

THE VARIANCE CONJECTURE ON PROJECTIONS OF THE CUBE

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ABSTRACT. We prove that the uniform probability measure μ on every $(n-k)$ -dimensional projection of the n -dimensional unit cube verifies the variance conjecture with an absolute constant C

$$\mathrm{Var}_\mu |x|^2 \leq C \sup_{\theta \in S^{n-1}} \mathbb{E}_\mu \langle x, \theta \rangle^2 \mathbb{E}_\mu |x|^2,$$

provided that $1 \leq k \leq \sqrt{n}$. We also prove that if $1 \leq k \leq n^{\frac{2}{3}} (\log n)^{-\frac{1}{3}}$, the conjecture is true for the family of uniform probabilities on its projections on random $(n-k)$ -dimensional subspaces.

1. INTRODUCTION AND NOTATION

The (generalized) variance conjecture states that there exists an absolute constant C such that for every centered log-concave probability μ on \mathbb{R}^n (i.e. of the form $d\mu = e^{-v(x)} dx$ for some convex function $v : \mathbb{R}^n \rightarrow (-\infty, \infty]$)

$$\mathrm{Var}_\mu |x|^2 \leq C \lambda_\mu^2 \mathbb{E}_\mu |x|^2,$$

where \mathbb{E}_μ and Var_μ denote the expectation and the variance with respect to μ and λ_μ is the largest eigenvalue of the covariance matrix, i.e. $\lambda_\mu^2 = \max_{\theta \in S^{n-1}} \mathbb{E}_\mu \langle x, \theta \rangle^2$ where S^{n-1} denotes the unit Euclidean sphere in \mathbb{R}^n .

This conjecture was first considered in the context of the so called Central Limit Problem for isotropic convex bodies in [BK] and it is a particular case of a more general statement, known as the Kannan, Lovász, and Simonovits or KLS-conjecture, see [KLS], which conjectures the existence of an absolute constant C such that for any centered log-concave probability in \mathbb{R}^n and any locally Lipschitz function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\mathrm{Var}_\mu g(x)$ is finite

$$\mathrm{Var}_\mu g(x) \leq C \lambda_\mu^2 \mathbb{E}_\mu |\nabla g(x)|^2.$$

In recent years a number of families of measures have been proved to verify these conjectures (see [AB2] for a recent review on the subject). For instance, the KLS-conjecture is known to be true for the Gaussian probability and the uniform probability measures on the ℓ_p^n -balls, some revolution bodies, the simplex and, with an extra $\log n$ factor, on unconditional bodies and log-concave probabilities with many symmetries (see [BaC], [BaW], [B], [H], [K], [LW], [S]). The best general known result for the KLS-conjecture adds a factor \sqrt{n} and is due to Lee and Vempala (see [LV]). Besides, the variance conjecture is known to be true for uniform

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probabilities on unconditional bodies and on hyperplane projections of the cross-polytope and the cube (see [K] and [AB1]). The best general estimate for the variance conjecture is the one given by Lee and Vempala for the KLS-conjecture.

We would like to remark that, while in the case of the KLS-conjecture one can assume without loss of generality that μ is isotropic (since then every linear transformation of the measure verifies it) this is not the case when we restrict to the variance conjecture, as we are considering only the function $g(x) = |x|^2$.

Before stating our results let us introduce some more notation. Let

$$B_\infty^n := \{x \in \mathbb{R}^n : |x_i| \leq 1, \forall 1 \leq i \leq n\}$$

denote the n -dimensional unit cube and, for any $1 \leq k \leq n$, let $G_{n,n-k}$ be the set of all $(n-k)$ -dimensional subspaces of \mathbb{R}^n . For any $E \in G_{n,n-k}$ we will denote by $K := P_E B_\infty^n$ the orthogonal projection of B_∞^n onto E and by μ the uniform probability on K . $\{e_i\}_{i=1}^n$ will denote the standard canonical basis in \mathbb{R}^n . As mentioned before, it was proved in [AB1] that the family of uniform probabilities on any $(n-1)$ -dimensional projection of B_∞^n verifies the variance conjecture.

In this paper we will prove the following

Theorem 1.1. *There exists an absolute constant C such that for any $1 \leq k \leq \sqrt{n}$ and any $E \in G_{n,n-k}$, if μ denotes the uniform probability measure on $K = P_E B_\infty^n$, then*

$$\text{Var}_\mu |x|^2 \leq C \lambda_\mu^2 \mathbb{E}_\mu |x|^2.$$

We will also prove the following theorem, which shows that for k in a larger range, the variance conjecture is true for the family of uniform probabilities on the projections of B_∞^n on a random $(n-k)$ -dimensional subspace. For that matter, we denote by $\mu_{n,n-k}$ the Haar probability measure on $G_{n,n-k}$.

Theorem 1.2. *There exist absolute constants C, c_1, c_2 such that for any $1 \leq k \leq \frac{n^{\frac{2}{3}}}{(\log n)^{\frac{1}{3}}}$, if μ denotes the uniform probability measure on $K = P_E(B_\infty^n)$, the measure $\mu_{n,n-k}$ of the subspaces $E \in G_{n,n-k}$ for which*

$$\text{Var}_\mu |x|^2 \leq C \lambda_\mu^2 \mathbb{E}_\mu |x|^2$$

is greater than $1 - c_1 e^{-c_2 n^{\frac{2}{3}} (\log n)^{\frac{2}{3}}}$.

The main tool to prove both theorems will be to decompose an integral on K as the sum of the integrals on the projections of some $(n-k)$ -dimensional faces. It was proved in [ABBW] that for any $E \in G_{n,n-k}$ there exist F_1, \dots, F_l a set of $(n-k)$ -dimensional faces of B_∞^n such that for any integrable function f on K

$$\begin{aligned} \mathbb{E}_\mu f &:= \frac{1}{|K|} \int_K f(x) dx = \sum_{i=1}^l \frac{|P_E(F_i)|}{|K|} \mathbb{E}_{P_E(F_i)} f(x) \\ &= \sum_{i=1}^l \frac{|P_E(F_i)|}{|K|} \frac{1}{|P_E(F_i)|} \int_{P_E(F_i)} f(x) dx \\ &= \sum_{i=1}^l \frac{|P_E(F_i)|}{|K|} \frac{1}{|F_i|} \int_{F_i} f(P_E x) dx \\ (1) \quad &= \sum_{i=1}^l \frac{|P_E(F_i)|}{|K|} \mathbb{E}_{F_i} f(P_E x), \end{aligned}$$

where we have denoted by $|\cdot|$ the relative volume of a convex body to the affine subspace in which it lies, \mathbb{E}_{F_i} and by $\mathbb{E}_{P_E(F_i)}$ the expectation with respect to the uniform probability on the face F_i and on its projection $P_E(F_i)$. In particular

$$\sum_{i=1}^l \frac{|P_E(F_i)|}{|K|} = 1.$$

In this way, $\text{Var}_\mu |x|^2$ is written as the sum of two averages on some faces (see Lemma 2.1 below). The restrictions on the range of k in both theorems arise from the fact that in order to give estimates for these averages we do it for *each* face of the cube separately. With this method we provide an upper bound for $\text{Var}_\mu |x|^2$ and a lower bound for $\lambda_\mu^2 \mathbb{E}_\mu |x|^2$ of the same order. It would be interesting to know whether one could bound these two terms from above directly by $\lambda_\mu^2 \mathbb{E}_\mu |x|^2$.

Notice that the $(n-k)$ -dimensional faces of B_∞^n are the sets of the form

$$F_{(i_1, \varepsilon_1, \dots, i_k, \varepsilon_k)} = \{x \in B_\infty^n : x_{i_j} = \varepsilon_j, j = 1, \dots, k\},$$

where $1 \leq i_j \leq n$, $i_{j_1} \neq i_{j_2}$ and $\varepsilon_j = \pm 1$. In other words, if we divide $\{1, \dots, n\}$ into two disjoint sets $\{i_j\}_{j=1}^k$ and $\{l_j\}_{j=1}^{n-k}$ of cardinality k and $n-k$, then the $(n-k)$ -dimensional face $F_{(i_1, \varepsilon_1, \dots, i_k, \varepsilon_k)}$ is

$$(2) \quad F_{(i_1, \varepsilon_1, \dots, i_k, \varepsilon_k)} = \sum_{j=1}^k \varepsilon_j e_{i_j} + I_F(B_\infty^{n-k}),$$

where I_F is the linear map from \mathbb{R}^{n-k} to \mathbb{R}^n given by $I_F x = \sum_{j=1}^{n-k} x_j e_{l_j}$.

For any $E \in G_{n, n-k}$ we write $S_E = S^{n-1} \cap E$ and denote by σ_E the Haar probability measure on S_E .

2. THE VARIANCE CONJECTURE ON $(n-k)$ -DIMENSIONAL PROJECTIONS OF THE CUBE

In this section we shall prove Theorem 1.1. We will use the aforementioned representation of $\mathbb{E}_\mu f$ in order to write $\text{Var}_\mu |x|^2$ as the sum of two terms. One of them will be an average of the variances of $|x|^2$ on the projections of the faces involved in (1). The other one will be an average of the distances from the expected value of $|x|^2$ on the projections of such faces to $\mathbb{E}_\mu |x|^2$. We will give estimates for these terms valid for every face and prove

$$\text{Var}_\mu |x|^2 \leq Cn \leq C\lambda_\mu^2 \mathbb{E}_\mu |x|^2.$$

We start with the following lemma, which can be proved by direct computation:

Lemma 2.1. *Let $E \in G_{n, n-k}$, μ the uniform probability on $K = P_E(B_\infty^n)$ and $\{F_i\}_{i=1}^l$ the set of $(n-k)$ -dimensional faces described in (1). Then*

$$(3) \quad \text{Var}_\mu |x|^2 = \sum_{i=1}^l \frac{|P_E(F_i)|}{|K|} \text{Var}_{P_E(F_i)} |x|^2 + \sum_{i=1}^l \frac{|P_E(F_i)|}{|K|} (\mathbb{E}_{P_E(F_i)} |x|^2 - \mathbb{E}_\mu |x|^2)^2.$$

We will estimate the two summands appearing in (3). The following lemma provides upper and lower bounds to some of parameters involved.

Lemma 2.2. *Let $E \in G_{n,n-k}$. Then, for any $\theta \in S_E$ and any $(n-k)$ -dimensional face $F = F_{(i_1, \varepsilon_1, \dots, i_k, \varepsilon_k)}$ of B_∞^n we have,*

$$\mathbb{E}_{P_E(F)} \langle x, \theta \rangle^2 = \mathbb{E}_F \langle P_E x, \theta \rangle^2 = \frac{1}{3} + \left(\sum_{j=1}^k \varepsilon_j \theta_{i_j} \right)^2 - \frac{1}{3} \sum_{j=1}^k \theta_{i_j}^2$$

and

$$\mathbb{E}_{P_E(F)} |x|^2 = \mathbb{E}_F |P_E x|^2 = \frac{n-k}{3} + \left| P_E \left(\sum_{j=1}^k \varepsilon_j e_{i_j} \right) \right|^2 - \frac{1}{3} \sum_{j=1}^k |P_E(e_{i_j})|^2.$$

Consequently,

$$\frac{n-2k}{3} \leq \mathbb{E}_{P_E(F)} |x|^2 \leq \frac{n+2k}{3}, \quad \frac{n-2k}{3} \leq \mathbb{E}_\mu |x|^2 \leq \frac{n+2k}{3},$$

and

$$\lambda_\mu^2 \geq \frac{n-2k}{3(n-k)}.$$

Proof. Using (2), we have a random vector x uniformly distributed in F can be written as

$$x = \sum_{j=1}^k \varepsilon_j e_{i_j} + I_F(y)$$

where y is uniformly distributed in B_∞^{n-k} . Then, for every $\theta \in S_E$, straightforward computations yield

$$\begin{aligned} \mathbb{E}_F \langle P_E x, \theta \rangle^2 &= \mathbb{E}_F \langle x, \theta \rangle^2 \\ &= \frac{1}{|B_\infty^{n-k}|} \int_{B_\infty^{n-k}} \left(\sum_{j=1}^k \varepsilon_j \theta_{i_j} + \sum_{j=1}^{n-k} y_j \theta_{l_j} \right)^2 dy \\ &= \left(\sum_{j=1}^k \varepsilon_j \theta_{i_j} \right)^2 + \frac{1}{3} \left(\sum_{j=1}^{n-k} \theta_{l_j}^2 \right) \\ &= \left(\sum_{j=1}^k \varepsilon_j \theta_{i_j} \right)^2 + \frac{1}{3} \left(1 - \sum_{j=1}^k \theta_{i_j}^2 \right). \end{aligned}$$

This proves the first identity. Now, by integrating on $\theta \in S_E$ with respect to the uniform probability measure on S_E , using the fact that for every $x \in \mathbb{R}^n$

$$|P_E x|^2 = (n-k) \int_{S_E} \langle P_E x, \theta \rangle^2 d\sigma_E(\theta) = (n-k) \int_{S_E} \langle x, \theta \rangle^2 d\sigma_E(\theta),$$

and using Fubini's theorem, we obtain

$$\frac{1}{n-k} \mathbb{E}_F |P_E x|^2 = \frac{1}{3} + \frac{1}{n-k} \left| P_E \left(\sum_{j=1}^k \varepsilon_j e_{i_j} \right) \right|^2 - \frac{1}{3(n-k)} \sum_{j=1}^k |P_E(e_{i_j})|^2,$$

since $\left(\sum_{j=1}^k \varepsilon_j \theta_{i_j} \right)^2 = \left\langle \sum_{j=1}^k \varepsilon_j e_{i_j}, \theta \right\rangle^2$ and $\theta_{i_j}^2 = \langle e_{i_j}, \theta \rangle^2$. This proves the second identity.

The bounds

$$0 \leq \left| P_E \left(\sum_{j=1}^k \varepsilon_j e_{i_j} \right) \right|^2 \leq \left| \sum_{j=1}^k \varepsilon_j e_{i_j} \right|^2 = k$$

and

$$0 \leq |P_E(e_{i_j})|^2 \leq |e_{i_j}|^2 = 1,$$

prove the upper and lower bound for $\mathbb{E}_F |P_E x|^2$ and by using formula (1) we deduce the estimates for $\mathbb{E}_\mu |x|^2$. Finally, notice that

$$\lambda_\mu^2 \geq \int_{S_E} \mathbb{E}_\mu \langle x, \theta \rangle^2 d\sigma_E(\theta) = \frac{1}{n-k} \mathbb{E}_\mu |x|^2 \geq \frac{n-2k}{3(n-k)},$$

which proves the last inequality. \square

We now focus on the first summand in (3). We take into account the fact that for any $(n-k)$ -dimensional face $F = F_{(i_1, \varepsilon_1, \dots, i_k, \varepsilon_k)}$ we can write

$$P_E(F) = a_F + T_F(B_\infty^{n-k}),$$

where $a_F = P_E \left(\sum_{j=1}^k \varepsilon_j e_{i_j} \right)$ and $T_F : \mathbb{R}^{n-k} \rightarrow E$ is the linear map obtained as the composition of I_F in (2) and P_E .

The effect of the translation map in our problem is the content of the next

Lemma 2.3. *Let ν be a symmetric measure in \mathbb{R}^n (i.e., $\nu(A) = \nu(-A)$ for every measurable set A), $a \in \mathbb{R}^n$ and ν_a the translate measure $\nu_a(A) := \nu(A - a)$ (or equivalently, $\int_{\mathbb{R}^n} f(x) d\nu_a(x) = \int_{\mathbb{R}^n} f(x + a) d\nu(x)$ for any integrable measurable function f). Then,*

$$\text{Var}_{\nu_a} |x|^2 = \text{Var}_\nu |x|^2 + 4\mathbb{E}_\nu \langle a, x \rangle^2.$$

Proof.

$$|x + a|^2 = |x|^2 + 2\langle a, x \rangle + |a|^2$$

and

$$|x + a|^4 = |x|^4 + 4|x|^2 \langle a, x \rangle + 4|a|^2 \langle a, x \rangle + 2|x|^2 |a|^2 + 4\langle a, x \rangle^2 + |a|^4.$$

Taking expectations and using symmetry we have,

$$\begin{aligned} \text{Var}_{\nu_a} |x|^2 &= \mathbb{E}_{\nu_a} |x|^4 - (\mathbb{E}_{\nu_a} |x|^2)^2 = \mathbb{E}_\nu |x + a|^4 - (\mathbb{E}_\nu |x + a|^2)^2 = \\ &= \mathbb{E}_\nu |x + a|^4 - (\mathbb{E}_\nu |x|^2 + |a|^2)^2 = \text{Var}_\nu |x|^2 + 4\mathbb{E}_\nu \langle a, x \rangle^2. \end{aligned}$$

\square

Taking in the previous lemma ν as the uniform probability measure on $T_F(B_\infty^{n-k})$ we have

Corollary 2.1. *Let $F = F_{(i_1, \varepsilon_1, \dots, i_k, \varepsilon_k)}$ be an $(n-k)$ -dimensional face of B_∞^n and let $P_E(F) = a_F + T_F(B_\infty^{n-k})$ as above. Then*

$$\text{Var}_{P_E(F)} |x|^2 = \text{Var}_{T_F(B_\infty^{n-k})} |x|^2 + 4\mathbb{E}_{T_F(B_\infty^{n-k})} \langle a_F, x \rangle^2.$$

Lemma 2.4. *Let $E \in G_{n,n-k}$. Then, for any $\theta \in S_E$ and any $(n-k)$ -dimensional face of B_∞^n , $F = F_{(i_1, \varepsilon_1, \dots, i_k, \varepsilon_k)}$, if $P_E(F) = a_F + T_F(B_\infty^{n-k})$ as above, we have*

$$\mathbb{E}_{T_F(B_\infty^{n-k})} \langle x, \theta \rangle^2 = \frac{1}{3} - \frac{1}{3} \sum_{j=1}^k \theta_{i_j}^2 \quad \left(\leq \frac{1}{3} \right),$$

and

$$\mathbb{E}_{T_F(B_\infty^{n-k})} |x|^2 = \frac{n-k}{3} - \frac{1}{3} \sum_{j=1}^k |P_E(e_{i_j})|^2 \quad \left(\leq \frac{n-k}{3} \right).$$

Proof. Notice that

$$\begin{aligned} \mathbb{E}_{P_E(F)} \langle x, \theta \rangle^2 &= \mathbb{E}_{T_F(B_\infty^n)} \langle a_F + x, \theta \rangle^2 = \langle a_F, \theta \rangle^2 + \mathbb{E}_{T_F(B_\infty^n)} \langle x, \theta \rangle^2 \\ &= \left(\sum_{j=1}^k \varepsilon_j \theta_{i_j} \right)^2 + \mathbb{E}_{T_F(B_\infty^n)} \langle x, \theta \rangle^2. \end{aligned}$$

On the other hand, by Lemma 2.2

$$\mathbb{E}_{P_E(F)} \langle x, \theta \rangle^2 = \mathbb{E}_F \langle P_E x, \theta \rangle^2 = \frac{1}{3} + \left(\sum_{j=1}^k \varepsilon_j \theta_{i_j} \right)^2 - \frac{1}{3} \sum_{j=1}^k \theta_{i_j}^2,$$

and we obtain the result. By integrating in $\theta \in S_E$ with respect to the uniform measure and using Fubini's theorem we obtain the second identity. \square

As a consequence we have the following lemma, which gives an upper bound for the first term in (3) of the right order for the variance conjecture to be true as long as $k \leq \frac{n}{3}$.

Lemma 2.5. *Let $E \in G_{n,n-k}$. Then, for any $(n-k)$ -dimensional face F of B_∞^n we have,*

$$\text{Var}_{P_E(F)} |x|^2 \leq Cn.$$

Consequently, there exists an absolute constant C such that if $k \leq \frac{n}{3}$ and $\{F_i\}_{i=1}^l$ is the set of $(n-k)$ -dimensional faces described in (1) then

$$\sum_{i=1}^l \frac{|P_E(F_i)|}{|K|} \text{Var}_{P_E(F_i)} |x|^2 \leq C \lambda_\mu^2 \mathbb{E}_\mu |x|^2.$$

Proof. By Corollary 2.1, we have that for any such F

$$\text{Var}_F |P_E x|^2 = \text{Var}_{T_F(B_\infty^{n-k})} |x|^2 + 4 \mathbb{E}_{T_F(B_\infty^{n-k})} \langle a_F, x \rangle^2.$$

Since B_∞^{n-k} verifies the Kannan-Lovász-Simonovits conjecture (see [LW]), every linear transform of it verifies the variance conjecture and therefore there exists an absolute constant C such that

$$\text{Var}_{T_F(B_\infty^{n-k})} |x|^2 \leq C \lambda_{T_F(B_\infty^{n-k})}^2 \mathbb{E}_{T_F(B_\infty^{n-k})} |x|^2.$$

Since by Lemma 2.4 the two factors involved are bounded by $\frac{1}{3}$ and $\frac{n-k}{3}$ respectively, we have $\text{Var}_{T_F(B_\infty^{n-k})} |x|^2 \leq C(n-k)$.

On the other hand, by Lemma 2.4,

$$\mathbb{E}_{T_F(B_\infty^{n-k})} \langle a_F, x \rangle^2 \leq \frac{1}{3} |a_F|^2 = \frac{1}{3} \left| P_E \left(\sum_{j=1}^k \varepsilon_j e_{i_j} \right) \right|^2 \leq \frac{1}{3} \left| \left(\sum_{j=1}^k \varepsilon_j e_{i_j} \right) \right|^2 = \frac{1}{3} k.$$

Therefore, there exists an absolute constant C such that

$$\text{Var}_{P_E(F)} |x|^2 \leq C(n - k + k) = Cn,$$

which proves the first part of the Lemma.

For the second part, notice that by Lemma 2.2 we have

$$\lambda_\mu^2 \mathbb{E}_\mu |x|^2 \geq \frac{(n - 2k)^2}{9(n - k)} \geq \frac{n}{54}$$

when $1 \leq k \leq \frac{n}{3}$ and now the second part of the Lemma easily follows. \square

For the second summand of (3) we invoke once again Lemma 2.2. The estimates therein provide an upper bound of the right order for the variance conjecture to hold as long as $k \leq \sqrt{n}$.

Lemma 2.6. *Let $E \in G_{n,n-k}$ and let μ be the uniform probability on $K = P_E(B_\infty^n)$. Then for any $(n - k)$ -dimensional face F of B_∞^n we have,*

$$|\mathbb{E}_F |P_E x|^2 - \mathbb{E}_\mu |x|^2| \leq \frac{4k}{3}.$$

Consequently, there exists an absolute constant C such that if $k \leq \sqrt{n}$ and $\{F_i\}_{i=1}^l$ is the set of $(n - k)$ -dimensional faces described in (1),

$$\sum_{i=1}^l \frac{|P_E(F_i)|}{|K|} (\mathbb{E}_{F_i} |P_E x|^2 - \mathbb{E}_\mu |x|^2)^2 \leq C \lambda_\mu^2 \mathbb{E}_\mu |x|^2.$$

Proof. By Lemma 2.2 we have that for any $(n - k)$ dimensional face F

$$-\frac{4k}{3} \leq \mathbb{E}_F |P_E x|^2 - \mathbb{E}_\mu |x|^2 \leq \frac{4k}{3}.$$

Therefore,

$$\sum_{i=1}^l \frac{|P_E(F_i)|}{|K|} (\mathbb{E}_{F_i} |P_E x|^2 - \mathbb{E}_\mu |x|^2)^2 \leq \frac{16k^2}{9}.$$

On the other hand, using as above the bound $\lambda_\mu^2 \mathbb{E}_\mu |x|^2 \geq \frac{n}{54}$ we have that if $k \leq \sqrt{n}$

$$\sum_{i=1}^l \frac{|P_E(F_i)|}{|K|} (\mathbb{E}_{F_i} |P_E x|^2 - \mathbb{E}_\mu |x|^2)^2 \leq C \lambda_\mu^2 \mathbb{E}_\mu |x|^2.$$

\square

Lemmas 2.5 and 2.6, together with formula (3) prove Theorem 1.1.

3. THE VARIANCE CONJECTURE ON RANDOM $(n - k)$ -DIMENSIONAL PROJECTIONS OF THE CUBE

We will show that we can improve the range of the codimension k for which the variance conjecture remains true on a random subspace $E \in G_{n,n-k}$. In order to do that we will consider, for any $(n - k)$ -dimensional face F of B_∞^n , the function $f : G_{n,n-k} \rightarrow \mathbb{R}$ given by $f(E) = \mathbb{E}_F |P_E x|^2$ and make use of the concentration of measure theorem, proved by Gromov and Milman, on $G_{n,n-k}$ (see, for instance, [MS]). As a consequence, since the value of $\mathbb{E}_{\mu_{n,n-k}} f(E)$ does not depend on F , we will obtain that, as long as k is in the range considered in the statement of Theorem 1.2, for a set of subspaces with large measure $f(E)$ is very close to its expected value and, therefore, to $\mathbb{E}_\mu |x|^2$ for every $(n - k)$ -dimensional face F . This will imply that the second term in (3) is bounded by Cn for every subspace E in this set. Then, since we have seen before that the first term in (3) is also bounded by Cn for a larger range of k , we have that

$$\text{Var}_\mu |x|^2 \leq Cn \leq C\lambda_\mu^2 \mathbb{E}_\mu |x|^2.$$

We will denote by $O(n)$ the orthogonal group equipped with the Hilbert-Schmidt distance $\|\cdot\|_{HS}$ and we represent any $U \in O(n)$ by $U = (u_1, \dots, u_n)$, where (u_i) is an orthonormal basis of \mathbb{R}^n . Let us recall that for any linear map $T = (t_{ij})_{i,j=1}^n$, its Hilbert-Schmidt norm is defined by

$$\|T\|_{HS} := \left(\sum_{i,j=1}^n t_{ij}^2 \right)^{\frac{1}{2}}$$

and that $\|T\| \leq \|T\|_{HS}$, where $\|T\|$ denotes the usual operator norm $\|T\| = \sup_{x \in S^{n-1}} |Tx|$.

Theorem 3.1 (Concentration of measure). *Let $f : G_{n,n-k} \rightarrow \mathbb{R}$ be a Lipschitz function with Lipschitz constant σ with respect to the distance*

$$d(E_1, E_2) = \inf \{ \|U - V\|_{HS} : U, V \in O(n), E_1 = \text{span}\{u_1, \dots, u_{n-k}\}, \\ E_2 = \text{span}\{v_1, \dots, v_{n-k}\} \}.$$

Then, for every $\lambda > 0$

$$\mu_{n,n-k} \{ E \in G_{n,n-k} : |f(E) - \mathbb{E}f(E)| > \lambda \} \leq c_1 e^{-\frac{c_2 \lambda^2 n}{\sigma^2}},$$

where c_1 and c_2 are positive absolute constants.

In the following lemma we compute the expected value of f . Let us point out that what matters to us for our purposes is that, due to the symmetries of B_∞^n , its value does not depend on the face F . Nevertheless, we compute its exact value.

Lemma 3.1. *Let F be an $(n - k)$ -dimensional face of B_∞^n . Then*

$$\int_{G_{n,n-k}} \mathbb{E}_F |P_E x|^2 d\mu_{n,n-k}(E) = \frac{(n - k)(n + 2k)}{3n}.$$

Proof. Notice that, by Fubini's theorem and the uniqueness of the Haar measure σ on S^{n-1} we have

$$\int_{G_{n,n-k}} \mathbb{E}_F |P_E x|^2 d\mu_{n,n-k}(E) = (n - k) \int_{G_{n,n-k}} \mathbb{E}_F \int_{S_E} \langle P_E x, \theta \rangle^2 d\sigma_E(\theta) d\mu_{n,n-k}(E)$$

$$\begin{aligned}
&= (n-k) \int_{G_{n,n-k}} \mathbb{E}_F \int_{S_E} \langle x, \theta \rangle^2 d\sigma_E(\theta) d\mu_{n,n-k}(E) \\
&= (n-k) \mathbb{E}_F \int_{G_{n,n-k}} \int_{S_E} \langle x, \theta \rangle^2 d\sigma_E(\theta) d\mu_{n,n-k}(E) \\
&= (n-k) \mathbb{E}_F \int_{S^{n-1}} \langle x, \theta \rangle^2 d\sigma(\theta) \\
&= \frac{n-k}{n} \mathbb{E}_F |x|^2.
\end{aligned}$$

Using the description of any $(n-k)$ dimensional face F given in (2), a random vector uniformly distributed in the face F has k coordinates equal to ± 1 and the other $n-k$ coordinates are given by a random vector uniformly distributed on B_∞^{n-k} . Thus, the latter expectation equals

$$\begin{aligned}
\frac{n-k}{n} \left(k + \mathbb{E}_{B_\infty^{n-k}} |x|^2 \right) &= \frac{n-k}{n} \left(k + \frac{n-k}{3} \right) \\
&= \frac{n-k}{n} \frac{n+2k}{3}.
\end{aligned}$$

□

In the following lemma we estimate the Lipschitz constant of f with respect to the distance defined in Theorem 3.1. Notice that, as before, its value does not depend on F .

Lemma 3.2. *Let $F = F_{(i_1, \varepsilon_1, \dots, i_k, \varepsilon_k)}$ be an $(n-k)$ -dimensional face of B_∞^n and let $f : G_{n,n-k} \rightarrow \mathbb{R}$ be the function defined as $f(E) = \mathbb{E}_F |P_E x|^2$. For any $E_1, E_2 \in G_{n,n-k}$ we have*

$$|f(E_1) - f(E_2)| \leq \frac{8\sqrt{2}k}{3} d(E_1, E_2).$$

Proof. Let $E_1, E_2 \in G_{n,n-k}$. By Lemma 2.2 we have

$$\begin{aligned}
&|f(E_1) - f(E_2)| = |\mathbb{E}_F |P_{E_1} x|^2 - \mathbb{E}_F |P_{E_2} x|^2| \\
&= \left| \left| P_{E_1} \left(\sum_{j=1}^k \varepsilon_j e_{i_j} \right) \right|^2 - \left| P_{E_2} \left(\sum_{j=1}^k \varepsilon_j e_{i_j} \right) \right|^2 - \frac{1}{3} \left(\sum_{j=1}^k |P_{E_1}(e_{i_j})|^2 - |P_{E_2}(e_{i_j})|^2 \right) \right| \\
&\leq \left| \left| P_{E_1} \left(\sum_{j=1}^k \varepsilon_j e_{i_j} \right) \right|^2 - \left| P_{E_2} \left(\sum_{j=1}^k \varepsilon_j e_{i_j} \right) \right|^2 \right| \\
&+ \frac{1}{3} \left| \left(\sum_{j=1}^k |P_{E_1}(e_{i_j})|^2 - |P_{E_2}(e_{i_j})|^2 \right) \right| \\
&= \left| \left| P_{E_1} \left(\sum_{j=1}^k \varepsilon_j e_{i_j} \right) \right| + \left| P_{E_2} \left(\sum_{j=1}^k \varepsilon_j e_{i_j} \right) \right| \right| \left| \left| P_{E_1} \left(\sum_{j=1}^k \varepsilon_j e_{i_j} \right) \right| - \left| P_{E_2} \left(\sum_{j=1}^k \varepsilon_j e_{i_j} \right) \right| \right| \\
&+ \frac{1}{3} \left(\sum_{j=1}^k \left| |P_{E_1}(e_{i_j})| + |P_{E_2}(e_{i_j})| \right| \left| |P_{E_1}(e_{i_j})| - |P_{E_2}(e_{i_j})| \right| \right)
\end{aligned}$$

$$\begin{aligned}
&\leq 2\sqrt{k} \left| (P_{E_1} - P_{E_2}) \left(\sum_{j=1}^k \varepsilon_j e_{i_j} \right) \right| \\
&+ \frac{2}{3} \sum_{j=1}^k |(P_{E_1} - P_{E_2})(e_{i_j})| \\
&\leq 2k \|P_{E_1} - P_{E_2}\| + \frac{2k}{3} \|P_{E_1} - P_{E_2}\| \\
&= \frac{8k}{3} \|P_{E_1} - P_{E_2}\|.
\end{aligned}$$

Notice that for any $U, V \in O(n)$ such that $E_1 = \text{span}\{u_1, \dots, u_{n-k}\}$ and $E_2 = \text{span}\{v_1, \dots, v_{n-k}\}$, the vectors $\{u_j\}_{j=1}^{n-k}$ and $\{v_j\}_{j=1}^{n-k}$ form orthonormal basis of E_1 and E_2 respectively and for any $x \in \mathbb{R}^n$

$$\begin{aligned}
\bullet \quad P_{E_1} x &= \sum_{j=1}^{n-k} \langle P_{E_1} x, u_j \rangle u_j = \sum_{j=1}^{n-k} \langle x, u_j \rangle u_j, \\
\bullet \quad P_{E_2} x &= \sum_{j=1}^{n-k} \langle P_{E_2} x, v_j \rangle v_j = \sum_{j=1}^{n-k} \langle x, v_j \rangle v_j.
\end{aligned}$$

Then, for any such U, V , the projections onto E_1 and E_2 , P_{E_1} and P_{E_2} are given by the matrices (with respect to the canonical basis in \mathbb{R}^n) $(\sum_{j=1}^{n-k} \langle u_j, e_k \rangle \langle u_j, e_l \rangle)_{k,l=1}^n$ and $(\sum_{j=1}^{n-k} \langle v_j, e_k \rangle \langle v_j, e_l \rangle)_{k,l=1}^n$ respectively. Thus,

$$\begin{aligned}
\|P_{E_1} - P_{E_2}\|^2 &\leq \|P_{E_1} - P_{E_2}\|_{HS}^2 = 2(n-k) - 2 \sum_{i,j=1}^{n-k} \langle u_i, v_j \rangle^2 \\
&\leq 2 \sum_{j=1}^{n-k} (1 - \langle u_j, v_j \rangle^2) \leq 2 \sum_{j=1}^{n-k} |u_j - v_j|^2 \leq 2 \sum_{j=1}^n |u_j - v_j|^2 \\
&= 2\|U - V\|_{HS}^2,
\end{aligned}$$

since $1 - \langle u_j, v_j \rangle^2 \leq 2(1 - \langle u_j, v_j \rangle) = |u_j - v_j|^2$. Consequently $\|P_{E_1} - P_{E_2}\| \leq \sqrt{2}d(E_1, E_2)$ and we obtain the result. \square

Lemma 3.3. *Let $1 \leq k \leq \frac{n^{\frac{2}{3}}}{(\log n)^{\frac{1}{3}}}$. There exist positive absolute constants C, c_1, c_2 such that the set*

$$\left\{ E \in G_{n,n-k} : \left| \mathbb{E}_F |P_E x|^2 - \frac{(n-k)(n+2k)}{3n} \right| > C\sqrt{n}, \text{ for some } F \right\}$$

has measure $\mu_{n,n-k}$ smaller than $c_1 e^{-c_2 n^{\frac{2}{3}} (\log n)^{\frac{2}{3}}}$.

Proof. Let F be a fixed $(n-k)$ -dimensional face of B_∞^n . Then, taking $\lambda = C\sqrt{n}$ we obtain, using Theorem 3.1 that

$$\mu_{n,n-k} \left\{ E \in G_{n,n-k} : \left| \mathbb{E}_F |P_E x|^2 - \frac{(n-k)(n+2k)}{3n} \right| > C\sqrt{n} \right\} \leq c_1 e^{-\frac{c_2 C^2 n^2}{k^2}}.$$

Since the number of $(n-k)$ -dimensional faces of B_∞^n equals $2^k \binom{n}{k} \leq 2^k \left(\frac{en}{k}\right)^k$, using the union bound we have that for any $C > 0$

$$\begin{aligned}
&\mu_{n,n-k} \left\{ E \in G_{n,n-k} : \left| \mathbb{E}_F |P_E x|^2 - \frac{(n-k)(n+2k)}{3n} \right| > C\sqrt{n}, \text{ for some } F \right\} \\
&\leq c_1 e^{-\frac{c_2 C^2 n^2}{k^2} + k \log 2 + k \log \frac{en}{k}} \leq c_1 e^{-\frac{c_2 C^2 n^2}{k^2} + c_3 k \log n} \\
&= c_1 e^{-\frac{c_2 C^2 n^2 - c_3 k^3 \log n}{k^2}} \leq c_1 e^{-(c_2 C^2 - c_3) n^{\frac{2}{3}} (\log n)^{\frac{2}{3}}},
\end{aligned}$$

Taking into account that $\frac{c_2 C^2 n^2 - c_3 k^3 \log n}{k^2}$ is decreasing in k and that $1 \leq k \leq \frac{n^{\frac{2}{3}}}{(\log n)^{\frac{1}{3}}}$, the latter term is bounded above by $c_1 e^{-(c_2 C^2 - c_3) n^{\frac{2}{3}} (\log n)^{\frac{2}{3}}}$. Choosing C a constant big enough we obtain the result. \square

As a consequence, we obtain the following lemma, which gives an estimate of the right order for most subspaces, for the second term in (3).

Lemma 3.4. *There exists absolute constants C, c_1, c_2 such that for any $1 \leq k \leq \frac{n^{\frac{2}{3}}}{(\log n)^{\frac{1}{3}}}$, the measure $\mu_{n,n-k}$ of the set of subspaces $E \in G_{n,n-k}$ for which*

$$|\mathbb{E}_{P_E(F)}|x|^2 - \mathbb{E}_\mu|x|^2| \leq C\sqrt{n}$$

for every $(n-k)$ -dimensional face F of B_∞^n is greater than $1 - c_1 e^{-c_2 n^{\frac{2}{3}} (\log n)^{\frac{2}{3}}}$.

Proof. Recall that μ denotes the uniform probability measure on $K = P_E(B_\infty^n)$ and $\mathbb{E}_{P_E(F)}$ denotes the expectation with respect to the uniform probability on $P_E(F)$. By Lemma 3.3 there exists positive absolute constants C, c_1, c_2 and a set of subspaces with measure $\mu_{n,n-k}$ greater than $1 - c_1 e^{-c_2 n^{\frac{2}{3}} (\log n)^{\frac{2}{3}}}$ such that for every E in such set and every $(n-k)$ -dimensional face F of B_∞^n ,

$$\left| \mathbb{E}_{P_E(F)}|x|^2 - \frac{(n-k)(n+2k)}{3n} \right| = \left| \mathbb{E}_F |P_E x|^2 - \frac{(n-k)(n+2k)}{3n} \right| \leq C\sqrt{n}.$$

Then, for every F_1, F_2 , $(n-k)$ -dimensional faces

$$|\mathbb{E}_{P_E(F_1)}|x|^2 - \mathbb{E}_{P_E(F_2)}|x|^2| \leq 2C\sqrt{n}.$$

Consequently, since $\mathbb{E}_\mu|x|^2 = \sum_{i=1}^l \frac{|P_E(F_i)|}{|K|} \mathbb{E}_{P_E F_i}|x|^2$, we have that for every E in this set and every face F

$$\mathbb{E}_{P_E(F)}|x|^2 - \mathbb{E}_\mu|x|^2 \leq \mathbb{E}_{P_E(F)}|x|^2 - \min_{i=1,\dots,l} \mathbb{E}_{P_E(F_i)}|x|^2 \leq 2C\sqrt{n}$$

and

$$\mathbb{E}_F |P_E x|^2 - \mathbb{E}_\mu|x|^2 \geq \mathbb{E}_F |P_E x|^2 - \max_{i=1,\dots,l} \mathbb{E}_{F_i}|x|^2 \geq -2C\sqrt{n}.$$

\square

Now we are able to prove Theorem 1.2.

Proof of Theorem 1.2. If $1 \leq k \leq \frac{n^{\frac{2}{3}}}{(\log n)^{\frac{1}{3}}}$ by Lemma 2.2 we have that there exists an absolute constant C such that $\lambda_\mu^2 \mathbb{E}_\mu|x|^2 \geq Cn$. By equation (3), if $\{F_i\}_{i=1}^l$ are the $(n-k)$ -dimensional faces of B_∞^n described in (1) we have,

$$\text{Var}_\mu|x|^2 \leq \max_{i=1,\dots,l} \text{Var}_{P_E(F_i)}|x|^2 + \max_{i=1,\dots,l} (\mathbb{E}_{P_E(F_i)}|x|^2 - \mathbb{E}_\mu|x|^2)^2.$$

By Lemma 2.5 the first maximum is bounded from above by $C_1 n$ and by Lemma 3.4 there exists a set of $(n-k)$ -dimensional subspaces with measure larger than $1 - c_1 e^{-c_2 n^{\frac{2}{3}} (\log n)^{\frac{2}{3}}}$ such that the second maximum is bounded from above by $C_2 n$, where C_1 and C_2 are absolute constants. This proves Theorem 1.2. \square

REFERENCES

- [AB1] D. ALONSO-GUTIÉRREZ, J. BASTERO *The variance conjecture on some polytopes*. In Asymptotic Geometric Analysis, Proceedings of the Fall 2010, Fields Institute Thematic Program, pp. 1–20, (Springer, 2013)
- [AB2] D. ALONSO-GUTIÉRREZ, J. BASTERO *Approaching the Kannan-Lovász-Simonovits and Variance Conjectures*, Lecture Notes in Math. **2131**, Springer, (2015).
- [ABBW] D. ALONSO-GUTIÉRREZ, J. BASTERO, J. BERNUÉS, P. WOLFF *On the isotropy constant of projections of polytopes*. Journal of Functional Analysis **258**, no. 5 (2010), pp. 1452–1465.
- [BaC] F. BARTHE, D. CORDERO-ERAUSQUIN *Invariances in variance estimates*. Proc. London Math. Soc. **106**, no. 1 (2013), pp. 33–64.
- [BaW] F. BARTHE, P. WOLFF *Remarks on non-interacting conservative spin systems: The case of gamma distributions*. pp. 37–43. Stochastic Process. App. **119** (2009), pp. 2711–2723.
- [B] S. BOBKOV *Spectral gap and concentration for some spherically symmetric probability measures*. Geometric Aspects of Functional Analysis, Lecture Notes in Math. **1807** (2003), pp. 37–43
- [BK] S. BOBKOV, A. KOLDOBSKY *On the central limit property of convex bodies*. Geometric Aspects of Functional Analysis, Lecture Notes in Math. **1807** Springer (2003), pp. 44–52
- [H] N. HUET *Spectral gap for some invariant log-concave probability measures*. Mathematika **57**, no. 1 (2011), pp. 51–62.
- [KLS] R. KANNAN, L. LOVSZ, M. SIMONOVITS *Isoperimetric problems for convex bodies and a localization lemma*. Discrete Comput. Geom. **13**, no. 3-4 (1995), pp. 541–559.
- [K] B. KLARTAG *A Berry-Esseen type inequality for convex bodies with an unconditional basis*. Probab. Theory Related Fields **145**, no. 1-2 (2009), pp. 1–33.
- [LW] R. LATAŁA AND J. O. WOJTASZCZYK *On the infimum convolution inequality*. Studia Math. **189**, no. 2 (2008), pp. 147–187.
- [LV] Y.T. LEE AND S.S. VEMPALA *Eldans Stochastic Localization and the KLS Hyperplane Conjecture: An Improved Lower Bound for Expansion* arXiv:1612.01507
- [MS] V. MILMAN, G. SCHECHTMAN. *Asymptotic theory of finite dimensional normed spaces*, Lecture Notes in Math. **1200**, Springer, (1986).
- [S] S. SODIN *An isoperimetric inequality on the ℓ_p balls*. Ann. Inst. H. Poincaré Probab. Statist. **44**, no. 2 (2008), pp. 362–373.

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