COMPLEX STRUCTURES OF SPLITTING TYPE

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Abstract. We study the six-dimensional solvmanifolds that admit complex structures of splitting type classifying the underlying solvable Lie algebras. In particular, many complex structures of this type exist on the Nakamura manifold X, and they allow us to construct a countable family of compact complex non-$\partial\bar{\partial}$ manifolds X_k, $k \in \mathbb{Z}$, that admit a small holomorphic deformation $\{(X_k)_t\}_{t\in\Delta_k}$ satisfying the $\partial\bar{\partial}$-Lemma for any $t \in \Delta_k$ except for the central fibre. Moreover, a study of the existence of special Hermitian metrics is also carried out on six-dimensional solvmanifolds with splitting-type complex structures.

Introduction

Let $\mathfrak{g}$ be a real Lie algebra of even dimension. A complex structure on $\mathfrak{g}$ is an endomorphism $J\colon \mathfrak{g} \to \mathfrak{g}$ satisfying $J^2 = -\mathrm{Id}_{\mathfrak{g}}$ and the *Nijenhuis* condition

$$\text{(1)} \qquad \mathrm{Nij}_J(X,Y) := [JX,JY] - J[JX,Y] - J[X,JY] - [X,Y] = 0, \quad \text{for all } X,Y \in \mathfrak{g}.$$

An important problem is to find the Lie algebras that admit such a structure. They allow to construct many interesting examples of compact complex manifolds whenever the simply-connected Lie group G of $\mathfrak{g}$ has a lattice Γ of maximal rank. Indeed, by extending J to the group G and then passing J to the quotient G/Γ one obtains nilmanifolds, resp. solvmanifolds, when G is nilpotent, resp. solvable, endowed with G-left-invariant complex structures. In real dimension four, the solvable Lie algebras admitting a complex structure have been classified by Ovando in [28], however no general result is known in higher dimension. Focused in six dimensions, Salamon [33] classifies the nilpotent Lie algebras that admit a complex structure, finding eighteen non-isomorphic Lie algebras (see also [6]). In [1] Andrada, Barberis and Dotti obtain the Lie algebras endowed with a complex structure J of *abelian* type, i.e. J satisfies $[JX,JY] = [X,Y]$ for all $X,Y \in \mathfrak{g}$. More recently, Fino and the second and third authors [11] classify the 6-dimensional unimodular solvable Lie algebras admitting a complex structure J with non-zero closed $(3,0)$-form Ψ.

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The existence of a closed nowhere vanishing $(n, 0)$-form Ψ on a $2n$-dimensional almost complex manifold automatically implies the Nijenhuis condition (1), and such complex manifolds have holomorphically trivial canonical bundle. Nilmanifolds with G-left-invariant complex structures are examples of this kind; in fact, by [33, Theorem 1.3], for any basis $\{\omega^j\}_{j=1}^n$ of (1,0)-forms on the underlying nilpotent Lie algebra, the $(n, 0)$-form $\Psi = \omega^1 \wedge \cdots \wedge \omega^n$ is closed. However, this is no longer true for general G-left-invariant complex structures on solvmanifolds. In [11, Proposition 2.1] it is proved that for solvmanifolds, the existence of a G-left-invariant complex structure with holomorphically trivial canonical bundle is equivalent to the existence of a non-zero closed $(n, 0)$-form on the Lie algebra underlying the solvmanifold. The second author finds in [27, Chapter 4] that several of the complex structures with holomorphically trivial canonical bundle on 6-dimensional solvmanifolds are also of splitting type, i.e. they satisfy [18, Assumption 1.1] (see Definition 1.1 for details), but that there are other complex structures that are not of splitting type.

In addition to providing an important source of examples of compact complex manifolds with unusual and interesting properties, the complex structures of splitting type have also interest because they constitute a natural solvable extension of complex nilmanifolds, as they are certain semi-direct products of the latter by $\mathbb{C}^n$. In this sense, they allow to investigate to what extent geometric properties of nilmanifolds still survive in this larger class of homogeneous spaces. See, e.g., the deformation limits constructed in [4]; compare also the observation [19] that Oeljeklaus-Toma manifolds are solvmanifolds of real splitting type endowed with a left-invariant complex structure, and as such they do not admit Vaisman metrics. Furthermore, some complex cohomological invariants of the manifold can be obtained explicitly, which allows to study several aspects of their complex [21, 18, 22, 3] and Hermitian [19, 20, 10] geometry.

One of these invariants are the Dolbeault cohomology groups. For nilmanifolds, several steps have been done in [7, 9, 31, 32] towards the (still open) conjecture that the Dolbeault cohomology of a nilmanifold with G-left-invariant complex structure J can be computed in terms of invariant forms on G, i.e. in terms of the pair $(\mathfrak{g}, J)$. Concerned with the calculus of the Dolbeault cohomology of solvmanifolds, Kasuya [18] provides a technique to compute such complex invariants when the complex structure is of splitting type. The Dolbeault cohomology groups are obtained by means of a certain finite-dimensional subalgebra of the de Rham complex and, more recently, the first author and Kasuya develop in [3] a technique to compute the Bott-Chern cohomology by means of another finite-dimensional subalgebra. These techniques have allowed to study the deformation limits of compact complex $\partial\bar{\partial}$-manifolds [4] and of compact balanced manifolds [11].

Our objective in this paper is the complex geometry of 6-dimensional solvmanifolds endowed with a (G-left-invariant) complex structure of splitting type. The paper is structured as follows. In Section 1, we obtain the solvable Lie algebras that may support a splitting-type complex structure. More concretely, in Theorem 1.7 we prove that if G/Γ is a 6-dimensional solvmanifold endowed with a complex structure J of splitting type, then the Lie algebra $\mathfrak{g}$ of G is isomorphic to $\mathfrak{s}_k$ for some $1 \leq k \leq 12$ (see the list in Theorem 1.7 for a description of the Lie algebras $\mathfrak{s}_k$). Since six of the Lie algebras $\mathfrak{s}_k$ have parameters in their description, the number of non-isomorphic Lie algebras underlying the solvmanifolds with splitting-type complex structure is not finite. In Remark 1.17 we discuss the existence of lattices.

In Section 2, we investigate the existence of Hermitian metrics, with special attention to strong Kähler with torsion (SKT) and balanced metrics. In particular, we obtain SKT structures on solvmanifolds corresponding to $\mathfrak{s}_1$ and we show the existence of balanced structures on the other Lie algebras $\mathfrak{s}_k$ for $2 \leq k \leq 12$ (see Table 6). A conjecture of Fino and Vezzoni [13] states that in the compact non-Kähler case it is never possible to find an SKT metric and also a balanced one. In [14] they prove the conjecture for nilmanifolds and in [13] for 6-dimensional solvmanifolds having holomorphically trivial canonical bundle. As a consequence of our study in Section 2, it turns out that the conjecture also holds for any splitting-type complex structure on a 6-dimensional solvmanifold. On the other hand, Popovici proposes in [30] a conjecture relating the balanced and the Gauduchon cones of $\partial\bar{\partial}$-manifolds, and he observes that, if proved to hold, the conjecture would imply the existence of a balanced metric on any $\partial\bar{\partial}$-manifold. Since solvmanifolds corresponding to $\mathfrak{s}_1$ do not satisfy the $\partial\bar{\partial}$-Lemma, as another consequence of our study in Section 2, one has that balanced metrics exist on any $\partial\bar{\partial}$-solvmanifold of dimension 6 endowed with a splitting-type complex structure (see Corollary 2.8).

Finally, Section 3 is devoted to the complex geometry of the Nakamura manifold and to the construction of some analytic families of compact complex structures on it. The Lie algebra underlying the Nakamura manifold is $\mathfrak{s}_{12}$, and the complex-parallelizable structure given in [26] and the abelian complex structure found in [1] are particular examples of splitting-type complex structures. After classifying, up to equivalence, the splitting-type complex structures on the Nakamura manifold (see Proposition 3.1), we prove in Theorem 3.3, by an appropriate deformation of its abelian complex structure, that the property of *having holomorphically trivial canonical bundle* and the property of *being of splitting type* are not stable under holomorphic deformations.

Moreover, in Theorem 3.8 we construct, for each $k \in \mathbb{Z}$, a compact complex manifold X_k that does not satisfy the $\partial\bar{\partial}$-Lemma, and we prove that X_k admits a small holomorphic deformation $\{(X_k)_t\}_{t\in\Delta_k}$, Δ_k being an open disc in $\mathbb{C}$ around 0, such that $(X_k)_t$ is a compact complex $\partial\bar{\partial}$-manifold for any $t \neq 0$. For the proof of this result we make use of the complex geometry on $\mathfrak{s}_{12}$, since the compact complex manifolds X_k, $k \in \mathbb{Z}$, and all of their small holomorphic deformations $(X_k)_t$, $t \in \Delta_k$, are solvmanifolds corresponding to $\mathfrak{s}_{12}$ endowed with complex structures of splitting type. Furthermore, they all have holomorphically trivial canonical bundle and admit a balanced metric.

When we consider the case $k = -1$, then we recover the main result in [4] because it corresponds precisely to the complex-parallelizable structure. So our Theorem 3.8 shows that the result extends to a countable family of complex structures. Since one of the complex structures (concretely $k = 0$) is the abelian one [1], we have in particular that the abelian complex structure can be deformed to complex structures satisfying the $\partial\bar{\partial}$-Lemma. In other words, the abelian complex structure on the Nakamura manifold (which does not satisfy the $\partial\bar{\partial}$-Lemma) is the central limit of an analytic family of compact complex $\partial\bar{\partial}$-manifolds.

1. The Lie algebras underlying the solvmanifolds with complex structures of splitting type

We are concerned with solvmanifolds $X = G/\Gamma$ endowed with a complex structure of *splitting type* in the following sense:

Definition 1.1. [18, Assumption 1.1] *A solvmanifold $X = G/\Gamma$ endowed with a G-left-invariant complex structure J is said to be of* splitting type *if G is a semi-direct product $G = \mathbb{C}^n \ltimes_\varphi N$ such that:*

1. *N is a connected simply-connected $2k$-dimensional nilpotent Lie group endowed with an N-left-invariant complex structure J_N;*
2. *for any $\mathbf{z} \in \mathbb{C}^n$, it holds that $\varphi(\mathbf{z}) \in \mathrm{Aut}(N)$ is a holomorphic automorphism of N with respect to J_N;*
3. *φ induces a semi-simple action on the Lie algebra $\mathfrak{n}$ associated to N;*

(4) *G has a lattice Γ (then Γ can be written as $\Gamma = \Gamma_{\mathbb{C}^n} \ltimes_\varphi \Gamma_N$ such that $\Gamma_{\mathbb{C}^n}$ and Γ_N are lattices of $\mathbb{C}^n$ and N, respectively, and, for any $\mathbf{z} \in \Gamma_{\mathbb{C}^n}$, it holds $\varphi(\mathbf{z})(\Gamma_N) \subseteq \Gamma_N$);*

(5) *the inclusion $\wedge^{\bullet,\bullet}(\mathfrak{n} \otimes_{\mathbb{R}} \mathbb{C})^* \hookrightarrow \wedge^{\bullet,\bullet}(N/\Gamma_N)$ induces the isomorphism in cohomology*

$$H^{\bullet,\bullet}_{\bar\partial}\left(\wedge^{\bullet,\bullet}(\mathfrak{n} \otimes_{\mathbb{R}} \mathbb{C})^*\right) \stackrel{\cong}{\to} H^{\bullet,\bullet}_{\bar\partial}(N/\Gamma_N) \ .$$

We recall the construction of the complex structure (for further details see [18]). Let $G = \mathbb{C}^n \ltimes_\varphi N$; taking $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$, we consider $\{dz_1, \ldots, dz_n\}$ the standard $(1,0)$-basis of $\mathbb{C}^n$. Consider $\{\varphi^1, \ldots, \varphi^k\}$ the N-invariant $(1,0)$-basis such that the induced action is given by the diagonal matrix

$$\varphi(\mathbf{z}) = \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_k \end{pmatrix},$$

where $\alpha_j \in \mathrm{Hom}(\mathbb{C}^n; \mathbb{C}^*)$ are characters of $\mathbb{C}^n$, $j = 1, \ldots, k$. Then $\{dz_1, \ldots, dz_n, \alpha_1^{-1}\varphi^1, \ldots, \alpha_k^{-1}\varphi^k\}$ is a G-invariant $(1,0)$-basis for the complex structure on $G = \mathbb{C}^n \ltimes_\varphi N$.

1.1. **Reduced equations of splitting-type complex structures in dimension 6.** If the complex dimension of the solvmanifold is $n + k = 3$, then we have the following cases: $G = \mathbb{C}^2 \ltimes_\varphi \mathbb{C}$ or $G = \mathbb{C} \ltimes_\varphi N$, where the nilpotent factor N in the semi-direct product has real dimension 4 and it is endowed with a left-invariant complex structure. There are only two possibilities for N, namely the complex surface $\mathbb{C}^2$ or the real 4-dimensional nilpotent Lie group KT with Lie algebra $\mathfrak{Kt} = \mathfrak{h}_3 \oplus \mathbb{R}$ (we denote by $\mathfrak{h}_3$ the real 3-dimensional Heisenberg Lie algebra) endowed with the left-invariant complex structure defined by a basis of $(1,0)$-forms $\{\tau, \sigma\}$ satisfying

$$\begin{cases} d\tau = 0, \\ d\sigma = \tau \wedge \bar\tau. \end{cases} \tag{2}$$

The nilmanifold KT/Γ endowed with the complex structure (2) is the well-known Kodaira-Thurston manifold.

For the case $\mathbb{C} \ltimes_\varphi N$, either for $N = \mathbb{C}^2$ or KT, the action $\varphi\colon \mathbb{C} \to \mathrm{Aut}(N)$ will be represented for every $z_3 \in \mathbb{C}$ by a diagonal matrix of the form

$$\varphi(z_3) = \begin{pmatrix} e^{Az_3 + B\bar z_3} & 0 \\ 0 & e^{Cz_3 + D\bar z_3} \end{pmatrix}, \tag{3}$$

where $A, B, C, D \in \mathbb{C}$. For the case $\mathbb{C}^2 \ltimes_\varphi \mathbb{C}$, the action is given for every $(z_2, z_3) \in \mathbb{C}^2$ by

$$\varphi(z_2, z_3) = e^{Az_2 + B\bar z_2 + Cz_3 + D\bar z_3},$$

where $A, B, C, D \in \mathbb{C}$.

Proposition 1.2. *Let $X = G/\Gamma$ be a 6-dimensional solvmanifold endowed with a complex structure of splitting type, and suppose that $G = \mathbb{C}^2 \ltimes_\varphi \mathbb{C}$ or $G = \mathbb{C} \ltimes_\varphi \mathbb{C}^2$. If $\mathfrak{g}$ is the Lie algebra of G, then there is a basis $\{\omega^1, \omega^2, \omega^3\}$ for $(\mathfrak{g}^{1,0})^*$ satisfying the complex structure equations*

$$\begin{cases} d\omega^1 = A\,\omega^{13} + B\omega^{1\bar3}, \\ d\omega^2 = -(A + \bar B + \varepsilon)\,\omega^{23} + \varepsilon\,\omega^{2\bar3}, \\ d\omega^3 = 0, \end{cases}$$

for some $A, B \in \mathbb{C}$ and $\varepsilon \in \{0, 1\}$. (Here, and in what follows, $\omega^{\bar k}$ stands for $\overline{\omega^k}$.)

Proof. Let $G = \mathbb{C} \ltimes_\varphi N$ be the semi-direct product where the action $\varphi\colon \mathbb{C} \to \mathrm{Aut}(N)$ is given by the matrix (3), once fixed a $(1,0)$-coframe for N. We are considering the case $N = \mathbb{C}^2$. Hence, $\varphi(z_3)$ is automatically an automorphism of $\mathbb{C}^2$ and the complex structure on G is determined by the global G-invariant $(1,0)$-basis $\{\omega^1 = e^{-Az_3 - B\bar z_3} dz_1,\ \omega^2 = e^{-Cz_3 - D\bar z_3} dz_2,\ \omega^3 = dz_3\}$. The complex structure equations in the basis $\{\omega^1, \omega^2, \omega^3\}$ are

$$d\omega^1 = A\omega^{13} + B\omega^{1\bar 3}, \quad d\omega^2 = C\omega^{23} + D\omega^{2\bar 3}, \quad d\omega^3 = 0.$$

The unimodularity of G is equivalent to the condition $d(\wedge^{3,2}\mathfrak{g}^* \oplus \wedge^{2,3}\mathfrak{g}^*) = \{0\}$, which forces $A + \bar B + C + \bar D = 0$. Clearly, if $D = 0$, then $C = -A - \bar B$. Now, if $D \neq 0$ then, up to scaling ω^3, we can suppose that D is equal to 1 and so $C = -A - \bar B - 1$, arriving at the desired structure equations.

Consider next the case $G = \mathbb{C}^2 \ltimes_\varphi \mathbb{C}$. In this case we have a $(1,0)$-coframe $\{\eta^1, \eta^2, \eta^3\}$ given by $\{\eta^1 = e^{-Az_2 - B\bar z_2 - Cz_3 - D\bar z_3} dz_1,\ \eta^2 = dz_2,\ \eta^3 = dz_3\}$. Hence, the structure equations are

$$d\eta^1 = A\eta^{12} + B\eta^{1\bar 2} + C\eta^{13} + D\eta^{1\bar 3}, \qquad d\eta^2 = d\eta^3 = 0.$$

The unimodularity condition is equivalent to $A + \bar B = 0$ and $C + \bar D = 0$. Thus, we can consider $(A, C) \neq (0, 0)$, because otherwise φ is trivial. Now, if $A \neq 0$ (similarly for $C \neq 0$ when $A = 0$) then the change of basis $\{\omega^1 = \eta^1,\ \omega^2 = \eta^3,\ \omega^3 = A\eta^2 + C\eta^3\}$ provides the structure equations $d\omega^1 = \omega^{13} - \omega^{1\bar 3}$ and $d\omega^2 = d\omega^3 = 0$. □

Proposition 1.3. *Let $X = G/\Gamma$ be a 6-dimensional solvmanifold endowed with a complex structure of splitting type, and suppose that $G = \mathbb{C} \ltimes_\varphi KT$. Then, there is a $(1,0)$-basis $\{\omega^1, \omega^2, \omega^3\}$ satisfying the complex structure equations*

$$\begin{cases} d\omega^1 = \varepsilon\,(\omega^{13} - \omega^{1\bar 3}),\\ d\omega^2 = \omega^{1\bar 1},\\ d\omega^3 = 0, \end{cases}$$

where $\varepsilon \in \{0, 1\}$.

Proof. The semisimple action induced by φ on $\mathfrak{Kt}$ assures the existence of a basis for $\mathfrak{Kt}$ such that the action is diagonal. So, we can take a basis of the form

$$P \cdot \begin{pmatrix} \tau \\ \sigma \end{pmatrix}, \qquad \text{where} \qquad P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C})$$

and $\{\tau, \sigma\}$ is the preferred basis of $(1,0)$-forms with structure equations (2).

Denote also

$$Q := P^{-1} = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} = \frac{1}{p_{11}p_{22} - p_{12}p_{21}} \begin{pmatrix} p_{22} & -p_{12} \\ -p_{21} & p_{11} \end{pmatrix}.$$

With respect to this basis, we can assume that the action φ is diagonal and given by the inverse of the matrix (3), which we will denote simply by α. So, the invariant basis we choose is

$$\{\omega^1, \omega^2, \omega^3 := dz_3\}, \qquad \text{where} \qquad \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} = \alpha \cdot P \cdot \begin{pmatrix} \tau \\ \sigma \end{pmatrix}.$$

Since

$$d\alpha = -\alpha \cdot \mathcal{E}, \qquad \text{where} \qquad \mathcal{E} := \begin{pmatrix} A\,\omega^3 + B\,\omega^{\bar 3} & 0 \\ 0 & C\,\omega^3 + D\,\omega^{\bar 3} \end{pmatrix},$$

whence we get the structure equations (here $\wedge$ is intended componentwise):

$$\begin{aligned} d\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix} &= d\alpha\wedge P\cdot\begin{pmatrix}\tau\\ \sigma\end{pmatrix}+\alpha\cdot P\cdot d\begin{pmatrix}\tau\\ \sigma\end{pmatrix}\\ &= -\alpha\cdot\mathcal{E}\wedge\alpha^{-1}\cdot\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix}+\alpha\cdot P\cdot\begin{pmatrix}0\\ \tau\wedge\bar\tau\end{pmatrix}\\ &= -\begin{pmatrix}\left(A\omega^3+B\omega^{\bar 3}\right)\wedge\omega^1\\ \left(C\omega^3+D\omega^{\bar 3}\right)\wedge\omega^2\end{pmatrix}+\alpha\cdot P\cdot\left(\begin{pmatrix}0\\ \tau\end{pmatrix}\wedge\begin{pmatrix}0\\ \bar\tau\end{pmatrix}\right)\\ &= \begin{pmatrix}\omega^1\wedge\left(A\omega^3+B\omega^{\bar 3}\right)\\ \omega^2\wedge\left(C\omega^3+D\omega^{\bar 3}\right)\end{pmatrix}+\alpha\cdot P\cdot\left(\begin{pmatrix}0\\ \tau\end{pmatrix}\wedge\begin{pmatrix}0\\ \bar\tau\end{pmatrix}\right). \end{aligned}$$

Since there is no dependence on α in the first term, it is well-defined for any value of the parameters $A,B,C,D\in\mathbb{C}$.

As for the second term:

$$\begin{aligned} &\alpha\cdot P\cdot\left(\begin{pmatrix}0\\ \tau\end{pmatrix}\wedge\begin{pmatrix}0\\ \bar\tau\end{pmatrix}\right)\\ &= \alpha\cdot\left(P\cdot\begin{pmatrix}0\\ q_{11}\alpha_1^{-1}\omega^1+q_{12}\alpha_2^{-1}\omega^2\end{pmatrix}\right)\wedge\left(\bar P\cdot\begin{pmatrix}0\\ \bar q_{11}\bar\alpha_1^{-1}\omega^{\bar 1}+\bar q_{12}\bar\alpha_2^{-1}\omega^{\bar 2}\end{pmatrix}\right)\\ &= \alpha\cdot\left(\begin{pmatrix}p_{12}\cdot\left(q_{11}\alpha_1^{-1}\omega^1+q_{12}\alpha_2^{-1}\omega^2\right)\\ p_{22}\cdot\left(q_{11}\alpha_1^{-1}\omega^1+q_{12}\alpha_2^{-1}\omega^2\right)\end{pmatrix}\wedge\begin{pmatrix}\bar p_{12}\cdot\left(\bar q_{11}\bar\alpha_1^{-1}\omega^{\bar 1}+\bar q_{12}\bar\alpha_2^{-1}\omega^{\bar 2}\right)\\ \bar p_{22}\cdot\left(\bar q_{11}\bar\alpha_1^{-1}\omega^{\bar 1}+\bar q_{12}\bar\alpha_2^{-1}\omega^{\bar 2}\right)\end{pmatrix}\right)\\ &= \frac{1}{\left|p_{11}p_{22}-p_{12}p_{21}\right|^2}\begin{pmatrix}\alpha_1\cdot|p_{12}|^2\\ \alpha_2\cdot|p_{22}|^2\end{pmatrix}\cdot\omega\ , \end{aligned}$$

where

$$\omega=\alpha_1^{-1}\bar\alpha_1^{-1}|p_{22}|^2\omega^{1\bar 1}-\alpha_1^{-1}\bar\alpha_2^{-1}p_{22}\bar p_{12}\omega^{1\bar 2}-\bar\alpha_1^{-1}\alpha_2^{-1}p_{12}\bar p_{22}\omega^{2\bar 1}+\alpha_2^{-1}\bar\alpha_2^{-1}|p_{12}|^2\omega^{2\bar 2}.$$

So,

$$d\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix}=\begin{pmatrix}\left(A\omega^3+B\omega^{\bar 3}\right)\wedge\omega^1\\ \left(C\omega^3+D\omega^{\bar 3}\right)\wedge\omega^2\end{pmatrix}+\frac{1}{\left|p_{11}p_{22}-p_{12}p_{21}\right|^2}\begin{pmatrix}\alpha_1\cdot|p_{12}|^2\\ \alpha_2\cdot|p_{22}|^2\end{pmatrix}\cdot\omega.$$

Now, we have to take care about the dependence on z_3 of the terms in the expression above. Note that the case $(A,B,C,D)=(0,0,0,0)$ is trivial, that is, it yields just the product. Let us assume $(A,B)\neq(0,0)$. The term

$$\alpha_1\cdot\frac{|p_{12}|^2}{\left|p_{11}p_{22}-p_{12}p_{21}\right|^2}\cdot\alpha_1^{-1}\bar\alpha_1^{-1}|p_{22}|^2\omega^{1\bar 1}\ =\ \bar\alpha_1^{-1}\cdot\frac{|p_{12}|^2|p_{22}|^2}{\left|p_{11}p_{22}-p_{12}p_{21}\right|^2}\omega^{1\bar 1}$$

contains $\bar\alpha_1^{-1}$ that depends on z_3. So either $p_{12}=0$ or $p_{22}=0$. If $p_{22}=0$, then $\omega=\alpha_2^{-1}\bar\alpha_2^{-1}|p_{12}|^2\omega^{2\bar 2}$ and therefore we have to assume $\alpha_1\alpha_2^{-1}\bar\alpha_2^{-1}$ to be constant. Up to rescale p_{12}, we may assume

$$\alpha_1\alpha_2^{-1}\bar\alpha_2^{-1}\cdot\frac{|p_{12}|^2}{|p_{21}|^2}\ =\ 1\ ,$$

getting in this case the structure equations

$$d\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix}=\begin{pmatrix}\left(A\omega^3+B\omega^{\bar 3}\right)\wedge\omega^1\\ \left(C\omega^3+D\omega^{\bar 3}\right)\wedge\omega^2\end{pmatrix}+\begin{pmatrix}\omega^{2\bar 2}\\ 0\end{pmatrix}\ . \tag{4}$$

On the other hand, if $p_{12} = 0$, then $\omega = \alpha_1^{-1}\bar{\alpha}_1^{-1}|p_{22}|^2\omega^{1\bar{1}}$ and we get the necessary assumption that $\alpha_2\alpha_1^{-1}\bar{\alpha}_1^{-1}$ is constant. Moreover, up to rescaling, we may assume

$$\alpha_2\alpha_1^{-1}\bar{\alpha}_1^{-1} \cdot \frac{|p_{22}|^2}{|p_{11}|^2} \;=\; 1\;,$$

which reduces the structure equations to

$$d\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix} = \begin{pmatrix}\left(A\omega^3 + B\omega^{\bar{3}}\right)\wedge\omega^1\\ \left(C\omega^3 + D\omega^{\bar{3}}\right)\wedge\omega^2\end{pmatrix} + \begin{pmatrix}0\\ \omega^{1\bar{1}}\end{pmatrix}\;. \tag{5}$$

In case $(C, D) \neq (0, 0)$, we look at the term

$$\alpha_2 \cdot \frac{|p_{22}|^2}{|p_{11}p_{22} - p_{12}p_{21}|^2} \cdot \alpha_2^{-1}\bar{\alpha}_2^{-1}|p_{12}|^2\omega^{2\bar{2}} \;=\; \bar{\alpha}_2^{-1} \cdot \frac{|p_{12}|^2|p_{22}|^2}{|p_{11}p_{22} - p_{12}p_{21}|^2}\omega^{2\bar{2}}$$

and we argue in the same way as before. If $p_{22} = 0$, then we are reduced to the structure equations (4), whereas if $p_{12} = 0$, then we are reduced to the structure equations (5).

Note that, with reference, e.g., to the second case (5), the Jacobi condition yields the equations

$$A + \bar{B} - C \;=\; D - \bar{C} \;=\; 0\;.$$

Now the unimodularity condition is then equivalent to the equation $\bar{A} + B = 0$. Finally, if $A \neq 0$ then we can suppose that it is equal to 1 after rescaling ω^3. □

For the sake of clearness, we summarize Proposition 1.2 and Proposition 1.3 in the following statement.

Theorem 1.4. *Let $X = G/\Gamma$ be a 6-dimensional solvmanifold endowed with a complex structure of splitting type. Then, there is a co-frame $\{\omega^1, \omega^2, \omega^3\}$ of invariant $(1, 0)$-forms satisfying the complex structure equations*

$$\begin{cases} d\omega^1 = A\,\omega^{13} + B\,\omega^{1\bar{3}},\\ d\omega^2 = -(A + \bar{B} + \varepsilon)\,\omega^{23} + \varepsilon\,\omega^{2\bar{3}},\\ d\omega^3 = 0 \end{cases} \tag{6}$$

or

$$\begin{cases} d\omega^1 = \varepsilon\,(\omega^{13} - \omega^{1\bar{3}}),\\ d\omega^2 = \omega^{1\bar{1}},\\ d\omega^3 = 0, \end{cases} \tag{7}$$

where $A, B \in \mathbb{C}$ and $\varepsilon \in \{0, 1\}$.

Remark 1.5. We note that for a complex structure in (6), the canonical bundle is holomorphically trivial if and only if $B = -\varepsilon$. Indeed, by [11, Proposition 2.1], since the complex structure is left-invariant, a nowhere vanishing holomorphic $(3, 0)$-form on $X = G/\Gamma$ is necessarily invariant, but a direct calculation shows that $d\omega^{123} = (B + \varepsilon)\omega^{123\bar{3}}$. Similarly, for a complex structure in (7), one has that $d\omega^{123} = -\varepsilon\,\omega^{123\bar{3}}$, so the canonical bundle is holomorphically trivial if and only if $\varepsilon = 0$. We show below which are the Lie algebras underlying such solvmanifolds.

1.2. **Six-dimensional solvable Lie algebras with complex structures of splitting type.** In this section we determine the 6-dimensional real Lie algebras underlying the reduced equations of splitting-type complex structures obtained in the previous section. For simplicity, we introduce the following definition.

Definition 1.6. We will say that $\mathfrak{g}$ *admits a complex structure of splitting type* if $\mathfrak{g}$ is a real Lie algebra underlying the complex equations (6) or (7) in Theorem 1.4.

Recall that those Lie algebras underlying the complex equations (7) correspond to Lie groups of the form $\mathbb{C} \ltimes_\varphi KT$, whereas the Lie algebras underlying (6) correspond to $\mathbb{C}^2 \ltimes_\varphi \mathbb{C}$ or $\mathbb{C} \ltimes_\varphi \mathbb{C}^2$.

The main result in this section is the following theorem.

Theorem 1.7. *Let $\mathfrak{g}$ be a unimodular (non-nilpotent) solvable Lie algebra of dimension 6. Then, $\mathfrak{g}$ admits a complex structure of splitting type if and only if it is isomorphic to one in the following list:*

$$\begin{aligned}
&\mathfrak{s}_1 = (e^{23}, e^{34}, -e^{24}, 0, 0, 0),\\
&\mathfrak{s}_2 = (0, -e^{13}, e^{12}, 0, 0, 0),\\
&\mathfrak{s}_3 = (0, -e^{13}, e^{12}, 0, -e^{46}, e^{45}),\\
&\mathfrak{s}_4 = (e^{15}, -e^{25}, -e^{35}, e^{45}, 0, 0),\\
&\mathfrak{s}_5^{\alpha} = (e^{15}, e^{25}, -e^{35} + \alpha\, e^{45}, -\alpha\, e^{35} - e^{45}, 0, 0), \quad \alpha > 0,\\
&\mathfrak{s}_6^{\alpha,\beta} = (\alpha\, e^{15} + e^{25}, -e^{15} + \alpha\, e^{25}, -\alpha\, e^{35} + \beta\, e^{45}, -\beta\, e^{35} - \alpha\, e^{45}, 0, 0),\\
&\qquad \alpha > 0,\ \ 0 < \beta < 1,\\
&\mathfrak{s}_7^{\alpha} = (e^{25}, -e^{15}, \alpha\, e^{45}, -\alpha\, e^{35}, 0, 0), \quad 0 < \alpha \leq 1,\\
&\mathfrak{s}_8^{\alpha} = (\alpha\, e^{15} + e^{25}, -e^{15} + \alpha\, e^{25}, -\alpha\, e^{35} + e^{45}, -e^{35} - \alpha\, e^{45}, 0, 0), \quad \alpha > 0,\\
&\mathfrak{s}_9 = (-e^{16}, -e^{26}, e^{36} - e^{45}, e^{35} + e^{46}, 0, 0),\\
&\mathfrak{s}_{10}^{\alpha,\beta} = (e^{15} + \beta\, e^{16} - e^{26}, e^{16} + e^{25} + \beta\, e^{26}, -e^{35} - \beta\, e^{36} - \alpha\, e^{45}, \alpha\, e^{35} - e^{45} - \beta\, e^{46}, 0, 0),\\
&\qquad \alpha \neq 0, \quad \beta \in \mathbb{R},\\
&\mathfrak{s}_{11}^{\alpha} = (e^{16} - e^{25}, e^{15} + e^{26}, -e^{36} - \alpha\, e^{45}, \alpha\, e^{35} - e^{46}, 0, 0), \quad \alpha \in (0, 1),\\
&\mathfrak{s}_{12} = (e^{16} - e^{25}, e^{15} + e^{26}, -e^{36} + e^{45}, -e^{35} - e^{46}, 0, 0).
\end{aligned}$$

Here we follow the notation in [33]. For example, by writing $(e^{23}, e^{34}, -e^{24}, 0, 0, 0)$ we mean that there exists a basis $\{e^1, \ldots, e^6\}$ of the dual of the Lie algebra satisfying $de^1 = e^2 \wedge e^3$, $de^2 = e^3 \wedge e^4$, $de^3 = -e^2 \wedge e^4$, and $de^4 = de^5 = de^6 = 0$.

Remark 1.8. For detailed explanations on the values of the parameters in the list above, see Appendix A.

Let us start by determining the Lie algebras underlying the equations (7).

Proposition 1.9. *The Lie algebras $\mathfrak{g}$ that admit a complex structure of splitting type corresponding to $\mathbb{C} \ltimes_\varphi KT$ are $(0, 0, 0, 0, 0, e^{12})$ if $\mathfrak{g}$ is nilpotent, and $\mathfrak{s}_1$ if $\mathfrak{g}$ is solvable but non-nilpotent.*

Proof. It is clear that one obtains $(0,0,0,0,0,e^{12})$ if $\varepsilon = 0$ in (7). For $\varepsilon = 1$, if we consider the basis $\{e^1,\ldots,e^6\}$ given by $\omega^1 = e^3 - i\,e^2$, $\omega^2 = 2(e^5 - i\,e^1)$ and $\omega^3 = \frac{1}{2}(e^6 - i\,e^4)$, then it is immediate to see that the real Lie algebra is $\mathfrak{s}_1$. □

We divide the study of equations (6) according to the vanishing of coefficient ε. As a result we present several tables (see Tables 1, 2, 3, 4 and 5). There, the real basis $\{e^1,\ldots,e^6\}$ is the one that corresponds to the real structure equations in Theorem 1.7.

Proposition 1.10. *The Lie algebras underlying equations* (6) *with* $\varepsilon = 0$ *are* $\mathfrak{s}_2$, $\mathfrak{s}_9$, $\mathfrak{s}_{10}^{\alpha,\beta}$, $\mathfrak{s}_{11}^{\alpha}$, $\mathfrak{s}_{12}$.

Proof. Suppose first that $B = -\bar{A}$ in (6), so we can suppose that $A \neq 0$. Moreover, taking $\{\omega'^1 = \omega^1, \omega'^2 = \omega^2, \omega'^3 = A\omega^3\}$ we can suppose that $A = -B = 1$. If we set $\omega^1 = e^3 + ie^2$, $\omega^2 = e^4 + ie^5$, $\omega^3 = e^6 + \frac{i}{2}e^1$, then we obtain the structure equations of $\mathfrak{s}_2$.

On the other hand, if $B \neq -\bar{A}$, observe that we can normalize the coefficient in ω^{23} just by taking a new basis $\{\omega'^1 = \omega^1, \omega'^2 = \omega^2, \omega'^3 = -(A+\bar{B})\omega^3\}$.

If we denote $\omega'^1 = \alpha^1 + i\alpha^2$, $\omega'^2 = \alpha^3 + i\alpha^4$, $\omega'^3 = \alpha^5 + i\alpha^6$, then the real structure equations become $d\alpha^5 = d\alpha^6 = 0$ and

$$\begin{cases} d\alpha^1 &=& -\alpha^{15} - 2\,\mathfrak{Im}\,B\,\alpha^{25} + (1 + 2\,\mathfrak{Re}\,B)\,\alpha^{26}, \\ d\alpha^2 &=& 2\,\mathfrak{Im}\,B\,\alpha^{15} - (1 + 2\,\mathfrak{Re}\,B)\,\alpha^{16} - \alpha^{25}, \\ d\alpha^3 &=& \alpha^{35} - \alpha^{46}, \\ d\alpha^4 &=& \alpha^{36} + \alpha^{45}. \end{cases}$$

It is straightforward to see that if $B = -\frac{1}{2}$, the Lie algebra is isomorphic to $\mathfrak{s}_9$ (take $e^i = \alpha^i$, $i = 1,2,3,4$, $e^5 = \alpha^6$, $e^6 = \alpha^5$). If we consider the real basis $e^1 = \alpha^3$, $e^2 = \alpha^4$, $e^3 = \alpha^1$, $e^4 = \alpha^2$, $e^5 = \alpha^6$, $e^6 = \alpha^5$, then the Lie algebra is isomorphic to $\mathfrak{s}_{12}$ if $B = 0$ and $\mathfrak{s}_{11}^{\alpha'}$, for $\alpha' = -1 - 2B$, if $B \in \mathbb{R} \setminus \{-\frac{1}{2}, 0\}$. Notice that $\alpha' \in \mathbb{R} \setminus \{-1, 0\}$ and hence the Lie algebra $\mathfrak{s}_{11}^{\alpha'}$ is isomorphic to the Lie algebra $\mathfrak{s}_{11}^{\alpha}$ for some $\alpha \in (0,1)$, as it appears in Theorem 1.7 (see Appendix A for details). If $B = -1$, taking $e^1 = -\alpha^3$, $e^2 = -\alpha^4$, $e^3 = \alpha^2$, $e^4 = \alpha^1$, $e^5 = \alpha^6$, $e^6 = \alpha^5$ we obtain the Lie algebra $\mathfrak{s}_{12}$. Finally, if $\mathfrak{Im}\,B \neq 0$, then with respect to the real basis $e^1 = \alpha^3$, $e^2 = \alpha^4$, $e^3 = \alpha^1$, $e^4 = \alpha^2$, $e^5 = \alpha^5 - \frac{1+2\,\mathfrak{Re}\,B}{2\,\mathfrak{Im}\,B}\alpha^6$, $e^6 = \alpha^6$, we obtain $\mathfrak{s}_{10}^{\alpha,\beta}$ where $\alpha = 2\,\mathfrak{Im}\,B \neq 0$ and $\beta = \frac{1+2\,\mathfrak{Re}\,B}{2\,\mathfrak{Im}\,B}$. □

In Table 1 we summarize the results obtained in the previous proposition.

From now on, we focus on the equations (6) with $\varepsilon = 1$. Let us consider the basis of real 1-forms $\{\alpha^1,\ldots,\alpha^6\}$ given by

$$\omega^1 = \alpha^1 + i\alpha^2, \quad \omega^2 = \alpha^3 + i\alpha^4, \quad \omega^3 = \alpha^5 + i\alpha^6. \tag{8}$$

<table>
<tr><th colspan="3">$A, B \in \mathbb{C}$</th><th>Real basis $\{e^1, \ldots, e^6\}$</th><th>Lie algebra</th></tr>
<tr><td colspan="3">$A = -\bar{B} \neq 0$</td><td>$\omega^1 = e^3 + ie^2$, $\omega^2 = e^4 + ie^5$,
$\omega^3 = e^6 + \frac{i}{2}e^1$</td><td>$\mathfrak{s}_2$</td></tr>
<tr><td rowspan="5">$A = -1 - \bar{B}$</td><td rowspan="4">$B \in \mathbb{R}$</td><td>$B = -1$</td><td>$\omega^1 = e^4 + ie^3$, $\omega^2 = -e^1 - ie^2$,
$\omega^3 = e^6 + ie^5$</td><td>$\mathfrak{s}_{12}$</td></tr>
<tr><td>$B = -\frac{1}{2}$</td><td>$\omega^1 = e^1 + ie^2$, $\omega^2 = e^3 + ie^4$,
$\omega^3 = e^6 + ie^5$</td><td>$\mathfrak{s}_9$</td></tr>
<tr><td>$B = 0$</td><td>$\omega^1 = e^3 + ie^4$, $\omega^2 = e^1 + ie^2$,
$\omega^3 = e^6 + ie^5$</td><td>$\mathfrak{s}_{12}$</td></tr>
<tr><td>$B \neq -1, -\frac{1}{2}, 0$</td><td>$\omega^1 = e^3 + ie^4$, $\omega^2 = e^1 + ie^2$,
$\omega^3 = e^6 + ie^5$</td><td>$\mathfrak{s}_{11}^{\alpha'}$
$\alpha' = -1 - 2B$</td></tr>
<tr><td colspan="2">$\mathfrak{Im}\, B \neq 0$</td><td>$\omega^1 = e^3 + ie^4$, $\omega^2 = e^1 + ie^2$,
$\omega^3 = \left(e^5 + \frac{1+2\,\mathfrak{Re}\, B}{2\,\mathfrak{Im}\, B}\, e^6\right) + ie^6$</td><td>$\mathfrak{s}_{10}^{\alpha,\beta}$
$\alpha = 2\,\mathfrak{Im}\, B$, $\beta = \frac{1+2\,\mathfrak{Re}\, B}{2\,\mathfrak{Im}\, B}$</td></tr>
</table>

TABLE 1. Lie algebras underlying equations (6) with $\varepsilon = 0$ (Proposition 1.10).

Hence, in terms of this basis the real structure equations become $d\alpha^5 = d\alpha^6 = 0$ and

$$
(9) \qquad \begin{cases}
d\alpha^1 &= (\mathfrak{Re}\, A + \mathfrak{Re}\, B)\, \alpha^{15} - (\mathfrak{Im}\, A - \mathfrak{Im}\, B)\, \alpha^{16} \\
&\quad -(\mathfrak{Im}\, A + \mathfrak{Im}\, B)\, \alpha^{25} - (\mathfrak{Re}\, A - \mathfrak{Re}\, B)\, \alpha^{26}, \\
d\alpha^2 &= (\mathfrak{Im}\, A + \mathfrak{Im}\, B)\, \alpha^{15} + (\mathfrak{Re}\, A - \mathfrak{Re}\, B)\, \alpha^{16} \\
&\quad +(\mathfrak{Re}\, A + \mathfrak{Re}\, B)\, \alpha^{25} - (\mathfrak{Im}\, A - \mathfrak{Im}\, B)\, \alpha^{26}, \\
d\alpha^3 &= -(\mathfrak{Re}\, A + \mathfrak{Re}\, B)\, \alpha^{35} + (\mathfrak{Im}\, A - \mathfrak{Im}\, B)\, \alpha^{36} \\
&\quad +(\mathfrak{Im}\, A - \mathfrak{Im}\, B)\, \alpha^{45} + (2 + \mathfrak{Re}\, A + \mathfrak{Re}\, B)\, \alpha^{46}, \\
d\alpha^4 &= -(\mathfrak{Im}\, A - \mathfrak{Im}\, B)\, \alpha^{35} - (2 + \mathfrak{Re}\, A + \mathfrak{Re}\, B)\, \alpha^{36} \\
&\quad -(\mathfrak{Re}\, A + \mathfrak{Re}\, B)\, \alpha^{45} + (\mathfrak{Im}\, A - \mathfrak{Im}\, B)\, \alpha^{46}.
\end{cases}
$$

We need to consider different cases in order to identify all the possible real Lie algebras underlying these equations. Concretely, we focus our attention at the expression $\mathfrak{Im}\, A - \mathfrak{Im}\, B$ in (9) distinguishing three cases, namely: $\mathfrak{Im}\, A = \mathfrak{Im}\, B = 0$, $\mathfrak{Im}\, A = \mathfrak{Im}\, B \neq 0$, or $\mathfrak{Im}\, A \neq \mathfrak{Im}\, B$.

1.2.1. *Case* $\varepsilon = 1$, $\mathfrak{Im}\, A = \mathfrak{Im}\, B = 0$.

Lemma 1.11. *The Lie algebras underlying equations* (6) *with* $\varepsilon = 1$ *and* $A, B \in \mathbb{R}$ *are* $\mathfrak{s}_2$, $\mathfrak{s}_4$, $\mathfrak{s}_7^{\alpha}$, $\mathfrak{s}_9$, $\mathfrak{s}_{11}^{\alpha}$, $\mathfrak{s}_{12}$.

Proof. Imposing condition $\mathfrak{Im}\, A = \mathfrak{Im}\, B = 0$ in (9), the equations simplify as

$$d\alpha^1 = (A + B)\, \alpha^{15} - (A - B)\, \alpha^{26}, \quad d\alpha^3 = -(A + B)\, \alpha^{35} + (2 + A + B)\, \alpha^{46},$$

$$d\alpha^2 = (A - B)\, \alpha^{16} + (A + B)\, \alpha^{25}, \quad d\alpha^4 = -(2 + A + B)\, \alpha^{36} - (A + B)\, \alpha^{45}.$$

Now, it suffices to consider different cases depending on the vanishing of the coefficients in the previous structure equations. Concretely, we divide our analysis in the subcases

$A=-B$, $A=B\neq 0$ and $A\neq \pm B$. The results appear in Table 2. Notice that in the case of the Lie algebra $\mathfrak{s}_7^{\alpha'}$, if $\alpha'=|A|>1$ then it is isomorphic to $\mathfrak{s}_7^{\alpha}$ with $\alpha=1/\alpha'$, so that $0<\alpha\leq 1$ according to Theorem 1.7. Similarly, $\mathfrak{s}_{11}^{\alpha'}$ is isomorphic to the Lie algebra $\mathfrak{s}_{11}^{\alpha}$ for some $\alpha\in(0,1)$, as it appears in Theorem 1.7.

For each case in Table 2, we need to apply a change of real basis between the initial one $\{\alpha^1,\dots,\alpha^6\}$ and the final one $\{e^1,\dots,e^6\}$. These changes are given simply by equalling the expression of ω^i's given in (8) and their corresponding expressions given in Table 2.

□

$A,B\in\mathbb{R}$		Real basis $\{e^1,\dots,e^6\}$	Lie algebra
$A=-B$	$A=0$	$\omega^1=e^4+ie^5,\ \omega^2=e^3+ie^2,$ $\omega^3=-e^6-\frac{i}{2}e^1$	$\mathfrak{s}_2$
	$A\neq 0$	$\omega^1=-\frac{A}{\lvert A\rvert}\,e^3+ie^4,$ $\omega^2=e^1+ie^2,\ \omega^3=e^6+\frac{i}{2}e^5$	$\mathfrak{s}_7^{\alpha'}$ $\alpha'=\lvert A\rvert$
$A=B$	$A=-1$	$\omega^1=e^1+ie^4,\ \omega^2=e^3+ie^2,$ $\omega^3=-\frac{1}{2}e^5+ie^6$	$\mathfrak{s}_4$
	$A\neq 0,-1$	$\omega^1=e^1+ie^2,\ \omega^2=e^3+ie^4,$ $\omega^3=-\frac{1}{2A}\,e^6-\frac{i}{2(A+1)}\,e^5$	$\mathfrak{s}_9$
$A\neq\pm B$	$A=-1$	$\omega^1=e^1+ie^2,\ \omega^2=e^4+ie^3,$ $\omega^3=\frac{1}{B-1}\,e^6-\frac{i}{B+1}\,e^5$	$\mathfrak{s}_{12}$
	$B=-1$	$\omega^1=e^1+ie^2,\ \omega^2=e^3+ie^4,$ $\omega^3=\frac{1}{A-1}\,e^6+\frac{i}{A+1}\,e^5$	$\mathfrak{s}_{12}$
	$A+B=-2$	$\omega^1=e^3+ie^4,\ \omega^2=e^1+ie^2,$ $\omega^3=-\frac{1}{2}\,e^6+\frac{i}{2(A+1)}\,e^5$	$\mathfrak{s}_9$
	$A+B\neq-2$ $A,B\neq -1$	$\omega^1=e^1+ie^2,\ \omega^2=e^3+ie^4,$ $\omega^3=\frac{1}{A+B}\,e^6+\frac{i}{A-B}\,e^5$	$\mathfrak{s}_{11}^{\alpha'}$ $\alpha'=\frac{2+A+B}{B-A}$

TABLE 2. Lie algebras underlying equations (6) with $\varepsilon=1$ and $\mathfrak{Im}\,A=\mathfrak{Im}\,B=0$ (Lemma 1.11).

1.2.2. *Case* $\varepsilon=1$, $\mathfrak{Im}\,A=\mathfrak{Im}\,B\neq 0$.

Lemma 1.12. *The Lie algebras underlying equations* (6) *with* $\varepsilon=1$ *and* $\mathfrak{Im}\,A=\mathfrak{Im}\,B\neq 0$ *are* $\mathfrak{s}_3$, $\mathfrak{s}_5^{\alpha}$, $\mathfrak{s}_9$, $\mathfrak{s}_{10}^{\alpha,\beta}$.

Proof. Taking $\mathfrak{Im}\, A = \mathfrak{Im}\, B \neq 0$, the equations (9) transform into

$$\begin{cases} d\alpha^1 &= (\mathfrak{Re}\, A + \mathfrak{Re}\, B)\, \alpha^{15} - 2\,\mathfrak{Im}\, A\, \alpha^{25} - (\mathfrak{Re}\, A - \mathfrak{Re}\, B)\, \alpha^{26}, \\ d\alpha^2 &= 2\,\mathfrak{Im}\, A\, \alpha^{15} + (\mathfrak{Re}\, A - \mathfrak{Re}\, B)\, \alpha^{16} + (\mathfrak{Re}\, A + \mathfrak{Re}\, B)\, \alpha^{25}, \\ d\alpha^3 &= -(\mathfrak{Re}\, A + \mathfrak{Re}\, B)\, \alpha^{35} + (2 + \mathfrak{Re}\, A + \mathfrak{Re}\, B)\, \alpha^{46}, \\ d\alpha^4 &= -(2 + \mathfrak{Re}\, A + \mathfrak{Re}\, B)\, \alpha^{36} - (\mathfrak{Re}\, A + \mathfrak{Re}\, B)\, \alpha^{45}. \end{cases}$$

We consider the following cases according to the vanishing of some coefficients in the equations above, namely $\mathfrak{Re}\, A = -\mathfrak{Re}\, B$, $\mathfrak{Re}\, A = \mathfrak{Re}\, B \neq 0$ and $\mathfrak{Re}\, A \neq \pm\mathfrak{Re}\, B$, obtaining the results that appear in Table 3. The changes of basis between $\{\alpha^i\}_{i=1}^6$ and $\{e^i\}_{i=1}^6$ follow directly from Table 3, taking into account (8). ☐

$\mathfrak{Im}\, A = \mathfrak{Im}\, B \neq 0$	Real basis $\{e^1, \ldots, e^6\}$	Lie algebra
$\mathfrak{Re}\, A = -\mathfrak{Re}\, B$	$\omega^1 = e^2 - ie^3$, $\omega^2 = e^5 + ie^6$, $\omega^3 = \frac{1}{2\,\mathfrak{Im}\, A}(e^1 - \mathfrak{Re}\, A\, e^4) + \frac{i}{2} e^4$	$\mathfrak{s}_3$
$\mathfrak{Re}\, A = \mathfrak{Re}\, B = -1$	$\omega^1 = -\frac{\mathfrak{Im}\, A}{\lvert\mathfrak{Im}\, A\rvert} e^3 + ie^4$, $\omega^2 = e^1 + ie^2$, $\omega^3 = \frac{1}{2} e^5 + ie^6$	$\mathfrak{s}_5^{\alpha}$ $\alpha = \lvert\mathfrak{Im}\, A\rvert$
$\mathfrak{Re}\, A = \mathfrak{Re}\, B \neq 0, -1$	$\omega^1 = e^3 + ie^4$, $\omega^2 = e^1 + ie^2$, $\omega^3 = -\frac{1}{2\,\mathfrak{Re}\, A} e^5 - \frac{i}{2(\mathfrak{Re}\, A + 1)} e^6$	$\mathfrak{s}_{10}^{\alpha,0}$ $\alpha = -\frac{\mathfrak{Im}\, A}{\mathfrak{Re}\, A}$
$\mathfrak{Re}\, A \neq \pm\mathfrak{Re}\, B$ $\mathfrak{Re}\, A + \mathfrak{Re}\, B = -2$	$\omega^1 = e^3 + ie^4$, $\omega^2 = e^1 + ie^2$, $\omega^3 = -\frac{1}{2} e^6 + \frac{i}{2(\mathfrak{Re}\, A + 1)} (e^5 + \mathfrak{Im}\, A\, e^6)$	$\mathfrak{s}_9$
$\mathfrak{Re}\, A \neq \pm\mathfrak{Re}\, B$ $\mathfrak{Re}\, A + \mathfrak{Re}\, B \neq -2$	$\omega^1 = e^1 + ie^2$, $\omega^2 = e^3 + ie^4$, $\omega^3 = \frac{1}{\mathfrak{Re}\, A + \mathfrak{Re}\, B} e^5 + \frac{1}{2\,\mathfrak{Im}\, A} e^6 - i\, \frac{2\,\mathfrak{Im}\, A}{\mathfrak{Re}^2 A - \mathfrak{Re}^2 B}\, e^5$	$\mathfrak{s}_{10}^{\alpha,\beta}$ $\alpha = \frac{2\,\mathfrak{Im}\, A(2 + \mathfrak{Re}\, A + \mathfrak{Re}\, B)}{\mathfrak{Re}^2 A - \mathfrak{Re}^2 B}$ $\beta = \frac{\mathfrak{Re}\, A + \mathfrak{Re}\, B}{2\,\mathfrak{Im}\, A}$

TABLE 3. Lie algebras underlying equations (6) with $\varepsilon = 1$ and $\mathfrak{Im}\, A = \mathfrak{Im}\, B \neq 0$ (Lemma 1.12).

1.2.3. *Case* $\varepsilon = 1$, $\mathfrak{Im}\, A \neq \mathfrak{Im}\, B$. Starting from (9), let us consider the new basis $\{\beta^1, \ldots, \beta^6\}$ given by

$$\beta^i = \alpha^i,\ i = 1, 2, 3, 4, \quad \beta^5 = (\mathfrak{Im}\, A - \mathfrak{Im}\, B)\, \alpha^5, \quad \beta^6 = (\mathfrak{Im}\, A - \mathfrak{Im}\, B)\, \alpha^6.$$

In terms of this basis, the structure equations (9) are

$$(10)\quad \begin{cases} d\beta^1 &= -\beta^1 \wedge \left(\beta^6 - \frac{\mathfrak{Re}\, A + \mathfrak{Re}\, B}{\mathfrak{Im}\, A - \mathfrak{Im}\, B}\, \beta^5\right) - \beta^2 \wedge \left(\frac{\mathfrak{Im}\, A + \mathfrak{Im}\, B}{\mathfrak{Im}\, A - \mathfrak{Im}\, B}\, \beta^5 + \frac{\mathfrak{Re}\, A - \mathfrak{Re}\, B}{\mathfrak{Im}\, A - \mathfrak{Im}\, B}\, \beta^6\right), \\ d\beta^2 &= -\beta^2 \wedge \left(\beta^6 - \frac{\mathfrak{Re}\, A + \mathfrak{Re}\, B}{\mathfrak{Im}\, A - \mathfrak{Im}\, B}\, \beta^5\right) + \beta^1 \wedge \left(\frac{\mathfrak{Im}\, A + \mathfrak{Im}\, B}{\mathfrak{Im}\, A - \mathfrak{Im}\, B}\, \beta^5 + \frac{\mathfrak{Re}\, A - \mathfrak{Re}\, B}{\mathfrak{Im}\, A - \mathfrak{Im}\, B}\, \beta^6\right), \\ d\beta^3 &= \beta^3 \wedge \left(\beta^6 - \frac{\mathfrak{Re}\, A + \mathfrak{Re}\, B}{\mathfrak{Im}\, A - \mathfrak{Im}\, B}\, \beta^5\right) + \beta^4 \wedge \left(\beta^5 + \frac{2 + \mathfrak{Re}\, A + \mathfrak{Re}\, B}{\mathfrak{Im}\, A - \mathfrak{Im}\, B}\, \beta^6\right), \\ d\beta^4 &= -\beta^3 \wedge \left(\beta^5 + \frac{2 + \mathfrak{Re}\, A + \mathfrak{Re}\, B}{\mathfrak{Im}\, A - \mathfrak{Im}\, B}\, \beta^6\right) + \beta^4 \wedge \left(\beta^6 - \frac{\mathfrak{Re}\, A + \mathfrak{Re}\, B}{\mathfrak{Im}\, A - \mathfrak{Im}\, B}\, \beta^5\right). \end{cases}$$

We define the 1-forms

$$\nu^5 = \beta^5 + \frac{2 + \mathfrak{Re}\, A + \mathfrak{Re}\, B}{\mathfrak{Im}\, A - \mathfrak{Im}\, B}\, \beta^6, \quad \nu^6 = \beta^6 - \frac{\mathfrak{Re}\, A + \mathfrak{Re}\, B}{\mathfrak{Im}\, A - \mathfrak{Im}\, B}\, \beta^5.$$

The linear dependence of ν^5 and ν^6 will play a key role in our study of the underlying Lie algebras. Let us define

$$\begin{aligned} \Delta = \Delta(A, B) &= (\mathfrak{Im}\, A - \mathfrak{Im}\, B)^2 + (2 + \mathfrak{Re}\, A + \mathfrak{Re}\, B)(\mathfrak{Re}\, A + \mathfrak{Re}\, B) \\ &= |A|^2 + |B|^2 + 2(\mathfrak{Re}\, A + \mathfrak{Re}\, B + \mathfrak{Re}\, A \mathfrak{Re}\, B - \mathfrak{Im}\, A \mathfrak{Im}\, B). \end{aligned}$$

It is straightforward to check that ν^5 and ν^6 are linearly independent if and only if $\Delta \neq 0$. In the following lemmata we study the cases $\Delta = 0$ and $\Delta \neq 0$.

Lemma 1.13. *The Lie algebras underlying equations* (6) *with* $\varepsilon = 1$, $\mathfrak{Im}\, A \neq \mathfrak{Im}\, B$ *and* $\Delta(A, B) = 0$ *are* $\mathfrak{s}_5^{\alpha}$, $\mathfrak{s}_6^{\alpha,\beta}$, $\mathfrak{s}_8^{\alpha}$, $\mathfrak{s}_{10}^{\alpha,0}$.

Proof. Notice first that the condition $\Delta = 0$ implies that $\mathfrak{Re}\, A + \mathfrak{Re}\, B \neq 0, -2$. Since ν^5 and ν^6 are linearly dependent, we have that

$$\nu^5 = \theta\, \nu^6, \quad \text{where} \quad \theta = \frac{2 + \mathfrak{Re}\, A + \mathfrak{Re}\, B}{\mathfrak{Im}\, A - \mathfrak{Im}\, B} = -\frac{\mathfrak{Im}\, A - \mathfrak{Im}\, B}{\mathfrak{Re}\, A + \mathfrak{Re}\, B} \neq 0.$$

Let us consider the new basis $\{\gamma^1, \ldots, \gamma^6\}$ given by $\gamma^i = \beta^i$, $1 \leq i \leq 5$, and $\gamma^6 = \nu^6 = \beta^6 + \frac{1}{\theta}\, \beta^5$. With respect to this basis, the structure equations (10) are

$$(11) \qquad \begin{cases} d\gamma^1 = -\gamma^{16} + \gamma^2 \wedge \left[\left(\frac{|B|^2 - |A|^2}{(\mathfrak{Im}\, A - \mathfrak{Im}\, B)^2} \right) \gamma^5 - \left(\frac{\mathfrak{Re}\, A - \mathfrak{Re}\, B}{\mathfrak{Im}\, A - \mathfrak{Im}\, B} \right) \gamma^6 \right], \\ d\gamma^2 = -\gamma^{26} - \gamma^1 \wedge \left[\left(\frac{|B|^2 - |A|^2}{(\mathfrak{Im}\, A - \mathfrak{Im}\, B)^2} \right) \gamma^5 - \left(\frac{\mathfrak{Re}\, A - \mathfrak{Re}\, B}{\mathfrak{Im}\, A - \mathfrak{Im}\, B} \right) \gamma^6 \right], \\ d\gamma^3 = \gamma^{36} + \theta\, \gamma^{46}, \\ d\gamma^4 = \gamma^{46} - \theta\, \gamma^{36}. \end{cases}$$

In order to determine the Lie algebras underlying the equations (11), we distinguish the cases when $|A| = |B|$ or $|A| \neq |B|$ (see Table 4 for details). Notice that the Lie algebras $\mathfrak{s}_5^{\alpha'}$, $\mathfrak{s}_6^{\alpha',\beta'}$ and $\mathfrak{s}_8^{\alpha'}$ in Table 4 are isomorphic to the Lie algebras $\mathfrak{s}_5^{\alpha}$, $\mathfrak{s}_6^{\alpha,\beta}$ and $\mathfrak{s}_8^{\alpha}$ with the values of the parameters α and β that appear in Theorem 1.7.

Observe that the relation between the bases $\{\gamma^i\}_{i=1}^6$ and $\{e^i\}_{i=1}^6$ can be deduced from the following diagram:

$$\alpha \longrightarrow \beta \longrightarrow \gamma \longrightarrow e, \qquad \alpha \xrightarrow{\text{Table 4}} e.$$

□

Lemma 1.14. *The Lie algebras underlying equations* (6) *with* $\varepsilon = 1$, $\mathfrak{Im}\, A \neq \mathfrak{Im}\, B$ *and* $\Delta(A, B) \neq 0$ *are* $\mathfrak{s}_9$, $\mathfrak{s}_{10}^{\alpha,\beta}$, $\mathfrak{s}_{11}^{\alpha}$, $\mathfrak{s}_{12}$.

Proof. Since $\Delta \neq 0$, the 1-forms ν^5 and ν^6 are linearly independent. Hence, we consider the basis $\{\nu^1, \ldots, \nu^6\}$ given by

$$\nu^i = \beta^i,\ i = 1, 2, 3, 4, \quad \nu^5 = \beta^5 + \frac{2 + \mathfrak{Re}\, A + \mathfrak{Re}\, B}{\mathfrak{Im}\, A - \mathfrak{Im}\, B}\, \beta^6, \quad \nu^6 = \beta^6 - \frac{\mathfrak{Re}\, A + \mathfrak{Re}\, B}{\mathfrak{Im}\, A - \mathfrak{Im}\, B}\, \beta^5.$$

$\mathfrak{Im}\,A \neq \mathfrak{Im}\,B,\ \ \Delta(A,B)=0$			Real basis $\{e^1,\dots,e^6\}$	Lie algebra
$\lvert A\rvert = \lvert B\rvert$	$B=\bar A$		$\omega^1=e^1+ie^2,\quad \omega^2=e^3+ie^4,$ $\omega^3=\frac{1}{2\,\mathfrak{Im}\,A}\left[e^6-i\left(e^5+\frac{\mathfrak{Im}\,A}{1+\mathfrak{Re}\,A}\,e^6\right)\right]$	$\mathfrak{s}_5^{\alpha'}$ $\alpha'=-\frac{1+\mathfrak{Re}\,A}{\mathfrak{Im}\,A}$
$\lvert A\rvert = \lvert B\rvert$	$B\neq\bar A$	$B=-1$ $\mathfrak{Im}\,A\neq 0$	$\omega^1=e^1+ie^2,\quad \omega^2=e^4+ie^3,$ $\omega^3=\frac{1}{\mathfrak{Im}\,A}e^6-\frac{i}{1+\mathfrak{Re}\,A}(e^5+e^6)$	$\mathfrak{s}_8^{\alpha'}$ $\alpha'=\frac{\mathfrak{Im}\,A}{1+\mathfrak{Re}\,A}$
$\lvert A\rvert = \lvert B\rvert$	$B\neq\bar A$	$A=-1$ $\mathfrak{Im}\,B\neq 0$	$\omega^1=e^1+ie^2,\quad \omega^2=e^3+ie^4,$ $\omega^3=\frac{-1}{\mathfrak{Im}\,B}e^6+\frac{i}{1+\mathfrak{Re}\,B}(e^5-e^6)$	$\mathfrak{s}_8^{\alpha'}$ $\alpha'=\frac{\mathfrak{Im}\,B}{1+\mathfrak{Re}\,B}$
$\lvert A\rvert = \lvert B\rvert$	$B\neq\bar A$	$\mathfrak{Re}\,A\neq\mathfrak{Re}\,B$ $\mathfrak{Re}\,A,\mathfrak{Re}\,B\neq -1$ $(\mathfrak{Im}\,A)(\mathfrak{Im}\,B)\neq 0$	$\omega^1=e^1+ie^2,\quad \omega^2=e^3+ie^4,$ $\omega^3=\frac{1}{\mathfrak{Im}\,A-\mathfrak{Im}\,B}e^6-i\frac{1}{\mathfrak{Re}\,A-\mathfrak{Re}\,B}e^5$ $+i\frac{\mathfrak{Re}\,A+\mathfrak{Re}\,B}{(\mathfrak{Im}\,A-\mathfrak{Im}\,B)^2}\,e^6$	$\mathfrak{s}_6^{\alpha',\beta'}$ $\alpha'=\frac{\mathfrak{Im}\,A-\mathfrak{Im}\,B}{\mathfrak{Re}\,A-\mathfrak{Re}\,B}$ $\beta'=\frac{-(2+\mathfrak{Re}\,A+\mathfrak{Re}\,B)}{\mathfrak{Re}\,A-\mathfrak{Re}\,B}$
$\lvert A\rvert \neq \lvert B\rvert$			$\omega^1=e^1+ie^2,\quad \omega^2=e^3+ie^4,$ $\omega^3=\frac{\mathfrak{Re}\,A-\mathfrak{Re}\,B}{\lvert A\rvert^2-\lvert B\rvert^2}\,e^5+\frac{\mathfrak{Im}\,A-\mathfrak{Im}\,B}{\lvert A\rvert^2-\lvert B\rvert^2}\,e^6$ $-\,i\frac{\mathfrak{Im}\,A+\mathfrak{Im}\,B}{\lvert A\rvert^2-\lvert B\rvert^2}\,e^5+i\frac{\mathfrak{Re}\,A+\mathfrak{Re}\,B}{\lvert A\rvert^2-\lvert B\rvert^2}\,e^6$	$\mathfrak{s}_{10}^{\alpha,0}$ $\alpha=\frac{2+\mathfrak{Re}\,A+\mathfrak{Re}\,B}{\mathfrak{Im}\,A-\mathfrak{Im}\,B}$

TABLE 4. Lie algebras underlying equations (6) with $\varepsilon=1$, $\mathfrak{Im}\,A\neq\mathfrak{Im}\,B$ and $\Delta(A,B)=0$ (Lemma 1.13).

The structure equations (10) transform into

$$\begin{cases} d\nu^1 &= -\nu^{16}-\frac{\mathfrak{Im}\,A-\mathfrak{Im}\,B}{\Delta}\left(X\,\nu^{25}-Y\,\nu^{26}\right), \\ d\nu^2 &= -\nu^{26}+\frac{\mathfrak{Im}\,A-\mathfrak{Im}\,B}{\Delta}\left(X\,\nu^{15}-Y\,\nu^{16}\right), \\ d\nu^3 &= \nu^{36}+\nu^{45}, \\ d\nu^4 &= -\nu^{35}+\nu^{46}, \end{cases}$$

where

$$X=\frac{|A|^2-|B|^2}{\mathfrak{Im}\,A-\mathfrak{Im}\,B},\qquad Y=2\,\frac{\mathfrak{Im}\,A(1+\mathfrak{Re}\,B)+\mathfrak{Im}\,B(1+\mathfrak{Re}\,A)}{\mathfrak{Im}\,A-\mathfrak{Im}\,B}.$$

Now, the study is divided according to the vanishing of coefficients X and Y (see Table 5 for details). For the sake of clarity, we see what happens when $X=Y=0$: let us define $p=\frac{\mathfrak{Im}\,A+\mathfrak{Im}\,B}{\mathfrak{Im}\,A-\mathfrak{Im}\,B}$ and $q=\frac{\mathfrak{Re}\,A-\mathfrak{Re}\,B}{\mathfrak{Im}\,A-\mathfrak{Im}\,B}$, and consider the following system of equations in variables p and q:

$$\begin{cases} X=p(\mathfrak{Im}\,A-\mathfrak{Im}\,B)+q(\mathfrak{Re}\,A+\mathfrak{Re}\,B), \\ Y=p(2+\mathfrak{Re}\,A+\mathfrak{Re}\,B)-q(\mathfrak{Im}\,A-\mathfrak{Im}\,B). \end{cases}$$

Observe that the determinant associated to the system is $-\Delta$. Since $\Delta\neq 0$, if $X=Y=0$, the system has trivial solution and therefore $B=\bar A$ and, in particular, $\Delta=4(|A|^2+\mathfrak{Re}\,A)\neq 0$.

Finally, the relation between the bases $\{\nu^i\}_{i=1}^6$ and $\{e^i\}_{i=1}^6$ can be deduced from

$$\alpha \longrightarrow \beta \longrightarrow \nu \longrightarrow e. \qquad (\alpha \xrightarrow{\text{Table 5}} e)$$

∎

<table>
<tr><th colspan="3">$\mathfrak{Im}\,A \neq \mathfrak{Im}\,B,\ \ \Delta(A,B) \neq 0$</th><th>Real basis $\{e^1,\dots,e^6\}$</th><th>Lie algebra</th></tr>
<tr><td rowspan="2">$|A| = |B|$</td><td colspan="2">$Y = 0$</td><td>$\omega^1 = e^1 + ie^2,\quad \omega^2 = e^3 + ie^4,$
$\omega^3 = \frac{-\mathfrak{Im}\,A}{2(|A|^2+\mathfrak{Re}\,A)}\left(e^5 + \frac{1+\mathfrak{Re}\,A}{\mathfrak{Im}\,A}\,e^6\right)$
$- i\,\frac{\mathfrak{Im}\,A}{2(|A|^2+\mathfrak{Re}\,A)}\left(\frac{\mathfrak{Re}\,A}{\mathfrak{Im}\,A}\,e^5 - e^6\right)$</td><td>$\mathfrak{s}_9$</td></tr>
<tr><td colspan="2">$Y \neq 0$</td><td>$\omega^1 = e^3 + ie^4,\quad \omega^2 = e^1 - ie^2,$
$\omega^3 = \frac{\mathfrak{Im}\,A-\mathfrak{Im}\,B}{\Delta}\left(e^6 - \frac{2+\mathfrak{Re}\,A+\mathfrak{Re}\,B}{\mathfrak{Im}\,A-\mathfrak{Im}\,B}\,e^5\right)$
$+\frac{i(\mathfrak{Im}\,A-\mathfrak{Im}\,B)}{\Delta}\left(e^5 + \frac{\mathfrak{Re}\,A+\mathfrak{Re}\,B}{\mathfrak{Im}\,A-\mathfrak{Im}\,B}\,e^6\right)$</td><td>$\mathfrak{s}_{10}^{\alpha,0}$
$\alpha = \frac{-Y(\mathfrak{Im}\,A-\mathfrak{Im}\,B)}{\Delta}$</td></tr>
<tr><td rowspan="3">$|A| \neq |B|$</td><td rowspan="2">$Y = 0$</td><td>$\Delta = \pm(|A|^2 - |B|^2)$</td><td>$\omega^1 = e^3 + ie^4,\quad \omega^2 = e^1 - ie^2,$
$\omega^3 = \frac{\mathfrak{Im}\,A-\mathfrak{Im}\,B}{\Delta}\left(e^5 - \frac{2+\mathfrak{Re}\,A+\mathfrak{Re}\,B}{\mathfrak{Im}\,A-\mathfrak{Im}\,B}\,e^6\right)$
$+\frac{i(\mathfrak{Im}\,A-\mathfrak{Im}\,B)}{\Delta}\left(\frac{\mathfrak{Re}\,A+\mathfrak{Re}\,B}{\mathfrak{Im}\,A-\mathfrak{Im}\,B}\,e^5 + e^6\right)$</td><td>$\mathfrak{s}_{12}$</td></tr>
<tr><td>$\Delta \neq \pm(|A|^2 - |B|^2)$</td><td>$\omega^1 = e^3 + ie^4,\quad \omega^2 = e^1 - ie^2,$
$\omega^3 = \frac{\mathfrak{Im}\,A-\mathfrak{Im}\,B}{\Delta}\left(e^5 - \frac{2+\mathfrak{Re}\,A+\mathfrak{Re}\,B}{\mathfrak{Im}\,A-\mathfrak{Im}\,B}\,e^6\right)$
$+\frac{i(\mathfrak{Im}\,A-\mathfrak{Im}\,B)}{\Delta}\left(\frac{\mathfrak{Re}\,A+\mathfrak{Re}\,B}{\mathfrak{Im}\,A-\mathfrak{Im}\,B}\,e^5 + e^6\right)$</td><td>$\mathfrak{s}_{11}^{\alpha'}$
$\alpha' = \frac{-X(\mathfrak{Im}\,A-\mathfrak{Im}\,B)}{\Delta}$</td></tr>
<tr><td colspan="2">$Y \neq 0$</td><td>$\omega^1 = e^1 - ie^2,\quad \omega^2 = e^3 + ie^4,$
$\omega^3 = \frac{X(2+\mathfrak{Re}\,A+\mathfrak{Re}\,B)-Y(\mathfrak{Im}\,A-\mathfrak{Im}\,B)}{X\,\Delta}\,e^5$
$+\frac{2+\mathfrak{Re}\,A+\mathfrak{Re}\,B}{Y(\mathfrak{Im}\,A-\mathfrak{Im}\,B)}\,e^6 - \frac{i}{Y}\,e^6$
$-i\,\frac{X(\mathfrak{Im}\,A-\mathfrak{Im}\,B)+Y(\mathfrak{Re}\,A+\mathfrak{Re}\,B)}{X\,\Delta}\,e^5$</td><td>$\mathfrak{s}_{10}^{\alpha,\beta}$
$\alpha = \frac{Y}{X}$
$\beta = \frac{\Delta}{Y(\mathfrak{Im}\,A-\mathfrak{Im}\,B)}$</td></tr>
</table>

TABLE 5. Lie algebras underlying equations (6) with $\varepsilon = 1$, $\mathfrak{Im}\,A \neq \mathfrak{Im}\,B$ and $\Delta(A,B) \neq 0$ (Lemma 1.14).

The previous lemmata provide the following

Proposition 1.15. *The unimodular solvable* 6*-dimensional Lie algebras underlying equations* (6) *with* $\varepsilon = 1$ *are* $\mathfrak{s}_2$, $\mathfrak{s}_3$, $\mathfrak{s}_4$, $\mathfrak{s}_5^{\alpha}$, $\mathfrak{s}_6^{\alpha,\beta}$, $\mathfrak{s}_7^{\alpha}$, $\mathfrak{s}_8^{\alpha}$, $\mathfrak{s}_9$, $\mathfrak{s}_{10}^{\alpha,\beta}$, $\mathfrak{s}_{11}^{\alpha}$, $\mathfrak{s}_{12}$.

As a consequence of the previous propositions, we prove the main result of this section:

Proof of Theorem 1.7. The "only if" part of the theorem follows from Propositions 1.9, 1.10 and 1.15.

For the proof of the "if" part, we must show that all the Lie algebras in the list admit a splitting-type complex structure. This is clear for the Lie algebras $\mathfrak{s}_1$, $\mathfrak{s}_2$, $\mathfrak{s}_3$, $\mathfrak{s}_4$, $\mathfrak{s}_9$ and $\mathfrak{s}_{12}$ from Proposition 1.9 and Tables 1, 2 and 3. The remaining Lie algebras in the list depend on parameters, so we will show next particular appropriate values of A and B that define a complex structure of splitting type on each one of the Lie algebras $\mathfrak{s}_5^\alpha$, $\mathfrak{s}_6^{\alpha,\beta}$, $\mathfrak{s}_7^\alpha$, $\mathfrak{s}_8^\alpha$, $\mathfrak{s}_{10}^{\alpha,\beta}$ and $\mathfrak{s}_{11}^\alpha$ in the list.

For the Lie algebra $\mathfrak{s}_5^\alpha$, $\alpha > 0$, we consider A and B given by $A = B = -1 + i\,\alpha$. These values of the parameters A and B lie in Table 3, since $\mathfrak{Im}\,A = \mathfrak{Im}\,B = \alpha \neq 0$ and $\mathfrak{Re}\,A = \mathfrak{Re}\,B = -1$. Hence, the (1,0)-forms $\omega^1 = -\frac{\mathfrak{Im}\,A}{|\mathfrak{Im}\,A|}e^3 + ie^4 = -e^3 + ie^4$, $\omega^2 = e^1 + ie^2$, $\omega^3 = \frac{1}{2}\,e^5 + ie^6$ define a splitting-type complex structure on $\mathfrak{s}_5^\alpha$ according to Table 3.

For the other Lie algebras the argument is similar. We show below particular appropriate values of A, B and the table where the corresponding basis of (1,0)-forms is given:

- For $\mathfrak{s}_6^{\alpha,\beta}$, $\alpha > 0$ with $\alpha \neq 1$, $0 < \beta < 1$, it suffices to take $A = \frac{-2}{1+\beta} + i\frac{1-\alpha^2}{\alpha(1+\beta)}$ and $B = i\frac{1+\alpha^2}{\alpha(1+\beta)}$ in Table 4;
- For $\mathfrak{s}_6^{1,\beta}$, $0 < \beta < 1$, we can take $A = -\frac{1+\beta}{1+\beta^2} + i\frac{1-\beta}{1+\beta^2}$ and $B = -\frac{1-\beta}{1+\beta^2} + i\frac{1+\beta}{1+\beta^2}$ in Table 4;
- For $\mathfrak{s}_7^\alpha$, $0 < \alpha \leq 1$, we take $A = -B = \alpha$ in Table 2;
- For $\mathfrak{s}_8^\alpha$, $\alpha > 0$, we take $A = \frac{1}{1+\alpha^2}(1 - \alpha^2 + 2i\alpha)$ and $B = -1$ in Table 4;
- For $\mathfrak{s}_{10}^{\alpha,\beta}$, $\alpha \neq 0$, $\beta \in \mathbb{R}$, we can take $A = -1 - \bar{B}$ with $B = \frac{1}{2}(\alpha\beta - 1 + i\beta)$ in Table 1;
- Finally, for the Lie algebra $\mathfrak{s}_{11}^\alpha$, $\alpha \in (0,1)$, we take $A = -1 - B$ with $B = -\frac{1}{2}(1+\alpha)$ in Table 1. □

Remark 1.16. In view of Remark 1.5, a 6-dimensional unimodular (non-nilpotent) solvable Lie algebra admits a complex structure of splitting type with a non-zero closed $(3,0)$-form if and only if $B = -\varepsilon$ in the structure equations (6). Looking at the tables above, it is easy to check that this condition is satisfied if and only if the Lie algebra is isomorphic to $\mathfrak{s}_4$, $\mathfrak{s}_7^1$, $\mathfrak{s}_8^\alpha$ or $\mathfrak{s}_{12}$, which is in accord with [11, Theorem 2.8] (notice that these Lie algebras correspond, respectively, to the Lie algebras labeled as $\mathfrak{g}_1$, $\mathfrak{g}_2^\alpha$ and $\mathfrak{g}_8$ in [11]).

On the other hand, the Lie algebras obtained in Theorem 1.7 appear with different notations in previous papers. Next, we make explicit the correspondence with [5, 35]:

$$\begin{array}{llll} \mathfrak{s}_1 = \mathfrak{g}_{4,9}^0 \oplus \mathbb{R}^2, & \mathfrak{s}_2 = \mathfrak{g}_{3,5}^0 \oplus \mathbb{R}^3, & \mathfrak{s}_3 = \mathfrak{g}_{3,5}^0 \oplus \mathfrak{g}_{3,5}^0, & \mathfrak{s}_4 = \mathfrak{g}_{5,7}^{-1,-1,1} \oplus \mathbb{R}, \\ \mathfrak{s}_5^\alpha = \mathfrak{g}_{5,13}^{1,-1,\alpha} \oplus \mathbb{R}, & \mathfrak{s}_6^{\alpha,\beta} = \mathfrak{g}_{5,17}^{\alpha,-\alpha,\beta} \oplus \mathbb{R}, & \mathfrak{s}_7^\alpha = \mathfrak{g}_{5,17}^{0,0,\alpha} \oplus \mathbb{R}, & \mathfrak{s}_8^\alpha = \mathfrak{g}_{5,17}^{\alpha,-\alpha,1} \oplus \mathbb{R}, \\ \mathfrak{s}_9 = N_{6,13}^{0,-1,0,-1}, & \mathfrak{s}_{10}^{\alpha,\beta} = N_{6,15}^{-1,\alpha,\beta,-\beta}, & \mathfrak{s}_{11}^\alpha = N_{6,18}^{0,\alpha,-1}, & \mathfrak{s}_{12} = N_{6,18}^{0,-1,-1}. \end{array}$$

It turns out that the only Lie algebra that is completely solvable is $\mathfrak{s}_4$.

Remark 1.17. As regards solvmanifolds of splitting type, we notice that the condition (5) in Definition 1.1 is satisfied by the Kodaira-Thurston manifold; see [7, 9, 31, 32] for general results on the Dolbeault cohomology of nilmanifolds. Therefore, we need to study the existence of lattices in the connected and simply-connected solvable Lie groups G_k corresponding to the Lie algebras $\mathfrak{s}_k$ in Theorem 1.7. The Lie groups G_1, G_2 and G_3 admit lattices (see [5, Table 8]). Also G_4 admits lattices by [11, 27]. Moreover, by [8, page 13] we have:

- $G_6^{\alpha,\beta}$ admits lattices if and only if $\beta = \frac{r_1}{r_2} \in \mathbb{Q}$ and α satisfies $\exp(2\pi\alpha^{-1}r_2) + \exp(-2\pi\alpha^{-1}r_2) \in \mathbb{Z}$, that is, α is the form $\alpha_n := \frac{2\pi r_2}{\log\left(\frac{1}{2}\left(n\pm\sqrt{n^2-4}\right)\right)}$ with $n \in \mathbb{N}$;
- G_7^α admits lattices if and only if $\alpha \in \mathbb{Q}$;
- G_8^α admits lattices if and only if $\exp(2\pi\alpha^{-1}) + \exp(-2\pi p\alpha^{-1}) \in \mathbb{Z}$, that is, for any α of the form $\alpha_n := \frac{2\pi}{\log\left(\frac{1}{2}\left(n\pm\sqrt{n^2-4}\right)\right)}$ with $n \in \mathbb{N}$.

In Proposition 1.18 below we show the existence of lattices for a countable family of G_5^α. Note that the results on the existence of lattices are consistent with [36, Proposition 8.7], where it is shown that only countably many non-isomorphic simply-connected solvable Lie groups admit a lattice. Therefore, one cannot expect a lattice to exist on G_5^α, $G_6^{\alpha,\beta}$, G_7^α or G_8^α for every value of α, β, and so, in this sense, our proposition below completes the cases when the Lie algebra is decomposable.

The indecomposable case is more difficult to treat, but in Section 3 we will provide explicit lattices on the Lie group associated to $\mathfrak{s}_{12}$ (which is the Lie algebra underlying the Nakamura manifold [26], see also [37]) with interesting properties with respect to the $\partial\bar{\partial}$-Lemma.

Proposition 1.18. *There is a countable family $\{\alpha_{s,n}\} \subset \mathbb{R}^+$ such that the connected and simply-connected Lie group $G_5^{\alpha_{s,n}}$ admits a lattice.*

Proof. The Lie algebra of G_5^α, $\alpha > 0$, can be written as $\mathfrak{s}_5^\alpha = \mathfrak{g}_{5,13}^{1,-1,\alpha} \oplus \mathbb{R}$ with $\mathfrak{g}_{5,13}^{1,-1,\alpha} = \mathbb{R} \ltimes_{\mathrm{ad}_{e_5}} \mathbb{R}^4$. Since the simply-connected Lie group H_α corresponding to $\mathfrak{g}_{5,13}^{1,-1,\alpha}$ is almost-nilpotent [5], it admits a lattice if and only if there exists $\tau \neq 0$ such that the matrix $\exp(\tau\,\mathrm{ad}_{e_5})$ belongs to the conjugation class of an integer matrix. We have (see [5, p. 41]) that $\exp(t\,\mathrm{ad}_{e_5})$ is given by

$$\exp(t\,\mathrm{ad}_{e_5}) = \begin{pmatrix} e^{-t} & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ 0 & 0 & e^t\cos\alpha t & -e^t\sin\alpha t \\ 0 & 0 & e^t\sin\alpha t & e^t\cos\alpha t \end{pmatrix}. \tag{12}$$

Let $\tau \neq 0$ be such that $\sin\alpha\tau = 0$, that is, $\tau = \frac{s\pi}{\alpha}$ with $0 \neq s \in \mathbb{Z}$. In this case the matrix (12) is diagonal and its characteristic polynomial is

$$p(\lambda) = \left(\lambda^2 - (e^{-\tau} + (-1)^s e^\tau)\lambda + (-1)^s\right)^2. \tag{13}$$

Now, if $\exp(\tau\,\mathrm{ad}_{e_5})$ lies in the conjugation class of an integer matrix, then $p(\lambda) \in \mathbb{Z}[\lambda]$, that is, $e^{-\tau} + (-1)^s e^\tau = n$, for some $n \in \mathbb{Z}$. Solving this equation, we get

$$\tau_{s,n} = -\log\left(\frac{n + \sqrt{n^2 - 4(-1)^s}}{2}\right), \qquad \alpha_{s,n} = -\frac{s\pi}{\log\left(\frac{n+\sqrt{n^2-4(-1)^s}}{2}\right)}, \text{ for } n \geq 3.$$

Substituting these values in (13), we get $p(\lambda) = \left(\lambda^2 - n\lambda + (-1)^s\right)^2 \in \mathbb{Z}[\lambda]$, which is also the characteristic polynomial of the integer matrix

$$B_s = \begin{pmatrix} 0 & (-1)^{s+1} & 0 & 0 \\ 1 & n & 0 & 0 \\ 0 & 0 & 0 & (-1)^{s+1} \\ 0 & 0 & 1 & n \end{pmatrix} \in \mathrm{Aut}(4, \mathbb{Z}).$$

In addition, it turns out that $Q\exp(\tau_{s,n}\,\mathrm{ad}_{e_5})\,Q^{-1} = B_s$, where

$$Q = \begin{pmatrix} 0 & \beta_+ & 0 & \beta_- \\ 0 & 1 & 0 & 1 \\ \beta_+ & 0 & \beta_- & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \qquad \beta_\pm = \frac{1}{2}\left(-n \pm \sqrt{n^2 - 4(-1)^s}\right),$$

concluding the proof. ∎

2. Hermitian geometry of splitting-type complex structures

In this section we study the existence of special Hermitian metrics on solvmanifolds endowed with a complex structure of splitting type. From now on, F denotes the fundamental $(1,1)$-form associated to a Hermitian metric g, and n is the complex dimension of the complex manifold.

It is well-known that the *Kähler condition* "$dF = 0$" can be weakened in the "*geometry with torsion*" direction, and the main classes of Hermitian structures that arise are:

- *Hermitian-symplectic* (or *holomorphic-tamed*), that is, F is the $(1,1)$-component of a d-closed 2-form;
- *SKT* (*strong Kähler with torsion* or *pluri-closed*), that is, $\partial\bar{\partial}F = 0$;
- *k-Gauduchon* [15], that is, $\partial\bar{\partial}F^k \wedge F^{n-k-1} = 0$, where $k = 1, \ldots, n-2$.

The following implications are clear from the definitions:

$$\text{Kähler} \quad \Rightarrow \quad \text{Hermitian-symplectic} \quad \Rightarrow \quad \text{SKT} \quad \Rightarrow \quad \text{1-Gauduchon}.$$

So far, no example of compact complex non-Kähler manifold admitting Hermitian-symplectic structure is known, see [23, page 678], [34, Question 1.7].

Other interesting and well-known classes of Hermitian metrics on compact complex manifolds are:

- *balanced* (in the sense of Michelsohn [25]), that is, $dF^{n-1} = 0$;
- *strongly Gauduchon* [29], that is, F^{n-1} is the $(n-1,n-1)$-component of a d-closed $(2n-2)$-form; equivalently, the $(n,n-1)$-form ∂F^{n-1} is $\bar{\partial}$-exact;
- *Gauduchon* [16], that is, $\partial\bar{\partial}F^{n-1} = 0$.

It is clear that

$$\text{Kähler} \quad \Rightarrow \quad \text{balanced} \quad \Rightarrow \quad \text{strongly Gauduchon} \quad \Rightarrow \quad \text{Gauduchon}.$$

We recall also that any conformal class of Hermitian structures admits a Gauduchon representative by the foundational theorem by Gauduchon [16, Théorème 1]. A recent conjecture of Fino and Vezzoni [13] states that in the compact non-Kähler case it is never possible to find an SKT metric and also a balanced one, and they prove the conjecture for nilmanifolds [14] and for 6-dimensional solvmanifolds having holomorphically trivial canonical bundle [13]. On the other hand, Popovici [30] proposes, for $\partial\bar{\partial}$-manifolds, a conjecture relating their balanced and Gauduchon cones, and he observes that, if proved to hold, the conjecture would imply the existence of a balanced structure on any $\partial\bar{\partial}$-manifold. Recall that a $\partial\bar{\partial}$-manifold is a compact complex manifold X satisfying the $\partial\bar{\partial}$-Lemma, that is, if for any d-closed form γ of pure type on X, the following exactness properties are equivalent:

$$\gamma \text{ is } d\text{-exact} \iff \gamma \text{ is } \partial\text{-exact} \iff \gamma \text{ is } \bar{\partial}\text{-exact} \iff \gamma \text{ is } \partial\bar{\partial}\text{-exact}.$$

We have the following general result.

Proposition 2.1. *Let $X = G/\Gamma$ be a solvmanifold endowed with a complex structure of splitting type, i.e., $G = \mathbb{C} \ltimes_{\varphi} N$, where N is nilpotent. Then, X admits a balanced (respectively, strongly Gauduchon) Hermitian structure if and only if N admits an invariant balanced (respectively, strongly Gauduchon) Hermitian structure.*

Proof. First of all, by the well-known symmetrization process, X admits a balanced (respectively, strongly Gauduchon) Hermitian structure if and only if the Lie group G admits an invariant balanced (respectively, strongly Gauduchon) Hermitian structure. Let n be the complex dimension of X, and denote by $\{\omega^n\}$ a co-frame of $(1,0)$-forms for the factor $\mathbb{C}$ in

G. First, notice that, if we have an invariant Hermitian structure F_G on G, (respectively, an invariant Hermitian structure F_N on N) then we can construct an invariant Hermitian structure F_N on N (respectively, an invariant Hermitian structure F_G on G) such that

$$F_G^{n-1} \;=\; F_N^{n-1} + F_N^{n-2} \wedge \omega^{n\bar{n}} \;,$$

with abuse of notations. Indeed, as a vector space, the Lie algebra $\mathfrak{g}$ of G splits as $\mathfrak{g} = \mathfrak{n} \oplus \mathbb{R}^2$, where $\mathfrak{n}$ is the Lie algebra of N. Invariant structures on G (respectively, on N) are identified with linear structures on $\mathfrak{g}$ (respectively, on $\mathfrak{n}$). If we start from a Hermitian structure F_N on N, then we can take $F_G := \sqrt[n-1]{F_N^{n-1} + F_N^{n-2} \wedge \omega^{n\bar{n}}}$, which is a Hermitian structure on G. On the other hand, if we start from a Hermitian structure F_G on G, then it induces a Hermitian structure F_N on N and the Hermitian structure $\omega^{n\bar{n}}$ on $\mathbb{R}^2$, up to multiplicative positive constants, such that $F_G = F_N + \omega^{n\bar{n}}$, which yields the above identity.

Since $d\omega^n = 0$, we have

$$dF_G^{n-1} \;=\; dF_N^{n-1} + dF_N^{n-2} \wedge \omega^{n\bar{n}} \;=\; dF_N^{n-1} + d_N F_N^{n-2} \wedge \omega^{n\bar{n}} \;, \tag{14}$$

where d_N denotes the differential over N.

We notice also that $dF_N^{n-1} = 0$ by unimodularity. Otherwise, if $dF_N^{n-1} \neq 0$, then either $d\left(F_N^{n-1} \wedge \omega^n\right)$ or $d\left(F_N^{n-1} \wedge \omega^{\bar{n}}\right)$ would be non-trivial d-exact $2n$-forms.

Then, (14) reduces to

$$dF_G^{n-1} \;=\; d_N F_N^{n-2} \wedge \omega^{n\bar{n}} \;.$$

It follows that $dF_G^{n-1} = 0$ if and only if $d_N F_N^{n-2} = 0$. Analogously, it follows that ∂F_G^{n-1} is $\bar{\partial}$-exact if and only if ∂F_N^{n-2} is $\bar{\partial}$-exact. □

In [10] it is studied the existence of Hermitian-symplectic structures on complex solvmanifolds (see [10, Theorem 1.1] for case when G is not of type (I) and [10, Theorem 1.2] for other cases). We recall that a Lie group G is said to be of type (I) if for any $X \in \mathfrak{g}$, all the eigenvalues of the adjoint operator ad_X are pure imaginary. Some of the Lie algebras in the list of Theorem 1.7 are of type (I) but other not, however for all of them (except $\mathfrak{s}_1$) the Lie group is of the form $G = \mathbb{C} \ltimes_\varphi \mathbb{C}^{n-1}$, so we give in the following result an alternative direct proof about existence of special Hermitian metrics in this concrete case.

Proposition 2.2. *Let $X = G/\Gamma$ be a solvmanifold endowed with a complex structure of splitting type, such that $G = \mathbb{C} \ltimes_\varphi \mathbb{C}^{n-1}$. Then, for X it is equivalent: to admit SKT structures; to admit Hermitian-symplectic structures; to admit Kähler structures.*

Proof. By the symmetrization process, X admits SKT, Hermitian-symplectic or Kähler structure if and only if the Lie group G admits an invariant SKT, invariant Hermitian-symplectic or invariant Kähler structure. Fix a co-frame $\{\omega^1, \ldots, \omega^{n-1}\}$ of $(1,0)$-forms on $\mathbb{C}^{n-1}$ and a co-frame $\{\omega^n\}$ of $(1,0)$-forms on $\mathbb{C}$, such that the complex structure equations are of the form

$$\begin{cases} d\omega^j &=& A^j\,\omega^{nj} + B^j\,\omega^{\bar{n}j}, \qquad j \in \{1, \ldots, n-1\}, \\ d\omega^n &=& 0, \end{cases}$$

for suitable $A^j, B^j \in \mathbb{C}$. Notice that the Jacobi identity is satisfied for any value of the structure constants, while the unimodularity condition corresponds to the requirement

$$\sum_{j=1}^{n-1} (A^j + \bar{B}^j) \;=\; 0 \;.$$

Consider the general invariant metric on G given by

$$F \;:=\; \sum_{h,k=1}^{n} \alpha_{h\bar{k}}\,\omega^{h\bar{k}}$$

where $(\alpha_{h\bar{k}})_{h,k}$ is a Hermitian matrix with entries in $\mathbb{C}$. By noticing that

$$\partial\bar{\partial}\omega^{h\bar{k}} = (B^h + \bar{A}^k)(A^h + \bar{B}^k)\omega^{n\bar{n}h\bar{k}}, \qquad d\omega^{h\bar{k}} = (A^h + \bar{B}^k)\omega^{nh\bar{k}} + (\bar{A}^k + B^h)\omega^{\bar{n}h\bar{k}},$$

we get

$$\partial\bar{\partial}F \;=\; \sum_{h,k=1}^{n-1} \alpha_{h\bar{k}}(B^h + \bar{A}^k)(A^h + \bar{B}^k)\omega^{n\bar{n}h\bar{k}} \;.$$

So, if F is SKT, then every coefficients must vanish. In particular, for any $j \in \{1,\dots,n-1\}$,

$$|B^j + \bar{A}^j|^2 = 0 \;,$$

since $\alpha_{j\bar{j}} \neq 0$. But this implies that the diagonal Hermitian structure $\tilde{F} := \frac{i}{2}\sum_{h=1}^{n}\omega^{h\bar{h}}$ is Kähler, since $2d\tilde{F} = i\sum_{h=1}^{n-1}\Big((A^h + \bar{B}^h)\omega^{nh\bar{h}} + (\bar{A}^h + B^h)\omega^{\bar{n}h\bar{h}}\Big) = 0$. □

2.1. **Hermitian structures in dimension 6.** Next we consider the case when the (real) dimension of X is 6. As we reminded in the proofs of Propositions 2.1 and 2.2, the existence of Kähler, Hermitian-symplectic, SKT, balanced and strongly Gauduchon structures is reduced to their existence at the Lie algebra level, so we will study the spaces of such Hermitian structures on each $\mathfrak{s}_k$, for $1 \le k \le 12$. We also study the existence of 1-Gauduchon structures on the Lie algebras $\mathfrak{s}_k$, although as it is pointed out in [12], the symmetrization process does not hold for this kind of Hermitian structures on solvmanifolds, and so our study covers only the space of *invariant* 1-Gauduchon structures. The existence results are summarized in Table 6.

A generic Hermitian structure on $\mathfrak{s}_k$ is given, with respect to any coframe $\{\omega^1,\omega^2,\omega^3\}$ of $(1,0)$-forms, by

$$\begin{pmatrix} ir^2 & u & z \\ -\bar{u} & is^2 & v \\ -\bar{z} & -\bar{v} & it^2 \end{pmatrix}$$

or equivalently, by the expression

$$2F \;=\; ir^2\omega^{1\bar{1}} + is^2\omega^{2\bar{2}} + it^2\omega^{3\bar{3}} + u\omega^{1\bar{2}} - \bar{u}\omega^{2\bar{1}} + v\omega^{2\bar{3}} - \bar{v}\omega^{3\bar{2}} + z\omega^{1\bar{3}} - \bar{z}\omega^{3\bar{1}}, \tag{15}$$

where $r,s,t \in \mathbb{R}\setminus\{0\}$ and $u,v,z \in \mathbb{C}$ satisfy the conditions that ensure that F is positive-definite: $r^2s^2 > |u|^2$, $s^2t^2 > |v|^2$, $r^2t^2 > |z|^2$ and $r^2s^2t^2 + 2\,\mathfrak{Re}\,(i\bar{u}\bar{v}z) > t^2|u|^2 + r^2|v|^2 + s^2|z|^2$.

Let us consider first the Lie algebra $\mathfrak{s}_1$, which corresponds to the structure equations (7) for $\varepsilon = 1$, and for which we can apply Proposition 2.1 because the Lie group G is of the form $\mathbb{C} \ltimes_\varphi KT$. By [29, Observation 4.4], every strongly Gauduchon compact complex surface is Kähler, so in particular the Kodaira-Thurston manifold does not admit strongly Gauduchon structures. Hence, by Proposition 2.1, we conclude that $\mathfrak{s}_1$ does not admit either strongly Gauduchon or balanced structures. A direct calculation shows that it does not admit Hermitian-symplectic structures. However, there always exist SKT and 1-Gauduchon structures, since for a metric F given by (15) we have

$$2\,\partial\bar{\partial}F = u\,\omega^{13\bar{2}\bar{3}} - \bar{u}\,\omega^{23\bar{1}\bar{3}}, \qquad 2\,\partial\bar{\partial}F \wedge F = |u|^2\,\omega^{123\bar{1}\bar{2}\bar{3}}.$$

More precisely, F is SKT if and only if F is 1-Gauduchon, if and only if $u = 0$.

The remaining Lie algebras $\mathfrak{s}_k$, $2 \leq k \leq 12$, correspond to the complex structure equations (6), and we can apply Proposition 2.2 because the Lie group G is of the form $\mathbb{C} \ltimes_\varphi \mathbb{C}^2$. As a matter of notation, let us denote such complex structures simply as $J = (A, B, \varepsilon) \in \mathbb{C}^2 \times \{0, 1\}$. Given a generic Hermitian structure (15), we first note that one can always normalize the metric coefficients r and s, i.e. we can suppose $r = s = 1$. Therefore, we will identify the Hermitian structures simply by a tuple $F = (t^2, u, v, z) \in \mathbb{R}^+ \times \mathbb{C}^3$, where $1 > |u|^2$, $t^2 > |v|^2$, $t^2 > |z|^2$ and $t^2 + 2\,\mathfrak{Re}\,(i\bar{u}\bar{v}z) > t^2|u|^2 + |v|^2 + |z|^2$, in order for F to be positive-definite.

Now, by Proposition 2.2, there exists a Kähler structure if and only if there is a Hermitian-symplectic structure, if and only if there exists an SKT structure. A direct calculation from (6) shows that the existence of one of these types of structures implies

$$A + \bar{B} \;=\; 0,$$

that is, the complex structure must be of the form $J = (A, -\bar{A}, \varepsilon)$, where $A \in \mathbb{C}$ and $\varepsilon \in \{0, 1\}$. According to the classification given in Section 1.2, the Lie algebras admitting such a complex structure are $\mathfrak{s}_2$, $\mathfrak{s}_3$, $\mathfrak{s}_7^\alpha$. Indeed,

- if $\varepsilon = 0$ then from Table 1 we get $\mathfrak{s}_2$ (notice that we can take $A = 1$ in this case);
- if $\varepsilon = 1$ and $A \in \mathbb{R}$, then by Table 2 the possibilities are $\mathfrak{s}_2$, $\mathfrak{s}_7^\alpha$;
- if $\varepsilon = 1$ and $\mathfrak{Im}\,A \neq 0$, then from Table 3 we get $\mathfrak{s}_3$.

Next we give a detailed description of the spaces of Kähler structures.

Proposition 2.3. *Let $\mathfrak{g}$ be a 6-dimensional solvable Lie algebra with a complex structure J of splitting type. Then, $\mathfrak{g}$ admits a Kähler structure if and only if $\mathfrak{g}$ is isomorphic to $\mathfrak{s}_2$, $\mathfrak{s}_3$ or $\mathfrak{s}_7^\alpha$, and the Kähler structures (J, F) are the following:*

(K.i) $(\mathfrak{s}_2, J, F)$, *where* $J = (1, -1, 0)$ *and* $F = (t^2, 0, v, 0)$.

(K.ii) $(\mathfrak{s}_3, J, F)$, *where* $J = (A, -\bar{A}, 1)$, $\mathfrak{Im}\,A \neq 0$, *and* $F = (t^2, 0, 0, 0)$.

(K.iii) $(\mathfrak{s}_7^\alpha, J, F)$, *where* $J = (A, -A, 1)$, $A \in \mathbb{R} \setminus \{0, -1\}$, *and* $F = (t^2, 0, 0, 0)$. *(Notice that* $\alpha = |A|$ *or* $\alpha = |\frac{1}{A}|$.*)*

(K.iv) $(\mathfrak{s}_7^1, J, F)$, *where* $J = (-1, 1, 1)$ *and* $F = (t^2, u, 0, 0)$.

Proof. A direct computation shows that

$$2\,\bar{\partial}F \;=\; (\bar{A} + \varepsilon)\,(u\,\omega^{1\bar{2}\bar{3}} + \bar{u}\,\omega^{2\bar{1}\bar{3}}) - \varepsilon\bar{v}\,\omega^{3\bar{2}\bar{3}} + \bar{A}\bar{z}\,\omega^{3\bar{1}\bar{3}}\,. \tag{16}$$

Hence the conditions to be satisfied for F being Kähler are

$$u(A + \varepsilon) \;=\; 0\,, \qquad \varepsilon v \;=\; 0\,, \qquad Az \;=\; 0\,.$$

If $\varepsilon = 0$, then we may assume that $A = 1$ (see the proof of Proposition 1.10 for details) and therefore $u = z = 0$. The Kähler structures are then given by $(t^2, 0, v, 0)$ and we obtain case (K.i).

If $\varepsilon = 1$, then $v = 0$ and several cases appear:

- If $A = 0$, it is equivalent to the previous case (K.i).
- If $A = -1$, then $z = 0$. So, $J = (-1, 1, 1)$ and $F = (t^2, u, 0, 0)$, which corresponds to (K.iv).
- If $A \neq 0$ and $A \neq -1$, then $u = v = z = 0$. Depending on the values of A (see Tables 2 and 3), we get the remaining cases (K.ii) or (K.iii). □

Remark 2.4. In [17, Example 4], it is shown that the complex structures corresponding to cases (K.i) and (K.iv) in Proposition 2.3 admit Kähler metrics. In the recent paper [2] it is shown that $\mathfrak{s}_3$ admits a Kähler structure and, moreover, solvmanifolds constructed from the Lie algebra $\mathfrak{s}_7^1$ give rise to new supersymmetric vacua. Notice that $\mathfrak{s}_2, \mathfrak{s}_3$ and $\mathfrak{s}_7^\alpha$ are the only (non abelian) solvable Lie algebras in six dimensions admitting Ricci flat metrics (see [2] and the references therein). By Proposition 2.3 all these Lie algebras admit a Kähler structure, although by [11] only $\mathfrak{s}_7^1$ with $J = (1, -1, 1)$ admits a Calabi-Yau structure.

In the following proposition we compare the spaces of Hermitian-symplectic, SKT and 1-Gauduchon structures with the space of Kähler structures.

Proposition 2.5. *Let $\mathfrak{g}$ be a 6-dimensional solvable Lie algebra with a complex structure J of splitting type that admits Kähler structures. Any Hermitian structure (J, F) on $\mathfrak{g}$ is 1-Gauduchon if and only if it is Hermitian-symplectic, if and only if it is SKT. Moreover, any SKT structure (J, F) on $\mathfrak{g}$ is one of the following:*

(SKT.i) $(\mathfrak{s}_2, J, F)$, *where* $J = (1, -1, 0)$ *and* $F = (t^2, 0, v, z)$.

(SKT.ii) $(\mathfrak{s}_3, J, F)$, *where* $J = (A, -\bar{A}, 1)$, $\mathfrak{Im}\, A \neq 0$, *and* $F = (t^2, 0, v, z)$.

(SKT.iii) $(\mathfrak{s}_7^\alpha, J, F)$, *where* $J = (A, -A, 1)$, $A \in \mathbb{R} \setminus \{0, -1\}$, *and* $F = (t^2, 0, v, z)$.

(SKT.iv) $(\mathfrak{s}_7^1, J, F)$, *where* $J = (-1, 1, 1)$ *and* $F = (t^2, u, v, z)$.

Proof. Using (16), we have

$$2\,\partial\bar{\partial}F = |A+\varepsilon|^2(u\,\omega^{13\bar{2}\bar{3}} - \bar{u}\,\omega^{23\bar{1}\bar{3}}), \qquad 2\,\partial\bar{\partial}F \wedge F = |u|^2\,|A+\varepsilon|^2\,\omega^{123\bar{1}\bar{2}\bar{3}}.$$

Therefore, the SKT condition is equivalent to the 1-Gauduchon condition, and they are equivalent to $u(A+\varepsilon) = 0$.

On the other hand, the structure F is Hermitian-symplectic if

$$\bar{\partial}F = \partial\beta, \quad \bar{\partial}\beta = 0, \quad \text{where} \quad \beta \in \mathfrak{g}^{0,2}\ .$$

Since $\partial\beta \in \langle A\,\omega^{3\bar{1}\bar{3}},\, \varepsilon\,\omega^{3\bar{2}\bar{3}}\rangle$, it follows from (16) that F is Hermitian-symplectic if and only if there exist $\lambda,\,\mu \in \mathbb{C}$ satisfying

$$u(A+\varepsilon) = 0\,, \quad v\,\varepsilon = \lambda\,\varepsilon\,, \qquad z\,A = \mu\,\bar{A}.$$

It is always possible to find λ, μ satisfying the last two equations. The first one is precisely the SKT condition.

Now, depending on the vanishing of the metric coefficient u, the possibilities for a Hermitian structure (J, F) to satisfy the SKT condition are:

- $u \neq 0$. Then, $\varepsilon = 1$ and $A = -1$, which corresponds to the case (SKT.iv).
- $u = 0$. If $\varepsilon = 0$, then we can suppose $A = 1$, which leads to the case (SKT.i). The remaining cases (SKT.ii) and (SKT.iii) are obtained when $\varepsilon = 1$.

□

Remark 2.6. A complex structure J as above admits SKT structures if and only if it admits Kähler ones, however, for any fixed J, there exist SKT structures which are not Kähler. Indeed, by Propositions 2.3 and 2.5, any SKT structure with metric coefficient $z \neq 0$ is not Kähler. Similarly, there exist Hermitian-symplectic structures and 1-Gauduchon structures which are not Kähler.

Now, with respect to balanced and strongly Gauduchon Hermitian structures, we can apply Proposition 2.1 for $N = \mathbb{C}^2$ and so any complex structure corresponding to the equations (6) admits balanced structures. Indeed, for any value of the tuple $(A, B, \varepsilon) \in \mathbb{C}^2 \times \{0, 1\}$, the Hermitian structures given by $(t^2, u, 0, 0)$ are balanced. Notice that there exist strongly Gauduchon Hermitian structures that are not balanced, for instance, consider a complex structure $J = (A, B, 1)$, i.e. with $\varepsilon = 1$, and a Hermitian structure F given by $(t^2, 0, v, z)$ with $v \neq 0$.

We summarize all the results about Hermitian structures in Table 6. Here, the symbol "✓" means that the corresponding kind of Hermitian metrics exists *for any complex structure of splitting type* on the Lie algebra (see Tables 1–5), whereas "−" means that none of the complex structures admits such kind of metrics. Here "H-symplectic" means Hermitian-symplectic and "sG" refers to strongly Gauduchon metrics.

	Kähler	H-symplectic	SKT	invariant 1-G	balanced	sG
$\mathfrak{s}_1$	−	−	✓	✓	−	−
$\mathfrak{s}_2$	✓	✓	✓	✓	✓	✓
$\mathfrak{s}_3$	✓	✓	✓	✓	✓	✓
$\mathfrak{s}_4$	−	−	−	−	✓	✓
$\mathfrak{s}_5^{\alpha}$	−	−	−	−	✓	✓
$\mathfrak{s}_6^{\alpha,\beta}$	−	−	−	−	✓	✓
$\mathfrak{s}_7^{\alpha}$	✓	✓	✓	✓	✓	✓
$\mathfrak{s}_8^{\alpha}$	−	−	−	−	✓	✓
$\mathfrak{s}_9$	−	−	−	−	✓	✓
$\mathfrak{s}_{10}^{\alpha,\beta}$	−	−	−	−	✓	✓
$\mathfrak{s}_{11}^{\alpha}$	−	−	−	−	✓	✓
$\mathfrak{s}_{12}$	−	−	−	−	✓	✓

TABLE 6. Existence of Hermitian metrics for any complex structure of splitting type.

Remark 2.7. Note that the Lie algebra $\mathfrak{s}_1 = \mathfrak{g}_{4,9}^0 \oplus \mathbb{R}^2$ (see Remark 1.16) admits SKT Hermitian structures because the 4-dimensional Lie algebra $\mathfrak{g}_{4,9}^0$ admit them by [24], and so the product complex structure on $\mathfrak{s}_1$ admits SKT structures. However, the Hermitian structures that we have obtained on $\mathfrak{s}_1$ are different because the splitting-type complex structure is not a product, and in this sense, our study above provides a new example of SKT metrics in dimension 6.

Finally, we notice also that our results provide (up to our knowledge) new families of non-Kähler balanced solvmanifolds (see also Remark 1.17). The $\mathfrak{s}_{12}$ case is especially rich, as Section 3 below shows.

In relation to the conjectures in [13] and in [30] mentioned above, as a consequence of the results of this section one has the following result.

Corollary 2.8. *Let $X = G/\Gamma$ be a 6-dimensional solvmanifold endowed with a complex structure of splitting type. We have:*

(i) *If X has an SKT metric and also a balanced metric, then X is Kähler.*
(ii) *If X satisfies the $\partial\bar\partial$-Lemma, then X is balanced.*

Proof. If X has an SKT metric and also a balanced metric, then by symmetrization, there is an SKT structure and also a balanced structure on the Lie algebra $\mathfrak{g}$ underlying X. Now, by Table 6, the Lie algebra is isomorphic to $\mathfrak{s}_2$, $\mathfrak{s}_3$ or $\mathfrak{s}_7^\alpha$. In any case, there is a Kähler structure on $\mathfrak{g}$ and so X is Kähler, which completes the proof of (i).

For the proof of (ii), in view of Table 6 it is enough to prove that for any lattice Γ on the connected and simply-connected Lie group G_1 corresponding to $\mathfrak{s}_1$, the solvmanifold G_1/Γ does not satisfy the $\partial\bar\partial$-Lemma with respect to any complex structure of splitting type J. In addition, by the symmetrization process, it suffices to check that the $\partial\bar\partial$-Lemma is not satisfied at the Lie algebra level. Now, for any complex structure of splitting type J we have a basis $\{\omega^1,\omega^2,\omega^3\}$ of (1,0)-forms satisfying (7) with $\varepsilon=1$, therefore the (1,1)-form

$$\omega^{1\bar1} = d\omega^2 = \partial(-\omega^{\bar2}) = \bar\partial\omega^2$$

is d-exact, ∂-exact and $\bar\partial$-exact, but it is not $\partial\bar\partial$-exact. □

3. Complex structures on the Nakamura manifold

In this section we focus on the complex geometry of splitting type on the Nakamura manifold [26], whose underlying Lie algebra is $\mathfrak{s}_{12}$. Firstly, we classify the complex structures of splitting type, which allows us to produce analytic families of complex solvmanifolds with holomorphically trivial canonical bundle satisfying interesting properties in relation to the $\partial\bar\partial$-Lemma.

3.1. Moduli of complex structures of splitting type on the Nakamura manifold.

Next we study the space of complex structures of splitting type on the Lie algebra $\mathfrak{s}_{12}$ up to equivalence.

Proposition 3.1. *On the Lie algebra $\mathfrak{s}_{12}$, there exist the following non-equivalent complex structures of splitting type:*

(i) $(\mathfrak{s}_{12},\tilde J): d\omega^1=-\omega^{13},\quad d\omega^2=\omega^{23},\quad d\omega^3=0$;

(ii) $$(\mathfrak{s}_{12},J_A): \begin{cases} d\omega^1 = A\,\omega^{13}-\omega^{1\bar3}, \\ d\omega^2 = -A\,\omega^{23}+\omega^{2\bar3}, \quad A\in\mathbb{C},\quad |A|\neq 1, \\ d\omega^3=0; \end{cases}$$

(iii) $$(\mathfrak{s}_{12},J_B): \begin{cases} d\omega^1 = -\omega^{13}+B\,\omega^{1\bar3}, \\ d\omega^2 = -\bar B\,\omega^{23}+\omega^{2\bar3}, \quad B\in\mathbb{C},\quad |B|< 1, \\ d\omega^3=0. \end{cases}$$

Proof. Here the equivalence between the complex structures is in the usual sense: two complex structures J and J' on a Lie algebra $\mathfrak{g}$ are equivalent if there exists an automorphism $F\colon \mathfrak{g}\longrightarrow\mathfrak{g}$ such that $J=F^{-1}\circ J'\circ F$. We first observe the following property of the complex structures defined by equations (6) with $A=-1$ and $\varepsilon=1$: if we denote by J_B such a complex structure, then, for $B\neq0$, J_B is equivalent to $J_{1/B}$ (indeed, it suffices to multiply ω^3 by $\bar B$, and change ω^1 with ω^2). This property explains the condition $|B|<1$ in the equations (iii) above.

Now, according to our classification in Section 1 of complex structures of splitting type, the Lie algebra $\mathfrak{s}_{12}$ appears only in some specific cases in the Tables 1, 2 and 5. First, from Table 1, in the case $(A,B)=(-1,0)$ we obtain equations (i), and in the case $(A,B)=(0,-1)$

equations (iii) for $B = 0$, just considering a new basis $\{\tau^1 = \omega^2,\, \tau^2 = \omega^1,\, \tau^3 = -\omega^3\}$. The case $B = -1$, $A \in \mathbb{R} - \{\pm 1\}$, of Table 2 lies in equations (ii), whereas the case $A = -1$, $B \in \mathbb{R} - \{\pm 1\}$, of Table 2 lies in the equations (iii).

With respect to Table 5, the complex structures on $\mathfrak{s}_{12}$ satisfy the conditions

$$\mathfrak{Im}\, A \neq \mathfrak{Im}\, B, \qquad \mathfrak{Im}\, A(1 + \mathfrak{Re}\, B) = -\mathfrak{Im}\, B(1 + \mathfrak{Re}\, A), \qquad \Delta = \pm(|B|^2 - |A|^2) \neq 0,$$

where $\Delta = |A|^2 + |B|^2 + 2(\mathfrak{Re}\, A + \mathfrak{Re}\, B + \mathfrak{Re}\, A\, \mathfrak{Re}\, B - \mathfrak{Im}\, A\, \mathfrak{Im}\, B)$. Next we study the solutions of this set of equations:

- If $\mathfrak{Im}\, A = 0$, since $\mathfrak{Im}\, B \neq 0$, then $A = -1$ and $\Delta = |B|^2 - 1 \neq 0$. In this case, the structures belong to case (iii). Similarly, if $\mathfrak{Im}\, B = 0$ then $B = -1$ and $\Delta = 1 - |A|^2 \neq 0$, so we are in case (ii).

- Suppose now that $(\mathfrak{Im}\, A)(\mathfrak{Im}\, B) \neq 0$ and $\mathfrak{Re}\, A = \mathfrak{Re}\, B = -1$. It is straightforward to verify that $\Delta = (\mathfrak{Im}\, A - \mathfrak{Im}\, B)^2$. But $\Delta = \pm(|B|^2 - |A|^2) = \pm(\mathfrak{Im}^2 B - \mathfrak{Im}^2 A)$ implies $\mathfrak{Im}\, A = \mathfrak{Im}\, B$, which is a contradiction to $\Delta \neq 0$.

- Finally, if $(\mathfrak{Im}\, A)(\mathfrak{Im}\, B) \neq 0$ and $(1 + \mathfrak{Re}\, A)(1 + \mathfrak{Re}\, B) \neq 0$, then we can take $\mathfrak{Im}\, B = -\mathfrak{Im}\, A \left(\frac{1+\mathfrak{Re}\, B}{1+\mathfrak{Re}\, A}\right)$. Now, the condition $\Delta = |B|^2 - |A|^2$ is equivalent to $B = -A\left(\frac{1+\bar{A}}{1+A}\right)$, which implies $|B| = |A|$. The case $\Delta = -(|B|^2 - |A|^2)$ is similar. In conclusion, we do not get complex structures in these cases.

Let us study now the equivalences of complex structures. Observe first that all the complex structures in the cases (i) and (ii) satisfy $\dim H^{3,0}_{\bar\partial}(\mathfrak{g}) = 1$, but $\dim H^{3,0}_{\bar\partial}(\mathfrak{g}) = 0$ for the complex structures in case (iii). Therefore, there are no equivalences between the case (iii) and cases (i)-(ii). A direct calculation allows to show that the complex structure (i) is not equivalent to any complex structure in (ii), and moreover, two complex structures J and J' in (ii), respectively in (iii), are equivalent if and only if $A = A'$, respectively $B = B'$. ☐

Remark 3.2. Observe that $\tilde{J}$ given by (i) is the complex-parallelizable structure on the Nakamura manifold [26], and the complex structure given by $A = 0$ in the family (ii) corresponds to the abelian complex structure, see [1]. In addition, a complex structure of splitting type on $\mathfrak{s}_{12}$ gives rise to a complex solvmanifold with holomorphically trivial canonical bundle if and only if it belongs to (i) or (ii), accordingly to Remark 1.5.

The following theorem reveals that the Nakamura manifold has a rich space of complex structures. The result is based on an appropriate deformation of its abelian complex structure.

Theorem 3.3. *The property of having holomorphically trivial canonical bundle and the property of being of splitting type are not stable under holomorphic deformations.*

Proof. Although the first part of the theorem was firstly shown by Nakamura [26], we provide other proof based on the invariant complex geometry described in Proposition 3.1.

Let Γ be any lattice on the Lie group G_{12} corresponding to $\mathfrak{s}_{12}$, and consider the complex solvmanifold $X_0 = (G_{12}/\Gamma, J_0)$ endowed with its abelian complex structure J_0, which is given by taking $A = 0$ in the family (ii) of Proposition 3.1. Consider an open disc $\Delta := \Delta(0, \epsilon)$ around 0 in $\mathbb{C}$ for $\epsilon > 0$ small enough, and the family $\{X_t\}_{t\in\Delta}$ of complex solvmanifolds given by the solvmanifold G_{12}/Γ endowed with the complex structure J_t defined by the (1,0)-co-frame $\{\omega^1_t := \omega^1,\ \omega^2_t := \omega^2,\ \omega^3_t := \omega^3 - t\omega^{\bar 1}\}$. Notice that the form $\omega^{\bar 1}$ defines a non-zero Dolbeault cohomology class on X_0, and so the previous family X_t provides a small holomorphic deformation of X_0. The complex structure equations of the invariant complex structure J_t are

$$d\omega^1_t = -\omega^{1\bar 3}_t, \qquad d\omega^2_t = -\bar{t}\,\omega^{12}_t + \omega^{2\bar 3}_t, \qquad d\omega^3_t = -t\,\omega^{3\bar 1}_t. \tag{17}$$

Now, since $d\omega_t^{123} = -t\,\omega_t^{123\bar{1}} \neq 0$ for any $t \in \Delta^*$, by [11, Proposition 2.1] the solvmanifold X_t does not have holomorphically trivial canonical bundle for any $t \neq 0$. Indeed, J_t does not belong to (i) or (ii) for $t \neq 0$, see Remark 3.2. Moreover, from the complex structure equations (17) one also has that J_t does not belong to the family (iii), because there are not non-zero invariant holomorphic (1,0)-form for $t \neq 0$. In conclusion, J_t is not of splitting type for any $t \neq 0$. □

Remark 3.4. All the complex structures J_t given in the proof of Theorem 3.3 admit balanced metrics.

3.2. **The $\partial\bar{\partial}$-Lemma on a family of splitting-type complex structures on the Nakamura manifold.** In [11, Proposition 3.7] the complex structures on the Lie algebra $\mathfrak{s}_{12}$ giving rise to complex solvmanifolds with holomorphically trivial canonical bundle are classified. There are two complex structures, denoted in the aforementioned paper by J' and J'', and a family J_C parametrized by $C \in \mathbb{C}$ with $\mathfrak{Im}\,C \neq 0$ which can be represented by a $(1,0)$-co-frame $\{\omega_C^1, \omega_C^2, \omega_C^3\}$ with structure equations:

$$J_C: \quad \begin{cases} d\omega_C^1 &= -(C-i)\,\omega_C^{13} - (C+i)\,\omega_C^{1\bar{3}}, \\ d\omega_C^2 &= (C-i)\,\omega_C^{23} + (C+i)\,\omega_C^{2\bar{3}}, \\ d\omega_C^3 &= 0. \end{cases} \tag{18}$$

Observe that all the structures J_C are of splitting type, whereas J' and J'' are not.

Moreover, the family (18) unifies the complex structures (i) and (ii) of Proposition 3.1. Concretely, if $C = -i$ in (18) then we obtain the complex-parallelizable structure $\tilde{J}$ in Proposition 3.1, whereas if $C \neq -i$ then the complex structure J_C corresponds to the complex structure J_A in the family (ii) of Proposition 3.1 for $A = -(C-i)/(\bar{C}-i)$. Thus, the connected and simply-connected solvable Lie group G_{12} with Lie algebra $\mathfrak{s}_{12}$, endowed with a left-invariant complex structure J_C given by (18), may be written as a semi-direct product $(G_{12}, J_C) = \mathbb{C} \ltimes_{\varphi_C} \mathbb{C}^2$, where the action φ_C is described by a diagonal matrix (3) and the characters $\alpha_1^C, \alpha_2^C : \mathbb{C} \to \mathbb{C}^*$ are

$$\alpha_1^C(z_3) \;=\; e^{-(C-i)z_3-(C+i)\bar{z}_3}, \qquad \alpha_2^C(z_3) \;=\; \alpha_1^C(z_3)^{-1}. \tag{19}$$

Now, we are concerned with the construction of lattices Γ in (G_{12}, J_C) compatible with the splitting. They are of the form $\Gamma = \Gamma' \ltimes_{\varphi_C} \Gamma''$, where Γ' and Γ'' are lattices of $\mathbb{C}$ and $\mathbb{C}^2$ respectively and Γ' is compatible with the splitting, in other words, $\varphi_C(z)\,(\Gamma'') \subseteq \Gamma''$ for any $z \in \Gamma'$. The former condition implies that $\varphi_C|_{\Gamma'}$ must be in the conjugation class of a matrix in $\mathrm{GL}(2,\mathbb{Z})$.

Lemma 3.5. *For every $C \in \mathbb{C}$ with $\mathfrak{Im}\,C \neq 0$, the lattice $\Gamma'_C := \frac{\pi}{2\,\mathfrak{Im}\,C}(1 - i\mathfrak{Re}\,C)\mathbb{Z} \oplus \frac{i}{2}\log(\frac{3+\sqrt{5}}{2})\mathbb{Z}$ of $\mathbb{C}$ is compatible with the splitting φ_C given by the characters (19). Thus, the complex solvmanifold $X_C := (G_{12}/\Gamma_C, J_C)$ is of splitting type, where $\Gamma_C := \Gamma'_C \ltimes_{\varphi_C} \Gamma''$ and Γ'' is a lattice of $\mathbb{C}^2$.*

Proof. After computing its characteristic polynomial, it turns out that the diagonal matrix (3) with characters (19) is in the conjugation class of a matrix in $\mathrm{GL}(2,\mathbb{Z})$, if the condition $(C-i)z_3 + (C+i)\bar{z}_3 = \log(\frac{n+\sqrt{n^2-4}}{2})$ holds for some $n \in \mathbb{Z}$. In particular, fixed $C \in \mathbb{C}$ with $\mathfrak{Im}\,C \neq 0$, we get $z_3 = \frac{\pi}{2\,\mathfrak{Im}\,C}(1 - i\mathfrak{Re}\,C)$ for $n = -2$ and $z_3 = \frac{i}{2}\log(\frac{3+\sqrt{5}}{2})$ for $n = 3$. Therefore, $\Gamma'_C = \frac{\pi}{2\,\mathfrak{Im}\,C}(1 - i\mathfrak{Re}\,C)\mathbb{Z} \oplus \frac{i}{2}\log(\frac{3+\sqrt{5}}{2})\mathbb{Z}$ is a lattice of $\mathbb{C}$ compatible with the splitting. □

As a consequence of the previous lemma, we have a family $\{X_C\}_{\mathfrak{Im}\, C\neq 0}$ of complex solvmanifolds of splitting type with underlying real Lie algebra $\mathfrak{s}_{12}$. We are interested in knowing which of them satisfy the $\partial\bar\partial$-Lemma. The following result states a sufficient condition in order to satisfy the $\partial\bar\partial$-Lemma. This condition is stated and proved in terms of the differential complexes $(B_\Gamma^{\bullet,\bullet}, \bar\partial)$ and $(C_\Gamma^{\bullet,\bullet}, \partial, \bar\partial)$ defined by Kasuya [18], respectively, by Kasuya and the first author [3]. Recall that such complexes allow to compute the Dolbeault, respectively the Bott-Chern cohomology of complex solvmanifolds of splitting type.

Lemma 3.6. *Let $X = (G/\Gamma, J)$ be a complex solvmanifold of splitting type. If $\partial|_{B_\Gamma^{\bullet,\bullet}} = \bar\partial|_{B_\Gamma^{\bullet,\bullet}} = 0$ and $B_\Gamma^{q,p} = \overline{B_\Gamma^{p,q}}$ for all $p, q \in \mathbb{N}$, then X satisfies the $\partial\bar\partial$-Lemma.*

Proof. If the complex $(B_\Gamma^{\bullet,\bullet}, \partial, \bar\partial)$ satisfies $B_\Gamma^{q,p} = \overline{B_\Gamma^{p,q}}$ for all $p, q \in \mathbb{N}$ then, since $C_\Gamma^{\bullet,\bullet} := B_\Gamma^{\bullet,\bullet} + \overline{B_\Gamma^{\bullet,\bullet}}$, it holds $C_\Gamma^{\bullet,\bullet} = B_\Gamma^{\bullet,\bullet}$. Furthermore, the condition $\partial|_{B_\Gamma^{\bullet,\bullet}} = \bar\partial|_{B_\Gamma^{\bullet,\bullet}} = 0$ forces the natural isomorphisms

$$H_{\mathrm{BC}}^{\bullet,\bullet}(X) \cong H_{\mathrm{BC}}^{\bullet,\bullet}(C_\Gamma) = C_\Gamma^{\bullet,\bullet} = B_\Gamma^{\bullet,\bullet} = H_{\bar\partial}^{\bullet,\bullet}(B_\Gamma) \cong H_{\bar\partial}^{\bullet,\bullet}(X).$$

Hence, X satisfies the $\partial\bar\partial$-Lemma. □

Now we recall the construction of the differential complex $(B_{\Gamma_C}^{\bullet,\bullet}, \bar\partial)$ defined in [18, Corollary 4.2] in order to compute the Dolbeault cohomology of X_C. For any complex solvmanifold in the family $\{X_C\}_{\mathfrak{Im}\, C\neq 0}$, consider a set $\{z_1, z_2\}$ of local coordinates on $\mathbb{C}^2$ and z_3 a local coordinate on $\mathbb{C}$. We have the following basis $\{\omega_C^1, \omega_C^2, \omega_C^3\}$ of left-invariant $(1,0)$-forms, where $\omega_C^3 = dz_3$ and

$$\omega_C^1 = \left(\alpha_1^C\right)^{-1} dz_1 = e^{(C-i)z_3+(C+i)\bar z_3} dz_1, \quad \omega_C^2 = \left(\alpha_2^C\right)^{-1} dz_2 = e^{-(C-i)z_3-(C+i)\bar z_3} dz_2,$$

satisfying the complex structure equations (18). Now, the unitary characters $\beta_1^C, \beta_2^C, \gamma_1^C, \gamma_2^C \colon \mathbb{C} \to \mathbb{C}^*$ satisfying that $\alpha_1^C\left(\beta_1^C\right)^{-1}$, $\alpha_2^C\left(\beta_2^C\right)^{-1}$, $\bar\alpha_1^C\left(\gamma_1^C\right)^{-1}$, $\bar\alpha_2^C\left(\gamma_2^C\right)^{-1}$ are holomorphic and required to construct the double complex $(B_{\Gamma_C}^{\bullet,\bullet}, \bar\partial)$ are:

$$\begin{array}{lrclrclcl} (20) \qquad & \beta_1^C(z_3) &=& e^{(\bar C-i)z_3-(C+i)\bar z_3}, \qquad & \beta_2^C(z_3) &=& \beta_1^C(z_3)^{-1} &=& e^{-(\bar C-i)z_3+(C+i)\bar z_3}, \\ & \gamma_1^C(z_3) &=& e^{(C-i)z_3-(\bar C+i)\bar z_3}, \qquad & \gamma_2^C(z_3) &=& \gamma_1^C(z_3)^{-1} &=& e^{-(C-i)z_3+(\bar C+i)\bar z_3}. \end{array}$$

Following [18, Corollary 4.2], [3, Theorem 2.16], and defining for the sake of simplicity that $\beta_3^C = \gamma_3^C \equiv 1$, we have that for X_C the complexes $B_{\Gamma_C}^{\bullet,\bullet}$ and $C_{\Gamma_C}^{\bullet,\bullet}$ are generated by:

$$\begin{array}{lrcl} (21) \qquad & B_{\Gamma_C}^{p,q} &=& \left\langle \beta_I^C \omega_C^I \wedge \gamma_J^C \bar\omega_C^J \;\middle|\; \begin{array}{l} \text{the restriction of } \beta_I\gamma_J \text{ on } \Gamma_C \text{ is trivial} \\ |I| = p,\ |J| = q \end{array} \right\rangle, \\ & C_{\Gamma_C}^{p,q} &=& B_{\Gamma_C}^{p,q} + \bar B_{\Gamma_C}^{p,q}, \end{array}$$

where $(p,q) \in \mathbb{N}^2$. Taking into account the expressions in (20), it turns out that the restrictions induced by the characters on the generators in (21) reduce in our case to satisfy one of the following conditions:

$$\beta_1^C|_{\Gamma_C} = 1, \quad \gamma_1^C|_{\Gamma_C} = 1, \quad \left(\beta_1^C\gamma_1^C\right)|_{\Gamma_C} = 1, \quad \left(\beta_1^C\left(\gamma_1^C\right)^{-1}\right)|_{\Gamma_C} = 1.$$

From now on, we will express the generators of the complexes $B_{\Gamma_C}^{\bullet,\bullet}$ and $C_{\Gamma_C}^{\bullet,\bullet}$ in terms of the following:

$$(22)\quad \begin{cases} \varphi^1 := \beta_1^C\omega_C^1 = e^{(C+\bar{C}-2i)z_3}dz_1, \\ \varphi^2 := \beta_2^C\omega_C^2 = e^{-(C+\bar{C}-2i)z_3}dz_2, \\ \varphi^3 := dz_3, \end{cases} \qquad \begin{cases} \tilde{\varphi}^1 := \gamma_1^C\omega_C^{\bar{1}} = e^{(C+\bar{C}-2i)z_3}d\bar{z}_1, \\ \tilde{\varphi}^2 := \gamma_2^C\omega_C^{\bar{2}} = e^{-(C+\bar{C}-2i)z_3}d\bar{z}_2, \\ \tilde{\varphi}^3 := d\bar{z}_3, \end{cases}$$

where $\varphi^1, \varphi^2, \varphi^3$ have bidegree $(1,0)$ and $\tilde{\varphi}^1, \tilde{\varphi}^2, \tilde{\varphi}^3$ have bidegree $(0,1)$. The complex structure equations expressed in the co-frame $\{\varphi^1, \varphi^2, \varphi^3, \tilde{\varphi}^1, \tilde{\varphi}^2, \tilde{\varphi}^3\}$ are:

$$(23)\quad \begin{cases} d\varphi^1 = -(C+\bar{C}-2i)\varphi^{13}, \\ d\varphi^2 = (C+\bar{C}-2i)\varphi^{23}, \\ d\varphi^3 = 0, \end{cases} \qquad \begin{cases} d\tilde{\varphi}^1 = (C+\bar{C}-2i)\varphi^{3\tilde{1}}, \\ d\tilde{\varphi}^2 = -(C+\bar{C}-2i)\varphi^{3\tilde{2}}, \\ d\tilde{\varphi}^3 = 0. \end{cases}$$

In the tables below, we shorten, e.g., $\varphi^{1\tilde{2}} := \varphi^1 \wedge \tilde{\varphi}^2$.

Proposition 3.7. *Let $X_C = (G_{12}/\Gamma_C, J_C)$ be a complex solvmanifold according to Lemma 3.5. Then, X_C satisfies the $\partial\bar{\partial}$-Lemma if and only if $C \neq \frac{i}{k} \in \mathbb{C}$, for $0 \neq k \in \mathbb{Z}$.*

Proof. Let $C \in \mathbb{C}$ with $\mathfrak{Im}\, C \neq 0$ and Γ'_C be the lattice of $\mathbb{C}$ provided in Lemma 3.5. The triviality of the products of the characters restricted to Γ'_C behaves as follows:

$$\begin{cases} \left(\beta_1^C\gamma_1^C\right)|_{\Gamma'_C} = 1, & \text{for any } C, \\ \left(\beta_1^C\left(\gamma_1^C\right)^{-1}\right)|_{\Gamma'_C} = 1, & \text{if and only if } C = \frac{i}{k} \text{ with } 0 \neq k \in \mathbb{Z}, \\ \beta_1^C|_{\Gamma'_C} = \gamma_1^C|_{\Gamma'_C} = 1, & \text{if and only if } C = \frac{i}{2k+1} \text{ with } k \in \mathbb{Z}. \end{cases}$$

The computations of the double complex $B^{\bullet,\bullet}_{\Gamma_C}$ and of the Hodge and the Betti numbers for the solvmanifolds X_C can be found in Table 8. The computations of these numbers reveal that if $C = \frac{i}{k}$ for $0 \neq k \in \mathbb{Z}$ then $h^{2,0}_{\bar{\partial}}(X_C) + h^{1,1}_{\bar{\partial}}(X_C) + h^{0,2}_{\bar{\partial}}(X_C) \neq b_2(X_C)$, thus X_C does not satisfy the $\partial\bar{\partial}$-Lemma. However, when $C \neq \frac{i}{k}$ it turns out that the hypothesis of Lemma 3.6 are satisfied by using the relations

$$\tilde{\varphi}^1 \wedge \tilde{\varphi}^2 = \bar{\varphi}^1 \wedge \bar{\varphi}^2, \quad \varphi^2 \wedge \tilde{\varphi}^1 = -\bar{\varphi}^1 \wedge \bar{\tilde{\varphi}}^2, \quad \varphi^1 \wedge \tilde{\varphi}^2 = -\bar{\varphi}^2 \wedge \bar{\tilde{\varphi}}^1,$$

of the generators (22), and the complex structure equations (23). Hence, all the corresponding complex solvmanifolds X_C for $C \neq \frac{i}{k}$ satisfy the $\partial\bar{\partial}$-Lemma. □

3.3. **The $\partial\bar{\partial}$-Lemma under holomorphic deformations.** In this section we construct complex solvmanifolds of splitting type with holomorphically trivial canonical bundle that satisfy the $\partial\bar{\partial}$-Lemma by deforming structures that do not satisfy this last condition.

We consider the differential complexes $(B^{\bullet,\bullet}_{\Gamma,t}, \bar{\partial})$ and $(C^{\bullet,\bullet}_{\Gamma,t}, \partial, \bar{\partial})$ and the techniques introduced in [4] to compute the Dolbeault and Bott-Chern cohomologies of small deformations. In particular, by means of the computation of the cohomologies of the complex-parallelizable structure on the Nakamura manifold, the non-closedness of the $\partial\bar{\partial}$-Lemma property under holomorphic deformations is proved in [4, Corollary 6.1]. Using the splitting-type complex geometry on $\mathfrak{s}_{12}$, we extend this result to the following:

Theorem 3.8. *There is an infinite family of complex solvmanifolds $\{X_k\}_{k\in\mathbb{Z}}$ not satisfying the $\partial\bar{\partial}$-Lemma and admitting a small holomorphic deformation $\{(X_k)_t\}_{t\in\Delta_k}$ such that $(X_k)_t$ does satisfy the $\partial\bar{\partial}$-Lemma for every $t \neq 0$.*

Moreover, the solvmanifolds $\{(X_k)_t\}_{t\in\Delta_k}$, $k \in \mathbb{Z}$ have holomorphically trivial canonical bundle and are balanced.

Proof. Consider the infinite family $\{X_k\}_{k\in\mathbb{Z}}$ where $X_k := X_{C_k}$, $C_k = \frac{i}{2k+1}$ and X_C is the complex solvmanifold described in Lemma 3.5. By Proposition 3.7, X_k does not satisfy the $\partial\bar{\partial}$-Lemma for any $k \in \mathbb{Z}$.

We consider an open disc $\Delta_k := \Delta(0,\epsilon_k) \subset \mathbb{C}$ for $\epsilon_k > 0$ small enough, and the family $\{(X_k)_t\}_{t\in\Delta_k}$, $k \in \mathbb{Z}$, of holomorphic deformations of X_k given by the $(1,0)$-co-frame $\left\{\omega^1_{C_k,t} := \omega^1_{C_k}\ ,\ \omega^2_{C_k,t} := \omega^2_{C_k}\ ,\ \omega^3_{C_k,t} := \omega^3_{C_k} + t\,\bar{\omega}^3_{C_k}\right\}$. For simplicity, we will denote $\omega^i_{C_k,t}$ as $\omega^i_{k,t}$. The structure equations become:

$$\begin{cases} d\omega^1_{k,t} &=& -\frac{(C_k-i)+(C_k+i)\,\bar{t}}{1-|t|^2}\,\omega^{13}_{k,t} - \frac{(C_k+i)+(C_k-i)\,t}{1-|t|^2}\,\omega^{1\bar{3}}_{k,t}, \\ d\omega^2_{k,t} &=& \frac{(C_k-i)+(C_k+i)\,\bar{t}}{1-|t|^2}\,\omega^{23}_{k,t} + \frac{(C_k+i)+(C_k-i)\,t}{1-|t|^2}\,\omega^{2\bar{3}}_{k,t}, \\ d\omega^3_{k,t} &=& 0. \end{cases}$$

It is easy to see that the previous complex structures are of splitting type, and therefore, there exist balanced metrics (see Table 6). Moreover, since $d\omega^{123}_{k,t} = 0$, the solvmanifolds have holomorphically trivial canonical bundle.

Taking into account the characters α^C_1, α^C_2, β^C_1, β^C_2, γ^C_1, γ^C_2 described in (19) and (20), we define the generators of the complex $B^{\bullet,\bullet}_{\Gamma_{C_k},t} = \wedge^{\bullet,\bullet}\langle\varphi^1_t,\varphi^2_t,\varphi^3_t,\tilde{\varphi}^1_t,\tilde{\varphi}^2_t,\tilde{\varphi}^3_t\rangle$ associated to the complex solvmanifold $(X_k)_t$:

$$\begin{cases} \varphi^1_t &:=& \beta^{C_k}_1\,\omega^1_{k,t} &=& \exp(-2\,i\,z_3)\,dz_1, \\ \varphi^2_t &:=& \beta^{C_k}_2\,\omega^2_{k,t} &=& \exp(2\,i\,z_3)\,dz_2, \\ \varphi^3_t &:=& \omega^3_{k,t} &=& dz_3 + t\,d\bar{z}_3, \end{cases} \qquad \begin{cases} \tilde{\varphi}^1_t &:=& \gamma^{C_k}_1\,\omega^{\bar{1}}_{k,t} &=& \exp(-2\,i\,z_3)\,d\bar{z}_1, \\ \tilde{\varphi}^2_t &:=& \gamma^{C_k}_2\,\omega^{\bar{2}}_{k,t} &=& \exp(2\,i\,z_3)\,d\bar{z}_2, \\ \tilde{\varphi}^3_t &:=& \omega^{\bar{3}}_{k,t} &=& d\bar{z}_3 + \bar{t}\,dz_3, \end{cases}$$

where φ^1_t, φ^2_t, and φ^3_t have bi-degree $(1,0)$ and $\tilde{\varphi}^1_t$, $\tilde{\varphi}^2_t$, and $\tilde{\varphi}^3_t$ have bi-degree $(0,1)$, as explicitly described in Table 9. Consider also the bi-differential bi-graded double complex

$$C^{\bullet,\bullet}_{\Gamma_{C_k},t} := B^{\bullet,\bullet}_{\Gamma_{C_k},t} + \overline{B^{\bullet,\bullet}_{\Gamma_{C_k},t}}$$

of vector spaces, where

$$\begin{array}{llll} \bar{\varphi}^3_t = \tilde{\varphi}^3_t, & \tilde{\varphi}^1_t\wedge\tilde{\varphi}^2_t = \bar{\varphi}^1_t\wedge\bar{\varphi}^2_t, & \varphi^1_t\wedge\bar{\tilde{\varphi}}^1_t = 0, & \varphi^2_t\wedge\bar{\tilde{\varphi}}^2_t = 0, \\ \varphi^1_t\wedge\tilde{\varphi}^2_t = \bar{\tilde{\varphi}}^1_t\wedge\bar{\varphi}^2_t, & \varphi^2_t\wedge\tilde{\varphi}^1_t = \bar{\tilde{\varphi}}^2_t\wedge\bar{\varphi}^1_t, & \varphi^1_t\wedge\bar{\varphi}^1_t = \bar{\tilde{\varphi}}^1_t\wedge\tilde{\varphi}^1_t, & \varphi^2_t\wedge\bar{\varphi}^2_t = \bar{\tilde{\varphi}}^2_t\wedge\tilde{\varphi}^2_t, \end{array}$$

as explicitly described in Table 9. We compute the structure equations:

$$\begin{cases} d\varphi^1_t &=& \frac{2\,i}{1-|t|^2}\,\varphi^1_t\wedge\varphi^3_t - \frac{2\,t\,i}{1-|t|^2}\,\varphi^1_t\wedge\bar{\varphi}^3_t, \\ d\varphi^2_t &=& -\frac{2\,i}{1-|t|^2}\,\varphi^2_t\wedge\varphi^3_t + \frac{2\,t\,i}{1-|t|^2}\,\varphi^2_t\wedge\bar{\varphi}^3_t, \\ d\varphi^3_t &=& 0, \\ d\bar{\tilde{\varphi}}^1_t &=& -\frac{2\,i}{1-|t|^2}\,\bar{\tilde{\varphi}}^1_t\wedge\bar{\varphi}^3_t + \frac{2\,\bar{t}\,i}{1-|t|^2}\,\bar{\tilde{\varphi}}^1_t\wedge\varphi^3_t, \\ d\bar{\tilde{\varphi}}^2_{A_k,t} &=& \frac{2\,i}{1-|t|^2}\,\bar{\tilde{\varphi}}^2_t\wedge\bar{\varphi}^3_t - \frac{2\,\bar{t}\,i}{1-|t|^2}\,\bar{\tilde{\varphi}}^2_t\wedge\varphi^3_t, \end{cases} \qquad \begin{cases} d\tilde{\varphi}^1_t &=& -\frac{2\,i}{1-|t|^2}\,\varphi^3_t\wedge\tilde{\varphi}^1_t - \frac{2\,t\,i}{1-|t|^2}\,\tilde{\varphi}^1_t\wedge\bar{\varphi}^3_t, \\ d\tilde{\varphi}^2_t &=& \frac{2\,i}{1-|t|^2}\,\varphi^3_t\wedge\tilde{\varphi}^2_t + \frac{2\,t\,i}{1-|t|^2}\,\tilde{\varphi}^2_t\wedge\bar{\varphi}^3_t, \\ d\tilde{\varphi}^3_t &=& 0, \\ d\bar{\varphi}^1_t &=& -\frac{2\,i}{1-|t|^2}\,\bar{\varphi}^1_t\wedge\bar{\varphi}^3_t - \frac{2\,\bar{t}\,i}{1-|t|^2}\,\varphi^3_t\wedge\bar{\varphi}^1_t, \\ d\bar{\varphi}^2_t &=& \frac{2\,i}{1-|t|^2}\,\bar{\varphi}^2_t\wedge\bar{\varphi}^3_t + \frac{2\,\bar{t}\,i}{1-|t|^2}\,\varphi^3_t\wedge\bar{\varphi}^2_t. \end{cases}$$

By [18, Corollary 1.3], [3, Theorem 2.16] and [4, Theorem 1.1, Theorem 1.2] (with respect to the Hermitian metric $g := \varphi^1_t\odot\tilde{\varphi}^1_t + \varphi^2_t\odot\tilde{\varphi}^2_t + \varphi^3_t\odot\bar{\varphi}^3_t$), such complexes allow to compute the Dolbeault cohomology and the Bott-Chern cohomology of $(X_k)_t$.

Note that the differential bi-graded algebra $\left(B^{\bullet,\bullet}_{\Gamma_{C_k},t},\, \bar\partial\right)$ and the bi-differential double complex $\left(C^{\bullet,\bullet}_{\Gamma_{C_k},t},\, \partial,\, \bar\partial\right)$ of vector spaces do not depend on C_k; in particular, for any $C_k = \frac{i}{2k+1}$ varying $k \in \mathbb{Z}$, they are isomorphic to the corresponding object with $k = -1$, that is, $C = -i$. Hence, it follows that the computations and the results in [4, §4] still hold for any C_k. More precisely, we recall in Table 10 the harmonic representatives in the Dolbeault cohomology, respectively Bott-Chern cohomology, with respect to the metric g.

In Table 7, we summarize the results of the computations by giving the Betti, Hodge, and Bott-Chern numbers of the complex solvmanifolds X_k and of its small deformations $(X_k)_t$. From the results summarized in Table 10, we get that $(X_k)_t$ satisfies the $\partial\bar\partial$-Lemma for any $t \neq 0$. □

$\dim_{\mathbb{C}} H^{\bullet,\bullet}_{\sharp}(X_k)_t$	dR	$t=0$ $\bar\partial$	$t=0$ BC	$t\neq 0$ $\bar\partial$	$t\neq 0$ BC
$(\mathbf{0},\mathbf{0})$	1	1	1	1	1
$(\mathbf{1},\mathbf{0})$	2	3	1	1	1
$(\mathbf{0},\mathbf{1})$		3	1	1	1
$(\mathbf{2},\mathbf{0})$	5	3	3	1	1
$(\mathbf{1},\mathbf{1})$		9	7	3	3
$(\mathbf{0},\mathbf{2})$		3	3	1	1
$(\mathbf{3},\mathbf{0})$	8	1	1	1	1
$(\mathbf{2},\mathbf{1})$		9	9	3	3
$(\mathbf{1},\mathbf{2})$		9	9	3	3
$(\mathbf{0},\mathbf{3})$		1	1	1	1
$(\mathbf{3},\mathbf{1})$	5	3	3	1	1
$(\mathbf{2},\mathbf{2})$		9	11	3	3
$(\mathbf{1},\mathbf{3})$		3	3	1	1
$(\mathbf{3},\mathbf{2})$	2	3	5	1	1
$(\mathbf{2},\mathbf{3})$		3	5	1	1
$(\mathbf{3},\mathbf{3})$	1	1	1	1	1

Table 7. Summary of the dimensions of the de Rham, Dolbeault, and Bott-Chern cohomologies of the complex solvmanifolds $(X_k)_t$.

	$B^{\bullet,\bullet}_{\Gamma_C}(C=\frac{i}{2k+1},\, k\in\mathbb{Z})$	$\mathbf{h}^{\bullet,\bullet}_{\bar\partial}$	$\mathbf{b}_\bullet$	$B^{\bullet,\bullet}_{\Gamma_C}(C=\frac{i}{2k},\, 0\neq k\in\mathbb{Z})$	$\mathbf{h}^{\bullet,\bullet}_{\bar\partial}$	$\mathbf{b}_\bullet$	$B^{\bullet,\bullet}_{\Gamma_C}(C\neq\frac{i}{k},\, 0\neq k\in\mathbb{Z})$	$\mathbf{h}^{\bullet,\bullet}_{\bar\partial}$	$\mathbf{b}_\bullet$
$(\mathbf{1},\mathbf{0})$	$\mathbb{C}\langle\varphi^1,\varphi^2,\varphi^3\rangle$	$\mathbf{3}$	2	$\mathbb{C}\langle\varphi^3\rangle$	$\mathbf{1}$	2	$\mathbb{C}\langle\varphi^3\rangle$	$\mathbf{1}$	2
$(\mathbf{0},\mathbf{1})$	$\mathbb{C}\langle\bar\varphi^1,\bar\varphi^2,\varphi^{\bar3}\rangle$	$\mathbf{3}$		$\mathbb{C}\langle\varphi^{\bar3}\rangle$	$\mathbf{1}$		$\mathbb{C}\langle\varphi^{\bar3}\rangle$	$\mathbf{1}$	
$(\mathbf{2},\mathbf{0})$	$\mathbb{C}\langle\varphi^{12},\varphi^{13},\varphi^{23}\rangle$	$\mathbf{3}$	5	$\mathbb{C}\langle\varphi^{12}\rangle$	$\mathbf{1}$	5	$\mathbb{C}\langle\varphi^{12}\rangle$	$\mathbf{1}$	5
$(\mathbf{1},\mathbf{1})$	$\mathbb{C}\left\langle\varphi^{1\bar1},\varphi^{1\bar2},\varphi^{1\bar3},\varphi^{2\bar1},\varphi^{2\bar2},\varphi^{2\bar3},\varphi^{3\bar1},\varphi^{3\bar2},\varphi^{3\bar3}\right\rangle$	$\mathbf{9}$		$\mathbb{C}\left\langle\varphi^{1\bar1},\varphi^{1\bar2},\varphi^{2\bar1},\varphi^{2\bar2},\varphi^{3\bar3}\right\rangle$	$\mathbf{5}$		$\mathbb{C}\left\langle\varphi^{1\bar2},\varphi^{2\bar1},\varphi^{3\bar3}\right\rangle$	$\mathbf{3}$	
$(\mathbf{0},\mathbf{2})$	$\mathbb{C}\left\langle\varphi^{\bar1\bar2},\varphi^{\bar1\bar3},\varphi^{\bar2\bar3}\right\rangle$	$\mathbf{3}$		$\mathbb{C}\left\langle\varphi^{\bar1\bar2}\right\rangle$	$\mathbf{1}$		$\mathbb{C}\left\langle\varphi^{\bar1\bar2}\right\rangle$	$\mathbf{1}$	
$(\mathbf{3},\mathbf{0})$	$\mathbb{C}\langle\varphi^{123}\rangle$	$\mathbf{1}$	8	$\mathbb{C}\langle\varphi^{123}\rangle$	$\mathbf{1}$	8	$\mathbb{C}\langle\varphi^{123}\rangle$	$\mathbf{1}$	8
$(\mathbf{2},\mathbf{1})$	$\mathbb{C}\left\langle\varphi^{12\bar1},\varphi^{12\bar2},\varphi^{12\bar3},\varphi^{13\bar1},\varphi^{13\bar2},\varphi^{13\bar3},\varphi^{23\bar1},\varphi^{23\bar2},\varphi^{23\bar3}\right\rangle$	$\mathbf{9}$		$\mathbb{C}\left\langle\varphi^{12\bar3},\varphi^{13\bar1},\varphi^{13\bar2},\varphi^{23\bar1},\varphi^{23\bar2}\right\rangle$	$\mathbf{5}$		$\mathbb{C}\left\langle\varphi^{12\bar3},\varphi^{13\bar2},\varphi^{23\bar1}\right\rangle$	$\mathbf{3}$	
$(\mathbf{1},\mathbf{2})$	$\mathbb{C}\left\langle\varphi^{1\bar1\bar2},\varphi^{1\bar1\bar3},\varphi^{1\bar2\bar3},\varphi^{2\bar1\bar2},\varphi^{2\bar1\bar3},\varphi^{2\bar2\bar3},\varphi^{3\bar1\bar2},\varphi^{3\bar1\bar3},\varphi^{3\bar2\bar3}\right\rangle$	$\mathbf{9}$		$\mathbb{C}\left\langle\varphi^{1\bar1\bar3},\varphi^{1\bar2\bar3},\varphi^{2\bar1\bar3},\varphi^{2\bar2\bar3},\varphi^{3\bar1\bar2}\right\rangle$	$\mathbf{5}$		$\mathbb{C}\left\langle\varphi^{1\bar2\bar3},\varphi^{2\bar1\bar3},\varphi^{3\bar1\bar2}\right\rangle$	$\mathbf{3}$	
$(\mathbf{0},\mathbf{3})$	$\mathbb{C}\left\langle\varphi^{\bar1\bar2\bar3}\right\rangle$	$\mathbf{1}$		$\mathbb{C}\left\langle\varphi^{\bar1\bar2\bar3}\right\rangle$	$\mathbf{1}$		$\mathbb{C}\left\langle\varphi^{\bar1\bar2\bar3}\right\rangle$	$\mathbf{1}$	
$(\mathbf{3},\mathbf{1})$	$\mathbb{C}\left\langle\varphi^{123\bar1},\varphi^{123\bar2},\varphi^{123\bar3}\right\rangle$	$\mathbf{3}$	5	$\mathbb{C}\langle\varphi^{123\bar3}\rangle$	$\mathbf{1}$	5	$\mathbb{C}\langle\varphi^{123\bar3}\rangle$	$\mathbf{1}$	5
$(\mathbf{2},\mathbf{2})$	$\mathbb{C}\left\langle\varphi^{12\bar1\bar2},\varphi^{12\bar1\bar3},\varphi^{12\bar2\bar3},\varphi^{13\bar1\bar2},\varphi^{13\bar1\bar3},\varphi^{13\bar2\bar3},\varphi^{23\bar1\bar2},\varphi^{23\bar1\bar3},\varphi^{23\bar2\bar3}\right\rangle$	$\mathbf{9}$		$\mathbb{C}\left\langle\varphi^{12\bar1\bar2},\varphi^{13\bar1\bar3},\varphi^{13\bar2\bar3},\varphi^{23\bar1\bar3},\varphi^{23\bar2\bar3}\right\rangle$	$\mathbf{5}$		$\mathbb{C}\left\langle\varphi^{12\bar1\bar2},\varphi^{13\bar2\bar3},\varphi^{23\bar1\bar3}\right\rangle$	$\mathbf{3}$	
$(\mathbf{1},\mathbf{3})$	$\mathbb{C}\left\langle\varphi^{1\bar1\bar2\bar3},\varphi^{2\bar1\bar2\bar3},\varphi^{3\bar1\bar2\bar3}\right\rangle$	$\mathbf{3}$		$\mathbb{C}\left\langle\varphi^{3\bar1\bar2\bar3}\right\rangle$	$\mathbf{1}$		$\mathbb{C}\left\langle\varphi^{3\bar1\bar2\bar3}\right\rangle$	$\mathbf{1}$	
$(\mathbf{3},\mathbf{2})$	$\mathbb{C}\left\langle\varphi^{123\bar1\bar2},\varphi^{123\bar1\bar3},\varphi^{123\bar2\bar3}\right\rangle$	$\mathbf{3}$	2	$\mathbb{C}\left\langle\varphi^{123\bar1\bar2}\right\rangle$	$\mathbf{1}$	2	$\mathbb{C}\left\langle\varphi^{123\bar1\bar2}\right\rangle$	$\mathbf{1}$	2
$(\mathbf{2},\mathbf{3})$	$\mathbb{C}\left\langle\varphi^{12\bar1\bar2\bar3},\varphi^{13\bar1\bar2\bar3},\varphi^{23\bar1\bar2\bar3}\right\rangle$	$\mathbf{3}$		$\mathbb{C}\left\langle\varphi^{12\bar1\bar2\bar3}\right\rangle$	$\mathbf{1}$		$\mathbb{C}\left\langle\varphi^{12\bar1\bar2\bar3}\right\rangle$	$\mathbf{1}$	
$(\mathbf{3},\mathbf{3})$	$\mathbb{C}\left\langle\varphi^{123\bar1\bar2\bar3}\right\rangle$	$\mathbf{1}$	$\mathbf{1}$	$\mathbb{C}\left\langle\varphi^{123\bar1\bar2\bar3}\right\rangle$	$\mathbf{1}$	$\mathbf{1}$	$\mathbb{C}\left\langle\varphi^{123\bar1\bar2\bar3}\right\rangle$	$\mathbf{1}$	$\mathbf{1}$

TABLE 8. The double complex $B^{\bullet,\bullet}_{\Gamma_C}$ for computing the Dolbeault cohomology of the complex solvmanifolds X_k, described in Lemma 3.5.

	$B^{\bullet,\bullet}_{\Gamma_{C_k},t}$	$C^{\bullet,\bullet}_{\Gamma_{C_k},t}$
$(\mathbf{0},\mathbf{0})$	$\mathbb{C}\langle 1\rangle$	$\mathbb{C}\langle 1\rangle$
$(\mathbf{1},\mathbf{0})$	$\mathbb{C}\langle \varphi_t^{1},\ \varphi_t^{2},\ \varphi_t^{3}\rangle$	$\mathbb{C}\left\langle \varphi_t^{1},\ \varphi_t^{2},\ \varphi_t^{3},\ \varphi_t^{\bar{\tilde{1}}},\ \varphi_t^{\bar{\tilde{2}}}\right\rangle$
$(\mathbf{0},\mathbf{1})$	$\mathbb{C}\left\langle \varphi_t^{\tilde{1}},\ \varphi_t^{\tilde{2}},\ \varphi_t^{\bar{3}}\right\rangle$	$\mathbb{C}\left\langle \varphi_t^{\tilde{1}},\ \varphi_t^{\tilde{2}},\ \varphi_t^{\bar{3}},\ \varphi_t^{\bar{1}},\ \varphi_t^{\bar{2}}\right\rangle$
$(\mathbf{2},\mathbf{0})$	$\mathbb{C}\langle \varphi_t^{12},\ \varphi_t^{13},\ \varphi_t^{23}\rangle$	$\mathbb{C}\left\langle \varphi_t^{12},\ \varphi_t^{13},\ \varphi_t^{23},\ \varphi_t^{\bar{\tilde{1}}3},\ \varphi_t^{\bar{\tilde{2}}3}\right\rangle$
$(\mathbf{1},\mathbf{1})$	$\mathbb{C}\left\langle \varphi_t^{1\tilde{1}},\ \varphi_t^{1\tilde{2}},\ \varphi_t^{1\bar{3}},\ \varphi_t^{2\tilde{1}},\ \varphi_t^{2\tilde{2}},\ \varphi_t^{2\bar{3}},\ \varphi_t^{3\tilde{1}},\ \varphi_t^{3\tilde{2}},\ \varphi_t^{3\bar{3}}\right\rangle$	$\mathbb{C}\left\langle \varphi_t^{1\tilde{1}},\ \varphi_t^{1\tilde{2}},\ \varphi_t^{1\bar{3}},\ \varphi_t^{2\tilde{1}},\ \varphi_t^{2\tilde{2}},\right.$ $\left.\varphi_t^{2\bar{3}},\ \varphi_t^{3\tilde{1}},\ \varphi_t^{3\tilde{2}},\ \varphi_t^{3\bar{3}},\ \varphi_t^{\bar{\tilde{1}}\bar{1}},\ \varphi_t^{3\bar{1}},\ \varphi_t^{\bar{\tilde{2}}\bar{2}},\ \varphi_t^{3\bar{2}},\ \varphi_t^{\bar{\tilde{1}}\bar{3}},\ \varphi_t^{\bar{\tilde{2}}\bar{3}}\right\rangle$
$(\mathbf{0},\mathbf{2})$	$\mathbb{C}\left\langle \varphi_t^{\tilde{1}\tilde{2}},\ \varphi_t^{\tilde{1}\bar{3}},\ \varphi_t^{\tilde{2}\bar{3}}\right\rangle$	$\mathbb{C}\left\langle \varphi_t^{\tilde{1}\tilde{2}},\ \varphi_t^{\tilde{1}\bar{3}},\ \varphi_t^{\tilde{2}\bar{3}},\ \varphi_t^{\bar{1}\bar{3}},\ \varphi_t^{\bar{2}\bar{3}}\right\rangle$
$(\mathbf{3},\mathbf{0})$	$\mathbb{C}\langle \varphi_t^{123}\rangle$	$\mathbb{C}\langle \varphi_t^{123}\rangle$
$(\mathbf{2},\mathbf{1})$	$\mathbb{C}\left\langle \varphi_t^{12\tilde{1}},\ \varphi_t^{12\tilde{2}},\ \varphi_t^{12\bar{3}},\ \varphi_t^{13\tilde{1}},\ \varphi_t^{13\tilde{2}},\ \varphi_t^{13\bar{3}},\ \varphi_t^{23\tilde{1}},\ \varphi_t^{23\tilde{2}},\ \varphi_t^{23\bar{3}}\right\rangle$	$\mathbb{C}\left\langle \varphi_t^{12\tilde{1}},\ \varphi_t^{12\tilde{2}},\ \varphi_t^{12\bar{3}},\ \varphi_t^{13\tilde{1}},\ \varphi_t^{13\tilde{2}},\ \varphi_t^{13\bar{3}},\ \varphi_t^{23\tilde{1}},\ \varphi_t^{23\tilde{2}},\right.$ $\left.\varphi_t^{23\bar{3}},\ \varphi_t^{\bar{\tilde{1}}\bar{\tilde{2}}\bar{1}},\ \varphi_t^{\bar{\tilde{1}}3\bar{1}},\ \varphi_t^{\bar{\tilde{1}}\bar{\tilde{2}}\bar{2}},\ \varphi_t^{\bar{\tilde{2}}3\bar{2}},\ \varphi_t^{\bar{\tilde{1}}3\bar{3}},\ \varphi_t^{\bar{\tilde{2}}3\bar{3}}\right\rangle$
$(\mathbf{1},\mathbf{2})$	$\mathbb{C}\left\langle \varphi_t^{1\tilde{1}\tilde{2}},\ \varphi_t^{1\tilde{1}\bar{3}},\ \varphi_t^{1\tilde{2}\bar{3}},\ \varphi_t^{2\tilde{1}\tilde{2}},\ \varphi_t^{2\tilde{1}\bar{3}},\ \varphi_t^{2\tilde{2}\bar{3}},\ \varphi_t^{3\tilde{1}\tilde{2}},\ \varphi_t^{3\tilde{1}\bar{3}},\ \varphi_t^{3\tilde{2}\bar{3}}\right\rangle$	$\mathbb{C}\left\langle \varphi_t^{1\tilde{1}\tilde{2}},\ \varphi_t^{1\tilde{1}\bar{3}},\ \varphi_t^{1\tilde{2}\bar{3}},\ \varphi_t^{2\tilde{1}\tilde{2}},\ \varphi_t^{2\tilde{1}\bar{3}},\ \varphi_t^{2\tilde{2}\bar{3}},\ \varphi_t^{3\tilde{1}\tilde{2}},\ \varphi_t^{3\tilde{1}\bar{3}},\ \varphi_t^{3\tilde{2}\bar{3}},\right.$ $\left.\varphi_t^{\bar{\tilde{1}}\bar{1}\bar{2}},\ \varphi_t^{\bar{\tilde{2}}\bar{1}\bar{2}},\ \varphi_t^{\bar{\tilde{1}}\bar{1}\bar{3}},\ \varphi_t^{3\bar{1}\bar{3}},\ \varphi_t^{\bar{\tilde{2}}\bar{2}\bar{3}},\ \varphi_t^{3\bar{2}\bar{3}}\right\rangle$
$(\mathbf{0},\mathbf{3})$	$\mathbb{C}\left\langle \varphi_t^{\tilde{1}\tilde{2}\bar{3}}\right\rangle$	$\mathbb{C}\left\langle \varphi_t^{\tilde{1}\tilde{2}\bar{3}}\right\rangle$
$(\mathbf{3},\mathbf{1})$	$\mathbb{C}\left\langle \varphi_t^{123\tilde{1}},\ \varphi_t^{123\tilde{2}},\ \varphi_t^{123\bar{3}}\right\rangle$	$\mathbb{C}\left\langle \varphi_t^{123\tilde{1}},\ \varphi_t^{123\tilde{2}},\ \varphi_t^{123\bar{3}},\ \varphi_t^{\bar{\tilde{1}}\bar{\tilde{2}}3\bar{1}},\ \varphi_t^{\bar{\tilde{1}}\bar{\tilde{2}}3\bar{2}}\right\rangle$
$(\mathbf{2},\mathbf{2})$	$\mathbb{C}\left\langle \varphi_t^{12\tilde{1}\tilde{2}},\ \varphi_t^{12\tilde{1}\bar{3}},\ \varphi_t^{12\tilde{2}\bar{3}},\ \varphi_t^{13\tilde{1}\tilde{2}},\ \varphi_t^{13\tilde{1}\bar{3}},\ \varphi_t^{13\tilde{2}\bar{3}},\ \varphi_t^{23\tilde{1}\tilde{2}},\right.$ $\left.\varphi_t^{23\tilde{1}\bar{3}},\ \varphi_t^{23\tilde{2}\bar{3}}\right\rangle$	$\mathbb{C}\left\langle \varphi_t^{12\tilde{1}\tilde{2}},\ \varphi_t^{12\tilde{1}\bar{3}},\ \varphi_t^{12\tilde{2}\bar{3}},\ \varphi_t^{13\tilde{1}\tilde{2}},\ \varphi_t^{13\tilde{1}\bar{3}},\ \varphi_t^{13\tilde{2}\bar{3}},\ \varphi_t^{23\tilde{1}\tilde{2}},\ \varphi_t^{23\tilde{1}\bar{3}},\right.$ $\left.\varphi_t^{23\tilde{2}\bar{3}},\ \varphi_t^{\bar{\tilde{1}}3\bar{1}\bar{2}},\ \varphi_t^{\bar{\tilde{2}}3\bar{1}\bar{2}},\ \varphi_t^{\bar{\tilde{1}}\bar{\tilde{2}}\bar{1}\bar{3}},\ \varphi_t^{\bar{\tilde{1}}\bar{\tilde{2}}\bar{2}\bar{3}},\ \varphi_t^{\bar{\tilde{1}}3\bar{1}\bar{3}},\ \varphi_t^{\bar{\tilde{2}}3\bar{2}\bar{3}}\right\rangle$
$(\mathbf{1},\mathbf{3})$	$\mathbb{C}\left\langle \varphi_t^{1\tilde{1}\tilde{2}\bar{3}},\ \varphi_t^{2\tilde{1}\tilde{2}\bar{3}},\ \varphi_t^{3\tilde{1}\tilde{2}\bar{3}}\right\rangle$	$\mathbb{C}\left\langle \varphi_t^{1\tilde{1}\tilde{2}\bar{3}},\ \varphi_t^{2\tilde{1}\tilde{2}\bar{3}},\ \varphi_t^{3\tilde{1}\tilde{2}\bar{3}},\ \varphi_t^{\bar{\tilde{1}}\bar{1}\bar{2}\bar{3}},\ \varphi_t^{\bar{\tilde{2}}\bar{1}\bar{2}\bar{3}}\right\rangle$
$(\mathbf{3},\mathbf{2})$	$\mathbb{C}\left\langle \varphi_t^{123\tilde{1}\tilde{2}},\ \varphi_t^{123\tilde{1}\bar{3}},\ \varphi_t^{123\tilde{2}\bar{3}}\right\rangle$	$\mathbb{C}\left\langle \varphi_t^{123\tilde{1}\tilde{2}},\ \varphi_t^{123\tilde{1}\bar{3}},\ \varphi_t^{123\tilde{2}\bar{3}},\ \varphi_t^{\bar{\tilde{1}}\bar{\tilde{2}}3\bar{1}\bar{3}},\ \varphi_t^{\bar{\tilde{1}}\bar{\tilde{2}}3\bar{2}\bar{3}}\right\rangle$
$(\mathbf{2},\mathbf{3})$	$\mathbb{C}\left\langle \varphi_t^{12\tilde{1}\tilde{2}\bar{3}},\ \varphi_t^{13\tilde{1}\tilde{2}\bar{3}},\ \varphi_t^{23\tilde{1}\tilde{2}\bar{3}}\right\rangle$	$\mathbb{C}\left\langle \varphi_t^{12\tilde{1}\tilde{2}\bar{3}},\ \varphi_t^{13\tilde{1}\tilde{2}\bar{3}},\ \varphi_t^{23\tilde{1}\tilde{2}\bar{3}},\ \varphi_t^{\bar{\tilde{1}}3\bar{1}\bar{2}\bar{3}},\ \varphi_t^{\bar{\tilde{2}}3\bar{1}\bar{2}\bar{3}}\right\rangle$
$(\mathbf{3},\mathbf{3})$	$\mathbb{C}\left\langle \varphi_t^{123\tilde{1}\tilde{2}\bar{3}}\right\rangle$	$\mathbb{C}\left\langle \varphi_t^{123\tilde{1}\tilde{2}\bar{3}}\right\rangle$

TABLE 9. The (double-)complexes $B^{\bullet,\bullet}_{\Gamma_{C_k},t}$ in [18, Corollary 4.2] and $C^{\bullet,\bullet}_{\Gamma_{C_k},t}$ in [3, Theorem 2.16] for computing the Dolbeault and the Bott-Chern cohomology of the complex solvmanifolds $(X_k)_t$.

	$H^{\bullet,\bullet}_{\bar\partial_{J_{C_k,t=0}}}(X)$	$H^{\bullet,\bullet}_{\bar\partial_{J_{C_k,t\neq 0}}}(X)$	$H^{\bullet,\bullet}_{BC_{J_{C_k,t=0}}}(X)$	$H^{\bullet,\bullet}_{BC_{J_{C_k,t\neq 0}}}(X)$
$(\mathbf{0},\mathbf{0})$	$\mathbb{C}\langle 1\rangle$	$\mathbb{C}\langle 1\rangle$	$\mathbb{C}\langle 1\rangle$	$\mathbb{C}\langle 1\rangle$
$(\mathbf{1},\mathbf{0})$	$\mathbb{C}\langle \varphi_0^{1}, \varphi_0^{2}, \varphi_0^{3}\rangle$	$\mathbb{C}\langle \varphi_t^{3}\rangle$	$\mathbb{C}\langle \varphi_0^{3}\rangle$	$\mathbb{C}\langle \varphi_t^{3}\rangle$
$(\mathbf{0},\mathbf{1})$	$\mathbb{C}\left\langle \varphi_0^{\tilde{1}}, \varphi_0^{\tilde{2}}, \varphi_0^{\bar{3}}\right\rangle$	$\mathbb{C}\langle \varphi_t^{\bar{3}}\rangle$	$\mathbb{C}\langle \varphi_0^{\bar{3}}\rangle$	$\mathbb{C}\langle \varphi_t^{\bar{3}}\rangle$
$(\mathbf{2},\mathbf{0})$	$\mathbb{C}\langle \varphi_0^{12}, \varphi_0^{13}, \varphi_0^{23}\rangle$	$\mathbb{C}\langle \varphi_t^{12}\rangle$	$\mathbb{C}\langle \varphi_0^{12}, \varphi_0^{13}, \varphi_0^{23}\rangle$	$\mathbb{C}\langle \varphi_t^{12}\rangle$
$(\mathbf{1},\mathbf{1})$	$\mathbb{C}\left\langle \varphi_0^{1\tilde{1}}, \varphi_0^{1\tilde{2}}, \varphi_0^{1\bar{3}}, \varphi_0^{2\tilde{1}}, \varphi_0^{2\tilde{2}}, \varphi_0^{2\bar{3}}, \varphi_0^{3\tilde{1}}, \varphi_0^{3\tilde{2}}, \varphi_0^{3\bar{3}}\right\rangle$	$\mathbb{C}\left\langle \varphi_t^{1\tilde{2}}, \varphi_t^{2\tilde{1}}, \varphi_t^{3\bar{3}}\right\rangle$	$\mathbb{C}\left\langle \varphi_0^{1\tilde{2}}, \varphi_0^{2\tilde{1}}, \varphi_0^{3\tilde{1}}, \varphi_0^{3\tilde{2}}, \varphi_0^{3\bar{3}}, \varphi_0^{\bar{\tilde{1}}\bar{3}}, \varphi_0^{\bar{\tilde{2}}\bar{3}}\right\rangle$	$\mathbb{C}\left\langle \varphi_t^{1\tilde{2}}, \varphi_t^{2\tilde{1}}, \varphi_t^{3\bar{3}}\right\rangle$
$(\mathbf{0},\mathbf{2})$	$\mathbb{C}\left\langle \varphi_0^{\tilde{1}\tilde{2}}, \varphi_0^{\tilde{1}\bar{3}}, \varphi_0^{\tilde{2}\bar{3}}\right\rangle$	$\mathbb{C}\left\langle \varphi_t^{\tilde{1}\tilde{2}}\right\rangle$	$\mathbb{C}\left\langle \varphi_0^{\tilde{1}\tilde{2}}, \varphi_0^{\tilde{1}\bar{3}}, \varphi_0^{\tilde{2}\bar{3}}\right\rangle$	$\mathbb{C}\left\langle \varphi_t^{\tilde{1}\tilde{2}}\right\rangle$
$(\mathbf{3},\mathbf{0})$	$\mathbb{C}\langle \varphi_0^{123}\rangle$	$\mathbb{C}\langle \varphi_t^{123}\rangle$	$\mathbb{C}\langle \varphi_0^{123}\rangle$	$\mathbb{C}\langle \varphi_t^{123}\rangle$
$(\mathbf{2},\mathbf{1})$	$\mathbb{C}\left\langle \varphi_0^{12\tilde{1}}, \varphi_0^{12\tilde{2}}, \varphi_0^{12\bar{3}}, \varphi_0^{13\tilde{1}}, \varphi_0^{13\tilde{2}}, \varphi_0^{13\bar{3}}, \varphi_0^{23\tilde{1}}, \varphi_0^{23\tilde{2}}, \varphi_0^{23\bar{3}}\right\rangle$	$\mathbb{C}\left\langle \varphi_t^{12\bar{3}}, \varphi_t^{13\tilde{2}}, \varphi_t^{23\tilde{1}}\right\rangle$	$\mathbb{C}\left\langle \varphi_0^{12\bar{3}}, \varphi_0^{13\tilde{1}}, \varphi_0^{13\tilde{2}}, \varphi_0^{13\bar{3}}, \varphi_0^{23\tilde{1}}, \varphi_0^{23\tilde{2}}, \varphi_0^{23\bar{3}}, \varphi_0^{\bar{\tilde{1}}3\bar{3}}, \varphi_0^{\bar{\tilde{2}}3\bar{3}}\right\rangle$	$\mathbb{C}\left\langle \varphi_t^{12\bar{3}}, \varphi_t^{13\tilde{2}}, \varphi_t^{23\tilde{1}}\right\rangle$
$(\mathbf{1},\mathbf{2})$	$\mathbb{C}\left\langle \varphi_0^{1\tilde{1}\tilde{2}}, \varphi_0^{1\tilde{1}\bar{3}}, \varphi_0^{1\tilde{2}\bar{3}}, \varphi_0^{2\tilde{1}\tilde{2}}, \varphi_0^{2\tilde{1}\bar{3}}, \varphi_0^{2\tilde{2}\bar{3}}, \varphi_0^{3\tilde{1}\tilde{2}}, \varphi_0^{3\tilde{1}\bar{3}}, \varphi_0^{3\tilde{2}\bar{3}}\right\rangle$	$\mathbb{C}\left\langle \varphi_t^{1\tilde{2}\bar{3}}, \varphi_t^{2\tilde{1}\bar{3}}, \varphi_t^{3\tilde{1}\tilde{2}}\right\rangle$	$\mathbb{C}\left\langle \varphi_0^{1\tilde{2}\bar{3}}, \varphi_0^{2\tilde{1}\bar{3}}, \varphi_0^{3\tilde{1}\tilde{2}}, \varphi_0^{3\tilde{1}\bar{3}}, \varphi_0^{3\tilde{2}\bar{3}}, \varphi_0^{\bar{\tilde{1}}\tilde{1}\bar{3}}, \varphi_0^{3\tilde{1}\bar{3}}, \varphi_0^{\bar{\tilde{2}}\tilde{2}\bar{3}}, \varphi_0^{3\tilde{2}\bar{3}}\right\rangle$	$\mathbb{C}\left\langle \varphi_t^{1\tilde{2}\bar{3}}, \varphi_t^{2\tilde{1}\bar{3}}, \varphi_t^{3\tilde{1}\tilde{2}}\right\rangle$
$(\mathbf{0},\mathbf{3})$	$\mathbb{C}\left\langle \varphi_0^{\tilde{1}\tilde{2}\bar{3}}\right\rangle$	$\mathbb{C}\left\langle \varphi_t^{\tilde{1}\tilde{2}\bar{3}}\right\rangle$	$\mathbb{C}\left\langle \varphi_0^{\tilde{1}\tilde{2}\bar{3}}\right\rangle$	$\mathbb{C}\left\langle \varphi_t^{\tilde{1}\tilde{2}\bar{3}}\right\rangle$
$(\mathbf{3},\mathbf{1})$	$\mathbb{C}\left\langle \varphi_0^{123\tilde{1}}, \varphi_0^{123\tilde{2}}, \varphi_0^{123\bar{3}}\right\rangle$	$\mathbb{C}\langle \varphi_t^{123\bar{3}}\rangle$	$\mathbb{C}\left\langle \varphi_0^{123\tilde{1}}, \varphi_0^{123\tilde{2}}, \varphi_0^{123\bar{3}}\right\rangle$	$\mathbb{C}\langle \varphi_t^{123\bar{3}}\rangle$
$(\mathbf{2},\mathbf{2})$	$\mathbb{C}\left\langle \varphi_0^{12\tilde{1}\tilde{2}}, \varphi_0^{12\tilde{1}\bar{3}}, \varphi_0^{12\tilde{2}\bar{3}}, \varphi_0^{13\tilde{1}\tilde{2}}, \varphi_0^{13\tilde{1}\bar{3}}, \varphi_0^{13\tilde{2}\bar{3}}, \varphi_0^{23\tilde{1}\tilde{2}}, \varphi_0^{23\tilde{1}\bar{3}}, \varphi_0^{23\tilde{2}\bar{3}}\right\rangle$	$\mathbb{C}\left\langle \varphi_t^{12\tilde{1}\tilde{2}}, \varphi_t^{13\tilde{2}\bar{3}}, \varphi_t^{23\tilde{1}\bar{3}}\right\rangle$	$\mathbb{C}\left\langle \varphi_0^{12\tilde{1}\tilde{2}}, \varphi_0^{13\tilde{1}\tilde{2}}, \varphi_0^{13\tilde{1}\bar{3}}, \varphi_0^{13\tilde{2}\bar{3}}, \varphi_0^{23\tilde{1}\tilde{2}}, \varphi_0^{23\tilde{1}\bar{3}}, \varphi_0^{23\tilde{2}\bar{3}}, \varphi_0^{\bar{\tilde{1}}\bar{\tilde{2}}\tilde{1}\bar{3}}, \varphi_0^{\bar{\tilde{1}}\bar{\tilde{2}}\tilde{2}\bar{3}}, \varphi_0^{\bar{\tilde{1}}3\tilde{1}\bar{3}}, \varphi_0^{\bar{\tilde{2}}3\tilde{2}\bar{3}}\right\rangle$	$\mathbb{C}\left\langle \varphi_t^{12\tilde{1}\tilde{2}}, \varphi_t^{13\tilde{2}\bar{3}}, \varphi_t^{23\tilde{1}\bar{3}}\right\rangle$
$(\mathbf{1},\mathbf{3})$	$\mathbb{C}\left\langle \varphi_0^{1\tilde{1}\tilde{2}\bar{3}}, \varphi_0^{2\tilde{1}\tilde{2}\bar{3}}, \varphi_0^{3\tilde{1}\tilde{2}\bar{3}}\right\rangle$	$\mathbb{C}\left\langle \varphi_t^{3\tilde{1}\tilde{2}\bar{3}}\right\rangle$	$\mathbb{C}\left\langle \varphi_0^{1\tilde{1}\tilde{2}\bar{3}}, \varphi_0^{2\tilde{1}\tilde{2}\bar{3}}, \varphi_0^{3\tilde{1}\tilde{2}\bar{3}}\right\rangle$	$\mathbb{C}\left\langle \varphi_t^{3\tilde{1}\tilde{2}\bar{3}}\right\rangle$
$(\mathbf{3},\mathbf{2})$	$\mathbb{C}\left\langle \varphi_0^{123\tilde{1}\tilde{2}}, \varphi_0^{123\tilde{1}\bar{3}}, \varphi_0^{123\tilde{2}\bar{3}}\right\rangle$	$\mathbb{C}\left\langle \varphi_t^{123\tilde{1}\tilde{2}}\right\rangle$	$\mathbb{C}\left\langle \varphi_0^{123\tilde{1}\tilde{2}}, \varphi_0^{123\tilde{1}\bar{3}}, \varphi_0^{123\tilde{2}\bar{3}}, \varphi_0^{\bar{\tilde{1}}23\tilde{1}\bar{3}}, \varphi_0^{\bar{\tilde{1}}23\tilde{2}\bar{3}}\right\rangle$	$\mathbb{C}\left\langle \varphi_t^{123\tilde{1}\tilde{2}}\right\rangle$
$(\mathbf{2},\mathbf{3})$	$\mathbb{C}\left\langle \varphi_0^{12\tilde{1}\tilde{2}\bar{3}}, \varphi_0^{13\tilde{1}\tilde{2}\bar{3}}, \varphi_0^{23\tilde{1}\tilde{2}\bar{3}}\right\rangle$	$\mathbb{C}\left\langle \varphi_t^{12\tilde{1}\tilde{2}\bar{3}}\right\rangle$	$\mathbb{C}\left\langle \varphi_0^{12\tilde{1}\tilde{2}\bar{3}}, \varphi_0^{13\tilde{1}\tilde{2}\bar{3}}, \varphi_0^{23\tilde{1}\tilde{2}\bar{3}}, \varphi_0^{\bar{\tilde{1}}3\tilde{1}\tilde{2}\bar{3}}, \varphi_0^{\bar{\tilde{2}}3\tilde{1}\tilde{2}\bar{3}}\right\rangle$	$\mathbb{C}\left\langle \varphi_t^{12\tilde{1}\tilde{2}\bar{3}}\right\rangle$
$(\mathbf{3},\mathbf{3})$	$\mathbb{C}\left\langle \varphi_0^{123\tilde{1}\tilde{2}\bar{3}}\right\rangle$	$\mathbb{C}\left\langle \varphi_t^{123\tilde{1}\tilde{2}\bar{3}}\right\rangle$	$\mathbb{C}\left\langle \varphi_0^{123\tilde{1}\tilde{2}\bar{3}}\right\rangle$	$\mathbb{C}\left\langle \varphi_t^{123\tilde{1}\tilde{2}\bar{3}}\right\rangle$

TABLE 10. The harmonic representatives with respect to the metric $g := \varphi_t^1 \odot \tilde{\varphi}_t^1 + \varphi_t^2 \odot \tilde{\varphi}_t^2 + \varphi_t^3 \odot \tilde{\varphi}_t^3$ of the Dolbeault and the Bott-Chern cohomology of the complex solvmanifolds $(X_k)_t$.

Appendix A. Reduction of parameters

In this appendix we show how to reduce the value of the parameters in the algebras $\mathfrak{s}_5^{\alpha}$, $\mathfrak{s}_6^{\alpha,\beta}$, $\mathfrak{s}_7^{\alpha}$, $\mathfrak{s}_8^{\alpha}$, $\mathfrak{s}_{10}^{\alpha,\beta}$, $\mathfrak{s}_{11}^{\alpha}$ according to Theorem 1.7.

Let us consider the following changes from a basis $\{e^1,\ldots,e^6\}$ to another real basis $\{f^1,\ldots,f^6\}$, where λ is a non-zero real number:

ChA $f^i = e^i$, $i=1,3,5,6$, $\quad f^2=e^4$, $\quad f^4=e^2$.

ChB $f^i = e^i$, $i=1,2,5,6$, $\quad f^3=e^4$, $\quad f^4=e^3$.

ChC $f^i = e^i$, $i=1,3,6$, $\quad f^2=-e^2$, $\quad f^4=-e^4$, $\quad f^5=-e^5$.

ChD $f^i = e^i$, $i=1,2,3,5,6$, $\quad f^4=-e^4$.

ChE $f^1=e^3$, $\quad f^2=-e^4$, $\quad f^3=e^1$, $\quad f^4=-e^2$, $\quad f^5=-\lambda e^5$, $\quad f^6=e^6$.

ChF $f^i = e^i$, $i=2,4,6$, $\quad f^1=-e^5$, $\quad f^3=e^1$, $\quad f^5=e^3$.

ChG $f^1=e^3$, $\quad f^2=e^4$, $\quad f^3=e^1$, $\quad f^4=e^2$, $\quad f^i=e^i$, $\quad i=5,6$.

ChH $f^1=e^3$, $\quad f^2=e^4$, $\quad f^3=e^1$, $\quad f^4=e^2$, $\quad f^5=\lambda e^5$, $\quad f^6=-e^6$.

Case $\mathfrak{s}_5^{\alpha}$: Consider $\mathfrak{s}_5^{\alpha}$ where $\alpha\in\mathbb{R}$, with structure equations

$$\mathfrak{s}_5^{\alpha} = (e^{15}, e^{25}, -e^{35}+\alpha\, e^{45}, -\alpha\, e^{35}-e^{45}, 0, 0), \quad \alpha\in\mathbb{R}.$$

Then:

- If $\alpha=0$, change ChA gives the isomorphism $\mathfrak{s}_5^{0}\cong\mathfrak{s}_4$.
- Change ChB gives the isomorphism $\mathfrak{s}_5^{\alpha}\cong\mathfrak{s}_5^{-\alpha}$.

Therefore, we can suppose $\alpha>0$.

Case $\mathfrak{s}_6^{\alpha,\beta}$: Consider $\mathfrak{s}_6^{\alpha,\beta}$ where $\alpha,\beta\in\mathbb{R}$, with structure equations

$$\mathfrak{s}_6^{\alpha,\beta} = (\alpha\, e^{15}+e^{25}, -e^{15}+\alpha\, e^{25}, -\alpha\, e^{35}+\beta\, e^{45}, -\beta\, e^{35}-\alpha\, e^{45}, 0, 0), \quad \alpha,\beta\in\mathbb{R}.$$

Then:

- Change ChC gives the isomorphism $\mathfrak{s}_6^{\alpha,\beta}\cong\mathfrak{s}_6^{-\alpha,\beta}$.
- Change ChD gives the isomorphism $\mathfrak{s}_6^{\alpha,\beta}\cong\mathfrak{s}_6^{\alpha,-\beta}$.
- If $\alpha=0$, $\mathfrak{s}_6^{0,\beta}\cong\mathfrak{s}_7^{\beta}$.
- If $\beta=0$ and $\alpha\neq0$, change ChE with $\lambda=\alpha$ gives the isomorphism $\mathfrak{s}_6^{\alpha,0}\cong\mathfrak{s}_5^{\frac{1}{\alpha}}$.
- If $\beta=1$, $\mathfrak{s}_6^{\alpha,1}\cong\mathfrak{s}_8^{\alpha}$.
- If $\beta\neq0,1$, change ChE with $\lambda=\beta$ gives the isomorphism $\mathfrak{s}_6^{\alpha,\beta}\cong\mathfrak{s}_6^{\frac{\alpha}{\beta},\frac{1}{\beta}}$.

Therefore, we can suppose $\alpha>0$ and $\beta\in(0,1)$.

Case $\mathfrak{s}_7^{\alpha}$: Consider $\mathfrak{s}_7^{\alpha}$ where $\alpha\in\mathbb{R}$, with structure equations

$$\mathfrak{s}_7^{\alpha} = (e^{25}, -e^{15}, \alpha\, e^{45}, -\alpha\, e^{35}, 0, 0), \quad \alpha\in\mathbb{R}.$$

Then:

- If $\alpha=0$, change ChF gives the isomorphism $\mathfrak{s}_7^{0}\cong\mathfrak{s}_2$.
- Change ChD gives the isomorphism $\mathfrak{s}_7^{\alpha}\cong\mathfrak{s}_7^{-\alpha}$.

- Change ChE with $\lambda = \alpha$ gives the isomorphism $\mathfrak{s}_7^{\alpha} \cong \mathfrak{s}_7^{\frac{1}{\alpha}}$.

Therefore, we can suppose $0 < \alpha \leq 1$.

Case $\mathfrak{s}_8^{\alpha}$: Consider $\mathfrak{s}_8^{\alpha}$ where $\alpha \in \mathbb{R}$, with structure equations

$$\mathfrak{s}_8^{\alpha} = (\alpha e^{15} + e^{25}, -e^{15} + \alpha e^{25}, -\alpha e^{35} + e^{45}, -e^{35} - \alpha e^{45}, 0, 0), \quad \alpha \in \mathbb{R}.$$

Then:

- Observe that $\mathfrak{s}_8^0 \cong \mathfrak{s}_7^1$.
- Change ChC gives the isomorphism $\mathfrak{s}_8^{\alpha} \cong \mathfrak{s}_8^{-\alpha}$.

Therefore, we can suppose $\alpha > 0$.

Case $\mathfrak{s}_{11}^{\alpha}$: Consider $\mathfrak{s}_{11}^{\alpha}$ where $\alpha \in \mathbb{R}$, with structure equations

$$\mathfrak{s}_{11}^{\alpha} = (e^{16} - e^{25}, e^{15} + e^{26}, -e^{36} - \alpha e^{45}, \alpha e^{35} - e^{46}, 0, 0), \quad \alpha \in \mathbb{R}.$$

Then:

- If $\alpha = 0$, change ChG gives the isomorphism $\mathfrak{s}_{11}^0 \cong \mathfrak{s}_9$.
- Change ChB gives the isomorphism $\mathfrak{s}_{11}^{\alpha} \cong \mathfrak{s}_{11}^{-\alpha}$.
- Change ChH with $\lambda = \alpha$ gives the isomorphism $\mathfrak{s}_{11}^{\alpha} \cong \mathfrak{s}_{11}^{\frac{1}{\alpha}}$.
- Change ChB gives the isomorphism $\mathfrak{s}_{11}^1 \cong \mathfrak{s}_{12}$.

Therefore, we can suppose $\alpha \in (0, 1)$.

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