.

Lemma (V.1.6). *The number of prime (or irreducible) factors of the polynomial $t^e - x_0^{m_0} \cdots x_k^{m_k}$ regarded as an element in $\mathbb{C}[x_0, \ldots, x_n]^{\mu_{\mathbf{d}}} \otimes_{\mathbb{C}} \mathbb{C}[t]$ is $m(g^*(0), P)$. Hence this number also coincides with the cardinality of the fiber over P of the covering $\varrho : \widetilde{X} \to X$.*

Proof. Let us denote $\ell = \gcd(m_0, \ldots, m_k)$ and $C_i = \sum_{j=0}^{k} a_{ij} m_j$ for $i = 0, \ldots, r$. The polynomial $t^e - x_0^{m_0} \cdots x_k^{m_k} \in \mathbb{C}[x_0, \ldots, x_n, t]$ factorizes into ℓ different components as

$$t^e - x_0^{m_0} \cdots x_k^{m_k} = \prod_{i=0}^{\ell-1} \left(t^{\frac{e}{\ell}} - \zeta_\ell^i \, x_0^{\frac{m_0}{\ell}} \cdots x_k^{\frac{m_k}{\ell}} \right),$$

where ζ_ℓ is a primitive ℓ-th root of unity. However, this factors are not invariant under the group $\mu_{\mathbf{d}}$, since they are mapped to

$$t^{\frac{e}{\ell}} - \zeta_\ell^i \, x_0^{\frac{m_0}{\ell}} \cdots x_k^{\frac{m_k}{\ell}} \quad \longmapsto \quad t^{\frac{e}{\ell}} - \xi_{d_0}^{\frac{C_0}{\ell}} \cdots \xi_{d_r}^{\frac{C_r}{\ell}} \cdot \zeta_\ell^i \, x_0^{\frac{m_0}{\ell}} \cdots x_k^{\frac{m_k}{\ell}},$$

by the action of $(\xi_{d_0}, \ldots, \xi_{d_r}) \in \mu_{\mathbf{d}}$. Recall that $\mathbb{C}^{n+1}/\mu_{\mathbf{d}} = X(\mathbf{d}; A)$.

Let H_i be the cyclic group defined by $H_i := \{\xi_{d_i}^{C_i/\ell} \mid \xi_{d_i} \in \mu_{d_i}\}$, for $i = 0, \ldots, r$, and consider $H = H_0 \cdots H_r$. Since $t^e - x_0^{m_0} \cdots x_k^{m_k}$ defines a function over $X(\mathbf{d}; A) \times \mathbb{C}$, then d_i must divide C_i and, consequently, all the previous groups are (normal) subgroups of μ_ℓ. The order of μ_ℓ / H is exactly the number of prime (or irreducible) components of the preceding polynomial regarded as an element in $\mathbb{C}[x_0, \ldots, x_n]^{\mu_{\mathbf{d}}} \otimes_{\mathbb{C}} \mathbb{C}[t]$.

The order of H_i is $|H_i| = \frac{d_i}{\gcd(d_i, C_i/\ell)} = \frac{\ell}{\gcd(\ell, C_i/d_i)}$. Then, one has

$$\begin{aligned} |H| &= |H_0 \cdots H_r| = \operatorname{lcm}\big(|H_0|, \ldots, |H_r|\big) \\ &= \frac{\ell}{\gcd\left(\ell, \frac{C_0}{d_0}, \ldots, \frac{C_r}{d_r}\right)} = \frac{\ell}{m(P)}. \end{aligned} \tag{25}$$

In the expression above, a general property about greatest common divisor and least common multiple already mentioned in (V.1.5) was used. □

Assume that $g^{-1}(0) = E_0 \cup \cdots \cup E_s$ and let us denote $D_i = \varrho^{-1}(E_i)$ for $i = 0, \ldots, s$ and $D = \bigcup_{i=0}^{s} D_i$. This commutative diagram illustrates the whole process of the semistable reduction.

$$\begin{array}{ccccccc} D_i & \hookrightarrow & \widetilde{X} & \xrightarrow{\nu} & X_1 & \xrightarrow{g_1} & D^2_{\eta^{1/e}} \\ \downarrow{\scriptstyle \varrho} & & \downarrow{\scriptstyle \varrho} & & \downarrow{\scriptstyle \sigma_1} & & \downarrow{\scriptstyle \sigma} \\ E_i & \hookrightarrow & X & = & X & \xrightarrow[g]{} & D^2_\eta \end{array} \qquad \widetilde{g} = g_1 \circ \nu : \widetilde{X} \to D^2_{\eta^{1/e}} \tag{26}$$

Consider the stratification of X associated with the normal crossing divisor $g^{-1}(0) \subset X$. That is, given a possibly empty set $I \subseteq \{0, 1, \ldots, s\}$, consider

$$E_I^\circ := \Big(\cap_{i \in I} E_i \Big) \setminus \Big(\cup_{i \notin I} E_i \Big).$$

Also, let $X = \bigsqcup_{j \in J} Q_j$ be a finite stratification of X given by its quotient singularities so that the local equation of g at $P \in E_I^\circ \cap Q_j$ is of the form

$$x_1^{m_1} \cdots x_k^{m_k} : B/G \longrightarrow \mathbb{C},$$

where B is an open ball around P, and G is an abelian group acting diagonally as in $(\mathbf{d}; A)$. The multiplicities m_i's and the action G are the same along each stratum $E_I^\circ \cap Q_j$, i.e. they do not depend on the chosen point $P \in E_I^\circ \cap Q_j$. Denote $m_{I,j} := m(E_I^\circ \cap Q_j)$. Finally, assume that $E_I^\circ \cap Q_j$ is connected.

Proposition (V.1.7). *The variety $\widetilde{X}$ only has abelian quotient singularities located at $\widetilde{g}^{-1}(0) = D$ which is a reduced divisor with normal crossings on $\widetilde{X}$. Also, $\varrho : \widetilde{X} \to X$ is a cyclic branched covering of e sheets unramified over $X \setminus g^{-1}(0)$. Moreover, for $\emptyset \neq I \subseteq S := \{0, 1, \ldots, s\}$ and $j \in J$, the following properties hold.*

1. *The restriction $\varrho| : \varrho^{-1}(\overline{E_I^\circ \cap Q_j}) \to \overline{E_I^\circ \cap Q_j}$ is a cyclic branched covering of $m_{I,j}$ sheets unramified over $E_I^\circ \cap Q_j$.*
2. *The variety $\varrho^{-1}(\overline{E_I^\circ \cap Q_j})$ is a V-manifold with abelian quotient singularities with $\gcd(\{m(P) \mid P \in \overline{E_I^\circ \cap Q_j}\})$ connected components.*
3. *Let $\varphi : \widetilde{X} \to \widetilde{X}$ be the canonical generator of the monodromy of the covering ϱ. Then, its restriction to $\varrho^{-1}(\overline{E_I^\circ \cap Q_j})$ is a generator of the monodromy of $\varrho| : \varrho^{-1}(\overline{E_I^\circ \cap Q_j}) \to \overline{E_I^\circ \cap Q_j}$.*
4. *The Euler characteristic of each connected component of D_i is*

$$\sum_{\substack{\{i\} \subset I \subset \{0,1,\ldots,s\} \\ j \in J}} m_{I,j} \cdot \chi(E_I^\circ \cap Q_j) \Big/ \gcd(\{m(P) \mid P \in E_i\}).$$

Proof. First note that the morphism $\varrho : \widetilde{X} \to X$ is a cyclic branched covering unramified over $X \setminus g^{-1}(0)$, since so is $\sigma_1 : X_1 \to X$ and the normalization $\nu : \widetilde{X} \to X_1$ does not change the normal points.

Let $P \in g^{-1}(0)$ and choose coordinates $x_0, \ldots, x_n$ as in (V.1.1) so that $X_1 \subset X(\mathbf{d}; A) \times \mathbb{C}$ is locally given by the polynomial $t^e - x_0^{m_0} \cdots x_k^{m_k}$. Let us denote for $i = 0, \ldots, k$,

$$m(P) = m(g^*(0), P), \qquad e' = e/m(P), \qquad m_i' = m_i/m(P).$$

Consider the ring

$$A = \frac{\mathbb{C}[x_0, \ldots, x_n, t]}{\langle t^e - x_0^{m_0} \cdots x_k^{m_k} \rangle}.$$

The action given by $X(\mathbf{d}; A)$ is extended to A so that the variable t is invariant. Then, by Lemma (V.1.6), the normalization $\overline{A^{\mu_\mathbf{d}}}$ of the ring $A^{\mu_\mathbf{d}}$ is isomorphic to the direct sum of $m(P)$ isomorphic copies of the normalization of

$$\frac{\mathbb{C}[x_0, \ldots, x_n]^{\mu_\mathbf{d}} \otimes_\mathbb{C} \mathbb{C}[t]}{\left\langle t^{e'} - x_0^{m'_0} \cdots x_k^{m'_k} \right\rangle} = \left(\frac{\mathbb{C}[x_0, \ldots, x_n, t]}{\left\langle t^{e'} - x_0^{m'_0} \cdots x_k^{m'_k} \right\rangle} \right)^{\mu_\mathbf{d}}.$$

Therefore to compute it we only need to consider the case $m(P) = 1$, for which the ring $A^{\mu_\mathbf{d}}$ is an integral domain. Now we plan to apply Lemma (V.1.2) to a ring extension $A^{\mu_\mathbf{d}} \subset B$, where B is a polynomial algebra.

Let $c_i = e/m_i$ for $i = 0, \ldots, k$. Denote $B = \mathbb{C}[y_0, \ldots, y_n]$ and consider $A^{\mu_\mathbf{d}}$ as subring of B by putting

$$\begin{cases} x_i = y_i^{c_i} & \text{if} \quad 0 \leq i \leq k, \\ x_i = y_i & \text{for} \quad i > k, \\ t = y_0 \cdots y_k \end{cases}$$

Note that A can not be embedded in B because it is not even a domain. Since $\mu_\mathbf{d}$ acts diagonally on $\mathbb{C}^{n+2}$, there exists $N \gg 0$ such that

$$y_0^{c_0 N}, \ldots, y_k^{c_k N},\ y_{k+1}^N, \ldots, y_n^N \ \in\ A^{\mu_\mathbf{d}}.$$

This implies that the extension $A^{\mu_\mathbf{d}} \subset B$ is integral. Also, B is a normal domain. It remains to prove that $Q(B)|Q(A^{\mu_\mathbf{d}})$ is a Galois field extension. One has

$$\mathbb{C}(y_0^{c_0 N}, \ldots, y_k^{c_k N}, y_{k+1}^N, \ldots, y_n^N) \subset Q(A^{\mu_\mathbf{d}}) \subset Q(B) = \mathbb{C}(y_0, \ldots, y_n).$$

Note that the largest extension is clearly Galois. Its Galois group is abelian and it is isomorphic to

$$\mu_{c_0 N} \times \cdots \times \mu_{c_k N} \times \mu_N \times \overset{n-k}{\cdots} \times \mu_N.$$

Thus $\overline{A^{\mu_\mathbf{d}}} = B^{\mathrm{Gal}(Q(B)|Q(A^{\mu_\mathbf{d}}))}$.

This shows that $\mathrm{Spec}(\overline{A^{\mu_\mathbf{d}}})$ and hence $\widetilde{X}$ are V-manifolds. Locally D is the quotient under the group $\mathrm{Gal}(Q(B)|Q(A^{\mu_\mathbf{d}}))$ of the reduced divisor $y_0 \cdots y_k = 0$. The rest of the statement follows from the fact that the branched coverings involved are cyclic. For the last part, use the classical Riemann-Hurwitz formula. □

Remark **(V.1.8).** Assume $\mathbb{C}[x_0,\ldots,x_n]^{\mu_{\mathbf{d}}} = \mathbb{C}[\{x_0^{\alpha_0}\cdots x_n^{\alpha_n}\}_{\alpha\in\Lambda}]$. Then $A^{\mu_{\mathbf{d}}}$ is identified with the subring

$$\mathbb{C}[\{y_0^{\alpha_0 c_0}\cdots y_k^{\alpha_k c_k}\cdot y_{k+1}^{\alpha_{k+1}}\cdots y_n^{\alpha_n}\}_{\alpha\in\Lambda},\, y_0\cdots y_k] \subset \mathbb{C}[y_0,\ldots,y_n].$$

Hence the Galois extension

$$\operatorname{Gal}(Q(B)|Q(A^{\mu_{\mathbf{d}}})) \subset \mu_{c_0 N}\times\cdots\times\mu_{c_k N}\times\mu_N\times\overset{(n-k)}{\cdots}\times\mu_N$$

is given by the elements $(\xi_0,\ldots,\xi_k,\eta_{k+1},\ldots,\eta_n)$ such that

$$\forall\alpha\in\Lambda,\quad \begin{cases} \xi_0^{\alpha_0 c_0}\cdots\xi_k^{\alpha_k c_k}\cdot\eta_{k+1}^{\alpha_{k+1}}\cdots\eta_n^{\alpha_n}=1,\\ \xi_0\cdots\xi_k=1.\end{cases}$$

In general, this group is not a small subgroup of $GL(n+1,\mathbb{C})$, that is, there may exist elements of the group having 1 as an eigenvalue of multiplicity precisely n.

Remark **(V.1.9).** Note that $\varrho| : \varrho^{-1}(\overline{E_i^\circ\cap Q_j}) \to \overline{E_i^\circ\cap Q_j}$ is an isomorphism when $I=\{i\}$ and the multiplicity of E_i (at the smooth points) is equal to one.

In what follows this construction is applied to $g=f\circ\pi$, where the map $f:(M,p)\to(\mathbb{C},0)$ is the germ of a non-constant analytic function and $\pi: X\to(M,p)$ is an embedded $\mathbf{Q}$-resolution of $\{f=0\}\subset(M,p)$, cf. Section V.3. Let us see an example.

Example (V.1.10). We continue here with Example (IV.2.3) where the plane curve $f=x^p+y^q$ is considered. Recall that after the (q_1,p_1)-weighted blow-up at the origin, one obtains an embedded $\mathbf{Q}$-resolution with only one exceptional divisor $\mathcal{E}$ of multiplicity $\operatorname{lcm}(p,q)$, where $p=p_1\gcd(p,q)$ and $q=q_1\gcd(p,q)$.

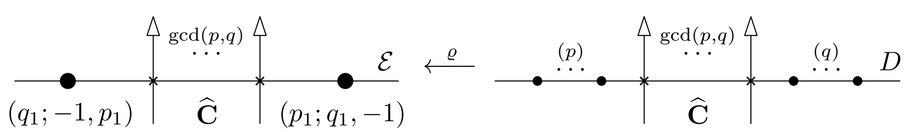


FIGURE V.1. Semistable reduction of x^p+y^q.

Following Proposition (V.1.7), $D=\varrho^{-1}(\mathcal{E})$ is irreducible and the restriction $\varrho: D\to\mathcal{E}$ is a branched covering of $\operatorname{lcm}(p,q)$ sheets. Also, the singular point of type $(q_1;-1,p_1)$ (resp. $(p_1;q_1,-1)$) is converted into p (resp. q) smooth points in the semistable reduction. Finally, $\varrho| : \varrho^{-1}(\widehat{\mathbf{C}})\to\widehat{\mathbf{C}}$ is an isomorphism. This implies that the Euler characteristic of D is

$$\chi(D)=p+q+\gcd(p,q)-pq=\gcd(p,q)+1-\mu.$$

The p points in D which are lift over the point of type $(q_1; -1, p_1)$ are smooth. Of course, the same happens for the point of type $(p_1; q_1, -1)$. Also, the intersection of the strict transform with D gives rise to $\gcd(p, q)$ smooth points. As we shall see the smoothness is not relevant for providing a mixed Hodge structure on the cohomology of the Milnor fiber.

Section § V.2
Monodromy Filtration

This exposition is extracted from [**Art94b**], which is in turn based on the book [**AGV88**].

Let H be a $\mathbb{C}$-vector space of finite dimension. Consider a nilpotent endomorphism $N : H \to H$, i.e. there exists $k \in \mathbb{N}$ such that $N^k = 0$. Its Jordan canonical form is determined by the sequence of integers formed by the size of the Jordan blocks.

There is an alternative way to encode the Jordan form giving instead an increasing filtration on H. Let us fix $k \in \mathbb{Z}$; it will be called the *central index* of the filtration. Consider a basis of H such that the matrix of N in this basis is the Jordan matrix. It is a direct sum of Jordan blocks of the following form:

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Each Jordan block determines a subfamily $\{v_1, \ldots, v_r\}$ of the basis such that $N(v_i) = v_{i-1}$ for $i = 2, \ldots, r$ and $N(v_1) = 0$. Let us denote by $l(v_i)$ the unique integer determined by the following two conditions:

(1) $l(v_i) = l(v_{i-1}) + 2$, $\forall i = 2, \ldots, r$.
(2) $\{l(v_1), \ldots, l(v_r)\}$ is symmetric with respect to k.

In fact, this integer is $l(v_i) = k - r + 2i - 1$, $\forall i = 1, \ldots, r$, as one can check directly.

Applying this construction to all the Jordan blocks, one defines W_l as the vector subspace of H generated by $\{v \mid v \text{ in the basis, } l(v) \leq l\}$. This gives rise to an increasing filtration $\{W_l\}_{l\in\mathbb{Z}}$ on H. Its graded part is denoted by $\operatorname{Gr}_l^W(H) := W_l/W_{l-1}$ for $l \in \mathbb{Z}$.

Also, denote by $J_l(N)$ the number of Jordan blocks in N of size l. Then, it is satisfied that

$$J_l(N) = \dim(\operatorname{Gr}_{k-l+1}^W(H)) - \dim(\operatorname{Gr}_{k-l-1}^W(H)).$$

Proposition (V.2.1) ([**Sch73**]). *There exists a unique increasing filtration* $\{W_l\}_{l\in\mathbb{Z}}$ *such that:*

(1) $N(W_l) \subset W_{l-2}$.

(2) $N^l : \mathrm{Gr}^W_{k+l}(H) \to \mathrm{Gr}^W_{k-l}(H)$ *is an isomorphism.* □

This filtration is called the *weight filtration* of N with central index k. One checks that the filtration $\{W_l\}_{l\in\mathbb{Z}}$ defined above satisfies these two properties. In particular, the description of $\{W_l\}_{l\in\mathbb{Z}}$ does not depend on the chosen basis.

(V.2.2). Using this construction, the Jordan form of an arbitrary automorphism $M : H \to H$ can be described too. Let $M = M_u M_s$ be the decomposition of M into its unipotent and semisimple components. It is known that $M_u M_s = M_s M_u$ and that the decomposition is unique, see [**Ser66**]. Recall that the semisimple part contains the information about the eigenvalues and the unipotent one, the information about the size of the Jordan blocks.

Note that the endomorphism $N := \log(M_u)$ is nilpotent and the number of Jordan blocks of size l is $J_l(N) = J_l(M_u) = J_l(M)$.

For a given $k \in \mathbb{Z}$, consider the weight filtration associated with N with central index k. Due to the properties of the decomposition, the subspaces W_l are invariant by the action of M_s, and thus by the action of M.

The endomorphism induced by M_u on each graded part $\mathrm{Gr}^W_l(H)$ is semisimple and, since M_u is unipotent, it is indeed the identity. Hence the actions of M and M_s on $\mathrm{Gr}^W_l(H)$ coincide.

The conclusion is that the Jordan form of M is determined by the filtration $\{W_l\}_{l\in\mathbb{Z}}$ and the action of M over $\mathrm{Gr}^W_l(H)$ for $l \in \mathbb{Z}$.

Example (V.2.3). The decomposition above for a Jordan block of size 3 is given by

$$\underbrace{\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}}_{M} = \underbrace{\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}}_{M_s} \underbrace{\begin{pmatrix} 1 & \lambda^{-1} & 0 \\ 0 & 1 & \lambda^{-1} \\ 0 & 0 & 1 \end{pmatrix}}_{M_u}.$$

Let us denote $K = \left(\begin{smallmatrix} 0 & \lambda^{-1} & 0 \\ 0 & 0 & \lambda^{-1} \\ 0 & 0 & 0 \end{smallmatrix}\right)$. Then, since $K^3 = 0$, the nilpotent endomorphism N is

$$N = \log(M_u) = K - \frac{K^2}{2} = \begin{pmatrix} 0 & \frac{1}{\lambda} & \frac{-1}{2\lambda^2} \\ 0 & 0 & \frac{1}{\lambda} \\ 0 & 0 & 0 \end{pmatrix}.$$

One clearly sees that the Jordan normal forms of the matrices N, M_u, and M are all the same type but with different eigenvalues.

(V.2.4). Let $(V,0) \subset (\mathbb{C}^{n+1},0)$ be a germ of an isolated hypersurface singularity at the origin. Denote by $\varphi : H^n(F,\mathbb{C}) \to H^n(F,\mathbb{C})$ its complex monodromy.

Consider the decomposition of $H^n(F,\mathbb{C})$ as a direct sum of two subspaces invariant under φ, $H^{\neq 1}$ and H^1, such that $Id - \varphi$ is invertible over $H^{\neq 1}$ and nilpotent over H^1.

Let $W^{\neq 1}$ be the weight filtration of $\varphi|_{H^{\neq 1}}$ with central index n. Analogously, denote by W^1 the weight filtration of $\varphi|_{H^1}$ with central index $n+1$. These filtrations satisfy $W^{\neq 1}_{-1} = W^1_1 = 0$, $W^1_{2n} = H^1$, and $W^{\neq 1}_{2n} = H^{\neq 1}$.

Definition (V.2.5). The *monodromy filtration* of the cohomology of the Milnor fiber is $W := W^1 \oplus W^{\neq 1}$.

Note that the Jordan form of the complex monodromy is completely determined by the action of φ over the graded parts of the monodromy filtration W. Let us fix the notation for the characteristic polynomials of φ acting on the following vector spaces:

Vector space	**Characteristic polynomial**
$H := H^n(F,\mathbb{C})$	$\Delta(t)$
$\mathrm{Gr}^{W^{\neq 1}}_{n-l}(H)$	$\Delta^{\neq 1}_l(t)$
$\mathrm{Gr}^{W^1}_{n-l+1}(H)$	$\Delta^1_l(t)$
$\mathrm{Gr}^{W^{\neq 1}}_{n-l}(H) \oplus \mathrm{Gr}^{W^1}_{n-l+1}(H)$	$\Delta_l(t)$

Observe that the Jordan blocks of size l are given by the polynomial $\frac{\Delta_{l-1}(t)}{\Delta_{l+1}(t)}$. More precisely, the multiplicity of $\zeta \in \mathbb{C}$ as root is this polynomial equals the number of Jordan blocks of size l for the eigenvalue ζ.

Section § V.3
The Spectral Sequence by Steenbrink

The Jordan form of the complex monodromy is closely related to the theory of mixed Hodge structures (MHS), first introduced in [**Del71a, Del71b, Del74**]. By different methods, Steenbrink and Varčenko proved that the cohomology of the Milnor fiber admits a MHS compatible with the monodromy, see [**Ste77**] and [**Var80, Var81**].

Definition (V.3.1). A *Hodge structure* of weight n is a pair $(H_{\mathbb{Z}}, F)$ consisting of a finitely generated abelian group $H_{\mathbb{Z}}$ and a decreasing filtration $F = \{F^p\}_{p\in\mathbb{Z}}$ on $H_{\mathbb{C}} := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ satisfying $H_{\mathbb{C}} = F^p \oplus \overline{F^{n-p+1}}$ for all $p \in \mathbb{Z}$. One calls F the *Hodge filtration*.

An equivalent definition is obtained replacing the Hodge filtration by a decomposition of $H_{\mathbb{C}}$ into a direct sum of complex subspaces $H^{p,q}$, where $p+q=n$, with the property that $\overline{H^{p,q}} = H^{q,p}$. The relation between these two descriptions is given by

$$H_{\mathbb{C}} = \bigoplus_{p+q=n} H^{p,q}, \qquad F^p = \bigoplus_{i\geq p} H^{i,n-i}, \qquad H^{p,q} = F^p \cap \overline{F^q}.$$

The typical example of a pure Hodge structure of weight n is the cohomology $H^n(X,\mathbb{Z})$ where X is a compact Kähler manifold. In the sequel, we will use the fact that, for compact Kähler V-manifold, $H^n(X,\mathbb{Z})$ can also be endowed with a pure Hodge structure of weight n. Deligne proved that the same is true for smooth compact algebraic varieties, see [**Del71b**].

Above, one may replace $\mathbb{Z}$ by any ring A contained in $\mathbb{R}$ such that $A\otimes_{\mathbb{Z}}\mathbb{Q}$ is a field and obtain A-Hodge structures. In particular, one uses $A=\mathbb{Q}$ or $\mathbb{R}$. In this way the primitive cohomology groups of a compact Kähler manifold are $\mathbb{R}$-Hodge structures.

Definition (V.3.2). A *mixed Hodge structure* is a triple $(H_{\mathbb{Z}}, W, F)$ where $H_{\mathbb{Z}}$ is a finitely generated abelian group, $W = \{W_n\}_{n\in\mathbb{Z}}$ is an increasing filtration on $H_{\mathbb{Q}} := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$, and $F = \{F^p\}_{p\in\mathbb{Z}}$ is a decreasing filtration on $H_{\mathbb{C}} := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$, such that F induces a $\mathbb{Q}$-Hodge structure of weight n on each graded part $\operatorname{Gr}_n^W(H_{\mathbb{Q}})$, $\forall n \in \mathbb{Z}$. One calls F the *Hodge filtration* and W the *weight filtration*.

Let us denote again by the same letter the filtration induced by W on the complexification $H_{\mathbb{C}}$, i.e. $W_n(H_{\mathbb{C}}) = W_n \otimes \mathbb{C}$. Then, the filtration induced by F on $\operatorname{Gr}_n^W(H_{\mathbb{C}})$ is defined by

$$F^p\left(\operatorname{Gr}_n^W(H_{\mathbb{C}})\right) = \frac{F^p \cap (W_n\otimes\mathbb{C}) + W_{n-1}\otimes\mathbb{C}}{W_{n-1}\otimes\mathbb{C}}.$$

Thus the condition above on the weight and Hodge filtrations can be stated as, $\forall n, p \in \mathbb{Z}$,

$$F^p\left(\operatorname{Gr}_n^W(H_{\mathbb{C}})\right) \oplus \overline{F^{n-p+1}\left(\operatorname{Gr}_n^W(H_{\mathbb{C}})\right)} = \operatorname{Gr}_n^W(H_{\mathbb{C}}).$$

Example (V.3.3). Let D be a divisor with normal crossings whose irreducible components are smooth and Kähler. Then, $H^*(D,\mathbb{Z})$ admits a functorial MHS, see [**GS75**]. This results is extended to V-manifolds with $\mathbb{Q}$-normal crossings whose irreducible components are Kähler.

In [**Del71b**], it is proven that if X is the complement in a compact Kähler manifold of a normal crossing divisor, then $H^*(X,\mathbb{Z})$ has a functorial MHS which does not depend on the ambient variety.

From now on, let us fix π an embedded **Q**-resolution of the singularity. The following result can be proven as in [**Ste77**] repeating exactly the same arguments. The main reason is that, starting with an embedded **Q**-resolution, the total space produced after the semistable reduction is again a V-manifold with abelian quotient singularities, see Proposition (V.1.7).

Theorem (V.3.4). *There exists a spectral sequence $\{E_n^{p,q}\}$ constructed from the embedded **Q**-resolution π that verifies:*

(1) *It converges to the cohomology of the Milnor fiber and degenerates at E_2.*

(2) *The spaces $E_1^{p,q}$ has a pure Hodge structure of weight p respected by the differentials. In particular, $E_2^{p,q} = E_\infty^{p,q}$ also has a pure Hodge structure of weight p.*

(3) *There exists a Hodge filtration on the cohomology of the Milnor fiber which induces a Hodge filtration on $E_\infty^{p,q}$. One constructs a weight filtration using the filtration with respect to the first index:*

$$\operatorname{Gr}_l^W(H^k(F,\mathbb{C})) \cong E_\infty^{l,k-l} \cong E_2^{l,k-l}.$$

Therefore, these two filtrations provide a MHS on the cohomology of the Milnor fibration. This structure is an invariant of the singularity which only depends on the resolution π. □

In [**Var81**], there is another construction of the MHS on the cohomology of the Milnor fiber, using asymptotic integration. The weight filtration of both MHS coincide. Varčenko's definition does not depend on the resolution. Although both Hodge filtrations do not coincide, they induce the same pure Hodge structure on the graded part of the weight filtration.

Theorem (V.3.5). *The complexification of the weight filtration of the MHS of the cohomology of the Milnor fiber is exactly the monodromy filtration.*

Moreover, the complex monodromy φ acts over the first term E_1 of the spectral sequence and commutes with the differentials. The action induced on the complexification of $E_2 = E_\infty$ coincides with the action induced on the graded parts of the monodromy filtration. □

We finish this section with the explicit description of Steenbrink's spectral sequence. As we shall see, it is constructed from the divisors associated with the semistable reduction of $g := f \circ \pi : X \to \mathbb{C}$.

(V.3.6). Consider the divisor D associated with the semistable reduction of the embedded **Q**-resolution π. Let us decompose $D = D_0 \cup D_1 \cup \cdots \cup D_s$ so that D_0 corresponds to the strict transform of the singularity and the divisor $D_+ := D_1 \cup \cdots \cup D_s$ corresponds to the exceptional components.

Let us introduce some notation.

- Let $I = (i_0, \ldots, i_k)$ with $0 \leq i_0 < \cdots < i_k \leq s$.

$$D_I = D_{i_0,\ldots,i_k} := D_{i_0} \cap \cdots \cap D_{i_k},$$

$$\check{D}_I = \check{D}_{i_0,\ldots,i_k} := D_I \setminus \bigcup_{j \neq i_0,\ldots,i_k} (D_j \cap D_I).$$

  The first one is a projective V-manifold of dimension[1] $n-k$. The second one is a smooth complex variety of the same dimension.

- Let $0 \leq i_0 < \cdots < i_k \leq s$ and $i_j < i'_j < i_{j+1}$ with $-1 \leq j \leq k$. Denote by

$$\kappa^{i'_j}_{i_0,\ldots,i_j,i_{j+1},\ldots,i_k} : D_{i_0,\ldots,i_j,i'_j,i_{j+1},\ldots,i_k} \hookrightarrow D_{i_0,\ldots,i_j,i_{j+1},\ldots,i_k},$$

  the natural inclusion.

- Let $D^{[k]} := \bigsqcup_{0 \leq i_0 < \cdots < i_k \leq s} D_{i_0,\ldots,i_k}$.

- Let $D^{[k]}_+ := \bigsqcup_{1 \leq i_0 < \cdots < i_k \leq s} D_{i_0,\ldots,i_k}$.

Definition (V.3.7). Let $k \in \mathbb{Z}$ with $0 \leq k \leq n$ and let $i, j \in \mathbb{Z}$ with $i, j \geq 0$. Define ${}^kE_1^{i,k-j}$ as

$$ {}^kE_1^{i,k-j} := \begin{cases} H^i(D^{[k]}_+, \mathbb{Q}) & \text{if} \quad j = 0, \\ H^{i-2j}(D^{[k+j]}, \mathbb{Q}) & \text{if} \quad j > 0. \end{cases} $$

Note that for $j = 0$ the divisor D_+ is used, while for $j > 0$ it is taken the divisor D. All the spaces whose cohomology is considered are compact.

(V.3.8). These spaces give rise to the first term E_1 of our spectral sequence $E = \{E_n^{p,q}\}$:

$$E_1^{p,q} := \bigoplus_{k=0}^{n} {}^kE_1^{p,q},$$

where ${}^kE_1^{p,q} = 0$ if it is not defined previously.

Note that the space ${}^pE_1^{i,k-j}$ possesses a natural pure Hodge structure of weight $i - 2j$, since it is defined as the cohomology of degree $i - 2j$ of a compact Kähler V-manifold. Performing an index shifting $\widetilde{H}^{p+j,q+j} := H^{p,q}$, ${}^pE_1^{i,k-j}$ also has a pure Hodge structure of weight i, cf. Theorem (V.3.4).

---

[1] Recall the convection on indices, e.g. for plane curve $n = 1$.

It still remains to define the differentials. In the first term E_1 the differentials are of type $(0,1)$, i.e. upward vertical arrows.

(V.3.9). Let us resume the notation of (V.3.6). Let

$$\left(\kappa^{i'_j}_{i_0,\ldots,i_k}\right)_* : H_*\left(D_{i_0,\ldots,i_j,i'_j,i_{j+1},\ldots,i_k},\mathbb{Q}\right) \longrightarrow H_*\left(D_{i_0,\ldots,i_j,i_{j+1},\ldots,i_k},\mathbb{Q}\right)$$

be the homomorphism induced by the inclusion on the homology groups. Using Poincaré duality for compact V-manifolds, one has the following Gysin-type maps:

$$\begin{array}{ccc}
H_*\left(D_{i_0,\ldots,i_j,i'_j,i_{j+1},\ldots,i_k},\mathbb{Q}\right) & \xrightarrow{\left(\kappa^{i'_j}_{i_0,\ldots,i_k}\right)_*} & H_*\left(D_{i_0,\ldots,i_j,i_{j+1},\ldots,i_k},\mathbb{Q}\right) \\
{\scriptstyle DP}\downarrow\cong & \# & \cong\downarrow{\scriptstyle DP} \\
H^{2(n-k-1)-*}\left(D_{i_0,\ldots,i_j,i'_j,i_{j+1},\ldots,i_k},\mathbb{Q}\right) & \dashrightarrow & H^{2(n-k)-*}\left(D_{i_0,\ldots,i_j,i_{j+1},\ldots,i_k},\mathbb{Q}\right)
\end{array}$$

These arrows are only possible if the spaces are compact. It is always the case except for $k=0$ and $i_0=0$, where the corresponding map is defined as zero.

By abuse of notation, the morphism associated with the dashed arrow that completes the previous diagram is again denoted by $(\kappa^{i'_j}_{i_0,\ldots,i_k})_*$.

Definition (V.3.10). The differentials on ${}^kE_1$, ${}^k\delta : {}^kE_1^{i,k-j-1} \to {}^kE_1^{i,k-j}$ are defined by

$${}^k\delta\,|_{H^{i-2(j+1)}(D_{i_0,\ldots,i_{k+j+1}},\mathbb{Q})} := \sum_{l=0}^{k+j+1} (-1)^l \left(\kappa^{i_l}_{i_0,\ldots,\widehat{i_l},\ldots,i_{k+j+1}}\right)_*.$$

Remark **(V.3.11).** The pair $({}^kE_1, {}^k\delta)$ is the term E_1 of the spectral sequence that provides the MHS of

$$\bigsqcup_{0\leq i_0<\cdots<i_k\leq s} \check{D}_{i_0,\ldots,i_k},$$

which is the complement of a divisor with normal crossings on a projective variety.

To finish with the description of the differentials, the interactions between different ${}^kE_1$ have to be taken into account. These differentials are of Mayer-Viétoris type. Denote by

$$\left(\kappa^{i_l}_{i_0,\ldots,i_{k+j}}\right)^*$$

the corresponding homomorphism on the cohomology groups.

Definition (V.3.12). The morphisms ${}^{k,k+1}\delta : {}^{k}E_1^{i,k-j} \to {}^{k+1}E_1^{i,k-k+1}$ are defined as

$$ {}^{k,k+1}\delta\,|_{H^{i-2j}(D_{i_0,\ldots,i_{k+j}},\mathbb{Q})} := \sum_{\ell \neq i_0,\ldots,i_{k+j}} (-1)^{e(l;\, i_0,\ldots,i_{k+j})} \left(\kappa^{i_l}_{i_0,\ldots,i_{k+j}} \right)^*, $$

where $e(l;\, i_0, \ldots, i_{k+j})$ is the number of coefficients $i_0, \ldots, i_{k+j}$ less than l.

Remark **(V.3.13).** The pair $({}^{k}E_1^{i,k}, {}^{k,k+1}\delta)$ is exactly the term E_1 of the spectral sequence providing the MHS of the divisor with normal crossings D_+ which appears in [**Del71b**]. Observe that the first two columns of this spectral sequence for $k = 0$ coincides with the first two columns of the term E_1 of $\{E_n^{p,q}\}$.

Definition (V.3.14). The direct sum of the differentials ${}^{k}\delta$ and ${}^{k,k+1}\delta$ is the differential δ of the term E_1.

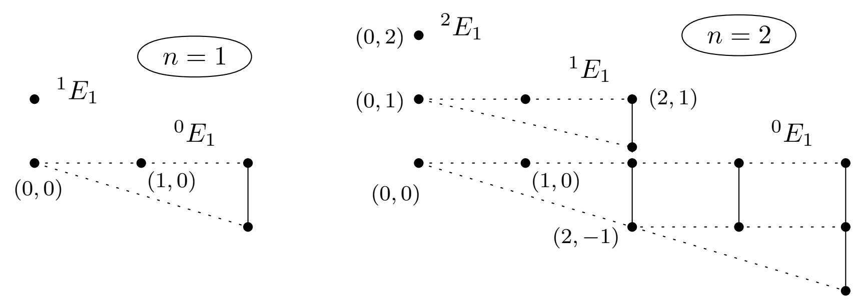


FIGURE V.2. Decomposition of $E = \{E_n^{p,q}\}$ for $n = 1, 2$.

This section ends with the explicit description of the spectral sequence $\{E_n^{p,q}\} \otimes_{\mathbb{Q}} \mathbb{C}$ for the cases $n = 1, 2$. For $n = 1$, let us denote with a triangle the terms belonging to ${}^{1}E_1$ and with a circle the ones belonging to ${}^{0}E_1$.

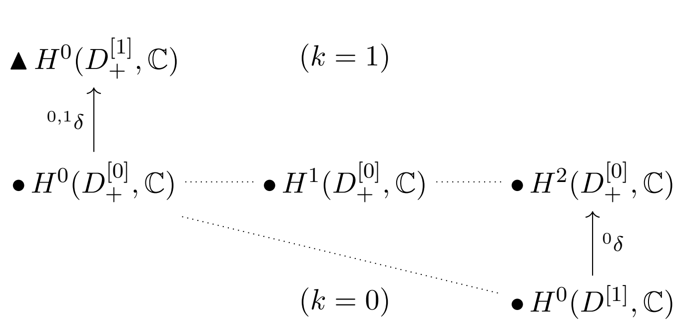


FIGURE V.3. Steenbrink's spectral sequence for plane curves, i.e. $n = 1$, with its decomposition $E_1 = {}^{0}E_1 \oplus {}^{1}E_1$.

For surfaces, that is $n = 2$, denote with a square the terms belonging to ${}^2E_1$, with a triangle the ones belonging to ${}^1E_1$, and finally with a circle those coming from ${}^0E_1$.

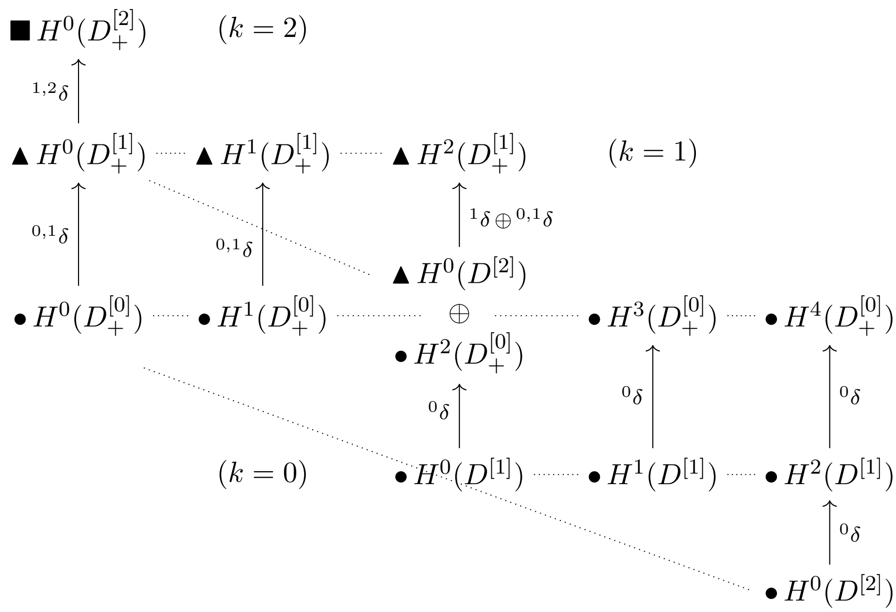


FIGURE V.4. Steenbrink's spectral sequence for surfaces, i.e. $n = 2$, with its decomposition $E_1 = {}^0E_1 \oplus {}^1E_1 \oplus {}^2E_1$.

Section § **V.4**

Example of a Plane Curve

Assume $\gcd(p, q) = \gcd(r, s) = 1$ and $\frac{p}{q} < \frac{r}{s}$. Let $f = (x^p + y^q)(x^r + y^s)$ and consider $\mathbf{C}_1 = \{x^p + y^q = 0\}$ and $\mathbf{C}_2 = \{x^r + y^s = 0\}$. In Example (I.3.15), an embedded $\mathbf{Q}$-resolution of $\{f = 0\} \subset \mathbb{C}^2$ is computed, see Figure V.5 below.

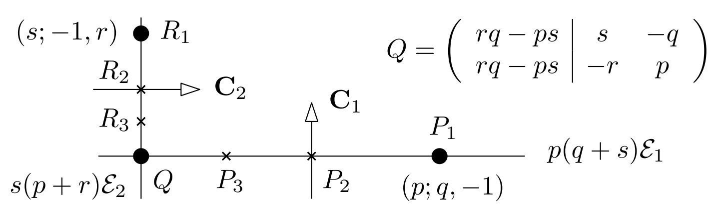


FIGURE V.5. Embedded $\mathbf{Q}$-resolution of $f = (x^p + y^q)(x^r + y^s)$.

Let us calculate here the MHS associated with the Milnor fiber of f and its complex monodromy. Before that, the notion of (weighted) dual graph in this setting is introduced.

Usually one encodes a normal crossing divisor with its *dual complex*: one vertex for each irreducible component, one edge for the intersection of two irreducible components, one triangle for the intersection of three irreducible components, etc. This is particularly useful for normal crossing divisors on surfaces where the dual complex is converted into a (weighted) graph.

(V.4.1). Let us explain in detail how to encode a $\mathbb{Q}$-divisor with $\mathbb{Q}$-normal crossings on a V-manifold using its weighted dual graph. We are interested in the following cases: the divisor $\pi^*(\mathbf{C}) = (f \circ \pi)^*(0) \subset \widehat{X}$ in an embedded $\mathbf{Q}$-resolution π of a plane curve $\mathbf{C} = f^*(0)$, and also in its corresponding semistable reduction. Their *weighted dual graph* Γ is defined as follows:

- The set V_Γ of vertices of Γ is the ordered set of irreducible components of $\pi^*(\mathbf{C})$ (for some arbitrary order). It is decomposed in two subsets $V_\Gamma = V_\Gamma^0 \coprod V_\Gamma^{\mathbf{C}}$; the first one corresponds to the exceptional components and the second one to the strict transforms, using arrow-ends.
- The set E_Γ of edges of Γ is in bijection with the double points of $\pi^*(\mathbf{C})$.
- Each $E \in V_\Gamma^0$ is weighted by its genus g_E (omitted if $g_E = 0$). It is also weighted by its self-intersection number $e_E \in \mathbb{Q}$, see Definition (III.1.2).
- Each $E \in V_\Gamma$ is weighted by m_E defined as follows: given a generic point in E, one can choose local analytic coordinates (x_E, y_E) centered at this point such that $y_E = 0$ is a local equation of E and $(f \circ \pi)(x_E, y_E) = y_E^{m_E}$.
- For $E \in V_\Gamma$, let $\operatorname{Sing}^0(E)$ be the set of singular points of $\widehat{X}$ in E which are not double points. Then, together with E, the sequence of normalized types $\{(d_P; a_P, b_P)\}_{P \in \operatorname{Sing}^0(E)}$, where E is the image of $y = 0$, is given. Note that d_P divides m_E.
- If the double point $P_\gamma = E_1 \cap E_2$, $E_1 < E_2$, associated with $\gamma \in E_\Gamma$ is singular, we provide a normalized type $(d; a, b)$, where E_1 is the image of $x = 0$ and E_2 is the image of $y = 0$. Note that d divides $am_{E_1} + bm_{E_2}$.

Remark **(V.4.2).** The weighted dual graph can also be considered associated with an abstract good $\mathbf{Q}$-resolution. This is especially useful for describing a $\mathbf{Q}$-resolution via Jung method, see [**AMO11b**] for details.

Example (V.4.3). The embedded **Q**-resolution of the preceding example, see Figure V.5 above, is computed with the (q,p)-blow-up at the origin of $\mathbb{C}^2$ followed by the $(s, qr-ps)$-blow-up at a point of type $(q;-1,p)$. Its weighted dual graph is shown in Figure V.6.

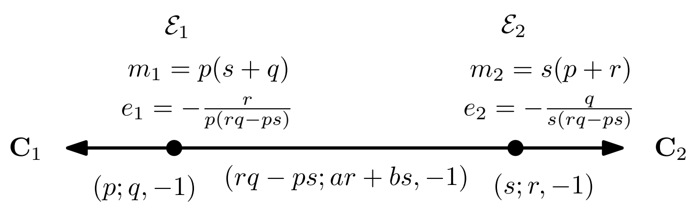


FIGURE V.6. Dual graph of the embedded **Q**-resolution of $\{(x^p+y^q)(x^r+y^s)=0\} \subset \mathbb{C}^2$, where $ap+bq=1$.

The self-intersection numbers are calculated using (III.3.2). The point Q is also of type $(rq-ps; ar+bs, -1)$ where $ap+bq=1$. In fact, it is normalized since $\gcd(rq-ps, ar+bs)=1$.

Now is the time to study the semistable reduction of the embedded **Q**-resolution obtained in Example (V.4.3). Denote by P_1 the point in $\mathcal{E}_1$ of type $(p;q,-1)$, P_2 the intersection of $\mathbf{C}_1$ with $\mathcal{E}_1$, and P_3 a generic point in $\mathcal{E}_1$. Analogously, denote by R_1 the point in $\mathcal{E}_2$ of type $(s;-1,r)$, R_2 the intersection of $\mathbf{C}_2$ with $\mathcal{E}_2$, and R_3 a generic point in $\mathcal{E}_2$, cf. Figure V.5.

Let $E = \mathbf{C}_1 \cup \mathbf{C}_2 \cup \mathcal{E}_1 \cup \mathcal{E}_2 \subset X$ be the total transform of the plane curve $\{f=0\} \subset \mathbb{C}^2$ for the embedded **Q**-resolution $\pi : X \to \mathbb{C}^2$. Also, write $g := f \circ \pi$ and use the notation in (V.1.1) and (26) so that $E = g^*(0)$. Following Definition (V.1.4), the numbers $m(E,P)$, where $P \in E$ is one of the previous points, are calculated below:

$$
\begin{array}{ll}
m(E,P_1) = q+s, & m(E,R_1) = p+r, \\
m(E,P_2) = 1, & m(E,R_2) = 1, \\
m(E,P_3) = p(q+s), & m(E,R_3) = s(p+r).
\end{array}
$$

On the other hand, by Lemma (V.1.6), the cardinality of the fiber over $Q \in \mathcal{E}_1 \cap \mathcal{E}_2$ of the covering $\varrho : \widetilde{X} \to X$ (i.e. the semistable reduction) is

$$
m(E,Q) = \gcd\Big(p(q+s), s(p+r), A, B\Big),
$$

where

$$
A = \frac{p(q+s)\cdot s + s(p+r)\cdot(-q)}{rq-ps} = -s,
$$

$$
B = \frac{p(q+s)\cdot(-r) + s(p+r)\cdot p}{rq-ps} = -p.
$$

Consequently, $m(E,Q) = \gcd(p,s)$.

From Proposition (V.1.7), one deduces the following statements. The divisors $D_1 := \varrho^{-1}(\mathcal{E}_1)$ and $D_2 := \varrho^{-1}(\mathcal{E}_2)$ have just one connected component. Their Euler characteristics are

$$\begin{aligned} \chi(D_1) &= q + s + \gcd(p,s) + 1 - p(q+s), \\ \chi(D_2) &= p + r + \gcd(p,s) + 1 - s(p+r). \end{aligned}$$

The preimage of the strict transforms, $\varrho^{-1}(\mathbf{C}_1)$ and $\varrho^{-1}(\mathbf{C}_2)$, are isomorphic to $\mathbf{C}_1$ and $\mathbf{C}_2$ respectively, and thus denoted again by the same letter.

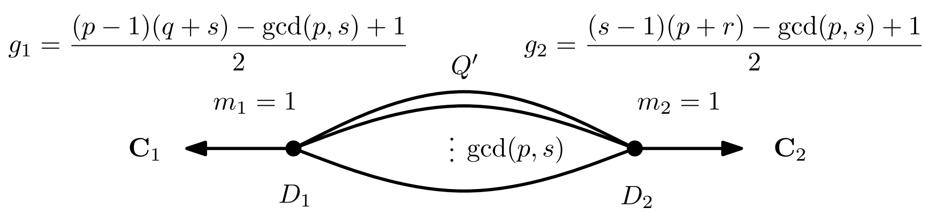


FIGURE V.7. Dual graph of the semistable reduction of f.

Since the singularity defined by f is isolated, the generalized Steenbrink's spectral sequence gives rise the exact sequences

$$0 \longrightarrow \underbrace{\operatorname{Ker}\left({}^{0,1}\delta\right)}_{\mathbb{C}} \longrightarrow H^0(D_+^{[0]}) \xrightarrow{{}^{0,1}\delta} H^0(D_+^{[1]}) \longrightarrow \underbrace{\operatorname{Coker}\left({}^{0,1}\delta\right)}_{\operatorname{Gr}_0^W H^1(F,\mathbb{C})} \longrightarrow 0,$$

and

$$0 \longrightarrow \underbrace{\operatorname{Ker}\left({}^{0}\delta\right)}_{\operatorname{Gr}_2^W H^1(F,\mathbb{C})} \longrightarrow H^0(D^{[1]}) \xrightarrow{{}^{0}\delta} H^2(D_+^{[0]}) \longrightarrow \underbrace{\operatorname{Coker}\left({}^{0}\delta\right)}_{0} \longrightarrow 0.$$

Moreover, $\operatorname{Gr}_1^W H^1(F,\mathbb{C}) = H^1(D_+^{[0]})$.

The divisor $D_+^{[0]}$ is the disjoint union of D_1 and D_2, and $D^{[1]}$ (resp. $D_+^{[1]}$) consists of $\gcd(p,s)+2$ (resp. $\gcd(p,s)$) points. Hence,

$$\left.\begin{aligned} H^0(D_+^{[0]}) &= 2 \\ H^0(D_+^{[1]}) &= \gcd(p,s) \end{aligned}\right\} \Longrightarrow H^{0,0} = \operatorname{Gr}_0^W H^1(F,\mathbb{C}) = \mathbb{C}^{\gcd(p,s)-1}.$$

Analogously, $H^0(D^{[1]}) = \gcd(p,s) + 2$ and $H^2(D_+^{[0]}) = 2$, which implies that $H^{1,1} = \operatorname{Gr}_2^W H^1(F,\mathbb{C}) = \mathbb{C}^{\gcd(p,s)}$.

As for the (pure) Hodge structure of weight 1 associated with the cohomology $H^1(D_+^{[0]}) = H^{0,1} \oplus H^{1,0}$, it is known to be determined by the genus of the corresponding real surface. In this case, $H^{0,1} = \mathbb{C}^{g_1} \oplus \mathbb{C}^{g_2}$ and $H^{1,0} = \mathbb{C}^{g_1} \oplus \mathbb{C}^{g_2}$.

Remark **(V.4.4).** It must be satisfied that $\sum_{p,q} \dim_{\mathbb{C}} H^{p,q} = \mu$. In fact, the Milnor number is the degree of the characteristic polynomial, which is by Theorem (IV.3.14) equal to

$$\Delta(t) = \frac{\big(t-1\big)\big(t^{p(q+s)}-1\big)\big(t^{s(p+r)}-1\big)}{\big(t^{q+s}-1\big)\big(t^{p+r}-1\big)}.$$

Summarizing, the mixed Hodge structure of the cohomology of the Milnor fiber $H^1(F,\mathbb{C})$ obtained is

$$H^1(F,\mathbb{C}) = \underbrace{H^{0,0}}_{\mathrm{Gr}_0^W H^1(F,\mathbb{C})} \oplus \underbrace{H^{0,1} \oplus H^{1,0}}_{\mathrm{Gr}_1^W H^1(F,\mathbb{C})} \oplus \underbrace{H^{1,1}}_{\mathrm{Gr}_2^W H^1(F,\mathbb{C})},$$

where

$$\begin{aligned} H^{0,0} &= \mathbb{C}^{\gcd(p,s)-1}, \\ H^{0,1} &= \mathbb{C}^{g_1} \oplus \mathbb{C}^{g_2}, \\ H^{1,0} &= \mathbb{C}^{g_1} \oplus \mathbb{C}^{g_2} = \overline{H^{0,1}}, \\ H^{1,1} &= \mathbb{C}^{\gcd(p,s)}. \end{aligned}$$

The genera g_1 and g_2 are calculated in Figure V.7. The action of the monodromy on $\mathrm{Gr}_0^W H^1(F,\mathbb{C})$ is given by the polynomial

$$\frac{t^{\gcd(p,s)}-1}{t-1}.$$

Note that this provides the eigenvalues of the monodromy with Jordan blocks of size 2.

VI

An Embedded Q-Resolution for Superisolated Singularities

Let $(V,0) \subset (\mathbb{C}^3,0)$ be a germ of surface singularity in $\mathbb{C}^3$. By definition, V is the zero set of a holomorphic function $f: U \to \mathbb{C}$, where $U \subset \mathbb{C}^3$ is a small neighborhood of the origin and $f(0)=0$. Denote also by f the germ at the origin of this function; it is an element of the local ring $\mathbb{C}\{x,y,z\}$.

Consider the decomposition of f into homogeneous parts,

$$f(x,y,z) = f_m(x,y,z) + f_{m+1}(x,y,z) + \cdots,$$

where f_i is homogeneous of degree i and $f_m \neq 0$. The integer m is the multiplicity of the singularity and the order of the series f. Denote by $\mathbf{C} := V(f_m) \subset \mathbb{P}^2$ the projective plane curve defined by the tangent cone of the singularity. The following families are considered in this work:

(1) *Superisolated singularity* (or, shortly, SIS): the local equation f satisfies $\mathbb{P}^2 \supset \operatorname{Sing}(\mathbf{C}) \cap V(f_{m+1}) = \emptyset$.

(2) *Yomdin-Lê singularity* (YS): the decomposition of f into homogeneous polynomials is of the form $f = f_m + f_{m+k} + \cdots$ and the condition $\operatorname{Sing}(\mathbf{C}) \cap V(f_{m+k}) = \emptyset$ holds.

(3) *Weighted Yomdin-Lê singularity* (WYS): let $\omega := (a,b,c) \in \mathbb{N}^3$ be three positive numbers such that $\gcd(a,b,c)=1$. The sum $f = f_m + f_{m+k} + \cdots$ is the decomposition of f into (a,b,c)-homogeneous parts and $\operatorname{Sing}(\mathbf{C}) \cap V(f_{m+k}) = \emptyset$ in $\mathbb{P}^2_\omega$.

Remark **(VI.0.5).** Recall that when f_i is a quasi-homogeneous polynomial with respect to ω, it defines a curve in the weighted projective plane $\mathbb{P}^2_\omega$. The notion of singular point in this setting is given in Chapter III. Now, the third definition above makes sense.

These singularities have been studied by many authors. We are content to cite merely the survey [**ALM06**], where part of the theory of these singularities and their applications including some new and recent developments are reviewed.

Although these three families can be studied simultaneously, for better exposition they are presented and treated separately. In this chapter, a detailed description of an embedded **Q**-resolution of superisolated surface singularities in terms of an embedded **Q**-resolution of its tangent cone is given. In particular, it is proven that only weighted blow-ups at points are needed.

Also, we shall see that an exceptional divisor in the resolution of $(V,0)$ contributes to the complex monodromy if and only if so does the corresponding divisor in the tangent cone. Thus the weights can be chosen so that every exceptional divisor in the **Q**-resolution of $(V,0)$ contributes to the monodromy.

The generalized A'Campo's formula applies and the characteristic polynomial and the Milnor number are calculated as an application. Other more sophisticated invariants, including mixed Hodge structure of the cohomology of the Milnor fiber, are the subject of our study for the future.

As we will see, the previous chapters are essential for describing the embedded **Q**-resolution. More precisely, the following sections and results will be very useful: §I.3–1, §I.3–2, (III.2.1), (III.3.2), (III.4.3).

Section § VI.1
Preparations for the Q-Resolution

These singularities have been introduced by Luengo and also appear in a paper by Stevens, where the μ-constant stratum is studied, see [**Lue87**] and [**Ste89**] respectively. Afterward Artal described in his PhD thesis [**Art94b**] an embedded resolution of such singularities using blow-ups at points and rational curves.

Here an embedded **Q**-resolution is given and particularly it is proven that only weighted blow-ups at points are needed. By contrast, the final ambient space obtained has abelian quotient singularities.

(VI.1.1). Let $(V,0)$ be a SIS in $(\mathbb{C}^3,0)$ defined by a holomorphic function $f:U\to\mathbb{C}$. As above, denote by m the multiplicity of V, and $\mathbf{C}$ the tangent cone. Let $\pi_0:\widehat{U}\to U$ be the blow-up at the origin. Recall that the total transform is the divisor $\pi_0^*(V)=\widehat{V}+mE_0$, where $\widehat{V}$ is the strict transform of V, and E_0 is the exceptional divisor of π_0. The intersection $\widehat{V}\cap E_0$ is identified with the tangent cone of the singularity.

Let us consider $P \in \widehat{V} \cap E_0 = \mathbf{C}$. After linear change of coordinates we can assume that $P = ((0,0,0),[0:0:1]) \equiv [0:0:1] \in \mathbf{C}$. Take a chart of $\widehat{U}$ around P where $z=0$ is the equation of E_0 and the blowing-up takes the form

$$(x,y,z) \stackrel{\pi_0}{\longmapsto} (xz,yz,z).$$

Then the equation of $\widehat{V}$ is

$$\widehat{V}: \quad f_m(x,y,1) + z\Big[f_{m+1}(x,y,1) + z f_{m+2}(x,y,1) + \cdots\Big] = 0.$$

Two cases arise: if P is smooth in the tangent cone, then $\widehat{V}$ is also smooth at P and the intersection with E_0 at that point is transverse; otherwise, i.e $P \in \operatorname{Sing}(\mathbf{C})$, the SIS condition $\operatorname{Sing}(\mathbf{C}) \cap V(f_{m+1}) = \emptyset$ implies that the previous expression in brackets is a unit in the local ring $\mathbb{C}\{x,y,z\}$ and, in particular, $\widehat{V}$ is still smooth. Now the order of $f_m(x,y,1)$ is greater than or equal to 2 and the intersection $\widehat{V} \cap E_0$ is not transverse at P.

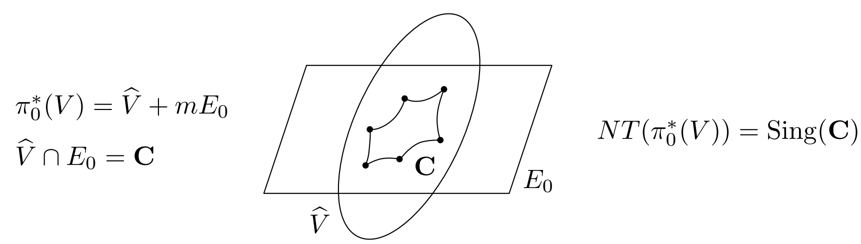


Figure VI.1. Step 0 in the embedded **Q**-resolution of $(V,0)$.

We summarize the previous discussion in the following result, which is actually the step zero in the resolution of [**Art94b**].

Lemma (VI.1.2) (Step 0)**.** *Let $P \in \mathbf{C}$ be a point in the tangent cone. Then $\widehat{V}$ is smooth in a neighborhood of P.*

Moreover, the surfaces $\widehat{V}$ and E_0 intersect transversely at P if and only if P is a smooth point in $\mathbf{C}$. Otherwise, i.e. $P \in \operatorname{Sing}(\mathbf{C})$, there exist local analytic coordinates around P such that the equations of the exceptional divisor and the strict transform are of the form

$$\begin{aligned} E_0 &: \quad z = 0\,; \\ \widehat{V} &: \quad z + h(x,y) = 0\,, \end{aligned}$$

where $h(x,y)=0$ is an equation of $\mathbf{C}$ and its order is at least 2.

Note that in the (weighted) Yomdin-Lê surface singularity case the step zero is very similar, cf. Lemma (VII.0.2) and (VII.3.1).

Section § VI.2
Construction of the Embedded Q-Resolution

We now proceed to construct the full **Q**-resolution of $(V,0)$. By the preceding lemma, the set of points where $\pi_0^*(V)$ is not a normal crossing divisor is finite, namely $\operatorname{Sing}(\mathbf{C})$. Therefore the next step in the resolution of $(V,0)$ is to blow up those points. Let us fix $P\in\operatorname{Sing}(\mathbf{C})$ and consider local coordinates as in Lemma (VI.1.2). Even though many objects that appear in this section depend on P, to simplify notation, it is omitted if no confusion seems likely to arise.

Definition (VI.2.1). Given a divisor D, the set of points where D is not a normal crossing divisor is called the *locus of non-transversality* of D and it is denote by $NT(D)$. In our case, the locus of non-transversality after the blowing-up at the origin of $(V,0)$ is $NT(\pi_0^*(V))=\operatorname{Sing}(\mathbf{C})$.

The following result is the first step in a sequence of blow-ups. We adopt the convention of writing the exceptional divisors appearing in the tangent cone in calligraphy letter, while normal letter is used for the divisors in the resolution of $(V,0)$.

Also, the objects coming from the blowing-up at $P_a\neq P$ (resp. P) are indexed by the corresponding subindex a (resp. the number 1). Finally, recall that the strict transform of a divisor is denoted again by the same letter as the own divisor.

Lemma (VI.2.2) (Step 1)**.** *Let $(p_1,q_1)\in\mathbb{N}^2$ be two positive coprime numbers. Let ϖ_1 be the weighted blow-up at $P\in\mathbf{C}$ with respect to (p_1,q_1). Denote by $\mathcal{E}_1$ its exceptional divisor and by ν_1 the (p_1,q_1)-multiplicity of $\mathbf{C}$ at P.*

Consider π_1 the (p_1,q_1,ν_1)-weighted blow-up at P in dimension 3 and E_1 the corresponding exceptional divisor. Then, the total transform of $\pi_0^*(V)$ verifies:

(1) $\pi_1^*\pi_0^*(V)=\widehat{V}+mE_0+(m+1)\nu_1E_1$,

(2) $NT(\pi_1^*\pi_0^*(V))=NT(\varpi_1^*(\mathbf{C}))$.

Proof. Let us start by blowing up the point $P\in\mathbf{C}$ with respect to the weight vector (p_1,q_1), $\gcd(p_1,q_1)=1$, in the tangent cone. Consider the local coordinates of Lemma (VI.1.2) around P so that the equation of $\mathbf{C}$ is $h(x,y)=0$; thus $\nu_1=\operatorname{ord}_{(p_1,q_1)}h(x,y)$.

Recall that the ambient space obtained has two cyclic quotient singular points corresponding to the origin of each chart and located at the exceptional divisor $\mathcal{E}_1$. The latter can be identified with the usual projective line $\mathbb{P}^1(p_1,q_1) \simeq \mathbb{P}^1$ under the map $[x:y] \mapsto [x^{q_1} : x^{p_1}]$, and it has self-intersection $\frac{-1}{p_1 q_1}$ by (III.3.2). Using the charts described in Section I.3–1,

$$\text{1st chart} \quad \left| \begin{array}{rcl} X(p_1;-1,q_1) & \longrightarrow & \widehat{\mathbb{C}}^2(p_1,q_1), \\ [(x,y)] & \mapsto & \big((x^{p_1}, x^{q_1}y), [1:y]_{(p_1,q_1)}\big); \end{array} \right.$$

$$\text{2nd chart} \quad \left| \begin{array}{rcl} X(q_1;p_1,-1) & \longrightarrow & \widehat{\mathbb{C}}^2(p_1,q_1), \\ [(x,y)] & \mapsto & \big((xy^{p_1}, y^{q_1}), [x:1]_{(p_1,q_1)}\big); \end{array} \right.$$

one obtains the following equations for the divisor $\varpi_1^*(\mathbf{C}) = \mathbf{C} + \nu_1 \mathcal{E}_1$.

$$X(p_1;-1,q_1) \supseteq \begin{cases} \mathcal{E}_1 : & x = 0; \\ \mathbf{C} : & h_1(x,y) = 0, \end{cases}$$

$$X(q_1;p_1,-1) \supseteq \begin{cases} \mathcal{E}_1 : & y = 0; \\ \mathbf{C} : & h_2(x,y) = 0. \end{cases}$$

Note that $h_1(x,y)$ and $h_2(x,y)$ are not functions on the previous quotient spaces but they define a zero set, since they satisfy

$$\begin{aligned} h_1(\xi_{p_1}^{-1}x, \xi_{p_1}^{q_1}y) &= \xi_{p_1}^{\nu_1} h_1(x,y), \\ h_2(\xi_{q_1}^{p_1}x, \xi_{q_1}^{-1}y) &= \xi_{q_1}^{\nu_1} h_2(x,y). \end{aligned} \tag{27}$$

Also, if the sum $h = h_{\nu_1} + h_{\nu_1+l} + \cdots$ is the decomposition of $h(x,y)$ into (p_1,q_1)-homogeneous parts, then $h_1(0,y) = h_{\nu_1}(1,y)$, $h_2(x,0) = h_{\nu_1}(x,1)$, and the (global) equation of $\mathbf{C} \cap \mathcal{E}_1 \subset \mathbb{P}^1(p_1,q_1)$ is of the form

$$h_{\nu_1}(x,y) = x^a y^b \prod_i (x^{q_1} - \gamma_i^{q_1} y^{p_1})^{e_i} = 0.$$

Thus the intersection multiplicity of $\mathcal{E}_1$ and $\mathbf{C}$ at the point $[\gamma_i : 1]$ is e_i, while it is $\frac{a}{q_1}$ (resp. $\frac{b}{p_1}$), not necessarily an integer, at the singular point $[0:1]$ (resp. $[1:0]$), see (III.3.4) and Remark (VI.2.3) below.

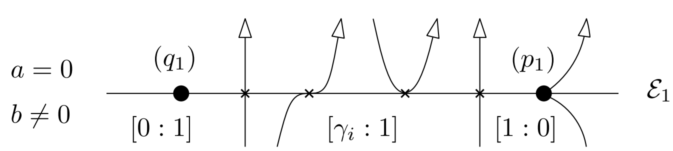


Figure VI.2. Step 1 in the embedded **Q**-resolution of $(\mathbf{C}, P)$.

Now describe the weighted blow-up at P with respect to (p_1, q_1, ν_1) in dimension 3. The new space has in general two (not three because p_1 and q_1 are coprime) cyclic quotient singular lines, each of them isomorphic to $\mathbb{P}^1$, and located at the new exceptional divisor E_1. They correspond to the lines at infinity $x = 0$ and $y = 0$ of $E_1 = \mathbb{P}^2(p_1, q_1, \nu_1)$.

As an abstract space, E_1 contains two singular points and it is isomorphic to another weighted projective plane as the following expression shows, see Proposition (I.2.5),

$$\begin{array}{ccc} \mathbb{P}^2(p_1, q_1, \nu_1) & \longrightarrow & \mathbb{P}^2\Big(\frac{p_1}{(p_1,\nu_1)}, \frac{q_1}{(q_1,\nu_1)}, \frac{\nu_1}{(p_1,\nu_1)(q_1,\nu_1)}\Big), \\ [x:y:z] & \mapsto & [x^{(q_1,\nu_1)} : y^{(p_1,\nu_1)} : z]. \end{array}$$

The multiplicity of E_1 is the sum of the (p_1, q_1, ν_1)-multiplicities, in our local coordinates, of the components of the divisor $\pi_0^*(V)$ that pass through P, that is $\nu_1 m + \nu_1 = (m+1)\nu_1$. Hence the total transform is the divisor

$$\pi_1^* \pi_0^*(V) = \widehat{V} + mE_0 + (m+1)\nu_1 E_1.$$

The equations in the three charts are given in the table below. Note that the cyclic quotient spaces are written in their normalized form, since $\gcd(p_1, q_1, \nu_1) = 1$, see Section I.3–2 for details.

$(x,y,z) \stackrel{\pi_1}{\longmapsto}$	$X(p_1; -1, q_1, \nu_1)$ $(x^{p_1}, x^{q_1}y, x^{\nu_1}z)$	$X(q_1; p_1, -1, \nu_1)$ $(xy^{p_1}, y^{q_1}, y^{\nu_1}z)$
E_0	$z = 0$	$z = 0$
E_1	$x = 0$	$y = 0$
$\widehat{V}$	$z + h_1(x,y) = 0$	$z + h_2(x,y) = 0$

$(x,y,z) \stackrel{\pi_1}{\longmapsto}$	$X(\nu_1; p_1, q_1, -1)$ $(xz^{p_1}, yz^{q_1}, z^{\nu_1})$
E_0	$-$
E_1	$z = 0$
$\widehat{V}$	$1 + h_{\nu_1}(x,y) + z^l h_{\nu_1+l}(x,y) + \cdots = 0$

Using the automorphism on $X(p_1; -1, q_1, \nu_1)$ defined by $[(x,y,z)] \mapsto [(x, y, z + h_1(x,y))]$, which is well defined due to (27), one sees that both E_0 and $\widehat{V}$ intersect transversely E_1. The equations of these intersections are given by

$$\begin{aligned} E_0 \cap E_1 &= \{z = 0\}, \\ \widehat{V} \cap E_1 &= \{z + h_{\nu_1}(x,y) = 0\}, \end{aligned}$$

as projective subvarieties in $E_1 = \mathbb{P}^2(p_1, q_1, \nu_1)$.

By (III.4.3), these smooth projective curves are two sections of E_1 with self-intersection $\frac{\nu_1}{p_1 q_1}$. They meet at $\#(\mathbf{C}\cap\mathcal{E}_1)$ points with exactly the same intersection number as in $\mathbf{C}\cap\mathcal{E}_1$, that is, for $P\in\mathbf{C}\cap\mathcal{E}_1\equiv\widehat{V}\cap E_0\cap E_1$, one has

$$\Big(E_0\cap E_1,\widehat{V}\cap E_1;\,E_1\Big)_P=\Big(\mathbf{C},\mathcal{E}_1;\,\widehat{\mathbb{C}}^2_{(p_1,q_1)}\Big)_P. \tag{28}$$

On the other hand, the intersection of the total transform with E_0 produces an identical situation to the tangent cone. All these statements follow from the equations above. In Figure VI.3, we see the intersection of the divisor $\pi_1^*\pi_0^*(V)$ with E_0 and E_1, respectively.

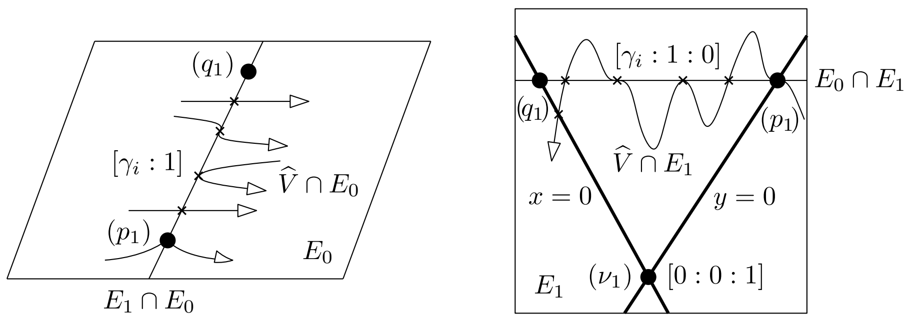


FIGURE VI.3. Step 1 in the **Q**-resolution of $(V,0)$.

Finally, the triple points of the total transform in dimension 3 are identified with the points of $\mathbf{C}\cap\mathcal{E}_1$ and, by (28), the intersection at one of those points is transverse if and only if so is it in dimension 2. This concludes the proof. □

Remark **(VI.2.3).** To study the curves $\{z=0\}$ and $\{z+h_{\nu_1}(x,y)=0\}$ in $\mathbb{P}^2(p_1,q_1,\nu_1)$ at the point $[0:1:0]$, one chooses the second chart of the weighted projective plane and obtains the local equations $z=0$ and $z+x^a=0$ around the origin of $X(q_1;p_1,\nu_1)$.

The intersection multiplicity at that point is a/q_1, although the quotient space is not written in its normalized form, see (III.2.1). Analogous considerations follow for the points $[\gamma_i:1:0]$ and $[1:0:0]$. This fact was used to prove (28).

Remark **(VI.2.4).** The curve $\widehat{V}\cap E_1$ meets the line $x=0$ (resp. $y=0$) in the projective plane $\mathbb{P}^2(p_1,q_1,\nu_1)$ at exactly one point and the intersection is always transverse. If $a=0$ (resp. $b=0$), then $\gcd(q_1,\nu_1)=q_1$ (resp. $\gcd(p_1,\nu_1)=p_1$) and that point is different from the origins, see table with the equations. This is important to obtain transversality in the next steps of the resolution of $(V,0)$.

After the first blow-up a very similar situation to Lemma (VI.1.2) is produced, except that there is a new divisor to be considered and the points where the total transform does not have normal crossings could be singular in the ambient space. The main advantage compared with Artal's resolution [**Art94b**] is that in the latter ν_1 blow-ups at points and rational curves were needed to achieve a similar situation.

(VI.2.5). Before going on with the second step let us give the natural stratification of each exceptional divisor associated with its quotient singularities.

In the tangent cone, the only exceptional divisor is decomposed as

$$\mathcal{E}_1 = \mathcal{E}_{1,1} \cup \mathcal{E}_{1,q_1} \cup \mathcal{E}_{1,p_1},$$

where $\mathcal{E}_{1,1}$ is isomorphic to $\mathbb{P}^1 \setminus \{[0:1],[1:0]\}$ and $\mathcal{E}_{1,q_1}$, $\mathcal{E}_{1,p_1}$ are the two origins of the projective line.

In dimension 3, we have two exceptional divisors. The first one is decomposed as

$$E_0 = E_{0,1} \cup E_{0,q_1} \cup E_{0,p_1},$$

where the smooth part $E_{0,1}$ is a weighted projective plane with a weighted blow-up (at a point) minus two points, and E_{0,q_1}, E_{0,p_1} are those two points. The stratification of the second one is

$$E_1 = E_{1,1} \cup E_{1,\gcd(q_1,\nu_1)} \cup E_{1,\gcd(p_1,\nu_1)} \cup E_{1,\nu_1} \cup E_{1,q_1} \cup E_{1,p_1}.$$

To describe it, denote $L_1 := \{x=0\}$, $L_2 := \{y=0\} \subset \mathbb{P}^2(p,q,\nu)$. The smooth part is the previous projective plane minus $L_1 \cup L_2$. The strata of dimension 1 are

$$\begin{aligned} E_{1,\gcd(q_1,\nu_1)} &= L_1 \setminus \{[0:0:1],[0:1:0]\}, \\ E_{1,\gcd(p_1,\nu_1)} &= L_2 \setminus \{[0:0:1],[1:0:0]\}. \end{aligned}$$

Finally, the zero-dimensional strata E_{1,ν_1}, E_{1,q_1}, E_{1,p_1} are the three origins.

See proof of Lemma (VI.2.2) and its figures.

The next result is the second step in the resolution of $(V,0)$ and it corresponds to the second step in the resolution of $(\mathbf{C},P)$. Fix a point $P_a \in NT(\varpi_1^*(\mathbf{C}))$ and, to cover all cases, assume P_a is possibly not smooth in the ambient space.

Lemma (VI.2.6) (Step 2). *Let $(p_a,q_a) \in \mathbb{N}^2$ be two positive coprime numbers. Let ϖ_a be the weighted blow-up at P_a with respect to (p_a,q_a). Denote by $\mathcal{E}_a$ its exceptional divisor, ν_a the (p_a,q_a)-multiplicity of $\mathbf{C}$ at P_a, and m_a the multiplicity of $\mathcal{E}_a$.*

Consider π_a the (p_a, q_a, ν_a)-weighted blow-up at P_a in dimension 3 and let E_a be the corresponding exceptional divisor. Then, the new total transforms satisfy:

(1) $m_a = \dfrac{\nu_a + p_a\nu_1}{\gcd(p_1, q_a + p_a q_1)}$,

(2) $\varpi_a^*\varpi_1^*(\mathbf{C}) = \mathbf{C} + \nu_1\mathcal{E}_1 + m_a\mathcal{E}_a$,

(3) $\pi_a^*\pi_1^*\pi_0^*(V) = \widehat{V} + mE_0 + (m+1)\nu_1 E_1 + (m+1)m_a E_a$,

(4) $NT(\pi_a^*\pi_1^*\pi_0^*(V)) = NT(\varpi_a^*\varpi_1^*(\mathbf{C}))$.

Proof. To fix ideas assume that $P_a = [1:0] \in \mathbf{C} \cap \mathcal{E}_1$. The other cases follow analogously. Let us first describe the (p_a, q_a)-weighted blow-up at the point P_a in the tangent cone. Consider local coordinates around P_a so that the equation of $\varpi_1(\mathbf{C}) = \mathbf{C} + \nu_1\mathcal{E}_1$ is given by the well-defined function

$$x^{\nu_1} h_1(x,y) \,:\, X(p_1; -1, q_1) \longrightarrow \mathbb{C},$$

where $x = 0$ is the exceptional divisor $\mathcal{E}_1$ and $h_1(x,y) = 0$ is the strict transform of the curve as in the proof of Lemma (VI.1.2). Hence the order at P_a is $\nu_a = \operatorname{ord}_{(p_a,q_a)} h_1(x,y)$.

Also, take α_1, β_1 satisfying the Bézout's identity $\alpha_1 p_1 + \beta_1 q_1 = 1$ so that $X(p_1; -1, q_1) = X(p_1; \beta_1, -1)$ and thus $x^{\nu_1} h_1(x,y)$ also defines a function on the latter quotient space.

Denote $d := \gcd(p_1, q_a + p_a q_1)$. Two new cyclic quotient singularities of orders $\frac{p_1 p_a}{d}$ and $\frac{p_1 q_a}{d}$ appear in the ambient space. They correspond to the origin of each chart and thus located at the new exceptional divisor

$$\mathcal{E}_a = \frac{\mathbb{P}^1_{(p_a,q_a)}}{\mu_{p_1}} = \mathbb{P}^1_{(p_a,q_a)}(p_1; -1, q_1),$$

which has self-intersection $\frac{-d^2}{p_1 p_a q_a}$, see (III.3.2).

Let $h_1 = h_{\nu_a} + h_{\nu_a + l} + \cdots$ be the decomposition of $h_1(x,y)$ into (p_a, q_a)-homogeneous parts. Denote by $g_1(x,y)$ and $g_2(x,y)$ the unique polynomials such that

$$h_1(x^{p_a}, x^{q_a} y) = x^{\nu_a} g_1(x,y),$$
$$h_1(x y^{p_a}, y^{q_a}) = y^{\nu_a} g_2(x,y).$$

Then,

$$g_1(x^{\frac{1}{d}}, y)|_{x=0} = g_1(0,y) = h_{\nu_a}(1,y),$$
$$g_2(x, y^{\frac{1}{d}})|_{y=0} = g_2(x,0) = h_{\nu_a}(x,1).$$

Hence the set of points $\mathbf{C} \cap \mathcal{E}_a$ is given by the (global) equation

$$\{h_{\nu_a}(x,y) = 0\} \subset \mathbb{P}^1_{(p_a,q_a)}(p_1; -1, q_1).$$

Note that $h_{\nu_a}(x,y)$ is not a function on the previous quotient space but it defines a zero set, since

$$
\begin{aligned}
h_{\nu_a}(\xi_{p_1}^{-1}x, \xi_{p_1}^{q_1}y) &= \xi_{p_1}^{\nu_1} h_{\nu_a}(x,y),\\
h_{\nu_a}(\xi_{p_1}^{\beta_1}x, \xi_{p_1}^{-1}y) &= \xi_{p_1}^{-\beta_1\nu_1} h_1(x,y).
\end{aligned}
\tag{29}
$$

The multiplicity of the new exceptional divisor $\mathcal{E}_a$ is $m_a = \frac{\nu_a + p_a\nu_1}{d}$. The equations of the total transform $\varpi_a^*\varpi_1^*(\mathbf{C})$ in the two charts are given in the table below, see Section I.3–1 and (I.3.14).

Equations of $\varpi_a^*\varpi_1^*(\mathbf{C})$	Chart
$\mathcal{E}_a:\ x=0$ $\mathcal{E}_1:\ -$ $\mathbf{C}:\ g_1(x^{\frac{1}{d}},y)=0$	$X\left(\frac{p_1p_a}{d}; -1, \frac{q_a+p_aq_1}{d}\right) \longrightarrow \widehat{\mathbb{C}}^2(p_a,q_a)\Big/\mu_{p_1}$ $\big[(x^d,y)\big] \mapsto \big[\big((x^{p_a}, x^{q_a}y), [1:y]_{(p_a,q_a)}\big)\big]$
$\mathcal{E}_a:\ y=0$ $\mathcal{E}_1:\ x=0$ $\mathbf{C}:\ g_2(x,y^{\frac{1}{d}})=0$	$X\left(\frac{p_1q_a}{d}; \frac{p_a+\beta_1q_a}{d}, -1\right) \longrightarrow \widehat{\mathbb{C}}^2(p_a,q_a)\Big/\mu_{p_1}$ $\big[(x,y^d)\big] \mapsto \big[\big((xy^{p_a}, y^{q_a}), [x:1]_{(p_a,q_a)}\big)\big]$

Now let us see the behavior of the (p_a,q_a,ν_a)-weighted blow-up at the point P_a in dimension 3. In our local coordinates around

$$P_a = [1:0:0] \in (\widehat{V} \cap E_0) \cap E_1,$$

the equation of the divisor $\pi_1^*\pi_0^*(V) = \widehat{V} + mE_0 + (m+1)\nu_1E_1$ is given by the function

$$z^m x^{(m+1)\nu_1}(z + h_1(x,y)) \,:\, X(p_1; -1, q_1, \nu_1) \longrightarrow \mathbb{C}.$$

Note that $X(p_1;-1,q_1,\nu_1) = X(p_1;\beta_1,-1,-\beta_1\nu_1)$. Now we use the charts described in Section I.3–2.

The ambient space has two new lines of singular points corresponding to the lines at infinity $\{x=0\}$ and $\{y=0\}$ of the exceptional divisor

$$E_a = \frac{\mathbb{P}^2_{(p_a,q_a,\nu_a)}}{\mu_{p_1}} = \mathbb{P}^2_{(p_a,q_a,\nu_a)}(p_1;-1,q_1,\nu_1).$$

Recall that $[0:0:1] \in E_a$ is a quotient singular point not necessarily cyclic.

The multiplicity of E_a is the sum of the (p_a,q_a,ν_a)-multiplicities of the components of the divisor $\pi_1^*\pi_0^*(V)$ that pass through P_a divided by $d = \gcd(p_1, q_a + p_aq_1)$, that is,

$$\frac{\nu_a m + p_a(m+1)\nu_1 + \nu_a}{d} = \frac{(m+1)(\nu_a + p_a\nu_1)}{d} = (m+1)m_a.$$

To study the locus of non-transversality in a neighborhood of E_a, the equations of the total transform are calculated in the following table. Note that the third chart is not given in a normalized form but, as we shall see, it is not needed for our purpose.

1st chart

$$E_a : x = 0 \qquad E_1 : - \qquad E_0 : z = 0$$

$$X\left(\frac{p_1 p_a}{d}; -1, \frac{q_a + p_a q_1}{d}, m_a\right)$$

$$\widehat{V} : z + g_1(x^{\frac{1}{d}}, y) = 0 \qquad \big[(x^d, y, z)\big] \mapsto \Big[\big((x^{p_a}, x^{q_a} y, x^{\nu_a} z), [1 : y : z]\big)\Big]$$

2nd chart

$$E_a : y = 0 \qquad E_1 : x = 0 \qquad E_0 : z = 0$$

$$X\left(\frac{p_1 q_a}{d}; \frac{p_a + \beta_1 q_a}{d}, -1, \frac{\nu_a - \beta_1 \nu_1 q_a}{d}\right)$$

$$\widehat{V} : z + g_2(x, y^{\frac{1}{d}}) = 0 \qquad \big[(x, y^d, z)\big] \mapsto \Big[\big((x y^{p_a}, y^{q_a}, y^{\nu_a} z), [x : 1 : z]\big)\Big]$$

3rd chart

$$E_a : z = 0 \qquad E_1 : x = 0 \qquad E_0 : -$$

$$X\left(\begin{array}{c|ccc} \nu_a & p_a & q_a & -1 \\ p_1 \nu_a & p_a \nu_1 + \nu_a & q_a \nu_1 - q_1 \nu_a & -\nu_1 \end{array}\right)$$

$$\widehat{V} : 1 + \frac{h_1(x z^{p_a}, y z^{q_a})}{z^{p_a}} = 0 \qquad \big[(x, y, z)\big] \mapsto \Big[\big((x z^{p_a}, y z^{q_a}, z^{\nu_a}), [x : y : 1]\big)\Big]$$

The divisor $mE_0 + (m+1)\nu_1 E_1 + (m+1) m_a E_a$ has clearly normal crossings. Since the polynomial $x^{\nu_1} y^{m_a} g_2(x, y^{\frac{1}{d}})$ defines a function on the quotient space $X(\frac{p_1 q_a}{d}; \frac{p_a + \beta_1 q_a}{d}, -1)$, the following map is a well-defined automorphism on the corresponding cyclic quotient space

$$X\left(\frac{p_1 q_a}{d}; \frac{p_a + \beta_1 q_a}{d}, -1, \frac{\nu_a - \beta_1 \nu_1 q_a}{d}\right)$$

$$[(x, y, z)] \longmapsto [(x, y, z + g_2(x, y^{\frac{1}{d}})]$$

and hence the divisor $\widehat{V} + (m+1)\nu_1 E_1 + (m+1) m_a E_a$ has also normal crossings.

Only the intersection $\widehat{V} \cap E_0 \cap E_a$ has to be studied. To do so, we consider the curves $E_0 \cap E_a = \{z = 0\}$ and $\widehat{V} \cap E_a = \{z + h_{\nu_a}(x, y) = 0\}$ as subvarieties in $E_a = \mathbb{P}^2_{(p_a, q_a, \nu_a)}(p_1; -1, q_1, \nu_1)$. The first two charts of the latter space are respectively isomorphic to

$$X\left(\frac{p_1 p_a}{d}; \frac{q_a + p_a q_1}{d}, m_a\right), \qquad X\left(\frac{p_1 q_a}{d}; \frac{p_a + \beta_1 q_a}{d}, \frac{\nu_a - \beta_1 \nu_1 q_a}{d}\right).$$

By Proposition (III.4.3), these smooth projective curves are two sections of E_a with self-intersection number $\frac{\nu_a d}{p_1 p_a q_a}$; note that

$$\gcd\Big(p_1,\, q_a + p_a q_1,\, \nu_a + p_a\nu_1,\, q_1\nu_a - \nu_1 q_a\Big) = d,$$

which is the greatest common divisor needed in the proposition mentioned above.

Now working as in Remark (VI.2.3), see also (III.2.1) and (III.3.4), one sees that they meet at $\#(\mathbf{C}\cap\mathcal{E}_a)$ points with exactly the same intersection multiplicity as in the latter, that is, for $P \in \mathbf{C}\cap\mathcal{E}_a \equiv \widehat{V}\cap E_0 \cap E_a$, one has

$$(30) \qquad \Big(E_0\cap E_a, \widehat{V}\cap E_a;\, E_a\Big)_P = \Big(\mathbf{C}, \mathcal{E}_a;\, \widehat{\mathbb{C}}^2_{(p_a,q_a)}/\mu_{p_1}\Big)_P .$$

As in the first step, the intersection of the total transform with E_0 produces an identical situation to the tangent cone. Also, note that Figures VI.2 and VI.3 can also be used to illustrate the general situation here. The main difference is that the line at infinity $\{x = 0\} \subset E_a$ coincides with $E_1 \cap E_a$ and thus the point $[0:0:1] \in E_a$ belongs to two divisors.

Now, to finish, observe that the triple points $\widehat{V}\cap E_0\cap E_a$ of the total transform in dimension 3 are identified with the points of $\mathbf{C}\cap\mathcal{E}_a$ and, by (30), the intersection at one of those points is transverse if and only if so is it in dimension 2. □

Remark **(VI.2.7).** Note that if $x^k g_1(x,y) : X(e;-1,r) \to \mathbb{C}$ defines a function and $x \nmid g_1(x,y)$, then $d := \gcd(e,r)$ divides k and $g_1(x^{\frac{1}{d}}, y)$ is a polynomial. This implies, in particular, that m_a is an integer since the polynomial $x^{\nu_a+p_a\nu_1} g_1(x,y)$ defines a function on $X(p_1p_a;-1,q_a+p_aq_1)$.

Remark **(VI.2.8).** If $y \nmid h_{\nu_a}(x,y)$, or equivalently $\mathcal{E}_a \ni [1:0] \notin \mathbf{C}$, then $p_a|\nu_a$ and $p_1|(\nu_1 + \frac{\nu_a}{p_a})$; consequently, $\gcd(\frac{p_1p_a}{d}, m_a) = \frac{p_1p_a}{d}$.

Indeed, assume that $h_{\nu_a}(x,y) = x^{e_0}y^{e_\infty}\prod_{i\ge1}(x^{q_a} - \gamma_i y^{p_a})^{e_i}$. Then, its order is $\nu_a = e_0p_a + e_\infty q_a + p_aq_a\sum_i e_i$. By (29), the following two expressions are equal:

$$\begin{aligned} h_{\nu_a}(\xi_{p_1}^{-1}x, \xi_{p_1}^{q_1}y) &= \xi_{p_1}^{-e_0+e_\infty q_1} x^{e_0}y^{e_\infty}\prod(\xi_{p_1}^{-q_a}x^{q_a} - \xi_{p_1}^{q_1p_a}\gamma_i y^{p_a})^{e_i} = \\ &= \xi_{p_1}^{-e_0+e_\infty q_1 - q_a\sum_i e_i} x^{e_0}y^{e_\infty}\prod(x^{q_a} - \xi_{p_1}^{q_1p_a+q_a}\gamma_i y^{p_a})^{e_i}, \end{aligned}$$

$$\xi_{p_1}^{\nu_1} h_{\nu_a}(x,y) = \xi_{p_1}^{\nu_1} x^{e_0}y^{e_\infty}\prod(x^{q_a} - \gamma_i y^{p_a})^{e_i}.$$

Hence p_1 divides $\nu_1 + e_0 - e_\infty q_1 + q_a\sum_i e_i$. In the case $e_\infty = 0$, the latter number is $\nu_1 + \frac{\nu_a}{p_a}$ and the claim follows.

Anologously, if $x \nmid h_{\nu_a}(x,y)$ ($\Leftrightarrow \mathcal{E}_a \ni [0:1] \notin \mathbf{C} \Leftrightarrow e_0 = 0$), then one has that $q_a|\nu_a$ and $p_1|(\frac{\nu_a}{q_a} - \beta_1\nu_1)$; consequently, $\gcd(\frac{p_1q_a}{d}, \frac{p_a+\beta_1q_a}{d}) = \frac{p_1q_a}{d}$.

Remark **(VI.2.9).** Although the third chart, say X_3, is not in general a cyclic quotient space, there are a couple of situations where it is.

- If $\gcd(\nu_1, \nu_a) = 1$, then the action given by the second row includes the first one and thus X_3 is just $\mathbb{C}^3$ under the second row action.
- Also if $\gcd(p_1, \nu_1) = 1$ and λ is the inverse of ν_1 modulo p_1, then $X(p_1; -1, q_1, \nu_1)$ can be written in the form $X(p_1; \lambda, -\lambda q_1, -1)$ and thus $X_3 = X(p_1\nu_a; p_a + \lambda\nu_a, q_a - \lambda q_1 \nu_a, -1)$.

Let Γ and Γ_+ be the dual graphs associated with the total transform and the exceptional divisor, after having computed an embedded **Q**-resolution of $(\mathbf{C}, P)$, respectively. Denote by $S(\Gamma)$ and $S(\Gamma_+)$ the sets of their vertices. The classical partial order on $S(\Gamma_+)$ is denoted by $\preccurlyeq$.

The locus of non-transversality after the last blow-up in dimension 3 is identified with the locus of non-transversality in the resolution of $(\mathbf{C}, P)$. Each of these points corresponds to a weighted blow-up in the resolution of the tangent cone, that is, to a vertex of Γ_+. Thus in the next step we need to blow-up those points to produce a similar situation. Again the same operation will be applied to the points where the total transform is not a normal crossing divisor. These points will also be associated with vertices of Γ_+.

The following result is proven by induction on $S(\Gamma_+)$ using the relation $\preccurlyeq$. Lemma (VI.2.2) is the first step in the induction. The proof of Lemma (VI.2.6) tells us the way to show the general case. Let $b \in S(\Gamma_+)$ be a vertex such that P_b belongs to the locus of non-transversality of the total transform. As usual, denote by $\mathcal{E}_b$ the exceptional divisor appearing after blowing up the point P_b.

Proposition (VI.2.10) (Step b)**.** *Let ϖ_b be the (p_b, q_b)-weighted blow-up at P_b with $b \in S(\Gamma_+)$. Denote by $\mathcal{E}_b$ its exceptional divisor, ν_b the (p_b, q_b)-multiplicity of $\mathbf{C} \subset \mathbb{C}^2$, and m_b the multiplicity of $\mathcal{E}_b$.*

Consider π_b the (p_b, q_b, ν_b)-weighted blow-up at P_b in dimension 3 and E_b the corresponding exceptional divisor. Then, after blowing up the point P_b, the new total transform verifies:

(1) *The exceptional divisor E_b is isomorphic to $\mathbb{P}^2(p_b, q_b, \nu_b)/\mu_e$ and its multiplicity equals $(m+1)m_b$. In general, the lines at infinity $\{x = 0\}$ and $\{y = 0\}$ are quotient singular in the ambient space and the point $[0:0:1]$ is the only one which may be non-cyclic. By contrast, the stratum $\{z = 0\} \setminus \{[0:1:0], [1:0:0]\} \subset E_b$ is always smooth.*

(2) *Let a be a vertex such that $a \prec b$. Then, $E_a \cap E_b \neq \emptyset$ if and only if $P_b \in \mathcal{E}_a$. In such a case, the curve $E_a \cap E_b$ is one of the two lines at infinity of E_b different from $\{z = 0\}$. If $P_b \in \mathcal{E}_a \cap \mathcal{E}_{a'}$, $a \neq a'$, then the corresponding lines are different and hence they meet at the point $[0 : 0 : 1]$.*

(3) *The intersection of the rest of components with E_0 produces an identical situation to the resolution of $(\mathbf{C}, P)$, after blowing up the point P_b. More precisely,*

$$\begin{aligned} \widehat{V} \cap E_0 &= \mathbf{C}, \\ E_b \cap E_0 &= \mathcal{E}_b, \\ E_a \cap E_0 &= \mathcal{E}_a, \quad \forall a \preccurlyeq b. \end{aligned}$$

(4) *The curves $E_0 \cap E_b = \{z = 0\}$ and $\widehat{V} \cap E_b = \{z + H_{\nu_b}(x, y) = 0\}$ are two $\left(\frac{-\mathcal{E}_b^2 \nu_b}{d}\right)$-sections of E_b and the intersecting points can be identified with $\mathbf{C} \cap \mathcal{E}_b$. Moreover, the intersection multiplicity of these two sections at one of those points is the same as in the latter, that is, for $P \in \mathbf{C} \cap \mathcal{E}_b \equiv \widehat{V} \cap E_0 \cap E_b$, one has*

$$\Big(E_0 \cap E_b, \widehat{V} \cap E_b;\, E_b\Big)_P = \Big(\mathbf{C}, \mathcal{E}_b;\, \widehat{\mathbb{C}}^2_{(p_b,q_b)}/\mu_e\Big)_P.$$

If $P_b \in \mathcal{E}_a$, then $E_a \cap E_b$ and $\widehat{V} \cap E_b$ always meet at exactly one point. This point passes through $E_0 \cap E_b$ if and only if $\mathbf{C} \cap \mathcal{E}_a \cap \mathcal{E}_b \neq \emptyset$. This is the case when there exist quadruple points.

(5) *The locus of non-transversality of the total transform in dimension 3 is identified with the one in the resolution of $(\mathbf{C}, P)$. These points belong to $\widehat{V} \cap E_0 \cap E_b = \mathbf{C} \cap \mathcal{E}_b$ and they correspond to the ones where the curves $E_0 \cap E_b$ and $\widehat{V} \cap E_b$, or equivalently $\mathcal{E}_b$ and $\mathbf{C}$, do not meet transversely.*

(6) *The strict transform $\widehat{V}$ never passes through $[0 : 0 : 1] \in E_b$. In particular, $\widehat{V}$ only contains cyclic quotient singularities.*

Proof. By induction on $S(\Gamma_+)$ with respect to $\preccurlyeq$. The base case is Lemma (VI.2.2). As for the inductive step, one proceeds as in the proof of (VI.2.6). Assume, by induction, that the local equation of the total transform in the resolution of the tangent cone around P_b is given by the function $(\gcd(e, r) = \gcd(e, s) = 1)$

$$x^{m_a} y^{m_{a'}} H(x, y) \,:\, X(e; r, s) \longrightarrow \mathbb{C}, \tag{31}$$

where $\mathbf{C} = \{H(x, y) = 0\}$ is the equation of the strict transform and the others correspond to the divisors $\mathcal{E}_a$ and $\mathcal{E}_{a'}$ (they may not appear if m_a or $m_{a'}$ equals zero).

Also, the equation of the total transform around P_b in dimension 3 is given by the function

$$x^{(m+1)m_a} \cdot y^{(m+1)m_{a'}} \cdot z^m \big[z + H(x,y)\big] \;:\; X(e;r,s,t) \longrightarrow \mathbb{C}, \tag{32}$$

where $\widehat{V} = \{z + H(x,y) = 0\}$ is the strict transform, $E_0 = \{z = 0\}$, and the others are the divisors E_a and $E_{a'}$ (if they exist). Using that both (31) and (32) are well-defined functions, one has

$$t + m_a \cdot r + m_{a'} \cdot s \in (e).$$

The verification of the statement is very simple once the local equations of the divisors appearing in the total transform are calculated. The main ideas behind are contained in the proof of Lemmas (VI.2.2) and (VI.2.6). The details are omitted to avoid repeating the same arguments; only the local equations are given, see below.

To do so, consider the following data and use the charts described in Sections I.3–1 and I.3–2. As auxiliary results (III.2.1), (III.3.2), and (III.4.3) are also needed.

$$\nu_b = \operatorname{ord}_{(p_b,q_b)} H(x,y) \qquad m_b = \frac{p_b \cdot m_a + q_b \cdot m_{a'} + \nu_b}{d}$$
$$d = \gcd(e,\, p_b \cdot s - q_b \cdot r)$$

$$s'r + s \equiv 0 \mod (e) \qquad r's + r \equiv 0 \mod (e)$$
$$H_1(x,y) = \frac{H(x^{p_b}, x^{q_b} y)}{x^{\nu_b}} \qquad H_2(x,y) = \frac{H(xy^{p_b}, y^{q_b})}{y^{\nu_b}}$$

These are the equations in the resolution of the tangent cone $\mathbf{C}$ presented as zero sets in the corresponding (abelian) quotient space, cf. proof of Lemma (VI.2.6).

Equations	Chart
$\mathcal{E}_b :\ x = 0$ $\mathcal{E}_a :\ -$ $\mathcal{E}_{a'}\ \ y = 0$ $\mathbf{C} :\ H_1(x^{\frac{1}{d}}, y) = 0$	$X\left(\frac{ep_b}{d}; -1, \frac{q_b + s'p_b}{d}\right) \longrightarrow \widehat{\mathbb{C}}^2(p_b,q_b)\big/\mu_e$ $\big[(x^d, y)\big] \mapsto \big[\big((x^{p_b}, x^{q_b}y), [1:y]_{(p_b,q_b)}\big)\big]$
$\mathcal{E}_b :\ y = 0$ $\mathcal{E}_a :\ x = 0$ $\mathcal{E}_{a'} :\ -$ $\mathbf{C} :\ H_2(x, y^{\frac{1}{d}}) = 0$	$X\left(\frac{eq_b}{d}; \frac{p_b + r'q_b}{d}, -1\right) \longrightarrow \widehat{\mathbb{C}}^2(p_b,q_b)\big/\mu_e$ $\big[(x, y^d)\big] \mapsto \big[\big((xy^{p_b}, y^{q_b}), [x:1]_{(p_b,q_b)}\big)\big]$

In dimension 3, the local equations of the total transform are presented as well-defined functions over the corresponding quotient spaces. The notation is self-explanatory to recognize the equation of each divisor.

1st chart

$$X\left(\frac{ep_b}{d}; -1, \frac{q_b + s'p_b}{d}, \frac{\nu_b + t'p_b}{d}\right) \longrightarrow \mathbb{C}$$

$$x^{(m+1)m_b} \cdot y^{(m+1)m_{a'}} \cdot z^m \big[z + H_1(x^{\frac{1}{d}}, y)\big]$$

2nd chart

$$X\left(\frac{eq_b}{d}; \frac{p_b + r'q_b}{d}, -1, \frac{\nu_b + t''q_b}{d}\right) \longrightarrow \mathbb{C}$$

$$x^{(m+1)m_a} \cdot y^{(m+1)m_b} \cdot z^m \big[z + H_2(x, y^{\frac{1}{d}})\big]$$

3rd chart

$$X\left(\begin{array}{c|ccc} \nu_b & p_b & q_b & -1 \\ e\nu_b & r\nu_b - tp_b & s\nu_b - tq_b & t \end{array}\right) \longrightarrow \mathbb{C}$$

$$x^{(m+1)m_a} \cdot y^{(m+1)m_{a'}} \cdot z^{(m+1)m_b \cdot d}\left[1 + \tfrac{H(xz^{p_b}, yz^{q_b})}{z^{\nu_b}}\right]$$

Here t' and t'' are taken so that $t'r + t \equiv 0$ and $t''s + t \equiv 0$ modulo (e). The exceptional divisor E_b is identified with $\mathbb{P}^2(p_b, q_b, \nu_b)/\mu_e$ where the action is of type $(e; r, s, t)$, i.e. $E_b = \mathbb{P}^2_{(p_b,q_b,\nu_b)}(e; r, s, t)$. □

Remark **(VI.2.11).** Note that the equations after the blowing-up at P_b around the points where the total transform is not a normal crossing divisor are of the same form as in (31) and (32). Hence, by induction, this fact holds for every stage of the resolution.

Remark **(VI.2.12).** Let us write $H_{\nu_b}(x, y) = x^{e_0} y^{e_\infty} \prod_{i\geq 1}(x^{q_b} - \gamma_i y^{p_b})^{e_i}$. As in Remark (VI.2.8), if

$$y \nmid H_{\nu_b}(x, y) \quad \big(\Longleftrightarrow \mathcal{E}_b \ni [1:0] \notin \mathbf{C} \Longleftrightarrow e_\infty = 0\,\big),$$

then $p_b | \nu_b$ and $e | (\frac{\nu_b}{p_b} + t')$; consequently, $\gcd(\frac{ep_b}{d}, \frac{\nu_b + t'p_b}{d}) = \frac{ep_b}{d}$. Analogously, $e_0 = 0$ implies $\gcd(\frac{eq_b}{d}, \frac{v_b + t''q_b}{d}) = \frac{eq_b}{d}$.

Theorem (VI.2.13). *Given an embedded* **Q**-*resolution of* $(\mathbf{C}, P)$ *for all* $P \in \mathrm{Sing}(\mathbf{C})$*, one can construct an embedded* **Q**-*resolution of* $(V, 0)$*, consisting of weighted blow-ups at points. Each of these blow-ups corresponds to a weighted blow-up in the resolution of* $(\mathbf{C}, P)$ *for some* $P \in \mathrm{Sing}(\mathbf{C})$*, that is, it corresponds to a vertex of* Γ_+^P. □

We shall see later that an exceptional divisor in the resolution of $(V, 0)$ obtained contributes to the monodromy if and only if so does the corresponding divisor in the tangent cone, see (VI.3.3) and (VI.3.5).

In particular, the weights can be chosen so that every exceptional divisor in the embedded **Q**-resolution of $(V,0)$ contributes to the monodromy.

Section § VI.3
The Characteristic Polynomial of the Monodromy

Here we plan to apply Theorem (IV.3.14) to compute the characteristic polynomial of the monodromy and the Milnor number of $(V,0)$ in terms of its tangent cone $(\mathbf{C},P)$. Some notation need to be introduced, concerning the stratification of each irreducible component of the exceptional divisor in terms of its quotient singularities.

(VI.3.1). Given a point $P\in\operatorname{Sing}(\mathbf{C})$, denote by $\varrho^P : Y^p\to(\mathbf{C},P)$ an embedded **Q**-resolution of the tangent cone. Assume that the total transform is given by

$$(\varrho^P)^*(\mathbf{C},P)=\mathbf{C}+\sum_{a\in S(\Gamma_+^P)} m_a^P\mathcal{E}_a^P,$$

where $\mathcal{E}_a^P$ is the exceptional divisor of the (p_a^P,q_a^P)-blow-up at a point P_a belonging to the locus of non-transversality. Denote by ν_a^P the (p_a^P,q_a^P)-multiplicity of $\mathbf{C}$ at P_a.

Recall that $\mathcal{E}_a^P$ is naturally isomorphic to $\mathbb{P}^1_{(p_a^P,q_a^P)}/\mu_e$. Using this identification, define

$$\mathcal{E}_{a,1}^P=\mathcal{E}_a^P\setminus\{[0:1],[1:0]\},\qquad \mathcal{E}_{a,x}^P=\{[0:1]\},\qquad \mathcal{E}_{a,y}^P=\{[1:0]\}.$$

The strata $\check{\mathcal{E}}_{a,j}^P:=\mathcal{E}_{a,j}^P\setminus\left(\mathcal{E}_{a,j}^P\cap\left(\bigcup_{b\neq a}\mathcal{E}_b^P\cup\mathbf{C}\right)\right)$ for $j=1,x,y$ (see notation just above Theorem (IV.3.14)) will be considered in Lemma (VI.3.3).

(VI.3.2). Let us see the situation in the superisolated singularity $(V,0)$. Denote by $\rho : X\to(V,0)$ the embedded **Q**-resolution obtained following Proposition (VI.2.10). Then, the total transform is

$$\rho^*(V,0)=\widehat{V}+mE_0+\sum_{\substack{P\in\operatorname{Sing}(\mathbf{C})\\ a\in S(\Gamma_+^P)}}(m+1)m_a^PE_a^P,$$

and E_a^P appears after the (p_a^P,q_a^P,ν_a^P)-blow-up at the point P_a (recall that the locus of non-transversality in dimension 2 and 3 are identified).

The divisor E_a^P is naturally isomorphic to $\mathbb{P}^2_{(p_a^P,q_a^P,\nu_a^P)}/\mu_e$. Using this identification, define

$$E_{a,1}^P=E_a^P\setminus\{xy=0\},\qquad E_{a,x}^P=\{x=0\}\setminus\{[0:1:0],[0:0:1]\},$$
$$E_{a,y}^P=\{y=0\}\setminus\{[1:0:0],[0:0:1]\},\qquad E_{a,xy}^P=\{[0:0:1]\}.$$

Analogously, one considers $E^P_{a,xz}$ and $E^P_{a,yz}$ so that $E^P_a = \bigsqcup_j E^P_{a,j}$ really defines a stratification. However, these two strata belong to more than one irreducible divisor in the total transform and hence they do not contribute to the characteristic polynomial.

As for E_0, according to its quotient singularities, no stratification need to be considered (it is smooth).

The Euler characteristic of $\check{E}_0$ and $\check{E}^P_{a,j} := E^P_{a,j} \setminus \big(E^P_{a,j} \cap \big(\bigcup_{b\neq a} E^P_b \cup \widehat{V}\big)\big)$ for $j = 1, x, y, xy$ (see notation just above Theorem (IV.3.14)) as well as its multiplicity are calculated in Lemma (VI.3.3).

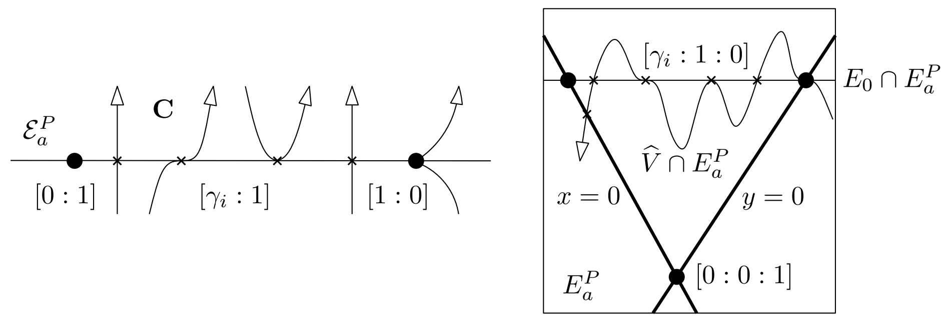


FIGURE VI.4. Stratification of $\mathcal{E}^P_a$ and E^P_a needed for the generalized A'Campo's formula.

Lemma (VI.3.3). *Using the previous notation, the Euler characteristic and the multiplicity of $\check{E}_0$ are*

$$\chi(\check{E}_0) = \chi(\mathbb{P}^2 \setminus \mathbf{C}), \qquad m(\check{E}_0) = m.$$

For the rest of strata of $\check{E}^P_a$, let us fix a point $P \in \operatorname{Sing}(\mathbf{C})$. Then, one has that

$$\chi(\check{E}^P_{a,j}) = \begin{cases} 1 & a = 1,\ j = xy \\ 0 & a \neq 1,\ j = xy \\ -\chi(\check{\mathcal{E}}^P_{a,j}) & \forall a,\ j = 1, x, y\,; \end{cases}$$

$$\chi(\check{\mathcal{E}}^P_{a,j}) \neq 0 \implies m(\check{E}^P_{a,j}) = \begin{cases} m+1 & a = 1,\ j = xy \\ m(\check{\mathcal{E}}^P_{a,j}) \cdot (m+1) & \forall a,\ j = 1, x, y. \end{cases}$$

In fact, $\forall a \in S(\Gamma^P_+)$, $a \neq 1$, the stratum $\check{E}^P_{a,xy}$ is empty and, in particular, its Euler characteristic is zero.

PROOF. Let E be an irreducible component of the exceptional divisor of ρ. Let us travel back in the history of the resolution until the time when E first appears. Consider the space defined at that moment by E minus the intersections with the other components.

Since all the weighted blow-ups have center in that intersections, this space is naturally isomorphic to $\check{E}$. Using these arguments, we will perform the calculations of the Euler characteristics at the moment when the component appears in the history of the resolution.

For E_0, the space $\check{E}_0$ is isomorphic to $E_0 \setminus (\widehat{V} \cap E_0)$ which is identified with $\mathbb{P}^2 \setminus \mathbf{C}$; its multiplicity is m, see discussion before Lemma (VI.1.2) and Figure VI.1.

For the rest of the proof the cases $j = 1, x, y, xy$ are treated separately. Let us fix a point $P \in \operatorname{Sing}(\mathbf{C})$ and omit the index "P" to simplify the notation.

Recall that $E_a = \mathbb{P}^2_{(p_a,q_a,\nu_a)}/\mu_e$, see Lemma (VI.2.10)(1). Also Figure VI.4 will be useful.

- $\underline{j = xy}$:

The stratum $E_{a,xy}$ is the point $[0:0:1] \in E_a$. By (VI.2.10), it belongs to just one divisor if and only if $a \neq 1$, see Lemma (VI.2.2) and its proof. This implies that $\chi(\check{E}_{1,xy}) = 1$ and that

$$\chi(\check{E}_{a,xy}) = 0, \qquad \forall a \in S(\Gamma_+) \setminus \{1\}.$$

Following Definition (IV.3.12), the multiplicity of $\check{E}_{1,xy}$ is $\frac{(m+1)\nu_1}{\nu_1}$, since the origin $[0:0:1] \in E_1$ is a cyclic quotient singular point of order ν_1, see Lemma (VI.2.2).

- $\underline{j = x}$:

The stratum $E_{a,x}$ is the line $\{x = 0\} \subset E_a$. If there is another component of the divisor that passes through $\mathcal{E}_{a,x} = [0:1] \in \mathcal{E}_a$, then one has $\check{\mathcal{E}}_{a,x} = \emptyset$, and either $\check{E}_{a,x} = E_{a,x} \setminus \{2 \text{ points}\}$ or $\check{E}_{a,x} = \emptyset$. Otherwise, $\check{\mathcal{E}}_{a,x} = [0:1]$ and $\check{E}_{a,x} = E_{a,x} \setminus \{3 \text{ points}\}$, see second part of (VI.2.10)(4). In the case when the Euler characteristic is different from zero, by Remark (VI.2.12), the multiplicity is

$$m(\check{E}_{b,x}) = \frac{(m+1)m_b}{\gcd(\frac{eq_b}{d}, \frac{\nu_b + t'' q_b}{d})} = \frac{(m+1)m_b}{\frac{eq_b}{d}} = (m+1)m(\check{\mathcal{E}}_{b,x}).$$

The case $j = y$ is exactly the same as $j = x$.

- $\underline{j = 1}$:

Consider the projection of $E_a \setminus E_{a,xy}$ onto the line $\{z = 0\} \equiv \mathcal{E}_a$. This map is identify with the morphism

$$\begin{array}{rcl} \tau \,:\, \mathbb{P}^2_{(p_a,q_a,\nu_a)}(e;r,s,t) \setminus \{[0:0:1]\} & \longrightarrow & \mathbb{P}^1_{(p_a,q_a)}(e;r,s), \\ {[x:y:z]} & \mapsto & [x:y]. \end{array}$$

Note that the restriction $\tau| : \check{E}_{a,1} \to \check{\mathcal{E}}_{a,1}$ is a fibration with fiber isomorphic to $\mathbb{C} \setminus \{2 \text{ points}\}$ and hence $\chi(\text{fiber}) = -1$.

The multiplicity of the smooth part is $(m+1)m_a$ in the superisolated singularity while it is m_a in the tangent cone.

To finish observe that in any case, one has that $\chi(\check{E}_{a,j}) = -\chi(\check{\mathcal{E}}_{a,j})$ and, if they are different from zero, $m(\check{E}_{a,j}) = (m+1)m(\check{\mathcal{E}}_{a,j})$. Now the proof is complete. □

Remark **(VI.3.4).** The Euler characteristic of the complement of a projective plane curve in $\mathbb{P}^2$ is known to be

$$\chi(\mathbb{P}^2 \setminus \mathbf{C}) = (m^2 - 3m + 3) - \sum_{P \in \operatorname{Sing}(\mathbf{C})} \mu_P,$$

see [**Esn82**], or [**Art94a**] for an elementary proof based on the additivity of the Euler characteristic.

Theorem (VI.3.5). *The characteristic polynomial of the complex monodromy of* $(V, 0)$ *is*

$$\Delta_{(V,0)}(t) = \frac{(t^m - 1)^{\chi(\mathbb{P}^2 \setminus \mathbf{C})}}{t-1} \prod_{P \in \operatorname{Sing}(\mathbf{C})} \Delta_{(\mathbf{C},P)}(t^{m+1}),$$

where $\Delta_{(\mathbf{C},P)}(t)$ *denotes the characteristic polynomial of the local complex monodromy of* $(\mathbf{C}, P)$*.*

Proof. Given a point $P \in \operatorname{Sing}(\mathbf{C})$, let us compute the characteristic polynomial of $(\mathbf{C}, P)$. Its total transform is

$$(\varrho^P)^*(\mathbf{C}, P) = \widehat{\mathbf{C}} + \sum_{a \in S(\Gamma_+^P)} m_a^P \mathcal{E}_a^P,$$

and the stratification associated with each exceptional divisor needed for applying A'Campo's formula is $\check{\mathcal{E}}_a = \check{\mathcal{E}}_{a,1} \sqcup \check{\mathcal{E}}_{a,x} \sqcup \check{\mathcal{E}}_{a,y}$. Then, by Theorem (IV.3.14),

$$\Delta_{(\mathbf{C},P)}(t) = (t-1) \prod_{\substack{a \in S(\Gamma_+^P) \\ j = 1, x, y}} \left(t^{m(\check{\mathcal{E}}_{a,j}^P)} - 1\right)^{-\chi(\check{\mathcal{E}}_{a,j}^P)}. \tag{33}$$

Let us see the situation in the superisolated singularity $(V, 0)$. The total transform is

$$\rho^*(V, 0) = \widehat{V} + mE_0 + \sum_{\substack{P \in \operatorname{Sing}(\mathbf{C}) \\ a \in S(\Gamma_+^P)}} (m+1) m_a^P E_a^P,$$

and the corresponding stratification is $\check{E}_a^P = \check{E}_{a,1}^P \sqcup \check{E}_{a,x}^P \sqcup \check{E}_{a,y}^P \sqcup \check{E}_{a,xy}^P$.

By Theorem (IV.3.14), the characteristic polynomial of $(V, 0)$ is

$$\Delta_{(V,0)}(t) = \frac{1}{t-1}(t^{m(\check{E}_0)} - 1)^{\chi(\check{E}_0)} \prod_{\substack{P \in \mathrm{Sing}(\mathbf{C}) \\ a \in S(\Gamma_+^P) \\ j=1,x,y,xy}} (t^{m(\check{E}_{a,j}^P)} - 1)^{\chi(\check{E}_{a,j}^P)}. \tag{34}$$

From Lemma (VI.3.3), $m(\check{E}_0) = m$ and $\chi(\check{E}_0) = \chi(\mathbb{P}^2 \setminus \mathbf{C})$, and the latter can be computed combinatorially as indicated in the statement. Let us calculate the contribution of the preceding product for a given point $P \in \mathrm{Sing}(\mathbf{C})$.

Again using Lemma (VI.3.3) and, in particular, the fact that $a \neq 1$ implies $\chi(\check{E}_{a,xy}^P) = 0$, one has that

$$\begin{aligned}
&\prod_{\substack{a \in S(\Gamma_+^P) \\ j=1,x,y,xy}} \left(t^{m(\check{E}_{a,j}^P)} - 1\right)^{\chi(\check{E}_{a,j}^P)} = \\
&= \underbrace{\left(t^{m(\check{E}_{1,xy}^P)} - 1\right)^{\chi(\check{E}_{1,xy}^P)}}_{a=1, \quad j=xy} \prod_{\substack{a \in S(\Gamma_+^P) \\ j=1,x,y}} \left(t^{m(\check{E}_{a,j}^P)} - 1\right)^{\chi(\check{E}_{a,j}^P)} \\
&= \left(t^{m+1} - 1\right)^1 \prod_{\substack{a \in S(\Gamma_+^P) \\ j=1,x,y}} \left(t^{(m+1)m(\check{\mathcal{E}}_{a,j}^P)} - 1\right)^{-\chi(\check{\mathcal{E}}_{a,j}^P)}.
\end{aligned}$$

By (33), the last expression is equal to $\Delta_{(\mathbf{C},P)}(t^{m+1})$ and hence (34) is exactly the formula of the statement. □

Remark **(VI.3.6).** Note that the first part of $\Delta(t)$ is closely related to the zeta function of the tangent cone $f_m(x, y, z)$ regarded as an function on $\mathbb{C}^3$. In fact, $Z(f_m : \mathbb{C}^3 \to \mathbb{C};\, t) = (1 - t^m)^{\chi(\mathbb{P}^2 \setminus \mathbf{C})}$.

This is a consequence of the fact that the monodromy zeta function of a homogeneous polynomial of degree d is $Z(t) = (1 - t^d)^{\chi(F)/d}$, where F is the corresponding Milnor fiber.

Corollary (VI.3.7). *The Milnor number of a superisolated surface singularity can be expressed in terms of the Milnor numbers of the singular points of the tangent cone, namely*

$$\mu(V, 0) = (m-1)^3 + \sum_{P \in \mathrm{Sing}(\mathbf{C})} \mu(\mathbf{C}, P).$$

PROOF. The Milnor number coincides with the degree of the characteristic polynomial. Then,

$$\begin{aligned}\deg \Delta(t) &= m\big(m^2 - 3m + 3 - \sum_P \mu_P\big) - 1 + \sum_P \underbrace{\deg \Delta_P(t)}_{\mu_P}(m+1)\\ &= m^3 - 3m^2 + 3m - m\sum_P \mu_P - 1 + (m+1)\sum_P \mu_P\\ &= (m-1)^3 + \sum_P \mu_P.\end{aligned}$$

Above, the sums are taken over $P \in \operatorname{Sing}(\mathbf{C})$. □

SECTION § VI.4 Higher Dimension

The family of singularities studied in this chapter can be easily generalized to higher dimension. In such a case, a (classical) embedded resolution of the tangent cone, consisting of blow-ups with smooth center, is used instead to describe the embedded **Q**-resolution of the singularity. In order not to repeat the same arguments of the preceding sections, only a sketch of the embedded **Q**-resolution is presented.

Let $f = f_m + f_{m+1} + \cdots \in \mathbb{C}\{x_0, \ldots, x_n\}$ be the decomposition of f into its homogeneous parts. Assume f defines a superisolated singularity $(V, 0)$, i.e. $\operatorname{Sing}(\mathbf{C}) \cap V(f_{m+1}) = \emptyset$. This implies that $\mathbf{C} := V(f_m) \subset \mathbb{P}^n$ only has a finite number of singular points, say $\{P_1, \ldots, P_r\}$.

One starts the resolution of $(V, 0)$ with the usual blow-up at the origin of $\mathbb{C}^{n+1}$, producing an identical situation to Lemma (VI.1.2), but in higher dimension. That is, the exceptional divisor E_0 and the strict transform intersect transversely at P if and only if P is smooth in $\mathbf{C}$.

As for the singular points, there exist local coordinates around P_i such that the equations of the exceptional divisor and the strict transform are

$$\text{Step } 0 \left\{ \begin{array}{ll} E_0: & x_0 = 0\,; \\ \widehat{V}: & x_0 + h(x_1, \ldots, x_n) = 0\,, \end{array} \right.$$

where $h(x_1, \ldots, x_n) = 0$ is an equation of the germ $(\mathbf{C}, P_i)$ and its order is at least 2. In this coordinates $P_i = [1 : 0 : \ldots : 0]$. Moreover, $\chi(\check{E}_0) = \chi(\mathbb{P}^n \setminus \mathbf{C})$ and $m(\check{E}_0) = m$.

Let P be one of the singular points of $\mathbf{C}$. Now one proceeds with the (usual) blow-up at P in the tangent cone and with the $(\nu_1, 1, \overset{(n)}{\ldots}, 1)$-blow-up at P in $(V, 0)$, where $\nu_1 = \operatorname{ord} h(\mathbf{x})$, see §I.3–3. Here are the equations of the total transforms:

$$\text{Step 1} \begin{cases} \text{Cone}: & x_1^{\nu_1} h_1(\mathbf{x}) = 0\,; \\ \text{SIS}: & x_0^m x_1^{(m+1)\nu_1}(x_0 + h_1(\mathbf{x})) = 0\,. \end{cases}$$

From these equations, the locus of non-transversality in dimension 2 and 3 are identified. Note that the new ambient space has just one singular point Q of type $(\nu_1; -1, 1, \ldots, 1)$ located at the exceptional divisor E_1. The strict transform $\widehat{V}$ and the divisor E_0 do not pass through Q. Hence the corresponding stratification is $\check{E}_1 = \check{E}_{1,1} \sqcup \check{E}_{1,Q}$, where the latter represents the singular point Q and the other one the smooth part of the divisor.

Moreover, $\chi(\check{E}_{1,1}) = -\chi(\check{\mathcal{E}}_1)$, $m(\check{E}_{1,1}) = (m+1)\nu_1$ and $\chi(\check{E}_{1,Q}) = 1$, $m(\check{E}_{1,Q}) = m + 1$. We use the following argument to compute the Euler characteristic of the smooth part. Denote by $h_{\nu_1}(\mathbf{x})$ the ν_1-homogeneous part of $h(\mathbf{x})$, $F_1 = \{x_0 = 0\}$, and $F_2 = \{x_0 + h_{\nu_1}(\mathbf{x}) = 0\} \subset E_1$. Then,

$$\begin{aligned} \check{\mathcal{E}}_1 &= \mathbb{P}^{n-1} \setminus \{h_{\nu_1}(\mathbf{x}) = 0\} \cong (F_1 \setminus (F_1 \cap F_2)), \\ \check{E}_{1,1} &= \mathbb{P}^n_{(\nu_1,1,\ldots,1)} \setminus \big[(F_1 \setminus (F_1 \cap F_2)) \sqcup F_2 \sqcup Q\big]. \end{aligned} \tag{35}$$

Consequently, $\chi(\check{E}_{1,1}) = n + 1 - \chi(\check{E}_1) - n - 1 = -\chi(\check{E}_1)$ as claimed.

Finally, to give the embedded $\mathbf{Q}$-resolution of $(V, 0)$, every time there is a (usual) blow-up in the tangent cone with center $Z = \{x_1 = \cdots = x_k = 0\}$, one considers the $(\nu, 1, \overset{(k)}{\ldots}, 1)$-blow-up with center $\{x_0 = \cdots = x_k = 0\}$, where ν is the order of $\mathbf{C}$ with respect to Z.

The new ambient space is covered by $k+1$ charts, namely $U_i = \mathbb{C}^{n+1}$ for $i = 1, \ldots, k$ and $U_0 = X(\nu; -1, 1, \overset{(k)}{\ldots}, 1) \times \mathbb{C}^{n-k}$. Hence the exceptional divisor $E_i = \mathbb{P}^k_{(\nu,1,\ldots,1)} \times \mathbb{C}^{n-k}$ contains the subset $[(0, \ldots, 0)] \times \mathbb{C}^{n-k}$ as quotient singularities, see §I.3–3. The equations of the total transform are:

$$\text{Step } i \begin{cases} \text{Cone}: & x_1^{m_1} \cdots x_l^{m_l} \cdot h_i(\mathbf{x}) = 0\,; \\ \text{SIS}: & x_0^m \cdot x_1^{(m+1)m_1} \cdots x_l^{(m+1)m_l} \cdot (x_0 + h_i(\mathbf{x})) = 0\,. \end{cases}$$

The main difference in the i-th step $(i \neq 0, 1)$ is that the singular points of E_i always belong to more than one exceptional divisor and thus they do not contribute to the characteristic polynomial. The stratification is therefore the usual one. Moreover, using the same arguments as in (35), one has that $\chi(\check{E}_i) = -\chi(\check{\mathcal{E}}_i)$ and $m(\check{E}_i) = (m+1)m(\check{\mathcal{E}}_i)$.

Now we have all the ingredients to apply Theorem (IV.3.14). The details are left to the reader because they do not provide any new idea. The characteristic polynomial of the monodromy of $(V,0) \subset (\mathbb{C}^{n+1},0)$ is

$$\Delta_{(V,0)}(t) = \left[\frac{(t^m-1)^{\chi(\mathbb{P}^n \setminus \mathbf{C})}}{t-1} \right]^{(-1)^n} \cdot \prod_{P \in \mathrm{Sing}(\mathbf{C})} \Delta_{(\mathbf{C},P)}(t^{m+1}),$$

where $\Delta_{(\mathbf{C},P)}(t)$ denotes the characteristic polynomial of the local complex monodromy of $(\mathbf{C},P)$.

The Euler characteristic of $\mathbb{P}^n \setminus \mathbf{C}$ is calculated combinatorially from the expression $m \cdot \chi(\mathbb{P}^n \setminus \mathbf{C}) = 1 + (-1)^n \left[(m-1)^{n+1} - m \sum_{P \in \mathrm{Sing}(P)} \mu_P\right]$ and thus the Milnor number is

$$\mu(V,0) = (m-1)^{n+1} + \sum_{P \in \mathrm{Sing}(\mathbf{C})} \mu(\mathbf{C},P).$$

VII

Yomdin-Lê Surface Singularities

The family of singularities studied in Chapter VI can be generalized as follows. Let $f = f_m + f_{m+k} + \cdots \in \mathbb{C}\{x,y,z\}$ be the decomposition of f into its homogeneous parts, $k \geq 1$. Denote $V := V(f) \subset \mathbb{C}^3$ and $\mathbf{C} := V(f_m) \subset \mathbb{P}^2$. Then, the germ $(V,0)$ is said to be a *Yomdin-Lê surface singularity* (YS) if the condition $\mathrm{Sing}(\mathbf{C}) \cap V(f_{m+k}) = \emptyset$ holds in $\mathbb{P}^2$.

The main difficulty in finding a (usual) embedded resolution of this kind of singularities is that after several blow-ups at points and rational curves, following the ideas of [**Art94b**], one eventually obtains a branch of resolutions depending on k. Thus the study of this singularities by using these tools seem to be very long and tedious.

However, an embedded $\mathbf{Q}$-resolution of $(V,0)$ can be computed exactly as in Chapter VI, i.e. by means of weighted blow-ups at points, see Ex. (IV.2.7). In fact, this is the main purpose of this chapter. Again, the weights at each step can be chosen so that every exceptional divisor in the $\mathbf{Q}$-resolution contributes to the monodromy. As an application, the characteristic polynomial and the Milnor number are calculated using Theorem (IV.3.14).

In order not to repeat the same arguments, the proofs of this chapter are sketched, commented, or simply omitted. Moreover, they are presented following the same structure as in Chapter VI so that one can easily compares the corresponding results with the SIS. In the discussion, one usual thinks that $k \neq 1$, since otherwise $(V,0)$ is a SIS, cf. [**Mar11a**].

As for notations and conventions, we use the same as in the previous chapter. It is extremely recommended to take a look at it before continuing because, in this sense, this chapter is not self-contained. In particular, see proof of Lemma (VI.2.2).

(VII.0.1). We start the **Q**-resolution of $(V,0)$ with the usual blow-up at the origin $\pi_0 : \widehat{\mathbb{C}}^3 \to \mathbb{C}^3$. The total transform is the divisor $\pi_0^*(V) = \widehat{V} + mE_0$, where $\widehat{V}$ is the strict transform and E_0 is the exceptional divisor. The intersection $\widehat{V} \cap E_0$ is identified with the tangent cone of the singularity.

Let us consider $P \in \widehat{V} \cap E_0 = \mathbf{C}$. After linear change of coordinates we can assume that $P = ((0,0,0),[0:0:1]) \equiv [0:0:1] \in \mathbf{C}$. Take a chart of $\widehat{\mathbb{C}}^3$ around P where $z=0$ is the equation of E_0 and the blowing-up takes the form

$$(x,y,z) \stackrel{\pi_0}{\longmapsto} (xz, yz, z).$$

Then, the equation of $\widehat{V}$ is

$$\widehat{V}: \quad f_m(x,y,1) + z^k\Big[f_{m+k}(x,y,1) + zf_{m+k+1}(x,y,1) + \cdots\Big] = 0.$$

Two cases arise: if P is smooth in the tangent cone, then $\widehat{V}$ is also smooth at P and the intersection with E_0 at that point is transverse; otherwise, i.e. $P \in \operatorname{Sing}(\mathbf{C})$, the YS condition $\operatorname{Sing}(\mathbf{C}) \cap V(f_{m+k}) = \emptyset$ implies that the previous expression in brackets is a unit in the local ring $\mathbb{C}\{x,y,z\}$ and $\widehat{V}$ is not smooth at P (unless $k=1$). Now the order of $f_m(x,y,1)$ is greater than or equal to 2 and the intersection $\widehat{V} \cap E_0$ is not transverse at P.

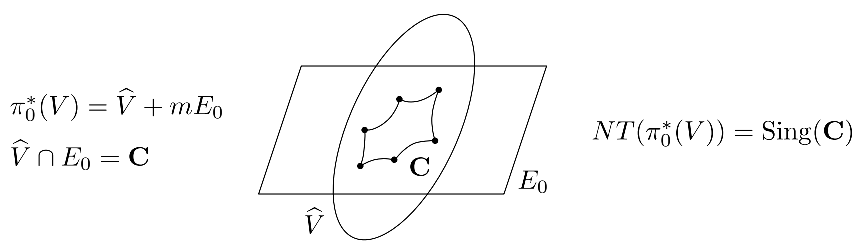


FIGURE VII.1. Step 0 in the embedded **Q**-resolution of $(V,0)$.

We summarize the previous discussion in the following result, which is the step zero in our **Q**-resolution of $(V,0)$.

Lemma (VII.0.2) (Step 0)**.** *Let $P \in \mathbf{C}$. The surfaces $\widehat{V}$ and E_0 intersect transversely at P if and only if P is a smooth point in $\mathbf{C}$. Otherwise, i.e. $P \in \operatorname{Sing}(\mathbf{C})$, there exist local analytic coordinates around P such that the equations of the exceptional divisor and the strict transform are*

$$E_0: \quad z = 0\,;$$
$$\widehat{V}: \quad z^k + h(x,y) = 0\,,$$

where $h(x,y)=0$ is an equation of $\mathbf{C}$ and its order is at least 2.

Remark **(VII.0.3).** Observe that the main difference at this stage is that $\widehat{V}$ is not smooth at the singular points of the tangent cone and its equation at those points has z^k as one of its terms.

Section § VII.1
An Embedded Q-Resolution for YS

After the step zero $NT(\pi_0^*(V))$ is identified with $\operatorname{Sing}(\mathbf{C})$. The next step in the $\mathbf{Q}$-resolution of $(V,0)$ is to blow up those points. Let us fix $P \in \operatorname{Sing}(\mathbf{C})$ and consider local coordinates as in Lemma (VII.0.2). The idea is to choose suitable weights so that the strict transform of $\widehat{V}$ has again an equation of the same form, namely $z^k + H(x,y) = 0$.

Given an exceptional divisor in the tangent cone $\mathcal{E}_a$, $a \in S(\Gamma_+)$, and m_a its multiplicity, denote $k_a := \gcd(k, m_a)$. When $a = 1$, then $m_1 = \nu_1$ and thus $k_1 = \gcd(k, \nu_1)$.

Lemma (VII.1.1) (Step 1)**.** *Let $(p_1, q_1) \in \mathbb{N}^2$ be two positive coprime numbers. Let ϖ_1 be the (p_1,q_1)-weighted blow-up at $P \in \mathbf{C}$. Denote by $\mathcal{E}_1$ its exceptional divisor and by ν_1 the (p_1,q_1)-multiplicity of $\mathbf{C}$ at P.*

Consider π_1 the $\left(\frac{kp_1}{k_1}, \frac{kq_1}{k_1}, \frac{\nu_1}{k_1}\right)$-weighted blow-up at P in dimension 3 and E_1 the corresponding exceptional divisor. Then, the total transforms verify:

(1) $\varpi_1^*(\mathbf{C}) = \mathbf{C} + \nu_1 \mathcal{E}_1,$

(2) $\pi_1^*\pi_0^*(V) = \widehat{V} + mE_0 + (m+k)\dfrac{\nu_1}{k_1}E_1,$

(3) $NT(\pi_1^*\pi_0^*(V)) = NT(\varpi_1^*(\mathbf{C})).$

Proof. The weighted blow-up at P in the tangent cone is described in detail in the first part of the proof of Lemma (VI.2.2). Thus we only consider here the weighted blow-up at P with respect to $\left(\frac{kp_1}{k_1}, \frac{kq_1}{k_1}, \frac{\nu_1}{k_1}\right)$ in dimension 3.

The new space has in general three cyclic quotient singular lines, see Remark (VII.1.2)(1) below, each of them isomorphic to $\mathbb{P}^1$, and located at the new exceptional divisor E_1. They correspond to the three lines at infinity of $E_1 = \mathbb{P}^2\left(\frac{kp_1}{k_1}, \frac{kq_1}{k_1}, \frac{\nu_1}{k_1}\right)$.

The multiplicity of E_1 is the sum of the multiplicities, in our local coordinates, of the components of the divisor $\pi_0^*(V)$ that pass through P, that is, $m\frac{\nu_1}{k_1} + k\frac{\nu_1}{k_1} = (m+k)\frac{\nu_1}{k_1}$.

Hence the total transform is the divisor

$$\pi_1^*\pi_0^*(V) = \widehat{V} + mE_0 + (m+k)\frac{\nu_1}{k_1}E_1.$$

To study the locus of non-transversality, the equations in the three charts are calculated in the table below. Note that the cyclic quotient spaces are represented by their normalized types, since $\gcd\left(\frac{kp_1}{k_1}, \frac{kq_1}{k_1}, \frac{\nu_1}{k_1}\right) = 1$, see Section I.3–2 for details.

$(x,y,z) \overset{\pi_1}{\longmapsto}$	$X\left(\frac{kp_1}{k_1}; -1, \frac{kq_1}{k_1}, \frac{\nu_1}{k_1}\right)$ $(x^{\frac{kp_1}{k_1}}, x^{\frac{kq_1}{k_1}} y, x^{\frac{\nu_1}{k_1}} z)$	$X\left(\frac{kq_1}{k_1}; \frac{kp_1}{k_1}, -1, \frac{\nu_1}{k_1}\right)$ $(xy^{\frac{kp_1}{k_1}}, y^{\frac{kq_1}{k_1}}, y^{\frac{\nu_1}{k_1}} z)$
E_0	$z=0$	$z=0$
E_1	$x=0$	$y=0$
$\widehat{V}$	$z^k + h_1(x^{\frac{k}{k_1}}, y) = 0$	$z^k + h_2(x, y^{\frac{k}{k_1}}) = 0$

$(x,y,z) \overset{\pi_1}{\longmapsto}$	$X\left(\frac{\nu_1}{k_1}; \frac{kp_1}{k_1}, \frac{kq_1}{k_1}, -1\right)$ $(xz^{\frac{kp_1}{k_1}}, yz^{\frac{kq_1}{k_1}}, z^{\frac{\nu_1}{k_1}})$
E_0	–
E_1	$z=0$
$\widehat{V}$	$1 + h_{\nu_1}(x,y) + z^{\frac{kl}{k_1}} h_{\nu_1+l}(x,y) + \cdots = 0$

Clearly E_1 and E_0 intersect transversely. The strict transform $\widehat{V}$ also cuts E_1 transversely except perhaps at $\{z=0\} \subset E_1$. The equations of these intersections are given by

$$E_0 \cap E_1 = \{z = 0\},$$
$$\widehat{V} \cap E_1 = \{z^k + h_{\nu_1}(x,y) = 0\},$$

as projective subvarieties in $E_1 = \mathbb{P}^2\left(\frac{kp_1}{k_1}, \frac{kq_1}{k_1}, \frac{\nu_1}{k_1}\right)$.

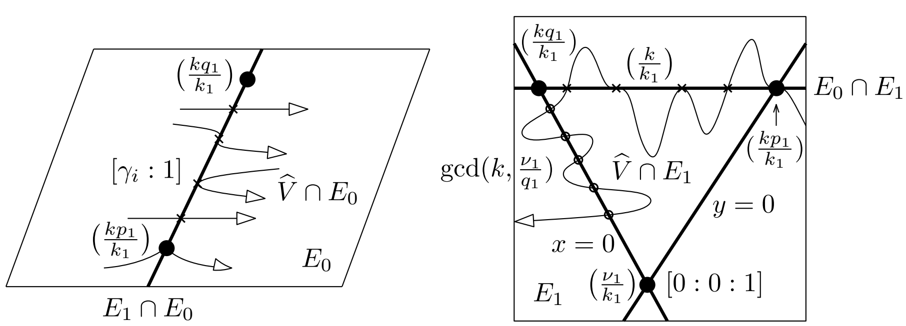


FIGURE VII.2. Step 1 in the embedded $\mathbf{Q}$-resolution of $(V, 0)$.

By (III.4.3), these smooth projective curves have self-intersection numbers $\frac{k_1\nu_1}{k^2p_1q_1}$ and $\frac{k_1\nu_1}{p_1q_1}$ respectively. They meet at $\#(\mathbf{C}\cap\mathcal{E}_1)$ points with intersection number k_1/k times the intersection number in $\mathbf{C}\cap\mathcal{E}_1$, that is, for $P\in\mathbf{C}\cap\mathcal{E}_1\equiv\widehat{V}\cap E_0\cap E_1$, one has

$$(36)\qquad \left(\widehat{V}\cap E_1, E_0\cap E_1;\, E_1\right)_P = \frac{k_1}{k}\cdot\left(\mathbf{C},\mathcal{E}_1;\,\mathbb{C}^2_{(p_1,q_1)}\right)_P.$$

On the other hand, the intersection of the total transform with E_0 produces an identical situation to the tangent cone, see Remark (VII.1.2)(2) for a more detailed explanation.

All these statements follow from the equations above. In Figure VII.2, we see the intersection of the divisor $\pi_1^*\pi_0^*(V)$ with E_0 and E_1, respectively. See also Figure VI.2 for the situation in $\mathbf{C}$.

Finally, the triple points of the total transform in dimension 3 are identified with the points of $\mathbf{C}\cap\mathcal{E}_1$ and, by (36), the intersection at one of those points is transverse if and only if so is it in dimension 2. This concludes the proof. □

Remark **(VII.1.2).** Just to emphasize, we collect below the main differences with the embedded **Q**-resolution of a superisolated surface singularity at this stage, cf. Lemma (VI.2.2) and its proof.

(1) The stratum $\{z=0\}\setminus\{[0:1:0],[1:0:0]\}\subset E_1$ is not smooth. In fact, the group acting on these points is of type $\left(\frac{k}{k_1};-1,0,\frac{\nu_1}{k_1}\right)$, see Figure VII.2.

(2) In principle, the intersection of E_0 with the rest of components seem to be different from the situation in the tangent cone, because in the first chart $E_1\cap E_0=\{x=0\}$ and $\widehat{V}\cap E_0=\{h_1(x^{k/k_1},y)=0\}$ on $X\left(\frac{kp_1}{k_1};-1,\frac{kq_1}{k_1}\right)$. After normalizing the latter type, one finds the equation of $\mathcal{E}_1$ and $\mathbf{C}$ on $X(p_1;-1,q_1)$, cf. (VII.1.3).

(3) Write $h_{\nu_1}(x,y)=x^ay^b\prod_i(x^{q_1}-\gamma_i^{q_1}y^{p_1})^{e_i}=0$. If $a=0$, or equivalently $\mathcal{E}_1\ni[0:1]\notin\mathbf{C}$, then $\{x=0\}\subset E_1$ cuts $\widehat{V}\cap E_1=\{z^k+h_{\nu_1}(x,y)=0\}$ in exactly $\gcd(k,\frac{\nu_1}{q_1})$ points different from the origins of E_1. Analogously, $\{y=0\}\subset E_1$ intersects in $\gcd(k,\frac{\nu_1}{p_1})$ points if $b=0$. This can be checked directly or applying Bézout's Theorem on E_1, see Proposition (III.4.3).

Let Γ and Γ_+ be the dual graphs associated with the total transform and the exceptional divisor, after having computed an embedded **Q**-resolution of $(\mathbf{C},P)$, respectively. Denote by $S(\Gamma)$ and $S(\Gamma_+)$ the sets of their vertices. The classical partial order on $S(\Gamma_+)$ is denoted by $\preccurlyeq$.

The locus of non-transversality after the last blow-up in dimension 3 is identified with the locus of non-transversality in the resolution of $(\mathbf{C}, P)$. Each of these points corresponds to a weighted blow-up in the resolution of the tangent cone, that is, to a vertex of Γ_+. Thus in the next step we need to blow-up those points to produce a similar situation. Again the same operation will be applied to the points where the total transform is not a normal crossing divisor. These points will be associated with vertices of Γ_+ as well.

Before describing a generic step, blowing up the point P_b as in Proposition (VI.2.10), let us clarify the justification for working with non-normalized spaces.

(VII.1.3). After the first blow-up the local equation of the total transform of $(\mathbf{C}, P)$ is given by $x^{\nu_1} h_1(x,y) : X(p_1; -1, q_1) \to \mathbb{C}$, see proof of Lemma (VI.2.2). The situation in dimension 3 is provided by

$$\underbrace{x^{(m+k)\frac{\nu_1}{k_1}}}_{E_1} \cdot \underbrace{z^m}_{E_0} \cdot \underbrace{\left[z^k + h_1(x^{\frac{k}{k_1}}, y)\right]}_{\widehat{V}} : X\left(\frac{kp_1}{k_1}; -1, \frac{kq_1}{k_1}, \frac{\nu_1}{k_1}\right) \longrightarrow \mathbb{C},$$

as we have just seen in the proof of Lemma (VII.1.1). The divisors E_1 and $\mathcal{E}_1$ are both represented by $x = 0$.

However, the equation of the strict transform of $\mathbf{C}$ and $\widehat{V}$ do not correspond to each other directly. This obstruction can be solved working with non-normalized types, since the function

$$x^{\frac{k\nu_1}{k_1}} h_1(x^{\frac{k}{k_1}}, y) : X\left(\frac{kp_1}{k_1}; -1, \frac{kq_1}{k_1}\right) \longrightarrow \mathbb{C}$$

also gives rise to the total transform of $\mathbf{C}$ on a space represented by a non-normalized type.

On the other hand, the embedded $\mathbf{Q}$-resolution of a Yomdin-Lê surface singularity will contain in general non-cyclic quotient singularities. Hence providing normalized types is long and tedious. Motivated by this fact and for better understanding of the relationship between $\mathbf{C}$ and $(V, 0)$, we present the embedded $\mathbf{Q}$-resolution without explicitly giving the normalized type of each quotient space.

The following result is proven by induction on $S(\Gamma_+)$ using the relation $\preccurlyeq$. Lemma (VII.1.1) and (VII.1.3) just above is the first step in the induction. Let $b \in S(\Gamma_+)$ be a vertex such that P_b belongs to the locus of non-transversality of the total transform. As usual, denote by $\mathcal{E}_b$ the exceptional divisor appearing after blowing up the point P_b.

Proposition (VII.1.4) (Step b)**.** *Let ϖ_b be the (p_b,q_b)-weighted blow-up at P_b with $b\in S(\Gamma_+)$. Denote by $\mathcal{E}_b$ its exceptional divisor, ν_b the (p_b,q_b)-multiplicity of $\mathbf{C}\subset\mathbb{C}^2$, and m_b the multiplicity of $\mathcal{E}_b$. Assume, if necessary, that $k|p_b$ and $k|q_b$ so that $k|\nu_b$ too.*

Consider π_b the $(p_b,q_b,\frac{\nu_b}{k})$-weighted blow-up at P_b in dimension 3 and E_b the corresponding exceptional divisor. Then, after blowing up the point P_b, the new total transform verifies:

(1) *The exceptional divisor E_b is isomorphic to $\mathbb{P}^2(p_b,q_b,\frac{\nu_b}{k})/\mu_{\mathbf{e}}$ and its multiplicity equals $(m+k)\frac{m_b}{k_b}$. In general, their three lines at infinity are quotient singular in the ambient space.*

(2) *Let a be a vertex such that $a\prec b$. Then, $E_a\cap E_b\neq\emptyset$ if and only if $P_b\in\mathcal{E}_a$. In such a case, the curve $E_a\cap E_b$ is one of the two lines at infinity of E_b different from $\{z=0\}$. If $P_b\in\mathcal{E}_a\cap\mathcal{E}_{a'}$, $a\neq a'$, then the corresponding lines are different and hence they meet at the point $[0:0:1]$.*

(3) *The intersection of the rest of components with E_0 produces an identical situation to the resolution of $(\mathbf{C},P)$, after blowing up the point P_b. More precisely,*

$$\begin{aligned}\widehat{V}\cap E_0&=\mathbf{C},\\ E_b\cap E_0&=\mathcal{E}_b,\\ E_a\cap E_0&=\mathcal{E}_a,\quad \forall a\preccurlyeq b.\end{aligned}$$

(4) *The curves $E_0\cap E_b=\{z=0\}$ and $\widehat{V}\cap E_b=\{z^k+H_{\nu_b}(x,y)=0\}$ have self-intersection numbers $\frac{-\mathcal{E}_b^2\nu_b k_b}{k^2\ell}$ and $\frac{-\mathcal{E}_b^2\nu_b k_b}{\ell}$ respectively, and the intersecting points can be identified with $\mathbf{C}\cap\mathcal{E}_b$.*

Moreover, the intersection multiplicity of these two curve at those points can be computed as follows. Let $P\in\widehat{V}\cap E_0\cap E_b\equiv\mathbf{C}\cap\mathcal{E}_b$, then one has

$$\left(\widehat{V}\cap E_b, E_0\cap E_b;\, E_b\right)_P=\frac{1}{O(E_{b,z})}\cdot\left(\mathbf{C},\mathcal{E}_b;\,\widehat{\mathbb{C}}^2_{(p_b,q_b)}/\mu_e\right)_P,$$

where $O(E_{b,z})$ denotes the order of the group acting on the natural stratum $E_{b,z}:=\{z=0\}\setminus\{[0:1:0],[1:0:0]\}\subset E_b$.

Let $P_b\in\mathcal{E}_a$ ($a\prec b$) and assume e.g. $E_a\cap E_b=\{x=0\}\subset E_b$. If $\mathbf{C}\cap\mathcal{E}_a\cap E_b=\emptyset$, then $E_a\cap E_b$ and $\widehat{V}\cap E_b$ meet transversely at exactly $\gcd(k,m(\check{\mathcal{E}}_{b,x}))$ points different from the origins of E_b. Otherwise, i.e. $\mathbf{C}\cap\mathcal{E}_a\cap E_b\neq\emptyset$, the letter curves only meet at one point, which besides passes through $E_0\cap E_b$. This is the case when there exist quadruple points.

(5) *The locus of non-transversality of the total transform in dimension 3 is identified with the one in the resolution of* $(\mathbf{C},P)$. *These points belong to* $\widehat{V}\cap E_0\cap E_b\equiv \mathbf{C}\cap\mathcal{E}_b$ *and they correspond to the ones where the curves* $E_0\cap E_b$ *and* $\widehat{V}\cap E_b$, *or equivalently* $\mathcal{E}_b$ *and* $\mathbf{C}$, *do not meet transversely.*

(6) *The strict transform* $\widehat{V}$ *never passes through* $[0:0:1]\in E_b$.

Proof. By induction on $S(\Gamma_+)$ with respect to the order $\preccurlyeq$. The base case is Lemma (VII.1.1) together with its modification explained in (VII.1.3). As for the inductive step, one proceeds as in the proof of (VI.2.6). Assume, by induction, that the local equation of the total transform in the resolution of the tangent cone around P_b is given by the function

$$x^{n_a}y^{n_{a'}}H(x,y)\,:\,X(\mathbf{e};\mathbf{r},\mathbf{s})\longrightarrow\mathbb{C}, \tag{37}$$

where $\mathbf{C}=\{H(x,y)=0\}$ is the equation of the strict transform and the others correspond to the divisors $\mathcal{E}_a$ and $\mathcal{E}_{a'}$ (they may not appear if n_a or $n_{a'}$ equals zero). In principle, the type $(\mathbf{e};\mathbf{r},\mathbf{s})$ is not assumed to be normalized. Hence n_a and $n_{a'}$ are not the multiplicities of $\mathcal{E}_a$ and $\mathcal{E}_{a'}$.

Also, the equation of the total transform around P_b in dimension 3 is given by the function

$$x^{\frac{(m+k)n_a}{k}}\cdot y^{\frac{(m+k)n_{a'}}{k}}\cdot z^m\cdot\left[z^k+H(x,y)\right]\,:\,X(\mathbf{e};\mathbf{r},\mathbf{s},\mathbf{t})\longrightarrow\mathbb{C}, \tag{38}$$

where $\widehat{V}=\{z^k+H(x,y)=0\}$ is the strict transform, $E_0=\{z=0\}$, and the others are the divisors E_a and $E_{a'}$ (if they exist). Using that both equations are well-defined functions on the corresponding quotient spaces, one has

$$\frac{n_a}{k}\cdot\mathbf{r}+\frac{n_{a'}}{k}\cdot\mathbf{s}+\mathbf{t}\equiv 0\pmod{\mathbf{e}}. \tag{39}$$

The verification of the statement is very simple once the local equations of the divisors appearing in the total transform are calculated. The main ideas behind are contained in the proof of Lemma (VII.1.1) and (VII.1.3). The details are omitted to avoid repeating the same arguments; only the local equations are given, see below. To do so, consider the following data and use the charts described in Sections I.3–1 and I.3–2. As auxiliary results (III.2.1), (III.3.2), and (III.4.3) are also needed.

$$\nu_b:=\operatorname{ord}_{(p_b,q_b)}H(x,y)\qquad n_b:=p_b\cdot n_a+q_b\cdot n_{a'}+\nu_b$$
$$H_1(x,y):=\tfrac{H(x^{p_b},x^{q_b}y)}{x^{\nu_b}}\qquad H_2(x,y):=\tfrac{H(xy^{p_b},y^{q_b})}{y^{\nu_b}}$$

Note that if $Q_1^{\mathbf{C}}$ denotes the quotient space of the first chart in the tangent cone (see below) and $(Q_1^{\mathbf{C}},[(0,1)])\cong(\mathbb{C}^2,(0,1))$, $[(x,y)]\mapsto(x^\ell,y)$ defines an isomorphism of germs, then the multiplicity of the new exceptional divisor $\mathcal{E}_b$ is $m_b=\frac{n_b}{\ell}$.

These are the equations in the resolution of the tangent cone. They are presented as zero sets omitting their multiplicities.

Equations	Chart
$\mathcal{E}_b:\ x=0$ $\mathcal{E}_a:\ -$ $\mathcal{E}_{a'}\ \ y=0$ $\mathbf{C}:\ H_1(x,y)=0$	$X\left(\begin{array}{c\|cc} p_b & -1 & q_b \\ p_b\mathbf{e} & \mathbf{r} & p_b\mathbf{s}-q_b\mathbf{r}\end{array}\right) \longrightarrow \widehat{\mathbb{C}}^2(p_b,q_b)\big/\mu_{\mathbf{e}}$ $\big[(x,y)\big] \mapsto \big[\big((x^{p_b},x^{q_b}y),[1:y]_{(p_b,q_b)}\big)\big]$
$\mathcal{E}_b:\ y=0$ $\mathcal{E}_a:\ x=0$ $\mathcal{E}_{a'}:\ -$ $\mathbf{C}:\ H_2(x,y)=0$	$X\left(\begin{array}{c\|cc} q_b & p_b & -1 \\ q_b\mathbf{e} & q_b\mathbf{r}-p_b\mathbf{s} & \mathbf{s}\end{array}\right) \longrightarrow \widehat{\mathbb{C}}^2(p_b,q_b)\big/\mu_{\mathbf{e}}$ $\big[(x,y)\big] \mapsto \big[\big((xy^{p_b},y^{q_b}),[x:1]_{(p_b,q_b)}\big)\big]$

In dimension 3, the local equations of the total transform are presented as well-defined functions over the corresponding quotient spaces. The notation is self-explanatory to recognize the equation of each divisor. In the first chart, however, it is indicated the divisor corresponding to each equation. Note that, for instance, the polynomial of the first chart has been obtained after performing the substitution $(x,y,z)\mapsto(x^{p_b},x^{q_b}y,x^{\frac{\nu_b}{k}}z)$.

1st chart

$$X\left(\begin{array}{c|ccc} p_b & -1 & q_b & \frac{\nu_b}{k} \\ p_b\mathbf{e} & \mathbf{r} & p_b\mathbf{s}-q_b\mathbf{r} & p_b\mathbf{t}-\frac{\nu_b}{k}\mathbf{r}\end{array}\right)\longrightarrow\mathbb{C}$$

$$\underbrace{x^{\frac{(m+k)n_b}{k}}}_{E_b}\cdot\underbrace{y^{\frac{(m+k)n_{a'}}{k}}}_{E_{a'}}\cdot\underbrace{z^m}_{E_0}\cdot\underbrace{\big[z^k+H_1(x,y)\big]}_{\widehat{V}}$$

2nd chart

$$X\left(\begin{array}{c|ccc} q_b & p_b & -1 & \frac{\nu_b}{k} \\ q_b\mathbf{e} & q_b\mathbf{r}-p_b\mathbf{s} & \mathbf{s} & q_b\mathbf{t}-\frac{\nu_b}{k}\mathbf{s}\end{array}\right)\longrightarrow\mathbb{C}$$

$$x^{\frac{(m+k)n_a}{k}}\cdot y^{\frac{(m+k)n_b}{k}}\cdot z^m\cdot\big[z^k+H_2(x,y)\big]$$

3rd chart

$$X\left(\begin{array}{c|ccc} \frac{\nu_b}{k} & p_b & q_b & -1 \\ \frac{\nu_b}{k}\mathbf{e} & \frac{\nu_b}{k}\mathbf{r}-p_b\mathbf{t} & \frac{\nu_b}{k}\mathbf{s}-q_b\mathbf{t} & \mathbf{t}\end{array}\right)\longrightarrow\mathbb{C}$$

$$x^{\frac{(m+k)n_a}{k}}\cdot y^{\frac{(m+k)n_{a'}}{k}}\cdot z^{\frac{(m+k)n_b}{k}}\cdot\left[1+\frac{H(xz^{p_b},yz^{q_b})}{z^{\nu_b}}\right]$$

Note that if $Q_1^{\mathbf{V}}$ denotes the quotient space of the first chart in dimension 3 (see above) and $(Q_1^{\mathbf{V}},[(0,1,1)])\cong(\mathbb{C}^3,(0,1,1))$, $[(x,y,z)]\mapsto(x^L,y,z)$ defines an isomorphism of germs, then the multiplicity of the new exceptional divisor E_b is $\frac{(m+k)n_b}{kL}$. □

Remark **(VII.1.5).** Observe that the columns of the new spaces satisfy a condition analogous to (39). For example, using (39), it can be checked that

$$\frac{n_b}{k}\cdot\begin{pmatrix}-1\\ \mathbf{r}\end{pmatrix}+\frac{n_{a'}}{k}\cdot\begin{pmatrix}q_b\\ p_b\mathbf{s}-q_b\mathbf{r}\end{pmatrix}+\begin{pmatrix}\frac{\nu_b}{k}\\ p_b\mathbf{t}-\frac{\nu_b}{k}\mathbf{r}\end{pmatrix}\equiv\begin{pmatrix}0\\ \mathbf{0}\end{pmatrix},\quad \bmod \begin{pmatrix}p_b\\ p_b\mathbf{e}\end{pmatrix}.$$

In other words, the third column is a linear combination of the first two ones, modulo the order of the corresponding group. This can be used to prove that $L=\gcd(\ell,\frac{n_b}{k})$ and hence the multiplicity of E_b is $\frac{(m+k)\cdot m_b}{\gcd(k,m_b)}$ indeed.

Theorem (VII.1.6). *Given an embedded* $\mathbf{Q}$*-resolution of* $(\mathbf{C},P)$ *for all* $P\in\operatorname{Sing}(\mathbf{C})$*, one can construct an embedded* $\mathbf{Q}$*-resolution of* $(V,0)$*, consisting of weighted blow-ups at points. Each of these blow-ups corresponds to a weighted blow-up in the resolution of* $(\mathbf{C},P)$ *for some* $P\in\operatorname{Sing}(\mathbf{C})$*, that is, it corresponds to a vertex of* Γ_+^P*.* □

By (VII.2.3) and (VII.2.4), an exceptional divisor in the $\mathbf{Q}$-resolution of $(V,0)$ contributes to the monodromy if and only if so does the corresponding divisor in $(\mathbf{C},P)$. Hence the weights can be chosen so that every exceptional divisor contributes to the monodromy.

Section § VII.2
The Characteristic Polynomial of the Monodromy

Here we plan to apply Theorem (IV.3.14) to compute the characteristic polynomial of the monodromy and the Milnor number of $(V,0)$ in terms of its tangent cone $(\mathbf{C},P)$. Some notation need to be introduced, concerning the stratification of each irreducible component of the exceptional divisor in terms of its quotient singularities.

(VII.2.1). Given a point $P\in\operatorname{Sing}(\mathbf{C})$, denote by $\varrho^P: Y^p\to(\mathbf{C},P)$ an embedded $\mathbf{Q}$-resolution of the tangent cone. Assume that the total transform is given by

$$(\varrho^P)^*(\mathbf{C},P)=\mathbf{C}+\sum_{a\in S(\Gamma_+^P)} m_a^P\mathcal{E}_a^P,$$

where $\mathcal{E}_a^P$ is the exceptional divisor of the (p_a^P,q_a^P)-blow-up at a point P_a belonging to the locus of non-transversality. Denote by ν_a^P the (p_a^P,q_a^P)-multiplicity of $\mathbf{C}$ at P_a.

Recall that $\mathcal{E}_a^P$ is naturally isomorphic to $\mathbb{P}^1_{(p_a^P,q_a^P)}/\mu_{\mathbf{e}}$. Using this identification, define

$$\mathcal{E}_{a,1}^P=\mathcal{E}_a^P\setminus\{[0:1],[1:0]\},\qquad \mathcal{E}_{a,x}^P=\{[0:1]\},\qquad \mathcal{E}_{a,y}^P=\{[1:0]\}.$$

The strata $\check{\mathcal{E}}_{a,j}^P:=\mathcal{E}_{a,j}^P\setminus\left(\mathcal{E}_{a,j}^P\cap\left(\bigcup_{b\neq a}\mathcal{E}_b^P\cup\mathbf{C}\right)\right)$ for $j=1,x,y$ (see notation just above Theorem (IV.3.14)) will be considered in Lemma (VII.2.3).

(VII.2.2). Let us see the situation in the Yomdin-Lê singularity $(V,0)$. Denote by $\rho : X \to (V,0)$ the embedded $\mathbf{Q}$-resolution obtained following Proposition (VII.1.4). Then, the total transform is (recall $k_a^P := \gcd(k, m_a^P)$)

$$\rho^*(V,0) = \widehat{V} + mE_0 + \sum_{\substack{P\in \mathrm{Sing}(\mathbf{C})\\ a\in S(\Gamma_+^P)}} (m+k)\frac{m_a^P}{k_a^P} E_a^P,$$

and E_a^P appears after the blow-up at the point P_a with suitable weights (recall that the locus of non-transversality in dimension 2 and 3 are identified).

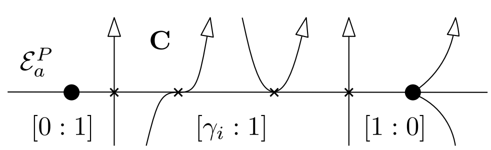
The divisor E_a^P is naturally isomorphic to $\mathbb{P}^2_\omega/\mu_{\mathbf{e}}$. Using this identification, define

$$E_{a,1}^P = E_a^P \setminus \{xyz=0\}, \qquad E_{a,x}^P = \{x=0\}\setminus\{[0:1:0],[0:0:1]\},$$
$$E_{a,y}^P = \{y=0\}\setminus\{[1:0:0],[0:0:1]\}, \qquad E_{a,xy}^P = \{[0:0:1]\}.$$

Analogously, one considers $E_{a,z}^P$, $E_{a,xz}^P$, and $E_{a,yz}^P$ so that $E_a^P = \bigsqcup_j E_{a,j}^P$ really defines a stratification. However, these three strata belong to more than one irreducible divisor in the total transform and hence they do not contribute to the characteristic polynomial.

As for E_0, according to its quotient singularities, no stratification need to be considered (it is smooth).

The Euler characteristic of $\check{E}_0$ and $\check{E}_{a,j}^P := E_{a,j}^P \setminus \big(E_{a,j}^P \cap \big(\bigcup_{b\neq a} E_b^P \cup \widehat{V}\big)\big)$ for $j = 1, x, y, xy$ (see notation just above Theorem (IV.3.14)) as well as its multiplicity are calculated in Lemma (VII.2.3).



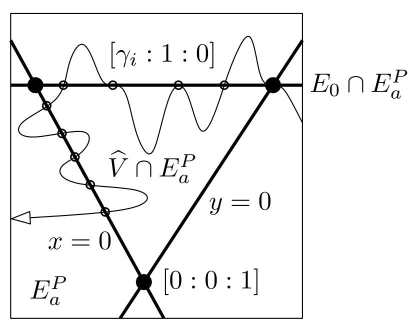


FIGURE VII.3. Stratification of $\mathcal{E}_a^P$ and E_a^P needed for applying the generalized A'Campo's formula.

The following three results are presented without their proofs because they do not provide any new idea. They are the analogous of Lemma (VI.3.3), Theorem (VI.3.5), and Corollary (VI.3.7), respectively. Anyway, recall that the Euler characteristic of $\mathbb{P}^2 \setminus \mathbf{C}$ is $m^2 - 3m + 3 - \sum_{P\in \mathrm{Sing}(P)} \mu_P$.

Lemma (VII.2.3). *Using the previous notation, the Euler characteristic and the multiplicity of* $\check{E}_0$ *are*

$$\chi(\check{E}_0) = \chi(\mathbb{P}^2 \setminus \mathbf{C}), \qquad m(\check{E}_0) = m.$$

For the rest of strata of $\check{E}^P_a$*, let us fix a point* $P \in \operatorname{Sing}(\mathbf{C})$*. Then, one has that*

$$\chi(\check{E}^P_{a,j}) = \begin{cases} 1 & a = 1,\ j = xy \\ 0 & a \neq 1,\ j = xy \\ -\gcd\left(k, m(\check{\mathcal{E}}^P_{a,j})\right) \cdot \chi(\check{\mathcal{E}}^P_{a,j}) & \forall a,\ j = 1, x, y\,; \end{cases}$$

$$\chi(\check{\mathcal{E}}^P_{a,j}) \neq 0 \implies m(\check{E}^P_{a,j}) = \begin{cases} m + k & a = 1,\ j = xy \\ \dfrac{(m+k) \cdot m(\check{\mathcal{E}}^P_{a,j})}{\gcd\left(k, m(\check{\mathcal{E}}^P_{a,j})\right)} & \forall a,\ j = 1, x, y. \end{cases}$$

In fact, $\forall a \in S(\Gamma^P_+)$, $a \neq 1$*, the stratum* $\check{E}^P_{a,xy}$ *is empty and, in particular, its Euler characteristic is zero.* □

Theorem (VII.2.4). *The characteristic polynomial of the complex monodromy of* $(V, 0)$ *is*

$$\Delta_{(V,0)}(t) = \frac{(t^m - 1)^{\chi(\mathbb{P}^2 \setminus \mathbf{C})}}{t - 1} \prod_{P \in \operatorname{Sing}(\mathbf{C})} \Delta^k_{(\mathbf{C},P)}(t^{m+k}),$$

where $\Delta_{(\mathbf{C},P)}(t)$ *denotes the characteristic polynomial of the local complex monodromy of* $(\mathbf{C}, P)$ *and if* $\Delta(t) = \prod_i (t^{m_i} - 1)^{a_i}$*, then* $\Delta^k(t)$ *denotes*

$$\Delta^k(t) = \prod_i \left(t^{\frac{m_i}{\gcd(m_i,k)}} - 1 \right)^{\gcd(m_i,k) a_i}. \qquad \square$$

Corollary (VII.2.5). *The Milnor number of a Yomdin-Lê surface singularity can be expressed in terms of the Milnor numbers of the singular points of the tangent cone, namely*

$$\mu(V, 0) = (m-1)^3 + k \sum_{P \in \operatorname{Sing}(\mathbf{C})} \mu(\mathbf{C}, P). \qquad \square$$

Section § VII.3
Weighted Yomdin-Lê Surface Singularities

There is still another generalization of a SIS. Let $\omega := (a, b, c) \in \mathbb{N}^3$ with $\gcd(a, b, c) = 1$. Let $f = f_m + f_{m+k} + \cdots \in \mathbb{C}\{x, y, z\}$ be the decomposition of f into its ω-homogeneous parts, $k \geq 1$. Denote $V := V(f) \subset \mathbb{C}^3$ and $\mathbf{C} := V(f_m) \subset \mathbb{P}^2_\omega$. Then, $(V, 0)$ is said to be a *weighted Yomdin-Lê surface singularity* (WYS) if the condition $\operatorname{Sing}(\mathbf{C}) \cap V(f_{m+k}) = \emptyset$ holds in $\mathbb{P}^2_\omega$.

This family can be treated the same way. However, it is not the purpose of this work to describe in detail an embedded **Q**-resolution of a WYS. Instead, only the step zero is presented. As we shall see, one can continue the process, following in a natural way the ideas explained in this chapter, to find an embedded **Q**-resolution of $(V,0)$.

(VII.3.1). We start the **Q**-resolution of $(V,0)$ with the ω-blow-up at the origin $\pi_0 : \widehat{\mathbb{C}}^3_\omega \to \mathbb{C}^3$, see §I.3–2. The total transform is the divisor $\pi_0^*(V) = \widehat{V} + mE_0$, where $\widehat{V}$ is the strict transform and E_0 is the exceptional divisor. The intersection $\widehat{V} \cap E_0$ is identified with the tangent cone of the singularity.

Let us consider $P \in \widehat{V} \cap E_0 = \mathbf{C}$. To simplify the exposition one assumes that $P = ((0,0,0),[0:0:1]) \equiv [0:0:1] \in \mathbf{C}$. Take a chart of $\widehat{\mathbb{C}}^3_\omega$ around P where $z=0$ is the equation of E_0 and the blowing-up takes the form

$$(x,y,z) \stackrel{\pi_0}{\longmapsto} (xz^a, yz^b, z^c).$$

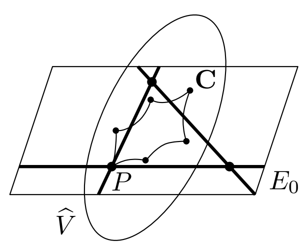
Then, the equation of $\widehat{V}$ on $X(c;a,b,-1)$ is

$$\widehat{V}: \quad f_m(x,y,1) + z^k \Big[f_{m+k}(x,y,1) + z^l f_{m+k+l}(x,y,1) + \cdots \Big] = 0.$$

Two cases arise: if P is smooth in the tangent cone, then $\widehat{V}$ is also smooth at P and the intersection with E_0 at that point is transverse; otherwise, i.e. $P \in \mathrm{Sing}(\mathbf{C})$, the WYS condition $\mathrm{Sing}(\mathbf{C}) \cap V(f_{m+k}) = \emptyset$ implies that the previous expression in brackets is a unit in the corresponding local ring and $\widehat{V}$ is not smooth at P. Now the order of $f_m(x,y,1)$ is greater than or equal to 2 and the intersection $\widehat{V} \cap E_0$ is not transverse at P.

$\pi_0^*(V) = \widehat{V} + mE_0$

$\widehat{V} \cap E_0 = \mathbf{C}$



$NT(\pi_0^*(V)) = \mathrm{Sing}(\mathbf{C})$

FIGURE VII.4. Step 0 in the embedded **Q**-resolution of $(V,0)$.

To achieve a similar situation after the step zero, one proceeds with the (p_1,q_1)-weighted blow-up at a point of type $(c;a,b)$ in the tangent cone and with the $\left(\frac{kp_1}{k_1}, \frac{kq_1}{k_1}, \frac{\nu_1}{k_1}\right)$-weighted blow-up at a point of type $(c;a,b,-1)$ in dimension 3, cf. Lemma (VII.1.1). As it is said, an embedded **Q**-resolution of $(V,0)$ can be computed in this way.

VIII

Algorithms for Checking Rational Roots of b-Functions and their Applications

The content of this chapter has already been submitted for publication in a joint work with Viktor Levandovskyy. There is a preliminary version available at [**LM10**], see also [**LM08, ALM09, ABL**$^+$**10**].

Bernstein-Sato polynomial of a hypersurface is an important object with numerous applications. However, its computation is hard, as a number of open questions and challenges indicate. In this chapter we propose a family of algorithms called `checkRoot` for optimized checking whether a given rational number is a root of Bernstein-Sato polynomial and in the affirmative case, computing its multiplicity.

This algorithms are used in the new approach to compute the whole global or local Bernstein-Sato polynomial and b-function of a holonomic ideal with respect to a weight vector. They can be applied in numerous situations, where an upper bound for the Bernstein-Sato polynomial can be established. Namely, it can be achieved by means of embedded resolution, for topologically equivalent singularities or using the formula of A'Campo and spectral numbers. We also present approaches to the logarithmic comparison problem and the intersection homology D-module.

Several applications are presented as well as solutions to some challenges which were intractable with the classical methods. One of the main applications consists of computing of a stratification of affine space with the local b-function being constant on each stratum. Notably, the algorithm we propose does not employ primary decomposition. Also we apply our results for the computation of Bernstein-Sato polynomials for varieties.

The methods from this chapter have been implemented in Singular as libraries `dmod.lib` and `bfun.lib`, see [**LM06**] and other related packages. All the examples have been computed with this implementation.

Section § VIII.1 Introduction

Through the chapter we assume $\mathbb{K}$ to be a field of characteristic 0. By R_n we denote the ring of polynomials $\mathbb{K}[x_1, \dots, x_n]$ in n variables over $\mathbb{K}$ and by D_n we denote the ring of $\mathbb{K}$-linear partial differential operators with coefficients in R_n, that is the *n-th Weyl algebra*. The ring D_n is the associative $\mathbb{K}$-algebra generated by the partial differential operators ∂_i and the multiplication operators x_i subject to relations

$$\{\partial_i x_j = x_j \partial_i + \delta_{ij}, x_j x_i = x_i x_j, \partial_j \partial_i = \partial_i \partial_j \mid 1 \le i, j \le n\}.$$

That is, the only non-commuting pairs of variables are (x_i, ∂_i); they satisfy the relation $\partial_i x_i = x_i \partial_i + 1$. We use the Lie bracket notation $[a, b] := ab - ba$ for operators a, b, then e.g. the latter relation can be written as $[\partial_i, x_i] = 1$. Finally, we denote by $D_n[s]$ the ring of polynomials in one variable s with coefficients in the n-th Weyl algebra, i.e. $D_n[s] = D_n \otimes_{\mathbb{K}} \mathbb{K}[s]$.

Let us recall Bernstein's construction. Given a non-constant polynomial $f \in R_n$ in n variables, consider $M = R_n[s, \frac{1}{f}] \cdot f^s$ which is by definition the free $R_n[s, \frac{1}{f}]$-module of rank one generated by the formal symbol f^s. Then M has a natural structure of left $D_n[s]$-module. Here the differential operators act in a natural way,

$$\partial_i\big(g(s,x) \cdot f^s\big) = \left(\frac{\partial g}{\partial x_i} + s g(s,x) \frac{\partial f}{\partial x_i} \frac{1}{f} \right) \cdot f^s \ \in \ M.$$

Theorem (VIII.1.1) ([**Ber72**])**.** *Given a non-constant polynomial $f \in R_n$, there exists a non-zero polynomial $b(s) \in \mathbb{K}[s]$ and a differential operator $P(s) \in D_n[s]$ such that*

$$P(s) f \cdot f^s = b(s) \cdot f^s \ \in \ R_n\big[s, \frac{1}{f}\big] \cdot f^s = M. \tag{40}$$

The monic polynomial $b(s)$ of minimal degree satisfying (40) is called the *Bernstein-Sato polynomial* or the *global b-function* of f.

This chapter is organized as follows. In Section VIII.2, the `checkRoot` family of algorithms for checking rational roots of the global and local Bernstein-Sato polynomial is developed. We also show how to compute the b-function of a holonomic ideal with respect to a certain weight vector.

In Section VIII.3, we show how to obtain an upper bound in various situations: by using an embedded resolution, for topologically equivalent singularities, by using A'Campo's formula and spectral numbers. In particular, we demonstrate a complicated example of (non-isolated) quasi-ordinary singularity.

In Section VIII.4, we discuss the possibilities to obtain integral roots of the b-function and apply it to the computation of the minimal integral root in the context of Intersection Homology D-module and Logarithmic Comparison Theorem. In Section VIII.5, we present a new method for computing the stratification of the affine space according to local Bernstein-Sato polynomials.

We want to stress, that Bernstein-Sato polynomials for most of the examples, presented in this chapter, cannot be computed by direct methods with any computer algebra system including SINGULAR:PLURAL [**GLS06**]. Indeed, these examples were known as open challenges in the community and here we present their solutions for the first time.

The examples of this chapter have been performed on a PC with Intel Core i3–540 Processor (4M Cache, 3.06 GHz) equipped with 4 GB RAM running Ubuntu 10.04 LTS Linux.

SECTION § VIII.2
The `checkRoot` Family of Algorithms

For the sake of completeness, some of the ideas coming from [**LM08**], as well as some results and their proofs have been included here. Several algorithms for computing the b-function associated with a polynomial are known, see e.g. [**Oak97a, Oak97b, Oak97c**], [**SST00**], [**BM02**], [**Nor02**], [**Sch04a**], [**LM08**]. However, from the computational point of view, it is very hard to obtain this polynomial in general. Despite significant recent progress, only restricted number of examples can actually be treated. In order to enhance the computation of the Bernstein-Sato polynomial via Gröbner bases, we study the following computational problems.

(1) Obtain an upper bound for $b_f(s)$, that is, find a nonzero polynomial $B(s) \in \mathbb{K}[s]$ such that $b_f(s)$ divides $B(s)$. Write

$$B(s) = \prod_{i=1}^{d} (s - \alpha_i)^{m_i}.$$

(2) Check whether α_i is a root of the b-function.
(3) Compute the multiplicity of α_i as a root of $b_f(s)$.

There exist some well-known methods to obtain an upper bound for the Bernstein-Sato polynomial of a hypersurface singularity once we know, for instance, an embedded resolution of such singularity [**Kas77**], see Section VIII.3. However, as far as we know, there is no algorithm for computing the b-function from this upper bound. In this section we present algorithms for checking whether a given rational number is a root of the b-function and for computing its multiplicity. As a first application, using this idea, we could obtain $b_f(s)$ for some interesting non-isolated singularities, see Example (VIII.3.3) below.

From the definition of the b-function it is clear that

$$\langle b_f(s)\rangle = (\operatorname{Ann}_{D_n[s]}(f^s) + \langle f\rangle) \cap \mathbb{K}[s]. \tag{41}$$

In fact, this is another way to define the Bernstein-Sato polynomial. This equation was used to prove the main result of this section, namely Theorem (VIII.2.1).

Theorem (VIII.2.1). *Let R be a $\mathbb{K}$-algebra, whose center contains $\mathbb{K}[s]$. Let $q(s) \in \mathbb{K}[s]$ be a polynomial in one variable and I a left ideal in R satisfying $I \cap \mathbb{K}[s] \neq 0$. The following equalities hold:*

(1) $\big(I + R\langle q(s)\rangle\big) \cap \mathbb{K}[s] = I \cap \mathbb{K}[s] + \mathbb{K}[s]\langle q(s)\rangle$,

(2) $\big(I : q(s)\big) \cap \mathbb{K}[s] = \big(I \cap \mathbb{K}[s]\big) : q(s)$,

(3) $\big(I : q(s)^\infty\big) \cap \mathbb{K}[s] = \big(I \cap \mathbb{K}[s]\big) : q(s)^\infty$.

In particular, using the ideal $I = \operatorname{Ann}_{D_n[s]}(f^s) + \langle f\rangle \subseteq D_n[s]$ in the previous equation (41)*, one has*

- $\big[\operatorname{Ann}_{D_n[s]}(f^s) + D_n[s]\langle f, q(s)\rangle\big] \cap \mathbb{K}[s] = \big\langle \operatorname{gcd}(b_f(s), q(s)) \big\rangle$,
- $\big[(\operatorname{Ann}_{D_n[s]}(f^s) + D_n[s]\langle f\rangle) : q(s)\big] \cap \mathbb{K}[s] = \big\langle \frac{b_f(s)}{\operatorname{gcd}(b_f(s), q(s))} \big\rangle$,
- $\big[(\operatorname{Ann}_{D_n[s]}(f^s) + D_n[s]\langle f\rangle) : q(s)^\infty\big] \cap \mathbb{K}[s] = \langle b_f(s)\rangle : q(s)^\infty$.

Proof. Let $b(s) \neq 0$ be a generator of $I \cap \mathbb{K}[s]$. At first, suppose that $h(s) \in (I + R\langle q(s)\rangle) \cap \mathbb{K}[s]$. Then one can write

$$h(s) = P(s) + Q(s)q(s), \tag{42}$$

where $P(s) \in I$ and $Q(s) \in R$. Let $d(s)$ be the greatest common divisor of $b(s)$ and $q(s)$. There exist $b_1(s)$ and $q_1(s)$ such that $d(s)b_1(s) = b(s)$ and $d(s)q_1(s) = q(s)$, and hence $b_1(s)q(s) = q_1(s)b(s)$. Since s commutes with all elements in R, multiplying in (42) by $b_1(s)$, one obtains

$$b_1(s)h(s) = b_1(s)P(s) + Q(s)q_1(s)b(s) \;\in\; I.$$

Thus $b_1(s)h(s) \in I \cap \mathbb{K}[s] = \langle b(s)\rangle$ and therefore $h(s) \in \langle b(s)\rangle : \langle b_1(s)\rangle = \langle d(s)\rangle = I \cap \mathbb{K}[s] + \langle q(s)\rangle$. The other inclusion follows obviously. The second and the third parts can be shown directly and now the proof is complete. □

Note that the second (resp. third) part of the previous theorem can be used to heuristically find an upper bound for $b_f(s)$ (resp. the roots of $b_f(s)$). Since $q(s)$ is in the center of $D_n[s]$, the quotient and saturation ideals can be computed effectively e.g. via the kernel of a module homomorphism procedures, cf. [**Lev05**]. More classical but less effective approach is to use the extra commutative variable, say T, and the formula

$$I : q(s)^\infty = D_n[s,T]\langle I, 1 - Tq(s)\rangle \cap D_n[s].$$

Let us see an example to illustrate how useful could be Th. (VIII.2.1).

Example (VIII.2.2). Let $f \in \mathbb{C}[x,y]$ be the polynomial $x(x^2+y^3)$. The annihilator of f^s in $D[s]$ can be generated by the operators $P_1(s) = 3xy^2\partial_x - y^3\partial_y - 3x^2\partial_y$ and $P_2(s) = 3x\partial_x + 2y\partial_y - 9s$. Consider the univariate polynomial

$$q(s) = (s+1)(s+5/9)(s+8/9)(s+10/9)(s+7/9)(s+11/9)(s+13/9).$$

Computing a Gröbner basis, one can see that the ideal in $D[s,T]$ generated by $\{P_1(s), P_2(s), f, 1-Tq(s)\}$ is the whole ring. From Theorem (VIII.2.1)(3), one deduces that $q(s)$ contains all the roots of $b_f(s)$.

Using this approach we only have to check whether an ideal is the whole ring or not. Therefore any admissible monomial ordering can be chosen, hence the one, which is generically fast.

Given an arbitrary rational number α, consider the ideal $I_\alpha \subseteq D_n[s]$ generated by the annihilator of f^s, the polynomial f, and $s+\alpha$. Theorem (VIII.2.1)(1) says that the equality $I_\alpha = D_n[s]$ holds generically (this is clarified in Corollary (VIII.2.3) below). Hence the roots of the Bernstein-Sato polynomial are the rational numbers for which the condition $I_\alpha \neq D_n[s]$ is satisfied.

This allows one to work out with parameters, that is over $\mathbb{K}(\alpha)\langle x, \partial_x\rangle[s]$, and find the corresponding complete set of special parameters. The latter procedure is algorithmic [**LZ07**] and implemented in Singular. Note, that the set of candidates to obstructions, returned by the latter algorithm is in general bigger, than the set of real obstructions.

Corollary (VIII.2.3). *Let $\{P_1(s), \ldots, P_k(s)\}$ be a system of generators of the annihilator of f^s in $D_n[s]$. The following conditions are equivalent:*

(1) *$\alpha \in \mathbb{Q}_{>0}$ is a root of $b_f(-s)$.*

(2) *$D_n[s]\langle P_1(s), \ldots, P_k(s), f, s+\alpha\rangle \neq D_n[s]$.*

(3) *$D_n\langle P_1(-\alpha), \ldots, P_k(-\alpha), f\rangle \neq D_n$.*

Moreover, in such a case $D_n[s]\langle P_1(s), \ldots, P_k(s), f, s+\alpha\rangle \cap \mathbb{K}[s] = \mathbb{K}[s]\langle s+\alpha\rangle$.

Proof. Let $J = D_n[s]\langle P_1(s), \ldots, P_k(s), f, s+\alpha\rangle$ and denote by K the ideal $J \cap D_n$. Then, one clearly has $K = D_n\langle P_1(-\alpha), \ldots P_k(-\alpha), f\rangle$. Now since

$$J = D[s] \iff J \cap \mathbb{K}[s] = \mathbb{K}[s] \iff K = D_n,$$

and $\gcd(b_f(s), s+\alpha) = 1$ if and only if $b_f(-\alpha) \neq 0$, the result follows from applying Theorem (VIII.2.1) using $q(s) = s + \alpha$. $\square$

Once we know a system of generators of the annihilator of f^s in $D_n[s]$, the last corollary provides an algorithm for checking whether a given rational number is a root of the b-function of f, using Gröbner bases in the Weyl algebra.

Algorithm 1 checkRoot1 (checks whether $\alpha \in \mathbb{Q}_{>0}$ is a root of $b_f(-s)$)

Input 1: $\{P_1(s), \ldots, P_k(s)\} \subseteq D_n[s]$, generators of $\mathrm{Ann}_{D_n[s]}(f^s)$;
Input 2: f, a polynomial in R_n; α, a number in $\mathbb{Q}_{>0}$;
Output: **true**, if α is a root of $b_f(-s)$; **false**, otherwise;

$K := \langle P_1(-\alpha), \ldots, P_k(-\alpha), f\rangle$; $\quad \triangleright\ K = J \cap D_n \subseteq D_n$
$G :=$ reduced Gröbner basis of K w.r.t. ANY term ordering;
return $(G \neq \{1\})$;

VIII.2–1. Multiplicities

Two approaches to deal with multiplicities are presented. We start with a natural generalization of Corollary (VIII.2.3).

Corollary (VIII.2.4). *Let m_α be the multiplicity of α as a root of $b_f(-s)$ and let us consider the ideals $J_i = \mathrm{Ann}_{D_n[s]}(f^s) + \langle f, (s+\alpha)^{i+1}\rangle \subseteq D_n[s]$, for $i = 0, \ldots, n$. Then, the following conditions are equivalent:*

(1) $m_\alpha > i$.

(2) $J_i \cap \mathbb{K}[s] = \langle (s+\alpha)^{i+1}\rangle$.

(3) $(s+\alpha)^i \notin J_i$.

Moreover, if $D_n[s] \supsetneq J_0 \supsetneq J_1 \supsetneq \cdots \supsetneq J_{m-1} = J_m$, then $m_\alpha = m$. In particular, $m \leq n$ and $J_{m-1} = J_m = \cdots = J_n$.

Proof. Let us first see $1 \iff 2$. Since the multiplicity $m_\alpha > i$ if and only if $\gcd(b_f(s), (s+\alpha)^{i+1}) = (s+\alpha)^{i+1}$, the equivalence follows from applying Theorem (VIII.2.1)(1) using $q(s) = (s+\alpha)^{i+1}$.

Note that if $(s+\alpha)^i \in J_i \cap \mathbb{K}[s]$, then clearly $J_i \cap \mathbb{K}[s] \supsetneq \langle (s+\alpha)^{i+1} \rangle$, that is, the implication $2 \Longrightarrow 3$ holds.

For $3 \Longrightarrow 2$, let $h(s) \in \mathbb{K}[s]$ be the monic generator of the ideal $J_i \cap \mathbb{K}[s]$. Since $(s+\alpha)^{i+1} \in J_i \cap \mathbb{K}[s] = \langle h(s) \rangle$, there exists $j \leq i+1$ such that $h(s) = (s+\alpha)^j$. Suppose that $j \leq i$. Then one has

$$(s+\alpha)^i = (s+\alpha)^{i-j}(s+\alpha)^j = (s+\alpha)^{i-j}h(s) \in J_i.$$

That, however, contradicts 3 and thus $j = i+1$.

The rest of the assertion follows by applying the previous result using $i = m$ and $i = m-1$, since $(s+\alpha)^m \in J_m$ and $(s+\alpha)^{m-1} \notin J_{m-1}$ from the hypothesis. □

Again once we know a system of generators of the annihilator of f^s in $D_n[s]$, the last corollary provides an algorithm for checking whether a given rational number is a root of the b-function of f and for computing its multiplicity, using Gröbner bases for differential operators.

Algorithm 2 CHECKROOT2 (computes the multiplicity of $\alpha \in \mathbb{Q}_{>0}$ as a root of $b_f(-s)$)

Input 1: $\{P_1(s), \ldots, P_k(s)\} \subseteq D_n[s]$, generators of $\mathrm{Ann}_{D_n[s]}(f^s)$;
Input 2: f, a polynomial in R_n; α, a number in $\mathbb{Q}_{>0}$;
Output: m_α, the multiplicity of α as a root of $b_f(-s)$;

for $i = 0$ to n **do**
  $J := D_n[s]\langle P_1(s), \ldots, P_k(s), f, (s+\alpha)^{i+1} \rangle$; ▷ J_i
  $G :=$ Gröbner basis of J w.r.t. ANY term ordering;
  $r :=$ normal form of $(s+\alpha)^i$ with respect to G;
  if $r = 0$ **then**
    $m_\alpha := i$; ▷ $r = 0 \Longrightarrow (s+\alpha)^i \in J_i$
    break ▷ leave the **for** block
  end if
end for
return m_α;

PROOF. (of Algorithm 2).
Termination: The algorithm CHECKROOT2 clearly terminates and one only has to consider the loop from 0 to n because the multiplicity of a root of $b_f(s)$ is at most n, see [**Sai94**].
Correctness: Corollary (VIII.2.4) implies the correctness of the method. □

Remark **(VIII.2.5).** There exists another version of `checkRoot2` with just one step, due to the formula, see Corollary (VIII.2.4) above,

$$\big(\operatorname{Ann}_{D_n[s]}(f^s) + D_n[s]\langle f, (s+\alpha)^n\rangle\big) \cap \mathbb{K}[s] = \langle (s+\alpha)^{m_\alpha}\rangle.$$

However, this method only seems to be useful when the multiplicity is close to n, otherwise `checkRoot2` is more effective. The reason is that in general, the multiplicity is far lower than the number of variables.

This algorithm is much faster, than the computation of the whole Bernstein polynomial via Gröbner bases, because no elimination ordering is needed for computing a Gröbner basis of J. Also, the element $(s+\alpha)^{i+1}$, added as a generator, seems to simplify tremendously such a computation. Actually, when $i = 0$ it is possible to eliminate the variable s in advance and we can perform the whole computation in D_n, see Corollary (VIII.2.3)(3) above.

Nevertheless, Algorithm 2 meets the problem to calculate on each step a Gröbner basis G_i for an ideal of the form $I + \langle (s+\alpha)^{i+1}\rangle$ and the set G_{i-1} is not used at all for such computation. A completely new Gröbner basis has to be performed instead. The classical idea of quotient and saturation are used to solve this obstruction. In particular, the following result holds.

Corollary (VIII.2.6). *Let m_α be the multiplicity of α as a root of $b_f(-s)$ and let us consider the ideal $I = \operatorname{Ann}_{D_n[s]}(f^s) + D_n[s]\langle f\rangle$. The following conditions are equivalent:*

1. $m_\alpha > i$.
2. $\big(I : (s+\alpha)^i\big) + D_n[s]\langle s+\alpha\rangle \neq D_n[s]$.
3. $\big(I : (s+\alpha)^i\big)|_{s=-\alpha} \neq D_n$.

Proof. Given $J \subseteq D_n[s]$ an ideal, we denote by $b_J(s)$ the monic generator of the ideal $J \cap \mathbb{K}[s]$. Then, from Theorem (VIII.2.1)(1), condition 2 is satisfied if and only if $-\alpha$ is a root of $b_{I:(s+\alpha)^i}(s)$. This univariate polynomial is nothing but $b_f(s)/\gcd(b_f(s),(s+\alpha)^i)$, due to Theorem (VIII.2.1)(2). On the other hand, one has the obvious equivalence

$$m_\alpha > i \iff (s+\alpha) \,\Big|\, \frac{b_f(s)}{\gcd(b_f(s),(s+\alpha)^i)},$$

and hence the claim follows. □

Since $s+\alpha$ belongs to the center of $D_n[s]$, the ideal $I : (s+\alpha)^i$ can recursively be computed by the formulas

$$I : (s+\alpha) = (I \cap D_n[s]\langle s+\alpha\rangle)/(s+\alpha),$$
$$I : (s+\alpha)^i = (I : (s+\alpha)^{i-1}) : (s+\alpha).$$

The following is a sketch of another algorithm for computing multiplicities using quotient ideals. The termination and correctness follow from Corollary (VIII.2.6).

Algorithm 3 CHECKROOT3 (computes the multiplicity of $\alpha \in \mathbb{Q}_{>0}$ as a root of $b_f(-s)$)

Input 1: $\{P_1(s), \ldots, P_k(s)\} \subseteq D_n[s]$, generators of $\mathrm{Ann}_{D_n[s]}(f^s)$;
Input 2: f, a polynomial in R_n; α, a number in $\mathbb{Q}_{>0}$;
Output: m_α, the multiplicity of α as a root of $b_f(-s)$;

$m := 0$; $I := D_n[s]\langle P_1(s), \ldots, P_k(s), f\rangle$; $J := I + D_n[s]\langle s+\alpha\rangle$;
while $G \neq \{1\}$ **do**
  $m := m + 1$;
  $I := I : (s+\alpha)$; ▷ $I : (s+\alpha)^i$
  $J := I + D_n[s]\langle s+\alpha\rangle$; (or $J := I|_{s=-\alpha}$)
  $G :=$ reduced Gröbner basis of J w.r.t. ANY term ordering;
end while
return m;

Remark **(VIII.2.7).** Several obvious modifications of the presented algorithms can be useful depending on the context. Assume, for instance, that $q(s)$ is a known factor of the Bernstein-Sato polynomial and one is interested in computing the rest of $b_f(s)$. Then the ideal $I : q(s)$ contains such information. This simple observation can help us in some special situations.

Remark **(VIII.2.8).** Define the *reduced Bernstein-Sato polynomial* of $f \in R_n$ to be $b'_f(s) = b_f(s)/(s+1)$. The *Jacobian ideal* of f is $J_f = \langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\rangle$. It is known, that taking $\langle f\rangle + J_f$ instead of $\langle f\rangle$ has the following consequence

$$(\mathrm{Ann}_{D[s]}\, f^s + \langle f, \tfrac{\partial f}{\partial x_1}, \ldots, \tfrac{\partial f}{\partial x_n}\rangle) \cap \mathbb{K}[s] = \langle b'_f(s)\rangle = \langle \tfrac{b_f(s)}{s+1}\rangle.$$

Hence, all the algorithms above can be modified to this setting, resulting in more effective computations. This is the way it should be done in the implementation. We decided, however, not to modify the description of algorithms in order to keep the exposition easier.

VIII.2–2. Local versus global b-functions

Here we are interested in what kind of information one can obtain from the global b-function for computing the local ones and conversely. In order to avoid theoretical problems we will assume in this paragraph that the ground field is $\mathbb{C}$.

Several algorithms to obtain the local b-function of a hypersurface f have been known without any Gröbner bases computation but under strong conditions on f. For instance, it was shown in [**Mal75**] that the minimal polynomial of $-\partial_t t$ acting on some vector space of finite dimension coincides with the reduced local Bernstein polynomial, assuming that the singularity is isolated.

Definition (VIII.2.9). Let $p \in \mathbb{C}^n$ be a point and $\mathfrak{m}_p = \langle \{x_1 - p_1, \ldots, x_n - p_n\} \rangle \subset R_n$ the corresponding maximal ideal. Let D_p be the local Weyl algebra at p, that is the n-th Weyl algebra with coefficients from $\mathbb{C}[x_1, \ldots, x_n]_p$ instead of $R_n = \mathbb{C}[x_1, \ldots, x_n]$. Define the *local b-function* or *local Bernstein-Sato polynomial* to be the univariate monic polynomial $b_{f,p}(s)$ of the minimal degree, such that the identity $P(s) f \cdot f^s = b(s) \cdot f^s$ holds for $P(s) \in D_p[s]$.

Theorem (VIII.2.10). *(Briançon-Maisonobe (unpublished) and* [**MN91**]*). Let $b_{f,p}(s)$ be the local b-function of f at the point $p \in \mathbb{C}^n$ and $b_f(s)$ the global one. Then, $b_f(s) = \operatorname{lcm}_{p\in\mathbb{C}^n} b_{f,p}(s) = \operatorname{lcm}_{p\in \operatorname{Sing}(f)} b_{f,p}(s)$.* □

The previous theorem can be very useful for computing the global b-function using the local ones. Let us see an example.

Example (VIII.2.11). Let $\mathcal{C}$ be the curve in $\mathbb{C}^2$ given by the equation $f = (y^2 - x^3)(3x - 2y - 1)(x + 2y)$. This curve has three isolated singular points $(0, 0)$, $(1, 1)$, and $(1/4, -1/8)$. The following is its real picture.

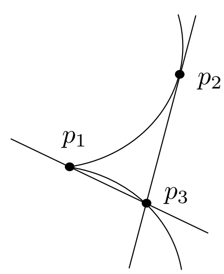


FIGURE VIII.1. The cup $(2, 3)$ with two lines.

The library `gmssing.lib` contains a procedure `bernstein`, which computes the local b-function at the origin. Moving to the corresponding points we can also compute $b_{f,p_i}(s)$.

$$b_{f,p_1}(s) = (s+1)^2(s+5/8)(s+7/8)(s+9/8)(s+11/8)$$
$$b_{f,p_2}(s) = (s+1)^2(s+3/4)(s+5/4)$$
$$b_{f,p_3}(s) = (s+1)^2(s+2/3)(s+4/3)$$

From this information and using Theorem (VIII.2.10), the global b-function is

$$b_f(s) = (s+1)^2(s+2/3)(s+5/8)(s+3/4)(s+7/8)$$
$$(s+4/3)(s+5/4)(s+9/8)(s+11/8).$$

The computation of the global b-function with Theorem (VIII.2.10) is effective, when the singular locus consists of finitely many isolated singular points. The Singular library `gmssing.lib` implemented by M. Schulze [**Sch04b**] and based on his work [**Sch04a**] allows one to compute invariants related to the the Gauss-Manin system of an isolated hypersurface singularity. In the non-isolated case the situation is more complicated, since no explicit algebraic description of the Gauss-Manin connection exists. For computing the local b-function in this case (which is important on its own) we suggest using the global b-function as an upper bound and a local version of the `checkRoot` algorithm, see Section VIII.2–2 below.

T. Oaku presented algorithms for computing the local b-function in [**Oak97a**] and [**Oak97c**]. In these algorithms, no knowledge of a global b-function is needed. However, these algorithms are quite hard from the computational point of view. Namely, more complicated elimination in Weyl algebra together with numerous computations of quotient ideals in a commutative ring need to be executed. An intersection of a left ideal with a principal subalgebra needs to be performed as well, and for the local case this has to be done within the localized ring.

In [**Nak09**], H. Nakayama presented an algorithm for computing local b-functions. One step in his algorithm uses a bound for the multiplicity of a given rational root of the global b-function. Then the algorithm checks if this multiplicity agrees with the local one. This approach is very similar to our `checkRoot` algorithm.

Localization of non-commutative rings

We recall some properties of rings of fractions in non-commutative setting. The reader is referred to [**GW04**] and [**MR01**] for further details.

Definition (VIII.2.12). Let R be a ring and $S \subseteq R$ a multiplicatively closed set. A *left ring of fractions* (analogously for *right rings of fractions*) for R with respect to S is a ring homomorphism $\phi : R \to Q$ such that:

(1) $\phi(s)$ is a unit of Q for all $s \in S$.
(2) Each element of Q can be written in the form for $\phi(s)^{-1}\phi(r)$ for some $r \in R$ and $s \in S$.
(3) $\ker(\phi) = \{r \in R \mid sr = 0 \text{ for some } s \in S\}$.

Theorem (VIII.2.13). *There exists a left ring of fractions for R with respect to S if and only if S is a left denominator set, that is, the following conditions hold:*

- Left Ore condition*: for each $r \in R$ and $s \in S$, there exist $r' \in R$ and $s' \in S$ such that $s'r = r's$, that is, $Sr \cap Rs \neq \emptyset$.*
- Left reversible*: if $rs = 0$ for some $r \in R$ and $s \in S$, then $\exists r' \in R$ such that $sr' = 0$.*

In such a case, the pair (Q, ϕ) is universal for homomorphisms $\varphi : R \to T$ such that $\varphi(S)$ consists of units of T and therefore Q is unique up to unique isomorphism. Moreover, if R also has a right ring of fractions Q' with respect to S, then $Q \simeq Q'$. □

Because of the uniqueness, the left ring of fractions Q (when it exists) is often denoted by $S^{-1}R$, and the natural map $\phi : R \to S^{-1}R$ is called the *localization map*. To simplify notation the elements of $S^{-1}R$ are denoted by $s^{-1}r$, even when $\ker \phi \neq 0$. Two quotients $s_1^{-1}r_1$ and $s_2^{-1}r_2$ are equal if and only if there exist $s \in S$ and $a \in R$ such that $as_1 = ss_2$ and $ar_1 = sr_2$. Actually $S^{-1}R$ could be described as $S \times R$ modulo the previous equivalence relation. The localization for left (resp. right) modules can be generalized in the obvious way and it is verified $S^{-1}M \cong S^{-1}R \otimes_R M$ (resp. $MS^{-1} \cong M \otimes_R RS^{-1}$).

Remark **(VIII.2.14).** If S satisfies the left Ore condition and it is left reversible, then the previous equivalence relation is the same as the following one, $(s_1, r_1) \sim (s_2, r_2) \iff \exists a, b \in R \mid as_1 = bs_2 \in S,\ ar_1 = br_2$.

Recall the following two classical results on localizations.

Lemma (VIII.2.15). *Let $R_1 \overset{i}{\hookrightarrow} R_2$ be a ring extension and $S \subset R_1$ a multiplicatively closed set. Assume $S^{-1}R_1$ and $S^{-1}R_2$ exist and consider the corresponding localization maps $\phi_1 : R_1 \to S^{-1}R_1$ and $\phi_2 : R_2 \to S^{-1}R_2$. Let $j : S^{-1}R_1 \to S^{-1}R_2$ be the map induced by i. Then, j is injective and for every left ideal $I \subseteq R_2$ one has $S^{-1}I \cap S^{-1}R_1 = S^{-1}(I \cap R_1)$.* □

Lemma (VIII.2.16). *Let R be a ring, $S \subseteq R$ a multiplicatively closed set and $I \subseteq R$ a left ideal. Assume $S^{-1}R$ exists. Then, $S^{-1}I$ is not the whole ring $S^{-1}R$ if and only if $I \cap S = \emptyset$.* □

Example (VIII.2.17). Let $R = D$ be the classical n-Weyl algebra and $S = \mathbb{K}[\mathbf{x}] \setminus \mathfrak{m}_p$, where $p \in \mathbb{K}^n$ is an arbitrary point, cf. Definition (VIII.2.9). Then S is a left and right denominator set as in the statement of Theorem (VIII.2.13), and the localization $(\mathbb{K}[\mathbf{x}] \setminus \mathfrak{m}_p)^{-1}D$ is naturally isomorphic to D_p. Analogous construction also holds for the extension $D[s] = \mathbb{K}[s] \otimes_{\mathbb{K}} D$.

Local version of the `checkRoot1` algorithm

Theorem (VIII.2.1) is general enough to be applied for checking rational roots of local Bernstein-Sato polynomials. To simplify the exposition, we concentrate our attention on the local version of `checkRoot1` algorithm. See Section VIII.5 for other generalizations.

Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial, $p \in \mathbb{C}^n$, and $\alpha \in \mathbb{Q}$. Then the first part of Theorem (VIII.2.1), see also Corollary (VIII.2.3), tells us that $(s+\alpha)$ is a factor of the local b-function at p if and only if the left ideal

$$\operatorname{Ann}_{D_p[s]}(f^s) + D_p[s]\langle f, s+\alpha\rangle \tag{43}$$

is not the whole ring $D_p[s]$. From Lemma (VIII.2.15) using $R_1 = D[s]$, $R_2 = D\langle t, \partial t\rangle := D \otimes_{\mathbb{K}} \mathbb{K}\langle t, \partial t \mid \partial t \cdot t = t \cdot \partial t + 1\rangle$, $S = \mathbb{C}[\mathbf{x}] \setminus \mathfrak{m}_p$, and $I = \operatorname{Ann}_{D\langle t,\partial t\rangle}(f^s) = I_f$ the Malgrange ideal associated with f, one obtains

$$\operatorname{Ann}_{D_p[s]}(f^s) = D_p[s] \operatorname{Ann}_{D[s]}(f^s).$$

Proposition (VIII.2.18). *Let $\{P_1(s), \ldots, P_k(s)\}$ be a system of generator of $\operatorname{Ann}_{D[s]}(f^s)$ and consider the ideal $I = D[s]\langle P_1(s), \ldots, P_k(s), f, s+\alpha\rangle$. Then we have*

$$(s+\alpha) \mid b_{f,p}(s) \iff p \in V(I \cap \mathbb{C}[\mathbf{x}]).$$

Proof. From the previous discussion, $D_p[s]I$ equals the ideal (43) and thus $s+\alpha$ is a factor of $b_{f,p}(s)$ if and only if $D_p[s]I \neq D_p[s]$. Now, by Lemma (VIII.2.16) using $R = D[s]$ and $S = \mathbb{C}[\mathbf{x}] \setminus \mathfrak{m}_p$,

$$D_p[s]I \neq D_p[s] \iff I \cap (\mathbb{C}[\mathbf{x}] \setminus \mathfrak{m}_p) = \emptyset \iff I \cap \mathbb{C}[\mathbf{x}] \subseteq \mathfrak{m}_p,$$

and the claim follows. □

There are several ways to check whether an ideal $I \subseteq D_p[s]$ is proper or not. However, it is an open problem to decide which one is more efficient. Mora division and standard bases techniques seem to be more suitable in this case, since otherwise a (global) elimination ordering is needed. On the other hand, using this approach, such orderings are unavoidable for obtaining the stratification associated with local b-functions, see Section VIII.5 where several examples are shown.

VIII.2–3. b-functions with respect to weights and `checkRoot`

The b-function associated with a holonomic ideal with respect to a weight is presented. We refer [**SST00**] for details. Let $0 \neq w \in \mathbb{R}^n_{\geq 0}$ and consider the V-filtration with respect to w, $\{V_m \mid m \in \mathbb{Z}\} = V$ on D, where V_m is spanned by $\left\{x^\alpha \partial^\beta \mid -w\alpha + w\beta \leq m\right\}$ over $\mathbb{K}$. That is, x_i and ∂_i get weights $-w_i$ and w_i respectively.

Note that the relation $\partial_i x_i = x_i \partial_i + 1$ is homogeneous of degree 0 with respect to such weights. The associated graded ring

$$\mathrm{Gr}^V(D) := \bigoplus_{m \in \mathbb{Z}} V_m / V_{m-1}$$

is isomorphic to D, which allows us to identify them.

For a non-zero operator

$$P = \sum_{\alpha,\beta \in \mathbb{N}^n} a_{\alpha\beta} x^\alpha \partial^\beta \in D,$$

the maximum $\max_{\alpha,\beta}\{-w\alpha + w\beta \mid c_{\alpha\beta} \neq 0\} \in \mathbb{R}$ is denoted by $\mathrm{ord}^V(P)$ and the principal symbol of P is the V-homogeneous operator given by

$$\sigma^V(P) := \sum_{-w\alpha + w\beta = \mathrm{ord}_V(P)} a_{\alpha\beta} x^\alpha \partial^\beta.$$

In addition, for a given ideal $I \subseteq D$, the associated graded ideal is defined as the vector space spanned by all its principal symbols, that is, $\mathrm{Gr}^V(I) := \mathbb{K} \cdot \{\sigma^V(P) \mid P \in I\}$. Sometimes, the principal symbol (resp. associated graded ideal) is called the initial form (resp. initial ideal) and it is denoted by $\mathrm{in}_{(-w,w)}(P)$ (resp. $\mathrm{in}_{(-w,w)}(I)$).

Definition (VIII.2.19). Let $I \subset D$ be a holonomic ideal. Consider $0 \neq w \in \mathbb{R}^n_{\geq 0}$ and $s := \sum_{i=1}^n w_i x_i \partial_i$. Then $\mathrm{Gr}^V(I) \cap \mathbb{K}[s] \neq 0$ is a principal ideal in $\mathbb{K}[s]$. Its monic generator is called the *global b-function of I with respect to the weight w*.

Although Theorem (VIII.2.1) can not be applied in this setting, since $s = \sum_i w_i x_i \partial_i$ does not belong to the center of the algebra, a similar result still holds, due to the properties of the V-filtration, see Proposition (VIII.2.20) below. Also Corollaries (VIII.2.3), (VIII.2.4), and (VIII.2.6) can be established using initial parts instead of annihilators.

Proposition (VIII.2.20). $\big(\, \mathrm{Gr}^V(I) + \mathrm{Gr}^V(D)\langle q(s)\rangle \,\big) \cap \mathbb{K}[s] = \mathrm{Gr}^V(I) \cap \mathbb{K}[s] + \mathbb{K}[s]\langle q(s)\rangle$.

Proof. Actually it is an easy consequence of being treated with V-homogeneous ideals. Consider $h(s) = Q + R \cdot q(s)$, where $Q \in \mathrm{Gr}^V(I)$ and $R \in \mathrm{Gr}^V(D)$. Taking V-homogeneous parts in the previous expression, one finds $Q_0 \in \mathrm{Gr}^V(I)$ and $R_0 \in \mathrm{Gr}^V(D)$ of degree 0 such that $h(s) = Q_0 + R_0 \cdot q(s)$. Now, since $q(s)$ commutes with Q_0, one can proceed as in the proof of Theorem (VIII.2.1)(1). □

Many algorithms in the realm of D-modules are based on the computation of such b-functions. For some applications like integration and restriction, only the maximal and the minimal integral roots have to be computed.

However, the previous proposition can not be used to find the set of all integral roots, since neither upper nor lower bound is known in advance. For instance, N. Takayama used the following simple example to show the general unboundness: $I = \langle x\partial_1 + k\rangle$, $k \in \mathbb{Z}$ is D_1-holonomic and $\mathrm{in}_{(-1,1)}(I) \cap \mathbb{C}[s] = \langle s+k\rangle$ with $s = t\partial_t$. Nevertheless, there is the natural possibility to check a particular root of a b-function with respect to the non-negative weight w.

Section § VIII.3
Computing b-Functions via Upper Bounds

As different possible ways to find upper bounds, we present embedded resolutions, topologically equivalent singularities, and A'Campo's formula. Depending on the context local or global version of our algorithm is used.

VIII.3–1. Embedded resolutions

In this part we will work again over the field $\mathbb{C}$ of the complex numbers. However, in actual computation we can assume that the ground field is generated by a finite number of (algebraic or transcendental) elements over the field $\mathbb{Q}$ and that the algebraic relations among these elements are specified.

Definition (VIII.3.1). Let $h : Y \to \mathbb{C}^n$ be a proper birational morphism. We say that h is a *global embedded resolution* of the hypersurface defined by a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$, $X := V(f)$, if the following conditions are satisfied:

(1) Y is a non-singular variety.
(2) $h : Y \setminus h^{-1}(X) \to \mathbb{C}^n \setminus X$ is an isomorphism.
(3) $h^{-1}(X)$ is a normal crossing divisor.

Since $h^{-1}(X)$ is a normal crossing divisor, the morphism $F = f \circ h : Y \to \mathbb{C}$ is locally given by a monomial. Hence, we can define the b-function of F as the least common multiple of the local ones. If F is locally given by the monomial $\mathbf{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ at the point p, then one has

$$b_{F,p}(s) = \prod_{i=1}^{\alpha_1}\left(s + \frac{i}{\alpha_1}\right) \cdots \prod_{i=1}^{\alpha_n}\left(s + \frac{i}{\alpha_n}\right) = \prod_{1 \le i_j \le \alpha_j}\ \prod_{1 \le k \le n}\left(s + \frac{i_k}{s_k}\right).$$

The following is the global version of the classical result by Kashiwara [**Kas77**]. The upper bound statement is due to Varchenko ([**Var81**]) and Saito ([**Sai93, Sai94**]).

Theorem (VIII.3.2). *For $f \in R_n$, there exists an integer k such that $b_f(s)$ is a divisor of the product $b_F(s)b_F(s+1)\cdots b_F(s+k)$. Moreover, $0 \le k \le n-1$.*

PROOF. Since h is a global embedded resolution of $X = V(f)$, h induces a local embedded resolution of the germ (X, p) at every point $p \in X$. Now, the existence of $k \ge 0$ with the divisibility property follows from the theorem by Kashiwara [**Kas77**] and from the fact that the global b-function is the least common multiple of the local ones, see Theorem (VIII.2.10). The proof for the upper bound can be found in the references above. □

This theorem allows one to find upper bounds also for the global case. Let us see an example to show how one can apply the algorithm `checkRoot` in order to compute the b-function.

Example (VIII.3.3). Let $f = (xz + y)(x^4 + y^5 + xy^4) \in \mathbb{Q}[x, y, z]$ and consider the univariate polynomial $B_1(s) = b_{x^5}(s)b_{y^{18}}(s)b_{z^{24}}(s)$. Since every root of $b_f(-s)$ belongs to the real interval $(0, 3)$, see Theorem (VIII.3.2), computing an embedded resolution of the singularity and using Kashiwara's result [**Kas77**], we obtain that $B(s) = B_1(s)B_1(s+1)B_1(s+2)$ is an upper bound for $b_f(s)$.

Once we know a system of generators of $\operatorname{Ann}_{D_n[s]} f^s$, checking whether each root of the upper bound is a root of the Bernstein-Sato polynomial was simple. It took less than 5 seconds except for those which appear in the table below. We also observe that when a candidate is not a root indeed, the computation is very fast. To the best of our knowledge, this example (first appeared in [**CU05**]) is intractable by any computer algebra system.

$$\begin{aligned} b_f(s) = {} & (s+1)^2(s+17/24)(s+\mathbf{5/4})(s+11/24)(s+5/8) \\ & (s+\mathbf{31/24})(s+13/24)(s+13/12)(s+7/12)(s+23/24) \\ & (s+5/12)(s+3/8)(s+11/12)(s+\mathbf{9/8})(s+7/8) \\ & (s+19/24)(s+3/4)(s+\mathbf{29/24})(s+25/24) \end{aligned}$$

The running time is given in the format *minutes:seconds*.

Root of $B(-s)$	Running time		Root of $b_f(-s)$?
	`checkRoot2`	`checkRoot1`	
5/4	18:47	12:42	Yes
31/24	47:31	31:05	Yes
9/8	0:56	0:24	Yes
29/24	17:41	7:57	Yes

Remark **(VIII.3.4).** Choosing the lexicographical ordering with $\partial_x > x$ in D, when using the `checkRoot` algorithm, reduced the running time to just 25 second. We ignore whether the lexicographical ordering is suitable for other families of singularities.

Let us give a brief description for computing a global embedded resolution of f. Denote by $V_1 := V(xz+y)$ and $V_2 := V(x^4+y^5+xy^4)$ the two components of $V(f)$. Note that $\mathrm{Sing}(V_2) \subset V_1 \cap V_2$ and the singular locus $\mathrm{Sing}(f) = V_1 \cap V_2$ can be decomposed into two disjoint algebraic sets as

$$\mathrm{Sing}(f) = V(xz+y, yz^4 - yz^3 + 1) \cup V(x,y) =: Y \sqcup Z.$$

The varieties V_1 and V_2 intersect transversely at every point of Y. Indeed, let us consider $P = (a,b,c) \in Y$. Then V_1 and V_2 are smooth at P and their tangent spaces

$$\{cx + y + az = 0\} \quad \text{and} \quad \{(4a^3 + b^4)x + (5b^4 + 4ab^3)y = 0\}$$

can not be the same ($a \neq 0$ holds).

Consider $\pi : \widehat{\mathbb{C}}^3 \to \mathbb{C}^3$ the blow-up of $\mathbb{C}^3$ with center in Z. Denote by $\widehat{V_1}$ and $\widehat{V_2}$ the corresponding strict transforms of V_1 and V_2. The exceptional divisor E_1 has multiplicity 5 and, $\widehat{V_1}$ and $\widehat{V_2}$ do not meet in a small neighborhood of E_1. Moreover, $\widehat{V_1}$ and E intersect transversely. The local equation of $\widehat{V_2} \cup E_1$ is given by the polynomial $y^5(x^4+y+xy)$.

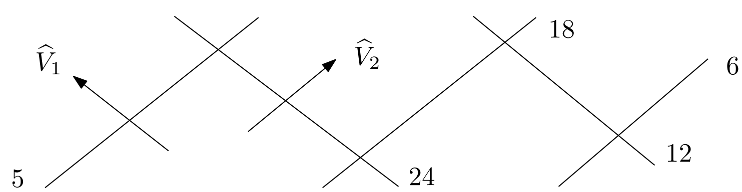


FIGURE VIII.2. Embedded resolution of $V((xz+y)(x^4+y^5+xy^4))$

Now, one can proceed as in the case of plane curves, since the local equation involves just two variables. Finally, we obtain seven divisors with normal crossings, see Figure VIII.2. This method can also be applied to the family $(xz+y)g(x,y)$ under some extra conditions on $g(x,y)$.

Remark **(VIII.3.5).** To the best of our knowledge, resolution of singularities has never been used before for computing Bernstein-Sato polynomials in an algorithmic way. Recall that an embedded resolution can be computed algorithmically in any dimension and for any affine algebraic variety [**BEV05**]. There is a sophisticated implementation by A. Frühbis-Krüger and G. Pfister [**FP05**] in SINGULAR.

One can find upper bounds for the case of hyperplane arrangements by computing an embedded resolution. This allows one among other to test formulas for Bernstein-Sato polynomials of non-generic arrangements. A formula for the Bernstein-Sato polynomial of a generic hyperplane arrangement was given by Walther in [**Wal05**].

VIII.3–2. Topologically equivalent singularities

Let f, g be two topologically equivalent singularities and assume that $b_f(s)$ is known. Since the set $E_f = \{e^{2\pi i\alpha} \mid b_{f,0}(\alpha) = 0\}$ is a topological invariant of the singularity $\{f = 0\}$ at the origin [**Mal75, Mal83**] and every root belongs to $(-n, 0)$ (Theorem (VIII.3.2)), one can find an upper bound for $b_g(s)$ from the roots of $b_f(s)$ and use our algorithms for computing $b_g(s)$. The upper bound is constructed as $\prod_{\beta\in E}(s - \beta)$, where

$$E := \{\alpha + k \mid \alpha \in E_f,\ k \in \mathbb{Z},\ \alpha + k \in (-n, 0)\}.$$

In general it is complicated to check whether two singularities are equivalent. However, there are some special families for which this can be done. This is the case of quasi-ordinary singularities, see e.g. [**Lip88**]. Consider an example of a non-isolated one.

Example (VIII.3.6). Let $f = z^4 + x^6y^5$ and $g = f + x^5y^4z$. Since the corresponding discriminants with respect to z are normal crossing divisors, the associated germs at the origin define quasi-ordinary singularities. Moreover, the characteristic exponents are in both cases the same and hence they are topologically equivalent, see e.g. [**Lip88**].

The Bernstein-Sato polynomial of f at the origin has 27 roots, all of them with multiplicity one except for $\alpha = -1$ which has multiplicity two. Here is the list in positive format.

$$1, \tfrac{5}{6}, \tfrac{9}{10}, \tfrac{4}{3}, \tfrac{13}{10}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{19}{20}, \tfrac{5}{12}, \tfrac{11}{10}, \tfrac{17}{12}, \tfrac{17}{20}, \tfrac{11}{12}, \tfrac{7}{10},$$

$$\boxed{\tfrac{19}{12}}, \tfrac{13}{20}, \tfrac{27}{20}, \tfrac{7}{6}, \tfrac{21}{20}, \tfrac{9}{20}, \tfrac{13}{12}, \tfrac{5}{4}, \mathbf{\tfrac{3}{2}}, \tfrac{7}{12}, \mathbf{\tfrac{31}{20}}, \boxed{\tfrac{7}{4}}, \tfrac{23}{20}$$

The exponential of the previous set has 24 elements. Each of them gives three candidates for $b_{g,0}(-s)$ except for $-\alpha = 1$ which gives just two. For instance $-\alpha = 1/2$ gives rise the following three possible roots,

$$\frac{1}{2} \to \left\{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}\right\}.$$

There are 71 possible roots in total. Note that using this approach we do not have any information about the multiplicities. Finally one obtains the roots for $b_{g,0}(-s)$.

$$1, \tfrac{5}{6}, \tfrac{9}{10}, \tfrac{4}{3}, \tfrac{13}{10}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{19}{20}, \tfrac{5}{12}, \tfrac{11}{10}, \tfrac{17}{12}, \tfrac{17}{20}, \tfrac{11}{12}, \tfrac{7}{10},$$
$$\tfrac{13}{20}, \tfrac{27}{20}, \tfrac{7}{6}, \tfrac{21}{20}, \tfrac{9}{20}, \tfrac{13}{12}, \tfrac{5}{4}, \boldsymbol{\tfrac{1}{2}}, \tfrac{7}{12}, \boldsymbol{\tfrac{11}{20}}, \qquad \tfrac{23}{20}$$

Observe that the Bernstein polynomials are very similar. The roots of $b_{f,0}(-s)$ marked with a box have disappeared in $b_{g,0}(-s)$ and the ones in bold 3/2, 31/20 have become 1/2, 11/20.

We have selected this example to show the topologically equivalent approach to keep the exposition as simple as possible. However, there is a family of examples depending on three indices

$$f_{m,p,q} = x^m + x^p y^q, \qquad g_{m,p,q} = x^m + x^p y^q + x^{p-1} y^{q-1} z$$

where the polynomials define topologically equivalent singularities if $m \leq p$, $m \leq q$, and at least one of the two inequalities is strict.

In the table we put the information on timings in *[hours:]minutes:seconds* format for the computation of the Bernstein-Sato polynomial of g. The symbol "–" means that the computation did not terminate (or full memory) after 5 hours.

	SINGULAR			Risa/Asir		$\deg b_g(s)$
(m,p,q)	`checkRoot`	`bfct`	`bfctAnn`	`bfct`	`bfunction`	
$(4,6,5)$	0:27	3:19	**0:18**	1:32	1:03	26
$(5,7,6)$	**7:22**	–	12:32	–	28:27	49
$(6,8,7)$	**51:15**	–	1:33:28	–	2:34:11	57

Observe that although `bfctAnn` and `bfunction` are competitive in this family of examples, we notice a better control of the memory due to the fact that many "small" Gröbner bases were needed for the `checkRoot` approach, while a "big" Gröbner basis is performed for the other methods. That is why our new algorithm is specially useful for extreme examples.

VIII.3–3. A'Campo's formula

The Jordan form of the local Picard-Lefschetz monodromy of superisolated surface singularities was calculated by Artal-Bartolo in [**Art94b**]. The main step in this computation was to present explicitly an embedded resolution for this family and study the mixed Hodge structure of the Milnor fibration.

Since every root of the Bernstein-Sato polynomial belongs to the interval $(-n, 0)$ (Theorem (VIII.3.2)) and the characteristic polynomial is a topological invariant, using the results by Malgrange [**Mal75, Mal83**], one can eventually provide an upper bound for the b-function. Let us see an example that was not feasible even with the powerful specialized implementation by Schulze [**Sch04b**].

Example (VIII.3.7). Let V be the superisolated singularity defined by $f = z^6 + (x^4 z + y^5 + xy^4)$. The characteristic polynomial is

$$\Delta(t) = \frac{(t^5 - 1)(t^6 - 1)(t^{120} - 1)}{(t - 1)(t^{30} - 1)(t^{24} - 1)}.$$

This polynomial has 76 different roots modulo $\mathbb{Z}$ and thus we know in advance that the Bernstein-Sato polynomial (resp. the reduced one) has at least 77 (resp. 76) different roots. Using the previous results in 230 possible candidates, only 77 of them are roots of the b-function indeed, all of them with multiplicity one:

$$1, \tfrac{27}{40}, \tfrac{101}{120}, \tfrac{41}{60}, \tfrac{17}{20}, \tfrac{83}{120}, \tfrac{103}{120}, \tfrac{43}{60}, \tfrac{53}{60}, \tfrac{29}{40}, \tfrac{107}{120}, \tfrac{23}{40}, \tfrac{89}{120}, \tfrac{109}{120}, \tfrac{71}{120}, \tfrac{91}{120}, \tfrac{37}{40}, \tfrac{73}{120}, \tfrac{31}{40},$$

$$\tfrac{113}{120}, \tfrac{37}{60}, \tfrac{47}{60}, \tfrac{19}{20}, \tfrac{77}{120}, \tfrac{97}{120}, \tfrac{39}{40}, \tfrac{13}{20}, \tfrac{49}{60}, \tfrac{59}{60}, \tfrac{79}{120}, \tfrac{33}{40}, \tfrac{119}{120}, \tfrac{3}{5}, \tfrac{4}{5}, \tfrac{121}{120}, \tfrac{47}{40}, \tfrac{161}{120}, \boldsymbol{\tfrac{181}{120}}, \tfrac{61}{60},$$

$$\tfrac{71}{60}, \tfrac{27}{20}, \boldsymbol{\tfrac{91}{60}}, \tfrac{41}{40}, \tfrac{143}{120}, \tfrac{163}{120}, \boldsymbol{\tfrac{61}{40}}, \tfrac{21}{20}, \tfrac{73}{60}, \tfrac{83}{60}, \boldsymbol{\tfrac{31}{20}}, \tfrac{127}{120}, \tfrac{49}{40}, \tfrac{167}{120}, \tfrac{187}{120}, \tfrac{43}{40}, \tfrac{149}{120}, \tfrac{169}{120}, \tfrac{131}{120},$$

$$\tfrac{151}{120}, \tfrac{57}{40}, \tfrac{133}{120}, \tfrac{51}{40}, \tfrac{173}{120}, \tfrac{67}{60}, \tfrac{77}{60}, \tfrac{29}{20}, \tfrac{137}{120}, \tfrac{157}{120}, \tfrac{59}{40}, \tfrac{23}{20}, \tfrac{79}{60}, \tfrac{89}{60}, \tfrac{139}{120}, \tfrac{53}{40}, \tfrac{179}{120}, \tfrac{6}{5}, \tfrac{7}{5}.$$

The total running time was 41.5 minutes. In the table below we show the candidates roots, for which computation ran more than 2 minutes. Again we observe that the detection of a non-root is very fast indeed. As usual, the running time is given in the format *minutes:seconds*.

Cadidate	Running time	Root of $b_f(-s)$?
181/20	2:52	Yes
91/60	6:01	Yes
61/40	4:53	Yes
31/20	4:21	Yes

Remark **(VIII.3.8).** Spectral numbers are defined using the semi-simple part of the action of the monodromy on the mixed Hodge structure on the cohomology of the Milnor fiber [**Ste77**], [**Var81**]. In [**GH07**, Th. 3.3], [**Sai93**, Th. 0.7] it is proven, that some roots of the Bernstein-Sato polynomial of a germ with an isolated critical point at the origin, can be obtained from the knowledge of the spectral numbers of the germ.

Since spectral numbers do not change under μ-constant deformations, this also gives a set of common roots of the Bernstein-Sato polynomials associated with the members of a μ-constant deformation of a germ. Therefore, they provide a lower bound, as well as an upper bound, for $b_f(s)$.

Section § VIII.4
Integral Roots of b-Functions

For several applications only integral roots of the b-function are needed, e.g. [**SST00**]. We present here problems related to the so-called Logarithmic Comparison Theorem and Intersection Homology D-module. Depending on the context local or global version of our algorithm is used.

VIII.4–1. Upper bounds from different ideals

Consider a left ideal $I \subseteq \operatorname{Ann}_{D[s]} f^s$. Then $I + \langle f \rangle \subseteq \operatorname{Ann}_{D[s]} f^s + \langle f \rangle \subsetneq D[s]$, that is the former is a proper ideal. Then define the *relative b-polynomial* $b_f^I(s) \in \mathbb{K}[s]$ to be the monic generator of $\big(I + \langle f \rangle\big) \cap \mathbb{K}[s]$, then $b_f(s) \mid b_f^I(s)$. Note, that quite often $b_f^I(s) = 0$. But if $b_f^I(s) \neq 0$, it gives us an upper bound for $b_f(s)$. In particular, one can take I, giving rise to a holonomic $D[s]$-module, that is $\text{GK.dim} D[s]/I = \text{GK.dim} D[s]/\operatorname{Ann}_{D[s]} f^s = n + 1$.

Since $(s+1) \mid b_f(s) \mid b_f^I(s)$, one can consider the *reduced relative b-polynomial* $\widetilde{b_f^I(s)} \in \mathbb{K}[s]$ to be the monic generator of $\big(I + \langle f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \rangle\big) \cap \mathbb{K}[s]$. A prominent example of I as above is the logarithmic annihilator. Let $I = \operatorname{Ann}_{D[s]}^{(1)}(f^s)$ be the ideal in $D[s]$ generated by the operators $P(s) \in \operatorname{Ann}_{D[s]}$ of total degree at most one in ∂_i. Let us define $b_f^{(1)}(s) := b^I(f^s)_f(s) = \big(\operatorname{Ann}_{D[s]}^{(1)}(f^s) + D[s]\langle f \rangle\big) \cap \mathbb{K}[s]$. The reduced $\widetilde{b_f^{(1)}}(s)$ is useful as well.

VIII.4–2. Minimal integral root of $b_f(s)$ and LCT

Since every root of $b_f(s)$ belongs to the real interval $(-n, 0)$, integral roots are bounded and therefore the whole Bernstein-Sato polynomial is not needed. Let us see an example that could not be treated before with the classical methods.

Example (VIII.4.1). Let A be the matrix given by

$$A = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_5 & x_6 & x_7 & x_8 \\ x_9 & x_{10} & x_{11} & x_{12} \end{pmatrix}.$$

Let us denote by Δ_i, $i = 1, 2, 3, 4$, the determinant of the minor resulting from deleting the i-th column of A, and consider $f = \Delta_1\Delta_2\Delta_3\Delta_4$. The polynomial f defines a non-isolated hypersurface in $\mathbb{C}^{12}$. Following Theorem (VIII.3.2), the set of all possible integral roots of $b_f(-s)$ is

$$\{11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1\}.$$

Using the algorithm `checkRoot` with the logarithmic annihilator, see Section VIII.4–1 above, instead of the classical one, we have proven, for $\alpha = 2, \ldots, 11$, that

$$\mathrm{Ann}^{(1)}_{D_n[s]}(f^s) + D_n[s]\langle f, s+\alpha\rangle = D_n[s],$$

and hence -1 is the minimal integral root of $b_f(s)$. The following is the timing information of the whole procedure. Of course, -1 is always a root, but it is interesting to compare the timings of confirming this fact.

Possible integral roots	1	2 ... 11
Root of $b_f^{(1)}(s)$?	Yes	No
Running time	1:19:31	$\approx$ 0:03:24

This example was suggested by F. Castro-Jiménez and J.-M. Ucha for testing the Logarithmic Comparison Theorem, see e.g. [**Tor07**]. The use of logarithmic annihilator allowed us to reduce the computation time. However, for f from this example it is known, that $\mathrm{Ann}_{D_n[s]}(f^s) = \mathrm{Ann}^{(1)}_{D_n[s]}(f^s)$ and this fact together with some homogeneous properties were used to compute other roots of $b_f(s)$, see Example (VIII.4.4) below.

Quasi-homogeneous polynomials

Assume $F \in R_n$ is a w-quasi-homogeneous polynomial with $w_i \neq 0$, that is, there are numbers $w_1, \ldots, w_n$ such that with $\xi = \sum_{i=1}^n w_i x_i \partial_i$ one has $F = \xi(F)$. Take $c \in \mathbb{K}^*$ and let us denote $f = F_{|x_k=1}$ for some fixed k. We are interested in studying the relationship between the Bernstein-Sato polynomials of f and F.

Proposition (VIII.4.2). *Let $F \in R_n$ be a quasi-homogeneous polynomial with respect to the weight vector $w = (w_1, \ldots, w_n)$. Assume $w_k \neq 0$ for some $k \in \{1, \ldots, n\}$ and define $f = F|_{x_k=c}$ for $c \in \mathbb{K}^*$. Then, $b_f(s)$ divides $b_F(s)$.*

PROOF. Consider the V-filtration on D_n given by the variable x_k. Let $P(s) \in D_n[s]$ be a differential operator satisfying the functional equation for F. There exists $d \geq 0$ such that $x_k^d P(s) \in \sum_{i\geq 0} x_k^i \cdot V_0$. From the quasi-homogeneity of F one can deduce that

$$x_k\partial_k \bullet F^{s+1} = \frac{1}{w_k}\Big(s+1-\sum_{i\neq k} w_i x_i \partial_i\Big) \bullet F^{s+1}.$$

Let D' be the $(n-1)$-th Weyl algebra in the variables $x_1, \ldots, \hat{x_k}, \ldots, x_n$. Thus $V_0 = D'[x_k\partial_k]$ and $x_k^d P(s)\cdot F^{s+1}$ can be written in the form $Q(s)\cdot F^{s+1}$ where the operator ∂_k does not appear in $Q(s) \in D_n[s]$. The functional equation for F has been converted in the following one:

$$x_k^d P(s) \bullet F^{s+1} = Q(s) \bullet F^{s+1} = x_k^d\, b_F(s) \bullet F^s.$$

Now the substitution $x_k = c \in \mathbb{K}^*$ can be made and the claim follows. $\square$

Example (VIII.4.3). The Bernstein-Sato polynomials of $F = x^2z + y^3$ and $f = F_{|z=1} = x^2 + y^3$ are

$$b_F(s) = \underbrace{(s+1)(s+5/6)(s+7/6)}_{b_f(s)}(s+4/3)(s+5/3).$$

From the result by Kashiwara [**Kas77**] one can see, blowing up the origin of F, that the last two factors are related to the b-function of $\{z^3 = 0\}$. This is a general fact.

Example (VIII.4.4). Now, we continue Example (VIII.4.1). Let g be the polynomial, resulting from f by substituting x_1, x_2, x_3, x_4, x_5, x_9 with 1. Using Proposition (VIII.4.2) several times, one can easily see that $b_g(s)$ divides $b_f(s)$. Finally, the `checkRoot` algorithm is used to obtain that

$$(s+1)^4(s+1/2)(s+3/2)(s+3/4)(s+5/4)$$

is a factor of $b_g(s)$ and therefore a factor of $b_f(s)$.

Factor of $b_g(s)$	$(s+1/2)$	$(s+3/4)$	$(s+3/2)$	$(s+1)^4$	$(s+5/4)$
Running time	0:02	0:04	0:10	3:45	4:46

VIII.4–3. Intersection homology $\mathcal{D}$-module

We introduce some new notation. We refer to [**Tor09**] for further details. Let X be a complex analytic manifold of dimension $n \geq 2$, $\mathcal{O}_X$ the sheaf of holomorphic function on X and $\mathcal{D}_X$ the sheaf of differential operators with holomorphic coefficients. At a point $x \in X$, we identify the stalks $\mathcal{O}_{X,x}$ with the ring $\mathcal{O} = \mathbb{C}\{x_1, \ldots, x_n\}$ of converging power series and $\mathcal{D}_{X,x}$ with $\mathcal{D} = \mathcal{O}\langle\partial_1, \ldots, \partial_n\rangle$.

Given a closed subspace $Y \subset X$ of pure codimension $p \geq 1$, we denote by $H^p_{[Y]}(\mathcal{O}_X)$ the sheaf of local algebraic cohomology with support in Y. Let $\mathcal{L}(Y,X) \subset H^p_{[Y]}(\mathcal{O}_X)$ be the intersection homology $\mathcal{D}_X$-Module of Brylinski-Kashiwara. This is the smallest $\mathcal{D}_X$-submodule of $H^p_{[Y]}(\mathcal{O}_X)$ which coincides with $H^p_{[Y]}(\mathcal{O}_X)$ at the generic points of Y.

A natural problem is to characterize the subspaces Y such that $\mathcal{L}(Y,X)$ coincides with $H^p_{[Y]}(\mathcal{O}_X)$. Indeed, from the Riemann-Hilbert correspondence of Kashiwara-Mebkhout, the regular holonomic D_X-module $H^p_{[Y]}(\mathcal{O}_X)$ corresponds to the perverse sheaf $\mathbf{C}_Y[p]$, while $\mathcal{L}(Y,X)$ corresponds to the intersection complex $IC^\bullet_Y$. This way, the condition $\mathcal{L}(Y,X) = H^p_{[Y]}(\mathcal{O}_X)$ is equivalent to the following one: the real link of Y at a point $x \in Y$ is a rational homology sphere. Torrelli proved, that the following connection to local Bernstein-Sato polynomial exists.

Theorem (VIII.4.5) (Theorem 1.2 in [**Tor09**]). *Let $Y \subset X$ be a hypersurface and $h \in \mathcal{O}_{X,x}$ a local equation of Y at a point $y \in Y$. The following conditions are equivalent:*

(1) *$\mathcal{L}(Y,X)_y$ coincides with $H^p_{[Y]}(\mathcal{O}_X)_y$.*
(2) *The reduced local Bernstein-Sato polynomial of h has no integral root.*

The proof of the theorem is based on a natural generalization of a classical result due to Kashiwara which links the roots of the b-function to some generators of $\mathcal{O}[\frac{1}{f}]f^\alpha$, $\alpha \in \mathbb{C}$.

Example (VIII.4.6). Let Y be the affine variety in $X = \mathbb{C}^3$ defined by the polynomial $f = z^7 + (x^2z + y^3)(x^3 + y^2z)$. The surface Y has the origin as its only singular point and thus the local b-function and the global one coincide.

The only possible integral roots are -2 and -1. Now consider J_f, the Jacobian ideal of f, cf. Remark (VIII.2.8). Since the reduced Bernstein-Sato polynomial is required, the ideal

$$\mathrm{Ann}_{D[s]}(f^s) + D[s]\langle f, J_f, s+\alpha\rangle$$

is used for checking rational roots, compare with Corollary (VIII.2.3)(2). We see that the previous ideal is not the whole ring for $\alpha = 1$ and hence the set of points $x \in Y$ such that $\mathcal{L}(Y,X)_x = H^p_{[Y]}(\mathcal{O}_X)_x$ is $Y \setminus \{0\}$.

Using the implementation by Schulze [**Sch04b**] (based on Gauss-Manin connection), the computation of the whole Bernstein-Sato polynomial took 123 seconds, while with our approach only 11 seconds were needed.

Remark **(VIII.4.7).** Given Y as above, the set of points $x \in Y$ for which the condition $\mathcal{L}(Y,X)_x = H^p_{[Y]}(\mathcal{O}_X)_x$ is satisfied, defines an open set in Y that can be effectively computed with the stratification associated with the integral roots of the reduced local b-functions, see the sequence of varieties (44) below. For instance, in Example (VIII.5.2), the open set is $V(f) \setminus V_1$.

Section § VIII.5
Stratification Associated with Local b-Functions

From Theorem (VIII.2.10), one can find a stratification of $\mathbb{C}^n$ so that $b_{f,p}(s)$ is constant on each stratum. The first method for computing such stratification was suggested by Oaku [**Oak97b**] (see also [**Oak97a, Oak97c**] and [**BO10**] for further information). However, this method relies on the primary decomposition of commutative ideals. Following the ideas started in Section VIII.2–2, we propose a new natural algorithm for computing such a stratification. At first, a stratification for each root of the global b-function is computed. Then one obtains a stratification, associated with the local b-function, notably without any primary ideal decomposition, see Examples (VIII.5.2), (VIII.5.3), and (VIII.5.5) below. We have created an experimental implementation, which was used for presented examples. The substitution of primary decomposition with elementary operations clearly decreases the total complexity of this algorithm.

This is a natural generalization of Proposition (VIII.2.18).

Theorem (VIII.5.1). *Let $\{P_1(s), \ldots, P_k(s), f\}$ be a system of generators of $\operatorname{Ann}_{D[s]}(f^s) + D[s]\langle f \rangle$ and consider the ideals $I_{\alpha,i} = \big(I : (s+\alpha)^i\big) + D[s]\langle s+\alpha\rangle$, for α root of $b_f(s)$ and $i = 0, \ldots, m_\alpha - 1$. Then, one has*

$$m_\alpha(p) > i \iff p \in V(I_{\alpha,i} \cap \mathbb{C}[\mathbf{x}]).$$

Proof. Repeat the same argument as in Corollary (VIII.2.6) and proceed as in the proof of Proposition (VIII.2.18), using Lemmas (VIII.2.15) and (VIII.2.16) when necessary. □

Using the notation of the previous theorem, let $V_{\alpha,i}$ be the affine variety corresponding to the ideal $I_{\alpha,i} \cap \mathbb{C}[\mathbf{x}]$. Then,

$$\emptyset =: V_{\alpha,m_\alpha} \subset V_{\alpha,m_\alpha-1} \subset \cdots \subset V_{\alpha,0} \subset V_{\alpha,-1} := \mathbb{C}^n, \tag{44}$$

and $m_\alpha(p) = i$ if and only if $p \in V_{\alpha,i-1} \setminus V_{\alpha,i}$. We call this sequence the *stratification associated with the root* α. Let us see some examples[1] to show how this result can be used to compute a stratification associated with local b-functions.

[1] The examples have been taken from `http://www.freigeist.cc/gallery.html`

Example (VIII.5.2). Consider $f = (x^2 + 9/4y^2 + z^2 - 1)^3 - x^2z^3 - 9/80y^2z^3 \in \mathbb{C}[x,y,z]$. The global b-function is

$$b_f(s) = (s+1)^2(s+4/3)(s+5/3)(s+2/3).$$

Take $V_1 = V(x^2+9/4y^2-1, z)$, $V_2 = V(x,y,z^2-1)$, and $V_3 = V(19x^2 + 1,171y^2 - 80, z)$. Then V_2 (resp. V_3) consists of two (resp. four) different points and $V_3 \subset V_1$, $V_1 \cap V_3 = \emptyset$. The singular locus of f is union of V_1 and V_2. The stratification associated with each root of $b_f(s)$ is given by

$$\begin{array}{rclcccc}
\alpha & = & -1, & \emptyset & \subset & V_1 \subset V(f) & \subset & \mathbb{C}^3\,; \\
\alpha & = & -4/3, & \emptyset & \subset & V_1 \cup V_2 & \subset & \mathbb{C}^3\,; \\
\alpha & = & -5/3, & \emptyset & \subset & V_2 \cup V_3 & \subset & \mathbb{C}^3\,; \\
\alpha & = & -2/3, & \emptyset & \subset & V_1 & \subset & \mathbb{C}^3.
\end{array}$$

From this, one can easily find a stratification of $\mathbb{C}^3$ into constructible sets such that $b_{f,p}(s)$ is constant on each stratum.

$$b_{f,p}(s) = \begin{cases} 1 & p \in \mathbb{C}^3 \setminus V(f), \\ s+1 & p \in V(f) \setminus (V_1 \cup V_2), \\ (s+1)^2(s+4/3)(s+2/3) & p \in V_1 \setminus V_3, \\ (s+1)^2(s+4/3)(s+5/3)(s+2/3) & p \in V_3, \\ (s+1)(s+4/3)(s+5/3) & p \in V_2. \end{cases}$$

The total running time including the computation of the global Bernstein polynomial was 8 min 7 sec. The system Risa/Asir needed more than 7 hours to obtain the same stratification.

Consider more interesting examples, which have already been studied in §VIII.3 when computing $b_f(s)$.

Example (VIII.5.3). Let us proceed with Example (VIII.3.3). The stratification associated with every root of $b_f(s)$ except for $\alpha = 1$ is given by the sequence $\emptyset \subset Z \subset \mathbb{C}^3$. For $\alpha = 1$ of multiplicity 2, the corresponding sequence is $\emptyset \subset Y \sqcup Z \subset V(f) \subset \mathbb{C}^3$. Hence the local b-function at $p \in \mathbb{C}^3$ is

$$b_{f,p}(s) = \begin{cases} 1 & p \in \mathbb{C}^3 \setminus V(f), \\ s+1 & p \in V(f) \setminus (Y \sqcup Z), \\ (s+1)^2 & p \in Y, \\ b_f(s) & p \in Z. \end{cases}$$

Using the lexicographical ordering with $\partial_x > x$ on D during the computation of the intersection with $\mathbb{C}[\mathbf{x}]$, cf. Remark (VIII.3.4), reduced the total running time to just 38 sec.

Remark **(VIII.5.4).** Note that one can define a stratification associated with the roots of the local b-functions, that is taking no multiplicities into account. We have observed that our algorithm is especially useful and very fast for computing this stratification. In particular, this is the case when each root has multiplicity one. Finally, also observe that in any case the global b-function is not actually needed, if a set containing the roots of $b_f(s)$ is used instead.

Example (VIII.5.5). Let us compute here the stratification associated with local b-functions of Example (VIII.3.6). Denote by V_1, V_2 the two axes $V_1 := V(x,z)$, $V_2 := V(y,z)$. The singular locus is in both cases the union of these varieties. Then, $b_{f,p}(s)$ is

$$\begin{cases} 1 & p \in \mathbb{C}^3 \setminus V(f), \\ s+1 & p \in V(f) \setminus (V_1 \cup V_2), \\ b_{z^4+x^6}(s) & p \in V_1 \setminus \{0\}, \\ b_{z^4+y^5}(s) & p \in V_2 \setminus \{0\}, \\ (s+3/2)(s+7/4)\,\mathrm{lcm}\left(b_{z^4+x^6}(s), b_{z^4+y^5}(s)\right) & p = 0. \end{cases}$$

The stratification given by the singularity $\{g=0\}$ is the same as above. The local b-functions in each stratum is obtained performing the replacements

$$\begin{aligned} z^4 + x^6 &\longmapsto z^4 + x^6 + x^5 z, \\ z^4 + y^5 &\longmapsto z^4 + y^5 + y^4 z, \\ (s+3/2)(s+7/4) &\longmapsto (s+1/2). \end{aligned}$$

This can be interpreted as follows. Let $P = (a,0,0) \in V_1 \setminus \{0\}$. The local equation of f (resp. g) is $z^4 + x^6 = 0$ (resp. $z^4 + x^6 + x^5 z$). By the semi-continuity of the Bernstien-Sato polynomial the local b-function at P divides the Bernstein-Sato polynomial at the origin. Analogous considerations hold for $P \in V_2 \setminus \{0\}$. The system Risa/Asir did not finish the computation of the stratification after more than 40 hours.

Remark **(VIII.5.6).** We see some common properties between the factorization of a Bernstein-Sato polynomial with the so-called *central character decomposition* by Levandovskyy [**Lev05**]. In particular, for $b_f(s) = \prod_{\alpha\in A}(s-\alpha)^{m_\alpha}$, where $A \subset \mathbb{Q}$ is the set of roots of $b_f(s)$, there is an algorithm for computing the following direct sum decomposition of the module

$$D[s]/(\mathrm{Ann}_{D[s]}(f^s) + \langle f \rangle) \cong \bigoplus_{\alpha\in A} D[s]/(\mathrm{Ann}_{D[s]}(f^s) + \langle f \rangle) : J(\alpha)^\infty,$$

where $J(\alpha) = \langle b_f(s)/(s-\alpha)^{m_\alpha} \rangle$. We plane to investigate this topic further and provide cyclic $D[s]$-modules, corresponding to different strata.

There is a very recent paper [**NN10**] by Nishiyama and Noro, where the authors build a stratification without using primary decomposition. The authors use initial ideals with respect to weight vectors in computations, which is a classical (cf. [**SST00**]) alternative to the methods, utilizing annihilators $\operatorname{Ann}_{D[s]}(f^s)$. In [**ALM09**] there is a comparison of performance of both approaches for the computation of Bernstein-Sato polynomials. Notably, no method is clearly superior over another. Rather there are classes of examples, where the difference is very distinct. In particular, initial-based method scores better results on hyperplane arrangements, while annihilator-based methods are better at complicated singularities, which are not hyperplane arrangements. A comparison of two methods for stratification is very interesting and it is an important task for the future. However, it seems to us that the method we presented will allow more thorough analysis of the algebraic situation due to the applicability of central character decomposition. At the moment it is not clear, whether such a decomposition exists for initial ideals.

Remark **(VIII.5.7).** The intersection $I \cap \mathbb{K}[t\partial_t, \mathbf{x}]$ suggested by Nishiyama and Noro in the computation of the stratification associated with local b-functions is very expensive from the computational point of view. Using our approach this elimination problem is solved. By contrast, once you have computed the intersection, Noro's approach seems to be faster. Therefore our methods is specially good for complicated and extreme examples.

Section § VIII.6
Other Applications

VIII.6–1. Bernstein-Sato polynomials for varieties

Let $f = (f_1, \ldots, f_r)$ be an r-tuple in $\mathbb{K}[x]^r$. Denote by $\mathbb{K}\langle S\rangle$ the universal enveloping algebra $U(\mathfrak{gl}_r)$, generated by the set of variables $S = (s_{ij})$, where $i, j = 1, \ldots, r$, subject to relations:

$$[s_{ij}, s_{kl}] = \delta_{jk} s_{il} - \delta_{il} s_{kj}.$$

Then, we denote by $D_n\langle S\rangle := D_n \otimes_{\mathbb{K}} \mathbb{K}\langle S\rangle$. Consider a free $\mathbb{K}[x, s, \frac{1}{f}]$-module of rank one generated by the formal symbol f^s and denote it by $M = \mathbb{K}[x, s_{11}, \ldots, s_{rr}, \frac{1}{f_1 \cdots f_r}] \cdot f^s$, where $f^s = f_1^{s_{11}} \cdot \ldots \cdot f_r^{s_{rr}}$. The module M has a natural structure of left $D_n\langle S\rangle$-module. Denote by $\operatorname{Ann}_{D_n\langle S\rangle}(f^s)$ the left ideal of all elements $P(S) \in D_n\langle S\rangle$ such that $P(S) \bullet f^s = 0$, that is the *annihilator* of f^s in $D_n\langle S\rangle$.

Theorem (VIII.6.1) (Budur, Mustaţă, Saito [**BMS06**])**.** *For every r-tuple $f = (f_1, \ldots, f_r) \in \mathbb{K}[x]^r$, there exists a non-zero polynomial in one variable $b(s) \in \mathbb{K}[s]$ and r differential operators $P_1(S), \ldots, P_r(S) \in D_n\langle S\rangle$ such that*

$$\sum_{k=1}^{r} P_k(S) f_k \cdot f^s = b(s_{11} + \cdots + s_{rr}) \cdot f^s \ \in \ M. \tag{45}$$

The *Bernstein-Sato polynomial* $b_f(s)$ of $f = (f_1, \ldots, f_r)$ is defined to be the monic polynomial of the lowest degree in the variable s satisfying the equation (45). It can be verified that $b_f(s)$ is independent of the choice of a system of generators of $\langle f_1, \ldots, f_r\rangle$.

Then the Bernstein-Sato polynomial of f can be computed as follows

$$(\mathrm{Ann}_{D_n\langle S\rangle}(f^s) + \langle f_1, \ldots, f_r\rangle) \cap \mathbb{K}[s_{11} + \cdots + s_{rr}] = \langle b_f(s_{11} + \ldots + s_{rr})\rangle.$$

In [**ALM09, ALM10**], an algorithm to find a system of generators of $\mathrm{Ann}_{D\langle S\rangle}(f^s)$ was given. Moreover, in computing the intersection of an ideal with the univariate subalgebra an optimized algorithm (which avoids elimination with Gröbner basis) was used.

The preceding formula together with Theorem (VIII.2.1) can be used to check rational roots of Bernstein-Sato polynomials also for affine algebraic varieties. Hence, following Corollary (VIII.2.6), a stratification associated with the local b-functions can be computed.

VIII.6–2. A remark in Narváez's paper

In [**Nar08**], Narváez introduces a polynomial denoted by $\beta(s)$ verifying $\beta(s)\,\mathrm{Ann}_{D[s]}(f^s) \subseteq \mathrm{Ann}^{(1)}_{D[s]}(f^s)$. For all the examples treated in [**Nar08**], he was able to compute an operator $P'(s) \in D[s]$ such that $b_f(s) - P'(s)f \in \mathrm{Ann}^{(1)}_{D[s]}(f^s)$. The last example in the paper is quite involved and could not be computed by using any computer algebra system directly. An iterated process for finding approximations of involutive bases was used instead. Indeed, for this propose the operator is not really needed, since

$$b_f(s) - P(s)f \in \mathrm{Ann}^{(1)}_{D[s]}(f^s) \iff b_f^{(1)}(s) = b_f(s) \iff b_f^{(1)}(s) \mid b_f(s),$$

and thus after computing $b_f^{(1)}(s)$, one only has to check whether each root of the latter polynomial is indeed a root of the b-function and the same with the multiplicities.

By definition, the following inclusions hold

$$\beta(s)\big(\,\mathrm{Ann}_{D[s]}(f^s) + \langle f\rangle\,\big) \ \subset\ \mathrm{Ann}^{(1)}_{D[s]}(f^s) + \langle f\rangle \ \subset\ \mathrm{Ann}_{D[s]}(f^s) + \langle f\rangle.$$

This implies that $b_f(s) \,|\, b_f^{(1)}(s) \,|\, \beta(s) b_f(s)$. In addition, if $\beta(s)$ divides $b_f(s)$, then the polynomials $b_f^{(1)}(s)$ and $b_f(s)$ both have the same roots and the previous condition is equivalent to $m_\alpha(b_f^{(1)}(s)) = m_\alpha(b_f(s))$ for every root α of $\beta(s)$.

Example (VIII.6.2). Let $f = (x_1x_3 + x_2)(x_1^7 - x_2^7)$ be the last example from [**Nar08**]. The Bernstein-Sato polynomial and the polynomial $\beta(s)$ are respectively

$$b_f = \Big(s+1\Big)^3 \Big(s+\frac{3}{4}\Big)\Big(s+\frac{3}{8}\Big)\Big(s+\frac{9}{8}\Big)\Big(s+\frac{1}{4}\Big)\Big(s+\frac{7}{8}\Big)\Big(s+\frac{1}{2}\Big)\Big(s+\frac{5}{8}\Big),$$

$$\beta = \Big(s+\frac{3}{4}\Big)\Big(s+\frac{5}{8}\Big)\Big(s+\frac{1}{2}\Big)\Big(s+\frac{3}{8}\Big)\Big(s+\frac{1}{4}\Big).$$

Now one only has to check that all roots of $\beta(s)$ have multiplicity 1 as a root of $b_f^{(1)}(s)$. This can be done using Theorem (VIII.2.1) with $I = \mathrm{Ann}_{D[s]}^{(1)}(f^s) + \langle f \rangle$. Using this approach, the computations become very easy (less than 5 seconds in this example).

CONCLUSION AND FUTURE WORK

As we have demonstrated in this work, embedded **Q**-resolutions are natural generalization of the standard embedded resolutions, for which the usual invariants are expected to be calculated effectively. Moreover, the combinatorial and computational complexity of embedded **Q**-resolutions is much simpler, but the keep as much information as needed for the comprehension of the topology of the singularity.

This reflects the good behavior of abelian quotient singularities with respect to normal crossing divisors. By contrast, non-abelian groups seem to work differently, see §IV.5 where it is shown that "double points" may contribute to $Z(f;t)$. In this sense abelian groups are the largest family for which these tools apply.

Here we list some specific open problems and questions, related to the topic, to be considered for future work. In fact, some of them are currently being studied.

(1) In Chapter V, following Steenbrink's approach [**Ste77**], we provide a mixed Hodge structure on the cohomology of the Milnor fiber using a spectral sequence that is constructed from the divisors associated with the semistable reduction of an embedded **Q**-resolution. On the other hand, in Chapter VI, we give a detailed description of an embedded **Q**-resolution for superisolated surface singularities in terms of its tangent cone. However, the corresponding semistable reduction and its associated spectral sequence has not been studied in this work. The same applies to (weighted) Yomdin-Lê surface singularities, see Chapter VII. This problem will be considered in the future so as to complete this work.

(2) Theorem (VII.1.6) says that only weighted blow-ups at points are needed to compute an embedded **Q**-resolution of a Yomdin-Lê surface singularity. On the other hand, there exist invariants associated with a hypersurface singularity which can be calculated from an embedded resolution obtained with just blow-ups at points. The generalization of the previous results to weighted blow-ups at points and embedded **Q**-resolutions will lead us to compute other invariants for Yomdin-Lê surface singularities. This includes, among others, the Poincaré series for which very little is known.

(3) In Chapter IV, we found the generalized A'Campo's formula for embedded **Q**-resolutions. In relation to the monodromy conjecture, it could be interesting to find a formula for the topological zeta function in terms of an embedded **Q**-resolution. In principle this will be feasible since the latter invariant has a good behavior with respect to a stratification. Hence only the topological zeta function $Z_{top}(s)$ for normal crossing divisors has to be computed.

(4) Using the fact that the characteristic polynomial is a topological invariant and the results of Malgrange [**Mal75**], one can find an upper bound for the Bernstein-Sato polynomial from an embedded **Q**-resolution of the singularity, assuming it is isolated. It could be nice to generalize Kashiwara's result [**Kas77**] for embedded **Q**-resolutions in order to find upper bounds for the non-isolated case as well. Moreover, this will give rise better upper bounds in the sense that less extra candidates will appear.

(5) There exist algorithms in D-modules for computing the cohomology of the complement $\mathbb{C}^2 \setminus V(f)$, where f is a polynomial in 2 variables. These algorithms use the notion of b-function of a holonomic ideal with respect to a weight vector. We hope that these new techniques can be generalized so as to compute the cohomology of $X(\mathbf{d}; A) \setminus V(f)$, where f is a polynomial defining a zero set on $X(\mathbf{d}; A)$. This is closely related to the computation of the cohomology of the complement $\mathbb{P}^2_\omega \setminus V(F)$, where F is quasi-homogeneous with respect to ω.

`checkRoot`

As for the last part of this work (Chapter VIII), we have demonstrated that the family of `checkRoot` algorithms (implemented in the Singular library `dmod.lib`) has many useful applications in the realm of D-modules. Nowadays, it is the only method that allows one to obtain some roots of the b-function without computing the whole Bernstein-Sato polynomial. The latter is often infeasible despite all the recent progress in computational D-module theory.

We emphasize, that presented techniques are elementary (by utilizing the principal ideal domain of the center $\mathbb{K}[s]$ of $D_n[s]$) but very powerful from the computational point of view. Many intractable examples and conjectures could be treated with this new method, as we have partially illustrated. Moreover, a stratification associated with the local b-functions can be obtained without primary decomposition [**Oak97b**] as in the very recent paper [**NN10**]. It is very interesting to study these algorithms further and compare our approach with the one of [**NN10**].

Unfortunately, these techniques cannot be generalized for Bernstein-Sato ideals, since such ideals lie in $\mathbb{K}[s_1, \ldots, s_m]$ for $m \geq 2$.

We have shown that one can use the idea of `checkRoot` for checking rational roots of b-function of a holonomic ideal with respect to a weight vector [**SST00**]. This gives an easier method for computing, among other, integral roots of such b-functions, if an upper bound is known in advance. In this context, it would be very interesting to have a version of Kashiwara's result for some holonomic ideals and certain weights, since many algorithms in D-modules theory are based on integrations and restrictions which need minimal and/or maximal roots.

CONCLUSIÓN (Spanish)

Como hemos demostrado en este trabajo, las **Q**-resoluciones encajadas son generalizaciones naturales de las estándares para las que se espera que los invariantes usuales se puedan calcular efectivamente. Además, la complejidad combinatoria y computacional de las **Q**-resoluciones encajadas son mucho más sencillas pero conservan la misma información necesaria para la comprensión de la topología de la singularidad.

Esto refleja el buen comportamiento de las singularidades cocientes abelianas respecto de los cruces normales. Por el contrario, los grupos no abelianos funcionan de otra manera, ver §IV.5 donde se muestra que los "puntos dobles" pueden contribuir a $Z(f;t)$. En este sentido los grupos abelianos son la familia más grande para las que estas técnicas se aplican.

Aquí se listan algunos problemas y cuestiones abiertas, relacionados con el tema, que será consideradas para el futuro. De hecho, algunos de ellos está siendo estudiados actualmente.

(1) En el capítulo V, siguiendo las ideas de Steenbrink, proporcionamos una estructura de Hodge mixta sobre la cohomología de la fibra de Milnor usando una sucesión espectral que se construye a partir de los divisores de la normalización semiestable de una **Q**-resolución encajada. Por otro lado, en el capítulo VI, damos una descripción detallada de una **Q**-resolución encajada de las singularidades superaisladas de superficie en términos de su cono tangente. Sin embargo, la correspondiente normalización semiestable y la sucesión espectral asociada no ha sido estudiada en este trabajo. Lo mismo se aplica para las singularidades de Yomdin-Lê (ponderadas) de superficies, ver capítulo VII. Este problema será considerado en el futuro para completar este trabajo.

(2) El teorema (VII.1.6) dice que solo hace falta explosiones ponderadas de puntos para calcular una **Q**-resolución encajada de una singularidad de Yomdin-Lê de superficie. Por otro lado, existen invariantes que se pueden calcular a partir de una resolución encajada obtenida solamente con explosiones de puntos. La generalización de los resultados anteriores para explosiones ponderadas de puntos y **Q**-resoluciones encajadas permitirá calcular otros invariantes de estas singularidades. Esto incluye, entre otros, las series de Poincaré de las cuales se conoce muy poco.

(3) En el capítulo IV, hemos encontrado la generalización de la fórmula de A'Campo para **Q**-resoluciones encajadas. En relación con la conjetura de la monodromía, sería interesante encontrar una fórmula para la función zeta topológica en términos de una **Q**-resolución encajada. En principio esto es factible puesto que el invariante anterior se comporta bien con respecto a estratificaciones. Así, solamente tenemos que calcular la función zeta topológica $Z_{top}(s)$ de un divisor con cruces normales.

(4) Usando que el polinomio característico es un invariante topológico y los resultados de Malgrange [**Mal75**], se puede encontrar una cota superior del polinomio de Bernstein-Sato a partir de una **Q**-resolución encajada de la singularidad, suponiendo que es aislada. Estaría bien generalizar el resultado de [**Kas77**] para **Q**-resoluciones encajadas y así encontrar también cotas superiores para el caso no aislado. Además, esto dará lugar a mejores cotas superiores es el sentido de que menos candidatos extras aparecerán.

(5) Existen algoritmos en D-módulos para calcular la cohomología del complementario $\mathbb{C}^2 \setminus V(f)$, donde f es un polinomio en 2 variables. Estos algoritmos usan la noción de b-función de un ideal holónomo con respecto a un vector de pesos. Esperamos que estas nuevas técnicas se puedan generalizar para calcular la cohomología de $X(\mathbf{d}; A) \setminus V(f)$, donde f es un polinomio que define un conjunto de ceros en $X(\mathbf{d}; A)$. Esto está relacionado con el cálculo de la cohomología del complementario $\mathbb{P}^2_\omega \setminus V(F)$, donde F es cuasihomogéneo con respecto a ω.

`checkRoot`

En cuanto a la última parte de este trabajo (capítulo VIII), hemos demostrado que la familia de algoritmos `checkRoot` (implementados en la librería `dmod.lib` de SINGULAR) tiene muchas aplicaciones útiles en el campo de los D-módulos. Hoy en día, es el único método que nos permite obtener algunas raíces de la b-función sin calcular todo el polinomio de Bernstein-Sato. Esto último no es habitualmente factible, a pesar de todos los progresos recientes en teoría de D-módulos computacional.

Enfatizamos que las técnicas presentadas son elementales (utilizamos que $\mathbb{K}[s]$ es un dominio de ideales principales contenido en $D_n[s]$) pero muy potentes desde el punto de vista computacional. Muchos ejemplos intratables y conjeturas pudieron ser tratados con este nuevo método, como hemos parcialmente ilustrado. Además, se puede obtener una estratificación asociada a la b-función local sin usar descomposición primaria, como en el reciente artículo [**NN10**]. Es interesante estudiar estos algoritmos y compararlos con los nuestros.

Desafortunadamente, estas técnicas no se pueden generalizar para los ideales de Bernstein-Sato puesto que viven en $\mathbb{K}[s_1, \ldots, s_m]$ con $m \geq 2$.

Hemos mostrado que se puede usar la idea de `checkRoot` para comprobar raíces de la b-función de un ideal holónomo con respecto a un vector de pesos [**SST00**]. Esto da un método más fácil para calcular, entre otras, las raíces enteras de tales b-funciones, si una cota superior es conocida. En este contexto, sería muy interesante tener una versión del resultado de Kashiwara para ciertos ideales holónomos puesto que muchos algoritmos en teoría de D-módulos están basados en integración y restricción que necesitan la menor y/o mayor raíz entera.

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