TESIS DE LA UNIVERSIDAD
DE ZARAGOZA

2018

88

Alejandra Sarina Córdova Martínez

Gradings on a family of simple structurable algebras

Departamento

Instituto Universitario de Investigación en
Matemáticas y sus Aplicaciones

Director/es

ELDUQUE PALOMO, ALBERTO CARLOS

http://zaguan.unizar.es/collection/Tesis

ISSN 2254-7606

Prensas de la Universidad
Universidad Zaragoza

Reconocimiento – NoComercial – SinObraDerivada (by-nc-nd): No se permite un uso comercial de la obra original ni la generación de obras derivadas.

© Universidad de Zaragoza
Servicio de Publicaciones

ISSN 2254-7606

Tesis Doctoral

GRADINGS ON A FAMILY OF SIMPLE STRUCTURABLE ALGEBRAS

Autor

Alejandra Sarina Córdova Martínez

Director/es

ELDUQUE PALOMO, ALBERTO CARLOS

UNIVERSIDAD DE ZARAGOZA

Instituto Universitario de Investigación en Matemáticas y sus Aplicaciones

2018

Repositorio de la Universidad de Zaragoza – Zaguan http://zaguan.unizar.es

DOCTORAL THESIS

Gradings on a family of simple structurable algebras

Author

Alejandra Sarina Córdova Martínez

Supervisor

Alberto Elduque

UNIVERSIDAD DE ZARAGOZA

Departamento de Matemáticas

2018

Agradecimientos

Agradezco el apoyo económico otorgado por el Consejo Nacional de Ciencia y Tecnología (CONACyT, México) a través de la beca 420842/262964.

También agradezco a mi director de tesis Alberto Elduque, por sus enseñanzas, su tiempo y su paciencia en estos cuatro años, así como a los miembros del departamento de matemáticas quienes siempre buscaron ayudarme y hacerme sentir como en casa.

A mis queridos padres Julián y Perla y mis hermanos Amaranta y Arturo, por apoyarme incondicionalmente, por todo su amor, confianza y por darme ánimos sobre todo en épocas difíciles. Sé que no ha sido fácil estar lejos estos años pero simpre los siento muy cerca de mí.

A mis compañeros Adrián, Alonso (aunque sólo por unos meses) y Diego, con quienes compartí muchos buenos momentos.

Gracias al grupo de jóvenes de la parroquia Corazón de María quien me acogió recién llegué a Zaragoza y en el que hice muy buenos amigos, en especial al Padre Tomás por siempre estar al pendiente de mí.

Finalmente al resto de mis amigos y familia, por todo su apoyo y cariño.

Contents

Introducción

Las graduaciones por grupos abelianos en álgebras de Lie simples de dimensión finita son ubicuas, empezando con la descomposición como espacios de raíces con respecto a una subálgebra de Cartan en un álgebra de Lie simple que escinde. Sin embargo, un estudio sistemático de graduaciones empezó apenas en 1989 con Patera y Zassenhaus [PZ89]. Cualquier graduación es un engrosamiento de una graduación fina, por lo tanto estas se convirtieron en un objeto central de estudio. En particular, la descomposición como espacios de raíces antes mencionada es fina. La clasificación de graduaciones finas en las álgebras de Lie clásicas simples de dimensión finita sobre un cuerpo algebraicamente cerrado de característica 0 fue finalmente lograda en [Eld10]. Para los casos excepcionales esto se logró a través del trabajo de varios autores: Elduque, Draper, Martín-González, Bahturin, Tvalavadze, Viruel, Yu. (Ver la monografía [EK13] o el trabajo [DE16] para encontrar detalles y referencias.)

Las llamadas *graduaciones por sistemas de raíces* fueron introducidas por Berman y Moody [BM92] y estudiadas por muchos autores: Neher, Benkart, Zelmanov, Allison, Smirnov, etcétera. Estas graduaciones han sido usadas para estudiar familias interesantes de algebras de Lie de dimensión infinita que incluyen a las álgebras de Lie de Kac-Moody.

En 2015, en el esfuerzo por clasificar graduaciones finas en las álgebras de Lie simples excepcionales, Elduque [Eld15] probó que cualquier graduación fina en un álgebra de Lie simple de dimensión finita sobre un cuerpo algebraicamente cerrado de característica 0 se obtiene mezclando, de una forma precisa, una graduación por un sistema de raíces, no necesariamente reducido, de rango igual al rango libre del grupo universal de la graduación fina, y una graduación fina en el 'álgebra coordenada' asociada a la graduación por el sistema de raíces. Esta 'álgebra coordenada' es, en general, un álgebra no asociativa.

En particular, para las graduaciones por el sistema de raíces no reducido de tipo BC_1, el álgebra coordenada es un álgebra estructurable.

Las álgebras estructurables fueron estudiadas por primera vez en 1972 en [K72] por I. L. Kantor quien estaba estudiando una clase más general de álgebras llamada álgebras conservativas. B. N. Allison introdujo las álgebras estructurables, en 1978 en [Al78], como álgebras unitarias (no necesariamente asociativas) con involución que satisfacen ciertas identidades. Los ejemplos más conocidos de álgebras estructurables son las álgebras de Jordan (con la identidad como involución).

En [Al78] Allison dio un teorema de clasificación de álgebras estructurables simples centrales de dimensión finita sobre un cuerpo de característica 0 con un caso faltante. O. Smirnov probó en [Sm90, Teorema 2.1] que cualquier álgebra estructurable semisimple es la suma directa de álgebras simples. Las álgebras simples son simples centrales sobre su centro, entonces la descripción de las álgebras semisimples se reduce a la descripción de las álgebras simples centrales. Smirnov, en [Sm90, Teorema 3.8], también completó la clasificación de las álgebras estructurables simples centrales de dimensión finita sobre un cuerpo de característica distinta de 2, 3 y 5.

Las álgebras estructurables simples centrales $(\mathcal{A}, -)$ dan lugar a álgebras de Lie simples centrales a través de diferentes construcciones. Un ejemplo es la construcción modificada de Kantor-Koecher-Tits usada en [Al79] para obtener todas las álgebras de Lie simples isotrópicas sobre cuerpos de característica 0. Esta es la construcción que yace detrás de las álgebras de Lie graduadas por el sistema de raíces no reducido de tipo BC_1. Para un grupo G, partiendo de una G-graduación en un álgebra estructurable simple central, podemos obtener una $\mathbb{Z} \times G$-graduación en el álgebra de Lie simple central asociada. Si la graduación en el álgebra estructurable es fina también lo es la graduación obtenida en el álgebra de Lie simple.

El objetivo principal de esta tesis es la clasificación de las graduaciones (por grupos) en una de las familias de álgebras estructurables simples: el producto tensorial de un álgebra de Cayley y un álgebra de Hurwitz $(\mathcal{C}^1 \otimes \mathcal{C}^2, -)$ siendo la involución el producto tensorial de las involuciones estándar de $\mathcal{C}^1$ y $\mathcal{C}^2$ respectivamente. Sabemos, por [Al79], que podemos obtener las álgebras de Lie simples centrales de tipo F_4, E_6, E_7 y E_8, a través de una construcción Kantor-Koecher-Tits modificada, a partir de estas álgebras $(\mathcal{C}^1 \otimes \mathcal{C}^2, -)$.

Una graduación fina muy importante en el álgebra estructurable simple excepcional de dimensión 56, que es responsable de algunas graduaciones peculiares en las álgebras de Lie simples de tipo E fue estudiada por Diego Aranda-Orna en su tesis [Ara17] (ver también [AEK14]), y las graduaciones en las álgebras estructurables simples de dimensión 35 descubiertas por Smirnov [Sm90], las cuales faltaron en la clasificación inicial de Allison, también han sido clasificadas por Diego Aranda-Orna (aún no publicado).

En el proceso de obtención de las graduaciones en el producto tensorial $(\mathcal{C}^1 \otimes \mathcal{C}^2, -)$ encontramos que el problema se podía reducir a encontrar graduaciones en el producto cartesiano $\mathcal{C}^1 \times \mathcal{C}^2$. Por supuesto esta no es un álgebra simple, sino una semisimple, y no se ha trabajado mucho en graduaciones en dichas álgebras. Sin embargo, las herramientas necesarias para movernos de álgebras simples a semisimples (esta palabra significará una suma directa finita de álgebras simples) son ya conocidas, por lo tanto partimos de nuestro objetivo original para dar clasificaciones completas de graduaciones en álgebras semisimples, una vez que las graduaciones en álgebras simples son conocidas.

Con la clasificación de graduaciones en álgebras semisimples a la mano, pudimos finalmente completar la buscada clasificación de graduaciones en álgebras estructurables $(\mathcal{C}^1 \otimes \mathcal{C}^2, -)$.

Nos referiremos a graduaciones por grupos cuando pongamos solamente "graduaciones".

La estructura de esta tesis es la siguiente:

En el Capítulo 1 damos definiciones y resultados sobre graduaciones, esquemas y álgebras lazo que necesitaremos para el resto de la tesis.

El Capítulo 2 está dedicado a las álgebras semisimples las cuales, para nuestro propósito, definimos como sumas directas finitas de ideales simples de dimensión finita. Empezamos dando algunos resultados que relacionan estas álgebras con álgebras lazo. Luego definimos una graduación en el producto de álgebras lazo de álgebras simples y damos una clasificación, salvo isomorfismo, de tales graduaciones. Finalmente, definiendo una graduación en el producto de álgebras graduadas, damos la clasificación de graduaciones finas en álgebras semisimples salvo equivalencia.

En el Capítulo 3 obtenemos las graduaciones en la superálgebra de Jordan de Kac K_{10}. Probamos que, para determinar estas graduaciones, es suficiente obtener las graduaciones, salvo equivalencia e isomorfismo, en $\mathsf{K}_3 \times \mathsf{K}_3$ donde K_3 es la superálgebra de Kaplansky de dimensión 3, la cual es simple $(\operatorname{char} \mathbb{F} \neq 2)$. Esto sirve como ejemplo de los resultados dados en el Capítulo 2 y servirá como preparación para obtener graduaciones en $(\mathcal{C}^1 \otimes \mathcal{C}^2, -)$ ya que el proceso es similar.

En el Capítulo 4 recordamos las definiciones de las álgebras de Hurwitz así como la clasificación de las graduaciones en ellas. También probamos un resultado sobre el esquema en grupos de automorfismos de un producto tensorial de álgebras de Cayley que usaremos después para simplificar el cálculo de graduaciones en el producto tensorial de dos álgebras de Cayley.

En el Capítulo 5 determinamos las graduaciones en el producto tensorial de un álgebra de Cayley y un álgebra de Hurwitz $(\mathcal{C}^1 \otimes \mathcal{C}^2, -)$. Para el caso donde $\mathcal{C}^1$ y $\mathcal{C}^2$ son álgebras de Cayley empezamos calculando graduaciones en el producto directo de ellas, y para este propósito usamos resultados dados en el Capítulo 2.

Introduction

Gradings by abelian groups on finite-dimensional simple Lie algebras are ubiquitous, starting with the root space decomposition with respect to a Cartan subalgebra in a split simple Lie algebra. However, a systematic study of gradings was started only in 1989 by Patera and Zassenhaus [PZ89]. Any grading is a coarsening of a fine grading, so these became a central object of study. In particular, the root space decomposition mentioned above is fine. The classification of fine gradings on the finite dimensional simple classical Lie algebras over an algebraically closed field of characteristic 0 was finally achieved in [Eld10]. For the exceptional cases, this was achieved through the work of several authors: Elduque, Draper, Martín-González, Bahturin, Tvalavadze, Viruel, Yu. (See the monograph [EK13] or the survey [DE16] for details and references.)

The so called *gradings by root systems* were introduced by Berman and Moody [BM92], and studied by many authors: Neher, Benkart, Zelmanov, Allison, Smirnov, etcetera. These gradings have been used to study interesting families of infinite-dimensional Lie algebras that include the Kac-Moody Lie algebras.

In 2015, in the effort to classify fine gradings on the exceptional simple Lie algebras, Elduque [Eld15] proved that any fine grading on a finite dimensional simple Lie algebra over an algebraic closed field of characteristic 0 is obtained by mixing, in a precise way, a grading by a root system, not necessarily reduced, of rank equal to the free rank of the universal group of the fine grading, and a fine grading on the 'coordinate algebra' attached to the grading by the root system. This 'coordinate algebra' is, in general, a nonassociative algebra.

In particular, for the gradings by the nonreduced root system of type BC_1, the coordinate algebra is a structurable algebra.

Structurable algebras were first studied in 1972 in [K72] by I. L. Kantor who was studying a more general class of algebras called conservative algebras. B. N. Allison introduced structurable algebras, in 1978 in [Al78], as unital (no necessarily associative) algebras with involution satisfying some identities. The best known examples of structurable algebras are Jordan algebras (with the identity involution).

In [Al78] Allison gave a classification theorem of finite-dimensional central simple structurable algebras over a field of characteristic 0 with a missing item. O. Smirnov proved in [Sm90, Theorem 2.1] that any semisimple structurable algebra is the direct sum of simple algebras. The simple algebras are central simple over their centre, and thus the description of semisimple algebras is reduced to the description of central simple algebras. Smirnov, in [Sm90, Theorem 3.8], also completed the classification of finite-dimensional central simple structurable algebras over a field of characteristic different of 2, 3 and 5.

Central simple structurable algebras $(\mathcal{A}, -)$ give rise to central simple Lie algebras through different constructions. One example is the modified Kantor-Koecher-Tits construction used in [Al79] to get all the isotropic simple Lie algebras over fields of characteristic 0. This is the construction that lies behind the Lie algebras graded by the nonreduced root system of type BC_1. For a group G, starting from a G-grading on a central simple structurable algebra, we can obtain a $\mathbb{Z} \times G$-grading on the associated central simple Lie algebra. If the grading on the structurable algebra is fine, so is the grading obtained on the simple Lie algebra.

The main goal of this thesis is the classification of gradings (by groups) on one of the families of simple structurable algebras: the tensor product of a Cayley algebra and a Hurwitz algebra $(\mathcal{C}^1 \otimes \mathcal{C}^2, -)$ with the involution being the tensor product of the standard involutions of $\mathcal{C}^1$ and $\mathcal{C}^2$ respectively. We know, by [Al79], that we can obtain the central simple Lie algebras of type F_4, E_6, E_7 and E_8, through a modified Kantor-Koecher-Tits construction, from these algebras $(\mathcal{C}^1 \otimes \mathcal{C}^2, -)$.

A quite important fine grading on the exceptional simple structurable algebra of dimension 56, which is responsible of some peculiar gradings on the simple Lie algebras of type E was studied by Diego Aranda-Orna in his thesis [Ara17] (see also [AEK14]), and gradings on the 35-dimensional simple structurable algebras discovered by Smirnov [Sm90], which was missing in the initial classification by Allison, have been classified too by Diego Aranda-Orna (unpublished).

In the process of obtaining the gradings on the *tensor* product $(\mathcal{C}^1 \otimes \mathcal{C}^2, -)$ we found that the problem could be reduced to the problem of finding gradings on the *cartesian* product $\mathcal{C}^1 \times \mathcal{C}^2$. Of course this is not a simple algebra, but a semisimple one, and not much work has been done on gradings on such algebras. However, the necessary tools to move from simple to semisimple algebras (and here this word will mean a finite direct sum of simple algebras) are already known, so we departed from our original goal to give complete classifications of gradings on semisimple algebras, once the gradings on simple algebras are known.

With the classification of gradings on semisimple algebras at hand, we could finally complete the sought classification of gradings on the structurable algebras $(\mathcal{C}^1 \otimes \mathcal{C}^2, -)$.

We will refer to group-gradings by saying only “gradings”.

The structure of this thesis is the following:

In Chapter 1 we give definitions and results about gradings, schemes and loop algebras we will need for the rest of the thesis.

Chapter 2 is devoted to semisimple algebras which, for our purpose, we define as finite direct sums of simple finite-dimensional ideals. We start by giving some results that relate these algebras with loop algebras. Then we define a grading on the product of loop algebras of simple algebras and give a classification, up to isomorphism, of such gradings. Finally, defining a grading on the product of graded algebras, we give the classification of fine gradings on semisimple algebras, up to equivalence.

In Chapter 3 we obtain the gradings on the Kac's Jordan superalgebra K_{10}. We prove that, in order to determine these gradings, it is enough to obtain the gradings, up to equivalence and isomorphism, on $\mathsf{K}_3 \times \mathsf{K}_3$ where K_3 is the 3-dimensional Kaplansky superalgebra, which is simple ($\operatorname{char} \mathbb{F} \neq 2$). This works as an example of the results given in Chapter 2 and will work as preparation to obtain gradings on $(\mathcal{C}^1 \otimes \mathcal{C}^2, -)$ since the process is similar.

In Chapter 4 we recall the definitions of Hurwitz algebras as well as the classification of gradings on them. We also prove a result about the automorphism group scheme of a tensor product of Cayley algebras that we will use later on to simplify the computation of gradings on the tensor product of two Cayley algebras.

In Chapter 5 we determine the gradings on the tensor product of a Cayley algebra and a Hurwitz algebra $(\mathcal{C}^1 \otimes \mathcal{C}^2, -)$. For the case where $\mathcal{C}^1$ and $\mathcal{C}^2$ are Cayley algebras we start by computing gradings on the direct product of them, and for this purpose we use results given in Chapter 2.

Chapter 1

Preliminaries

In this chapter we give the definitions and results that we will need later on.

1.1 Gradings

Let V be a vector space over a field $\mathbb{F}$ and let G be a set.

Definition 1.1.1. *A G-**grading** Γ on V is any decomposition of V into a direct sum of subspaces indexed by G,*

$$\Gamma : V = \bigoplus_{g \in G} V_g.$$

Here we allow some of the subspaces V_g to be zero. The set

$$\operatorname{Supp}\Gamma := \{g \in G : V_g \neq 0\}$$

*is called the **support** of Γ. The grading is **nontrivial** if the support consists of more than one element. If $v \in V_g$, then we say that v is **homogeneous** of **degree** g and we write $\deg v = g$. The subspace V_g is called the **homogeneous component** of degree g. If a grading Γ is fixed, then V will be referred to as a **graded vector space**.*

*Any element $v \in V$ can be uniquely written as $\sum_{g \in G} v_g$ where $v_g \in V_g$ and all but finitely many of the elements v_g are zero. We will refer to the v_g's as the **homogeneous components** of v.*

There are two natural ways in which a linear map $f : V \to W$ can respect gradings on V and W.

Definition 1.1.2. *Let V be a G-graded vector space and let W be an H-graded vector space. A linear map $f : V \to W$ will be called **graded** if for* *any $g \in G$ there exists $h \in H$ such that $f(V_g) \subset W_h$. Clearly, if $f(V_g) \neq 0$, then h is uniquely determined.*

Definition 1.1.3. *Let V and W be G-graded vector spaces. A linear map $f : V \to W$ will be called a **homomorphism of G-graded spaces** if for all $g \in G$, we have $f(V_g) \subset W_g$.*

Definition 1.1.4. *A subspace $W \subset V$ is said to be a **graded subspace** if*

$$W = \bigoplus_{g \in G} (V_g \cap W).$$

This happens if and only if, for any element $v \in W$, all its homogeneous components v_g are also in W. Taking $W_g = V_g \cap W$, we turn W into a G-graded vector space so that the imbedding $W \hookrightarrow V$ is a homomorphism of G-graded spaces. In particular, if $H \subset G$, then

$$V_H := \bigoplus_{h \in H} V_h$$

is a graded subspace of V.

If U is a G-graded vector space and V is an H-graded vector space, then the tensor product $W = U \otimes V$ has a natural $G \times H$-grading given by $W_{(g,h)} = U_g \otimes V_h$. If both U and V are G-graded and G is a semigroup, then $W = U \otimes V$ can also be regarded as a G-graded vector space:

$$W_g := \bigoplus_{g_1, g_2 \in G : g_1 g_2 = g} U_{g_1} \otimes V_{g_2}.$$

Let $\mathcal{A}$ be a nonassociative algebra. The most general concept of grading on $\mathcal{A}$ is a decomposition of $\mathcal{A}$ into a direct sum of subspaces such that the product of any two subspaces is contained in a third subspace. Using the terminology we just introduced, we can state this as follows.

Definition 1.1.5. *Let S be a set. A **set S-grading** on $\mathcal{A}$ is a vector space grading such that the multiplication map $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is graded (Definition 1.1.2), where $\mathcal{A} \otimes \mathcal{A}$ has its natural $S \times S$ grading. If such a grading on $\mathcal{A}$ is fixed, then $\mathcal{A}$ will be referred to as a **set graded algebra**.*

For the following discussion, it will be convenient to discard the homogeneous components that are zero, i.e., to assume that S is the support of the grading:

$$\Gamma : \mathcal{A} = \bigoplus_{s \in S} \mathcal{A}_s \text{ where } \mathcal{A}_s \neq 0 \text{ for any } s \in S. \tag{1.1.1}$$

Then for any $s_1, s_2 \in S$ either $\mathcal{A}_{s_1}\mathcal{A}_{s_2} = 0$ or there is a unique $s_3 \in S$ with $\mathcal{A}_{s_1}\mathcal{A}_{s_2} \subset \mathcal{A}_{s_3}$. Thus the support S is equipped with a partially defined (nonassociative) binary operation $s_1 \cdot s_2 = s_3$.

Definition 1.1.6. *We will say that* Γ *as in* (1.1.1) *is a* ***(semi)group grading*** *if* $(S, \cdot)$ *can be imbedded into a (semi)group* G*.*

Regarding S as a subset of the (semi)group G and setting $\mathcal{A}_g = 0$ for $g \in G \setminus S$, we have the next definition.

Definition 1.1.7. *A* ***grading*** *by a (semi)group* G *on an algebra* $\mathcal{A}$ *(not necessarily associative) over a field* $\mathbb{F}$*, or a* G***-grading*** *on* $\mathcal{A}$*, is a vector space decomposition*

$$\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$$

satisfying $\mathcal{A}_{g_1}\mathcal{A}_{g_2} \subset \mathcal{A}_{g_1 g_2}$ *for all* $g_1, g_2 \in G$*. If such a decomposition is fixed we will referred to* $\mathcal{A}$ *as a* G*-graded algebra.*

Replacing G with a sub(semi)group if necessary, we can assume that G is generated by S.

Remark 1.1.8. *If the algebra* $\mathcal{A}$ *is unital then* $1 \in \mathcal{A}_e$ *where* e *is the neutral element of* G*. Let us show this. We have that* $1 = \sum_{g \in G} a_g$ *(finite sum) where* $a_g \in \mathcal{A}_g$*. Then for* $0 \neq x \in \mathcal{A}_h$ *and* $h \in G$

$$x = 1x = \sum_{g \in G} a_g x = a_e x + \sum_{g \in G \setminus \{e\}} a_g x,$$

so

$$x - a_e x - \sum_{g \in G \setminus \{e\}} a_g x = 0.$$

Hence $x - a_e x = 0$ *and then* $x = a_e x$ *and* $a_e \neq 0$*. Analogously* $x = x a_e$ *for all* $x \in \mathcal{A}_h$ *and* $h \in G$*. Therefore* $a_e = 1$ *and so* $1 \in \mathcal{A}_e$*.*

Definition 1.1.9. *We will say that a grading* Γ *as in* (1.1.1) *is* ***realized*** *as a* G*-grading if* G *is a (semi)group containing a bijective copy of* S*, the subspaces*

$$\mathcal{A}_g := \begin{cases} \mathcal{A}_s & \text{if } g = s \in S\ ; \\ 0 & \text{if } g \notin S; \end{cases}$$

form a G*-grading on* $\mathcal{A}$ *as in Definition 1.1.7, and* S *generates* G*. A* ***realization*** *of* Γ *is the* G*-grading determined by a (semi)group* G *and an imbedding* $S \hookrightarrow G$ *as above.*

Definition 1.1.10. *Let $\mathcal{A}$ be an algebra. $\mathcal{A}$ is said to be **simple** if $\mathcal{A}\mathcal{A} \neq 0$ and the only ideals of $\mathcal{A}$ are 0 and $\mathcal{A}$.*

Remark 1.1.11. *We are interested in studying gradings on Lie algebras so next result (Proposition 1.12 of [EK13]) tells us that in order to study semigroup gradings in such algebras we have to study their gradings by abelian groups. That is why we will work with gradings on abelian groups.*

Proposition 1.1.12. *Let $\mathcal{L}$ be a simple Lie algebra over any field. If G is a semigroup and $\mathcal{L} = \bigoplus_{g\in G} \mathcal{L}_g$ is a G-grading with support S where G is generated by S, then G is an abelian group.*

There are two natural ways to define an equivalence relation on group gradings. We will use the term "isomorphism" for the case when the grading group is a part of the definition and "equivalence" for the case when the grading group plays a secondary role. An **equivalence of graded vector spaces** $f : V \to W$ is a linear isomorphism such that both f and f^{-1} are graded maps (Definition 1.1.2). Let

$$\Gamma : \mathcal{A} = \bigoplus_{s\in S} \mathcal{A}_s \text{ and } \Gamma' : \mathcal{B} = \bigoplus_{t\in T} \mathcal{B}_t$$

be two gradings on algebras, with supports S and T, respectively.

Definition 1.1.13. *We say that Γ and Γ' are **equivalent** if there exists an **equivalence of graded algebras** $\varphi : \mathcal{A} \to \mathcal{B}$, i.e. an isomorphism of algebras that is also an equivalence of graded vector spaces. We will also say that φ is an **equivalence** of Γ and Γ'. It determines a bijection $\alpha : S \to T$ such that $\varphi(\mathcal{A}_s) = \mathcal{B}_{\alpha(s)}$ for all $s \in S$. We will also say that Γ and Γ' are **equivalent via** (φ, α).*

In particular, two equivalent gradings on the same algebra $\mathcal{A}$ can be obtained from one another by the action on $\mathrm{Aut}(\mathcal{A})$ and relabeling the components.

Definition 1.1.14. *We say that two G-graded algebras, $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ and $\mathcal{B} = \bigoplus_{g\in G} \mathcal{B}_g$, are **isomorphic** if there exists an isomorphism of algebras $\varphi : \mathcal{A} \to \mathcal{B}$ such that $\varphi(\mathcal{A}_g) = \mathcal{B}_g$ for all $g \in G$. We denote this by*

$$\mathcal{A} \simeq_G \mathcal{B}.$$

*We will also say that φ is an **isomorphism of G-graded algebras**.*

In particular, two isomorphic gradings on the same algebra $\mathcal{A}$ can be obtained from one another by the action of $\mathrm{Aut}(\mathcal{A})$ (without relabeling) and hence have the same support.

Definition 1.1.15. *The* ***automorphism group*** *of* Γ*, denoted* $\mathrm{Aut}(\Gamma)$*, consists of all self-equivalences of* Γ*, i.e., automorphisms of* $\mathcal{A}$ *that permute the components of* Γ*. Each* $\varphi \in \mathrm{Aut}(\Gamma)$ *determines a self-bijection* $\alpha = \alpha(\varphi)$ *of the support* S *such that* $\varphi(\mathcal{A}_s) = \mathcal{A}_{\alpha(s)}$ *for all* $s \in S$*. The* ***stabilizer*** *of* Γ*, denoted* $\mathrm{Stab}(\Gamma)$*, is the kernel of the homomorphism* $\mathrm{Aut}(\Gamma) \to \mathrm{Sym}(S)$ *given by* $\varphi \mapsto \alpha(\varphi)$*, where* $\mathrm{Sym}(S)$ *denotes the group of permutations of the elements of* S*. (In the case of a* G*-graded algebra this is the same as the group of automorphisms,* $\mathrm{Aut}_G(\mathcal{A})$*, in the category of* G*-graded algebras.) Finally, the* ***diagonal group*** *of* Γ*, denoted* $\mathrm{Diag}(\Gamma)$*, is the (abelian) subgroup of the stabilizer consisting of all automorphisms* φ *such that the restriction of* φ *to any homogeneous component of* Γ *is the multiplication by a (nonzero) scalar.*

As was pointed out earlier, a group grading Γ in general, can be realized as a G-grading for many groups G. It turns out [PZ89] that there is one distinguished group among them.

Definition 1.1.16. *Let* Γ *be a grading on an algebra* $\mathcal{A}$*. Suppose that* Γ *admits a realization as a* G_0*-grading for some group* G_0*. We will say that* G_0 *is a* ***universal group*** *of* Γ *if for any other realization of* Γ *as a* G*-grading, there exists a unique homomorphism* $G_0 \to G$ *that restricts to the identity on* $\mathrm{Supp}\,\Gamma$*.*

Note that, by definition, G_0 is a group with a distinguished generating set, $\mathrm{Supp}\,\Gamma$. A standard argument shows that, if a universal group exists it is unique up to an isomorphism over $\mathrm{Supp}\,\Gamma$. The universal group may not be abelian. For any grading Γ we can define the **universal abelian group** by the same generators and relations.

Since we are interested in abelian grading groups we will use the definition of universal abelian group to refer to the "universal group" and we will denote it by $U(\Gamma)$, from now on we will also assume that all grading groups are abelian.

The following result ([EK13, Proposition 1.18]) shows that $U(\Gamma)$ exists and depends only on the equivalence class of Γ.

Proposition 1.1.17. *Let* Γ *be a group grading on an algebra* $\mathcal{A}$*. Then there exists a universal group* $U(\Gamma)$*. Two group gradings,* Γ *on* $\mathcal{A}$ *and* Γ' *on* $\mathcal{B}$*, are equivalent if and only if there exist an algebra isomorphism* $\varphi : \mathcal{A} \to \mathcal{B}$ *and a group isomorphism* $\alpha : U(\Gamma) \to U(\Gamma')$ *such that* $\varphi(\mathcal{A}_g) = \mathcal{B}_{\alpha(g)}$ *for all* $g \in U(\Gamma)$*.*

Remark 1.1.18. *Let Γ^1 be a $U(\Gamma^1)$-grading and let Γ^2 be a $U(\Gamma^2)$-grading. Suppose Γ^1 and Γ^2 are equivalent gradings via (φ, α), then we will take α not as the bijection on the respective supports but as the group isomorphism $U(\Gamma^1) \to U(\Gamma^2)$.*

Definition 1.1.19. *Let $\Gamma : \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ and $\Gamma' : \mathcal{A} = \bigoplus_{h\in H} \mathcal{A}'_h$ be two gradings on $\mathcal{A}$ with supports S and T, respectively. We will say that Γ is a* ***refinement*** *of Γ', or that Γ' is a* ***coarsening*** *of Γ, if for any $s \in S$ there exists $t \in T$ such that $\mathcal{A}_s \subseteq \mathcal{A}'_t$. If, for some $s \in S$, the inclusion is strict, then we will speak of a* ***proper*** *refinement or coarsening. We say Γ is* ***fine*** *if it does not admit proper refinements.*

Definition 1.1.20. *Let G and H be (semi)groups and let $\alpha : G \to H$ be a (semi)group homomorphism. If $\Gamma : V = \bigoplus_{g\in G} V_g$ is a grading on a vector space V, then the decomposition ${}^{\alpha}\Gamma : V = \bigoplus_{h\in H} V'_h$ defined by*

$$V'_h = \bigoplus_{g\in G:\alpha(g)=h} V_g$$

is an H-grading on V. We will say that ${}^{\alpha}\Gamma$ is the ***grading induced from Γ by the homomorphism*** *α. Notice that the grading induced from Γ by α is a coarsening of Γ, not necessarily proper.*

Lemma 1.1.21. *Let $\mathcal{A}$ be an algebra and let $\Gamma : \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ be a grading by an abelian group G. If $\Gamma' : \mathcal{A} = \bigoplus_{k\in K} \mathcal{A}'_k$ is a refinement of Γ where $K = U(\Gamma')$, then there exists a group homomorphism between K and G given by*

$$\begin{array}{rccl} \varphi: & K & \longrightarrow & G \\ & k & \longmapsto & g, \ \textit{such that } \mathcal{A}'_k \subseteq \mathcal{A}_g. \end{array}$$

Proof. We have the application

$$\begin{array}{rccl} \Phi: & \operatorname{Supp}\Gamma' & \longrightarrow & G \\ & k & \longmapsto & g, \ \text{such that } \mathcal{A}'_k \subseteq \mathcal{A}_g. \end{array}$$

Then Φ extends to a $\varphi : K \longrightarrow G$ group homomorphism, where $\varphi(k) = \Phi(k)$ for $k \in \operatorname{Supp}\Gamma'$. It is enough to prove that φ is a homomorphism in $\operatorname{Supp}\Gamma'$. Let $k_1, k_2 \in \operatorname{Supp}\Gamma'$ such that $\mathcal{A}'_{k_1}\mathcal{A}'_{k_2} \neq 0$. Then

$\{0\} \neq \mathcal{A}'_{k_1}\mathcal{A}'_{k_2} \subseteq \mathcal{A}_{\varphi(k_1)}\mathcal{A}_{\varphi(k_2)} \subseteq \mathcal{A}_{\varphi(k_1)\varphi(k_2)}$ and

$\{0\} \neq \mathcal{A}'_{k_1}\mathcal{A}'_{k_2} \subseteq \mathcal{A}'_{k_1k_2} \subseteq \mathcal{A}_{\varphi(k_1k_2)}$.

So $\mathcal{A}_{\varphi(k_1)\varphi(k_2)} = \mathcal{A}_{\varphi(k_1k_2)}$, hence $\varphi(k_1)\varphi(k_2) = \varphi(k_1k_2)$.

□

The next three results ([EK13, Proposition 1.25], [EK13, Corollary 1.26 and 1.27], respectively) reduce the study of gradings on finite-dimensional algebras to the study of fine gradings by their universal groups on such algebras.

Proposition 1.1.22. *Let Γ be a grading on an algebra $\mathcal{A}$. Assume that Γ is a group grading and $G = U(\Gamma)$ is its universal group. If Γ' is a coarsening of Γ which its itself a group grading, then, for any realization of Γ' as an H-grading for some group H, there exists a unique epimorphism $\alpha : G \to H$ such that $\Gamma' = {}^{\alpha}\Gamma$. Moreover, if $S = \operatorname{Supp}\Gamma$, $T = \operatorname{Supp}\Gamma'$ and $\pi : S \to T$ is the map associated to the coarsening, then $U(\Gamma')$ is the quotient of G by the normal subgroup generated by the elements $s_1 s_2^{-1}$ for all $s_1, s_2 \in S$ with $\pi(s_1) = \pi(s_2)$.*

Corollary 1.1.23. *Let $\mathcal{A}$ be a finite-dimensional algebra. Then all group gradings on $\mathcal{A}$, up to equivalence, are obtained by taking, for each fine group grading Γ on $\mathcal{A}$, the coarsenings Γ_N induced by all quotient maps $U(\Gamma) \to U(\Gamma)/N$ where N is the normal subgroup generated by some elements of the form $s_1 s_2^{-1}, s_1, s_2 \in \operatorname{Supp}\Gamma$. Moreover, $U(\Gamma)/N$ is the universal group of Γ_N.*

Corollary 1.1.24. *Let $\mathcal{A}$ be a finite-dimensional algebra and let G be a group. Then all G-gradings on $\mathcal{A}$, up to isomorphism, are obtained by taking, for each fine group grading Γ on $\mathcal{A}$, the G-gradings induced by all homomorphisms $U(\Gamma) \to G$.*

Note that, in general, a given grading can be induced from many fine gradings, so the descriptions given in Corollaries 1.1.23 and 1.1.24 do not yet give classifications of gradings up to equivalence and up to isomorphism, respectively.

1.2 Schemes

Now we give a summary of definitions and results from the theory of affine group schemes that will be needed for future results. The following can be found in [EK13, Appendix A].

Let $\mathbb{F}$ be a field. Let $\mathrm{Alg}_{\mathbb{F}}$ be the category of commutative associative unital algebras over $\mathbb{F}$. For R and S in $\mathrm{Alg}_{\mathbb{F}}$ we will denote by $\mathrm{Alg}(R, S)$ the set of all morphisms in the category $\mathrm{Alg}_{\mathbb{F}}$, i.e., homomorphisms $R \to S$ of unital $\mathbb{F}$-algebras. Let Set be the category of sets.

Definition 1.2.1. *Let $\boldsymbol{F}$ and $\boldsymbol{G}$ be two functors from $\mathrm{Alg}_{\mathbb{F}}$ to Set. A **natural map** $\theta : \boldsymbol{F} \to \boldsymbol{G}$ is a collection of maps $\theta_R : \boldsymbol{F}(R) \to \boldsymbol{G}(R)$, one for each*

$R \in \mathrm{Alg}_{\mathbb{F}}$, that respects morphisms in $\mathrm{Alg}_{\mathbb{F}}$, i.e., for any homomorphism $\varphi : R \to S$, the following diagram commutes:

$$\begin{array}{ccc} \boldsymbol{F}(R) & \xrightarrow{\theta_R} & \boldsymbol{G}(R) \\ {\scriptstyle \boldsymbol{F}(\varphi)}\downarrow & & \downarrow{\scriptstyle \boldsymbol{G}(\varphi)} \\ \boldsymbol{F}(S) & \xrightarrow{\theta_S} & \boldsymbol{G}(S) \end{array}$$

*A functor $\boldsymbol{F} : \mathrm{Alg}_{\mathbb{F}} \to \mathrm{Set}$ is said to be **representable** if there exists an object $\mathcal{A}$ in $\mathrm{Alg}_{\mathbb{F}}$ such that $\boldsymbol{F}$ is naturally isomorphic to $\mathrm{Alg}_{\mathbb{F}}(\mathcal{A}, \_)$, i.e., for each object R in $\mathrm{Alg}_{\mathbb{F}}$, there is a bijection between $\boldsymbol{F}(R)$ and $\mathrm{Alg}(\mathcal{A}, R)$ that respects morphisms in $\mathrm{Alg}_{\mathbb{F}}$. The object $\mathcal{A}$ is called a **representing object** for $\boldsymbol{F}$ and it is unique up to isomorphism.*

Lemma 1.2.2. (Yoneda) *Let $\boldsymbol{F}$ and $\boldsymbol{G}$ be set-valued functors on $\mathrm{Alg}_{\mathbb{F}}$. Assume that $\mathcal{A}$ and $\mathcal{B}$ are representing objects for $\boldsymbol{F}$ and $\boldsymbol{G}$, respectively. Then the set of natural maps $\boldsymbol{F} \to \boldsymbol{G}$ is in one-to-one correspondence with the set of homomorphisms $\mathcal{B} \to \mathcal{A}$. Moreover, the composition of natural maps corresponds to the composition of homomorphisms in reversed order.*

Definition 1.2.3. *An **affine group scheme** over $\mathbb{F}$ is a functor $\boldsymbol{G} : \mathrm{Alg}_{\mathbb{F}} \to \mathrm{Grp}$ such that the induced functor $\boldsymbol{F} \circ \boldsymbol{G} : \mathrm{Alg}_{\mathbb{F}} \to \mathrm{Grp} \to \mathrm{Set}$ is representable where $\boldsymbol{F}$ is the forgetful functor and Grp is the category of groups. We denote the representing object of $\boldsymbol{G}$ by $\mathbb{F}[\boldsymbol{G}]$.*

*Let $\boldsymbol{G}$ and $\boldsymbol{H}$ be affine group schemes. We say that $\boldsymbol{H}$ is a **subgroup-scheme** of $\boldsymbol{G}$ if, for any R in $\mathrm{Alg}_{\mathbb{F}}$, the group $\boldsymbol{H}(R)$ is a subgroup of $\boldsymbol{G}(R)$ and the injections $\boldsymbol{H}(R) \hookrightarrow \boldsymbol{G}(R)$ respect morphisms in $\mathrm{Alg}_{\mathbb{F}}$, i.e., form a natural map $\boldsymbol{H} \to \boldsymbol{G}$.*

Since the sets $\boldsymbol{G}(R)$ are endowed with multiplication that makes them groups, the representing object $\mathcal{A} = \mathbb{F}[\mathbf{G}]$ should also carry some additional structure. Namely, group multiplication defines a natural map of (set-valued) functors $\boldsymbol{G} \times \boldsymbol{G} \to \boldsymbol{G}$, which in view of Yoneda's Lemma, gives rise to a homomorphism $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$. The associativity of group multiplication translates to the property $(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$. The existence of identity element in each $\boldsymbol{G}(R)$ can be expressed as a natural map from the trivial group scheme to $\boldsymbol{G}$, which gives rise to a homomorphism $\varepsilon : \mathcal{A} \to \mathbb{F}$. The definition of identity element translates to the property $(\varepsilon \otimes id) \circ \Delta = id = (id \otimes \varepsilon) \circ \Delta$ where we identified $\mathbb{F} \otimes \mathcal{A}$ and $\mathcal{A} \otimes \mathbb{F}$ with $\mathcal{A}$. The existence of inverses can be expressed as a natural map $\boldsymbol{G} \to \boldsymbol{G}$, which gives rise to a homomorphism $S : \mathcal{A} \to \mathcal{A}$. The definition of inverse translates to the property $m \circ (S \otimes id) \circ \Delta = \eta \circ \varepsilon = m \circ (id \otimes S) \circ \Delta$ where $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is the multiplication map and $\eta : \mathbb{F} \to \mathcal{A}$ is the map $\lambda \to \lambda 1_{\mathcal{A}}$.

Definition 1.2.4. *A (counital coassociative)* ***coalgebra*** *is a vector space $\mathcal{C}$ with linear maps $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ and $\varepsilon : \mathcal{C} \to \mathbb{F}$ called* ***comultiplication*** *and* ***counit****, respectively, such that the following equations hold:*
$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$ *(coassociativity);*
$(\varepsilon \otimes id) \circ \Delta = id = (id \otimes \varepsilon) \circ \Delta$ *(counit axiom).*
A coalgebra $\mathcal{C}$ is said to be ***cocommutative*** *if $\Delta = \tau \circ \Delta$ where $\tau : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ is the "flip" $a \otimes b \mapsto b \otimes a$. If $\mathcal{C}$ and $\mathcal{D}$ are coalgebras, then a linear map $f : \mathcal{C} \to \mathcal{D}$ is said to be a* ***homomorphism of coalgebras*** *if $(f \otimes f) \circ \Delta_{\mathcal{C}} = \Delta_{\mathcal{D}} \circ f$ and $\varepsilon_{\mathcal{C}} = \varepsilon_{\mathcal{D}} \circ f$. A* ***subcoalgebra*** *of $\mathcal{C}$ is a subspace $\mathcal{D}$ satisfying $\Delta(\mathcal{D}) \subset \mathcal{D} \otimes \mathcal{D}$. A* ***coideal*** *of $\mathcal{C}$ is a subspace I satisfying $\Delta(I) \subset I \otimes \mathcal{C} + \mathcal{C} \otimes I$ and $\varepsilon(I) = 0$.*

The notion of coalgebra is the formal dual of the notion of (unital associative) algebra. Indeed, the latter can be expressed in terms of multiplication map m and unit map η.

Definition 1.2.5. *A* ***bialgebra*** *is a unital associative algebra $\mathcal{B}$ with linear maps $\Delta : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$ and $\varepsilon : \mathcal{B} \to \mathbb{F}$ such that $(\mathcal{B}, \Delta, \varepsilon)$ is a coalgebra, and Δ and ε are homomorphisms of unital algebras (or, equivalently, m and η are homomorphisms of counital coalgebras). A bialgebra $\mathcal{B}$ is said to be a* ***Hopf algebra*** *if there exists a linear map $S : \mathcal{B} \to \mathcal{B}$, called* ***antipode****, such that the following equation holds:*

$$m \circ (S \otimes id) \circ \Delta = \eta \circ \varepsilon = m \circ (id \otimes S) \circ \Delta \text{ (antipode axiom).}$$

If an antipode exists, it is unique. It is automatically an algebra anti-homomorphism, i.e., $S(1) = 1$ and $S(ab) = S(a)S(b)$ for all $a, b \in \mathcal{B}$, and a coalgebra anti-homomorphism [Swe69].

In particular, if $\mathcal{B}$ is commutative, then $S : \mathcal{B} \to \mathcal{B}$ is a homomorphism. We see that the additional structure on the representing object $\mathbb{F}[\mathbf{G}]$ is precisely what is required to make it a commutative Hopf algebra. Conversely, if $\mathcal{A}$ is a commutative Hopf algebra, then, for any R in $\mathrm{Alg}_{\mathbb{F}}$, the set $\mathrm{Alg}(\mathcal{A}, R)$ can be endowed with multiplication. Namely, for $f, g \in \mathrm{Alg}(\mathcal{A}, R)$ we define fg as follows:

$$(fg)(a) := \sum_i f(a_i')g(a_i'') \text{for all } a \in \mathcal{A} \text{ where } \Delta(a) = \sum_i a_i' \otimes a_i'' \qquad (1.2.1)$$

This multiplication is associative (because the coassociativity of Δ), the map $a \mapsto \varepsilon(a)1_R$ is the identity element, and $f \circ S$ is the inverse of f. It follows that $\mathrm{Alg}(\mathcal{A}, \_)$ is an affine group scheme. Looking at the proof of Yoneda's Lemma [EK13, Lemma A.1], one can verify that, if $\boldsymbol{G}$ is an affine group scheme and $\mathcal{A} = \mathbb{F}[\boldsymbol{G}]$, then the multiplication defined by (1.2.1) coincides with the original multiplication in $\boldsymbol{G}(R)$, and, if $\mathcal{A}$ is a commutative Hopf algebra and $\boldsymbol{G} = \mathrm{Alg}(\mathcal{A}, \_)$, then the Hopf algebra structure on $\mathcal{A}$ as the representing object of $\boldsymbol{G}$ coincides with the original one. Thus we have a one-to-one correspondence (more precisely, a duality of categories) between affine group schemes and commutative Hopf algebras. An affine group scheme is abelian if and only if the corresponding Hopf algebra is cocommutative.

Definition 1.2.6. *Let G be a group. The group algebra $\mathbb{F}G$ becomes a Hopf algebra if we declare all elements of G group-like, i.e., define Δ by setting $\Delta(g) = g \otimes g$ for all $g \in G$. If G is abelian, then $\mathbb{F}G$ is commutative and hence gives rise to an affine group scheme, which we will denote by G^D and it is called the* ***Cartier dual*** *of G. For any R in $\mathrm{Alg}_{\mathbb{F}}$, we have*

$$G^D(R) = \mathrm{Hom}(G, R^\times),$$

(see [EK13, Chapter 1, 1.4]). Affine group schemes of this form are called ***diagonalizable****.*

Definition 1.2.7. *Let $\mathbf{G}$ and $\mathbf{H}$ be affine group schemes. A* ***morphism*** *$\theta : \mathbf{G} \to \mathbf{H}$ is a natural map such that, for all R, the map $\theta(R) =: \theta_R : \mathbf{G}(R) \to \mathbf{H}(R)$ is a homomorphism of groups. It follows from Yoneda's Lemma that there is a unique homomorphism of Hopf algebras $\theta^* : \mathbb{F}[\mathbf{H}] \to \mathbb{F}[\mathbf{G}]$ such that $\theta_R(f) = f \circ \theta^*$ for all $f \in \mathrm{Alg}(\mathbb{F}[\mathbf{G}], R)$. We will call θ^* the* ***comorphism*** *of θ. Note that $(\theta_1\theta_2)^* = \theta_2^*\theta_1^*$. Thus Yoneda's Lemma establishes a duality between the category of affine group schemes and the category of commutative Hopf algebras. If such map θ_R is a isomorphism of groups for all R in $\mathrm{Alg}_{\mathbb{F}}$ then we say that θ is an* ***isomorphism*** *of affine group schemes.*

Definition 1.2.8. *A morphism $\theta : \mathbf{G} \to \mathbf{H}$ is said to be a* ***closed imbedding*** *if θ^* is surjective. It follows that, for any R, the map θ_R is injective. A morphism $\theta : \mathbf{G} \to \mathbf{H}$ is said to be a* ***quotient map*** *if θ^* is injective. This does not imply, however, that all θ_R are surjective.*

Let $\mathcal{A}$ be a nonassociative algebra over $\mathbb{F}$ such that $\dim \mathcal{A} = n < \infty$. Then for any R in $\mathrm{Alg}_{\mathbb{F}}$, the tensor product $\mathcal{A} \otimes R$ is an algebra over R. Define

$$\mathbf{Aut}(\mathcal{A})(R) := \mathrm{Aut}_R(\mathcal{A} \otimes R),$$

this is, the group of automorphisms of $\mathcal{A} \otimes R$ as an R-algebra. This defines the affine group scheme $\mathbf{Aut}(\mathcal{A})$.

Let $\Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ be a G-grading on an algebra $\mathcal{A}$. Define the subgroupscheme $\mathbf{Diag}(\Gamma)$ of $\mathbf{Aut}(\mathcal{A})$ as follows:

$$\mathbf{Diag}(\Gamma)(R) := \{f \in \mathrm{Aut}_R(\mathcal{A} \otimes R) : f\mid_{\mathcal{A}_g \otimes R} \in R^\times id_{\mathcal{A}_g \otimes R} \text{ for all } g \in G\}.$$

Clearly, $\mathrm{Diag}(\Gamma) = \mathbf{Diag}(\Gamma)(\mathbb{F})$. We have that $\mathbf{Diag}(\Gamma)$ is diagonalizable, so $\mathbf{Diag}(\Gamma) = U^D$ for some finitely generated abelian group U. It results that $U = U(\Gamma)$ [EK13, p. 23].

Remark 1.2.9. *i) Let Γ be a U-grading on an algebra $\mathcal{A}$ where $U = U(\Gamma)$. Then, for all $R \in \mathrm{Alg}_{\mathbb{F}}$,*

$$\mathrm{Hom}(\mathbb{F}U, R) \simeq U^D(R) \simeq \mathrm{Hom}(U, R^\times).$$

The first isomorphism comes from the definition of group scheme and from the fact that $\mathbb{F}[U^D] = \mathbb{F}U$ and the second isomorphism comes from Definition 1.2.6.

ii) There is an isomorphism of schemes $U^D \to \mathbf{Diag}(\Gamma)$. For $R \in \mathrm{Alg}_{\mathbb{F}}$, $\chi : U \to R^\times$, $x_u \in \mathcal{A}_u$ and $u \in U$ we have the following isomorphism

$$\begin{array}{rcccccc} \Omega(R): & \mathrm{Hom}(U, R^\times) & \longrightarrow & \mathbf{Diag}(\Gamma)(R) & & & \\ & \chi & \longmapsto & f_\chi : & \mathcal{A} \otimes R & \to & \mathcal{A} \otimes R \\ & & & & x_u \otimes r & \longmapsto & x_u \otimes \chi(u) r. \end{array}$$

Then $\Omega(R)$ together with the isomorphism $U^D(R) \simeq \mathrm{Hom}(U, R^\times)$ from i) give us the isomorphism of schemes $U^D \to \mathbf{Diag}(\Gamma)$.

Definition 1.2.10. *For an algebra $\mathcal{A}$ in $\mathrm{Alg}_{\mathbb{F}}$, we will write $\mathrm{rad}\,\mathcal{A}$ for the **nilradical** of $\mathcal{A}$, i.e., the set of all nilpotent elements of $\mathcal{A}$. We will say that $\mathcal{A}$ is **reduced** if $\mathrm{rad}\,\mathcal{A} = 0$.*

*Let $\mathbb{F}$ be an arbitrary field. Let $\mathbf{G}$ be an affine group scheme over $\mathbb{F}$. Then $\overline{\mathbb{F}}[\mathbf{G}] := \mathbb{F}[\mathbf{G}] \otimes \overline{\mathbb{F}}$ is a commutative Hopf algebra over $\overline{\mathbb{F}}$. An affine group scheme $\mathbf{G}$ is said to be **smooth** if $\mathrm{rad}\,\overline{\mathbb{F}}[\mathbf{G}] = 0$.*

There are several results concerning to quotient maps and closed imbeddings using smoothness. Like the next one (Theorem A.50 of [EK13]) which we will use later on.

Theorem 1.2.11. *Let $\theta : \mathbf{G} \to \mathbf{H}$ be a morphism of affine algebraic group schemes. Assume that $\mathbf{G}$ or $\mathbf{H}$ is smooth. Then θ is an isomorphism if and only if*

1) $\theta_{\overline{\mathbb{F}}} : \mathbf{G}(\overline{\mathbb{F}}) \to \mathbf{H}(\overline{\mathbb{F}})$ is bijective and

2) $d\theta : \mathrm{Lie}(\mathbf{G}) \to \mathrm{Lie}(\mathbf{H})$ is bijective.

1.3 Loop algebras

The next definitions and results can be found in [ABFP]. Let $\mathbb{F}$ be an arbitrary field. All algebras are assumed to be algebras (not necessarily associative or unital) over $\mathbb{F}$. We also assume that G is an abelian group.

Definition 1.3.1. *Suppose that $\pi : G \to \overline{G}$ is a group epimorphism, where $\overline{G}$ is an abelian group. We write $\pi(g) = \overline{g}$ for $g \in G$. Suppose there is a $\overline{G}$-grading $\overline{\Gamma} : \mathcal{A} = \bigoplus_{\overline{g}\in\overline{G}} \mathcal{A}_{\overline{g}}$. Then the tensor product $\mathcal{A} \otimes \mathbb{F}G$ is a G-graded algebra over $\mathbb{F}$ where $(\mathcal{A} \otimes \mathbb{F}G)_g = \mathcal{A} \otimes g$ for $g \in G$. Let us set*

$$L_\pi(\mathcal{A}) = \sum_{g\in G} \mathcal{A}_{\overline{g}} \otimes g$$

in $\mathcal{A} \otimes \mathbb{F}G$. Then $L_\pi(\mathcal{A})$ is a G-graded subalgebra of $\mathcal{A} \otimes \mathbb{F}G$ with

$$L_\pi(\mathcal{A})_g = \mathcal{A}_{\overline{g}} \otimes g$$

*for $g \in G$. This algebra is called the **loop algebra** of $\mathcal{A}$ relative to π. If we denote the above G-grading on $L_\pi(\mathcal{A})$ by Γ we will say that Γ is the **grading induced by $\overline{\Gamma}$ on** $L_\pi(\mathcal{A})$.*

Definition 1.3.2. *Let $\mathcal{B}$ be an algebra. Let $\mathrm{Mult}_{\mathbb{F}}(\mathcal{B})$ be the unital subalgebra of $\mathrm{End}_{\mathbb{F}}(\mathcal{B})$ generated by $\{Id_{\mathcal{B}}\} \cup \{l_a : a \in \mathcal{B}\} \cup \{r_a : a \in \mathcal{B}\}$, where l_a (resp. r_a) denotes the left (resp. right) multiplication operator by a. $\mathrm{Mult}_{\mathbb{F}}(\mathcal{B})$ is called the **multiplication algebra** of $\mathcal{B}$. The **centroid** of $\mathcal{B}$ is the centralizer of $\mathrm{Mult}_{\mathbb{F}}(\mathcal{B})$ in $\mathrm{End}_{\mathbb{F}}(\mathcal{B})$ and it is denoted by $C(\mathcal{B})$. $C(\mathcal{B})$ is a unital subalgebra of $\mathrm{End}_{\mathbb{F}}(\mathcal{B})$.*

Definition 1.3.3. *For an algebra $\mathcal{B}$ over $\mathbb{F}$ and $x, y, z \in \mathcal{B}$ we have the **commutator** of x and y given by $[x, y] := xy - yx$ and the **associator** of x, y, z given by $(x, y, z) := (xy)z - x(yz)$. The **nucleus** $N(\mathcal{B})$ of $\mathcal{B}$ is defined to be the set*

$$N(\mathcal{B}) = \{x \in \mathcal{B} : (x, \mathcal{B}, \mathcal{B}) = (\mathcal{B}, x, \mathcal{B}) = (\mathcal{B}, \mathcal{B}, x) = 0\}.$$

The subset

$$Z(\mathcal{B}) := \{c \in N(\mathcal{B}) : [C, \mathcal{B}] = 0\}$$

*is called the **centre** of $\mathcal{B}$.*

Next remark is [ABFP, Remark 4.1.2].

Remark 1.3.4. *If $\mathcal{B}$ is a unital algebra, then the map $a \mapsto l_a$ is an algebra isomorphism of the centre of $\mathcal{B}$ onto $C(\mathcal{B})$. Suppose that $\mathcal{B} = \bigoplus_{g\in G} \mathcal{B}_g$ is a G-graded algebra. For $g \in G$, we let*

$$\mathrm{End}_{\mathbb{F}}(\mathcal{B})_g = \{f \in \mathrm{End}_{\mathbb{F}}(\mathcal{B}) : f(\mathcal{B}_k) \subseteq \mathcal{B}_{gk} \text{ for } k \in G\}.$$

Then $\bigoplus_{g\in G} \mathrm{End}_{\mathbb{F}}(\mathcal{B})_g$ is a G-graded subalgebra of $\mathrm{End}_{\mathbb{F}}(\mathcal{B})$. We set

$$\mathrm{Mult}_{\mathbb{F}}(\mathcal{B})_g = \mathrm{Mult}_{\mathbb{F}}(\mathcal{B}) \cap \mathrm{End}_{\mathbb{F}}(\mathcal{B})_g \text{ and } C(\mathcal{B})_g = C(\mathcal{B}) \cap \mathrm{End}_{\mathbb{F}}(\mathcal{B})_g \quad (1.3.1)$$

for $g \in G$. It is clear that $\mathrm{Mult}_{\mathbb{F}}(\mathcal{B}) = \bigoplus_{g\in G} \mathrm{Mult}_{\mathbb{F}}(\mathcal{B})_g$, and hence $\mathrm{Mult}_{\mathbb{F}}(\mathcal{B})$ is G-graded. Although the centroid is not in general G-graded, it does have this property in many important cases (see for example Lemma 1.3.7 below).

Definition 1.3.5. *Let $\mathcal{B}$ be an algebra. If $\mathcal{B}$ is a G-graded algebra we say that $\mathcal{B}$ is G-**graded-simple** if $\mathcal{B}\mathcal{B} \neq 0$ and the only graded ideals of $\mathcal{B}$ are $\{0\}$ and $\mathcal{B}$. If there is no confusion about the grading group we can simply write "graded-simple".*

If $\mathcal{B}$ is a G-graded algebra and $\mathcal{B}\mathcal{B} \neq 0$ then $\mathcal{B}$ is simple if and only if for each nonzero homogeneous element $x \in \mathcal{B}$ we have $\mathcal{B} = \mathrm{Mult}(\mathcal{B})x$. The next result is [ABFP, Lemma 4.2.2]

Lemma 1.3.6. *Suppose that $\mathcal{B}$ is a G-graded algebra. Then*

$$\mathcal{B} \text{ is simple} \Leftrightarrow \mathcal{B} \text{ is graded-simple and } C(\mathcal{B}) \text{ is a field.}$$

Consequently, if $C(\mathcal{B}) = \mathbb{F}1$ and $\mathcal{B}$ is graded-simple then $\mathcal{B}$ is simple.

The following result ([ABFP, Lemma 4.2.3]) gives us some properties for the centroid when we assume that the algebra is G-graded-simple.

Lemma 1.3.7. *Suppose that $\mathcal{B}$ is a G-graded-simple algebra. Then*
(i) *$\mathcal{B} = \mathcal{B}\mathcal{B}$ and so $C(\mathcal{B})$ is commutative.*
(ii) *$C(\mathcal{B}) = \bigoplus_{g\in G} C(\mathcal{B})_g$, and so $C(\mathcal{B})$ is a G-graded algebra.*
(iii) *Each nonzero homogeneous element of $C(\mathcal{B})$ is invertible in $C(\mathcal{B})$.*
(iv) *$C(\mathcal{B})_e$ is a field, for e the neutral element in G.*
(v) *$\mathcal{B}$ and $C(\mathcal{B})$ are naturally G-graded algebras over the field $C(\mathcal{B})_e$.*

Definition 1.3.8. *Let $\mathcal{B}$ be a G-graded-simple algebra. Define*

$$\mathrm{Supp}_G\, C(\mathcal{B}) := \{h \in G : C(\mathcal{B})_h \neq 0\}.$$

*We call $\mathrm{Supp}_G\, C(\mathcal{B})$ the **central grading group** of $\mathcal{B}$.*

Effectively $\operatorname{Supp}_G C(\mathcal{B})$ is a subgroup of G, this is because for every $h \in \operatorname{Supp}_G C(\mathcal{B})$ if we take $0 \neq x \in C(\mathcal{B})_h$ then by Lemma 1.3.7 (iii) there exists $0 \neq y \in C(\mathcal{B})$ such that $xy = 1$, where 1 denotes the unital element in $C(\mathcal{B})$, then $y \in C(\mathcal{B})_{h^{-1}}$, and therefore $h^{-1} \in \operatorname{Supp}_G C(\mathcal{B})$. Since $\deg 1 = e$ where e denotes the identity element of G (see Remark 1.1.8), then $e \in \operatorname{Supp}_G C(\mathcal{B})$.

By Lemma 1.3.7 (i) if $\mathcal{B}$ is a G-graded-simple algebra then it induces a G-grading on its centroid $C(\mathcal{B})$. We will denote this G-grading induced by Γ on the centroid as $\Gamma_{C(\mathcal{B})}$.

Lemma 1.3.9. *Let G and K be abelian groups. Let $\Gamma' : \mathcal{B} = \bigoplus_{k\in K} \mathcal{B}'_k$ be a K-grading where $K = U(\Gamma')$ and let $\Gamma : \mathcal{B} = \bigoplus_{g\in G} \mathcal{B}_g$ be a G-grading. Suppose $\mathcal{B}$ is a K-graded-simple algebra and a G-graded-simple algebra. If Γ' is a refinement of Γ then $\Gamma'_{C(\mathcal{B})}$ is a refinement of $\Gamma_{C(\mathcal{B})}$.*

Proof. Consider the homomorphism of groups, given in Lemma 1.1.21, induced by the refinement of Γ

$$\begin{array}{rccl} \varphi: & K & \longrightarrow & G \\ & k & \longmapsto & g, \text{ such that } \mathcal{B}'_k \subseteq \mathcal{B}_g. \end{array}$$

Let $\Gamma'_{C(\mathcal{B})} : C(\mathcal{B}) = \bigoplus_{k\in K} C(\mathcal{B})'_k$ and $\Gamma_{C(\mathcal{B})} : C(\mathcal{B}) = \bigoplus_{g\in G} C(\mathcal{B})_g$ be the induced gradings on the centroid $C(\mathcal{B})$ of $\mathcal{B}$ which exist by Lemma 1.3.7. Let $f \in C(\mathcal{B})'_{k'}$ for $k' \in K$. Since Γ' is a refinement of Γ we have $\mathcal{B}_g = \bigoplus_{k\in K:\ \varphi(k)=g} \mathcal{B}'_k$ for $g \in G$. Then

$$\begin{aligned} f(\mathcal{B}_g) &= f\left(\bigoplus_{k\in K:\ \varphi(k)=g} \mathcal{B}'_k\right) = \bigoplus_{k\in K:\ \varphi(k)=g} f(\mathcal{B}'_k) \subseteq \bigoplus_{k\in K:\ \varphi(k)=g} \mathcal{B}'_{kk'} \\ &\subseteq \bigoplus_{k\in K:\ \varphi(k)=g} \mathcal{B}_{\varphi(k)\varphi(k')} = \mathcal{B}_{g\varphi(k')}. \end{aligned}$$

Hence $f \in C(\mathcal{B})_{\varphi(k')}$, then $C(\mathcal{B})'_{k'} \subseteq C(\mathcal{B})_{\varphi(k')}$ and so $\Gamma'_{C(\mathcal{B})}$ is a refinement of $\Gamma_{C(\mathcal{B})}$.

□

Definition 1.3.10. *Let $\mathcal{B}$ be an algebra. Then $\mathbb{F}1 \subseteq C(\mathcal{B})$ and we say that $\mathcal{B}$ is **central** if $\mathbb{F}1 = C(\mathcal{B})$. We say that $\mathcal{B}$ is **central-simple** if $\mathcal{B}$ is central and simple.*

*Suppose that $\mathcal{B}$ is a G-graded algebra, then $\mathbb{F}1 \subseteq C(\mathcal{B})_e \subseteq C(\mathcal{B})$, where $C(\mathcal{B})_e = \{c \in C(\mathcal{B}) : c(\mathcal{B}_g) \subseteq \mathcal{B}_g \text{ for } g \in G\}$. We say that $\mathcal{B}$ is **graded-central** if $C(\mathcal{B})_e = \mathbb{F}1$. And we say that $\mathcal{B}$ is **graded-central-simple** if $\mathcal{B}$ is graded-central and graded-simple.*

The next result is [ABFP, Lemma 4.3.4].

Lemma 1.3.11. *Let $\mathcal{B}$ a G-graded-simple algebra. Suppose either that* $\dim \mathcal{B}_g = 1$ *for some $g \in G$ or that $\mathbb{F}$ is algebraically closed and $0 < \dim \mathcal{B}_g < \infty$ for some $g \in G$. Then $\mathcal{B}$ is graded-central-simple.*

The next result is [ABFP, Lemma 4.3.5].

Lemma 1.3.12. *Suppose that $\mathcal{B}$ is a G-graded-central-simple algebra. Then $C(\mathcal{B})$ has a basis $\{c_h\}_{h\in H}$, where $H = \operatorname{Supp}_G C(\mathcal{B})$, such that $c_h \in C(\mathcal{B})_h$ is a unit of $C(\mathcal{B})$ for $h \in H$. Hence if $h \in H$ and $g \in G$, then $\mathcal{B}_{hg} = c_h\mathcal{B}_g$.*

Definition 1.3.13. *Let $\mathcal{B}$ be a G-graded-central-simple algebra. We say that the centroid $C(\mathcal{B})$ of $\mathcal{B}$ is **split** if*

$$C(\mathcal{B}) \simeq_H \mathbb{F}H$$

where $H = \operatorname{Supp}_G C(\mathcal{B})$. Note that both $C(\mathcal{B})$ and $\mathbb{F}H$ are G-graded since H is a subgroup of G. Thus we can alternately write $C(\mathcal{B}) \simeq_G \mathbb{F}H$. Note also that $C(\mathcal{B})$ is split if and only if a basis *$\{c_g\}_{g\in G}$ for $C(\mathcal{B})$ can be chosen as in the previous lemma with the additional property that $c_gc_f = c_{gf}$ for $g, f \in G$. If $\mathcal{B}$ is an algebra, let*

$$\operatorname{Alg}(C(\mathcal{B}), \mathbb{F})$$

denote the set of all unital $\mathbb{F}$-algebra homomorphisms of $C(\mathcal{B})$ into $\mathbb{F}$.

The next result is [ABFP, Lemma 4.3.7].

Lemma 1.3.14. *Suppose that $\mathcal{B}$ is a G-graded-central-simple algebra. Then*

$$C(\mathcal{B}) \textit{ is split} \Leftrightarrow \operatorname{Alg}(C(\mathcal{B}), \mathbb{F}) \neq \emptyset.$$

There are two cases when $C(\mathcal{B})$ is always split and they are shown in the next result ([ABFP, Lemma 4.3.8]).

Lemma 1.3.15. *Suppose that G is finitely generated and free, or that $\mathbb{F}$ is algebraically closed. If $\mathcal{B}$ is a G-graded-central-simple algebra, then the centroid of $\mathcal{B}$ is split.*

The next result is [ABFP, Lemma 4.4.1].

Lemma 1.3.16. *Suppose that $\mathcal{B}$ is a G-graded-central-simple algebra. Choose a set Θ of coset representatives of $\operatorname{Supp}_G C(\mathcal{B})$ in G, and for $\theta \in \Theta$, choose a $\mathbb{F}$-basis X^θ for $\mathcal{B}_\theta$. Using these choices let*

$$X = \cup_{\theta\in\Theta} X^\theta.$$

Then X is a homogeneous $C(\mathcal{B})$-basis for $\mathcal{B}$. Hence $\mathcal{B}$ is a free $C(\mathcal{B})$-module of rank $\sum_{\theta\in\Theta} \dim_{\mathbb{F}}(\mathcal{B}_\theta)$ (where we interpret the sum on the right as ∞ if any of the terms in the sum is infinite or if there are infinitely many nonzero terms in the sum).

From now on suppose that H is a subgroup of the abelian group G. Suppose further that $\pi : G \to \overline{G}$ is a group epimorphism of G onto an abelian group $\overline{G}$ such that $\ker \pi = H$. We will see some results about centrality and simplicity of the loop algebra $L_\pi(\mathcal{A})$. The next result is [ABFP, Lemma 5.1.1].

Lemma 1.3.17. *Suppose that $\mathcal{A}$ is a $\overline{G}$-graded algebra. Then*

$$\mathcal{A} \text{ is graded-simple} \Leftrightarrow L_\pi(\mathcal{A}) \text{ is graded-simple.}$$

The next result is [ABFP, Lemma 5.1.3].

Lemma 1.3.18. *Suppose that $\mathcal{A}$ is a $\overline{G}$-graded algebra.*

(i) If $\mathcal{A}$ is graded-simple, then

$$\mathcal{A} \text{ is graded-central} \Leftrightarrow L_\pi(\mathcal{A}) \text{ is graded-central.}$$

(ii) If $\mathcal{A}$ is graded-central-simple, then

$$\mathcal{A} \text{ is central-simple} \Leftrightarrow \operatorname{Supp}_G C(L_\pi(\mathcal{A})) = \ker \pi.$$

(iii) If $\mathcal{A}$ is central-simple, then

$$C(L_\pi(\mathcal{A})) = \operatorname{span}_{\mathbb{F}}\{L_{1\otimes h} : h \in H := \operatorname{Supp}_G C(L_\pi(\mathcal{A}))\} \simeq_G \mathbb{F}H,$$

where $L_{1\otimes h}$ denotes the homomorphism $a_{\bar{g}} \otimes g \mapsto a_{\bar{g}} \otimes gh$, for $a_{\bar{g}} \otimes g \in L_\pi(\mathcal{A})_g$ and $g \in G$. In particular, the centroid of $L_\pi(\mathcal{A})$ is split.

Definition 1.3.19. (i) *We let $\mathfrak{U}(\overline{G})$ be the class of $\overline{G}$-graded algebras $\mathcal{A}$ such that $\mathcal{A}$ is central-simple as an algebra.*

(ii) *We let $\mathfrak{B}(G, H)$ be the class of G-graded algebras $\mathcal{B}$ such that $\mathcal{B}$ is graded-central-simple, the centroid of $\mathcal{B}$ is split and* $\operatorname{Supp}_G C(\mathcal{B}) = H$. *Equivalently $\mathfrak{B}(G, H)$ is the class of G-graded algebras $\mathcal{B}$ such that $\mathcal{B}$ is graded-central-simple and $C(\mathcal{B}) \simeq_G \mathbb{F}H$.*

The next remark is [ABFP, Remark 5.2.2].

Remark 1.3.20. *The class $\mathfrak{U}(\overline{G})$ is closed under graded-isomorphism. That is, if $\mathcal{A}$ and $\mathcal{A}'$ are $\overline{G}$-graded algebras such that $\mathcal{A} \in \mathfrak{U}(\overline{G})$ and $\mathcal{A} \simeq_{\overline{G}} \mathcal{A}'$, then we also have $\mathcal{A}' \in \mathfrak{U}(\overline{G})$. Similarly $\mathfrak{B}(G, H)$ is closed under graded-isomorphism.*

In the next result (Proposition 5.2.3 [ABFP]) it is used the loop construction and the previous lemmas to establish a relationship between the classes $\mathfrak{U}(\overline{G})$ and $\mathfrak{B}(G,H)$. This relationship will be explored in more detail in the Correspondence Theorem (Theorem 1.3.32).

Proposition 1.3.21. *Let $\mathcal{A}$ be a $\overline{G}$-graded algebra. Then the following statements are equivalent:*

a) $\mathcal{A} \in \mathfrak{U}(\overline{G})$.

b) $L_\pi(\mathcal{A}) \in \mathfrak{B}(G,H)$.

c) $L_\pi(\mathcal{A})$ *is a graded-central-simple with central grading group* H.

Lemma 1.3.22. *Let $\pi : G \to \overline{G}$ be a surjective group homomorphism with kernel H and let $\mathcal{A}$ be a central-simple $\overline{G}$-graded algebra. Then the associated loop algebra $L_\pi(\mathcal{A})$ is graded-central-simple and the map*

$$\begin{aligned} \mathbb{F}H &\longrightarrow C\big(L_\pi(\mathcal{A})\big) \\ h &\mapsto \Big(x \otimes g \mapsto x \otimes hg\Big) \end{aligned}$$

for any $g \in G$ and $x \in \mathcal{A}_{\overline{g}}$ where $H := \operatorname{Supp}_G C(L_\pi(\mathcal{A}))$, is an isomorphism of G-graded algebras.

Proof. It follows from Lemma 1.3.18 (iii) and Proposition 1.3.21. □

The construction of the loop algebra gives us a way to pass from a $\overline{G}$-graded algebra to a G-graded algebra. In order to provide an inverse for this construction, we will need certain algebra homomorphisms, called central specializations from G-graded algebras to $\overline{G}$-graded algebras. Again we will assume that H is a subgroup of an arbitrary abelian group G and that $\pi : G \to \overline{G}$ is a group epimorphism such that $H = \ker \pi$.

Definition 1.3.23. *Let $\mathcal{B}$ be a G-graded algebra and let $\rho \in \operatorname{Alg}(C(\mathcal{B}), \mathbb{F})$. A ρ-**specialization** of $\mathcal{B}$ is a nonzero algebra epimorphism $\varphi : \mathcal{B} \to \mathcal{A}$ onto a $\overline{G}$-graded algebra $\mathcal{A}$ such that the following conditions hold:*

a) $\varphi(\mathcal{B}_g) \subseteq \mathcal{A}_{\overline{g}}$ *for* $g \in G$.

b) $\varphi(cx) = \rho(c)\varphi(x)$ *for* $c \in C(\mathcal{B})$, $x \in \mathcal{B}$.

*If $\varphi : \mathcal{B} \to \mathcal{A}$ is a ρ-specialization, we call $\mathcal{A}$ a ρ-**image** of $\mathcal{B}$. A **central specialization** of $\mathcal{B}$ is a map $\varphi : \mathcal{B} \to \mathcal{A}$ that is a ρ-specialization of $\mathcal{B}$ for some $\rho \in \operatorname{Alg}(C(\mathcal{B}), \mathbb{F})$. Similarly a **central image** of $\mathcal{B}$ is a $\overline{G}$-graded algebra that is a ρ-image of $\mathcal{B}$ for some $\rho \in \operatorname{Alg}(C(\mathcal{B}), \mathbb{F})$.*

The next remark is [ABFP, Remark 6.1.3].

Remark 1.3.24. *Let $\mathcal{B}$ be a G-graded algebra and let $\rho \in \operatorname{Alg}(C(\mathcal{B}), \mathbb{F})$.*

(i) *Since $\mathcal{B}$ is G-graded, $\mathcal{B}$ has a natural $\overline{G}$-grading defined by*

$$\mathcal{B}_{\overline{g}} = \sum_{k \in G\colon\, \overline{k} = \overline{g}} \mathcal{B}_k$$

for $g \in G$. Then assumption a) in Definition 1.3.23 says that φ is a $\overline{G}$-graded map.

(ii) *Suppose $\mathcal{B}'$ is another G-graded algebras and $\beta : \mathcal{B}' \to \mathcal{B}$ is an isomorphism of G-graded algebras. Then β induces an algebra isomorphism*

$$C(\beta) : C(\mathcal{B}') \to C(\mathcal{B})$$

defined by $C(\beta)(c') = \beta \circ c' \circ \beta^{-1}$ for $c' \in C(\mathcal{B}')$. It follows that $C(\mathcal{B}') = \bigoplus_{h \in H} C(\mathcal{B}')_h$ and that $C(\beta)$ is an isomorphism of G-graded algebras. Moreover, if $\varphi : \mathcal{B} \to \mathcal{A}$ is a ρ-specialization of $\mathcal{B}$, then $\varphi \circ \beta$ is a $\rho \circ C(\beta)$-specialization of $\mathcal{B}'$. Consequently if $\mathcal{A}$ is a central image of $\mathcal{B}$ then $\mathcal{A}$ is also a central image of $\mathcal{B}'$.

(iii) *If $\mathcal{A}$ is a ρ-image of $\mathcal{B}$ and $\mathcal{A}'$ is a $\overline{G}$-graded algebra such that $\mathcal{A} \simeq_{\overline{G}} \mathcal{A}'$, then $\mathcal{A}'$ is also a ρ-image of $\mathcal{A}$.*

The next example ([ABFP, Example 6.1.4]) is the construction of a particular ρ-specialization which will be very useful in our study.

Example 1.3.25. *Let $\mathcal{B}$ be a G-graded algebra which satisfies the following conditions:*

a) $C(\mathcal{B})$ is commutative and $C(\mathcal{B}) = \bigoplus_{h \in H} C(\mathcal{B})_h$ (where $C(\mathcal{B})_h$ is defined by 1.3.1 for $h \in H$).

b) $\mathcal{B}$ is a nonzero free $C(\mathcal{B})$-module (under the natural action).

(Note that by Lemmas 1.3.7 (i) and (ii) and 1.3.16 these conditions are satisfied if $\mathcal{B} \in \mathfrak{B}(G, H)$.) Suppose $\rho \in \operatorname{Alg}(C(\mathcal{B}), \mathbb{F})$. Let

$$(\ker \rho)\mathcal{B} = \operatorname{span}_{\mathbb{F}}\{cx : c \in \ker \rho,\ \ x \in \mathcal{B}\}.$$

Then $(\ker \rho)\mathcal{B}$ is an ideal of $\mathcal{B}$ (as an algebra). Also, regarding $\mathcal{B}$ as $\overline{G}$-graded as in Remark 1.3.24 (i), we have, using assumption a), that

$$(\ker \rho)\mathcal{B}_{\overline{g}} \subseteq C(\mathcal{B})\mathcal{B}_{\overline{g}} \subseteq \left(\sum_{h \in H} C(\mathcal{B})_h \right) \left(\sum_{k \in G\colon\, \overline{k} = \overline{g}} \mathcal{B}_k \right) \subseteq \mathcal{B}_{\overline{g}}$$

for $\overline{g} \in \overline{G}$. *It follows from this that* $(\ker \rho)\mathcal{B}$ *is a* $\overline{G}$*-graded ideal of* $\mathcal{B}$. *Thus the quotient algebra*

$$\mathcal{B}/(\ker \rho)\mathcal{B}$$

has the natural structure of a $\overline{G}$*-graded algebra. Observe also that* $\ker \rho \neq C(\mathcal{B})$ *and so, by assumption b),* $(\ker \rho)\mathcal{B} \neq \mathcal{B}$. *(Actually it would be enough to assume in place of b) that* $\mathcal{B}$ *is a faithfully flat* $C(\mathcal{B})$*-module.) Thus* $\mathcal{B}/(\ker \rho)\mathcal{B} \neq 0$. *Finally, let* $p_\rho : \mathcal{B} \to \mathcal{B}/(\ker \rho)\mathcal{B}$ *be the canonical projection defined by*

$$p_\rho(x) = x + (\ker \rho)\mathcal{B}$$

for $x \in \mathcal{B}$. *Note that since* $(c - \rho(c)1)x \in (\ker \rho)\mathcal{B}$, *we have*

$$p_\rho(cx) = \rho(c)p_\rho(x)$$

for $c \in C(\mathcal{B})$, $x \in \mathcal{B}$. *Thus* p_ρ *is a* ρ*-specialization of* $\mathcal{B}$. *We call* p_ρ *the* ***universal*** $\boldsymbol{\rho}$***-specialization*** *of* $\mathcal{B}$.

If $\mathcal{B}$ is in $\mathfrak{B}(G, H)$, then by Lemma 1.3.14 we have $\mathrm{Alg}(C(\mathcal{B})\mathbb{F}) \neq \emptyset$. Moreover, $\mathcal{B}$ satisfies assumptions a) and b) of Example 1.3.25, and so, for $\rho \in \mathrm{Alg}(C(\mathcal{B})\mathbb{F})$, we can construct the universal ρ-specialization $p_\rho : \mathcal{B} \to \mathcal{B}/(\ker \rho)\mathcal{B}$ of $\mathcal{B}$. In part (i) of the next result ([ABFP, Proposition 6.2.1]), we see that p_ρ is the unique ρ-specialization of $\mathcal{B}$ and we see why the name of "universal" ρ-specialization.

Proposition 1.3.26. *Suppose that* $\mathcal{B} \in \mathfrak{B}(G, H)$, $\rho \in \mathrm{Alg}(C(\mathcal{B}), \mathbb{F})$, $\mathcal{A}$ *is a* $\overline{G}$*-graded algebra and* $\varphi : \mathcal{B} \to \mathcal{A}$ *is a* ρ*-specialization of* $\mathcal{B}$. *Then*

(i) *There exists a unique* $\overline{G}$*-graded isomorphism* $\kappa : \mathcal{B}/(\ker \rho)\mathcal{B} \to \mathcal{A}$ *such that* $\varphi = \kappa \circ p_\rho$.

(ii) *If* X *is a homogeneous* $C(\mathcal{B})$*-basis for* $\mathcal{B}$ *chosen as in Lemma 1.3.15, then* φ *maps* X *bijectively onto a* $\mathbb{F}$*-basis* $\varphi(X)$ *of* $\mathcal{A}$.

(iii) *For* $g \in G$, φ *restricts to a linear bijection of* $\mathcal{B}_g$ *onto* $\mathcal{A}_{\overline{g}}$.

(iv) $L_\pi(\mathcal{A}) \simeq_G \mathcal{B}$ *via the isomorphism*

$$\begin{array}{rccc} \omega : & \mathcal{B} & \to & L_\pi(\mathcal{A}) \\ & x & \mapsto & \varphi(x) \otimes g \end{array}$$

for $x \in \mathcal{B}_g$ *and* $g \in G$.

(v) $\mathcal{A} \in \mathfrak{U}(\overline{G})$.

Since $\pi : G \to \overline{G}$ is surjective, π has a right inverse as a map of sets. Fix a choice ξ of such a right inverse. So $\xi : \overline{G} \to G$ is a map of sets such that

$$\pi \circ \xi = 1_{\overline{G}}.$$

Definition 1.3.27. *Let χ be a character of H, i.e. $\chi \in \mathrm{Hom}(H, \mathbb{F}^\times)$. Also let $\mathcal{A}$ be a $\overline{G}$-graded algebra. We define a $\overline{G}$-graded algebra $\mathcal{A}_\chi$ as follows. As a $\overline{G}$-graded vector space $\mathcal{A}_\chi = \mathcal{A}$. Further, the product $\cdot_\chi$ on $\mathcal{A}_\chi$ is defined by*

$$u \cdot_\chi v = \chi(\xi(\overline{g_1}) + \xi(\overline{g_2}) - \xi(\overline{g_1} + \overline{g_2}))uv$$

*for $\overline{g_1}, \overline{g_2} \in \overline{G}$, $u \in \mathcal{A}_{\overline{g_1}}$, $v \in \mathcal{A}_{g_2}$. We call $\mathcal{A}_\chi$ the **twist** of $\mathcal{A}$ by χ.*

Suppose that χ and $\mathcal{A}$ are as in the above definition. It can be checked that, up to $\overline{G}$-graded isomorphism, $\mathcal{A}_\chi$ is independent of the choice of the right inverse ξ for π. Twists of $\mathcal{A}$ have the following properties ([ABFP, Lemma 6.3.4]).

Lemma 1.3.28. *Suppose that $\mathcal{A}$ is a $\overline{G}$-graded algebra.*

(i) *If $\mathcal{A}'$ is a $\overline{G}$-graded algebra such that $\mathcal{A} \simeq_{\overline{G}} \mathcal{A}'$, then $\mathcal{A}_\chi \simeq_{\overline{G}} \mathcal{A}'_\chi$ for $\chi \in \mathrm{Hom}(H, \mathbb{F}^\times)$.*

(ii) *If $1 \in \mathrm{Hom}(H, \mathbb{F}^\times)$ is the trivial character (that is $1(h) = 1$ for all $h \in H$), then $\mathcal{A}_1 = \mathcal{A}$.*

(iii) *If $\chi_1, \chi_2 \in \mathrm{Hom}(H, \mathbb{F}^\times)$ then $(\mathcal{A}_{\chi_1})_{\chi_2} = \mathcal{A}_{\chi_1\chi_2}$.*

(iv) *If $\chi \in \mathrm{Hom}(H, \mathbb{F}^\times)$ extends to a character of G, then $\mathcal{A}_\chi \simeq_{\overline{G}} \mathcal{A}$.*

(v) *If $\mathbb{F}$ is algebraically closed, then $\mathcal{A}_\chi \simeq_{\overline{G}} \mathcal{A}$ for any $\chi \in \mathrm{Hom}(H, \mathbb{F}^\times)$.*

Definition 1.3.29. *Let $\mathcal{A}$ and $\mathcal{A}'$ be $\overline{G}$-graded algebras, we say that $\mathcal{A}$ and $\mathcal{A}'$ are **similar relative** to π, written $\mathcal{A} \sim_\pi \mathcal{A}'$, if $\mathcal{A}' \simeq_{\overline{G}} \mathcal{A}_\chi$ for some $\chi \in \mathrm{Hom}(H, \mathbb{F}^\times)$.*

The next remark is [ABFP, Remark 6.3.7].

Remark 1.3.30. (i) *The relation $\sim_\pi$ depends on the group epimorphism $\pi : G \to \overline{G}$ (with kernel H) but not on the choice of the right inverse ξ for π.*

(ii) *It follows form parts (i)-(iii) of Lemma 1.3.28 that $\sim_\pi$ is an equivalence relation on the class of $\overline{G}$-graded algebras.*

(iii) *Suppose that $\mathcal{A}$ and $\mathcal{A}'$ are $\overline{G}$-graded algebras. By Lemma 1.3.28 (ii) we see that*

$$\mathcal{A} \simeq_{\overline{G}} \mathcal{A}' \Rightarrow \mathcal{A} \sim_\pi \mathcal{A}'.$$

Moreover, if $\mathbb{F}$ is algebraically closed, then by Lemma 1.3.28 (v), we have

$$\mathcal{A} \simeq_{\overline{G}} \mathcal{A}' \Leftrightarrow \mathcal{A} \sim_\pi \mathcal{A}'.$$

The next result is [ABFP, Proposition 6.5.2].

Proposition 1.3.31. *Suppose that $\mathcal{B} \in \mathfrak{B}(G, H)$ and $\mathcal{A}$ is a $\overline{G}$-graded algebra. Then the following statements are equivalent:*

a) $L_\pi(\mathcal{A}) \simeq_G \mathcal{B}$.

b) $\mathcal{A}$ *is a central image of* $\mathcal{B}$.

c) $\mathcal{A} \simeq_{\overline{G}} \mathcal{B}/(\ker\rho)\mathcal{B}$ *for some* $\rho \in \mathrm{Alg}(C(\mathcal{B}), \mathbb{F})$.

Moreover, if a), b) or c) hold, then $\mathcal{A} \in \mathfrak{U}(\overline{G})$.

Now we use the results to prove the main theorem about the loop algebra construction. This theorem tells that the loop construction induces a correspondence between similarity classes of $\overline{G}$-graded algebras in $\mathfrak{U}(\overline{G})$ and graded-isomorphism classes of G-graded algebras in $\mathfrak{B}(G, H)$. The inverse correspondence is induced by central specialization. The next result is [ABFP, Theorem 7.1.1].

Theorem 1.3.32. (Correspondence Theorem)
Let H *be a subgroup of* G *and let* $\pi : G \to \overline{G}$ *be a group epimorphism such that* $\ker\pi = H$. *For a* $\overline{G}$*-graded algebra* $\mathcal{A}$, *let* $L_\pi(\mathcal{A}) = \sum_{g\in G} \mathcal{A}_{\overline{g}} \otimes g$, *where* $\overline{g} = \pi(g)$.

(i) *If* $\mathcal{A} \in \mathfrak{U}(\overline{G})$, *then* $L_\pi(\mathcal{A}) \in \mathfrak{B}(G, H)$.

(ii) *If* $\mathcal{B} \in \mathfrak{B}(G, H)$, *then there exists* $\mathcal{A} \in \mathfrak{U}(\overline{G})$ *such that* $L_\pi(\mathcal{A}) \simeq_G \mathfrak{B}$. *Moreover the* $\overline{G}$*-graded algebras* $\mathcal{A}$ *with this property are precisely the central images of* $\mathcal{B}$.

(iii) *If* $\mathcal{A}, \mathcal{A}' \in \mathfrak{U}(\overline{G})$, *then* $L_\pi(\mathcal{A}) \simeq_G L_\pi(\mathcal{A}')$ *if and only if* $\mathcal{A} \sim_\pi \mathcal{A}'$.

(iv) *If* $\mathcal{A} \in \mathfrak{U}(\overline{G})$, $\mathcal{B} \in \mathfrak{B}(G, H)$ *and* $\mathcal{B} \simeq_G L_\pi(\mathcal{A})$, *then* $\mathcal{A}$ *is finite dimensional if and only if* $\mathcal{B}$ *is finitely generated as a module over its centroid.*

Chapter 2

Gradings on Semisimple algebras

Recall that we are interested in gradings by abelian groups (see Remark 1.1.11), we will assume in the whole chapter that **all the algebras are finite-dimensional over an algebraically closed field $\mathbb{F}$ and all the grading groups are abelian**.

In this chapter a *semisimple* algebra will be defined as a finite direct sum of simple ideals. In Section 1 we will prove that a semisimple G-graded-simple algebra is isomorphic, as G-graded algebra, to the loop algebra of any of its simple ideals (Theorem 2.1.7). We will show that for a G-grading on a semisimple algebra, there is a decomposition of it as direct sum of G-graded-simple minimal ideals (Lemma 2.1.15). We define a group-grading on the product of loop algebras of simple algebras which we will call *loop product group-grading*. This last grading will be important because any group-grading on a semisimple algebra is isomorphic to a loop product group-grading. So, the study of group-gradings on semisimple algebras is equivalent to the study of loop product group-gradings. In Section 2 we give the classification, up to isomorphism, of group-gradings on loop product group-gradings. In Section 3 we prove that there is a direct relation between the universal group of a simple algebra and the one of its loop algebra (Lemma 2.3.2). We also give the classification, up to equivalence, of group-gradings on loop product group-gradings. In Section 4 we define the *product group-grading* which is a group-grading on the direct sum of group-graded algebras by the direct product of their grading groups. Using this definition, we give a classification of fine group-gradings (Theorem 2.4.12). Finally we give a classification of product group-gradings up to equivalence (Corollary 2.4.13).

2.1 Gradings on semisimple algebras

We will start giving the definition of semisimple algebra that will be used here.

Definition 2.1.1. *An algebra $\mathcal{B}$ over a field $\mathbb{F}$ will be called **semisimple** if $\mathcal{B} = \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_n$ where $\mathcal{A}_1, ..., \mathcal{A}_n$ are simple ideals of $\mathcal{B}$. We will call **simple factor** of $\mathcal{B}$ to each $\mathcal{A}_i$ for $i = 1, ..., n$. This is an odd definition of semisimplicity, however it is the most practical for the purposes of this work. Analogously, if we have simple algebras $\mathcal{A}_1, ..., \mathcal{A}_n$ over a field $\mathbb{F}$ we can consider the cartesian product of them $\mathcal{B} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ as a semisimple algebra. Notice that each $0 \times \cdots \times \mathcal{A}_i \times \cdots \times 0$ is a simple ideal of $\mathcal{B}$ for $i = 1, ..., n$.*

Remark 2.1.2. *Consider the $\mathbb{F}$-algebra $\mathcal{B}_1 = \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_n$ where the $\mathcal{A}_i$ are ideals of $\mathcal{B}_1$ for $i = 1, ..., n$ and the algebra $\mathcal{B}_2 = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ constructed as the cartesian product of the $\mathbb{F}$-algebras $\mathcal{A}_i$, $i = 1, ..., n$. We have the following isomorphism of algebras*

$$\begin{array}{rccc} \varphi: & \mathcal{A}_1 \times \cdots \times \mathcal{A}_n & \longrightarrow & \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_n \\ & (a_1, ..., a_n) & \longmapsto & a_1 + \cdots + a_n. \end{array}$$

It is straightforward to check that for a group G, any G-grading on $\mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ induces a G-grading on $\mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_n$, and vice versa, by means of

$$(\mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_n)_g := \varphi((\mathcal{A}_1 \times \cdots \times \mathcal{A}_n)_g).$$

We will identify both algebras and both group-gradings.

Remark 2.1.3. *If $\mathbb{F}$ is algebraically closed, then any finite-dimensional simple algebra is automatically central-simple by [Jac78, Theorem 10.1].*

Lemma 2.1.4. *Let H be a group such that $|H| = n$. Let $\chi_1, ..., \chi_n : H \to \mathbb{F}^\times$ be distinct homomorphisms. Then $\operatorname{char}\mathbb{F}$ does not divide n and $\widehat{H} = \{\chi_1, ..., \chi_n\}$.*

Proof. By Dedekind's Lemma ([Jac1, Dedekind Independence Theorem, Chapter 4.14]) χ_1, ..., χ_n are linearly independent. Since $|H| = n$, we have that $\dim_{\mathbb{F}}(\mathbb{F}H) = n$ and then $(\mathbb{F}H)^* = \operatorname{Hom}_{\mathbb{F}}(\mathbb{F}H, \mathbb{F})$ has dimension n. Therefore $\widehat{H} = \{\chi_1, ..., \chi_n\}$.

Suppose $\operatorname{char}\mathbb{F} = p$ divides $n = p^r \cdot p_2^{r_2} \cdots p_s^{r_s}$ where $p, p_2, ..., p_s$ are prime numbers and $r, r_2, ..., r_s \in \mathbb{N}$. Then there exists a Sylow p-subgroup of H which we will denote by P, that is, $|P| = p^r$. For all $\chi \in \widehat{H}$ and $g \in P$ we have $g^{p^r} = e$, applying χ we have $\chi(g)^{p^r} = 1$ and since $\operatorname{char}\mathbb{F} = p$, we get $\chi(g) = 1$. Then χ is character of H/P, so $\widehat{H} \simeq \widehat{(H/P)}$ has at most n/p^r elements which leads to a contradiction. $\square$

The next remark can be deduced from [EM94, Chapter I, section 2].

Remark 2.1.5. *Let $\mathcal{A}$ be a simple algebra over an algebraically closed field $\mathbb{F}$. Then $C(\mathcal{A}) \simeq \mathbb{F}$. Moreover if $\mathcal{A}$ is a semisimple algebra with decomposition $\mathcal{A} = \mathcal{S}_1 \oplus \cdots \oplus \mathcal{S}_n$ into simple ideals $\mathcal{S}_i$ of $\mathcal{A}$, then $C(\mathcal{A}) \simeq C(\mathcal{S}_1) \oplus \cdots \oplus C(\mathcal{S}_n)$.*

Lemma 2.1.6. *Let $\mathcal{B} = \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_n$ be an algebra where $\mathcal{A}_1$, ..., $\mathcal{A}_n$ are simple ideals of $\mathcal{B}$. Suppose that $\mathcal{B}$ is a G-graded-simple algebra. Then* $\operatorname{char}\mathbb{F}$ *does not divide n and the group of characters of $H = \operatorname{Supp}_G C(\mathcal{B})$ has exactly n elements.*

Proof. By Remark 2.1.5 we have that $C(\mathcal{B}) \simeq C(\mathcal{A}_1) \oplus \cdots \oplus C(\mathcal{A}_n) \simeq \mathbb{F} \oplus \cdots \oplus \mathbb{F}$. From this and the fact that $\mathbb{F} \oplus \cdots \oplus \mathbb{F} \simeq \mathbb{F} \times \cdots \times \mathbb{F}$ we have an isomorphism $f : C(\mathcal{B}) \to \mathbb{F} \times \cdots \times \mathbb{F}$, which we can compose with the canonical projections $\pi_i : \mathbb{F} \times \cdots \times \mathbb{F} \to \mathbb{F}$ for $i = 1, ..., n$ to get n different homomorphisms of unital algebras

$$\pi_i \circ f = \widehat{\pi}_i : C(\mathcal{B}) \to \mathbb{F}.$$

By Lemma 1.3.7 $C(\mathcal{B})$ is a G-graded algebra and each nonzero homogeneous element of $C(\mathcal{B})$ is invertible in $C(\mathcal{B})$ (that is $C(\mathcal{B})$ is a graded field). Let $H := \operatorname{Supp}_G C(\mathcal{B})$, notice that $|H| = n$. By Lemma 1.3.11 we have that $\mathcal{B}$ is graded-central-simple. By Lemma 1.3.14 $C(\mathcal{B})$ is split, then we have an isomorphism of H-graded algebras

$$\begin{array}{rlll} \varphi : & \mathbb{F}H & \to & C(\mathcal{B}) \\ & h & \mapsto & c_h \in C(\mathcal{B})_h. \end{array}$$

This implies that $\dim C(\mathcal{B})_h = 1$ for all $h \in H$. In particular, for all $h \in H$ there exists a unique $c_h \in C(\mathcal{B})_h$ such that $\widehat{\pi}_1(c_h) = 1$.

We have that $\chi_1 = \widehat{\pi}_1 \circ \varphi\,|_H$, ..., $\chi_n = \widehat{\pi}_n \circ \varphi\,|_H$ are n different characters of H where χ_1 is the trivial character, i.e. $\chi_1(h) = 1$ for all $h \in H$. Then by Lemma 2.1.4 $\operatorname{char}\mathbb{F}$ does not divide n and $\widehat{H} = \{\chi_1, ..., \chi_n\}$. ☐

The next result together with Remark 2.1.8 shows that a semisimple algebra which is graded-simple is isomorphic to the loop algebra of any of its simple factors. This is useful because sometimes it is easier to apply the theory of loop algebras.

Theorem 2.1.7. *Let $\mathcal{B} = \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_n$ be an algebra where $\mathcal{A}_1$, ..., $\mathcal{A}_n$ are simple ideals of $\mathcal{B}$. Suppose that $\mathcal{B}$ is a G-graded-simple algebra. Denote again by π_1 the natural projection $\mathcal{B} \to \mathcal{A}_1$. Let $H = \operatorname{Supp}_G C(\mathcal{B})$ and let π*

be the canonical projection $\pi : G \to \overline{G} = G/H$. Then $\mathcal{A}_1$ is a $\overline{G}$-graded ideal for the $\overline{G}$-grading on $\mathcal{B}$ induced by π and $\mathcal{B} \simeq_G L_\pi(\mathcal{A}_1)$ via the isomorphism

$$\begin{array}{rccc} \omega : & \mathcal{B} & \to & L_\pi(\mathcal{A}_1) \\ & x & \mapsto & \varphi(x) \otimes g \end{array}$$

for $x \in \mathcal{B}_g$ and $g \in G$.

Proof. By the proof of Lemma 2.1.6 we have that $\mathcal{B} \in \mathfrak{B}(G, H)$. Consider $\rho = \widehat{\pi}_1 \in \mathrm{Alg}(C(\mathcal{B}), \mathbb{F})$, then $(\ker \rho)\mathcal{B} = \mathcal{A}_2 \oplus \cdots \oplus \mathcal{A}_n$ and π_1 is a ρ-specialization. Then by Proposition 1.3.26 iv) ω is an isomorphism of G-graded algebras. □

Remark 2.1.8. *Notice that Theorem 2.1.7 still works if we replace $\mathcal{A}_1$ by $\mathcal{A}_i$ for $i = 1, 2, ..., n$. Taking $\rho = \widehat{\pi}_i$ the proof works on the same way.*

Now we give the characterization of the semisimple loop algebras.

Theorem 2.1.9. *Let $\pi : G \to \overline{G}$ be a surjective group homomorphism of abelian groups with finite kernel $H = \ker \pi$. Let $\mathcal{A}$ be a central-simple $\overline{G}$-graded algebra and let $L_\pi(\mathcal{A})$ be the associated loop algebra. Then $L_\pi(\mathcal{A})$ is semisimple if and only if the characteristic of $\mathbb{F}$ does not divide $|H|$.*

If this is the case then $L_\pi(\mathcal{A})$ is isomorphic to the cartesian product of $|H|$ copies of $\mathcal{A}$.

Proof. Since $\mathcal{A}$ is simple, it is graded-simple and by Lemma 1.3.17 $L_\pi(\mathcal{A})$ is also graded-simple. Then, by Lemma 1.3.11, $\mathcal{A}$ is graded-central-simple and by Lemma 1.3.18 (ii) $\mathrm{Supp}_G\, C(L_\pi(\mathcal{A})) = H$.

Assume $L_\pi(\mathcal{A})$ is semisimple, then by Lemma 2.1.6 we have that $\mathrm{char}\, \mathbb{F}$ does not divide $|H|$.

Suppose $\mathrm{char}\, \mathbb{F}$ does not divide $n = |H|$. In this case, the group of characters of H consists of n elements: $\widehat{H} = \{\chi_1, \ldots, \chi_n\}$. Also, as $\mathbb{F}$ is algebraically closed, $\mathbb{F}^\times$ is a divisible group and hence these characters may be extended to characters on the whole G. Consider the linear map:

$$\begin{gathered} \Phi : L_\pi(\mathcal{A}) \longrightarrow \mathcal{A} \times \cdots \times \mathcal{A} \quad (n \text{ copies}) \\ x_{\bar{g}} \otimes g \mapsto \big(\chi_1(g) x_{\bar{g}}, \ldots, \chi_n(g) x_{\bar{g}}\big) \end{gathered}$$

for $x_{\bar{g}} \otimes g \in L_\pi(\mathcal{A})_g$ and $g \in G$. The linear map Φ is a homomorphism of $\overline{G}$-graded algebras, where the $\overline{G}$-grading on $L_\pi(\mathcal{A})$ is given by the coarsening of Γ induced by π, and the $\overline{G}$-grading on $\mathcal{A} \times \cdots \times \mathcal{A}$ is given by $(\mathcal{A} \times \cdots \times \mathcal{A})_{\bar{g}} = \mathcal{A}_{\bar{g}} \times \cdots \times \mathcal{A}_{\bar{g}}$ for $\bar{g} \in \overline{G}$. For any $\bar{g} \in \overline{G}$ in the support of $\overline{\Gamma}$, fix a pre-image

$g \in G$, and let $H = \{h_1, \ldots, h_n\}$. The restriction of Φ to the homogeneous component of degree $\bar{g}$ is given by:

$$\begin{aligned} L_\pi(\mathcal{A})_{\bar{g}} = \bigoplus_{i=1}^{n} L_\pi(\mathcal{A})_{gh_i} = \bigoplus_{i=1}^{n} \mathcal{A}_{\bar{g}} \otimes gh_i &\longrightarrow \mathcal{A}_{\bar{g}} \times \cdots \times \mathcal{A}_{\bar{g}} \\ \sum_{i=1}^{n} a_i \otimes gh_i &\mapsto \sum_{i=1}^{n} \big(\chi_1(gh_i)a_i, \ldots, \chi_n(gh_i)a_i\big) \end{aligned}$$

where $a_1, \ldots, a_n \in \mathcal{A}_{\bar{g}}$. But the matrix $\Big(\chi_j(gh_i)\Big)_{1\le i,j\le n}$ is regular, and hence Φ is bijective on each nonzero homogeneous component of the $\overline{G}$-gradings. $\square$

Definition 2.1.10. *Given a group G and G-graded algebras $\mathcal{B}^i$ for $i = 1, ..., n$, there is a natural G-grading on the direct sum $\mathcal{B}^1 \oplus \cdots \oplus \mathcal{B}^n$ (analogously, on the cartesian product $\mathcal{B}^1 \times \cdots \times \mathcal{B}^n$) determined by*

$$(\mathcal{B}^1 \oplus \cdots \oplus \mathcal{B}^n)_g = \mathcal{B}^1_g \oplus \cdots \oplus \mathcal{B}^n_g$$

for any $g \in G$. This G-grading will be denoted by $\Gamma^1 \times_G \cdots \times_G \Gamma^n$ and will be called ***product G-grading*** *of $\Gamma^1, ..., \Gamma^n$ where Γ^i is the G-grading on $\mathcal{B}^i$ for $i = 1, ..., n$.*

Now we will define a particular group-grading and we will prove later (Theorem 2.1.16) that any group-grading on a semisimple algebra is isomorphic to one of this form. Then, in order to classify group-gradings on semisimple algebras, it is enough to classify this particular group-gradings. We will do such classification in Theorem 2.2.2 for isomorphisms and in Theorems 2.3.7 and 2.3.8 for equivalences.

Definition 2.1.11. *Let H_i be a subgroup of G for $i = 1, ..., n$ and let $\pi_i : G \to \overline{G}_i$ be a group epimorphism such that $\ker \pi_i = H_i$. Let $\mathcal{A}_i$ be a $\overline{G}_i$-graded-simple algebra and let $\Gamma^i_{\mathcal{A}_i}(\overline{G}_i)$ be a $\overline{G}_i$-grading on $\mathcal{A}_i$ for $i = 1, ..., n$. Denote by*

$$\Gamma(G, \Gamma^1_{\mathcal{A}_1}(\overline{G}_1), ..., \Gamma^n_{\mathcal{A}_n}(\overline{G}_n))$$

the G-grading on $\mathcal{B} := L_{\pi_1}(\mathcal{A}_1) \times \cdots \times L_{\pi_n}(\mathcal{A}_n)$ constructed by considering the loop algebras $L_{\pi_i}(\mathcal{A}_i)$ for $i = 1, ..., n$ and taking $\mathcal{B}_g := L_{\pi_1}(\mathcal{A}_1)_g \times \cdots \times L_{\pi_n}(\mathcal{A}_n)_g$ for $g \in G$. We will call this group-grading the ***loop product G-grading*** *of $\Gamma^1_{\mathcal{A}_1}(\overline{G}_1), ..., \Gamma^n_{\mathcal{A}_n}(\overline{G}_n)$.*

Now we give an example of such group-grading.

Example 2.1.12. *Let $\mathcal{A} = M_2(\mathbb{F})$ be the algebra of 2×2 matrices on $\mathbb{F}$. Consider the algebra $\mathcal{B} = \mathcal{A}^4$ and the group $G = \mathbb{Z}/4 \times \mathbb{Z}/2$, where $\mathbb{Z}/n := \mathbb{Z}/n\mathbb{Z}$. Denote by $\widehat{n}$ the class of n modulo 4 and by $\bar{n}$ the class of n modulo 2. Let $H_1 = \langle(\widehat{2}, \bar{0})\rangle$ and $H_2 = \langle(\widehat{0}, \bar{1})\rangle$ be subgroups of G. Consider the canonical projections*

$$\pi_1 : G \to G/H_1 \simeq \mathbb{Z}/2 \times \mathbb{Z}/2 \quad \textit{and} \quad \pi_2 : G \to G/H_2 \simeq \mathbb{Z}/4.$$

Let $\Gamma^1_{\mathcal{A}}(G/H_1)$ be the $\mathbb{Z}/2 \times \mathbb{Z}/2$-grading on $\mathcal{A}$ given by

$$\mathcal{A}_{(\bar{0},\bar{0})} = \left\{\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha \end{smallmatrix}\right) : \alpha \in \mathbb{F}\right\}, \mathcal{A}_{(\bar{1},\bar{0})} = \left\{\left(\begin{smallmatrix} \beta & 0 \\ 0 & -\beta \end{smallmatrix}\right) : \beta \in \mathbb{F}\right\},$$

$$\mathcal{A}_{(\bar{0},\bar{1})} = \left\{\left(\begin{smallmatrix} 0 & \gamma \\ \gamma & 0 \end{smallmatrix}\right) : \gamma \in \mathbb{F}\right\}, \mathcal{A}_{(\bar{1},\bar{1})} = \left\{\left(\begin{smallmatrix} 0 & \delta \\ -\delta & 0 \end{smallmatrix}\right) : \delta \in \mathbb{F}\right\}.$$

Let $\Gamma^2_{\mathcal{A}}(G/H_2)$ be the $\mathbb{Z}/4$-grading on $\mathcal{A}$ given by

$$\mathcal{A}_{\widehat{0}} = \left\{\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right) : \alpha, \beta \in \mathbb{F}\right\}, \mathcal{A}_{\widehat{1}} = \left\{\left(\begin{smallmatrix} 0 & \gamma \\ 0 & 0 \end{smallmatrix}\right) : \gamma \in \mathbb{F}\right\}, \mathcal{A}_{\widehat{3}} = \left\{\left(\begin{smallmatrix} 0 & 0 \\ \delta & 0 \end{smallmatrix}\right) : \delta \in \mathbb{F}\right\}.$$

We want to construct the G-grading on $\mathcal{B} = L_{\pi_1}(\mathcal{A}) \times L_{\pi_2}(\mathcal{A})$ given by $\Gamma(G, \Gamma^1_{\mathcal{A}}(G/H_1), \Gamma^2_{\mathcal{A}}(G/H_2))$. First construct the loop algebra of $\mathcal{A}$ associated to the projection $\pi_1 : G \to G/H_1$ to get

$$\begin{aligned}
L_{\pi_1}(\mathcal{A})_{(\widehat{0},\bar{0})} &= \left\{\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha \end{smallmatrix}\right) \otimes (\widehat{0}, \bar{0}) : \alpha \in \mathbb{F}\right\}, \\
L_{\pi_1}(\mathcal{A})_{(\widehat{1},\bar{0})} &= \left\{\left(\begin{smallmatrix} \beta & 0 \\ 0 & -\beta \end{smallmatrix}\right) \otimes (\widehat{1}, \bar{0}) : \beta \in \mathbb{F}\right\}, \\
L_{\pi_1}(\mathcal{A})_{(\widehat{2},\bar{0})} &= \left\{\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha \end{smallmatrix}\right) \otimes (\widehat{2}, \bar{0}) : \alpha \in \mathbb{F}\right\}, \\
L_{\pi_1}(\mathcal{A})_{(\widehat{3},\bar{0})} &= \left\{\left(\begin{smallmatrix} \beta & 0 \\ 0 & -\beta \end{smallmatrix}\right) \otimes (\widehat{3}, \bar{0}) : \beta \in \mathbb{F}\right\}, \\
L_{\pi_1}(\mathcal{A})_{(\widehat{0},\bar{1})} &= \left\{\left(\begin{smallmatrix} 0 & \gamma \\ \gamma & 0 \end{smallmatrix}\right) \otimes (\widehat{0}, \bar{1}) : \gamma \in \mathbb{F}\right\}, \\
L_{\pi_1}(\mathcal{A})_{(\widehat{1},\bar{1})} &= \left\{\left(\begin{smallmatrix} 0 & \delta \\ -\delta & 0 \end{smallmatrix}\right) \otimes (\widehat{1}, \bar{1}) : \delta \in \mathbb{F}\right\}, \\
L_{\pi_1}(\mathcal{A})_{(\widehat{2},\bar{1})} &= \left\{\left(\begin{smallmatrix} 0 & \gamma \\ \gamma & 0 \end{smallmatrix}\right) \otimes (\widehat{2}, \bar{1}) : \gamma \in \mathbb{F}\right\}, \\
L_{\pi_1}(\mathcal{A})_{(\widehat{3},\bar{1})} &= \left\{\left(\begin{smallmatrix} 0 & \delta \\ -\delta & 0 \end{smallmatrix}\right) \otimes (\widehat{3}, \bar{1}) : \delta \in \mathbb{F}\right\}.
\end{aligned}$$

Now we construct the loop algebra of $\mathcal{A}$ associated to the projection $\pi_2 : G \to$

G/H_2 to get

$$
\begin{aligned}
L_{\pi_2}(\mathcal{A})_{(\widehat{0},\overline{0})} &= \left\{ \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right) \otimes (\widehat{0},\overline{0}) : \alpha, \beta \in \mathbb{F} \right\}, \\
L_{\pi_2}(\mathcal{A})_{(\widehat{1},\overline{0})} &= \left\{ \left(\begin{smallmatrix} 0 & \gamma \\ 0 & 0 \end{smallmatrix}\right) \otimes (\widehat{1},\overline{0}) : \gamma \in \mathbb{F} \right\}, \\
L_{\pi_2}(\mathcal{A})_{(\widehat{2},\overline{0})} &= \{0\}, \\
L_{\pi_2}(\mathcal{A})_{(\widehat{3},\overline{0})} &= \left\{ \left(\begin{smallmatrix} 0 & 0 \\ \delta & 0 \end{smallmatrix}\right) \otimes (\widehat{3},\overline{0}) : \delta \in \mathbb{F} \right\}, \\
L_{\pi_2}(\mathcal{A})_{(\widehat{0},\overline{1})} &= \left\{ \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right) \otimes (\widehat{0},\overline{1}) : \alpha, \beta \in \mathbb{F} \right\}, \\
L_{\pi_2}(\mathcal{A})_{(\widehat{1},\overline{1})} &= \left\{ \left(\begin{smallmatrix} 0 & \gamma \\ 0 & 0 \end{smallmatrix}\right) \otimes (\widehat{1},\overline{1}) : \gamma \in \mathbb{F} \right\}, \\
L_{\pi_2}(\mathcal{A})_{(\widehat{2},\overline{1})} &= \{0\}, \\
L_{\pi_2}(\mathcal{A})_{(\widehat{3},\overline{1})} &= \left\{ \left(\begin{smallmatrix} 0 & 0 \\ \delta & 0 \end{smallmatrix}\right) \otimes (\widehat{3},\overline{1}) : \delta \in \mathbb{F} \right\}.
\end{aligned}
$$

Finally the G-grading on $\mathcal{B}$ associated to $\Gamma(G, \Gamma^1_{\mathcal{A}}(G/H_1), \Gamma^2_{\mathcal{A}}(G/H_2))$ is given by

$$
\begin{aligned}
\mathcal{B}_{(\widehat{0},\overline{0})} &= \left\{ \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha \end{smallmatrix}\right) \otimes (\widehat{0},\overline{0}) : \alpha \in \mathbb{F} \right\} \times \left\{ \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right) \otimes (\widehat{0},\overline{0}) : \alpha, \beta \in \mathbb{F} \right\}, \\
\mathcal{B}_{(\widehat{1},\overline{0})} &= \left\{ \left(\begin{smallmatrix} \beta & 0 \\ 0 & -\beta \end{smallmatrix}\right) \otimes (\widehat{1},\overline{0}) : \beta \in \mathbb{F} \right\} \times \left\{ \left(\begin{smallmatrix} 0 & \gamma \\ 0 & 0 \end{smallmatrix}\right) \otimes (\widehat{1},\overline{0}) : \gamma \in \mathbb{F} \right\}, \\
\mathcal{B}_{(\widehat{2},\overline{0})} &= \left\{ \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha \end{smallmatrix}\right) \otimes (\widehat{2},\overline{0}) : \alpha \in \mathbb{F} \right\} \times \{0\}, \\
\mathcal{B}_{(\widehat{3},\overline{0})} &= \left\{ \left(\begin{smallmatrix} \beta & 0 \\ 0 & -\beta \end{smallmatrix}\right) \otimes (\widehat{3},\overline{0}) : \beta \in \mathbb{F} \right\} \times \left\{ \left(\begin{smallmatrix} 0 & 0 \\ \delta & 0 \end{smallmatrix}\right) \otimes (\widehat{3},\overline{0}) : \delta \in \mathbb{F} \right\}, \\
\mathcal{B}_{(\widehat{0},\overline{1})} &= \left\{ \left(\begin{smallmatrix} 0 & \gamma \\ \gamma & 0 \end{smallmatrix}\right) \otimes (\widehat{0},\overline{1}) : \gamma \in \mathbb{F} \right\} \times \left\{ \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right) \otimes (\widehat{0},\overline{1}) : \alpha, \beta \in \mathbb{F} \right\}, \\
\mathcal{B}_{(\widehat{1},\overline{1})} &= \left\{ \left(\begin{smallmatrix} 0 & \delta \\ -\delta & 0 \end{smallmatrix}\right) \otimes (\widehat{1},\overline{1}) : \delta \in \mathbb{F} \right\} \times \left\{ \left(\begin{smallmatrix} 0 & \gamma \\ 0 & 0 \end{smallmatrix}\right) \otimes (\widehat{1},\overline{1}) : \gamma \in \mathbb{F} \right\}, \\
\mathcal{B}_{(\widehat{2},\overline{1})} &= \left\{ \left(\begin{smallmatrix} 0 & \gamma \\ \gamma & 0 \end{smallmatrix}\right) \otimes (\widehat{2},\overline{1}) : \gamma \in \mathbb{F} \right\} \times \{0\}, \\
\mathcal{B}_{(\widehat{3},\overline{1})} &= \left\{ \left(\begin{smallmatrix} 0 & \delta \\ -\delta & 0 \end{smallmatrix}\right) \otimes (\widehat{3},\overline{1}) : \delta \in \mathbb{F} \right\} \times \left\{ \left(\begin{smallmatrix} 0 & 0 \\ \delta & 0 \end{smallmatrix}\right) \otimes (\widehat{3},\overline{1}) : \delta \in \mathbb{F} \right\}.
\end{aligned}
$$

Next lemma shows the form of the ideals of a semisimple algebra. Lemma 2.1.14 says that the complement of a group-graded ideal of a semisimple algebra is also a group-graded ideal.

Lemma 2.1.13. *Let $\mathcal{B} = \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_n$ be an algebra where $\mathcal{A}_1$, ..., $\mathcal{A}_n$ are simple ideals of $\mathcal{B}$. Then the ideals of $\mathcal{B}$ are of the form $\mathcal{A}_{j_1} \oplus \cdots \oplus \mathcal{A}_{j_r}$ for $0 \le r \le n$ and $1 \le j_1 < \cdots < j_r \le n$.*

Proof. Take $0 \neq x \in \mathcal{B}$, then there exists $x_i \in \mathcal{A}_i$ for $i = 1, ..., n$ such that $x = x_1 + \cdots + x_n$. We have that

$$\mathcal{A}_1 x = \mathcal{A}_1(x_1 + \cdots + x_n) = \mathcal{A}_1 x_1 + \cdots + \mathcal{A}_1 x_n = \mathcal{A}_1 x_1$$

because for $j = 2, ..., n$ we have $\mathcal{A}_1 x_j \in \mathcal{A}_1 \cap \mathcal{A}_j = \{0\}$. Analogously $x\mathcal{A}_1 = x_1\mathcal{A}_1$. Then $\mathcal{A}_1 x + x\mathcal{A}_1 = \mathcal{A}_1 x_1 + x_1\mathcal{A}_1$. Suppose now that $x_1 \neq 0$, then $\mathcal{A}_1 x_1 + x_1\mathcal{A}_1 \neq 0$. This is because if $\mathcal{A}_1 x_1 + x_1\mathcal{A}_1 = 0$ then $\mathcal{A}_1 x_1 = \{0\} = x_1\mathcal{A}_1$, so $\mathbb{F}x_1$ would be an ideal of $\mathcal{A}_1$ which is simple. Therefore $\mathbb{F}x_1 = \mathcal{A}_1$ and then $x_1^2 \in x_1\mathcal{A}_1 = \{0\}$ which contradicts the fact that $\mathcal{A}_1^2 \neq 0$. We have $\mathrm{ideal}\langle x\rangle \supseteq \mathcal{A}_1 x_1 + x_1\mathcal{A}_1 \neq 0$, so $\mathrm{ideal}\langle x\rangle \cap \mathcal{A}_1 \neq 0$. Then using the fact that $\mathcal{A}_1$ is simple we get $\mathrm{ideal}\langle x\rangle \supseteq \mathcal{A}_1$. The same argument applies for $i \in \{2, ..., n\}$. Therefore $\mathrm{ideal}\langle x\rangle = \oplus\{\mathcal{A}_j : x_j \neq 0\}$. Now for any ideal $\mathcal{I}$, $\mathcal{I} = \sum_{x\in\mathcal{I}} \mathrm{ideal}\langle x\rangle$. □

In the situation of Lemma 2.1.13, for any $J \subseteq \{1, ..., n\}$, set $\mathcal{A}_J := \oplus_{j\in J}\mathcal{A}_j$.

Lemma 2.1.14. *Let $\mathcal{B} = \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_n$ be a G-graded algebra where $\mathcal{A}_1$, ..., $\mathcal{A}_n$ are simple ideals of $\mathcal{B}$. Take $J \subseteq \{1, ..., n\}$ and assume that $\mathcal{A}_J$ is a graded ideal of $\mathcal{B}$. Then $\widehat{\mathcal{A}}_J := \oplus_{j\notin J}\mathcal{A}_j$ is a graded ideal too.*

Proof. As a sum of ideals $\widehat{\mathcal{A}}_J$ is an ideal of $\mathcal{B}$. We have

$$\widehat{\mathcal{A}}_J := \bigoplus_{j\notin J} \mathcal{A}_j = \{x \in \mathcal{B} : x\mathcal{A}_J = \{0\} = \mathcal{A}_J x\}.$$

Then for $x \in \widehat{\mathcal{A}}_J$ there exist $x_{k_i} \in \mathcal{A}_{k_i}$ for $i = 1, ..., r$ and $k_i \in \{1, ..., n\} \setminus J$ such that $x = x_{k_1} + \cdots + x_{k_r}$. From $x\mathcal{A}_J = \{0\} = \mathcal{A}_J x$ we have $x(\mathcal{A}_J)_g = \{0\} = (\mathcal{A}_J)_g x$ for all $g \in G$ then $x_{k_i}(\mathcal{A}_J)_g = \{0\} = (\mathcal{A}_J)_g x_{k_i}$ for all $g \in G$ and $i = 1, ..., r$. Therefore $x_{k_i} \in \widehat{\mathcal{A}}_J$ for all $i = 1, ..., r$ and then $\widehat{\mathcal{A}}_J$ is graded. □

Last two lemmas lead to the next result which shows that for a G-graded semisimple algebra there is a unique decomposition as a direct sum of G-graded-simple ideals which are also semisimple. We will denote by "$\sqcup$" the disjoint union.

Lemma 2.1.15. *Let $\mathcal{B} = \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_n$ be an algebra where $\mathcal{A}_1$,...,$\mathcal{A}_n$ are simple ideals of $\mathcal{B}$. Let G be a group and let Γ be a G-grading on $\mathcal{B}$. Then there exist $J_i \subseteq \{1, ..., n\}$ for $i = 1, ..., m$ and $m \leq n$ such that $\{1, ..., n\} = J_1 \sqcup \cdots \sqcup J_m$ and $\mathcal{A}_{J_i} := \oplus_{j\in J_i}\mathcal{A}_j$ is a G-graded-simple semisimple ideal of $\mathcal{B}$ for all $i = 1, ..., m$. Therefore $\mathcal{B}$ is a direct sum of G-graded-simple semisimple ideals. Moreover, $\mathcal{A}_{J_1}, ..., \mathcal{A}_{J_m}$ are the minimal G-graded ideals of $\mathcal{B}$.*

Proof. Since $\mathcal{A}_1$, ..., $\mathcal{A}_n$ are simple, the ideals of $\mathcal{B}$ are direct sums of such algebras. If $\mathcal{B}$ is graded-simple we finish. Suppose $\mathcal{B}$ is not graded-simple.

Then there exists a G-graded ideal of $\mathcal{B}$. We can choose this ideal as graded-simple by taking a minimal proper G-graded ideal of $\mathcal{B}$. By Lemma 2.1.13 such G-graded ideal is $\mathcal{A}_{J_1}$ for some $J_1 \subset \{1,...,n\}$. By Lemma 2.1.14 $\widehat{\mathcal{A}}_{J_1} := \oplus_{i\notin J_1}\mathcal{A}_i$ is also a G-graded ideal of $\mathcal{B}$. If $\widehat{\mathcal{A}}_{J_1}$ is G-graded-simple we finish and $J_2 = \{1,...,n\} \setminus J_1$. If not then we repeat the process a finite number of times until we finish. Now suppose $\mathcal{I}$ is a minimal graded ideal of $\mathcal{B}$, then by Lemma 2.1.13 $\mathcal{I} = \mathcal{A}_J$ for some $J \subseteq \{1,...,n\}$. Then

$$\mathcal{I} = \mathcal{A}_J = \mathcal{A}_J \cap \mathcal{B} = \mathcal{A}_J \cap \left(\bigoplus_{j=1}^{m} \mathcal{A}_{J_j}\right) = \bigoplus_{j=1}^{m}(\mathcal{A}_J \cap \mathcal{A}_{J_j}).$$

Hence there exists $i \in \{1,...,m\}$ such that

$$\mathcal{I} = \mathcal{A}_J \cap \mathcal{A}_{J_i} = \mathcal{A}_{J_i},$$

therefore $\mathcal{A}_{J_1},...,\mathcal{A}_{J_m}$ are minimal. □

Now we will prove that any group-grading Γ on a semisimple algebra $\mathcal{B}$ is isomorphic to a loop product group-grading.

Theorem 2.1.16. *Let $\mathcal{B} = \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_n$ be an algebra where $\mathcal{A}_1$,...,$\mathcal{A}_n$ are simple ideals of $\mathcal{B}$. Let Γ be a G-grading on $\mathcal{B}$. Then there exist $J_i \subseteq \{1,...,n\}$ for $i = 1,...,m$ and $m \leq n$ such that $\{1,...,n\} = J_1 \sqcup \cdots \sqcup J_m$ and the algebras $\mathcal{A}_{J_i} := \oplus_{j\in J_i}\mathcal{A}_j$ are G-graded-simple ideals of $\mathcal{B}$ for all $i = 1,...,m$. Moreover for every $i = 1,...,m$ there exists a $\overline{G}_i := G/H_i$-grading $\Gamma^i_{\mathcal{A}_{j_i}}(\overline{G}_i)$ on $\mathcal{A}_{j_i}$, where $H_i = \mathrm{Supp}_G\, C(\mathcal{A}_{J_i})$ and $j_i := min\{j \in J_i\}$ such that Γ is isomorphic to the loop product G-grading $\Gamma(G, \Gamma^1_{\mathcal{A}_{j_1}}(\overline{G}_1), ..., \Gamma^m_{\mathcal{A}_{j_m}}(\overline{G}_m))$.*

Proof. By Lemma 2.1.15 there exists $J_i \subseteq \{1,...,n\}$ for $i = 1,...,m$ and $m \leq n$ such that $\{1,...,n\} = J_1 \sqcup \cdots \sqcup J_m$ and for all $i = 1,...,m$ the algebra $\mathcal{A}_{J_i}$ is a G-graded-simple ideal of $\mathcal{B}$. By Theorem 2.1.7 $\mathcal{A}_{j_i}$ is a $\overline{G}_i := G/H_i$-graded ideal of $\mathcal{A}_{J_i}$ and there exists an isomorphism of G-graded algebras

$$\omega_i : \mathcal{A}_{J_i} \to L_{\pi_i}(\mathcal{A}_{j_i})$$

where $\pi_i : G \to G/H_i$ is the canonical projection for $H_i = \mathrm{Supp}_G\, C(\mathcal{A}_{J_i})$ and $j_i = min\{j \in J_i\}$ for each $i \in \{1,...,m\}$. Denote by $\Gamma^i_{\mathcal{A}_{j_i}}(\overline{G}_i)$ the $\overline{G}_i$-grading on $\mathcal{A}_{j_i}$. Finally Γ is isomorphic to $\Gamma(G, \Gamma^1_{\mathcal{A}_{j_1}}(\overline{G}_1), ..., \Gamma^m_{\mathcal{A}_{j_m}}(\overline{G}_m))$ via the isomorphism

$$\begin{array}{ccc} L_{\pi_1}(\mathcal{A}_{j_1}) \times \cdots \times L_{\pi_m}(\mathcal{A}_{j_m}) & \to & \mathcal{A}_{J_1} \oplus \cdots \oplus \mathcal{A}_{J_m} \\ (a_1, ..., a_m) & \mapsto & \omega_1^{-1}(a_1) + \cdots + \omega_m^{-1}(a_m). \end{array}$$

for $a_i \in L_{\pi_i}(\mathcal{A}_{j_i})$ and $i = 1,...,m$.

□

2.2 Isomorphisms of gradings

In this section we give the classification, up to isomorphism, of loop product group-gradings (Theorem 2.2.2).

Lemma 2.2.1. *Let H be a subgroup of G and let $\pi : G \longrightarrow \overline{G}$ be a group epimorphism such that $\ker \pi = H$. Let $\mathcal{A}_i$ be simple $\overline{G}$-graded algebras for $i = 1, 2$. Let $L_\pi(\mathcal{A}_i) = \sum_{g \in G}((\mathcal{A}_i)_{\overline{g}} \otimes g)$, where $\overline{g} = \pi(g)$ for $i = 1, 2$. Then $L_\pi(\mathcal{A}_1) \simeq_G L_\pi(\mathcal{A}_2)$ if and only if $\mathcal{A}_1 \simeq_{\overline{G}} \mathcal{A}_2$.*

Proof. $L_\pi(\mathcal{A}_1) \simeq_G L_\pi(\mathcal{A}_2)$ if and only if $\mathcal{A}_1 \sim_\pi \mathcal{A}_2$ by Theorem 1.3.32 (iii) and $\mathcal{A}_1 \sim_\pi \mathcal{A}_2$ if and only if $\mathcal{A}_1 \simeq_{\overline{G}} \mathcal{A}_2$ by Remark 1.3.30 (iii). □

Theorem 2.2.2. *Let H_i and H'_j be subgroups of G and let $\pi_i : G \to \overline{G}_i$ and $\pi'_j : G \to \overline{G}'_j$ be group epimorphisms such that $\ker \pi_i = H_i$ and $\ker \pi'_j = H'_j$ for $i = 1, ..., m$ and $j = 1, ..., r$. Let $\Gamma^i_{\mathcal{A}_i}(\overline{G}_i)$ be a $\overline{G}_i$-grading on $\mathcal{A}_i$, where $\mathcal{A}_i$ is a simple algebra for $i = 1, ..., m$ and let $\Gamma'^j_{\mathcal{A}'_j}(\overline{G}'_j)$ be a $\overline{G}'_j$-grading on $\mathcal{A}'_j$, where $\mathcal{A}'_j$ is a simple algebra for $j = 1, ..., r$. Then*

$$\Gamma(G, \Gamma^1_{\mathcal{A}_1}(\overline{G}_1), ..., \Gamma^m_{\mathcal{A}_m}(\overline{G}_m)) \simeq \Gamma(G, \Gamma'^1_{\mathcal{A}'_1}(\overline{G}'_1), ..., \Gamma'^r_{\mathcal{A}'_r}(\overline{G}'_r))$$

if and only if $m = r$ and there exists $\sigma \in S_m$ such that $H_i = H'_{\sigma(i)}$ and $\Gamma^i_{\mathcal{A}_i}(\overline{G}_i) \simeq \Gamma'^{\sigma(i)}_{\mathcal{A}'_{\sigma(i)}}(\overline{G}'_{\sigma(i)})$ for all $i = 1, ..., m$.

Proof. $\Rightarrow$) We will denote by Γ^i the G-grading induced by $\Gamma^i_{\mathcal{A}_i}(\overline{G}_i)$ on $L_{\pi_i}(\mathcal{A}_i)$ for $i = 1, ..., m$ (see Definition 1.3.1). And we will denote by Γ'^j the G-grading induced by $\Gamma'^j_{\mathcal{A}'_j}(\overline{G}'_j)$ on $L_{\pi'_j}(\mathcal{A}'_j)$ for $j = 1, ..., r$. Since $\mathcal{A}_i$ is $\overline{G}_i$-graded-simple for $i = 1, ..., m$, by Lemma 1.3.17, $L_{\pi_i}(\mathcal{A}_i)$ is G-graded-simple. Analogously $L_{\pi'_j}(\mathcal{A}'_j)$ is G-graded-simple for $j = 1, ..., r$. Take the isomorphism of G-graded algebras

$$\Phi : L_{\pi_1}(\mathcal{A}_1) \times \cdots \times L_{\pi_m}(\mathcal{A}_m) \longrightarrow L_{\pi'_1}(\mathcal{A}'_1) \times \cdots \times L_{\pi'_r}(\mathcal{A}'_r)$$

from the hypothesis. Φ sends graded-simple ideals of $L_{\pi_1}(\mathcal{A}_1) \times \cdots \times L_{\pi_m}(\mathcal{A}_m)$ to graded-simple ideals of $L_{\pi'_1}(\mathcal{A}'_1) \times \cdots \times L_{\pi'_r}(\mathcal{A}'_r)$. Therefore for $i \in \{1, ..., m\}$ we have that

$$\Phi(0 \times \cdots \times L_{\pi_i}(\mathcal{A}_i) \times \cdots \times 0) = 0 \times \cdots \times L_{\pi'_j}(\mathcal{A}'_j) \times \cdots \times 0$$

for a unique $j \in \{1, ..., r\}$. Then we have a bijection between $\{1, ..., m\}$ and $\{1, ..., r\}$, so $m = r$ and there exists a $\sigma \in S_m$ such that

$$\Phi(0 \times \cdots \times L_{\pi_i}(\mathcal{A}_i) \times \cdots \times 0) = 0 \times \cdots \times L_{\pi'_{\sigma(i)}}(\mathcal{A}'_{\sigma(i)}) \times \cdots \times 0$$

for all $i = 1, ..., m$. Then Γ^i is isomorphic to $\Gamma'^{\sigma(i)}$ via

$$\Phi_i := P_{\sigma(i)} \circ \Phi\ |_{0\times\cdots\times L_{\pi_i}(\mathcal{A}_i)\times\cdots\times 0} \circ Q_i : L_{\pi_i}(\mathcal{A}_i) \to L_{\pi'_{\sigma(i)}}(\mathcal{A}'_{\sigma(i)})$$

where

$$\begin{array}{rccl} P_j : & L_{\pi'_1}(\mathcal{A}'_1) \times \cdots \times L_{\pi'_m}(\mathcal{A}'_m) & \longrightarrow & L_{\pi'_j}(\mathcal{A}'_j) \\ & (x_1, ..., x_m) & \longmapsto & x_j \end{array}$$

for $x_j \in L_{\pi'_j}(\mathcal{A}'_j)$ and $j = 1, ..., m$. And

$$\begin{array}{rccl} Q_i : & L_{\pi_i}(\mathcal{A}_i) & \longrightarrow & L_{\pi_1}(\mathcal{A}_1) \times \cdots \times L_{\pi_m}(\mathcal{A}_m) \\ & a & \longmapsto & (0, ..., \overset{i}{a}, ..., 0) \end{array}$$

for $a \in L_{\pi_i}(\mathcal{A}_i)$ and $i = 1, ..., m$. Since an isomorphism of G-graded algebras induces an isomorphism of G-graded algebras on their centroids (see Remark 1.3.24 (ii)), we have that $H_i = H'_{\sigma(i)}$ and therefore $\pi_i = \pi'_{\sigma(i)}$ for $i = 1, ..., m$. Finally since $L_{\pi_i}(\mathcal{A}_i) \simeq_G L_{\pi_i}(\mathcal{A}'_{\sigma(i)})$, we get by Lemma 2.2.1 that $\Gamma^i_{\mathcal{A}_i}(\overline{G}_i) \simeq \Gamma'^j_{\mathcal{A}'_{\sigma(i)}}(\overline{G}'_{\sigma(i)})$ for $i = 1, ..., m$.

$\Leftarrow$) Since $\Gamma^i_{\mathcal{A}_i}(\overline{G}_i) \simeq \Gamma'^{\sigma(i)}_{\mathcal{A}'_{\sigma(i)}}(\overline{G}'_{\sigma(i)})$, we have by Lemma 2.2.1 that there exist isomorphisms of G-graded algebras $\varphi_i : L_{\pi_i}(\mathcal{A}_i) \to L_{\pi'_{\sigma(i)}}(\mathcal{A}'_{\sigma(i)})$ for $i = 1, ..., m$. Then, up to permutations on the factors, we have the isomorphism of G-graded algebras $\varphi_1 \times \cdots \times \varphi_m$. □

We finish this section with a theorem that contains some of the results we have already proved. We just want to put them together in order to make easier their use in the future.

Theorem 2.2.3. *Let G be a group.*

1. *Let Γ be a G-grading on a semisimple algebra $\mathcal{B}$, then Γ is isomorphic to a product G-grading $\Gamma^1 \times_G \cdots \times_G \Gamma^n$ (see Definition 2.1.10) for some G-grading Γ^i on a semisimple and graded-simple algebra $\mathcal{B}^i$ for $i = 1, ..., n$. The factors $\mathcal{B}^i$ are uniquely determined up to reordering and G-graded isomorphisms.*

2. *Any G-graded-simple algebra $\mathcal{B}$ is isomorphic, as a G-graded algebra, to the loop algebra $L_\pi(\mathcal{A})$ associated to a surjective group homomorphism $\pi : G \to \overline{G}$ with finite kernel H, and a central-simple $\overline{G}$-graded algebra $\mathcal{A}$.*

   Moreover, in this situation $\mathcal{B}$ is semisimple if and only if $\operatorname{char}\mathbb{F}$ *does not divide $|H|$.*

Proof. Item 1 follows from Lemma 2.1.15. First part of item 2 follows from Theorem 1.3.32 (ii) and the second part from Theorem 2.1.9. □

2.3 Equivalence of gradings

In this section we give a relation between the universal group of a simple algebra and its loop algebra (Lemma 2.3.2). We also give the classification, up to equivalence, of loop product group-gradings (Theorems 2.3.7 and 2.3.8). Recall loop product group-gradings are isomorphic to group-gradings on semisimple algebras (Theorem 2.1.16).

Remark 2.3.1. *Let $\mathcal{A}$ be a simple $\overline{G}$-graded algebra with a $\overline{G}$-grading $\overline{\Gamma}$. Let H be a subgroup of G and let $\pi : G \to \overline{G}$ be a group epimorphism such that* $\ker \pi = H$. *Let Γ be the G-grading on $L_\pi(\mathcal{A})$ induced by $\overline{\Gamma}$ (see Definition 1.3.1). Notice that by Remark 2.1.3 we have that $\mathcal{A}$ is central-simple, and by Lemma 1.3.11 it is also graded-central-simple. Then, by Lemma 1.3.18 (ii) $H = \operatorname{Supp}_G C(L_\pi(\mathcal{A}))$.*

The next result shows a close relation between the universal group of a simple $\overline{G}$-graded algebra $\mathcal{A}$ and the one of its loop algebra related to some group epimorphism $\pi : G \to G/H := \overline{G}$ for some subgroup H of G. In order to do this, we will use an exact sequence of affine group schemes

$$\mathbf{1} \longrightarrow \mathbf{Diag}(\overline{\Gamma}) \longrightarrow \mathbf{Diag}(\Gamma) \longrightarrow \mathbf{Diag}(\Gamma_{C(\mathcal{B})}) \longrightarrow \mathbf{1}$$

where $\overline{\Gamma}$, Γ and $\Gamma_{C(\mathcal{B})}$ are group-gradings on $\mathcal{A}$, $\mathcal{B} := L_\pi(\mathcal{A})$ and $C(\mathcal{B})$ respectively. The affine group scheme $\mathbf{1}$ is the one with associated Hopf algebra $\mathbb{F}$. Then, using the fact that $\mathbf{Diag}(\Gamma') \simeq (U(\Gamma'))^D$ for any group-grading Γ', we will obtain an exact sequence of groups

$$1 \longrightarrow H \longrightarrow U(\Gamma) \longrightarrow U(\overline{\Gamma}) \longrightarrow 1.$$

Lemma 2.3.2. *Let $\mathcal{A}$ be a simple $\overline{G}$-graded algebra with a group-grading $\overline{\Gamma} : \mathcal{A} = \bigoplus_{\overline{g} \in \overline{G}} \mathcal{A}_{\overline{g}}$. Let H be a subgroup of G and let $\pi : G \to \overline{G}$ be a group epimorphism such that* $\ker \pi = H$. *Let $\Gamma : \mathcal{B} := L_\pi(\mathcal{A}) = \bigoplus_{g \in G} (\mathcal{A}_{\overline{g}} \otimes g)$ be the G-grading induced by $\overline{\Gamma}$. Then $\overline{G} = U(\overline{\Gamma})$ if and only if $G = U(\Gamma)$.*

Proof. By Proposition 1.3.21 $L_\pi(\mathcal{A})$ is a G-graded-simple algebra, so by Lemma 1.3.7 (ii) $C(\mathcal{B})$ is a G-graded algebra. We denote such group-grading induced by Γ on $C(\mathcal{B})$ by $\Gamma_{C(\mathcal{B})}$. Set $\overline{U} := U(\overline{\Gamma})$and $U := U(\Gamma)$. By Remark 2.3.1 we have that $H = \operatorname{Supp}_G C(\mathcal{B})$. First we will prove that the next sequence of schemes is exact

$$\mathbf{1} \;\longrightarrow\; \mathbf{Diag}(\overline{\Gamma}) \;\xrightarrow{\Psi}\; \mathbf{Diag}(\Gamma) \;\xrightarrow{\Phi}\; \mathbf{Diag}(\Gamma_{C(\mathcal{B})}) \;\longrightarrow\; \mathbf{1}$$

such that for $R \in \mathrm{Alg}_{\mathbb{F}}$

$$\begin{array}{ccccc} \mathbf{Diag}(\overline{\Gamma})(R) & \xrightarrow{\Psi(R)} & \mathbf{Diag}(\Gamma)(R) & \xrightarrow{\Phi(R)} & \mathbf{Diag}(\Gamma_{C(\mathcal{B})})(R) \\ \varphi & \longmapsto & \widehat{\varphi} & & \\ & & \delta & \longmapsto & \delta' \end{array}$$

where for $x_{\overline{g}} \in \mathcal{A}_{\overline{g}}$, $g \in G$, $r \in R$, $\alpha \in C(\mathcal{B})_h$ and $h \in H$ we have $\delta'(\alpha \otimes r) = \delta(\alpha \otimes r)\delta^{-1}$ and $\widehat{\varphi}((x_{\overline{g}} \otimes g) \otimes r) = (x_{\overline{g}} \otimes g) \otimes r r_{\overline{g}}$ where $r_{\overline{g}} \in R^\times$ is such that $\varphi \mid_{\mathcal{A}_{\overline{g}} \otimes R} = r_{\overline{g}} id_{\mathcal{A}_{\overline{g}} \otimes R}$. For $\varphi \in \mathbf{Diag}(\overline{\Gamma})(R)$ we denote by $r_{(\varphi,\overline{g})}$ the element in $R^\times$ such that $\varphi \mid_{\mathcal{A}_{\overline{g}} \otimes R} = r_{(\varphi,\overline{g})} id_{\mathcal{A}_{\overline{g}} \otimes R}$. Analogously for $\delta \in \mathbf{Diag}(\Gamma)(R)$ we denote by $r_{(\delta,g)}$ the element in $R^\times$ such that $\delta \mid_{L_\pi(\mathcal{A})_g \otimes R} = r_{(\delta,g)} id_{L_\pi(\mathcal{A})_g \otimes R}$. For $x_{\overline{g}} \otimes g \in \mathcal{A}_{\overline{g}} \otimes g$ and $s \in R$ we have

$$\begin{aligned} \delta \circ \delta^{-1}((x_{\overline{g}} \otimes g) \otimes s) &= \delta((x_{\overline{g}} \otimes g) \otimes s r_{(\delta^{-1},g)}) \\ &= (x_{\overline{g}} \otimes g) \otimes s r_{(\delta^{-1},g)} r_{(\delta,g)} \end{aligned}$$

So since $\delta \circ \delta^{-1} = id_{L_\pi(\mathcal{A}) \otimes R}$, we have $r_{(\delta^{-1},g)} r_{(\delta,g)} = 1$ and then $r^{-1}_{(\delta^{-1},g)} = r_{(\delta,g)}$. Notice also that for $x_{\overline{g}} \otimes g \in \mathcal{A}_{\overline{g}} \otimes R$, $s, t \in R$, $\alpha \in C(\mathcal{B})_h$, $h \in H$ we have

$$\begin{aligned} \delta\left((\alpha \otimes t)\delta^{-1}((x_{\overline{g}} \otimes g) \otimes s)\right) &= \delta\left((\alpha \otimes t)((x_{\overline{g}} \otimes g) \otimes s r_{(\delta^{-1},g))}\right) \\ &= \delta(\alpha(x_{\overline{g}} \otimes g) \otimes s r_{(\delta^{-1},g)} t) \\ &= \alpha(x_{\overline{g}} \otimes g) \otimes s r_{(\delta^{-1},g)} t r_{(\delta,gh)}. \end{aligned}$$

Then

$$\begin{aligned} \ker \Phi(R) &= \{\delta \in \mathbf{Diag}(\Gamma)(R) : \Phi(R)(\delta) = \delta' = id_{C(\mathcal{B}) \otimes R}\} \\ &= \{\delta \in \mathbf{Diag}(\Gamma)(R) : \delta(\alpha \otimes t)\delta^{-1} = \alpha \otimes t \text{ for all } \alpha \otimes t \in C(\mathcal{B})_h \otimes R, \\ &\qquad h \in H\} \\ &= \{\delta \in \mathbf{Diag}(\Gamma)(R) : r_{(\delta^{-1},g)} r_{(\delta,gh)} = 1 \text{ for all } g \in G \text{ and } h \in H\} \\ &= \{\delta \in \mathbf{Diag}(\Gamma)(R) : r_{(\delta,g)} = r_{(\delta,gh)} \text{ for all } g \in G \text{ and } h \in H\ \} \end{aligned}$$

and

$$\begin{aligned} \mathrm{im}\, \Psi(R) &= \{\Psi(\varphi) = \widehat{\varphi} : \varphi \in \mathbf{Diag}(\overline{\Gamma})(R)\} \\ &= \{\widehat{\varphi} \in \mathbf{Diag}(\Gamma)(R) : \widehat{\varphi}((x_{\overline{g}} \otimes g) \otimes t) = (x_{\overline{g}} \otimes g) \otimes t r_{(\varphi,\overline{g})} \\ &\qquad \text{for some } \varphi \in \mathbf{Diag}(\overline{\Gamma})(R), \forall t \in R, \forall g \in G, \forall x_{\overline{g}} \in \mathcal{A}_{\overline{g}}\}. \end{aligned}$$

For $R \in \mathrm{Alg}_{\mathbb{F}}$ take $\delta \in \ker \Phi(R)$. Then $r_{(\delta,g)} = r_{(\delta,gh)}$ for all $g \in G$ and $h \in H$. Consider $\varphi \in \mathbf{Diag}(\overline{\Gamma})(R)$ such that $r_{(\varphi,\overline{g})} = r_{(\delta,g)}$ which is well defined by the above argument. Then $\Psi(\varphi) = \delta$. Therefore $\ker \Phi(R) \subseteq \mathrm{im}\, \Psi(R)$.

For $\varphi \in \mathbf{Diag}(\overline{\Gamma})(R)$, $\alpha \otimes t \in C(\mathcal{A})_h \otimes R$, $x_{\overline{g}} \otimes g \in \mathcal{A} \otimes R$ and $s \in R$ we have

$$\begin{aligned}
\Phi(R) \circ \Psi(R)(\varphi)(\alpha \otimes t)((x_{\overline{g}} \otimes g) \otimes s) &= \widehat{\varphi}(\alpha \otimes t)(\widehat{\varphi})^{-1}((x_{\overline{g}} \otimes g) \otimes s) \\
&= \widehat{\varphi}(\alpha \otimes t)((x_{\overline{g}} \otimes g) \otimes s(r_{(\varphi,\overline{g})})^{-1}) \\
&= \widehat{\varphi}(\alpha(x_{\overline{g}} \otimes g) \otimes s(r_{(\varphi,\overline{g})})^{-1}t) \\
&= \alpha(x_{\overline{g}} \otimes g) \otimes s(r_{(\varphi,\overline{g})})^{-1}tr_{(\varphi,\overline{gh})} \\
&= \alpha(x_{\overline{g}} \otimes g) \otimes st(r_{(\varphi,\overline{g})})^{-1}r_{(\varphi,\overline{g})} \\
&= \alpha(x_{\overline{g}} \otimes g) \otimes st \\
&= (\alpha \otimes t)((x_{\overline{g}} \otimes g) \otimes s).
\end{aligned}$$

Then $\Phi(R) \circ \Psi(R)(\varphi) = id_{L_\pi(\mathcal{A})\otimes R}$. Therefore $\operatorname{im}\Psi(R) \subseteq \ker \Phi(R)$. This proves that $\ker \Phi(R) = \operatorname{im}\Psi(R)$. For $\varphi \in \mathbf{Diag}(\overline{\Gamma})(R)$ we have $r_{(\widehat{\varphi},g)} = r_{(\varphi,\overline{g})}$ for $g \in G$. Then

$$\begin{aligned}
\ker \Psi(R) &= \{\varphi \in \mathbf{Diag}(\overline{\Gamma})(R) : \Psi(R)(\varphi) = \widehat{\varphi} = id_{L_\pi(\mathcal{A})\otimes R}\} \\
&= \{\varphi \in \mathbf{Diag}(\overline{\Gamma})(R) : r_{(\widehat{\varphi},g)} = r_{(\varphi,\overline{g})} = 1 \text{ for all } \overline{g} \in \operatorname{Supp}\overline{\Gamma}\} \\
&= \{id_{\mathcal{A}\otimes R}\},
\end{aligned}$$

Therefore Ψ is injective. In order to prove that Φ is surjective let us prove first that the following diagram commutes

$$\begin{tikzcd}
\mathbf{Diag}(\Gamma) \arrow[r, "\Phi"] & \mathbf{Diag}(\Gamma_{C(\mathcal{B})}) \\
G^D \arrow[u, "\omega"] \arrow[ur, "\iota"'] &
\end{tikzcd}$$

where ω is defined as follows for $R \in \mathrm{Alg}_{\mathbb{F}}$ and $(x_{\overline{g}} \otimes g) \otimes s \in (\mathcal{A}_{\overline{g}} \otimes g) \otimes R$,

$$\begin{array}{llll}
\omega(R): & G^D(R) & \longrightarrow & \mathbf{Diag}(\Gamma)(R) \\
& \chi : G \to R^\times & \longmapsto & \begin{array}{llll} \omega(R)(\chi): & L_\pi(\mathcal{A}) \otimes R & \to & L_\pi(\mathcal{A}) \otimes R \\ & (x_{\overline{g}} \otimes g) \otimes s & \mapsto & (x_{\overline{g}} \otimes g) \otimes s\chi(g). \end{array}
\end{array}$$

Notice that

$$\begin{array}{llll}
(\omega(R)(\chi))^{-1}: & L_\pi(\mathcal{A}) \otimes R & \to & L_\pi(\mathcal{A}) \otimes R \\
& (x_{\overline{g}} \otimes g) \otimes s & \mapsto & (x_{\overline{g}} \otimes g) \otimes s(\chi(g))^{-1}.
\end{array}$$

And ι is defined as follows for $R \in \mathrm{Alg}_{\mathbb{F}}$, $\alpha \in C(\mathcal{B})_h$, $h \in H$ and $s \in R$

$$\begin{array}{llll}
\iota(R): & G^D(R) & \longrightarrow & \mathbf{Diag}(\Gamma_{C(\mathcal{B})})(R) \\
& \chi : G \to R^\times & \longmapsto & \begin{array}{llll} \iota(R)(\chi): & C(\mathcal{B}) \otimes R & \to & \mathcal{C}(\mathcal{B}) \otimes R \\ & \alpha \otimes s & \mapsto & \alpha \otimes s\chi(h). \end{array}
\end{array}$$

Note that $H = U(\Gamma_{C(\mathcal{B})})$. This follows from the fact that $H = \mathrm{Supp}_G(C(\mathcal{B}))$ and recall that a universal group of a group-grading is unique up to an isomorphism over the support of such group-grading (see Chapter 1, Section 1). So $\mathbf{Diag}(\Gamma_{C(\mathcal{B})}) \simeq H^D$ (see Chapter 1, Section 2). The expression above for $\iota(R)$ shows that the induced homomorphism $\iota^* : \mathbb{F}H \to \mathbb{F}G$ of Hopf algebras (see Definition 1.2.7) is the one induced by the inclusion $H \hookrightarrow G$. Hence ι^* is injective and therefore ι is surjective. For $\alpha \in C(\mathcal{B})_h$, $h \in H$, $x_{\overline{g}} \otimes g \in \mathcal{A}_{\overline{g}} \otimes g$, $g \in G$, $r, s \in R$ and $\chi : G \to R^\times \in G^D(R)$ we have

$$\begin{aligned}
&\Phi(R) \circ \omega(R)(\chi)(\alpha \otimes r)((x_{\overline{g}} \otimes g) \otimes s)\\
&\qquad = \omega(R)(\chi)(\alpha \otimes r)(\omega(R)(\chi))^{-1}((x_{\overline{g}} \otimes g) \otimes s)\\
&\qquad = \omega(R)(\chi)(\alpha \otimes r)((x_{\overline{g}} \otimes g) \otimes s(\chi(g))^{-1})\\
&\qquad = \omega(R)(\chi)(\alpha(x_{\overline{g}} \otimes g) \otimes sr(\chi(g))^{-1})\\
&\qquad = \alpha(x_{\overline{g}} \otimes g) \otimes sr(\chi(g))^{-1}\chi(gh)\\
&\qquad = \alpha(x_{\overline{g}} \otimes g) \otimes sr(\chi(g))^{-1}\chi(g)\chi(h)\\
&\qquad = \alpha(x_{\overline{g}} \otimes g) \otimes sr\chi(h)\\
&\qquad = (\alpha \otimes r\chi(h))((x_{\overline{g}} \otimes g) \otimes s)\\
&\qquad = \iota(R)(\chi)(\alpha \otimes r)((x_{\overline{g}} \otimes g) \otimes s),
\end{aligned}$$

therefore $\Phi(R) \circ \omega(R) = \iota(R)$, then the diagram commutes. Then we get the following commutative diagram involving the representing objects, i.e. a commutative diagram of Hopf algebra homomorphisms (see Definition 1.2.7).

$$\begin{tikzcd}
\mathbb{F}U \arrow[d, "\omega^*"'] & \mathbb{F}H \arrow[l, "\Phi^*"'] \arrow[dl, "\iota^*"] \\
\mathbb{F}G &
\end{tikzcd}$$

Since ι^* is injective, $\Phi^* \circ \omega^*$ is injective and therefore Φ^* is injective. Then Φ is surjective. This completes the proof that the affine group schemes sequence is exact. Recall that $\mathbf{Diag}(\overline{\Gamma}) \simeq \overline{U}^D$, $\mathbf{Diag}(\Gamma) \simeq U^D$ and $\mathbf{Diag}(\Gamma_{C(\mathcal{B})}) \simeq H^D$. By [Wat79, Theorem p. 15] we have an induced exact sequence of groups

$$1 \longrightarrow H \longrightarrow U \longrightarrow \overline{U} \longrightarrow 1.$$

The result follows from this. □

The following lemmas will help us to give the classification, up to equivalence, of loop product group-gradings (Theorems 2.3.7 and 2.3.8).

Let Γ be a G-grading on an algebra $\mathcal{B}$. For the next result we will write $(\mathcal{B}, \Gamma)$ to refer to the algebra $\mathcal{B}$ with the decomposition given by Γ on $\mathcal{B}$. Moreover, if $C(\mathcal{B})$ is G-graded with a G-grading induced by Γ then we will write $(C(\mathcal{B}), \Gamma)$ to refer to $C(\mathcal{B})$ with the decomposition given by $\Gamma_{C(\mathcal{B})}$.

Lemma 2.3.3. *Let G_1 and G_2 be groups and let H_i be a subgroup of G_i for $i = 1, 2$. Let $\pi_i : G_i \to \overline{G_i} := G_i/H_i$ be the canonical projection and let $\overline{\Gamma^i}$ be a $\overline{G_i}$-grading on a simple algebra $\mathcal{A}_i$, where $\overline{G_i} := U(\overline{\Gamma^i})$. Let Γ^i be the G_i-grading on $L_{\pi_i}(\mathcal{A}_i)$ induced by $\overline{\Gamma^i}$ for $i = 1, 2$. If Γ^1 is equivalent to Γ^2 then $\overline{\Gamma^1}$ is equivalent to $\overline{\Gamma^2}$.*

Proof. Notice that since $\overline{G_i} := U(\overline{\Gamma^i})$, we have by Lemma 2.3.2 that $G_i = U(\Gamma^i)$ for $i = 1, 2$. Since Γ^1 is equivalent to Γ^2, there exists an isomorphism of algebras $\varphi : (L_{\pi_1}(\mathcal{A}_1), \Gamma^1) \to (L_{\pi_2}(\mathcal{A}_2), \Gamma^2)$ and a group isomorphism $\alpha : G_1 \to G_2$ such that $\varphi(L_{\pi_1}(\mathcal{A}_1)_g) = L_{\pi_2}(\mathcal{A}_2)_{\alpha(g)}$ for $g \in G_1$ (by Proposition 1.1.17). That is Γ^1 is equivalent to Γ^2 via (φ, α) (see Remark 1.1.18). By Remark 2.3.1 we have that $H_i = \mathrm{Supp}_{G_i} C(L_{\pi_i}(\mathcal{A}_i))$ for $i = 1, 2$. Since an isomorphism of algebras induces an isomorphism in the centroids, we get $\alpha(H_1) = H_2$. So α induces an isomorphism $\overline{\alpha} : G_1/H_1 \longrightarrow G_2/H_2$ such that the following diagram commutes

$$\begin{tikzcd} G_1 \arrow[r, "\alpha"] \arrow[d, "\pi_1"'] & G_2 \arrow[d, "\pi_2"] \\ G_1/H_1 \arrow[r, "\overline{\alpha}"] & G_2/H_2. \end{tikzcd}$$

Denote $\pi_i(g_i)$ by $\overline{g_i}$ for $g_i \in G_i$ and $i = 1, 2$. Consider the G_1-grading ${}^{\alpha^{-1}}\Gamma^2$, the group-grading induced from Γ^2 by α^{-1} (see Definition 1.1.20), and we denote it by Γ^2_α. It is given by

$$\Gamma^2_\alpha := {}^{\alpha^{-1}}\Gamma^2 : L_{\pi_2}(\mathcal{A}_2) = \bigoplus_{g \in G_1} L_{\pi_2}(\mathcal{A}_2)'_g$$

where $L_{\pi_2}(\mathcal{A}_2)'_g := L_{\pi_2}(\mathcal{A}_2)_{\alpha(g)}$ for $g \in G_1$. Consider also the $\overline{G}_1$-grading ${}^{\overline{\alpha}^{-1}}\overline{\Gamma}^2$, the group-grading induced from $\overline{\Gamma}^2$ by $\overline{\alpha}^{-1}$ and denote it by $\overline{\Gamma}^2_\alpha$. It is given by

$$\overline{\Gamma}^2_\alpha := {}^{\overline{\alpha}^{-1}}\overline{\Gamma}^2 : \mathcal{A}_2 = \bigoplus_{\overline{g} \in \overline{G}_1} (\mathcal{A}_2)'_{\overline{g}}$$

where $(\mathcal{A}_2)'_{\overline{g}} := (\mathcal{A}_2)_{\overline{\alpha}(g)}$ for $\overline{g} \in \overline{G}_1$. By Lemma 1.3.17 $(L_{\pi_2}(\mathcal{A}_2), \Gamma^2_\alpha)$ is graded-simple and by Lemma 1.3.18 (i) it is also graded-central so it is graded-central-simple. Since $\mathbb{F} = \overline{\mathbb{F}}$, we have by Lemma 1.3.15 that $(C(L_{\pi_2}(\mathcal{A}_2)), \Gamma^2_\alpha)$ is split. Then $(C(L_{\pi_2}(\mathcal{A}_2)), \Gamma^2_\alpha) \simeq_{H_1} \mathbb{F}H_1$, indeed we have an isomorphism

$$\begin{array}{ccc} \mathbb{F}H_1 & \longrightarrow & (C(L_{\pi_2}(\mathcal{A}_2)), \Gamma^2_\alpha) \\ h & \longmapsto & c_h : a \otimes \alpha(g) \mapsto a \otimes \alpha(hg) \end{array}$$

for $h \in H_1$, $g \in G_1$ and $a \otimes \alpha(g) \in L_{\pi_2}(\mathcal{A}_2)'_g = (\mathcal{A}_2)_{\overline{\alpha(g)}} \otimes \alpha(g)$. Consider $\rho \in \mathrm{Alg}((C(L_{\pi_2}(\mathcal{A}_2)), \Gamma^2_\alpha), \mathbb{F})$ given by $\rho(c_h) = 1$ for all $h \in H_1$. Consider the following morphism

$$\begin{array}{rlcl} \phi: & (L_{\pi_2}(\mathcal{A}_2), \Gamma^2_\alpha) & \longrightarrow & (\mathcal{A}_2, \overline{\Gamma}^2_\alpha) \\ & a \otimes \alpha(g) & \longmapsto & a \end{array}$$

for $a \otimes \alpha(g) \in L_{\pi_2}(\mathcal{A}_2)'_g = (\mathcal{A}_2)_{\overline{\alpha(g)}} \otimes \alpha(g)$ and $g \in G_1$. Let us prove that ϕ is a ρ-specialization of $(L_{\pi_2}(\mathcal{A}_2), \Gamma^2_\alpha)$. The fact that ϕ is an epimorphism is clear. For $a \otimes \alpha(g) \in L_{\pi_2}(\mathcal{A})'_g$, $g \in G_1$, $c_h \in (C(L_{\pi_2}(\mathcal{A}_2)), \Gamma^2_\alpha)_h$ and $h \in H_1$ we have

a) $\phi(L_{\pi_2}(\mathcal{A}_2)'_g) = \phi((\mathcal{A}_2)_{\overline{\alpha(g)}} \otimes \alpha(g)) = (\mathcal{A}_2)_{\overline{\alpha(g)}}$ and

b) $\phi(c_h(a \otimes \alpha(g))) = \phi(a \otimes \alpha(hg)) = a = \rho(c_h)\phi(a \otimes \alpha(g))$.

Now take the isomorphism of algebras given by the equivalence between Γ^1 and Γ^2

$$\begin{array}{rlcl} \varphi: & (L_{\pi_1}(\mathcal{A}_1), \Gamma^1) & \longrightarrow & (L_{\pi_2}(\mathcal{A}_2), \Gamma^2) \\ & L_{\pi_1}(\mathcal{A}_1)_g & \longmapsto & L_{\pi_2}(\mathcal{A}_2)_{\alpha(g)} \ (=: L_{\pi_2}(\mathcal{A}_2)'_g) \end{array}$$

for $g \in G_1$. Observe that φ can be considered also as an isomorphism of G_1-gradings between Γ^1 and Γ^2_α, we will denote by φ' the same isomorphism of algebras but understood as isomorphism of G_1-gradings. Then by Remark 1.3.24 ii) φ' induces an isomorphism of algebras

$$C(\varphi') : (C(L_{\pi_1}(\mathcal{A}_1), \Gamma^1) \to (C(L_{\pi_2}(\mathcal{A}_2)), \Gamma^2_\alpha)$$

defined by $C(\varphi')(c) = \varphi' \circ c \circ (\varphi')^{-1}$ for $c \in (C(L_{\pi_1}(\mathcal{A}_1)), \Gamma^1)$. $C(\varphi')$ is an isomorphism of G_1-graded algebras. Then $\phi \circ \varphi'$ is a $\rho \circ C(\varphi')$-specialization of $(L_{\pi_1}(\mathcal{A}_1), \Gamma^1)$. Then by Proposition 1.3.26 (iv) we have that $(L_{\pi_1}(\mathcal{A}_1), \Gamma^1) \simeq_{G_1} (L_{\pi_1}(\mathcal{A}_2), \Gamma^2_\alpha)$. Finally by Lemma 2.2.1 we have that there exists an isomorphism of $\overline{G_1}$-graded algebras

$$\begin{array}{rlcl} \Psi: & (\mathcal{A}_1, \overline{\Gamma}^1) & \longrightarrow & (\mathcal{A}_2, \overline{\Gamma}^2_\alpha) \\ & (\mathcal{A}_1)_{\overline{g}} & \longmapsto & (\mathcal{A}_2)'_{\overline{g}} \ (=: (\mathcal{A}_2)_{\overline{\alpha}(\overline{g})}). \end{array}$$

Hence $\overline{\Gamma}^1$ is equivalent to $\overline{\Gamma}^2$ via $(\Psi, \overline{\alpha})$.

□

Lemma 2.3.4. *Let G_1 and G_2 be groups and let H_i be a subgroup of G_i for $i = 1, 2$. Let $\pi_i : G_i \to \overline{G_i} := G_i/H_i$ be the canonical projection and let $\overline{\Gamma^i}$ be a group-grading by $\overline{G_i}$ on a simple algebra $\mathcal{A}_i$, where $\overline{G_i} := U(\overline{\Gamma^i})$ for $i = 1, 2$. Let Γ^i be the G_i-grading on $L_{\pi_i}(\mathcal{A}_i)$ induced by $\overline{\Gamma^i}$ for $i = 1, 2$. Suppose there* *exists an equivalence between* $\overline{\Gamma^1}$ *and* $\overline{\Gamma^2}$ *given by an isomorphism* $\Phi : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ *and its associated group isomorphism* $\overline{\alpha} : G_1/H_1 \longrightarrow G_2/H_2$ *such that* $\overline{\alpha}$ *extends to a group morphism* $\alpha : G_1 \longrightarrow G_2$ *such that the following diagram commutes*

$$\begin{tikzcd} G_1 \arrow[r, "\alpha"] \arrow[d, "\pi_1"'] & G_2 \arrow[d, "\pi_2"] \\ G_1/H_1 \arrow[r, "\overline{\alpha}"] & G_2/H_2. \end{tikzcd}$$

Then there exists an isomorphism of algebras $\varphi : L_{\pi_1}(\mathcal{A}_1) \to L_{\pi_2}(\mathcal{A}_2)$ *such that* Γ^1 *is equivalent to* Γ^2 *via* (φ, α).

Proof. Notice that by Proposition 1.1.17 it makes sense to talk about the group isomorphism $\overline{\alpha}$ from the hypothesis and since $\overline{G_i} := U(\overline{\Gamma^i})$, we have by Lemma 2.3.2 that $G_i = U(\Gamma^i)$ for $i = 1, 2$. Denote $\pi_i(g_i)$ by $\overline{g}_i$ for $g_i \in G_i$ and $i = 1, 2$. We have the isomorphism of algebras

$$\begin{array}{llll} \varphi : & L_{\pi_1}(\mathcal{A}_1) & \longrightarrow & L_{\pi_2}(\mathcal{A}_2) \\ & L_{\pi_1}(\mathcal{A}_1)_g = (\mathcal{A}_1)_{\overline{g}} \otimes g & \longmapsto & \Phi((\mathcal{A}_1)_{\overline{g}}) \otimes \alpha(g) = (\mathcal{A}_2)_{\overline{\alpha}(\overline{g})} \otimes \alpha(g) \\ & & & = L_{\pi_2}(\mathcal{A}_2)_{\alpha(g)} \end{array}$$

for $g \in G_1$. Then Γ^1 is equivalent to Γ^2 via (φ, α). □

Remark 2.3.5. *i) Let* $\mathcal{A}_1$,..., $\mathcal{A}_n$ *be graded-simple algebras. Then the only nonzero graded-simple ideals of* $\mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ *are* $0 \times \cdots \times \mathcal{A}_i \times \cdots \times 0$ *for each* $i = 1, ..., n$. *Analogously, if* $\mathcal{B} = \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_n$ *where* $\mathcal{A}_1$,..., $\mathcal{A}_n$ *are graded-simple ideals of* $\mathcal{B}$. *Then the only nonzero graded-simple ideals of* $\mathcal{B}$ *are* $\mathcal{A}_i$ *for each* $i = 1, ..., n$. *(See Lemma 2.1.15.)*
ii) Let φ *be an equivalence of group-gradings between a* G_1*-grading* Γ^1 *on* $\mathcal{A}_1$ *and a* G_2*-grading* Γ^2 *on* $\mathcal{A}_2$ *where* $\mathcal{A}_i$ *is an algebra and* G_i *is a group for* $i = 1, 2$. *Let* $\alpha : \operatorname{Supp}\Gamma^1 \to \operatorname{Supp}\Gamma^2$ *be the bijection associated to* φ. *Let* $\mathcal{I}$ *be a* G_1*-graded ideal of* $\mathcal{A}_1$, *then* $\varphi(\mathcal{I})$ *is a* G_2*-graded ideal of* $\mathcal{A}_2$ *by means of* $\varphi(\mathcal{I}_g) = (\varphi(\mathcal{I}))_{\alpha(g)}$ *for* $g \in G_1$. *Analogously if* Φ *is an isomorphism between two* G*-gradings* Γ^1 *and* Γ^2 *on the algebras* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *respectively and* $\mathcal{I}$ *is a* G*-graded ideal of* $\mathcal{A}_1$. *Then* $\Phi(\mathcal{I})$ *is a* G*-graded ideal of* $\mathcal{A}_2$ *by means of* $\Phi(\mathcal{I}_g) = (\Phi(\mathcal{I}))_g$ *for* $g \in G$.

Lemma 2.3.6. *Let* φ *be an equivalence (or an isomorphism) between a* G_1*-grading on* $\mathcal{A}_1$ *and a* G_2*-grading on* $\mathcal{A}_2$ *where* $\mathcal{A}_i$ *is an algebra and* G_i *is a group for* $i = 1, 2$. *Then* φ *sends graded-simple ideals on* $\mathcal{A}_1$ *to graded-simple ideals on* $\mathcal{A}_2$.

Proof. Let $\mathcal{I}$ be a G_1-graded-simple ideal of $\mathcal{A}_1$. By Remark 2.3.5 ii) $\varphi(\mathcal{I})$ is a G_2-graded ideal of $\mathcal{A}_2$. Suppose $\mathcal{J}$ is a nonzero G_2-graded ideal of $\varphi(\mathcal{I})$, then again by 2.3.5 ii) $\varphi^{-1}(\mathcal{J})$ is a G_1-graded ideal of $\mathcal{I}$. Therefore $\varphi^{-1}(\mathcal{J}) = \mathcal{I}$ and then $\mathcal{J} = \varphi(\mathcal{I})$ so $\varphi(\mathcal{I})$ is a G_2-graded-simple ideal of $\mathcal{A}_2$. $\square$

The following two theorems give the classification, up to equivalence, of loop product group-gradings.

Theorem 2.3.7. *Let G and G' be groups. Let H_i (resp. H'_j) be subgroups of G (resp. G') and let $\pi_i : G \to G/H_i =: \overline{G}_i$ and $\pi'_j : G' \to G'/H'_j =: \overline{G}'_j$ be the canonical projections for $i = 1, ..., m$ and $j = 1, ..., r$. Let $\Gamma^i_{\mathcal{A}_i}(\overline{G}_i)$ be a $\overline{G}_i$-grading on $\mathcal{A}_i$, where $\mathcal{A}_i$ is a simple algebra for $i = 1, ..., m$ and let $\Gamma'^j_{\mathcal{A}'_j}(\overline{G}'_j)$ be a $\overline{G}'_j$-grading on $\mathcal{A}'_j$, where $\mathcal{A}'_j$ is a simple algebra for $j = 1, ..., r$. Assume $U(\Gamma^i_{\mathcal{A}_i}(\overline{G}_i)) = \overline{G}_i$ and $U(\Gamma'^j_{\mathcal{A}'_j}(\overline{G}'_j)) = \overline{G}'_j$ for $i = 1, ..., m$ and $j = 1, ..., r$. If $\Gamma(G, \Gamma^1_{\mathcal{A}_1}(\overline{G}_1), ..., \Gamma^m_{\mathcal{A}_m}(\overline{G}_m))$ is equivalent to $\Gamma(G', \Gamma'^1_{\mathcal{A}'_1}(\overline{G}'_1), ..., \Gamma'^r_{\mathcal{A}'_r}(\overline{G}'_r))$ then $m = r$ and there exist $\sigma \in S_m$ such that $\Gamma^i_{\mathcal{A}_i}(\overline{G}_i)$ is equivalent to $\Gamma'^{\sigma(i)}_{\mathcal{A}'_{\sigma(i)}}(\overline{G}'_{\sigma(i)})$ for $i = 1, ..., m$.*

Proof. Since $\mathcal{A}_i$ and $\mathcal{A}'_j$ are simple, we have by Proposition 1.3.21 that $L_{\pi_i}(\mathcal{A}_i)$ and $L_{\pi'_j}(\mathcal{A}'_j)$ are graded-simple for $i = 1, ..., m$ and $j = 1, ..., r$. Denote by Γ_i (resp. Γ'_j) the G-grading (resp G'-grading) on $L_{\pi_i}(\mathcal{A}_i)$ (resp. $L_{\pi'_j}(\mathcal{A}'_j)$) induced by $\Gamma^i_{\mathcal{A}_i}(\overline{G}_i)$ (resp. $\Gamma'^j_{\mathcal{A}'_j}(\overline{G}'_j)$) for $i = 1, ...m$ (resp. $j = 1, ..., r$) (see Definition 1.3.1).

We will denote by Γ and Γ' the group-gradings $\Gamma(G, \Gamma^1_{\mathcal{A}_1}(\overline{G}_1), ..., \Gamma^m_{\mathcal{A}_m}(\overline{G}_m))$ and $\Gamma(G', \Gamma'^1_{\mathcal{A}'_1}(\overline{G}'_1), ..., \Gamma'^r_{\mathcal{A}'_r}(\overline{G}'_r))$, respectively. Since Γ and Γ' are equivalent, there exists an isomorphism of algebras

$$\varphi : L_{\pi_1}(\mathcal{A}_1) \times \cdots \times L_{\pi_m}(\mathcal{A}_m) \to L_{\pi'_1}(\mathcal{A}'_1) \times \cdots \times L_{\pi'_r}(\mathcal{A}'_r)$$

and a bijection $\alpha : \operatorname{Supp}\Gamma \to \operatorname{Supp}\Gamma'$ such that

$$\varphi((L_{\pi_1}(\mathcal{A}_1) \times \cdots \times L_{\pi_m}(\mathcal{A}_m))_g) = (L_{\pi'_1}(\mathcal{A}'_1) \times \cdots \times L_{\pi'_r}(\mathcal{A}'_r))_{\alpha(g)}$$

for $g \in G$. By Lemma 2.3.6 we have that φ sends graded-simple ideals in $\mathcal{B} := L_{\pi_1}(\mathcal{A}_1) \times \cdots \times L_{\pi_m}(\mathcal{A}_m)$ to graded-simple ideals in $\mathcal{B}' := L_{\pi'_1}(\mathcal{A}'_1) \times \cdots \times L_{\pi'_r}(\mathcal{A}'_r)$. By Remark 2.3.5 i) the nonzero graded-simple ideals of $\mathcal{B}$ are $0 \times \cdots \times L_{\pi_i}(\mathcal{A}_i) \times \cdots \times 0$ for $i = 1, ..., m$ and the ones of $\mathcal{B}'$ are $0 \times \cdots \times L_{\pi'_j}(\mathcal{A}'_j) \times \cdots \times 0$ for $j = 1, ..., r$. Then for each $i \in \{1, ..., m\}$ there exists a unique $j \in \{1, ..., r\}$ such that

$$\varphi(0 \times \cdots \times L_{\pi_i}(\mathcal{A}_i) \times \cdots \times 0) = 0 \times \cdots \times L_{\pi'_j}(\mathcal{A}'_j) \times \cdots \times 0.$$

We get a bijection between $\{1,...,m\}$ and $\{1,...,r\}$. So $m = r$ and there exists $\sigma \in S_m$ such that

$$\varphi_i := P_{\sigma(i)} \circ \varphi\mid_{0\times\cdots\times L_{\pi_i}(\mathcal{A}_i)\times\cdots\times 0} \circ Q_i : L_{\pi_i}(\mathcal{A}_i) \to L_{\pi'_{\sigma(i)}}(\mathcal{A}'_{\sigma(i)})$$

for $i = 1,...,m$ where

$$\begin{array}{rccc} P_j : & L_{\pi'_1}(\mathcal{A}'_1) \times \cdots \times L_{\pi'_m}(\mathcal{A}'_m) & \longrightarrow & L_{\pi'_j}(\mathcal{A}'_j) \\ & (x_1,...,x_m) & \longmapsto & x_j \end{array}$$

for $x_j \in L_{\pi'_j}(\mathcal{A}'_j)$ and $j = 1,...,m$. And

$$\begin{array}{rccc} Q_i : & L_{\pi_i}(\mathcal{A}_i) & \longrightarrow & L_{\pi_1}(\mathcal{A}_1) \times \cdots \times L_{\pi_m}(\mathcal{A}_m) \\ & a & \longmapsto & (0,...,\overset{i}{a},...,0) \end{array}$$

for $a \in L_{\pi_i}(\mathcal{A}_i)$ and $i = 1,...,m$. Then φ_i is an isomorphism of algebras for $i = 1,...,m$ such that $\varphi_i(L_{\pi_i}(\mathcal{A}_i)_g) = L_{\pi'_{\sigma(i)}}(\mathcal{A}'_{\sigma(i)})_{\alpha(g)}$ for $g \in G$. Then Γ_i is equivalent to $\Gamma'_{\sigma(i)}$ via (φ_i, α) for $i = 1,...,m$. Then by Lemma 2.3.3 $\Gamma^i_{\mathcal{A}_i}(\overline{G}_i)$ is equivalent to $\Gamma'^{\sigma(i)}_{\mathcal{A}'_{\sigma(i)}}(\overline{G}'_{\sigma(i)})$ for $i = 1,...,m$. □

Theorem 2.3.8. *Let G and G' be groups. Let H_i be subgroups of G and let H'_i be subgroups of G' and let $\pi_i : G \to G/H_i =: \overline{G}_i$ and $\pi'_i : G' \to G'/H'_i =: \overline{G}'_i$ be the canonical projections for $i = 1,...,m$. Let $\Gamma^i_{\mathcal{A}_i}(\overline{G}_i)$ be a $\overline{G}_i$-grading on $\mathcal{A}_i$, where $\mathcal{A}_i$ is a simple algebra and let $\Gamma'^i_{\mathcal{A}'_i}(\overline{G}'_i)$ be a $\overline{G}'_i$-grading on $\mathcal{A}'_i$, where $\mathcal{A}'_i$ is a simple algebra for $i = 1,...,m$. Suppose $U(\Gamma^i_{\mathcal{A}_i}(\overline{G}_i)) = \overline{G}_i$ and $U(\Gamma'^i_{\mathcal{A}'_i}(\overline{G}'_i)) = \overline{G}'_i$ for $i = 1,...,m$. Suppose there exist equivalences of group-gradings $\Phi_i : \mathcal{A}_i \to \mathcal{A}'_i$ between $\Gamma^i_{\mathcal{A}_i}(\overline{G}_i)$ and $\Gamma'^i_{\mathcal{A}'_i}(\overline{G}'_i)$ for $i = 1,...,m$ such that the associated group isomorphisms $\overline{\alpha}_i : \overline{G}_i \to \overline{G}'_i$ extend to a group morphism $\alpha : G \to G'$ and such that the following diagrams commute for all $i = 1,...,m$*

$$\begin{array}{ccc} G & \xrightarrow{\alpha} & G' \\ {\scriptstyle \pi_i}\downarrow & & \downarrow{\scriptstyle \pi'_i} \\ G/H_i & \xrightarrow{\overline{\alpha}_i} & G'/H'_i. \end{array}$$

Then there exists an isomorphism of algebras

$$\varphi : L_{\pi_1}(\mathcal{A}_1) \times \cdots \times L_{\pi_m}(\mathcal{A}_m) \to L_{\pi'_1}(\mathcal{A}'_1) \times \cdots \times L_{\pi'_m}(\mathcal{A}'_m)$$

such that $\Gamma(G, \Gamma^1_{\mathcal{A}_1}(\overline{G}_1), ..., \Gamma^m_{\mathcal{A}_m}(\overline{G}_m))$ is equivalent to $\Gamma(G', \Gamma'^1_{\mathcal{A}'_1}(\overline{G}'_1), ..., \Gamma'^m_{\mathcal{A}'_m}(\overline{G}'_m))$ via (φ, α).

Proof. Notice that since $U(\Gamma^i_{\mathcal{A}_i}(\overline{G}_i)) = \overline{G}_i$ and $U(\Gamma'^i_{\mathcal{A}'_i}(\overline{G}'_i)) = \overline{G}'_i$ for $i = 1, ..., m$, by Proposition 1.1.17 it makes sense to talk about the group isomorphisms $\overline{\alpha}_i : \overline{G}_i \to \overline{G}'_i$ from the hypothesis. Let Γ_i (resp. Γ'_i) be the G_i-grading (resp. G'_i-grading) on $L_{\pi_i}(\mathcal{A}_i)$ (resp. $L_{\pi'_i}(\mathcal{A}'_i)$) induced by $\Gamma^i_{\mathcal{A}_i}(\overline{G}_i)$ (resp. $\Gamma'^i_{\mathcal{A}'_i}(\overline{G}'_i)$) for $i = 1, ..., m$. By Lemma 2.3.4 we get that there exists an isomorphism of algebras $\varphi_i : L_{\pi_i}(\mathcal{A}_i) \to L_{\pi'_i}(\mathcal{A}'_i)$ such that Γ_i is equivalent to Γ'_i via (φ_i, α) for $i = 1, ..., m$. Then we have the following isomorphism of algebras

$$\varphi : L_{\pi_1}(\mathcal{A}_1) \times \cdots \times L_{\pi_m}(\mathcal{A}_m) \to L_{\pi'_1}(\mathcal{A}'_1) \times \cdots \times L_{\pi'_m}(\mathcal{A}'_m)$$

where $\varphi = \varphi_1 \times \cdots \times \varphi_m$. And

$$\varphi((L_{\pi_1}(\mathcal{A}_1) \times \cdots \times L_{\pi_m}(\mathcal{A}_m))_g) = (L_{\pi'_1}(\mathcal{A}'_1) \times \cdots \times L_{\pi'_m}(\mathcal{A}'_m))_{\alpha(g)}$$

is satisfied for $g \in G$. Then we have the result.

□

2.4 Fine gradings

The main point of this section is giving a classification of fine group-gradings on semisimple algebras. In order to do this, we define the *product group-grading* which consists, as its name suggests, in doing the product of group-gradings (Definition 2.4.4). This results in a group-grading on the direct sum of group-graded algebras by the direct product of their grading groups. This definition is the natural one for a product of gradings, we can see it in Theorem A.4.4.

We give a characterization of fine product group-gradings (Theorem 2.4.12). Finally we give the classification, up to equivalence, of fine product group-gradings on semisimple algebras (Corollary 2.4.13) and some examples of how these results can be applied.

We will start proving that a group-grading on a simple algebra $\mathcal{A}$ is fine if and only if the group-grading induced by it on $L_\pi(\mathcal{A})$, for an epimorphism π, is fine. This shows a close relation between simple algebras and their loop algebras, as the result we saw for universal groups (Lemma 2.3.2).

Lemma 2.4.1. *Let $\mathcal{A}$ be a simple $\overline{G}$-graded algebra with a grading $\overline{\Gamma} : \mathcal{A} = \bigoplus_{\overline{g} \in \overline{G}} \mathcal{A}_{\overline{g}}$. Let H be a subgroup of G and let $\pi : G \to \overline{G}$ be a group epimorphism where* $\ker \pi = H$*. Let Γ be the G-grading induced by $\overline{\Gamma}$ on $L_\pi(\mathcal{A})$. Then Γ is fine if and only if $\overline{\Gamma}$ is fine.*

Proof. $\Rightarrow$) Assume $\overline{\Gamma}$ is not fine, then there exists a group-grading $\overline{\Gamma}' : \mathcal{A} = \bigoplus_{k\in K} \mathcal{A}'_k$ refining properly $\overline{\Gamma}$ for a group K. That is, for each $k \in K$ there exists $\overline{g} \in \overline{G}$ such that $\mathcal{A}'_k \subseteq \mathcal{A}_{\overline{g}}$. We set $\overline{g} := \pi(g)$ for $g \in G$. Consider the group-grading $\Gamma' : L_\pi(\mathcal{A}) = \bigoplus_{(k,g)\in K\times G} L_\pi(\mathcal{A})_{(k,g)}$ where

$$L_\pi(\mathcal{A})_{(k,g)} = \begin{cases} \mathcal{A}'_k \otimes g & \text{if } \mathcal{A}'_k \subseteq \mathcal{A}_{\overline{g}}; \\ 0 & \text{otherwise.} \end{cases}$$

Let us prove that Γ' is a group-grading that refines Γ properly. For $\overline{g} \in \overline{G}$ there exists $I_{\overline{g}} = \{k^1_{\overline{g}}, ..., k^n_{\overline{g}}\} \subseteq K$ such that $\mathcal{A}_{\overline{g}} = \bigoplus_{k\in I_{\overline{g}}} \mathcal{A}'_k$, then

$$L_\pi(\mathcal{A}) = \bigoplus_{g\in G} (\mathcal{A}_{\overline{g}} \otimes g) = \bigoplus_{g\in G} \left(\bigoplus_{k\in I_{\overline{g}}} \mathcal{A}'_k \otimes g \right) = \bigoplus_{(k,g)\in K\times G} L_\pi(\mathcal{A})_{(k,g)}.$$

Now, take $(k_i, g_i) \in \mathrm{Supp}(\Gamma')$ for $i = 1, 2$, such that $\mathcal{A}'_{k_1}\mathcal{A}'_{k_2} \neq \{0\}$. Using the properties of group-gradings we have $\mathcal{A}'_{k_1}\mathcal{A}'_{k_2} \subseteq \mathcal{A}'_{k_1k_2}$ and $\mathcal{A}'_{k_1}\mathcal{A}'_{k_2} \subseteq \mathcal{A}_{\overline{g_1}}\mathcal{A}_{\overline{g_2}} \subseteq \mathcal{A}_{\overline{g_1g_2}}$, so $\mathcal{A}'_{k_1k_2} \cap \mathcal{A}_{\overline{g_1g_2}} \neq \{0\}$ then $\mathcal{A}'_{k_1k_2} \subseteq \mathcal{A}_{\overline{g_1g_2}}$, that is $L_\pi(\mathcal{A})_{(k_1k_2,g_1g_2)} = \mathcal{A}'_{k_1k_2} \otimes g_1g_2 \neq \{0\}$. Then

$$\begin{aligned} L_\pi(\mathcal{A})_{(k_1,g_1)} L_\pi(\mathcal{A})_{(k_2,g_2)} &= (\mathcal{A}'_{k_1} \otimes g_1)(\mathcal{A}'_{k_2} \otimes g_2) \\ &= \mathcal{A}'_{k_1}\mathcal{A}'_{k_2} \otimes g_1g_2 \\ &\subseteq \mathcal{A}'_{k_1k_2} \otimes g_1g_2 \\ &= L_\pi(\mathcal{A})_{(k_1k_2,g_1g_2)}. \end{aligned}$$

So Γ' is a group-grading.

Take $(k, g) \in \mathrm{Supp}(\Gamma')$, then $L_\pi(\mathcal{A})_{(k,g)} = \mathcal{A}'_k \otimes g \subseteq \mathcal{A}_{\overline{g}} \otimes g = L_\pi(\mathcal{A})_g$. Finally, Γ' is a proper refinement of Γ which leads to a contradiction.

$\Leftarrow$) Assume that Γ is not fine, then there exists a proper refinement $\Gamma' : L_\pi(\mathcal{A}) = \bigoplus_{k\in K} L_\pi(\mathcal{A})'_k$ of Γ for a group K, i.e. for each $k \in K$ there exists $g \in G$ such that $L_\pi(\mathcal{A})'_k \subseteq L_\pi(\mathcal{A})_g$. Suppose $K := U(\Gamma')$. By Lemma 1.1.21 this refinement induces a homomorphism of groups

$$\begin{aligned} \varphi : \ & K \longrightarrow G \\ & k \ \mapsto \ g, \quad \text{such that } L_\pi(\mathcal{A})'_k \subseteq L_\pi(\mathcal{A})_g. \end{aligned}$$

For any $k \in K$ with $g = \varphi(k)$, since $L_\pi(\mathcal{A})_g = \mathcal{A}_{\overline{g}} \otimes g$, $L_\pi(\mathcal{A})'_k = \mathcal{A}'_k \otimes \varphi(g)$ for a vector space $\mathcal{A}'_k \subseteq \mathcal{A}_{\overline{g}}$. By Lemma 1.3.7 (ii) Γ' (resp. Γ) induces a group-grading on the centroid $C(L_\pi(\mathcal{A}))$ which we denote by

$$\Gamma'_{C(L_\pi(\mathcal{A}))} : C(L_\pi(\mathcal{A})) = \bigoplus_{k\in K} C(L_\pi(\mathcal{A}))'_k$$

(resp.

$$\Gamma_{C(L_\pi(\mathcal{A}))} : C(L_\pi(\mathcal{A})) = \bigoplus_{g\in G} C(L_\pi(\mathcal{A}))_g.)$$

By Lemma 1.3.9 $\Gamma'_{C(L_\pi(\mathcal{A}))}$ is a refinement of $\Gamma_{C(L_\pi(\mathcal{A}))}$. By Lemma 1.3.15 $C(L_\pi(\mathcal{A}))$ is split, then the refinement $\Gamma'_{C(L_\pi(\mathcal{A}))}$ is not proper. By Remark 2.3.1 $H = \mathrm{Supp}(\Gamma_{C(L_\pi(\mathcal{A}))})$ and we set $H' := \mathrm{Supp}(\Gamma'_{C(L_\pi(\mathcal{A}))})$ and we have $\varphi(H') = H$. Then φ induces the group homomorphism

$$\begin{array}{rccl} \overline{\varphi}: & K/H' & \longrightarrow & G/H \\ & kH' & \longmapsto & \varphi(k)H. \end{array}$$

Define $\mathcal{A}'_{kH'} := \mathcal{A}'_k$. Let us prove that this is well defined, i.e. $\mathcal{A}'_{kh'} = \mathcal{A}'_k$ for all $h' \in H'$ and $k \in K$. Take $k \in K$ and $h' \in H'$. Let $h = \varphi(h')$. We have

$$\begin{aligned} \mathcal{A}'_{kh'} \otimes \varphi(kh') &= L_\pi(\mathcal{A})'_{kh'} = C(L_\pi(\mathcal{A}))'_{h'} L_\pi(\mathcal{A})'_k = C(L_\pi(\mathcal{A}))_h L_\pi(\mathcal{A})'_k \\ &= C(L_\pi(\mathcal{A}))_h(\mathcal{A}'_k \otimes \varphi(k)) = \mathcal{A}'_k \otimes \varphi(k)h. \end{aligned}$$

Then $\mathcal{A}'_{kh'} = \mathcal{A}'_k$. Now we will prove that $\overline{\Gamma} : \mathcal{A} = \bigoplus_{kH'\in K/H'} \mathcal{A}'_{kH'}$ is a group-grading. The equality $\mathcal{A} = \bigoplus_{kH'\in K/H'} \mathcal{A}'_{kH'}$ follows from the fact that Γ' is a group-grading. Now take $k_1, k_2 \in K$ such that $\mathcal{A}'_{k_1}\mathcal{A}'_{k_1} \neq 0$. Then

$$L_\pi(\mathcal{A})'_{k_i} = \mathcal{A}'_{k_i} \otimes \varphi(k_i) \subseteq \mathcal{A}_{\overline{\varphi(k_i)}} \otimes \varphi(k_i) = L_\pi(\mathcal{A})_{\varphi(k_i)}$$

for $i = 1, 2$. Finally,

$$L_\pi(\mathcal{A})'_{k_1} L_\pi(\mathcal{A})'_{k_2} = \mathcal{A}'_{k_1}\mathcal{A}'_{k_2} \otimes \varphi(k_1)\varphi(k_2) \subseteq L_\pi(\mathcal{A})'_{k_1k_2} = \mathcal{A}'_{k_1k_2} \otimes \varphi(k_1k_2)$$

and then $\mathcal{A}'_{k_1}\mathcal{A}'_{k_2} \subseteq \mathcal{A}'_{k_1k_2}$. So we have a proper refinement of $\overline{\Gamma}$ which leads to a contradiction. □

We will give a couple of lemmas we will need later.

Lemma 2.4.2. *Let $\Gamma_{\mathcal{B}}$ be a G-grading on an algebra $\mathcal{B}$ and let $\Gamma_{\mathcal{D}}$ be a G-grading on an algebra $\mathcal{D}$. If $\Gamma_{\mathcal{B}}$ is isomorphic to $\Gamma_{\mathcal{D}}$ and one of the two G-gradings is fine then the other group-grading is also fine.*

Proof. Suppose $\Gamma_{\mathcal{B}}$ is a fine group-grading. Let $\Gamma'_{\mathcal{D}} : \mathcal{D} = \oplus_{k\in K}\mathcal{D}'_k$ be a refinement of $\Gamma_{\mathcal{D}}$ where $K = U(\Gamma'_{\mathcal{D}})$. Then by Lemma 1.1.21 there exists a group homomorphism

$$\begin{array}{rccl} \varphi: & K & \longrightarrow & G \\ & k & \longmapsto & g, \text{ such that } \mathcal{D}'_k \subseteq \mathcal{D}_g. \end{array}$$

Consider the isomorphism of G-graded algebras from the hypothesis

$$\begin{array}{rccc} \Phi: & \mathcal{B} & \longrightarrow & \mathcal{D} \\ & \mathcal{B}_g & \longmapsto & \mathcal{D}_g. \end{array}$$

Let us prove that $\Gamma'_{\mathcal{B}} = \oplus_{k\in K}\mathcal{B}'_k$ where $\mathcal{B}'_k := \Phi^{-1}(\mathcal{D}'_k)$ is a group-grading. We have

$$\mathcal{B} = \bigoplus_{g\in G}\mathcal{B}_g = \bigoplus_{g\in G}\Phi^{-1}(\mathcal{D}_g) = \bigoplus_{g\in G}\Phi^{-1}(\bigoplus_{k\in K:\varphi(k)=g}\mathcal{D}'_k) = \bigoplus_{k\in K}\Phi^{-1}(\mathcal{D}'_k) = \bigoplus_{k\in K}\mathcal{B}'_k.$$

And for $k_1, k_2 \in K$ we have

$$\mathcal{B}'_{k_1}\mathcal{B}'_{k_2} = \Phi^{-1}(\mathcal{D}'_{k_1})\Phi^{-1}(\mathcal{D}'_{k_2}) = \Phi^{-1}(\mathcal{D}'_{k_1}\mathcal{D}'_{k_2}) \subseteq \Phi^{-1}(\mathcal{D}'_{k_1k_2}) = \mathcal{B}'_{k_1k_2}.$$

Then $\Gamma'_{\mathcal{B}}$ is a group-grading. Let us prove that it is a refinement of $\Gamma_{\mathcal{B}}$. For $k \in K$

$$\mathcal{B}'_k = \Phi^{-1}(\mathcal{D}'_k) \subseteq \Phi^{-1}(\mathcal{D}_{\varphi(k)}) = \mathcal{B}_{\varphi(k)}.$$

Then $\Gamma'_{\mathcal{B}}$ is a refinement of $\Gamma_{\mathcal{B}}$. Since $\Gamma_{\mathcal{B}}$ is fine, we get that $\mathcal{B}'_k = \mathcal{B}_{\varphi(k)}$ for all $k \in \operatorname{Supp}\Gamma'_{\mathcal{B}}$, then $\Phi^{-1}(\mathcal{D}'_k) = \Phi^{-1}(\mathcal{D}_{\varphi(k)})$ and hence $\mathcal{D}'_k = \mathcal{D}_{\varphi(k)}$. Therefore $\Gamma_{\mathcal{D}}$ is fine.

□

Recall that for a G-graded-simple algebra $\mathcal{B} = \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_n$ where $\mathcal{A}_1$, ..., $\mathcal{A}_n$ are simple ideals of $\mathcal{B}$, we have a $\overline{G}$-grading on $\mathcal{A}_1$. We get this by making the coarsening associated to the canonical projection $\pi : G \to \overline{G} := G/H$ where $H = \operatorname{Supp}_G C(\mathcal{B})$ to obtain a $\overline{G}$-grading on $\mathcal{B}$ which induces a $\overline{G}$-grading on $\mathcal{A}_1$. Next theorem uses that a semisimple algebra is isomorphic to a loop algebra of one of its simple factors to show that if we have a fine group-grading on a semisimple algebra then the induced group-grading on its simple factors is also fine.

Lemma 2.4.3. *Let $\mathcal{B}$ be an algebra. Let $\Gamma : \mathcal{B} = \bigoplus_{g\in G}\mathcal{B}_g$ be a fine G-grading and let $\Gamma' : \mathcal{B} = \bigoplus_{k\in K}\mathcal{B}'_k$ be a K-grading refining Γ where G and K are groups. Then Γ is equivalent to Γ'.*

Proof. Since Γ' is a refinement of Γ and Γ is fine we have that for all $k \in K$ exists a unique $g \in G$ such that $\mathcal{B}'_k = \mathcal{B}_g$. Hence we have a bijection

$$\begin{array}{rlcl} \alpha: & \operatorname{Supp}\Gamma' & \longrightarrow & \operatorname{Supp}\Gamma \\ & k & \longmapsto & g, \text{ such that } \mathcal{B}'_k = \mathcal{B}_g. \end{array}$$

And the isomorphism of algebras

$$\begin{array}{rccc} id_{\mathcal{B}} = \varphi: & \mathcal{B} & \longrightarrow & \mathcal{B} \\ & \mathcal{B}'_k & \longmapsto & \mathcal{B}'_k = \mathcal{B}_{\alpha(k)}. \end{array}$$

This defines an equivalence of group-gradings between Γ and Γ'. □

The next definition results to be very convenient for the purpose of studying fine group-gradings. This is because we take a finite number of group-gradings and we construct a new one by "isolating" the components of degree different of the neutral element of each original group-grading in order to avoid, as possible, coarsenings. Such definition is the natural one on the product of group-gradings (see Definition A.4.3 and Theorem A.4.4).

Definition 2.4.4. *Give gradings on the algebras $\mathcal{A}_1, ..., \mathcal{A}_n$ by the groups $G^1, ..., G^n$ respectively, we can define the product group-grading of such gradings as in Definition A.4.3 which is a $G^1 \times \cdots \times G^n$-grading on the algebra $\mathcal{A}_1 \times \cdots \times \mathcal{A}_n$. We have the analogous definition:*

*Let $G_1, ..., G_n$ be groups and let $\mathcal{B}_1, ..., \mathcal{B}_n$ be graded ideals of the algebra $\mathcal{B} = \mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n$. Let Γ^i be a G_i-grading on $\mathcal{B}_i$ for $i = 1, ..., n$. We call **product group-grading** of Γ^1,..., Γ^n to the $G_1 \times \cdots \times G_n$-grading on $\mathcal{B}$ defined by*

$$\mathcal{B}_{(g_1,...,g_n)} = \begin{cases} (\mathcal{B}_j)_{g_j} & \text{if } g_j \neq e_j \text{ for } j \in \{1, ..., n\} \text{ and} \\ & g_k = e_k \text{ for all } k \in \{1, ..., n\} \setminus \{j\}; \\ (\mathcal{B}_1)_{e_1} \oplus \cdots \oplus (\mathcal{B}_n)_{e_n} & \text{if } (g_1, ..., g_n) = (e_1, ..., e_n); \\ 0 & \text{otherwise,} \end{cases}$$

where e_i denotes the neutral element of G_i for $i = 1, ..., n$ and $(g_1, ..., g_n) \in G_1 \times \cdots \times G_n$.

From Remark 2.1.2 we have that both definitions (the one on the direct sum and the one on the cartesian product) are equivalent. We will use indistinctly these both definitions and denote them by $\Gamma^1 \times \cdots \times \Gamma^n$.

It results from the definitions that $U(\Gamma^1 \times \cdots \times \Gamma^n) = U(\Gamma^1) \times \cdots \times U(\Gamma^n)$.

Now we will give some results we will use to classify fine product group-gradings (Theorem 2.4.12).

Lemma 2.4.5. *Let Γ be a G-grading on an algebra $\mathcal{B} = \mathcal{B}_1 \oplus \mathcal{B}_2$ where $\mathcal{B}_1$ and $\mathcal{B}_2$ are G-graded ideals of $\mathcal{B}$. We denote by Γ^i the G-grading induced by Γ on $\mathcal{B}_i$ for $i = 1, 2$. Suppose Γ is fine, then Γ^1 and Γ^2 are fine and* $\mathrm{Supp}\,\Gamma^1 \cap \mathrm{Supp}\,\Gamma^2 \subseteq \{e\}$, *where e denotes the neutral element of G. Moreover Γ and $\Gamma^1 \times \Gamma^2$ are equivalent.*

Proof. First notice that the homogeneous components of $\Gamma : \mathcal{B} = \bigoplus_{g \in G}(\mathcal{B}_1 \oplus \mathcal{B}_2)_g$ are $(\mathcal{B}_1 \oplus \mathcal{B}_2)_g := (\mathcal{B}_1)_g \oplus (\mathcal{B}_2)_g$. Suppose that Γ^1 is a proper coarsening of a K-grading $\Gamma'^1 : \mathcal{B}_1 = \bigoplus_{k \in K}(\mathcal{B}_1)'_k$ where $K = U(\Gamma'^1)$. By Lemma 1.1.21 there exists a group homomorphism

$$\begin{array}{rccl} \varphi : & K & \longrightarrow & G \\ & k & \longmapsto & g, \text{ such that } (\mathcal{B}_1)'_k \subseteq (\mathcal{B}_1)_g. \end{array}$$

Consider the product group-grading of Γ'^1 and Γ^2 by the group $K \times G$ which is a proper refinement of Γ since

$$(\mathcal{B}_1 \oplus \mathcal{B}_2)_{(k,g)} = \begin{cases} (\mathcal{B}_2)_g \subseteq (\mathcal{B}_1 \oplus \mathcal{B}_2)_g & \text{if } k = e_K \text{ and } g \neq e_G; \\ (\mathcal{B}'_1)_k \subseteq (\mathcal{B}_1 \oplus \mathcal{B}_2)_{\varphi(k)} & \text{if } k \neq e_K \text{ and } g = e_G; \\ (\mathcal{B}'_1)_{e_K} \oplus (\mathcal{B}_2)_{e_G} \subseteq (\mathcal{B}_1 \oplus \mathcal{B}_2)_{e_G} & \text{if } k = e_K \text{ and } g = e_G, \end{cases}$$

where e_K and e_G denote the neutral element of K and G respectively. This leads to a contradiction since Γ is fine. So Γ^1 is fine, analogously Γ^2 is also fine.

Consider the product group-grading of Γ^1 and Γ^2

$$\Gamma^1 \times \Gamma^2 : \mathcal{B}_1 \oplus \mathcal{B}_2 = \bigoplus_{(g_1,g_2)\in G\times G} (\mathcal{B}_1 \oplus \mathcal{B}_2)_{(g_1,g_2)},$$

given by

$$(\mathcal{B}_1 \oplus \mathcal{B}_2)_{(g_1,g_2)} = \begin{cases} 0 & \text{if } g_1 \neq e \neq g_2; \\ (\mathcal{B}_1)_{g_1} & \text{if } g_1 \neq e \text{ and } g_2 = e; \\ (\mathcal{B}_2)_{g_2} & \text{if } g_1 = e \text{ and } g_2 \neq e; \\ (\mathcal{B}_1)_e \oplus (\mathcal{B}_2)_e & \text{if } g_1 = e = g_2. \end{cases}$$

It is clear that $\Gamma^1 \times \Gamma^2$ refines Γ which is fine. Then by Lemma 2.4.3 $\Gamma^1 \times \Gamma^2$ is equivalent to Γ. For $g \neq e$ in G we have

$$(\mathcal{B}_1)_g = (\mathcal{B}_1 \oplus \mathcal{B}_2)_{(g,e)} \subseteq (\mathcal{B}_1 \oplus \mathcal{B}_2)_g \text{ and } (\mathcal{B}_2)_g = (\mathcal{B}_1 \oplus \mathcal{B}_2)_{(e,g)} \subseteq (\mathcal{B}_1 \oplus \mathcal{B}_2)_g$$

then $(\mathcal{B}_1 \oplus \mathcal{B}_2)_{(g,e)} = 0$ or $(\mathcal{B}_1 \oplus \mathcal{B}_2)_{(e,g)} = 0$. So for all $g \neq e$ we have that $(\mathcal{B}_1)_g = 0$ or $(\mathcal{B}_2)_g = 0$. Therefore $\operatorname{Supp}\Gamma^1 \cap \operatorname{Supp}\Gamma^2 \subseteq \{e\}$ and $\operatorname{Supp}\Gamma^1 \cup \operatorname{Supp}\Gamma^2 = \operatorname{Supp}\Gamma$. □

Corollary 2.4.6. *Let Γ be a G-grading on the algebra $\mathcal{B} = \mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n$ where $\mathcal{B}_1, ..., \mathcal{B}_n$ are G-graded ideals of $\mathcal{B}$. We denote by Γ^i the G-grading induced by Γ on $\mathcal{B}_i$ for $i = 1, ..., n$. Suppose Γ is fine, then Γ^1,..., Γ^n are fine and $\operatorname{Supp}\Gamma^i \cap \operatorname{Supp}\Gamma^j \subseteq \{e\}$ for $i, j \in \{1, ..., n\}$ and $i \neq j$ where e denotes the neutral element of G. Moreover Γ and $\Gamma^1 \times \cdots \times \Gamma^n$ are equivalent.*

Proof. Define $\mathcal{B}'_2 := \mathcal{B}_2 \oplus \cdots \oplus \mathcal{B}_n$ which is a G-graded ideal of $\mathcal{B}$. Denote by Γ'^2 the induced G-grading by Γ on $\mathcal{B}'_2$. Assume Γ is fine. We can apply Lemma 2.4.5 to $\mathcal{B}_1 \oplus \mathcal{B}'_2$ and we get that Γ^1 and Γ'^2 are fine. Moreover Γ is equivalent to $\Gamma^1 \times \Gamma'^2$. Since $\Gamma'^2 : \mathcal{B}_2 \oplus \cdots \oplus \mathcal{B}_n$ is fine we can apply this process again by taking $\Gamma'^2 : \mathcal{B}_2 \oplus \mathcal{B}'_3$ where $\mathcal{B}'_3 = \mathcal{B}_3 \oplus \cdots \oplus \mathcal{B}_n$.

After repeating this process a finite number of times we get that Γ^i is fine for all $i = 1, ..., n$. We also get that Γ is equivalent to $\Gamma^1 \times \Gamma'^2$ and Γ'^i is equivalent to $\Gamma^i \times \Gamma'^{i+1}$ for $i = 2, ..., n-1$ where $\Gamma'^n = \Gamma^n$ and therefore Γ is equivalent to $\Gamma^1 \times \cdots \times \Gamma^n$. Finally for $i, j \in \{1, ..., n\}$ where $i \neq j$ we take $\mathcal{B}_{ij} = \mathcal{B}_i \oplus \mathcal{B}_j$ and $\mathcal{B}'_{ij} = \oplus_{k \in \{1,...,n\} \setminus \{i,j\}} \mathcal{B}_k$. Consider the induced group-gradings by Γ on $\mathcal{B}_{ij}$ and $\mathcal{B}'_{ij}$ denoted by Γ_{ij} and Γ'_{ij} respectively. Then by applying again Lemma 2.4.5 we have that Γ_{ij} is fine and then by the same lemma $\operatorname{Supp} \Gamma^i \cap \operatorname{Supp} \Gamma^j \subseteq \{e\}$. □

Lemma 2.4.7. *Let $\mathcal{B} = \mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n$ be an algebra where $\mathcal{B}_i$ are semisimple ideals of $\mathcal{B}$ for $i = 1, ..., n$ (see Definition 2.1.1). Let Γ^i be a fine U_i-grading on $\mathcal{B}_i$ for a group U_i where $U_i = U(\Gamma^i)$ and assume $\mathcal{B}_i$ is graded-simple for $i = 1, ..., n$. Suppose* $\operatorname{Supp} \Gamma^i \neq \{e_i\}$ *where e_i is the neutral element of U_i for $i = 1, ..., n$, except for at most one $j \in \{1, ..., n\}$. Then the product group-grading $\Gamma^1 \times \cdots \times \Gamma^n$ is fine.*

Proof. Assume that for a group U,

$$\Gamma' : \mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n = \bigoplus_{u \in U} (\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n)'_u$$

is a group-grading that refines the $U_1 \times \cdots \times U_n$-grading $\Gamma^1 \times \cdots \times \Gamma^n$. Suppose $U = U(\Gamma')$. Define the set $J := \{i \in \{1, ..., n\} : \operatorname{Supp} \Gamma^i \neq \{e_i\}\}$. Let us prove that $\mathcal{B}_i$ is a U-graded ideal of $\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n$, regarding Γ', for all $i \in J$. We have for $u \in \operatorname{Supp} \Gamma'$ that

$$(\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n)'_u \subseteq \begin{cases} (\mathcal{B}_i)_{u_i} \text{ for some } i \in \{1, ..., n\} \text{ and } u_i \in \operatorname{Supp} \Gamma^i \setminus \{e_i\} \\ \textit{or} \\ (\mathcal{B}_1)_{e_1} \oplus \cdots \oplus (\mathcal{B}_n)_{e_n}. \end{cases}$$

For every $i \in J$ and $u_i \in \operatorname{Supp} \Gamma^i \setminus \{e_i\}$ the homogeneous component of degree $(e_1, ..., u_i, ..., e_n)$ in $\Gamma^1 \times \cdots \times \Gamma^n$ is refined by Γ' (not necessarily properly), i.e.

$$(\mathcal{B}_i)_{u_i} = \bigoplus_{g \in \operatorname{Supp} \Gamma'|_{(\mathcal{B}_i)_{u_i}}} (\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n)'_g.$$

Take $x \in (\mathcal{B}_i)_{u_i}$ a nonzero homogeneous element of Γ' and consider the ideal of $\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n$ generated by it: $\operatorname{ideal}_{\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n} \langle x \rangle =: \mathcal{I}$. Then

$$\mathcal{I} = \operatorname{ideal}_{\mathcal{B}_i} \langle x \rangle = \mathcal{B}_i$$

where the last equality holds from the fact that $\operatorname{ideal}_{\mathcal{B}_i} \langle x \rangle$ is a nonzero graded ideal of $\mathcal{B}_i$ which is graded-simple. This shows that $\mathcal{B}_i$ is U-graded ideal for all

$i \in J$. Now we want to prove that $(\mathcal{B}_i)_{e_i} \subseteq (\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n)'_e$ for all $i = 1, ..., n$, where e is the neutral element of U. For $i \in J$ we have that $\Gamma' \mid_{\mathcal{B}_i}$ refines Γ^i which is already fine, then $\Gamma' \mid_{\mathcal{B}_i} = \Gamma^i$, i.e. the homogeneous components are equal in each group-grading, just indexed by different groups. So for $i \in J$ there exists a unique homomorphism of groups

$$\varphi_i : U_i \to U$$

such that for all $u_i \in \operatorname{Supp} \Gamma^i \setminus \{e_i\}$ and $i \in J$ we have

$$1) \qquad (\mathcal{B}_i)_{u_i} = (\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n)'_{\varphi_i(u_i)}.$$

If $e_i \in \operatorname{Supp} \Gamma^i$ then

$$2) \qquad (\mathcal{B}_i)_{e_i} = \mathcal{B}_i \cap (\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n)'_{a_i}$$

for a unique $a_i \in U$. We will prove now that $a_i = e$ for all $i \in J$. Consider the homogeneous component $(\mathcal{B}_i)_{e_i}$ of Γ^i for $i \in J$. Suppose that $(\mathcal{B}_i)_{e_i}(\mathcal{B}_i)_{u_i} = 0 = (\mathcal{B}_i)_{u_i}(\mathcal{B}_i)_{e_i}$ for all $u_i \in \operatorname{Supp} \Gamma^i \setminus \{e_i\}$, then $(\mathcal{B}_i)_{e_i}$ is a graded ideal of $\mathcal{B}_i$ and this contradicts the fact that $\mathcal{B}_i$ is graded-simple. Then there exists $u_i \in \operatorname{Supp} \Gamma^i \setminus \{e_i\}$ such that $(\mathcal{B}_i)_{e_i}(\mathcal{B}_i)_{u_i} \neq 0$ or $(\mathcal{B}_i)_{u_i}(\mathcal{B}_i)_{e_i} \neq 0$. Without loss of generality suppose that $(\mathcal{B}_i)_{e_i}(\mathcal{B}_i)_{u_i} \neq 0$, then by 1)

$$3) \qquad 0 \neq (\mathcal{B}_i)_{e_i}(\mathcal{B}_i)_{u_i} \subseteq (\mathcal{B}_i)_{u_i} = (\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n)'_{\varphi_i(u_i)}.$$

From 1) and 2) we get

$$\begin{aligned} 4) \qquad 0 \neq (\mathcal{B}_i)_{e_i}(\mathcal{B}_i)_{u_i} &= (\mathcal{B}_i \cap (\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n)'_{a_i})(\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n)'_{\varphi_i(u_i)} \\ &\subseteq (\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n)'_{a_i}(\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n)'_{\varphi_i(u_i)} \subseteq (\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n)'_{a_i\varphi_i(u_i)}. \end{aligned}$$

Hence from 3) and 4) we get

$$(\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n)'_{\varphi_i(u_i)} = (\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n)'_{a_i\varphi_i(u_i)}$$

and therefore $a_i = e$ for all $i \in J$ and then

$$5) \qquad (\mathcal{B}_i)_{e_i} = \mathcal{B}_i \cap (\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n)'_e.$$

If there exists (a unique) $k \in \{1, ..., n\} \setminus J$ we have by Lemma 2.1.14 that $\mathcal{B}_k$ is also U-graded. Let us prove that $\mathcal{B}_k = (\mathcal{B}_k)_{e_k} \subseteq (\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n)'_e$. Since Γ' refines Γ^k which is already fine we have $\Gamma' \mid_{\mathcal{B}_k} = \Gamma^k$, that is, the homogeneous components are the same but indexed by different groups. Then $(\mathcal{B}_k)_{e_k} = \mathcal{B}_k = \mathcal{B}_k \cap (\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n)'_a$ for some $a \in U$. Using the fact that every element in $\mathcal{B}_k$ is homogeneous of degree a, regarding Γ', and $\mathcal{B}_k^2 = \mathcal{B}_k$ we get $a^2 = a$ and then $a = e$. This, together with 5), proves that

$$(\mathcal{B}_i)_{e_i} \subseteq (\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n)'_e$$

for all $i = 1, ..., n$, then

$$(\mathcal{B}_1)_{e_1} \oplus \cdots \oplus (\mathcal{B}_n)_{e_n} \subseteq (\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n)'_e.$$

Since Γ' refines $\Gamma^1 \times \cdots \times \Gamma^n$ we have the equality

$$(\mathcal{B}_1)_{e_1} \oplus \cdots \oplus (\mathcal{B}_n)_{e_n} = (\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n)'_e.$$

Considering the last equality together with 1) we get that Γ' is an improper refinement of $\Gamma^1 \times \cdots \times \Gamma^n$. Therefore $\Gamma^1 \times \cdots \times \Gamma^n$ is fine. $\square$

In the next example we see the group-gradings on $\mathfrak{sl}_2 \times \mathfrak{sl}_2$ obtained as product group-gradings of fine gradings on $\mathfrak{sl}_2$.

Example 2.4.8. *Consider the special linear Lie algebra of degree 2:*

$$\mathfrak{sl}_2 = \left\langle E = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), F = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), H = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right\rangle,$$

over a ground field $\mathbb{F}$ of characteristic not 2. This is a simple algebra, its bracket is determined by:

$$[E, F] = H, \quad [H, F] = -2F, \quad \textit{and} \quad [H, E] = 2E.$$

Up to equivalence, there are only two fine gradings on $\mathfrak{sl}_2$ (see [EK13, Theorem 3.55]):

- $\Gamma^1_{\mathfrak{sl}_2}$ *with universal group $\mathbb{Z}$ and homogeneous components:*

$$(\mathfrak{sl}_2)_{-1} = \mathbb{F}F, \quad (\mathfrak{sl}_2)_0 = \mathbb{F}H, \quad (\mathfrak{sl}_2)_1 = \mathbb{F}E.$$

- $\Gamma^2_{\mathfrak{sl}_2}$ *with universal group $(\mathbb{Z}/2)^2$ and homogeneous components:*

$$(\mathfrak{sl}_2)_{(\bar{1},\bar{0})} = \mathbb{F}H, \quad (\mathfrak{sl}_2)_{(\bar{0},\bar{1})} = \mathbb{F}(E + F), \quad (\mathfrak{sl}_2)_{(\bar{1},\bar{1})} = \mathbb{F}(E - F).$$

Denote by $\bar{n}$ the class of n modulo 2. The gradings on $\mathcal{L} = \mathfrak{sl}_2 \times \mathfrak{sl}_2$ obtained as product group-gradings of the fine gradings above are the following:

- $\Gamma^1_{\mathfrak{sl}_2} \times \Gamma^1_{\mathfrak{sl}_2}$ *with universal group $\mathbb{Z} \times \mathbb{Z}$ and homogeneous components*

$$\begin{array}{ll} \mathcal{L}_{(0,0)} = \mathbb{F}H \times \mathbb{F}H, & \\ \mathcal{L}_{(1,0)} = \mathbb{F}E \times 0, & \mathcal{L}_{(0,1)} = 0 \times \mathbb{F}E, \\ \mathcal{L}_{(-1,0)} = \mathbb{F}F \times 0, & \mathcal{L}_{(0,-1)} = 0 \times \mathbb{F}F. \end{array}$$

- $\Gamma^1_{\mathfrak{sl}_2}\times\Gamma^2_{\mathfrak{sl}_2}$ *with universal group* $\mathbb{Z}\times(\mathbb{Z}/2)^2$ *and homogeneous components:*

$$\begin{array}{ll} \mathcal{L}_{(0,(\bar 0,\bar 0))}=\mathbb{F}H\times 0, & \mathcal{L}_{(0,(1,\bar 0))}=0\times\mathbb{F}H, \\ \mathcal{L}_{(1,(\bar 0,\bar 0))}=\mathbb{F}E\times 0, & \mathcal{L}_{(0,(\bar 0,\bar 1))}=0\times\mathbb{F}(E+F), \\ \mathcal{L}_{(-1,(\bar 0,\bar 0))}=\mathbb{F}F\times 0, & \mathcal{L}_{(0,(\bar 1,\bar 1))}=0\times\mathbb{F}(E-F). \end{array}$$

- $\Gamma^2_{\mathfrak{sl}_2}\times\Gamma^2_{\mathfrak{sl}_2}$ *with universal group* $(\mathbb{Z}/2)^4$ *and homogeneous components:*

$$\begin{array}{ll} \mathcal{L}_{(\bar 0,\bar 0,\bar 1,\bar 0)}=0\times\mathbb{F}H, & \mathcal{L}_{(\bar 1,\bar 0,\bar 0,\bar 0)}=\mathbb{F}H\times 0, \\ \mathcal{L}_{(\bar 0,\bar 0,\bar 0,\bar 1)}=0\times\mathbb{F}(E+F), & \mathcal{L}_{(\bar 0,\bar 1,\bar 0,\bar 0)}=\mathbb{F}(E+F)\times 0, \\ \mathcal{L}_{(\bar 0,\bar 0,\bar 1,\bar 1)}=0\times\mathbb{F}(E-F), & \mathcal{L}_{(\bar 1,\bar 1,\bar 0,\bar 0)}=\mathbb{F}(E-F)\times 0. \end{array}$$

Let us see a counterexample of Lemma 2.4.7 omitting the hypothesis of $\operatorname{Supp}\Gamma^i\neq\{e_i\}$ for all $i=1,...,n$ except for at most one $j\in\{1,...,n\}$.

Example 2.4.9. *Assume* $\operatorname{char}\mathbb{F}\neq 2$ *and let* $\mathcal{B}=\mathbb{F}^2$ *be the algebra with the canonical basis* $\{e_1,e_2\}$*. Consider* $\mathcal{B}_1=\mathbb{F}e_1$ *and* $\mathcal{B}_2=\mathbb{F}e_2$ *trivially graded by* $\{e\}$ *and we denote such group-gradings by* Γ^1 *and* Γ^2 *respectively. Then the product group-grading* $\Gamma^1\times\Gamma^2$ *is given by*

$$\mathcal{B}=\mathcal{B}_{(e,e)}=\mathbb{F}e_1\oplus\mathbb{F}e_2$$

We can refine this group-grading by $\Gamma:\mathcal{B}=\oplus_{g\in\mathbb{Z}/2}\mathcal{B}'_g$ *which is given by*

$$\mathcal{B}'_g=\begin{cases}\mathbb{F}(e_1+e_2) & \text{if } g=\bar 0;\\ \mathbb{F}(e_1-e_2) & \text{if } g=\bar 1.\end{cases}$$

The next example shows that there are nontrivial examples of simple algebras for which the trivial grading is a fine group-grading.

Example 2.4.10. *Let* $\mathcal{A}=\mathbb{F}a+\mathbb{F}b$ *be an algebra with the following multiplication table*

	a	b
a	a	b
b	0	$a+b$.

It is easy to see that $\mathcal{A}$ *is simple. Let* $\mathcal{R}$ *be in* $\mathrm{Alg}_{\mathbb{F}}$*, we will write* a *(resp.* b*) to refer to* $a\otimes 1$ *(resp.* $b\otimes 1$*) in* $\mathcal{A}\otimes_{\mathbb{F}}\mathcal{R}$*. For any* $\mathcal{R}\in\mathrm{Alg}_{\mathbb{F}}$ *and any automorphism* $\varphi\in\mathrm{Aut}_{\mathcal{R}}(\mathcal{A}\otimes_{\mathbb{F}}\mathcal{R})$*,* $\varphi(a)=a$ *because* a *is the only left unity of* $\mathcal{A}\otimes_{\mathbb{F}}\mathcal{R}$*. If* $\varphi(b)=ra+sb$*,* $r,s\in\mathcal{R}$*, then from* $0=\varphi(ba)=\varphi(b)a=(ra+sb)a=ra$ *we obtain* $r=0$*. Now* $\varphi(b^2)=\varphi(a+b)=a+sb$*, while* $\varphi(b)^2=s^2(a+b)$*, so* $s^2=1=s$ *and* φ *is the identity. Therefore the affine group scheme* $\mathbf{Aut}_{\mathbb{F}}(\mathcal{A})$ *is trivial, and hence the only group-grading is the trivial one.*

We want to classify fine product group-gradings but first we need a previous result.

Lemma 2.4.11. *Let $\mathcal{A}$ be a central-simple algebra with no nontrivial group-gradings. Then the trivial group-grading on $\mathcal{A} \times \cdots \times \mathcal{A}$ ($n \geq 2$ copies of $\mathcal{A}$) is a fine group-grading if and only if $n = 2$ and $\operatorname{char} \mathbb{F} = 2$.*

Proof. If $n = 2$ and $\operatorname{char} \mathbb{F} \neq 2$, then $\mathcal{A} \times \mathcal{A}$ is isomorphic to $\mathcal{A} \otimes_{\mathbb{F}} (\mathbb{F} \times \mathbb{F})$. The $\mathbb{Z}/2$-grading on $\mathbb{F} \times \mathbb{F}$ in Example 2.4.9 induces a nontrivial $\mathbb{Z}/2$-grading on $\mathcal{A} \times \mathcal{A}$, with $(\mathcal{A} \times \mathcal{A})_{\bar{0}} = \{(x, x) : x \in \mathcal{A}\}$ and $(\mathcal{A} \times \mathcal{A})_{\bar{1}} = \{(x, -x) : x \in \mathcal{A}\}$. Therefore the trivial group-grading on $\mathcal{A} \times \mathcal{A}$ is not fine.

If $n \geq 3$ and $\operatorname{char} \mathbb{F} \neq 2$, we may use the above to define a nontrivial group-grading on $\mathcal{A} \times \mathcal{A}$ and hence take the product group-grading with the trivial group-grading on the remaining factors to get a nontrivial group-grading on $\mathcal{A} \times \mathcal{A} \times \cdots \times \mathcal{A}$.

If $n \geq 3$ and $\operatorname{char} \mathbb{F} = 2$, consider the group $\mathbb{Z}/3$, and its projection onto the trivial group $\pi : \mathbb{Z}/3 \to 1$. Let $G = \mathbb{Z}/3$, $\overline{G} = 1$ and $\overline{\Gamma}$ the trivial group-grading on $\mathcal{A}$. Consider the associated loop algebra $L_\pi(\mathcal{A})$. Its group-grading Γ is not trivial, and $L_\pi(\mathcal{A})$ is isomorphic to $\mathcal{A} \times \mathcal{A} \times \mathcal{A}$ (Theorem 2.1.9). Therefore there are nontrivial group-gradings on $\mathcal{A} \times \mathcal{A} \times \mathcal{A}$, and hence also on the cartesian product of $n \geq 3$ copies of $\mathcal{A}$.

On the other hand, suppose $n = 2$ and $\operatorname{char} \mathbb{F} = 2$. Suppose also that Γ is a nontrivial group-grading on $\mathcal{A} \times \mathcal{A}$ with universal group U, then $\mathcal{A} \times \mathcal{A}$ is a semisimple and graded-simple algebra, with centroid isomorphic to $\mathbb{F} \times \mathbb{F}$. By Theorem 2.2.3 2. $\mathcal{A} \times \mathcal{A}$ is isomorphic to a loop algebra of the form $L_\pi(\mathcal{A}')$, with $\pi : U \to \overline{U}$ a surjective group homomorphism with kernel H of order 2 and a central-simple $\overline{U}$-graded algebra $\mathcal{A}'$, then $L_\pi(\mathcal{A}')$ is semisimple, but this contradicts Theorem 2.1.9. □

Next result gives the characterization of fine product group-gradings.

Theorem 2.4.12. *Let Γ^i be a fine group-grading on the algebra $\mathcal{B}^i$ such that it is graded-central-simple. Then the product group-grading $\Gamma^1 \times \cdots \times \Gamma^n$ on $\mathcal{B} = \mathcal{B}^1 \oplus \cdots \oplus \mathcal{B}^n$, where $\mathcal{B}^i$ is an ideal of $\mathcal{B}$ for $i = 1, ..., n$, is a fine group-grading if and only if either:*

- $\operatorname{char} \mathbb{F} = 2$ *and for any index i such that Γ^i is trivial, there is at most one other index j such that Γ^i and Γ^j are equivalent (i.e., Γ^j is trivial and $\mathcal{B}^j$ is isomorphic to $\mathcal{B}^i$).*

- $\operatorname{char} \mathbb{F} \neq 2$ *and for any index i such that Γ^i is trivial, there is no other index j such that Γ^i and Γ^j are equivalent.*

Proof. If $\operatorname{char}\mathbb{F} = 2$ and there are three indices i, j, k with trivial group-gradings Γ^i, Γ^j and Γ^k and such that $\mathcal{B}^i$, $\mathcal{B}^j$ and $\mathcal{B}^k$ are isomorphic. Since these algebras are graded-simple then they are simple and by Remark 2.1.3 central-simple. Lemma 2.4.11 shows that there is a nontrivial group-grading on $\mathcal{B}^i \oplus \mathcal{B}^j \oplus \mathcal{B}^k$. Therefore the induced group-grading $(\Gamma^1 \times \cdots \times \Gamma^n)|_{\mathcal{B}^i \oplus \mathcal{B}^j \oplus \mathcal{B}^k}$ (which is the trivial group-grading) is not fine and Corollary 2.4.6 shows that $\Gamma^1 \times \cdots \times \Gamma^n$ is not fine. The situation for $\operatorname{char}\mathbb{F} \neq 2$, with two indices i, j with trivial group-gradings Γ^i and Γ^j and such that $\mathcal{B}^i$ and $\mathcal{B}^j$ are isomorphic, is similar.

On the other hand, assume that the hypotheses on the trivial group-gradings are satisfied. The arguments in the proof of Lemma 2.4.7 show that if Γ' is a group-grading on $\mathcal{B}$ that refines $\Gamma^1 \times \cdots \times \Gamma^n$, and if Γ^i is not trivial, then $\mathcal{B}^i$ is a graded ideal for Γ', and $\Gamma'|_{\mathcal{B}^i}$ coincides with Γ^i.

Consider the subset of indices

$$J = \{i \mid 1 \leq i \leq n \text{ and } \Gamma^i \text{ is the trivial group-grading: } \mathcal{B}^i = (\mathcal{B}^i)_e\}.$$

As $\mathcal{J} = \bigoplus_{i \notin J} \mathcal{B}^i$ is a graded ideal for Γ', by Lemma 2.1.14, $\mathcal{J}' = \bigoplus_{j \in J} \mathcal{B}^j$ is also a graded ideal of $\mathcal{B}$ for Γ'. For $j \in J$, $\mathcal{B}^j$ is central-simple, because it is graded-central-simple and Γ^j is trivial, so $\mathcal{J}'$ is semisimple. Thus each ideal of $\mathcal{J}'$ is of the form $\mathcal{B}^{j_1} \oplus \cdots \oplus \mathcal{B}^{j_r}$ for some indices $j_1 < \cdots < j_r$ in J. The centroid of such an ideal is the cartesian product of r copies of $\mathbb{F}$ (Remark 2.1.5). In particular its dimension is finite.

By Theorem 2.1.16, $\mathcal{J}'$ is a direct sum of graded-simple ideals $\mathcal{I}_1, ..., \mathcal{I}_r$ for $\Gamma'|_{\mathcal{J}'}$ and, by Lemma 1.3.11, such ideals are also graded-central-simple. Then, by Theorem 1.3.32 (ii), for each $i \in \{1, ..., r\}$ there exists a central-simple G/H-graded algebra $\mathcal{A}_i$ such that $\mathcal{I}_i$ is isomorphic, as G-graded algebras, to $L_{\pi_i}(\mathcal{A}_i)$ where $\pi_i : G \to G/H_i$ is the canonical projection for $H = \operatorname{Supp}_G C(I_i)$ and hence isomorphic to the cartesian product of a number of copies of a simple algebra, with $\operatorname{char}\mathbb{F}$ not dividing this number of copies (Theorem 2.1.9). Our hypotheses imply, due to Lemma 2.4.11, that the graded-simple ideals of $\Gamma'|_{\mathcal{J}'}$ are precisely the $\mathcal{B}^j$'s for $j \in J$ and, since Γ^j is fine, $\Gamma'|_{\mathcal{B}^j}$ is the trivial group-grading for any $j \in J$. We conclude that $\Gamma' = \Gamma^1 \times \cdots \times \Gamma^n$. □

The next result classifies fine group-gradings, up to equivalence, on semisimple algebras.

Corollary 2.4.13.

1. *Any fine group-grading on a finite-dimensional semisimple algebra is equivalent to a product group-grading* $\Gamma^1 \times \cdots \times \Gamma^n$*, with the* Γ^i*'s being*

fine group-gradings on a semisimple graded-simple ideal $\mathcal{B}^i$ *of an algebra* $\mathcal{B} = \mathcal{B}^1 \oplus \cdots \oplus \mathcal{B}^n$, *satisfying one of the following extra conditions:*

- $\operatorname{char} \mathbb{F} = 2$ *and for any index* i *such that* Γ^i *is trivial, there is at most one other index* j *such that* Γ^i *is equivalent to* Γ^j.
- $\operatorname{char} \mathbb{F} \neq 2$ *and for any index* i *such that* Γ^i *is trivial, there is no other index* j *such that* Γ^i *is equivalent to* Γ^j.

And conversely, any such product group-grading is a fine group-grading.

Moreover, the factors Γ^i *are uniquely determined, up to reordering and equivalence.*

2. *Any fine grading* Γ' *on a finite-dimensional graded-simple algebra* $\mathcal{B}$ *is equivalent to a* U*-grading* Γ *on the loop algebra* $L_\pi(\mathcal{A})$ *associated to a surjective group homomorphism* $\pi : U \to \overline{U}$ *with finite kernel, and a simple finite-dimensional* $\overline{U}$*-graded algebra* $\mathcal{A}$ *with* $\overline{\Gamma}$ *a fine group-grading with universal group* $\overline{U}$.

   Moreover, in this situation $\mathcal{B}$ *is semisimple if and only if* $\operatorname{char} \mathbb{F}$ *does not divide the order of* $\ker \pi$.

3. *For* $i = 1, 2$, *let* $\mathcal{A}^i$ *be a simple algebra endowed with a fine* $\overline{U}^i$*-grading* $\overline{\Gamma^i}$ *where* $U(\overline{\Gamma^i}) = \overline{U}^i$, *and let* $\pi^i : U^i \to \overline{U}^i$ *be a surjective group homomorphism for a group* U^i. *Let* Γ^i *be the group-grading induced by* $\overline{\Gamma^i}$ *on the associated loop algebra* $L_{\pi^i}(\mathcal{A}^i)$ *for* $i = 1, 2$. *Then* Γ^1 *and* Γ^2 *are equivalent if and only if the* $\overline{\Gamma}^1$ *and* $\overline{\Gamma}^2$ *are equivalent and there is an equivalence* $(\varphi : \mathcal{A}^1 \to \mathcal{A}^2, \alpha_\varphi : \overline{U}^1 \to \overline{U}^2)$ *such that* α_φ *extends to a group isomorphism* $\tilde{\alpha}_\varphi : U^1 \to U^2$. *(This means that the diagram*

$$\begin{array}{ccc} U^1 & \xrightarrow{\tilde{\alpha}_\varphi} & U^2 \\ {\scriptstyle \pi^1}\downarrow & & \downarrow{\scriptstyle \pi^2} \\ \overline{U}^1 & \xrightarrow{\alpha_\varphi} & \overline{U}^2 \end{array}$$

   is commutative.)

Proof. By Lemma 2.1.15 any semisimple graded algebra is uniquely, up to a permutation of the summands, a direct sum of graded-simple ideals. Now part 1 follows from Corollary 2.4.6 and Theorem 2.4.12.

Part 2 follows from Theorem 2.2.3 2 and Lemma 2.4.1.

For part 3 Lemma 2.3.3 and its proof shows the first part. The converse is Lemma 2.3.4.

□

In general, the group isomorphism α_φ^U at the end of the previous proof cannot be extended to a group isomorphism $U^1 \to U^2$, as the next example shows.

Example 2.4.14. *Let $\mathcal{J} = \mathbb{F}1 \oplus \mathbb{F}u \oplus \mathbb{F}v$ be the unital commutative algebra, with $u^2 = v^2 = 1$, $uv = 0$. This is the Jordan algebra of a two-dimensional quadratic form. It is simple. Consider the group-grading $\overline{\Gamma}^1$ on $\mathcal{J}$ by $\overline{U} = (\mathbb{Z}/2)^2$, with*

$$\mathcal{J}_{(\bar{0},\bar{0})} = \mathbb{F}1, \quad \mathcal{J}_{(\bar{1},\bar{0})} = \mathbb{F}u, \quad \mathcal{J}_{(\bar{1},\bar{1})} = \mathbb{F}v.$$

$\overline{U}$ is, up to isomorphism, the universal group of the group-grading. Consider also the group-grading $\overline{\Gamma}^2$ by the same group with

$$\mathcal{J}_{(\bar{0},\bar{0})} = \mathbb{F}1, \quad \mathcal{J}_{(\bar{0},\bar{1})} = \mathbb{F}u, \quad \mathcal{J}_{(\bar{1},\bar{1})} = \mathbb{F}v.$$

$\overline{U}$ is the universal group of Γ^2.

The identity map gives an equivalence between $\overline{\Gamma}^1$ and $\overline{\Gamma}^2$. The associated group isomorphism $\alpha_{\mathrm{id}} : \overline{U} \to \overline{U}$ is the map $(a, b) \mapsto (b, a)$.

Let $U = (\mathbb{Z}/4) \times (\mathbb{Z}/2)$, and let π be the natural projection map $U \to \overline{U}$ which is the identity on the second component and the projection $\mathbb{Z}/4 \to \mathbb{Z}/2$ on the first component. Then α_{id} does not extend to a group isomorphism $U \to U$, and therefore φ does not extend to an equivalence of the induced group-gradings Γ^1 and Γ^2 on the respective loop algebras.

If the characteristic of $\mathbb{F}$ is not 2, then $L_\pi(\mathcal{J})$ is isomorphic to $\mathcal{J} \times \mathcal{J}$ (Theorem 2.1.9), so we obtain two non-equivalent group-gradings on the semisimple algebra $\mathcal{J} \times \mathcal{J}$.

In the next example we compute the fine group-gradings on $\mathcal{B} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ such that $\mathcal{B}$ is graded-simple.

Example 2.4.15. *Let $\mathbb{F}$ be the base field with characteristic different of 2. We will compute the fine group-gradings on the algebra $\mathcal{B} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ where each copy of $\mathfrak{sl}_2$ is an ideal of $\mathcal{B}$ and such that $\mathcal{B}$ is graded-simple. In order to do this we will give a group G and a subgroup H of G such that $|H| = 2$ and $\overline{G} := G/H \simeq U(\overline{\Gamma})$ for $\overline{\Gamma} = \Gamma^1_{\mathfrak{sl}_2}(\mathbb{Z}, 1)$ and $\Gamma^2_{\mathfrak{sl}_2}((\mathbb{Z}/2)^2)$ given in Example 2.4.8. Then we will find the G-grading on $L_\pi(\mathfrak{sl}_2)$ associated to $\overline{\Gamma}$ for the canonical projection $\pi : G \to \overline{G}$ (Corollary 2.4.13 (2)). Notice that the universal group of this loop algebra is G (Lemma 2.3.2). Finally, using Theorem 2.1.9, we give the G-grading on $\mathfrak{sl}_2 \times \mathfrak{sl}_2$. Denote by $\bar{n}$ the class of n modulo 2.*

- *Set $G = \mathbb{Z} \times \mathbb{Z}/2$ and $H = 0 \times \mathbb{Z}/2$. Let $\pi : G \to G/H$ be the canonical projection. We have the G-grading on $L_\pi(\mathfrak{sl}_2)$ given by*

$$L_\pi(\mathfrak{sl}_2)_{(m,\bar{n})} = (\mathfrak{sl}_2)_m \otimes (m, \bar{n})$$

where $(\mathfrak{sl}_2)_m$ *is given by* $\Gamma^1_{\mathfrak{sl}_2}(\mathbb{Z}, 1)$*. Extending the characters of* H *to* G *we obtain*

$$\chi_1 \equiv 1 \qquad \text{and} \qquad \chi_2 : (m, \bar{n}) \mapsto (-1)^n.$$

Finally, we get the respective G*-grading on* $\mathfrak{sl}_2 \times \mathfrak{sl}_2$ *given by*

$$(\mathfrak{sl}_2 \times \mathfrak{sl}_2)_{(m,\bar{n})} = \{(x, (-1)^n x) : x \in (\mathfrak{sl}_2)_{\pi((m,\bar{n}))}\}.$$

- *Set* $G = (\mathbb{Z}/2)^3$ *and* $H = 0^2 \times \mathbb{Z}/2$*. Let* $\pi : G \to G/H$ *be the canonical projection. We have the* G*-grading on* $L_\pi(\mathfrak{sl}_2)$ *given by*

$$L_\pi(\mathfrak{sl}_2)_{(\bar{m},\bar{n},\bar{r})} = (\mathfrak{sl}_2)_{(\bar{m},\bar{n})} \otimes (\bar{m}, \bar{n}, \bar{r}).$$

where $(\mathfrak{sl}_2)_{(\bar{m},\bar{n})}$ *is given by* $\Gamma^2_{\mathfrak{sl}_2}((\mathbb{Z}/2)^2)$*. Extending the characters of* H *to* G *we obtain*

$$\chi_1 \equiv 1 \qquad \text{and} \qquad \chi_2 : (\bar{m}, \bar{n}, \bar{r}) \mapsto (-1)^r.$$

Finally, we get the respective group-grading on $\mathfrak{sl}_2 \times \mathfrak{sl}_2$ *given by*

$$(\mathfrak{sl}_2 \times \mathfrak{sl}_2)_{(\bar{m},\bar{n},\bar{r})} = \{(x, (-1)^r x) : x \in (\mathfrak{sl}_2)_{\pi((\bar{m},\bar{n},\bar{r}))}\}.$$

- *Set* $G = \mathbb{Z}/4 \times \mathbb{Z}/2$ *and* $H = \{\widehat{0}, \widehat{2}\} \times 0$ *where* $\widehat{m} := m \bmod 4$*. Let* $\pi : G \to G/H$ *be the canonical projection. We have the* G*-grading on* $L_\pi(\mathfrak{sl}_2)$ *given by*

$$L_\pi(\mathfrak{sl}_2)_{(\widehat{m},\bar{n})} = (\mathfrak{sl}_2)_{(\bar{m},\bar{n})} \otimes (\widehat{m}, \bar{n}).$$

where $(\mathfrak{sl}_2)_{(\bar{m},\bar{n})}$ *is given by* $\Gamma^2_{\mathfrak{sl}_2}((\mathbb{Z}/2)^2)$*. Extending the characters of* H *to* G *we obtain*

$$\chi_1 \equiv 1 \qquad \text{and} \qquad \chi_2 : (\widehat{m}, \bar{n}) \mapsto i^m$$

where i *denotes a square root of* -1 *in* $\mathbb{F}$*. Finally, we get the respective group-grading on* $\mathfrak{sl}_2 \times \mathfrak{sl}_2$ *given by*

$$(\mathfrak{sl}_2 \times \mathfrak{sl}_2)_{(\widehat{m},\bar{n})} = \{(x, i^m x) : x \in (\mathfrak{sl}_2)_{\pi((\widehat{m},\bar{n}))}\}.$$

Since we considered group-gradings by their universal groups, the above group-gradings are unique up to equivalence.

Therefore, by Corollary 2.4.13, the fine group-gradings on $\mathfrak{sl}_2 \times \mathfrak{sl}_2$, up to equivalence, are the ones given in Example 2.4.8 and 2.4.15.

Chapter 3

A Toy Example: Gradings on Kac's Jordan Superalgebra

This chapter is devoted to obtain gradings on the Kac's Jordan superalgebra K_{10} and is part of [CDE18]. We will see that in order to obtain the gradings, up to equivalence and isomorphism, on K_{10} it is enough to obtain the gradings, up to equivalence and isomorphism, on $\mathsf{K}_3 \times \mathsf{K}_3$ where K_3 is a simple algebra. Then by using an isomorphism we can obtain the gradings on K_{10}. The process we will follow to obtain the gradings on $\mathsf{K}_3 \times \mathsf{K}_3$ is an example of the theory given in Chapter 2 in the case when we have two simple factors, we see that such theory works for different types of algebras, such as superalgebras. The process shown here to obtain the gradings on K_{10} is analogous to the one we will use in Chapters 4 and 5 to obtain gradings on the tensor product of two Cayley algebras.

Kac's ten-dimensional superalgebra K_{10} is an exceptional Jordan superalgebra which appeared for the first time in Kac's classification [Kac77] of the finite dimensional simple Jordan superalgebras over an algebraically closed field of characteristic 0. It was constructed by Lie theoretical terms from a 3-grading of the exceptional simple Lie superalgebra $F(4)$.

A more conceptual definition was given in [BE02] over an arbitrary field $\mathbb{F}$ of characteristic not 2, using the three-dimensional Kaplansky superalgebra K_3. The even part $(\mathsf{K}_3)_{\bar{0}}$ is a copy of the ground field $\mathbb{F}$: $(\mathsf{K}_3)_{\bar{0}} = \mathbb{F}a$, with $a^2 = a$; while the odd part is a two-dimensional vector space W endowed with a nonzero skew-symmetric bilinear form $(\,|\,)$. The multiplication is determined as follows:

$$a^2 = a, \quad av = \frac{1}{2}v = va, \quad vw = (v|w)a,$$

for any $v, w \in W$. We extend $(\,|\,)$ to a supersymmetric bilinear form on K_3: $(\mathsf{K}_3)_{\bar{0}}$ and $(\mathsf{K}_3)_{\bar{1}}$ are orthogonal, with $(a|a) = \frac{1}{2}$.

The assumption on the ground field to be of characteristic not 2 will be kept throughout the chapter.

Then $\mathsf{K}_{10} = \mathbb{F}1 \oplus \big(\mathsf{K}_3 \otimes_{\mathbb{F}} \mathsf{K}_3\big)$, with $(\mathsf{K}_{10})_{\bar 0} = \mathbb{F}1 \oplus \big(\mathsf{K}_3 \otimes_{\mathbb{F}} \mathsf{K}_3\big)_{\bar 0} = \mathbb{F}1 \oplus \mathbb{F}(a \otimes a) \oplus (W \otimes W)$, and $(\mathsf{K}_{10})_{\bar 1} = \big(\mathsf{K}_3 \otimes_{\mathbb{F}} \mathsf{K}_3\big)_{\bar 1} = (W \otimes a) \oplus (a \otimes W)$. The multiplication is given by imposing that 1 is the unity, and for homogeneous elements $x, y, z, t \in \mathsf{K}_3$,

$$(x \otimes y)(z \otimes t) = (-1)^{yz}\Big(xz \otimes yt - \frac{3}{4}(x \mid z)(y \mid t)1\Big). \tag{3.0.1}$$

If the characteristic is 3, then $\mathsf{K}_9 := \mathsf{K}_3 \otimes_{\mathbb{F}} \mathsf{K}_3$ is a simple ideal in K_{10}. Otherwise K_{10} is simple.

It must be remarked that an 'octonionic' construction of K_{10} has been given in [RZ15].

Using the construction above of K_{10} in terms of K_3, the group of automorphisms $\mathrm{Aut}\big(\mathsf{K}_{10}\big)$ was computed in [ELS07]. This was used in [CDM10] to classify gradings on K_{10} over algebraically closed fields of characteristic zero.

We will compute the group scheme of automorphisms of K_{10}, and use it to revisit, extend, and simplify drastically, the known results on gradings on K_{10}.

Recall that the group scheme $\mathbf{Aut}\big(\mathsf{K}_{10}\big)$ is the functor that takes any object R in $\mathrm{Alg}_{\mathbb{F}}$ to the group $\mathrm{Aut}_R\big(\mathsf{K}_{10} \otimes_{\mathbb{F}} R)$ (the group of automorphisms of the R-superalgebra $\mathsf{K}_{10} \otimes_{\mathbb{F}} R$, i.e., the group of R-linear isomorphisms preserving the multiplication and the grading), with the natural definition on morphisms.

It turns out that $\mathbf{Aut}\big(\mathsf{K}_{10}\big)$ is isomorphic to a semidirect product $\big(\mathbf{SL}_2 \times \mathbf{SL}_2\big) \rtimes \mathbf{C}_2$ (Theorem 3.1.1), where $\mathbf{C}_2$ is the constant group scheme attached to the cyclic group of two elements (also denoted by $\mathbf{C}_2$). This extends the result in [ELS07].

A simple observation shows that $\big(\mathbf{SL}_2 \times \mathbf{SL}_2\big) \rtimes \mathbf{C}_2$ is also the automorphism group scheme of $\mathsf{K}_3 \times \mathsf{K}_3$.

Finally, given an abelian group G, a G-grading on a superalgebra $\mathcal{A}$ corresponds to a homomorphism of group schemes

$$G^D \longrightarrow \mathbf{Aut}(\mathcal{A}).$$

(See [EK13, Chapter 1, Section 1.4].)

The classification of G-gradings up to isomorphism on $\mathsf{K}_3 \times \mathsf{K}_3$ is an easy exercise (Proposition 3.2.2), and the classification of G-gradings, up to isomorphism, on K_{10} (Theorem 3.2.5) follows at once from this, thus, extending and simplifying widely the results in [CDM10].

3.1 The group scheme of automorphisms

Consider the two-dimensional vector space $W = (\mathsf{K}_3)_{\bar{1}}$, which is endowed with the nonzero skew-symmetric bilinear form $(\,|\,) : W \times W \to \mathbb{F}$. The special linear group scheme $\mathbf{SL}(W)$ coincides with the symplectic group scheme $\mathbf{Sp}(W)$, which is, up to isomorphism, the group scheme of automorphisms of K_3.

The constant group scheme $\mathbf{C}_2$ acts on $\mathbf{SL}(W) \times \mathbf{SL}(W)$ by swapping the arguments, and hence we get a natural semidirect product $\big(\mathbf{SL}(W) \times \mathbf{SL}(W)\big) \rtimes \mathbf{C}_2$. The group isomorphism Φ in [ELS07, p. 3809] extends naturally to a homomorphism of affine group schemes:

$$\begin{aligned} \Phi : \big(\mathbf{SL}(W) \times \mathbf{SL}(W)\big) \rtimes \mathbf{C}_2 &\longrightarrow \mathbf{Aut}\big(\mathsf{K}_{10}\big) \\ (f,g) \qquad &\mapsto \quad \Phi_{(f,g)} : \begin{cases} 1 \;\mapsto\; 1, \\ a \otimes a \mapsto a \otimes a, \\ v \otimes a \mapsto f(v) \otimes a, \\ a \otimes v \mapsto a \otimes g(v), \\ v \otimes w \mapsto f(v) \otimes g(w), \end{cases} \\ \text{generator of } \mathbf{C}_2 &\mapsto \quad \tau : \begin{cases} 1 \;\mapsto\; 1, \\ x \otimes y \mapsto (-1)^{xy} y \otimes x, \end{cases} \end{aligned} \tag{3.1.1}$$

for any R in $\mathrm{Alg}_{\mathbb{F}}$, $f, g \in \mathbf{SL}(W)(R)$ (i.e., $f, g \in \mathrm{End}_R(W \otimes_{\mathbb{F}} R) \simeq M_2(R)$ of determinant 1), $v, w \in W_R := W \otimes_{\mathbb{F}} R$, $x, y \in (\mathsf{K}_3)_R$. (Note that a representation $\rho : \mathbf{F} \to \mathbf{GL}(V)$ of a constant group scheme $\mathbf{F}$ is determined by its behavior over $\mathbb{F}$: $\rho_{\mathbb{F}} : \mathbf{F} \to GL(V)$.)

Theorem 3.1.1. Φ *is an isomorphism of affine group schemes.*

Proof. If $\overline{\mathbb{F}}$ denotes an algebraic closure of $\mathbb{F}$, then $\Phi_{\overline{\mathbb{F}}}$ is a group isomorphism [ELS07, Theorem 3.3]. Since $\big(\mathbf{SL}(W) \times \mathbf{SL}(W)\big) \rtimes \mathbf{C}_2 \simeq \big(\mathbf{SL}_2 \times \mathbf{SL}_2\big) \rtimes \mathbf{C}_2$ is smooth, it is enough to prove that the differential $\mathrm{d}\Phi$ is bijective (see, for instance [EK13, Theorem A.50]). The Lie algebra of $\big(\mathbf{SL}(W) \times \mathbf{SL}(W)\big) \rtimes \mathbf{C}_2$ is $\mathfrak{sl}(W) \times \mathfrak{sl}(W)$, while the Lie algebra of $\mathbf{Aut}\big(\mathsf{K}_{10}\big)$ is the even part of its Lie superalgebra of derivations, which is again, up to isomorphism, $\mathfrak{sl}(W) \times \mathfrak{sl}(W)$ identified naturally with a subalgebra of $\mathrm{End}_{\mathbb{F}}\big(\mathsf{K}_{10}\big)$ (see [BE02, Theorem 2.8]). Moreover, with the natural identification, $\mathrm{d}\Phi$ is the identity map. $\square$

The last result in this section is the simple observation that $\big(\mathbf{SL}(W) \times \mathbf{SL}(W)\big) \rtimes \mathbf{C}_2$ is also the group scheme of automorphisms of the Jordan superalgebra $\mathsf{K}_3 \times \mathsf{K}_3$. Its proof is straightforward, along the same lines as for Theorem 3.1.1.

Proposition 3.1.2. *The natural transformation defined by:*

$$\begin{aligned}
\Psi^1 : \big(\mathbf{SL}(W) \times \mathbf{SL}(W)\big) \rtimes \mathbf{C}_2 &\longrightarrow \mathbf{Aut}\big(\mathsf{K}_3 \times \mathsf{K}_3\big) \\
(f,g) \quad &\mapsto \quad \Psi^1_{(f,g)} : \begin{cases} (a,0) \mapsto (a,0), \\ (0,a) \mapsto (0,a), \\ (v,w) \mapsto \big(f(v), g(w)\big), \end{cases} \\
\textit{generator of } \mathbf{C}_2 &\mapsto \quad \tau : (x,y) \mapsto (y,x),
\end{aligned}$$

for $v, w \in W$ and $x, y \in \mathsf{K}_3$ is an isomorphism of group schemes.

3.2 Gradings

Given an abelian group G, a G-grading on a superalgebra $\mathcal{A} = \mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{1}}$ is a decomposition into a direct sum of subspaces: $\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$, such that $\mathcal{A}_g \mathcal{A}_h \subseteq \mathcal{A}_{gh}$ for any $g, h \in G$, and each homogeneous component is a subspace in the 'super' sense: $\mathcal{A}_g = (\mathcal{A}_g \cap \mathcal{A}_{\bar{0}}) \oplus (\mathcal{A}_g \cap \mathcal{A}_{\bar{1}})$.

A grading by G is equivalent to a homomorphism of affine group schemes $G^D \to \mathbf{Aut}(\mathcal{A})$ (see [EK13]), and this shows that two superalgebras with isomorphic group schemes of automorphisms have equivalent classifications of G-gradings up to isomorphism. Therefore, in order to classify gradings on K_{10} it is enough to classify gradings on $\mathsf{K}_3 \times \mathsf{K}_3$, and this is straightforward.

Actually, fix a symplectic basis $\{u, v\}$ of $W = (\mathsf{K}_3)_{\bar{1}}$.

Definition 3.2.1. *Given an abelian group G, consider the following gradings (e denotes the neutral element of G):*

- *For $g_1, g_2 \in G$, denote by $\Gamma^1(G; g_1, g_2)$ the G-grading given by:*

$$\begin{aligned}
&\deg(x) = e \textit{ for any } x \in \big(\mathsf{K}_3 \times \mathsf{K}_3)_{\bar{0}}, \\
&\deg(u,0) = g_1 = \deg(v,0)^{-1}, \quad \deg(0,u) = g_2 = \deg(0,v)^{-1}.
\end{aligned}$$

- *For $g, h \in G$ with $h^2 = e \neq h$, denote by $\Gamma^2(G; g, h)$ the G-grading given by:*

$$\begin{aligned}
&\deg(a,a) = e, \quad \deg(a,-a) = h, \\
&\deg(u,u) = g = \deg(v,v)^{-1}, \quad \deg(u,-u) = gh = \deg(v,-v)^{-1}.
\end{aligned}$$

Notice that $\Gamma^1(G; g_1, g_2)$ *is a product group-grading, while* $\Gamma^2(G; g, h)$ *is isomorphic to a loop grading.*

Proposition 3.2.2. *Any grading on* $\mathsf{K}_3 \times \mathsf{K}_3$ *by the abelian group* G *is isomorphic to either* $\Gamma^1(G; g_1, g_2)$ *or to* $\Gamma^2(G; g, h)$ *(for some* g_1, g_2 *or* g, h *in* G*).*

Moreover, no grading of the first type $\Gamma^1(G; g_1, g_2)$ *is isomorphic to a grading of the second type* $\Gamma^2(G; g, h)$*, and*

- $\Gamma^1(G; g_1, g_2)$ *is isomorphic to* $\Gamma^1(G; g_1', g_2')$ *if and only if the sets* $\{g_1, g_1^{-1}, g_2, g_2^{-1}\}$ *and* $\{g_1', (g_1')^{-1}, g_2', (g_2')^{-1}\}$ *coincide.*

- $\Gamma^2(G; g, h)$ *is isomorphic to* $\Gamma^2(G; g', h')$ *if and only if* $h' = h$ *and* $g' \in \{g, gh, g^{-1}, g^{-1}h\}$*.*

Proof. Any G-grading on $\mathcal{J} := \mathsf{K}_3 \times \mathsf{K}_3$ induces a G-grading on $\mathcal{J}_{\bar{0}}$, which is isomorphic to $\mathbb{F} \times \mathbb{F}$, and hence we are left with two cases:

1. The grading on $\mathcal{J}_{\bar{0}}$ is trivial, i.e., $\mathcal{J}_{\bar{0}}$ is contained in the homogeneous component $\mathcal{J}_e$. Then, with $W = (\mathsf{K}_3)_{\bar{1}}$, both $W \times 0 = (a, 0)\mathcal{J}_{\bar{1}}$ and $0 \times W = (0, a)\mathcal{J}_{\bar{1}}$ are graded subspaces of $\mathcal{J}_{\bar{1}}$. Hence we can take bases $\{u_i, v_i\}$ of W, $i = 1, 2$, such that $\{(u_1, 0), (v_1, 0), (0, u_2), (0, v_2)\}$ is a basis of $\mathcal{J}_{\bar{1}}$ consisting of homogeneous elements. We can adjust v_i, $i = 1, 2$, so that $(u_i \mid v_i) = 1$. If $\deg(u_i, 0) = g_i$, $i = 1, 2$, the grading is isomorphic to $\Gamma^1(G; g_1, g_2)$.

2. The grading on $\mathcal{J}_{\bar{0}}$ is not trivial. Then there is an element $h \in G$ of order 2 such that $\deg(a, a) = e$ and $\deg(a, -a) = h$. (Note that (a, a) is the unity element of $\mathcal{J}_{\bar{0}}$ ($\simeq \mathbb{F} \times \mathbb{F}$), so it is always homogeneous of degree e.) As $\mathcal{J}_{\bar{0}} = (\mathcal{J}_{\bar{1}})^2$, there are homogeneous elements $(u_1, u_2), (v_1, v_2) \in \mathcal{J}_{\bar{1}}$ such that $(u_1, u_2)(v_1, v_2) = (a, a)$. If $g = \deg(u_1, u_2)$, this grading is isomorphic to $\Gamma^2(G; g, h)$.

The conditions for isomorphism are clear. □

Any grading $\Gamma^1(G; g_1, g_2)$ is a coarsening of $\Gamma^1\big(\mathbb{Z}^2; (1, 0), (0, 1)\big)$, while any grading $\Gamma^2(G; g, h)$ is a coarsening of $\Gamma^2\big(\mathbb{Z} \times \mathbb{Z}/2; (1, \bar{0}), (0, \bar{1})\big)$, where $\mathbb{Z}/2 = \mathbb{Z}/2\mathbb{Z}$. As an immediate consequence, we obtain the next result.

Corollary 3.2.3. *Up to equivalence, there are exactly two fine gradings on* $\mathsf{K}_3 \times \mathsf{K}_3$*, with respective universal groups* $\mathbb{Z}^2$ *and* $\mathbb{Z} \times \mathbb{Z}/2$*:* $\Gamma^1\big(\mathbb{Z}^2; (1, 0), (0, 1)\big)$ *and* $\Gamma^2\big(\mathbb{Z} \times \mathbb{Z}/2; (1, \bar{0}), (0, \bar{1})\big)$*.*

In order to transfer these results to Kac's superalgebra K_{10}, take into account that K_{10} is generated by its odd part, as $\big((\mathsf{K}_{10})_{\bar{1}}\big)^2 = (\mathsf{K}_{10})_{\bar{0}}$, so any grading is determined by its restriction to the odd part, and use the commutativity of the diagram

$$\begin{tikzcd}
\mathbf{Aut}\big(\mathsf{K}_3 \times \mathsf{K}_3\big) \arrow[d, hook] & \big(\mathbf{SL}(W) \times \mathbf{SL}(W)\big) \rtimes \mathbf{C}_2 \arrow[l, "\Psi^1"'] \arrow[r, "\Phi"] & \mathbf{Aut}\big(\mathsf{K}_{10}\big) \arrow[d, hook] \\
\mathbf{GL}(W \times W) \arrow[rr, "\simeq"] & & \mathbf{GL}\big((W \otimes a) \oplus (a \otimes W)\big)
\end{tikzcd}$$

where the vertical arrows are given by the restrictions to the odd parts, and the bottom isomorphism is given by the natural identification $W \times W \to (W \otimes a) \oplus (a \otimes W)$, $(v, w) \mapsto v \otimes a + a \otimes w$.

Thus Definition 3.2.1 transfers to K_{10} as follows:

Definition 3.2.4. *Given an abelian group G, consider the following G-gradings on K_{10}:*

- *For $g_1, g_2 \in G$, denote by $\Gamma^1_{\mathsf{K}_{10}}(G; g_1, g_2)$ the G-grading determined by:*

$$\deg(u \otimes a) = g_1 = \deg(v \otimes a)^{-1}, \quad \deg(a \otimes u) = g_2 = \deg(a \otimes v)^{-1}.$$

- *For $g, h \in G$ with $h^2 = e \neq h$, denote by $\Gamma^2_{\mathsf{K}_{10}}(G; g, h)$ the G-grading determined by:*

$$\begin{aligned}
\deg(u \otimes a + a \otimes u) &= g = \deg(v \otimes a + a \otimes v)^{-1}, \\
\deg(u \otimes a - a \otimes u) &= gh = \deg(v \otimes a - a \otimes v)^{-1}.
\end{aligned}$$

And Proposition 3.2.2 and Corollary 3.2.3 are now transferred easily to K_{10}.

Theorem 3.2.5. *Any grading on K_{10} by the abelian group G is isomorphic to either $\Gamma^1_{\mathsf{K}_{10}}(G; g_1, g_2)$ or to $\Gamma^2_{\mathsf{K}_{10}}(G; g, h)$ (for some g_1, g_2 or g, h in G).*

No grading of the first type is isomorphic to a grading of the second type, and

- *$\Gamma^1_{\mathsf{K}_{10}}(G; g_1, g_2)$ is isomorphic to $\Gamma^1_{\mathsf{K}_{10}}(G; g_1', g_2')$ if and only if the sets $\{g_1, g_1^{-1}, g_2, g_2^{-1}\}$ and $\{g_1', (g_1')^{-1}, g_2', (g_2')^{-1}\}$ coincide.*
- *$\Gamma^2_{\mathsf{K}_{10}}(G; g, h)$ is isomorphic to $\Gamma^2_{\mathsf{K}_{10}}(G; g', h')$ if and only if $h' = h$ and $g' \in \{g, gh, g^{-1}, g^{-1}h\}$.*

Moreover, there are exactly two fine gradings on K_{10} *up to equivalence, namely* $\Gamma^1_{\mathsf{K}_{10}}\big(\mathbb{Z}^2;(1,0),(0,1)\big)$ *and* $\Gamma^2_{\mathsf{K}_{10}}\big(\mathbb{Z}\times\mathbb{Z}/2;(1,\bar{0}),(0,\bar{1})\big)$*.*

Remark 3.2.6. *Any grading on* K_{10} *by an abelian group* G *extends naturally to a grading by either* $\mathbb{Z}\times G$ *or* $(\mathbb{Z}/2)^2\times G$ *on the exceptional simple Lie superalgebra* $F(4)$*, which is obtained from* K_{10} *using the well-known Tits-Kantor-Koecher construction. However (see [DEM11]), not all gradings on* $F(4)$ *are obtained in this way.*

Chapter 4

Hurwitz algebras

Hurwitz algebras constitute a generalization of the classical algebras of the real $\mathbb{R}$, complex $\mathbb{C}$, quaternion $\mathbb{H}$ (1843) and octonion numbers $\mathbb{O}$ (1845). We are interested in the tensor product of two of these algebras because it is a particular case of structurable algebras (Chapter 5, Section 1). In Section 1 we give the definition of these algebras and their classification. In Section 2 we have the classification of group-gradings on Hurwitz algebras. Finally in Section 3 we prove a result about the tensor product of a finite number of Cayley algebras (octonion algebras) which will be used in Chapter 5 to get group-gradings on the tensor product of two Hurwitz algebras.

4.1 Definitions

The following definitions and results can be found in Chapter 4 of [EK13].

A **quadratic form** on a vector space V over a field $\mathbb{F}$ is a map $q : V \to \mathbb{F}$ satisfying $q(\lambda x) = \lambda^2 q(x)$ for any $\lambda \in \mathbb{F}$ and any $x \in V$, and such that its **polar form**, defined by $q(x, y) := q(x + y) - q(x) - q(y)$ for $x, y \in V$, is a bilinear form (necessarily symmetric). If $\operatorname{char} \mathbb{F} = 2$ then the form $q(x, y)$ is alternating, otherwise the quadratic form is determined by its polar form: $q(x) = \dfrac{1}{2} q(x, x)$.

Let $V^\perp = \{x \in V : q(x, y) = 0 \text{ for } y \in V\}$. The quadratic form q is called **nonsingular** if either $V^\perp = 0$ or $\dim V^\perp = 1$ and $q(V^\perp) \neq 0$. If the characteristic of $\mathbb{F}$ is not 2, then the quadratic form q is nonsingular if and only if its polar form is a nondegenerate symmetric bilinear form ($V^\perp = 0$).

Now let $\mathcal{C}$ be an algebra over $\mathbb{F}$. A quadratic form n on $\mathcal{C}$ is called **multiplicative** if

$$n(xy) = n(x)n(y) \tag{4.1.1}$$

for any $x, y \in \mathcal{C}$.

Definition 4.1.1. *An algebra $\mathcal{C}$ over a field $\mathbb{F}$ endowed with a nonsingular multiplicative quadratic form (the norm) $n : \mathcal{C} \to \mathbb{F}$ is called a* ***composition algebra****. The unital composition algebras are called* ***Hurwitz algebras****.*

Definition 4.1.2. *The linear form $\mathcal{C} \to \mathbb{F}$, $x \mapsto n(x, 1)$ is called the* ***trace*** *of the Hurwitz algebra $\mathcal{C}$, and the subspace of traceless elements: $\{x \in \mathcal{C} : n(x, 1) = 0\}$, is denoted by $\mathcal{C}_0$.*

By [EK13, Proposition 4.2] we have that the map $x \mapsto \overline{x} := n(x, 1)1 - x$ is an ***involution*** *of $\mathcal{C}$ (i.e., $\overline{xy} = \overline{y}\,\overline{x}$ and $\overline{\overline{x}} = x$ for any $x, y \in \mathcal{C}$), called the* ***standard conjugation****. Then an analogous definition for $\mathcal{C}_0$ is the subspace of $\mathcal{C}$ of antisymmetric elements: $\{x \in \mathcal{C} : \overline{x} = -x\}$.*

We will denote by $CD(Q, \alpha)$ the algebra obtained from a subalgebra Q of a Hurwitz algebra through the Cayley-Dickson doubling process where $0 \neq \alpha \in \mathbb{F}$ (see [EK13, p. 125]). The next result is Theorem 4.4 of [EK13].

Theorem 4.1.3. *Every Hurwitz algebra over a field $\mathbb{F}$ is isomorphic to one of the following types:*

(1) *The ground field $\mathbb{F}$ if its characteristic is different of 2.*

(2) *A quadratic commutative and associative separable algebra $\mathcal{K}[\mu] := \mathbb{F}1 \oplus \mathbb{F}v$, with $v^2 = v + \mu$ and $4\mu + 1 \neq 0$. Its norm is given by the generic norm: $n(a + bv) = a^2 - \mu b^2 + 2ab$ for $a, b \in \mathbb{F}$.*

(3) *A quaternion algebra $Q[\mu, \beta] := CD(\mathcal{K}[\mu], \beta)$ for $\mu \in \mathbb{F}$ and $0 \neq \beta \in \mathbb{F}$. (These are associative but not commutative.)*

(4) *A Cayley algebra (or* ***octonion algebra****) $\mathcal{C}[\mu, \beta, \gamma] := \mathfrak{CD}(Q[\mu, \beta), \gamma)$ for $\mu \in \mathbb{F}$ and $0 \neq \beta, \gamma \in \mathbb{F}$. (These are alternative but not associative.)*

In particular, the dimension of any Hurwitz algebra is finite and restricted to 1, 2, 4 or 8.

If the characteristic of the ground field $\mathbb{F}$ is not 2, then we can rephrase the theorem above as follows ([EK13, Corollary 4.6]):

Corollary 4.1.4. *Every Hurwitz algebra over a field $\mathbb{F}$ of characteristic not 2 is isomorphic to one of the following types:*

(1) *The ground field $\mathbb{F}$.*

(2) *A commutative and associative separable algebra $\mathcal{K}(\alpha) := \mathfrak{CD}(\mathbb{F}, \alpha)$, for $0 \neq \alpha \in \mathbb{F}$.*

(3) *A quaternion algebra $Q(\alpha, \beta) := CD(\mathbb{F}, \alpha, \beta)$ for $0 \neq \alpha, \beta \in \mathbb{F}$. (These are associative but not commutative.)*

(4) *A Cayley algebra (or* ***octonion algebra****) $\mathcal{C}(\alpha, \beta, \gamma) := CD(\mathbb{F}, \alpha, \beta, \gamma)$ for $0 \neq \alpha, \beta, \gamma \in \mathbb{F}$. (These are alternative but not associative.*

The next result is [EK13, Corollary 4.7].

Corollary 4.1.5. *Two Hurwitz algebras are isomorphic if and only if their norms are isometric.*

We will say that the norm n is **isotropic** if it represents 0. This is always the case if $\mathbb{F}$ is algebraically closed.

Up to isomorphism, there is a unique Cayley algebra whose norm is isotropic. It is called the **split Cayley algebra** and denoted by $\mathcal{C}_s$. One basis called the **good basis** of $\mathcal{C}_s$ is $\{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$ with multiplication table:

	e_1	e_2	u_1	u_2	u_3	v_1	v_2	v_3
e_1	e_1	0	u_1	u_2	u_3	0	0	0
e_2	0	e_2	0	0	0	v_1	v_2	v_3
u_1	0	u_1	0	v_3	$-v_2$	$-e_1$	0	0
u_2	0	u_2	$-v_3$	0	v_1	0	$-e_1$	0
u_3	0	u_3	v_2	$-v_1$	0	0	0	$-e_1$
v_1	v_1	0	$-e_2$	0	0	0	u_3	$-u_2$
v_2	v_2	0	0	$-e_2$	0	$-u_3$	0	u_1
v_3	v_3	0	0	0	$-e_2$	u_2	$-u_1$	0

The next result is Theorem 4.8 of [EK13].

Theorem 4.1.6. *There are, up to isomorphism, unique Hurwitz algebras of dimension 2, 4 and 8 whose norm is isotropic:*

(1) *The algebra $\mathbb{F} \times \mathbb{F}$ with norm $n((\alpha, \beta)) = \alpha\beta$.*

(2) *The algebra $M_2(\mathbb{F})$ with determinant as the norm.*

(3) *The split Cayley algebra $\mathcal{C}_s = CD(M_2(\mathbb{F}), 1)$ with the above multiplication table.*

4.2 Gradings on Hurwitz algebras

In dealing with group-gradings on Hurwitz algebras, it is enough to restrict ourselves to abelian group-gradings (see [EK13, Proposition 4.10]). Therefore we will keep working with abelian group-gradings.

The group-gradings **induced by the Cayley-Dickson doubling process** on a Cayley algebra $\mathcal{C}$, up to equivalence, are the following:

- If $\mathcal{C} = CD(Q, \alpha) = Q \oplus Qu$, this is a $\mathbb{Z}/2$-grading: $\mathcal{C}_{\bar{0}} = Q$, $\mathcal{C}_{\bar{1}} = Qu$.
- If, moreover, $Q = CD(K, \beta) = K \oplus Kv$, then $\mathcal{C} = K \oplus Kv \oplus Ku \oplus (Kv)u$ is a $(\mathbb{Z}/2)^2$-grading.

- Finally, if $K = CD(\mathbb{F}, \gamma) = \mathbb{F}1 \oplus \mathbb{F}w$, then $\mathcal{C}$ is $(\mathbb{Z}/2)^3$-graded.

The groups $(\mathbb{Z}/2)^r$ ($r = 1, 2, 3$) are the universal groups, respectively. The fine $(\mathbb{Z}/2)^3$-**grading induced by the Cayley-Dickson doubling process** on $\mathcal{C} = CD(\mathbb{F}, \alpha, \beta, \gamma)$ with the basis $\{1, w, v, vw, u, uw, vu, (wv)u\}$ is given by

$$\begin{array}{ll} \mathcal{C}_{(\bar{0},\bar{0},\bar{0})} = \mathbb{F}1, & \mathcal{C}_{(\bar{1},\bar{1},\bar{0})} = \mathbb{F}vu, \\ \mathcal{C}_{(\bar{1},\bar{0},\bar{0})} = \mathbb{F}u, & \mathcal{C}_{(\bar{1},\bar{0},\bar{1})} = \mathbb{F}wu, \\ \mathcal{C}_{(\bar{0},\bar{1},\bar{0})} = \mathbb{F}v, & \mathcal{C}_{(\bar{0},\bar{1},\bar{1})} = \mathbb{F}wv, \\ \mathcal{C}_{(\bar{0},\bar{0},\bar{1})} = \mathbb{F}w, & \mathcal{C}_{(\bar{1},\bar{1},\bar{1})} = \mathbb{F}(wv)u. \end{array} \tag{4.2.1}$$

The split Cayley algebra $\mathcal{C}_s$ (with the good basis) is $\mathbb{Z}^2$-graded with

$$\begin{array}{ll} (\mathcal{C}_s)_{(0,0)} = \mathbb{F}e_1 \oplus \mathbb{F}e_2, & \\ (\mathcal{C}_s)_{(1,0)} = \mathbb{F}u_1, & (\mathcal{C}_s)_{(-1,0)} = \mathbb{F}v_1, \\ (\mathcal{C}_s)_{(0,1)} = \mathbb{F}u_2, & (\mathcal{C}_s)_{(0,-1)} = \mathbb{F}v_2, \\ (\mathcal{C}_s)_{(1,1)} = \mathbb{F}v_3, & (\mathcal{C}_s)_{(-1,-1)} = \mathbb{F}u_3. \end{array} \tag{4.2.2}$$

This group-grading is called the **Cartan grading** and its universal group is $\mathbb{Z}^2$.

Remark 4.2.1.

1. *Let $\mathcal{C}$ be the Cayley algebra with the basis $\{1, w, v, vw, u, uw, vu, (wv)u\}$ given by the Cayley-Dickson doubling process. We have that $\{w, v, vw, u, uw, vu, (wv)u\}$ is a basis for the subspace of traceless elements $\mathcal{C}_0$ of $\mathcal{C}$.*

2. *Let $\mathcal{C}_s$ be the split Cayley algebra with the good basis $\{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$. We have that $\{e_1 - e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$ is a basis for the subspace of traceless elements $(\mathcal{C}_s)_0$ of $\mathcal{C}_s$.*

We can also see $\mathcal{C}_0$ and $(\mathcal{C}_s)_0$ as algebras with the multiplication given by the commutator. Observe that the subspace of traceless elements generates the whole Cayley algebra if we consider the usual multiplication, therefore there is enough to know the degrees of the traceless elements in order to know the grading on the whole Cayley algebra. Recall that $\deg 1 = e$ *where e is the neutral element of the group (Remark 1.1.8).*

The following result ([EK13, Theorem 4.12]) describes all possible group-gradings on Cayley algebras:

Theorem 4.2.2. *Any proper group-grading on a Cayley algebra is, up to equivalence, either a group-grading induced by the Cayley-Dickson doubling process or a coarsening of the Cartan grading on the split Cayley algebra.*

Next two corollaries are Corollary 4.13 and 4.14 of [EK13].

Corollary 4.2.3. *Let Γ be a group-grading on the Cayley algebra $\mathcal{C}$ over an algebraically closed field $\mathbb{F}$. Then, up to equivalence, either Γ is the $(\mathbb{Z}/2)^3$-grading induced by the Cayley-Dickson doubling process, or it is a coarsening of the Cartan grading. The first possibility does not occur if* $\operatorname{char}\mathbb{F} = 2$.

Corollary 4.2.4. *Let $\mathcal{C}$ be the Cayley algebra over an algebraically closed field. Then, up to equivalence, the fine gradings by abelian groups on $\mathcal{C}$ and their universal groups are the following:*

(1) *The Cartan grading, with universal group $\mathbb{Z}^2$.*

(2) *If* $\operatorname{char}\mathbb{F} \neq 2$, *the $(\mathbb{Z}/2)^3$-grading induced by the Cayley-Dickson doubling process, with universal group $(\mathbb{Z}/2)^3$.*

The next classification is Theorem 4.15 of [EK13].

Theorem 4.2.5. *Up to equivalence, the nontrivial group-gradings on the split Cayley algebra are:*

(1) *The $(\mathbb{Z}/2)^r$-gradings induced by the Cayley-Dickson doubling process, $r = 1, 2, 3$.*

(2) *The Cartan grading by $\mathbb{Z}^2$.*

(3) *The* 3*-grading: $\mathcal{C}_0 = \operatorname{span}\{e_1, e_2, u_3, v_3\}$, $\mathcal{C}_1 = \operatorname{span}\{u_1, v_2\}$, and $\mathcal{C}_{-1} = \operatorname{span}\{u_2, v_1\}$.*

(4) *The* 5*-grading: $\mathcal{C}_0 = \operatorname{span}\{e_1, e_2\}$, $\mathcal{C}_1 = \operatorname{span}\{u_1, u_2\}$, $\mathcal{C}_2 = \operatorname{span}\{v_3\}$, $\mathcal{C}_{-1} = \operatorname{span}\{v_1, v_2\}$, $\mathcal{C}_{-2} = \operatorname{span}\{u_3\}$.*

(5) *The $\mathbb{Z}/3$-grading: $\mathcal{C}_{\bar{0}} = \operatorname{span}\{e_1, e_2\}$, $\mathcal{C}_{\bar{1}} = \operatorname{span}\{u_1, u_2, u_3\}$, and $\mathcal{C}_{\bar{2}} = \operatorname{span}\{v_1, v_2, v_3\}$.*

(6) *The $\mathbb{Z}/4$-grading: $\mathcal{C}_{\bar{0}} = \operatorname{span}\{e_1, e_2\}$, $\mathcal{C}_{\bar{1}} = \operatorname{span}\{u_1, u_2\}$, and $\mathcal{C}_{\bar{2}} = \operatorname{span}\{u_3, v_3\}$ and $\mathcal{C}_{\bar{3}} = \operatorname{span}\{v_1, v_2\}$.*

(7) *The $\mathbb{Z} \times \mathbb{Z}/2$-grading defined by [EK13, (4.11)].*

The next remark ([EK13, Remark 4.16]) completes the classification, up to equivalence, of gradings on Hurwitz algebras.

Remark 4.2.6. *Up to equivalence, the only nontrivial group-gradings on a quaternion algebra are the Cartan grading on the split quaternion algebra $M_2(\mathbb{F})$, or either a $\mathbb{Z}/2$ or $(\mathbb{Z}/2)^2$-grading induced by the Cayley-Dickson doubling process. The last one does not appear in characteristic 2. For Hurwitz algebras of dimension 2, the only nontrivial group-gradings are, up to equivalence, the $\mathbb{Z}/2$-gradings over fields of characteristic not 2 induced by the Cayley-Dickson doubling process.*

Let G be an abelian group. To give the classification theorem of G-gradings on the Cayley algebra $\mathcal{C}$ over an algebraically closed field ([EK13, Theorem 4.21]) we have the following notation:

Denote by $\Gamma^1_{\mathcal{C}}$ the Cartan grading by $\mathbb{Z}^2$ and by $\Gamma^2_{\mathcal{C}}$ the $(\mathbb{Z}/2)^3$-grading induced by the Cayley-Dickson doubling process (assuming $\operatorname{char}\mathbb{F} \neq 2$ in this case).

- Let $\gamma = (g_1, g_2, g_3)$ be a triple of elements in G with $g_1 g_2 g_3 = e$. Denote by $\Gamma^1_{\mathcal{C}}(G,\gamma)$ the G-grading on $\mathcal{C}$ induced from $\Gamma^1_{\mathcal{C}}$ by the homomorphism $\mathbb{Z}^2 \to G$ sending $(1,0)$ to g_1 and $(0,1)$ to g_2. For two such triples, γ and γ' we will write $\gamma \sim \gamma'$ if there exists $\pi \in Sym(3)$ such that $g'_i = g_{\pi(i)}$ for all $i = 1,2,3$ or $g'_i = g^{-1}_{\pi(i)}$ for all $i = 1,2,3$.

- Let $H \subset G$ be a subgroup isomorphic to $(\mathbb{Z}/2)^3$. Then $\Gamma^2_{\mathcal{C}}$ may be regarded as a G-grading with support H. We denote this G-grading by $\Gamma^2_{\mathcal{C}}(G,H)$. (Since the Weyl group $W(\Gamma^2_{\mathcal{C}})$ is equal to $\operatorname{Aut}((\mathbb{Z}/2)^3)$, all induced group-gradings ${}^{\alpha}\Gamma^2_{\mathcal{C}}$ for various isomorphisms $\alpha : (\mathbb{Z}/2)^3 \to H$ are isomorphic, so $\Gamma^2_{\mathcal{C}}(G,H)$ is well-defined, see Chapter 4 of [EK13].)

Theorem 4.2.7. *Let $\mathcal{C}$ be the Cayley algebra over an algebraically closed field and let G be an abelian group. Then any G-grading on $\mathcal{C}$ is isomorphic to some $\Gamma^1_{\mathcal{C}}(G,\gamma)$ or $\Gamma^2_{\mathcal{C}}(G,H)$, but not both. Also,*

- *$\Gamma^1_{\mathcal{C}}(G,\gamma)$ is isomorphic to $\Gamma^1_{\mathcal{C}}(G,\gamma')$ if and only if $\gamma \sim \gamma'$;*

- *$\Gamma^2_{\mathcal{C}}(G,H)$ is isomorphic to $\Gamma^2_{\mathcal{C}}(G,H')$ if and only if $H = H'$.*

4.3 Automorphism scheme of the tensor product of Cayley algebras

In this section we will use definitions and results from [MPP] to prove that

$$\mathbf{Aut}(\mathcal{C}^1 \otimes \cdots \otimes \mathcal{C}^n) \simeq \mathbf{Aut}(\mathcal{C}^1 \times \cdots \times \mathcal{C}^n)$$

where $\mathcal{C}^i$ are Cayley algebras for $i = 1, ..., n$. This will reduce the problem of classifying group-gradings on $\mathcal{C}^1 \otimes \cdots \otimes \mathcal{C}^n$ to classify group-gradings on $\mathcal{C}^1 \times \cdots \times \mathcal{C}^n$. After classifying group-gradings on the direct product we will use the isomorphism of schemes to get the group-gradings on the tensor product. This is a similar process to the one followed in Chapter 3.

We will assume that the base field $\mathbb{F}$ has characteristic different of 2.

Definition 4.3.1. *The **generalized alternative nucleus** of an algebra $\mathcal{A}$ is defined by:*

$$N_{alt}(\mathcal{A}) := \{a \in \mathcal{A} : (a,x,y) = -(x,a,y) = (x,y,a)\ \forall x,y \in \mathcal{A}\}$$

([MPP, Definition 3.1]).

Remark 4.3.2. *Let $\mathcal{C} := \mathcal{C}^1 \otimes \cdots \otimes \mathcal{C}^n$ be the algebra where $\mathcal{C}^i$ is the Cayley algebra for $i = 1, ..., n$. Recall that $\mathcal{C}_0^i$ is the subspace of traceless elements of $\mathcal{C}^i$. Identify $\mathcal{C}^i$ with $1 \otimes \cdots \otimes \mathcal{C}^i \otimes \cdots \otimes 1$ for $i = 1, ..., n$. In [MPP] we find that for $\mathcal{C} := \mathcal{C}^1 \otimes \cdots \otimes \mathcal{C}^n$,*

$$N_{alt}(\mathcal{C}) = \mathbb{F}1 \oplus \mathcal{C}_0^1 \oplus \cdots \oplus \mathcal{C}_0^n = \mathcal{C}^1 + \cdots + \mathcal{C}^n.$$

*And the **derived algebra** of $N_{alt}(\mathcal{C})$ is*

$$N'_{alt}(\mathcal{C}) = [N_{alt}(\mathcal{C}), N_{alt}(\mathcal{C})] = \mathcal{C}_0^1 \oplus \cdots \oplus \mathcal{C}_0^n.$$

Remark 4.3.3. *In [EK13, p. 316 and 313] we have the following statements:*
i) For an affine algebraic group scheme $\mathbf{G}$,

$$\dim \mathrm{Lie}(\mathbf{G}) \geq \dim \mathbf{G} = \dim \mathbf{G}(\overline{\mathbb{F}}).$$

ii) For an algebra $\mathcal{A}$,

$$\mathrm{Lie}(\mathbf{Aut}(\mathcal{A})) = \mathrm{Der}(\mathcal{A}).$$

iii) $\mathbf{Aut}(\mathcal{A})$ is smooth if and only if $\dim \mathrm{Der}(\mathcal{A}) = \dim \mathrm{Aut}_{\overline{F}}(\mathcal{A} \otimes \overline{F})$.

Next result is Proposition 3.6 of [MPP], for $\mathcal{C} := \mathcal{C}^1 \otimes \cdots \otimes \mathcal{C}^n$ where $\mathcal{C}^i$ are Cayley algebras for $i = 1, ..., n$.

Proposition 4.3.4. *The restriction map gives the isomorphisms (of groups)*
$\mathrm{Aut}(\mathcal{C}) \simeq \mathrm{Aut}(N'_{alt}(\mathcal{C}))$,
$\mathrm{Der}(\mathcal{C}) \simeq \mathrm{Der}(N'_{alt}(\mathcal{C})) \simeq \mathrm{Der}(\mathcal{C}^1) \oplus \cdots \oplus \mathrm{Der}(\mathcal{C}^n)$.

From now on we will use the identification $\mathcal{A}_1 \times \cdots \times \mathcal{A}_n \simeq \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_n$ as in Remark 2.1.2.

Remark 4.3.5. *i) For the Cayley algebra $\mathcal{C}$ we have that $\mathbf{Aut}(\mathcal{C})$ is smooth ([EK13, p. 146]).*

ii) Let $\mathcal{C}$ be the Cayley algebra. Using Proposition 4.3.4 for $n = 1$ we have that the restriction map $\mathbf{Aut}(\mathcal{C}) \to \mathbf{Aut}(\mathcal{C}_0)$ satisfies conditions 1) and 2) of Theorem 1.2.11 and by i) we have that

$$\mathbf{Aut}(\mathcal{C}) \simeq \mathbf{Aut}(\mathcal{C}_0).$$

iii) Let $\mathcal{C}^i$ be Cayley algebras for $i = 1, ..., n$ and $R \in \mathrm{Alg}_{\mathbb{F}}$. We have the canonical imbedding

$$\begin{array}{rcl} \mathrm{Aut}(\mathcal{C}_0^1 \otimes R) \times \cdots \times \mathrm{Aut}(\mathcal{C}_0^n \otimes R) & \to & \mathrm{Aut}((\mathcal{C}_0^1 \times \cdots \times \mathcal{C}_0^n) \otimes R) \\ (f_1, ..., f_n) & \longmapsto & (f_1, ..., f_n) \end{array}$$

for $f_i \in \mathrm{Aut}_R(\mathcal{C}_0^i \otimes R)$ and $i = 1, ..., n$. Note that we have used the identification

$$\begin{array}{rcl} (\mathcal{C}_0^1 \times \cdots \times \mathcal{C}_0^n) \otimes R & \xrightarrow{\simeq} & (\mathcal{C}_0^1 \otimes R) \times \cdots \times (\mathcal{C}_0^n \otimes R) \\ (c_1, ..., c_n) \otimes r & \longmapsto & (c_1 \otimes r, ..., c_n \otimes r) \end{array}$$

where $c_i \in \mathcal{C}_0^i$ for $i = 1, ..., n$ and $r \in R$. Then $\mathbf{Aut}(\mathcal{C}_0^1) \times \cdots \times \mathbf{Aut}(\mathcal{C}_0^n)$ is subscheme of $\mathbf{Aut}(\mathcal{C}_0^1 \times \cdots \times \mathcal{C}_0^n)$.

Lemma 4.3.6. *Let $\mathcal{C}^1, ..., \mathcal{C}^n$ be Cayley algebras and let $\sigma = - \otimes \cdots \otimes -$ be the involution in $\mathcal{C}^1 \otimes \cdots \otimes \mathcal{C}^n$, i.e. the tensor product of the involutions in each $\mathcal{C}^i$ for $i = 1, ..., n$. Then*

$$\mathbf{Aut}(\mathcal{C}^1 \otimes \cdots \otimes \mathcal{C}^n) = \mathbf{Aut}(\mathcal{C}^1 \otimes \cdots \otimes \mathcal{C}^n, \sigma)$$

where, $\mathbf{Aut}(\mathcal{C}^1 \otimes \cdots \otimes \mathcal{C}^n, \sigma)(R) = \{\varphi \in \mathrm{Aut}_{R-\mathrm{alg}}(\mathcal{C}^1 \otimes \cdots \otimes \mathcal{C}^n \otimes R) : \varphi \circ (\sigma \otimes id_R) = (\sigma \otimes id_R) \circ \varphi\}$ for $R \in \mathrm{Alg}_{\mathbb{F}}$ (see Definition 5.1.2).

Proof. Let R be an arbitrary element in $\mathrm{Alg}_{\mathbb{F}}$. We want to prove that

$$\mathrm{Aut}_{R-\mathrm{alg}}(\mathcal{C}^1 \otimes \cdots \otimes \mathcal{C}^n \otimes R) = \mathrm{Aut}_{R-\mathrm{alg}}(\mathcal{C}^1 \otimes \cdots \otimes \mathcal{C}^n \otimes R, \sigma).$$

The containment "$\supseteq$" is trivial. We have

$$N_{alt}(\mathcal{C}^1 \otimes \cdots \otimes \mathcal{C}^n \otimes R) = (\mathcal{C}^1 + \cdots + \mathcal{C}^n) \otimes R$$

(see Remark 4.3.2). Then

$$\begin{aligned} &[N_{alt}(\mathcal{C}^1 \otimes \cdots \otimes \mathcal{C}^n \otimes R), N_{alt}(\mathcal{C}^1 \otimes \cdots \otimes \mathcal{C}^n \otimes R)] \\ &\quad = [(\mathcal{C}^1 + \cdots + \mathcal{C}^n) \otimes R, (\mathcal{C}^1 + \cdots + \mathcal{C}^n) \otimes R] \\ &\quad = (\mathcal{C}_0^1 \oplus \cdots \oplus \mathcal{C}_0^n) \otimes R, \end{aligned}$$

recall that this subspace generates $\mathcal{C}^1 \otimes \cdots \otimes \mathcal{C}^n \otimes R$ considering the usual multiplication (Remark 4.2.1). Consider $\varphi \in \mathrm{Aut}_{R-\mathrm{alg}}(\mathcal{C}^1 \otimes \cdots \otimes \mathcal{C}^n \otimes R)$. Notice that $[N_{alt}(\mathcal{C}^1 \otimes \cdots \otimes \mathcal{C}^n \otimes R), N_{alt}(\mathcal{C}^1 \otimes \cdots \otimes \mathcal{C}^n \otimes R)]$ is invariant under φ. We also have for $x_i \in \mathcal{C}_0^i$ and $r_i \in R$, $i = 1, ..., n$

$$\begin{aligned} &\sigma \otimes id_R(x_1 \otimes 1 \otimes \cdots \otimes r_1 + \cdots + 1 \otimes \cdots \otimes x_n \otimes r_n) \\ &\quad = \bar{x}_1 \otimes 1 \otimes \cdots \otimes r_1 + \cdots + 1 \otimes \cdots \otimes \bar{x}_n \otimes r_n \\ &\quad = -(x_1 \otimes 1 \otimes \cdots \otimes r_1 + \cdots + 1 \otimes \cdots \otimes x_n \otimes r_n), \end{aligned}$$

then $\sigma \otimes id_R = -id_{(\mathcal{C}_0^1 \oplus \cdots \oplus \mathcal{C}_0^n)\otimes R}$ in $(\mathcal{C}_0^1 \oplus \cdots \oplus \mathcal{C}_0^n) \otimes R$. Hence

$$\varphi \circ (\sigma \otimes id_R) = (\sigma \otimes id_R) \circ \varphi$$

in $(\mathcal{C}_0^1 \oplus \cdots \oplus \mathcal{C}_0^n) \otimes R$. Then $\varphi \circ (\sigma \otimes id_R) = (\sigma \otimes id_R) \circ \varphi$ in the whole $\mathcal{C}^1 \otimes \cdots \otimes \mathcal{C}^n \otimes R$, so $\varphi \in \mathrm{Aut}_{R-\mathrm{alg}}(\mathcal{C}^1 \otimes \cdots \otimes \mathcal{C}^n \otimes R, \sigma)$. Therefore

$$\mathbf{Aut}(\mathcal{C}^1 \otimes \cdots \otimes \mathcal{C}^n) = \mathbf{Aut}(\mathcal{C}^1 \otimes \cdots \otimes \mathcal{C}^n, \sigma).$$

□

Using the above results, we have the following:

Theorem 4.3.7. *Let $\mathcal{C}^i$ be the Cayley algebra for $i = 1, ..., n$. Then there exist isomorphisms of schemes Φ and φ*

$$\mathbf{Aut}(\mathcal{C}^1 \otimes \cdots \otimes \mathcal{C}^n) \xrightarrow{\Phi} \mathbf{Aut}(\mathcal{C}_0^1 \times \cdots \times \mathcal{C}_0^n) \xleftarrow{\varphi} \mathbf{Aut}(\mathcal{C}^1 \times \cdots \times \mathcal{C}^n)$$

where $\Phi(R)(f) = f|_{N'_{alt}(\mathcal{C}^1 \otimes \cdots \otimes \mathcal{C}^n)}$ and $\varphi(R)(g) = g|_{[\mathcal{C}^1 \times \cdots \times \mathcal{C}^n, \mathcal{C}^1 \times \cdots \times \mathcal{C}^n]}$ for $f \in \mathbf{Aut}(\mathcal{C}^1 \otimes \cdots \otimes \mathcal{C}^n)(R)$ and $g \in \mathbf{Aut}(\mathcal{C}^1 \times \cdots \times \mathcal{C}^n)(R)$ with $R \in \mathrm{Alg}_{\mathbb{F}}$.

Moreover,

$$\mathbf{Aut}(\mathcal{C}^1 \otimes \cdots \otimes \mathcal{C}^n) = \mathbf{Aut}(\mathcal{C}^1 \otimes \cdots \otimes \mathcal{C}^n, \sigma),$$

where σ is the involution in $\mathcal{C}^1 \otimes \cdots \otimes \mathcal{C}^n$.

Proof. By Theorem 1.2.11 we see that in order to prove that Φ is an isomorphism of schemes it is enough to show that

a) $\Phi(\overline{\mathbb{F}}) : \mathrm{Aut}_{\overline{\mathbb{F}}}((\mathcal{C}^1 \otimes \cdots \otimes \mathcal{C}^n) \otimes \overline{\mathbb{F}}) \to \mathrm{Aut}_{\overline{\mathbb{F}}}((\mathcal{C}_0^1 \times \cdots \times \mathcal{C}_0^n) \otimes \overline{\mathbb{F}})$ is bijective,

b) $d\Phi : \mathrm{Lie}(\mathbf{Aut}(\mathcal{C}^1 \otimes \cdots \otimes \mathcal{C}^n)) \to \mathrm{Lie}(\mathbf{Aut}(\mathcal{C}_0^1 \times \cdots \times \mathcal{C}_0^n))$ is bijective,

c) $\mathbf{Aut}(\mathcal{C}_0^1 \times \cdots \times \mathcal{C}_0^n)$ is smooth.

Since $\mathcal{A} \otimes \overline{\mathbb{F}} \simeq \mathcal{A}_{\overline{\mathbb{F}}}$, we can use Proposition 4.3.4 with $\mathbb{F} = \overline{\mathbb{F}}$ (since such proposition works for arbitrary fields of characteristic different of 2) and Remark 4.3.2 to get a). By Proposition 4.3.4 and Remark 4.3.3 ii) we have b).

To prove c), by Remark 4.3.3 iii), it is enough to show that $\dim \mathrm{Der}(\mathcal{C}_0^1 \times$

$\cdots \times \mathcal{C}_0^n) = \dim \mathrm{Aut}_{\overline{F}}((\mathcal{C}_0^1 \times \cdots \times \mathcal{C}_0^n) \otimes \overline{F})$. We have

$$\begin{aligned}
\sum_{i=1}^{n} \dim \mathbf{Aut}(\mathcal{C}^i) &= \sum_{i=1}^{n} \dim \mathbf{Aut}(\mathcal{C}_0^i) \quad \text{(by Remark 4.3.5 ii))} \\
&\leq \dim \mathbf{Aut}(\mathcal{C}_0^1 \times \cdots \times \mathcal{C}_0^n) \quad \text{(by Remark 4.3.5 iii))} \\
&\leq \dim \mathrm{Lie}(\mathbf{Aut}(\mathcal{C}_0^1 \times \cdots \times \mathcal{C}_0^n)) \quad \text{(by Remark 4.3.3 i))} \\
&= \dim \mathrm{Der}(\mathcal{C}_0^1 \times \cdots \times \mathcal{C}_0^n) \quad \text{(by Remark 4.3.3 ii))} \\
&= \sum_{i=1}^{n} \dim \mathrm{Der}(\mathcal{C}_0^i) \quad \text{(by Proposition 4.3.4 for } n=1) \\
&= \sum_{i=1}^{n} \dim \mathrm{Der}(\mathcal{C}^i) \quad \text{(by Proposition 4.3.4 for } n=1) \\
&= \sum_{i=1}^{n} \dim \mathbf{Aut}(\mathcal{C}^i) \quad \text{(by Remark 4.3.5 i) and 4.3.3 iii)).}
\end{aligned}$$

Then $\dim \mathbf{Aut}(\mathcal{C}_0^1 \times \cdots \times \mathcal{C}_0^n) = \dim \mathrm{Lie}(\mathbf{Aut}(\mathcal{C}_0^1 \times \cdots \times \mathcal{C}_0^n))$, from this and $\dim \mathbf{Aut}(\mathcal{C}_0^1 \times \cdots \times \mathcal{C}_0^n) = \dim \mathbf{Aut}(\mathcal{C}_0^1 \times \cdots \times \mathcal{C}_0^n)(\overline{\mathbb{F}})$ (Remark 4.3.3 i)) follows:

$$\begin{aligned}
\dim \mathbf{Aut}(\mathcal{C}_0^1 \times \cdots \times \mathcal{C}_0^n)(\overline{\mathbb{F}}) &= \dim \mathrm{Lie}(\mathbf{Aut}(\mathcal{C}_0^1 \times \cdots \times \mathcal{C}_0^n)) \\
&= \dim \mathrm{Der}(\mathcal{C}_0^1 \times \cdots \times \mathcal{C}_0^n) \text{ (Remark 4.3.3 ii))}
\end{aligned}$$

Therefore $\mathbf{Aut}(\mathcal{C}_0^1 \times \cdots \times \mathcal{C}_0^n)$ is smooth and then Φ is an isomorphism of schemes.

In order to prove that φ is an isomorphism of schemes we will use again Theorem 1.2.11. Since we have already proved that $\mathbf{Aut}(\mathcal{C}_0^1 \times \cdots \times \mathcal{C}_0^n)$ is smooth, we only have to prove that $\varphi(\overline{\mathbb{F}})$ and $d\varphi$ are bijective.

Consider the algebras $\mathcal{C}^1, ..., \mathcal{C}^n$ over an algebraically closed field. Take $g \in \mathbf{Aut}(\mathcal{C}^1 \times \cdots \times \mathcal{C}^n)(\overline{\mathbb{F}}) = \mathrm{Aut}(\mathcal{C}^1 \times \cdots \times \mathcal{C}^n)$, then

$$g|_{0\times\cdots\times\mathcal{C}^i\times\cdots\times 0} : 0 \times \cdots \times \mathcal{C}^i \times \cdots \times 0 \to \mathcal{C}^1 \times \cdots \times \mathcal{C}^n$$

is an homomorphism and applying the first theorem of isomorphisms we have

$$\mathrm{im}(g|_{0\times\cdots\times\mathcal{C}^i\times\cdots\times 0}) \simeq (0 \times \cdots \times \mathcal{C}^i \times \cdots \times 0)/(0 \times \cdots \times 0) \simeq \mathcal{C}^i.$$

Then for every $i \in \{1, ..., n\}$ there exists a unique $j \in \{1, ..., n\}$ such that

$$g(0 \times \cdots \times \mathcal{C}^i \times \cdots \times 0) = 0 \times \cdots \times \mathcal{C}^j \times \cdots \times 0$$

due to Lemma 2.1.13. So, there exists $\sigma \in S_n$ such that

$$g(0 \times \cdots \times \mathcal{C}^i \times \cdots \times 0) = 0 \times \cdots \times \mathcal{C}^{\sigma(i)} \times \cdots \times 0$$

for $i = 1, ..., n$. We have the isomorphism

$$\begin{array}{rccl} g_i: & \mathcal{C}^i & \longrightarrow & \mathcal{C}^{\sigma(i)} \\ & x & \longmapsto & P_{\sigma(i)}(g(0, ..., \overset{i}{x}, ..., 0)) \end{array}$$

where $P_{\sigma(i)}$ is the canonical projection in the $\sigma(i)$-th entry. We have

$$\begin{array}{rlclcl} g = & \mathcal{C}^1 \times \cdots \times \mathcal{C}^n & \to & \mathcal{C}^{\sigma(1)} \times \cdots \times \mathcal{C}^{\sigma(n)} & \to & \mathcal{C}^1 \times \cdots \times \mathcal{C}^n \\ & (c_1, ..., c_n) & \mapsto & (g_1(c_1), ..., g_n(c_n)) & & \\ & & & (a_1, ..., a_n) & \mapsto & (a_{\sigma^{-1}(1)}, ..., a_{\sigma^{-1}(n)}). \end{array}$$

Since $\mathcal{C}_0^i = [\mathcal{C}^i, \mathcal{C}^i]$ we have that $g_i|_{\mathcal{C}_0^i} : \mathcal{C}_0^i \to \mathcal{C}_0^{\sigma(i)}$ is an isomorphism. Then

$$g(\mathcal{C}_0^1 \times \cdots \times \mathcal{C}_0^n) = \mathcal{C}_0^1 \times \cdots \times \mathcal{C}_0^n,$$

i.e. $\mathcal{C}_0^1 \times \cdots \times \mathcal{C}_0^n$ is invariant under g and since $\mathcal{C}_0^1 \times \cdots \times \mathcal{C}_0^n$ generates $\mathcal{C}^1 \times \cdots \times \mathcal{C}^n$, we have that $\varphi(\overline{\mathbb{F}})$ is injective.

Take $f \in \mathbf{Aut}(\mathcal{C}_0^1 \times \cdots \times \mathcal{C}_0^n)(\overline{\mathbb{F}}) = \mathrm{Aut}(\mathcal{C}_0^1 \times \cdots \times \mathcal{C}_0^n)$. Notice that $0 \times \cdots \times \mathcal{C}_0^i \times \cdots \times 0$ is a minimal ideal of $\mathcal{C}_0^1 \times \cdots \times \mathcal{C}_0^n$ for $i = 1, ..., n$. Then there is a $\tau \in S_n$ such that

$$f(0 \times \cdots \times \mathcal{C}_0^i \times \cdots \times 0) = 0 \times \cdots \times \mathcal{C}_0^{\tau(i)} \times \cdots \times 0.$$

Then f induces the isomorphisms for $i = 1, ..., n$

$$\begin{array}{rccl} f_i: & \mathcal{C}_0^i & \to & \mathcal{C}_0^{\tau(i)} \\ & x & \mapsto & P_{\tau(i)} f(0, ..., \overset{i}{x}, ..., 0). \end{array}$$

We can extend f_i to $\mathcal{C}^i$ by defining

$$\begin{array}{rlcl} f_i': & \mathcal{C}^i & \to & \mathcal{C}^{\tau(i)} \\ & 1 & \mapsto & 1 \\ & x \in \mathcal{C}_0^i & \mapsto & f_i(x). \end{array}$$

Then, for the isomorphism

$$\begin{array}{rccl} f': & \mathcal{C}^1 \times \cdots \times \mathcal{C}^n & \to & \mathcal{C}^1 \times \cdots \times \mathcal{C}^n \\ & (0, ..., \overset{i}{1}, ..., 0) & \mapsto & (0, ..., \overset{\tau(i)}{1}, ..., 0) \\ & (0, ..., \overset{i}{x}, ..., 0) & \mapsto & (0, ..., \overset{\tau(i)}{f_i(x)}, ..., 0) \text{ for } x \in \mathcal{C}_0^i \end{array}$$

we have that $\varphi(\overline{\mathbb{F}})(f') = f$. Then $\varphi(\overline{\mathbb{F}})$ is surjective and therefore $\varphi(\overline{\mathbb{F}})$ is an isomorphism.

We will prove that $d\varphi$ is an isomorphism. Since $0 \times \cdots \times \mathcal{C}^i \times \cdots \times 0$ is an ideal of $\mathcal{C}^1 \times \cdots \times \mathcal{C}^n$ for all $i = 1, ..., n$ we have

$$\mathrm{Der}(\mathcal{C}^1 \times \cdots \times \mathcal{C}^n) = \mathrm{Der}(\mathcal{C}^1) \times \cdots \times \mathrm{Der}(\mathcal{C}^n).$$

By Proposition 4.3.4 we have

$$\mathrm{Der}(\mathcal{C}_0^1 \times \cdots \times \mathcal{C}_0^n) \simeq \mathrm{Der}(\mathcal{C}^1) \times \cdots \times \mathrm{Der}(\mathcal{C}^n).$$

Therefore

$$\mathrm{Der}(\mathcal{C}_0^1 \times \cdots \times \mathcal{C}_0^n) \simeq \mathrm{Der}(\mathcal{C}^1 \times \cdots \times \mathcal{C}^n).$$

Last part follows from Lemma 4.3.6. □

Chapter 5

Tensor product of composition algebras

In this chapter we study the group-gradings on the tensor product of two composition algebras, more specifically group-gradings on $\mathcal{C}^1 \otimes \mathcal{C}^2$ where $\mathcal{C}^1$ is a Cayley algebra and $\mathcal{C}^2$ is a Hurwitz algebra. In Section 1 we give the motivation for studying these particular type of algebras. This is because this tensor product is a particular case of structurable algebras which, through the TKK construction, result in Lie algebras. Moreover the group-gradings of structurable algebras (Definition 5.1.2) induce group-gradings on the respective Lie algebras. In Section 2 we give group-gradings (closed under involution) on such tensor products.

As before, **all grading groups considered will be assumed to be abelian**.

5.1 Motivation: Structurable algebras

A classification of finite-dimensional central simple structurable algebras over a field of characteristic zero was given in 1978 in [Al78, Theorem 25], with a missing item. Such classification was completed in 1990 in [Sm90, Theorem 3.8] for a base field of characteristic different of 2, 3 or 5. The importance of studying structurable algebras is their use in the construction of Lie algebras using, for example, a modified TKK-construction as in [Al79] where all the isotropic simple Lie algebras were obtained over an arbitrary field of characteristic zero.

We also have that from a G-grading on a central simple structurable algebra, where G is a group, we get a $G \times \mathbb{Z}$-grading in its corresponding central simple Lie algebra. We are interested in one point of the classification:

the tensor product of a Cayley algebra and a Hurwitz algebra of dimension 2, 4 or 8.

First we will give some definitions. For x, y in an algebra $\mathcal{A}$, define $V_{x,y} \in \mathrm{End}_{\mathbb{F}}(\mathcal{A})$ by

$$V_{x,y}(z) = (x\bar{y})z + (z\bar{y})x + (z\bar{x})y$$

for $z \in \mathcal{A}$. Put $T_z = V_{x,1}$, for $x \in \mathcal{A}$. Then,

$$T_x(z) = xz + zx - z\overline{x}$$

for $x, z \in \mathcal{A}$.

Definition 5.1.1. *[Al78] Let $\mathbb{F}$ be a field of characteristic different of 2 or 3. Let $(\mathcal{A}, -)$ be a finite-dimensional nonassociative unital algebra with involution over $\mathbb{F}$ (i.e. an antiautomorphism "-" of period 2, see Definition 4.1.2). We say that $(\mathcal{A}, -)$ is* ***structurable*** *if*

$$[T_a, V_{x,y}] = V_{T_z x,y} - V_{x,T_{\bar{z}} y}$$

for $x, y, z \in \mathcal{A}$.

We define the subspace of $(\mathcal{A}, -)$ of ***antisymmetric elements****:*

$$S(\mathcal{A}, -) := \{x \in \mathcal{A} : \overline{x} = -x\}$$

which is a subalgebra of $(\mathcal{A}, -)$ with the multiplication given by the commutator '[,]' since for $s, t \in S(\mathcal{A}, -)$ we have

$$\overline{[s,t]} = \overline{st - ts} = \overline{st} - \overline{ts} = \bar{t}\bar{s} - \bar{s}\bar{t} = ts - st = [t,s] = -[s,t].$$

Definition 5.1.2. *Let G be a group and let $(\mathcal{A}, -)$ be an algebra with involution. We will say that Γ is an* ***involution preserving grading*** *on $(\mathcal{A}, -)$ if Γ is a G-grading on the algebra $\mathcal{A}$ and it is closed under the involution, i.e., $\overline{\mathcal{A}_g} \subseteq \mathcal{A}_g$ for all $g \in G$.*

Let $(\mathcal{A}, -)$ and $(\mathcal{B}, -)$ be algebras with involution. We say that a homomorphism of algebras $\varphi : \mathcal{A} \to \mathcal{B}$ is an ***involution preserving homomorphism*** *if it commutes with the involution, i.e. $\varphi \circ - = - \circ \varphi$. Notice that the involution in the left (resp. right) side of the equality is the one in $\mathcal{A}$ (resp. $\mathcal{B}$). To make clear that it is a homomorphism that preserves involution we will write $\varphi : (\mathcal{A}, -) \to (\mathcal{B}, -)$.*

We will abuse of the notation by writing all the involutions as "$-$". We just have to take into account that each of them corresponds to a particular algebra.

Remark 5.1.3. *We are interested in studying involution preserving group-gradings on algebras with involution.*

Assume now that $\mathbb{F}$ is a field of characteristic different of 2, 3 or 5 and that all algebras are finite-dimensional. Smirnov proved in [Sm90, Theorem 2.1] that any semisimple structurable algebra is the direct sum of simple algebras. The simple algebras are central simple over their centre, and thus the description of semisimple algebras is reduced to the description of central simple algebras. We have the classification theorem ([Sm90, Theorem 3.8], see also [Al79, Theorem 11]):

Theorem 5.1.4. *Any central simple structurable algebra is isomorphic to one of the following:*
(a) a Jordan algebra (with the identity involution),
(b) an associative algebra with involution,
(c) a 2×2 matrix algebra constructed from the Jordan algebra $\mathcal{J}$ of an admissible cubic form with basepoint and a nonzero scalar or a form of such a 2×2 matrix algebra,
(d) an algebra with involution constructed from an hermitian form,
(e) a tensor product $(\mathcal{C}^1 \otimes \mathcal{C}^2, \sigma)$ where $\mathcal{C}^1$ is a Cayley algebra, $\mathcal{C}^2$ is a Hurwitz algebra and σ is the tensor product of the standard involutions, and the twisted tensor product algebra constructed from a Cayley algebra $\mathcal{C}$ over a quadratic field extension of the base field $\mathbb{F}$,
(f) a 35-dimensional central simple algebra $T(\mathcal{C}, -)$ constructed from an octonion algebra $(\mathcal{C}, -)$.

Gradings on Jordan algebras (see [EK13, Chapter 5]) and on associative algebras (see [EK13, Chapter 2]) are already known. Only some gradings on algebras of case c) are known. Gradings on algebras of case d) are unknown yet. Case f) has been studied by Diego Aranda-Orna but the results have not been published yet. And we are interested in the classification of group-gradings on algebras of the type (e) since they have not been studied.

One of the constructions of a central simple Lie algebra from a central simple structurable algebra $(\mathcal{A}, -)$ (used in [Al79]) is the following. Define $T_{\mathcal{A}} = \{T_x : x \in \mathcal{A}\}$ and let $\mathrm{Der}(\mathcal{A})$ be the set of derivations of $\mathcal{A}$ that commute with $-$. Denote $(\mathcal{A}, -)$ and $\mathcal{S}(\mathcal{A}, -)$ by $\mathcal{A}$ and $\mathcal{S}$ respectively. Then

$$\mathcal{F}(\mathcal{A}, -) = \mathcal{S}' \oplus \mathcal{A}' \oplus (\mathrm{Der}(\mathcal{A}) \oplus T_{\mathcal{A}}) \oplus \mathcal{A} \oplus \mathcal{S}$$

is a Lie algebra ([Al79, Theorem 3]) where $\mathcal{S}'$ and $\mathcal{A}'$ are copies of $\mathcal{S}$ and $\mathcal{A}$ respectively. Moreover we can give $\mathcal{F}(\mathcal{A}, -)$ the structure of a graded Lie algebra in the following way. For $j \in \mathbb{Z}$,

$$\begin{array}{ll} \mathcal{F}(\mathcal{A},-)_{-2} = \mathcal{S}', & \mathcal{F}(\mathcal{A},-)_{-1} = \mathcal{A}' \\ \mathcal{F}(\mathcal{A},-)_{2} = \mathcal{S}, & \mathcal{F}(\mathcal{A},-)_{1} = \mathcal{A} \\ \mathcal{F}(\mathcal{A},-)_{0} = \mathrm{Der}(\mathcal{A}) \oplus T_{\mathcal{A}}. & \end{array}$$

An involution preserving grading on $(\mathcal{A},-)$ by a group G induces a G-grading on $(\mathcal{S},-)$ and on $\mathrm{Der}(\mathcal{A}) \oplus T_{\mathcal{A}}$. Therefore if we start from an involution preserving grading Γ on $(\mathcal{A},-)$ we can obtain a $\mathbb{Z} \times G$-grading on $\mathcal{F}(\mathcal{A},-)$ by means of

$$\begin{array}{ll} \deg s = (-2, \deg_\Gamma s) & \deg a = (-1, deg_\Gamma a) \\ \deg s' = (2, \deg_\Gamma s') & \deg a' = (1, deg_\Gamma a') \\ \deg f = (0, \deg_\Gamma f) & \end{array}$$

for $s \in \mathcal{S}$, $a \in \mathcal{A}$, $s' \in \mathcal{S}'$, $a' \in \mathcal{A}'$ and $f \in \mathrm{Der}(\mathcal{A}) \oplus T_{\mathcal{A}}$ where $\deg_\Gamma$ is the degree in Γ.

We know, by [Al79], that we can obtain the central simple Lie algebras of type F_4, E_6, E_7 and E_8 through a construction related with the mentioned one from the algebras $(\mathcal{C}^1 \otimes \mathcal{C}^2, -)$ where $\mathcal{C}^1$ is a Cayley algebra and $\mathcal{C}^2$ is a Hurwitz algebra.

5.2 Gradings on the tensor product of two Hurwitz Algebras

We will assume that the characteristic of the ground field $\mathbb{F}$ is different of 2 and $\mathbb{F}$ is algebraically closed. We want to find involution preserving group-gradings on the algebra with involution $(\mathcal{C}^1 \otimes \mathcal{C}^2, - \otimes -)$ where $(\mathcal{C}^1, -)$ is a Cayley algebra and $(\mathcal{C}^2, -)$ is a Hurwitz algebra.

We prove that involution preserving group-gradings on the algebra $\mathcal{C}^1 \otimes \mathcal{C}^2$ where $\mathcal{C}^1$ is a Cayley algebra and $\mathcal{C}^2$ is a Hurwitz algebra of dimension 1, 4 or 8 do not really depend on the involution, that is, all group-gradings on such algebras preserve the involution (Remark 5.2.3). This does not happen in the case $\mathcal{C}^2$ is a Hurwitz algebra of dimension 2.

We give the classification, up to isomorphism, of group-gradings on $\mathcal{C}^1 \otimes \mathcal{C}^2$ where $\mathcal{C}^1$ is a Cayley algebra and $\mathcal{C}^2$ is a Hurwitz algebra of dimension 2 and 4 (Theorem 5.2.6) and the classification of the fine group-gradings up to equivalence.

For the case where $(\mathcal{C}^2, -)$ is also a Cayley algebra we will first compute, in Subsection 1, group-gradings on $\mathcal{C}^1 \times \mathcal{C}^2$ using results from Chapter 2 since the Cayley algebra is simple. Then, in Subsection 2, we use the automorphism schemes we mentioned in Chapter 4, Section 3 to obtain group-gradings on $\mathcal{C}^1 \otimes \mathcal{C}^2$.

First we will see some interesting graded subspaces.

Lemma 5.2.1. *Let Γ be a G-grading on an algebra $\mathcal{A}$ for a group G, then we have the following*

a) $N_{alt}(\mathcal{A})$ is a G-graded subspace of $\mathcal{A}$ (see Definition 4.3.1).

b) if $\mathcal{A} = (\mathcal{A}, -)$ is a structurable algebra, then the subspace of antisymmetric elements $(S(\mathcal{A}), -)$ (Definition 5.1.1) is a G-graded subspace of $\mathcal{A}$,

c) $[\mathcal{A}, \mathcal{A}]$ is a G-graded subspace of $\mathcal{A}$,

d) the center of $\mathcal{A}$ (i.e. $Z(\mathcal{A}) := \{x \in \mathcal{A} : xy = yx,\ (xy)z - x(yz) = (yx)z - y(xz) = (yz)x - y(zx) = 0,\ \forall y, z \in \mathcal{A}\}$, this is the definition of the center of nonassociative algebras) is a G-graded subspace of $\mathcal{A}$.

e) $J(\mathcal{A}, \mathcal{A}, \mathcal{A}) = \operatorname{span}\{[[x, y], z] + [[z, x], y] + [[y, z], x] : x, y, z \in \mathcal{A}\}$ is a G-graded subspace of $\mathcal{A}$,

f) $\mathcal{D} = \{x \in \mathcal{A} : J(x, \mathcal{A}, \mathcal{A}) = \{0\}\}$ is a G-graded subspace of $\mathcal{A}$.

Proof. a) Let a be in $N_{alt}(\mathcal{A})$ then there exist $a_i \in \mathcal{A}_{g_i}$ for $i = 1, ..., n$ and $g_i \in G$ with $g_i \neq g_j$ if $i \neq j$ such that $a = \sum_{i=1}^n a_i$. For all homogeneous elements $x, y \in \mathcal{A}$ we have

$$\left(\sum_{i=1}^n a_i, x, y \right) = - \left(x, \sum_{i=1}^n a_i, y \right) = \left(x, y, \sum_{i=1}^n a_i \right)$$

which is the same that

$$\sum_{i=1}^n (a_i x) y - \sum_{i=1}^n a_i (xy) = - \sum_{i=1}^n (x a_i) y + \sum_{i=1}^n x(a_i y) = \sum_{i=1}^n (xy) a_i - \sum_{i=1}^n x(y a_i).$$

Then we have

$$(a_i, x, y) = -(x, a_i, y) = (x, y, a_i)$$

for all $i = 1, ..., n$. Since this is satisfied for all homogeneous elements, it is satisfied for all $x, y \in \mathcal{A}$. Then $a_i \in N_{alt}(\mathcal{A})$ for all $i = 1, ..., n$, therefore $N_{alt}(\mathcal{A})$ is G-graded.

b) For $x \in (S(\mathcal{A}), -)$ there exist $x_i \in \mathcal{A}_{g_i}$ with $g_i \in G$ for $i = 1, ..., n$ such that $x = \sum_{i=1}^n x_i$. Then

$$\sum_{i=1}^n \overline{x_i} = \overline{x} = -x = -\sum_{i=1}^n x_i = \sum_{i=1}^n -x_i,$$

so $\overline{x_i} = -x_i$, that is $x_i \in (S(\mathcal{A}), -)$ for all $i = 1, ..., n$.

c) As $\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ we have

$$[\mathcal{A}, \mathcal{A}] = \sum_{g,h \in G} [\mathcal{A}_g, \mathcal{A}_h]$$

and each $[\mathcal{A}_g, \mathcal{A}_h]$ is a graded subspace because it is contained in $\mathcal{A}_{gh}$.

d) For $x \in Z(\mathcal{A})$ there exist $x_i \in \mathcal{A}_{g_i}$ for $g_i \in G$ and $i = 1, ..., n$ such that $x = \sum_{i=1}^n x_i$. For all $y \in \mathcal{A}_g$, where $g \in G$ we have

$$xy = \sum_{i=1}^n x_i y = y \sum_{i=1}^n x_i = yx,$$

then

$$0 = \sum_{i=1}^n x_i y - y \sum_{i=1}^n x_i = \sum_{i=1}^n (x_i y - y x_i)$$

where $x_i y - y x_i \in \mathcal{A}_{g_i g}$. So $x_i y - y x_i = 0$ for all homogeneous elements $y \in \mathcal{A}$ and $i = 1, ..., n$. Therefore $x_i y = y x_i$ for all $y \in \mathcal{A}$ and $i = 1, ..., n$.

Take $y \in \mathcal{A}_g$ and $z \in \mathcal{A}_h$ for $g, h \in G$. Since $(xy)z - x(yz) = 0$,

$$0 = \left(\sum_{i=1}^n x_i y \right) z - \sum_{i=1}^n x_i (yz) = \sum_{i=1}^n ((x_i y) z - x_i (yz))$$

where $(x_i y) z - x_i (yz) \in \mathcal{A}_{g_i g h}$. Then $(x_i y) z - x_i (yz) = 0$ for all homogeneous elements $y, z \in \mathcal{A}$ and $i = 1, ..., n$. Therefore $(x_i y) z - x_i (yz) = 0$ for all $y, z \in \mathcal{A}$ and $i = 1, ..., n$. The proof of $(y x_i) z - y(x_i z) = 0 = (yz) x_i - y(z x_i)$ is analogous. Then $x_i \in Z(\mathcal{A})$ for all $i = 1, ..., n$.

e) $J(\mathcal{A}, \mathcal{A}, \mathcal{A}) = \sum_{g,h,k \in G} J(\mathcal{A}_g, \mathcal{A}_h, \mathcal{A}_k)$, so $J(\mathcal{A}, \mathcal{A}, \mathcal{A})$ is a sum of graded subspaces and hence it is graded.

f) For $x \in \mathcal{D}$ there exist $x_i \in \mathcal{A}_{g_i}$ for $g_i \in G$ and $i = 1, ..., n$ such that $x = \sum_{i=1}^n x_i$. For all $y \in \mathcal{A}_g$ and $z \in \mathcal{A}_h$ with $g, h \in G$ we have

$$\begin{aligned} 0 =& [[x, y], z] + [[z, x], y] + [[y, z], x] \\ =& \left[\left[\sum_{i=1}^n x_i, y\right], z\right] + \left[\left[z, \sum_{i=1}^n x_i\right], y\right] + \left[[y, z], \sum_{i=1}^n x_i\right] \\ =& \sum_{i=1}^n ([[x_i, y], z] + [[z, x_i], y] + [[y, z], x_i]) \end{aligned}$$

where $[[x_i, y], z] + [[z, x_i], y] + [[y, z], x_i] \in \mathcal{A}_{g_igh}$. Then $[[x_i, y], z] + [[z, x_i], y] + [[y, z], x_i] = 0$ for all homogeneous elements $y, z \in \mathcal{A}$ and $i = 1, ..., n$. So $[[x_i, y], z] + [[z, x_i], y] + [[y, z], x_i] = 0$ for all $y, z \in \mathcal{A}$ and $i = 1, ..., n$. Therefore $x_i \in \mathcal{D}$ for all $i = 1, ..., n$.

□

Now we will give a result that relates involution preserving group-gradings and group-gradings on the algebra $(\mathcal{C} \otimes \mathcal{H}, - \otimes -)$.

Lemma 5.2.2. *Let $\mathcal{C}$ be a Cayley algebra and let $\mathcal{H}$ be a Hurwitz algebra of dimension 4. Then $N_{alt}(\mathcal{C} \otimes \mathcal{H}) = \mathcal{C} \otimes 1 + 1 \otimes \mathcal{H}$ and*

$$\mathbf{Aut}(\mathcal{C} \otimes \mathcal{H}) = \mathbf{Aut}(\mathcal{C} \otimes \mathcal{H}, \sigma)$$

where $\sigma = - \otimes -$ (the tensor product of the involution in $\mathcal{C}$ and the involution in $\mathcal{H}$).

Proof. Recall $\mathcal{H}$ is associative and observe that for $x, y, z \in \mathcal{C}$ and $u, v, w \in \mathcal{H}$ we have

$$\begin{aligned} (x \otimes u, y \otimes v, z \otimes w) &= (xy \otimes uv)(z \otimes w) - (x \otimes u)(yz \otimes vw) \\ &= (xy)z \otimes uvw - x(yz) \otimes uvw \\ &= ((xy)z - x(yz)) \otimes uvw \\ &= (x, y, z) \otimes uvw. \end{aligned}$$

For all $y, z \in \mathcal{C}$ and $u, v, w \in \mathcal{H}$ we have

$$\begin{aligned} &(1 \otimes u, y \otimes v, z \otimes w) = (1, y, z) \otimes uvw = 0, \\ &(y \otimes v, 1 \otimes u, z \otimes w) = 0, \\ &(y \otimes v, z \otimes w, 1 \otimes u) = 0. \end{aligned}$$

Then $1 \otimes \mathcal{H} \subseteq N_{alt}(\mathcal{C} \otimes \mathcal{H})$. For all $x, y, z \in \mathcal{C}$ and $v, w \in \mathcal{H}$ we have

$$\begin{aligned} (x \otimes 1, y \otimes v, z \otimes w) &= (x, y, z) \otimes vw, \\ (y \otimes v, x \otimes 1, z \otimes w) &= (y, x, z) \otimes vw, \\ (y \otimes v, z \otimes w, x \otimes 1) &= (y, z, x) \otimes vw. \end{aligned}$$

Since $\mathcal{C}$ is alternative we have $(x,y,z) = -(y,x,z) = (y,z,x)$ and then $\mathcal{C}\otimes 1 \subseteq N_{alt}(\mathcal{C}\otimes\mathcal{H})$. Therefore

$$1\otimes\mathcal{H}+\mathcal{C}\otimes 1 \subseteq N_{alt}(\mathcal{C}\otimes\mathcal{H}).$$

In order to prove the reverse containment we will consider the $(\mathbb{Z}/2)^5$-grading on $\mathcal{C}\otimes\mathcal{H}$ induced by the $(\mathbb{Z}/2)^3$-grading on $\mathcal{C}$ and the $(\mathbb{Z}/2)^2$-grading on $\mathcal{H}$ (both gradings induced by the Cayley-Dickson doubling process), such grading is explicitly given later in this section. Notice that each homogeneous component in such grading has dimension 1. By Lemma 5.2.1 a) $N_{alt}(\mathcal{C}\otimes\mathcal{H})$ is $(\mathbb{Z}/2)^5$-graded. The induced $(\mathbb{Z}/2)^5$-grading is given by

$$\Gamma : N_{alt}(\mathcal{C}\otimes\mathcal{H}) = \bigoplus_{g\in(\mathbb{Z}/2)^5} \left(N_{alt}(\mathcal{C}\otimes\mathcal{H})\cap(\mathcal{C}\otimes\mathcal{H})_g\right).$$

Therefore, each homogeneous component in such grading has dimension 1.

Consider the basis $\{1,i,j,k\}$ of $\mathcal{H}$ where every element of the basis is homogeneous and recall $\deg 1 = e$. Suppose there exist $e\neq a\in(\mathbb{Z}/2)^3$ and $e\neq b\in(\mathbb{Z}/2)^2$ such that $\mathcal{C}_a\otimes\mathcal{H}_b\subseteq N_{alt}(\mathcal{C}\otimes\mathcal{H})$. Without loss of generality suppose $\mathcal{H}_b=\mathbb{F}i$, then $x\otimes i\in N_{alt}(\mathcal{C}\otimes\mathcal{H})$ for $x\in\mathcal{C}\setminus\mathbb{F}1$. For all $y,z\in\mathcal{C}$ and $u,v\in\mathcal{H}$ we have

$$(x\otimes i, y\otimes u, z\otimes v) = -(y\otimes u, x\otimes i, z\otimes v) = (y\otimes u, z\otimes v, x\otimes i)$$

which is the same that

$$(x,y,z)\otimes iuv = -(y,x,z)\otimes uiv = (y,z,x)\otimes uvi.$$

If we take $u=v=j$ we have $(x,y,z)\otimes i = (y,x,z)\otimes i$ and since $\mathcal{C}$ is alternative we have $(x,y,z)=(y,x,z)=0$ for all $y,z\in\mathcal{C}$ and then $x\in\mathbb{F}1$ which is a contradiction. Therefore

$$1\otimes\mathcal{H}+\mathcal{C}\otimes 1 = N_{alt}(\mathcal{C}\otimes\mathcal{H}).$$

Now we will prove that for all $R\in\mathrm{Alg}_{\mathbb{F}}$ we have

$$\mathrm{Aut}_{R-\mathrm{alg}}(\mathcal{C}\otimes\mathcal{H}\otimes R) = \mathrm{Aut}_{R-\mathrm{alg}}(\mathcal{C}\otimes\mathcal{H}\otimes R,\sigma).$$

The containment "$\supseteq$" is clear. Take $\varphi\in\mathrm{Aut}_{R-\mathrm{alg}}(\mathcal{C}\otimes\mathcal{H}\otimes R)$. Observe that φ preserves $[N_{alt}(\mathcal{C}\otimes\mathcal{H}\otimes R), N_{alt}(\mathcal{C}\otimes\mathcal{H}\otimes R)]$ and

$$\begin{aligned}
&[N_{alt}(\mathcal{C}\otimes\mathcal{H}\otimes R), N_{alt}(\mathcal{C}\otimes\mathcal{H}\otimes R)]\\
&\qquad = [(\mathcal{C}\otimes 1+1\otimes\mathcal{H})\otimes R), (\mathcal{C}\otimes 1+1\otimes\mathcal{H})\otimes R)]\\
&\qquad = \mathcal{C}_0\otimes 1\otimes R + 1\otimes\mathcal{H}_0\otimes R.
\end{aligned}$$

By Remark 4.2.1 $\mathcal{C}_0 \otimes 1 \otimes R + 1 \otimes \mathcal{H}_0 \otimes R$ generates $\mathcal{C} \otimes \mathcal{H} \otimes R$ with the usual product. We have $\sigma \otimes id_R = -id_{\mathcal{C}_0 \otimes 1 \otimes R + 1 \otimes \mathcal{H}_0 \otimes R}$, then $\varphi \circ (\sigma \otimes id_R) = (\sigma \otimes id_R) \circ \varphi$ in $\mathcal{C}_0 \otimes 1 \otimes R + 1 \otimes \mathcal{H}_0 \otimes R$ and therefore in the whole $\mathcal{C} \otimes \mathcal{H} \otimes R$. Then

$$\mathbf{Aut}(\mathcal{C} \otimes \mathcal{H}) = \mathbf{Aut}(\mathcal{C} \otimes \mathcal{H}, \sigma).$$

□

Remark 5.2.3. *Involution preserving group-gradings on the algebras:*

$$(\mathcal{C}^1 \otimes \mathcal{C}^2, - \otimes -) \text{ and } (\mathcal{C} \otimes \mathcal{H}, - \otimes -)$$

are the same that the group-gradings on the algebras

$$\mathcal{C}^1 \otimes \mathcal{C}^2 \text{ and } \mathcal{C} \otimes \mathcal{H}$$

respectively. This follows from Theorem 4.3.7 and Lemma 5.2.2 since the automorphism scheme of such algebras does not depend on the involution. Notice that this is not the case for the tensor product of a Cayley algebra $\mathcal{C}$ and a Hurwitz algebra $\mathcal{K}$ of dimension 2, this is easy to see from the fact that

$$\mathcal{C} \otimes \mathcal{K} \simeq \mathcal{C} \times \mathcal{C} \text{ as algebras,}$$

and by Theorem 4.3.7 group-gradings on $\mathcal{C} \times \mathcal{C}$ are in correspondence with group-gradings on $\mathcal{C} \otimes \mathcal{C}$. So, if group-gradings on the structurable algebra $(\mathcal{C} \otimes \mathcal{K}, - \otimes -)$ would not depend on the involution we would have a correspondence between gradings on $(\mathcal{C} \otimes \mathcal{K}, - \otimes -)$ and $(\mathcal{C} \otimes \mathcal{C}, - \otimes -)$. Theorem 5.2.6 shows that this is not possible.

Remark 5.2.4. *It is straightforward to prove that the Hurwitz algebras of dimension 4 and 8 are simple. Moreover, since $\operatorname{char} \mathbb{F} \neq 2$, for the Hurwitz algebra $\mathcal{C}$ where $\dim(\mathcal{C}) = 4, 8$ we have that $\mathcal{C}_0$ (see Definition 4.1.2) is a simple subalgebra of $\mathcal{C}$ under the product given by the commutator $[\ ,\ ]$.*

Notice that the tensor product of a Cayley algebra $(\mathcal{C}, -)$ and the field $\mathbb{F}$ is isomorphic (as algebras with involution) to $(\mathcal{C}, -)$ and we already know the group-gradings on Cayley algebras (Chapter 4, Section 2). The next result (Corollary 4.25 of [EK13]) will be used in the characterization of group-gradings on the tensor product of Hurwitz algebras.

Corollary 5.2.5. *Let $\mathcal{C}$ be a Hurwitz algebra with $\dim \mathcal{C} \geq 4$ over a field $\mathbb{F}$, $\operatorname{char} \mathbb{F} \neq 2$. Let $\Gamma : \mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ be a group-grading on $\mathcal{C}$, then $\mathcal{C}_g$ is contained in $\mathcal{C}_0$ for any $g \neq e$ and $\Gamma' : \mathcal{C}_0 = \bigoplus_{g \in G} (\mathcal{C}_0 \cap \mathcal{C}_g)$ is a group-grading on the anticommutative algebra $\mathcal{C}_0$.*

Conversely, let $\Gamma' : \mathcal{C}_0 = \bigoplus_{g\in G}(\mathcal{C}_0)_g$ *be a group-grading on the anticommutative algebra* $\mathcal{C}_0$*, then with* $\mathcal{C}_e := \mathbb{F}1 \oplus (\mathcal{C}_0)_e$ *and* $\mathcal{C}_g := (\mathcal{C}_0)_g$ *for* $g \neq e$*, the decomposition* $\Gamma : \mathcal{C} = \bigoplus_{g\in G}\mathcal{C}_g$ *is a group-grading on* $\mathcal{C}$*.*

Moreover, two gradings Γ_1 *and* Γ_2 *on* $\mathcal{C}$ *are isomorphic (respectively, equivalent) if and only if so are the gradings* Γ_1' *and* Γ_2' *on* $\mathcal{C}_0$*.*

Next theorem tells us what the group-gradings on $\mathcal{C}^1 \otimes \mathcal{C}^2$ look like, where $\mathcal{C}^1$ is a Cayley algebra and $\mathcal{C}^2$ is a Hurwitz algebra of dimension 2 or 4.

Theorem 5.2.6. *Let* $(\mathcal{A}, -) = (\mathcal{C}^1 \otimes \mathcal{C}^2, -)$ *be the algebra with involution where* $\mathcal{C}^1$ *and* $\mathcal{C}^2$ *are Hurwitz algebras such that* $\dim(\mathcal{C}^1) = 8$ *and* $\dim(\mathcal{C}^2) = 2$ *or* 4 *and* $-$ *denotes the tensor product of the involutions in* $\mathcal{C}^1$ *and* $\mathcal{C}^2$*. Then* $\Gamma : \mathcal{A} = \oplus_{g\in G}\mathcal{A}_g$ *is a* G*-grading on* $(\mathcal{A}, -)$ *if and only if there exist* G*-gradings*

$$\Gamma^1 : \mathcal{C}^1 = \bigoplus_{g\in G}(\mathcal{C}^1)_g \;\; \textit{and} \;\; \Gamma^2 : \mathcal{C}^2 = \bigoplus_{g\in G}(\mathcal{C}^2)_g$$

such that for all $g \in G$

$$\mathcal{A}_g = \bigoplus_{g_1,g_2\in G: g_1g_2=g}(\mathcal{C}^1)_{g_1} \otimes (\mathcal{C}^2)_{g_2}.$$

Moreover, two G*-gradings* Γ *and* Γ' *on* $(\mathcal{A}, -)$ *are isomorphic if and only if so are* Γ^1 *and* $(\Gamma')^1$ *on* $\mathcal{C}^1$ *and* Γ^2 *and* $(\Gamma')^2$ *on* $\mathcal{C}^2$*.*

Proof. We have that $S(\mathcal{A}, -) = S(\mathcal{C}^1 \otimes \mathcal{C}^2, -) = \mathcal{C}_0^1 \otimes 1 \oplus 1 \otimes \mathcal{C}_0^2$ (see definitions 4.1.2 and 5.1.1).
$\Rightarrow$) By Remark 5.2.4 and Lemma 5.2.1 b) we get that $(\mathcal{S}, -) := S(\mathcal{A}, -)$ is a G-graded subalgebra of $(\mathcal{A}, -)$ with the product given by the commutator. Set

$$\Gamma_{\mathcal{S}} := \Gamma|_{\mathcal{S}} : \mathcal{S} = \bigoplus_{g\in G}(\mathcal{A}_g \cap \mathcal{S}).$$

Suppose $\dim(\mathcal{C}^2) = 2$. We will prove that $[\mathcal{S}, \mathcal{S}] = \mathcal{C}_0^1 \otimes 1$ and $Z(\mathcal{S}) = 1 \otimes \mathcal{C}_0^2$. We have

$$\begin{aligned}[\mathcal{S}, \mathcal{S}] =&[\mathcal{C}_0^1 \otimes 1 \oplus 1 \otimes \mathcal{C}_0^2, \mathcal{C}_0^1 \otimes 1 \oplus 1 \otimes \mathcal{C}_0^2]\\ =&[\mathcal{C}_0^1 \otimes 1, \mathcal{C}_0^1 \otimes 1] + [\mathcal{C}_0^1 \otimes 1, 1 \otimes \mathcal{C}_0^2] + [1 \otimes \mathcal{C}_0^2, \mathcal{C}_0^1 \otimes 1] + [1 \otimes \mathcal{C}_0^2, 1 \otimes \mathcal{C}_0^2]\\ =&[\mathcal{C}_0^1, \mathcal{C}_0^1] \otimes 1 + 1 \otimes [\mathcal{C}_0^2, \mathcal{C}_0^2]\\ =&\mathcal{C}_0^1 \otimes 1\end{aligned}$$

where the last equality follows from the fact that $[\mathcal{C}_0^2, \mathcal{C}_0^2] = \{0\}$ because $\dim(\mathcal{C}_0^2) = 1$. Take $a = a_1 \otimes 1 + 1 \otimes a_2 \in \mathcal{S}$ with $a_i \in \mathcal{C}_0^i$ for $i = 1, 2$, then

$$\begin{aligned} a \in Z(\mathcal{S}) &\Leftrightarrow [a, s] = 0 \text{ for all } s \in \mathcal{S}\\ &\Leftrightarrow [a_1 \otimes 1 + 1 \otimes a_2, s_1 \otimes 1 + 1 \otimes s_2] = 0 \text{ for all } s_i \in \mathcal{C}_0^i \text{ for } i = 1, 2\\ &\Leftrightarrow [a_1 \otimes 1, s_1 \otimes 1] = 0 \text{ for all } s_1 \in \mathcal{C}_0^1.\end{aligned}$$

Since $\mathcal{C}_0^1$ is simple, $Z(\mathcal{C}_0^1) = \{0\}$. Therefore $Z(\mathcal{S}) = 1 \otimes \mathcal{C}_0^2$. By Lemma 5.2.1 c) and d) $[\mathcal{S}, \mathcal{S}] = \mathcal{C}_0^1 \otimes 1$ and $Z(S) = 1 \otimes \mathcal{C}_0^2$ are G-graded subspaces of $\mathcal{S}$ with the group-gradings induced by $\Gamma_{\mathcal{S}}$

$$\Gamma_{\mathcal{C}_0^1 \otimes 1} := \Gamma_{\mathcal{S}}|_{\mathcal{C}_0^1 \otimes 1} \text{ and } \Gamma_{1 \otimes \mathcal{C}_0^2} := \Gamma_{\mathcal{S}}|_{1 \otimes \mathcal{C}_0^2}.$$

Consider the following isomorphisms

$$1) \quad \begin{array}{rccc} \varphi_1 : & \mathcal{C}_0^1 \otimes 1 & \to & \mathcal{C}_0^1 \\ & x \otimes 1 & \mapsto & x \end{array} \quad \text{and} \quad \begin{array}{rccc} \varphi_2 : & 1 \otimes \mathcal{C}_0^2 & \to & \mathcal{C}_0^2 \\ & 1 \otimes y & \mapsto & y. \end{array}$$

Then we have group-gradings on $\mathcal{C}_0^1$ and $\mathcal{C}_0^2$ given by

$$2) \quad \Gamma_{\mathcal{C}_0^1} : \mathcal{C}_0^1 = \bigoplus_{g \in G} (\mathcal{C}_0^1)_g, \text{ where } (\mathcal{C}_0^1)_g = \varphi_1((\mathcal{C}_0^1 \otimes 1)_g)$$

and

$$3) \quad \Gamma_{\mathcal{C}_0^2} : \mathcal{C}_0^2 = \bigoplus_{g \in G} (\mathcal{C}_0^2)_g, \text{ where } (\mathcal{C}_0^2)_g = \varphi_2((1 \otimes \mathcal{C}_0^2)_g).$$

By Corollary 5.2.5 we have that the decomposition

$$\Gamma_{\mathcal{C}^1} : \mathcal{C}^1 = \bigoplus_{g \in G} (\mathcal{C}^1)_g$$

where $(\mathcal{C}^1)_e := \mathbb{F}1 \oplus (\mathcal{C}_0^1)_e$ and $(\mathcal{C}^1)_g := (\mathcal{C}_0^1)_g$ for $g \neq e$, where e is the neutral element of G, is a G-grading on $\mathcal{C}^1$. We have the G-grading on $\mathcal{C}^2$

$$\Gamma_{\mathcal{C}^2} : \mathcal{C}^2 = \bigoplus_{g \in G} (\mathcal{C}^2)_g$$

given by $(\mathcal{C}^2)_e = \mathbb{F}1$ and $(\mathcal{C}^2)_g = (\mathcal{C}_0^2)_g$, notice that $g^2 = e$ for $g \in \operatorname{Supp} \Gamma_{\mathcal{C}_0^2}$.

Now suppose $\dim(\mathcal{C}^2) = 4$. We will prove that $J(\mathcal{S}, \mathcal{S}, \mathcal{S}) = \mathcal{C}_0^1 \otimes 1$ and $\mathcal{D} := \{x \in \mathcal{S} : J(x, \mathcal{S}, \mathcal{S}) = \{0\}\} = 1 \otimes \mathcal{C}_0^2$.

For $x = x_1 \otimes 1 + 1 \otimes x_2, y = y_1 \otimes 1 + 1 \otimes y_2, z = z_1 \otimes 1 + 1 \otimes z_2 \in \mathcal{S}$ where $x_1, y_1, z_1 \in \mathcal{C}_0^1$ and $x_2, y_2, z_2 \in \mathcal{C}_0^2$ we have

$$\begin{aligned}
&[[x, y], z] + [[z, x], y] + [[y, z], x] \\
&\quad = [[x_1 \otimes 1 + 1 \otimes x_2, y_1 \otimes 1 + 1 \otimes y_2], z_1 \otimes 1 + 1 \otimes z_2] \\
&\qquad + [[z_1 \otimes 1 + 1 \otimes z_2, x_1 \otimes 1 + 1 \otimes x_2], y_1 \otimes 1 + 1 \otimes y_2] \\
&\qquad + [[y_1 \otimes 1 + 1 \otimes y_2, z_1 \otimes 1 + 1 \otimes z_2], x_1 \otimes 1 + 1 \otimes x_2] \\
&\quad = ([[x_1, y_1], z_1] + [[z_1, x_1], y_1] + [[y_1, z_1], x_1]) \otimes 1 \\
&\qquad + 1 \otimes ([[x_2, y_2], z_2] + [[z_2, x_2], y_2] + [[y_2, z_2], x_2]).
\end{aligned}$$

We have that $J(\mathcal{C}_0^1,\mathcal{C}_0^1,\mathcal{C}_0^1)=\mathcal{C}_0^1$ because $\mathcal{C}_0^1$ is simple under the product given by the commutator and it is not a Lie algebra. It is clear that $J(\mathcal{C}_0^2,\mathcal{C}_0^2,\mathcal{C}_0^2)=\{0\}$. Therefore $J(\mathcal{S},\mathcal{S},\mathcal{S})=\mathcal{C}_0^1\otimes 1$. If we take $x,y,z\in\mathcal{S}$ we have

$$\begin{aligned} x\in\mathcal{D} &\Leftrightarrow [[x,y],z]+[[z,x],y]+[[y,z],x]=0 \text{ for all } y,z\in\mathcal{S}\\ &\Leftrightarrow ([[x_1,y_1],z_1]+[[z_1,x_1],y_1]+[[y_1,z_1],x_1])\otimes 1 \text{ for all } y_1,z_1\in\mathcal{C}_0^1\\ &\Leftrightarrow\ x_1=0\ . \end{aligned}$$

Therefore $\mathcal{D}=1\otimes\mathcal{C}_0^2$. By Lemma 5.2.1 e) and f) we have that $J(\mathcal{S},\mathcal{S},\mathcal{S})=\mathcal{C}_0^1\otimes 1$ and $\mathcal{D}=1\otimes\mathcal{C}_0^2$ are G-graded subspaces of $\mathcal{S}$

$$\Gamma_{\mathcal{C}_0^1\otimes 1}:=\Gamma_{\mathcal{S}}|_{\mathcal{C}_0^1\otimes 1} \text{ and } \Gamma_{1\otimes\mathcal{C}_0^2}:=\Gamma_{\mathcal{S}}|_{1\otimes\mathcal{C}_0^2}.$$

Consider again the isomorphisms φ_1 and φ_2 from 1) and we have the G-gradings on $\mathcal{C}_0^1\otimes 1$ and $1\otimes\mathcal{C}_0^2$ given by 2) and 3), respectively. By [EK13, Corollary 4.25] we have that the decomposition

$$\Gamma_{\mathcal{C}^i}:\mathcal{C}^i=\bigoplus_{g\in G}(\mathcal{C}^i)_g$$

where $(\mathcal{C}^i)_e:=\mathbb{F}1\oplus(\mathcal{C}_0^i)_e$ and $(\mathcal{C}^i)_g:=(\mathcal{C}_0^i)_g$ for $g\neq e$ is a G-grading on $\mathcal{C}^i$ for $i=1,2$.

We will prove that, effectively

$$(\mathcal{C}^1\otimes\mathcal{C}^2)_g=\bigoplus_{g_1,g_2\in G:g_1g_2=g}(\mathcal{C}^1)_{g_1}\otimes(\mathcal{C}^2)_{g_2}.$$

We have for $h,k\in G$:

- $(\mathcal{C}^1)_h\otimes(\mathcal{C}^2)_k=(\mathcal{C}_0^1)_h\otimes(\mathcal{C}_0^2)_k=(\mathcal{C}_0^1\otimes 1)_h(1\otimes\mathcal{C}_0^2)_k\subseteq(\mathcal{C}^1\otimes\mathcal{C}^2)_h(\mathcal{C}^1\otimes\mathcal{C}^2)_k\subseteq(\mathcal{C}^1\otimes\mathcal{C}^2)_{hk}$, if $h\neq e\neq k$;

- $(\mathcal{C}^1)_h\otimes(\mathcal{C}^2)_k=(\mathbb{F}1\oplus(\mathcal{C}_0^1)_e)\otimes(\mathcal{C}_0^2)_k=((\mathbb{F}1\oplus(\mathcal{C}_0^1)_e)\otimes 1)(1\otimes\mathcal{C}_0^2)_k=((\mathbb{F}1\otimes 1)\oplus((\mathcal{C}_0^1)_e)\otimes 1))(1\otimes\mathcal{C}_0^2)_k\subseteq(\mathcal{C}^1\otimes\mathcal{C}^2)_e(\mathcal{C}^1\otimes\mathcal{C}^2)_k\subseteq(\mathcal{C}^1\otimes\mathcal{C}^2)_k$, if $h=e$ and $k\neq e$;

- $(\mathcal{C}^1)_h\otimes(\mathcal{C}^2)_k=(\mathcal{C}_0^1)_h\otimes(\mathbb{F}1\oplus(\mathcal{C}_0^2)_e)=(\mathcal{C}_0^1\otimes 1)_h(1\otimes(\mathbb{F}1\oplus(\mathcal{C}_0^2)_e))=(\mathcal{C}_0^1\otimes 1)_h((1\otimes\mathbb{F}1)\oplus(1\otimes(\mathcal{C}_0^2)_e)))\subseteq(\mathcal{C}^1\otimes\mathcal{C}^2)_h(\mathcal{C}^1\otimes\mathcal{C}^2)_e\subseteq(\mathcal{C}^1\otimes\mathcal{C}^2)_h$ if $h\neq e$ and $k=e$.

Therefore $(\mathcal{C}^1)_h\otimes(\mathcal{C}^2)_k\subseteq(\mathcal{C}^1\otimes\mathcal{C}^2)_{hk}$. Since

$$\bigoplus_{h,k\in G}(\mathcal{C}^1)_h\otimes(\mathcal{C}^2)_k=\mathcal{C}^1\otimes\mathcal{C}^2=\bigoplus_{g\in G}(\mathcal{C}^1\otimes\mathcal{C}^2)_g$$

and

$$\bigoplus_{h,k\in G:hk=g} (\mathcal{C}^1)_h \otimes (\mathcal{C}^2)_k \subseteq (\mathcal{C}^1 \otimes \mathcal{C}^2)_g,$$

we get that

$$\mathcal{A}_g = \bigoplus_{g_1,g_2\in G:g_1g_2=g} (\mathcal{C}^1)_{g_1} \otimes (\mathcal{C}^2)_{g_2}.$$

$\Leftarrow$) We saw in Chapter 1, Section 1 that this defines a group-grading on a tensor product of algebras.

The last part is clear, because any G-grading Γ determines and is determined by the G-gradings $\Gamma_{\mathcal{C}^i}$, $i = 1, 2$. □

From now on we will not write down the involution of the Hurwitz algebras (recall Remark 5.2.3).

In order to give a more explicit description of group-gradings above we will recall the fine group-gradings on a Hurwitz algebra of dimension 2 and 4.

Remark 5.2.7. *By Remark 4.2.6 the fine gradings, up to equivalence, on a quaternion algebra $\mathcal{H}$ are:*

- *The Cartan grading on the split quaternion algebra over its universal group $\mathbb{Z}$. In this case $\mathcal{H}$ has a basis $\{e_1, e_2, u, v\}$ with multiplication table*

	e_1	e_2	u	v
e_1	e_1	0	u	0
e_2	0	e_2	0	v
u	0	u	0	$-e_1$
v	v	0	$-e_2$	0

  and the homogeneous components are given by

$$\mathcal{H}_0 = \mathbb{F}e_1 \oplus \mathbb{F}e_2, \quad \mathcal{H}_1 = \mathbb{F}u, \quad \mathcal{H}_{-1} = \mathbb{F}v. \tag{5.2.1}$$

- *The $(\mathbb{Z}/2)^2$-grading induced by the Cayley-Dickson doubling process. In this case $\mathcal{H} = CD(Q, \alpha) = Q \oplus Qv$ where $Q = CD(\mathbb{F}, \beta) = \mathbb{F}1 \oplus \mathbb{F}u$ and $\alpha, \beta \in \mathbb{F}$. The homogeneous components are given by*

$$\mathcal{H}_{(\bar{0},\bar{0})} = \mathbb{F}1, \quad \mathcal{H}_{(\bar{1},\bar{0})} = \mathbb{F}v, \quad \mathcal{H}_{(\bar{0},\bar{1})} = \mathbb{F}u, \quad \mathcal{H}_{(\bar{1},\bar{1})} = \mathbb{F}vu. \tag{5.2.2}$$

And the only nontrivial grading on a Hurwitz algebra $\mathcal{L}$ of dimension 2, up to equivalence, is the one induced by the Cayley-Dickson doubling process by $\mathbb{Z}/2$. In this case $\mathcal{L} = CD(\mathbb{F}, \alpha) = \mathbb{F}1 \oplus \mathbb{F}u$ for $\alpha \in \mathbb{F}$ and the homogeneous components are given by

$$\mathcal{L}_{\bar{0}} = \mathbb{F}1, \quad \mathcal{L}_{\bar{1}} = \mathbb{F}u. \tag{5.2.3}$$

The following two group-gradings are the only fine group-gradings, up to equivalence, on the tensor product of a Cayley algebra and a Hurwitz algebra of dimension 2. This follows from Theorem 5.2.6 and the fact that they are gradings by their universal groups.

1. The $(\mathbb{Z}/2)^4$-grading induced by the $(\mathbb{Z}/2)^3$-grading induced by the Cayley-Dickson doubling process on $\mathcal{C}$ (with basis $\{1, w, v, vw, u, uw, vu, (wv)u\}$ and homogeneous components given by Equation (4.2.1)) and the $\mathbb{Z}/2$-grading induced by the Cayley-Dickson doubling process on $\mathcal{L} = CD(\mathbb{F}, \alpha)$ $= \mathbb{F}1 \oplus \mathbb{F}u'$ for $\alpha \in \mathbb{F}$ (with basis $\{1, u'\}$ and homogeneous components given by Equation (5.2.3)), with homogeneous components given by

$$\begin{array}{ll}
(\mathcal{C}\otimes\mathcal{L})_{(\bar{0},\bar{0},\bar{0},\bar{0})} = \mathbb{F}1\otimes\mathbb{F}1, & (\mathcal{C}\otimes\mathcal{L})_{(\bar{1},\bar{0},\bar{0},\bar{0})} = \mathbb{F}u\otimes\mathbb{F}1, \\
(\mathcal{C}\otimes\mathcal{L})_{(\bar{0},\bar{0},\bar{0},\bar{1})} = \mathbb{F}1\otimes\mathbb{F}u', & (\mathcal{C}\otimes\mathcal{L})_{(\bar{1},\bar{0},\bar{0},\bar{1})} = \mathbb{F}u\otimes\mathbb{F}u', \\
(\mathcal{C}\otimes\mathcal{L})_{(\bar{0},\bar{0},\bar{1},\bar{0})} = \mathbb{F}w\otimes\mathbb{F}1, & (\mathcal{C}\otimes\mathcal{L})_{(\bar{1},\bar{0},\bar{1},\bar{0})} = \mathbb{F}wu\otimes\mathbb{F}1, \\
(\mathcal{C}\otimes\mathcal{L})_{(\bar{0},\bar{0},\bar{1},\bar{1})} = \mathbb{F}w\otimes\mathbb{F}u', & (\mathcal{C}\otimes\mathcal{L})_{(\bar{1},\bar{0},\bar{1},\bar{1})} = \mathbb{F}wu\otimes\mathbb{F}u', \\
(\mathcal{C}\otimes\mathcal{L})_{(\bar{0},\bar{1},\bar{0},\bar{0})} = \mathbb{F}v\otimes\mathbb{F}1, & (\mathcal{C}\otimes\mathcal{L})_{(\bar{1},\bar{1},\bar{0},\bar{0})} = \mathbb{F}vu\otimes\mathbb{F}1, \\
(\mathcal{C}\otimes\mathcal{L})_{(\bar{0},\bar{1},\bar{0},\bar{1})} = \mathbb{F}v\otimes\mathbb{F}u', & (\mathcal{C}\otimes\mathcal{L})_{(\bar{1},\bar{1},\bar{0},\bar{1})} = \mathbb{F}vu\otimes\mathbb{F}u', \\
(\mathcal{C}\otimes\mathcal{L})_{(\bar{0},\bar{1},\bar{1},\bar{0})} = \mathbb{F}wv\otimes\mathbb{F}1, & (\mathcal{C}\otimes\mathcal{L})_{(\bar{1},\bar{1},\bar{1},\bar{0})} = \mathbb{F}(wv)u\otimes\mathbb{F}1, \\
(\mathcal{C}\otimes\mathcal{L})_{(\bar{0},\bar{1},\bar{1},\bar{1})} = \mathbb{F}wv\otimes\mathbb{F}u', & (\mathcal{C}\otimes\mathcal{L})_{(\bar{1},\bar{1},\bar{1},\bar{1})} = \mathbb{F}(wv)u\otimes\mathbb{F}u'.
\end{array}$$

2. The $\mathbb{Z}^2 \times \mathbb{Z}/2$-grading induced by the Cartan $\mathbb{Z}^2$-grading on $\mathcal{C}$ (with basis $\{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$ and homogeneous components given by Equation (4.2.2)) and the $\mathbb{Z}/2$-grading induced by the Cayley-Dickson doubling process on $\mathcal{L} = CD(\mathbb{F}, \alpha) = \mathbb{F}1 \oplus \mathbb{F}u'$ for $\alpha \in \mathbb{F}$ (with basis $\{1, u'\}$ and homogeneous components given by Equation (5.2.3)), with homogeneous components given by

$$\begin{array}{ll}
(\mathcal{C}\otimes\mathcal{L})_{(0,0,\bar{0})} = (\mathbb{F}e_1\oplus\mathbb{F}e_2)\otimes\mathbb{F}1, & (\mathcal{C}\otimes\mathcal{L})_{(1,1,\bar{1})} = \mathbb{F}v_3\otimes\mathbb{F}u', \\
(\mathcal{C}\otimes\mathcal{L})_{(0,0,\bar{1})} = (\mathbb{F}e_1\oplus\mathbb{F}e_2)\otimes\mathbb{F}u', & (\mathcal{C}\otimes\mathcal{L})_{(-1,0,\bar{0})} = \mathbb{F}v_1\otimes\mathbb{F}1, \\
(\mathcal{C}\otimes\mathcal{L})_{(1,0,\bar{0})} = \mathbb{F}u_1\otimes\mathbb{F}1, & (\mathcal{C}\otimes\mathcal{L})_{(-1,0,\bar{1})} = \mathbb{F}v_1\otimes\mathbb{F}u', \\
(\mathcal{C}\otimes\mathcal{L})_{(1,0,\bar{1})} = \mathbb{F}u_1\otimes\mathbb{F}u', & (\mathcal{C}\otimes\mathcal{L})_{(0,-1,\bar{0})} = \mathbb{F}v_2\otimes\mathbb{F}1, \\
(\mathcal{C}\otimes\mathcal{L})_{(0,1,\bar{0})} = \mathbb{F}u_2\otimes\mathbb{F}1, & (\mathcal{C}\otimes\mathcal{L})_{(0,-1,\bar{1})} = \mathbb{F}v_2\otimes\mathbb{F}u', \\
(\mathcal{C}\otimes\mathcal{L})_{(0,1,\bar{1})} = \mathbb{F}u_2\otimes\mathbb{F}u', & (\mathcal{C}\otimes\mathcal{L})_{(-1,-1,\bar{0})} = \mathbb{F}u_3\otimes\mathbb{F}1, \\
(\mathcal{C}\otimes\mathcal{L})_{(1,1,\bar{0})} = \mathbb{F}v_3\otimes\mathbb{F}1, & (\mathcal{C}\otimes\mathcal{L})_{(-1,-1,\bar{1})} = \mathbb{F}u_3\otimes\mathbb{F}u'.
\end{array}$$

The following four group-gradings are the only fine group-gradings, up to equivalence, on the tensor product of a Cayley algebra and a Hurwitz algebra of dimension 4. This follows from Theorem 5.2.6 and the fact that they are gradings by their universal groups.

1. The $(\mathbb{Z}/2)^5$-grading induced by the $(\mathbb{Z}/2)^3$-grading induced by the Cayley-Dickson doubling process on $\mathcal{C}$ (with basis $\{1, w, v, vw, u, uw, vu, (wv)u\}$ and homogeneous components given by Equation (4.2.1)) and the $(\mathbb{Z}/2)^2$-grading induced by the Cayley-Dickson doubling process on $\mathcal{H}$ (with basis $\{1, v', u', v'u'\}$ and homogeneous components given by (5.2.2)), with homogeneous components given by

$$\begin{array}{ll}
(\mathcal{C}\otimes\mathcal{H})_{(\bar 0,\bar 0,\bar 0,\bar 0,\bar 0)} = \mathbb{F}1\otimes\mathbb{F}1, & (\mathcal{C}\otimes\mathcal{H})_{(\bar 1,\bar 0,\bar 0,\bar 0,\bar 0)} = \mathbb{F}u\otimes\mathbb{F}1,\\
(\mathcal{C}\otimes\mathcal{H})_{(\bar 0,\bar 0,\bar 0,\bar 1,\bar 0)} = \mathbb{F}1\otimes\mathbb{F}v', & (\mathcal{C}\otimes\mathcal{H})_{(\bar 1,\bar 0,\bar 0,\bar 1,\bar 0)} = \mathbb{F}u\otimes\mathbb{F}v',\\
(\mathcal{C}\otimes\mathcal{H})_{(\bar 0,\bar 0,\bar 0,\bar 0,\bar 1)} = \mathbb{F}1\otimes\mathbb{F}u', & (\mathcal{C}\otimes\mathcal{H})_{(\bar 1,\bar 0,\bar 0,\bar 0,\bar 1)} = \mathbb{F}u\otimes\mathbb{F}u',\\
(\mathcal{C}\otimes\mathcal{H})_{(\bar 0,\bar 0,\bar 0,\bar 1,\bar 1)} = \mathbb{F}1\otimes\mathbb{F}v'u', & (\mathcal{C}\otimes\mathcal{H})_{(\bar 1,\bar 0,\bar 0,\bar 1,\bar 1)} = \mathbb{F}u\otimes\mathbb{F}v'u',\\
(\mathcal{C}\otimes\mathcal{H})_{(\bar 0,\bar 0,\bar 1,\bar 0,\bar 0)} = \mathbb{F}w\otimes\mathbb{F}1, & (\mathcal{C}\otimes\mathcal{H})_{(\bar 1,\bar 0,\bar 1,\bar 0,\bar 0)} = \mathbb{F}wu\otimes\mathbb{F}1,\\
(\mathcal{C}\otimes\mathcal{H})_{(\bar 0,\bar 0,\bar 1,\bar 1,\bar 0)} = \mathbb{F}w\otimes\mathbb{F}v', & (\mathcal{C}\otimes\mathcal{H})_{(\bar 1,\bar 0,\bar 1,\bar 1,\bar 0)} = \mathbb{F}wu\otimes\mathbb{F}v',\\
(\mathcal{C}\otimes\mathcal{H})_{(\bar 0,\bar 0,\bar 1,\bar 0,\bar 1)} = \mathbb{F}w\otimes\mathbb{F}u', & (\mathcal{C}\otimes\mathcal{H})_{(\bar 1,\bar 0,\bar 1,\bar 0,\bar 1)} = \mathbb{F}wu\otimes\mathbb{F}u',\\
(\mathcal{C}\otimes\mathcal{H})_{(\bar 0,\bar 0,\bar 1,\bar 1,\bar 1)} = \mathbb{F}w\otimes\mathbb{F}v'u', & (\mathcal{C}\otimes\mathcal{H})_{(\bar 1,\bar 0,\bar 1,\bar 1,\bar 1)} = \mathbb{F}wu\otimes\mathbb{F}v'u',\\
(\mathcal{C}\otimes\mathcal{H})_{(\bar 0,\bar 1,\bar 0,\bar 0,\bar 0)} = \mathbb{F}v\otimes\mathbb{F}1, & (\mathcal{C}\otimes\mathcal{H})_{(\bar 1,\bar 1,\bar 0,\bar 0,\bar 0)} = \mathbb{F}vu\otimes\mathbb{F}1,\\
(\mathcal{C}\otimes\mathcal{H})_{(\bar 0,\bar 1,\bar 0,\bar 1,\bar 0)} = \mathbb{F}v\otimes\mathbb{F}v', & (\mathcal{C}\otimes\mathcal{H})_{(\bar 1,\bar 1,\bar 0,\bar 1,\bar 0)} = \mathbb{F}vu\otimes\mathbb{F}v',\\
(\mathcal{C}\otimes\mathcal{H})_{(\bar 0,\bar 1,\bar 0,\bar 0,\bar 1)} = \mathbb{F}v\otimes\mathbb{F}u', & (\mathcal{C}\otimes\mathcal{H})_{(\bar 1,\bar 1,\bar 0,\bar 0,\bar 1)} = \mathbb{F}vu\otimes\mathbb{F}u',\\
(\mathcal{C}\otimes\mathcal{H})_{(\bar 0,\bar 1,\bar 0,\bar 1,\bar 1)} = \mathbb{F}v\otimes\mathbb{F}v'u', & (\mathcal{C}\otimes\mathcal{H})_{(\bar 1,\bar 1,\bar 0,\bar 1,\bar 1)} = \mathbb{F}vu\otimes\mathbb{F}v'u',\\
(\mathcal{C}\otimes\mathcal{H})_{(\bar 0,\bar 1,\bar 1,\bar 0,\bar 0)} = \mathbb{F}wv\otimes\mathbb{F}1, & (\mathcal{C}\otimes\mathcal{H})_{(\bar 1,\bar 1,\bar 1,\bar 0,\bar 0)} = \mathbb{F}(wv)u\otimes\mathbb{F}1,\\
(\mathcal{C}\otimes\mathcal{H})_{(\bar 0,\bar 1,\bar 1,\bar 1,\bar 0)} = \mathbb{F}wv\otimes\mathbb{F}v', & (\mathcal{C}\otimes\mathcal{H})_{(\bar 1,\bar 1,\bar 1,\bar 1,\bar 0)} = \mathbb{F}(wv)u\otimes\mathbb{F}v',\\
(\mathcal{C}\otimes\mathcal{H})_{(\bar 0,\bar 1,\bar 1,\bar 0,\bar 1)} = \mathbb{F}wv\otimes\mathbb{F}u', & (\mathcal{C}\otimes\mathcal{H})_{(\bar 1,\bar 1,\bar 1,\bar 0,\bar 1)} = \mathbb{F}(wv)u\otimes\mathbb{F}u',\\
(\mathcal{C}\otimes\mathcal{H})_{(\bar 0,\bar 1,\bar 1,\bar 1,\bar 1)} = \mathbb{F}wv\otimes\mathbb{F}v'u', & (\mathcal{C}\otimes\mathcal{H})_{(\bar 1,\bar 1,\bar 1,\bar 1,\bar 1)} = \mathbb{F}(wv)u\otimes\mathbb{F}v'u'.
\end{array}$$

2. The $(\mathbb{Z}/2)^3\times\mathbb{Z}$-grading induced by the $(\mathbb{Z}/2)^3$-grading induced by the Cayley-Dickson doubling process on $\mathcal{C}$ (with basis $\{1, w, v, vw, u, uw, vu, (wv)u\}$ and homogeneous components given by Equation (4.2.1)) and the Cartan $\mathbb{Z}$-grading on $\mathcal{H}$ (with basis $\{e_1, e_2, u', v'\}$ and homogeneous components given by Equation (5.2.1)), with homogeneous components given

by

$$
\begin{aligned}
&(\mathcal{C}\otimes\mathcal{H})_{(\bar 0,\bar 0,\bar 0,0)} = \mathbb{F}1\otimes(\mathbb{F}e_1\otimes\mathbb{F}e_2), && (\mathcal{C}\otimes\mathcal{H})_{(\bar 1,\bar 0,\bar 0,1)} = \mathbb{F}u\otimes\mathbb{F}u',\\
&(\mathcal{C}\otimes\mathcal{H})_{(\bar 0,\bar 0,\bar 1,0)} = \mathbb{F}w\otimes(\mathbb{F}e_1\otimes\mathbb{F}e_2), && (\mathcal{C}\otimes\mathcal{H})_{(\bar 1,\bar 0,\bar 1,1)} = \mathbb{F}wu\otimes\mathbb{F}u',\\
&(\mathcal{C}\otimes\mathcal{H})_{(\bar 0,\bar 1,\bar 0,0)} = \mathbb{F}v\otimes(\mathbb{F}e_1\otimes\mathbb{F}e_2), && (\mathcal{C}\otimes\mathcal{H})_{(\bar 1,\bar 1,\bar 0,1)} = \mathbb{F}vu\otimes\mathbb{F}u',\\
&(\mathcal{C}\otimes\mathcal{H})_{(\bar 0,\bar 1,\bar 1,0)} = \mathbb{F}wv\otimes(\mathbb{F}e_1\otimes\mathbb{F}e_2), && (\mathcal{C}\otimes\mathcal{H})_{(\bar 1,\bar 1,\bar 1,1)} = \mathbb{F}(wv)u\otimes\mathbb{F}u',\\
&(\mathcal{C}\otimes\mathcal{H})_{(\bar 1,\bar 0,\bar 0,0)} = \mathbb{F}u\otimes(\mathbb{F}e_1\otimes\mathbb{F}e_2), && (\mathcal{C}\otimes\mathcal{H})_{(\bar 0,\bar 0,\bar 0,-1)} = \mathbb{F}1\otimes\mathbb{F}v',\\
&(\mathcal{C}\otimes\mathcal{H})_{(\bar 1,\bar 0,\bar 1,0)} = \mathbb{F}wu\otimes(\mathbb{F}e_1\otimes\mathbb{F}e_2), && (\mathcal{C}\otimes\mathcal{H})_{(\bar 0,\bar 0,\bar 1,-1)} = \mathbb{F}w\otimes\mathbb{F}v',\\
&(\mathcal{C}\otimes\mathcal{H})_{(\bar 1,\bar 1,\bar 0,0)} = \mathbb{F}vu\otimes(\mathbb{F}e_1\otimes\mathbb{F}e_2), && (\mathcal{C}\otimes\mathcal{H})_{(\bar 0,\bar 1,\bar 0,-1)} = \mathbb{F}v\otimes\mathbb{F}v',\\
&(\mathcal{C}\otimes\mathcal{H})_{(\bar 1,\bar 1,\bar 1,0)} = \mathbb{F}(wv)u\otimes(\mathbb{F}e_1\otimes\mathbb{F}e_2), && (\mathcal{C}\otimes\mathcal{H})_{(\bar 0,\bar 1,\bar 1,-1)} = \mathbb{F}wv\otimes\mathbb{F}v',\\
&(\mathcal{C}\otimes\mathcal{H})_{(\bar 0,\bar 0,\bar 0,1)} = \mathbb{F}1\otimes\mathbb{F}u', && (\mathcal{C}\otimes\mathcal{H})_{(\bar 1,\bar 0,\bar 0,-1)} = \mathbb{F}u\otimes\mathbb{F}v',\\
&(\mathcal{C}\otimes\mathcal{H})_{(\bar 0,\bar 0,\bar 1,1)} = \mathbb{F}w\otimes\mathbb{F}u', && (\mathcal{C}\otimes\mathcal{H})_{(\bar 1,\bar 0,\bar 1,-1)} = \mathbb{F}wu\otimes\mathbb{F}v',\\
&(\mathcal{C}\otimes\mathcal{H})_{(\bar 0,\bar 1,\bar 0,1)} = \mathbb{F}v\otimes\mathbb{F}u', && (\mathcal{C}\otimes\mathcal{H})_{(\bar 1,\bar 1,\bar 0,-1)} = \mathbb{F}vu\otimes\mathbb{F}v',\\
&(\mathcal{C}\otimes\mathcal{H})_{(\bar 0,\bar 1,\bar 1,1)} = \mathbb{F}wv\otimes\mathbb{F}u', && (\mathcal{C}\otimes\mathcal{H})_{(\bar 1,\bar 1,\bar 1,-1)} = \mathbb{F}(wv)u\otimes\mathbb{F}v'.
\end{aligned}
$$

3. The $\mathbb{Z}^2\times(\mathbb{Z}/2)^2$-grading induced by the Cartan $\mathbb{Z}^2$-grading on $\mathcal{C}$ (with basis $\{e_1,e_2,u_1,u_2,u_3,v_1,v_2,v_3\}$ and homogeneous components given by Equation (4.2.2)) and the $(\mathbb{Z}/2)^2$-grading induced by the Cayley-Dickson doubling process on $\mathcal{H}$ (with basis $\{1,v',u',v'u'\}$ and homogeneous components given by Equation (5.2.2)), with homogeneous components given by

$$
\begin{aligned}
&(\mathcal{C}\otimes\mathcal{H})_{(0,0,\bar 0,\bar 0)} = (\mathbb{F}e_1\otimes\mathbb{F}e_2)\otimes\mathbb{F}1, && (\mathcal{C}\otimes\mathcal{H})_{(1,1,\bar 0,\bar 1)} = \mathbb{F}v_3\otimes\mathbb{F}u',\\
&(\mathcal{C}\otimes\mathcal{H})_{(0,0,\bar 1,\bar 0)} = (\mathbb{F}e_1\otimes\mathbb{F}e_2)\otimes\mathbb{F}v', && (\mathcal{C}\otimes\mathcal{H})_{(1,1,\bar 1,\bar 1)} = \mathbb{F}v_3\otimes\mathbb{F}v'u',\\
&(\mathcal{C}\otimes\mathcal{H})_{(0,0,\bar 0,\bar 1)} = (\mathbb{F}e_1\otimes\mathbb{F}e_2)\otimes\mathbb{F}u', && (\mathcal{C}\otimes\mathcal{H})_{(-1,0,\bar 0,\bar 0)} = \mathbb{F}v_1\otimes\mathbb{F}1,\\
&(\mathcal{C}\otimes\mathcal{H})_{(0,0,\bar 1,\bar 1)} = (\mathbb{F}e_1\otimes\mathbb{F}e_2)\otimes\mathbb{F}v'u', && (\mathcal{C}\otimes\mathcal{H})_{(-1,0,\bar 1,\bar 0)} = \mathbb{F}v_1\otimes\mathbb{F}v',\\
&(\mathcal{C}\otimes\mathcal{H})_{(1,0,\bar 0,\bar 0)} = \mathbb{F}u_1\otimes\mathbb{F}1, && (\mathcal{C}\otimes\mathcal{H})_{(-1,0,\bar 0,\bar 1)} = \mathbb{F}v_1\otimes\mathbb{F}u',\\
&(\mathcal{C}\otimes\mathcal{H})_{(1,0,\bar 1,\bar 0)} = \mathbb{F}u_1\otimes\mathbb{F}v', && (\mathcal{C}\otimes\mathcal{H})_{(-1,0,\bar 1,\bar 1)} = \mathbb{F}v_1\otimes\mathbb{F}v'u',\\
&(\mathcal{C}\otimes\mathcal{H})_{(1,0,\bar 0,\bar 1)} = \mathbb{F}u_1\otimes\mathbb{F}u', && (\mathcal{C}\otimes\mathcal{H})_{(0,-1,\bar 0,\bar 0)} = \mathbb{F}v_2\otimes\mathbb{F}1,\\
&(\mathcal{C}\otimes\mathcal{H})_{(1,0,\bar 1,\bar 1)} = \mathbb{F}u_1\otimes\mathbb{F}v'u', && (\mathcal{C}\otimes\mathcal{H})_{(0,-1,\bar 1,\bar 0)} = \mathbb{F}v_2\otimes\mathbb{F}v',\\
&(\mathcal{C}\otimes\mathcal{H})_{(0,1,\bar 0,\bar 0)} = \mathbb{F}u_2\otimes\mathbb{F}1, && (\mathcal{C}\otimes\mathcal{H})_{(0,-1,\bar 0,\bar 1)} = \mathbb{F}v_2\otimes\mathbb{F}u',\\
&(\mathcal{C}\otimes\mathcal{H})_{(0,1,\bar 1,\bar 0)} = \mathbb{F}u_2\otimes\mathbb{F}v', && (\mathcal{C}\otimes\mathcal{H})_{(0,-1,\bar 1,\bar 1)} = \mathbb{F}v_2\otimes\mathbb{F}v'u',\\
&(\mathcal{C}\otimes\mathcal{H})_{(0,1,\bar 0,\bar 1)} = \mathbb{F}u_2\otimes\mathbb{F}u', && (\mathcal{C}\otimes\mathcal{H})_{(-1,-1,\bar 0,\bar 0)} = \mathbb{F}u_3\otimes\mathbb{F}1,\\
&(\mathcal{C}\otimes\mathcal{H})_{(0,1,\bar 1,\bar 1)} = \mathbb{F}u_2\otimes\mathbb{F}v'u', && (\mathcal{C}\otimes\mathcal{H})_{(-1,-1,\bar 1,\bar 0)} = \mathbb{F}u_3\otimes\mathbb{F}v',\\
&(\mathcal{C}\otimes\mathcal{H})_{(1,1,\bar 0,\bar 0)} = \mathbb{F}v_3\otimes\mathbb{F}1, && (\mathcal{C}\otimes\mathcal{H})_{(-1,-1,\bar 0,\bar 1)} = \mathbb{F}u_3\otimes\mathbb{F}u',\\
&(\mathcal{C}\otimes\mathcal{H})_{(1,1,\bar 1,\bar 0)} = \mathbb{F}v_3\otimes\mathbb{F}v', && (\mathcal{C}\otimes\mathcal{H})_{(-1,-1,\bar 1,\bar 1)} = \mathbb{F}u_3\otimes\mathbb{F}v'u'.
\end{aligned}
$$

4. The $\mathbb{Z}^3$-grading induced by the Cartan $\mathbb{Z}^2$-grading on $\mathcal{C}$ (with basis $\{e_1,e_2,u_1,u_2,u_3,v_1,v_2,v_3\}$ and homogeneous components given by Equation (4.2.2)) and the Cartan $\mathbb{Z}$-grading on $\mathcal{H}$ (with basis $\{e_1',e_2',u',v'\}$

and homogeneous components given by Equation (5.2.1)), with homogeneous components given by

$$
\begin{aligned}
&(\mathcal{C}\otimes\mathcal{H})_{(0,0,0)} = (\mathbb{F}e_1\otimes\mathbb{F}e_2)\otimes(\mathbb{F}e'_1\otimes\mathbb{F}e'_2),\\
&(\mathcal{C}\otimes\mathcal{H})_{(1,1,-1)} = \mathbb{F}v_3\otimes\mathbb{F}v',\\
&(\mathcal{C}\otimes\mathcal{H})_{(0,0,1)} = (\mathbb{F}e_1\otimes\mathbb{F}e_2)\otimes\mathbb{F}u',\\
&(\mathcal{C}\otimes\mathcal{H})_{(-1,0,0)} = \mathbb{F}v_1\otimes(\mathbb{F}e'_1\otimes\mathbb{F}e'_2),\\
&(\mathcal{C}\otimes\mathcal{H})_{(0,0,-1)} = (\mathbb{F}e_1\otimes\mathbb{F}e_2)\otimes\mathbb{F}v',\\
&(\mathcal{C}\otimes\mathcal{H})_{(-1,0,1)} = \mathbb{F}v_1\otimes\mathbb{F}u',\\
&(\mathcal{C}\otimes\mathcal{H})_{(1,0,0)} = \mathbb{F}u_1\otimes(\mathbb{F}e'_1\otimes\mathbb{F}e'_2),\\
&(\mathcal{C}\otimes\mathcal{H})_{(-1,0,-1)} = \mathbb{F}v_1\otimes\mathbb{F}v',\\
&(\mathcal{C}\otimes\mathcal{H})_{(1,0,1)} = \mathbb{F}u_1\otimes\mathbb{F}u',\\
&(\mathcal{C}\otimes\mathcal{H})_{(0,-1,0)} = \mathbb{F}v_2\otimes(\mathbb{F}e'_1\otimes\mathbb{F}e'_2),\\
&(\mathcal{C}\otimes\mathcal{H})_{(1,0,-1)} = \mathbb{F}u_1\otimes\mathbb{F}v',\\
&(\mathcal{C}\otimes\mathcal{H})_{(0,-1,1)} = \mathbb{F}v_2\otimes\mathbb{F}u',\\
&(\mathcal{C}\otimes\mathcal{H})_{(0,1,0)} = \mathbb{F}u_2\otimes(\mathbb{F}e'_1\otimes\mathbb{F}e'_2),\\
&(\mathcal{C}\otimes\mathcal{H})_{(0,-1,-1)} = \mathbb{F}v_2\otimes\mathbb{F}v',\\
&(\mathcal{C}\otimes\mathcal{H})_{(0,1,1)} = \mathbb{F}u_2\otimes\mathbb{F}u',\\
&(\mathcal{C}\otimes\mathcal{H})_{(-1,-1,0)} = \mathbb{F}u_3\otimes(\mathbb{F}e'_1\otimes\mathbb{F}e'_2),\\
&(\mathcal{C}\otimes\mathcal{H})_{(0,1,-1)} = \mathbb{F}u_2\otimes\mathbb{F}v',\\
&(\mathcal{C}\otimes\mathcal{H})_{(-1,-1,1)} = \mathbb{F}u_3\otimes\mathbb{F}u',\\
&(\mathcal{C}\otimes\mathcal{H})_{(1,1,0)} = \mathbb{F}v_3\otimes(\mathbb{F}e'_1\otimes\mathbb{F}e'_2),\\
&(\mathcal{C}\otimes\mathcal{H})_{(-1,-1,-1)} = \mathbb{F}u_3\otimes\mathbb{F}v',\\
&(\mathcal{C}\otimes\mathcal{H})_{(1,1,1)} = \mathbb{F}v_3\otimes\mathbb{F}u'.
\end{aligned}
$$

5.2.1 The direct product of two Cayley algebras

In order to find the gradings on the tensor product of two Cayley algebras we will start finding gradings on the direct product. Recall that in Chapter 4 Section 3 we proved that the automorphism group scheme of both algebras (the direct product and tensor product of two Cayley algebras) is isomorphic. After finding gradings on the direct product we will use an isomorphism of schemes to obtain gradings on the tensor product which are closed under the involution (Remark 5.2.3).

Let $\gamma = (g_1, g_2, g_3)$ be a triple of elements in a group G and let $H \subset G$ be a subgroup isomorphic to $(\mathbb{Z}/2)^3$. Recall the definitions of $\Gamma^1_{\mathcal{C}}(G,\gamma)$ and $\Gamma^2_{\mathcal{C}}(G,H)$ right before Theorem 4.2.7.

Theorems 2.2.3 and 2.1.9 show that given any abelian group G, any G-grading on $\mathcal{C}\times\mathcal{C}$ making it a graded-simple algebra (i.e., the two copies of $\mathcal{C}$

are not graded ideals) is isomorphic to the grading on a loop algebra $L_\pi(\mathcal{C})$, where $\pi : G \to \overline{G}$ is a surjective group homomorphism with $\ker\pi$ of order 2: $\ker\pi = \langle h\rangle$, h of order 2, obtained from a grading $\overline{\Gamma}$ on $\mathcal{C}$. We will denote $\overline{g} := \pi(g)$ for $g \in G$. The loop algebra is isomorphic to $\mathcal{C}\times\mathcal{C}$ by means of the isomorphism in Theorem 2.1.9 which allows us to transfer easily the grading on $L_\pi(\mathcal{C})$ to $\mathcal{C}\times\mathcal{C}$.

If $\overline{\Gamma}$ is isomorphic to $\Gamma^1_{\mathcal{C}}(\overline{G},\overline{\gamma})$, for a triple of elements $\overline{\gamma} = (\bar{g}_1, \bar{g}_2, \bar{g}_3)$ in $\overline{G}$, the corresponding grading on $\mathcal{C}\times\mathcal{C}$ will be denoted by $\Gamma^1_{\mathcal{C}\times\mathcal{C}}(G, h, \overline{\gamma})$, while if $\overline{\Gamma}$ is isomorphic to $\Gamma^2_{\mathcal{C}}(\overline{G},\overline{H})$ for $\overline{H} := \pi(H)$ where H is a subgroup of G such that $\overline{H}$ is isomorphic to $(\mathbb{Z}/2)^3$ the corresponding grading on $\mathcal{C}\times\mathcal{C}$ will be denoted by $\Gamma^2_{\mathcal{C}\times\mathcal{C}}(G, h, \overline{H})$.

The gradings $\Gamma^1_{\mathcal{C}\times\mathcal{C}}(G, h, \overline{\gamma})$ and $\Gamma^2_{\mathcal{C}\times\mathcal{C}}(G, h, \overline{H})$ are quite simple to describe if the surjective group homomorphism $\pi : G \to \overline{G}$ splits. That is, there is a section $s : \overline{G} \to G$ of π which is a group homomorphism. In this case, $G = s(\overline{G}) \times \langle h\rangle$ and the nontrivial character on $\ker\pi = \langle h\rangle$ ($\chi(h) = -1$) extends to a character χ on G with $\chi(g) = 1$ for any $g \in s(\overline{G})$. The isomorphism in Theorem 2.1.9 becomes the isomorphism

$$\begin{aligned} \Phi : L_\pi(\mathcal{C}) &\longrightarrow \mathcal{C}\times\mathcal{C} \\ x\otimes g &\mapsto \big(x, \chi(g)x\big) \end{aligned}$$

for $g \in G$ and $x \in \mathcal{C}_{\pi(g)}$. Thus, with $g_i = s(\overline{g_i})$ for $i = 1, 2, 3$, the G-grading $\Gamma^1_{\mathcal{C}\times\mathcal{C}}(G, h, \overline{\gamma})$ is determined by:

$$\begin{array}{ll} (\mathcal{C}\times\mathcal{C})_e = \mathbb{F}(e_1,e_1)\oplus\mathbb{F}(e_2,e_2), & (\mathcal{C}\times\mathcal{C})_h = \mathbb{F}(e_1,-e_1)\oplus\mathbb{F}(e_2,-e_2),\\ (\mathcal{C}\times\mathcal{C})_{g_1} = \mathbb{F}(u_1,u_1), & (\mathcal{C}\times\mathcal{C})_{g_1h} = \mathbb{F}(u_1,-u_1),\\ (\mathcal{C}\times\mathcal{C})_{g_2} = \mathbb{F}(u_2,u_2), & (\mathcal{C}\times\mathcal{C})_{g_2h} = \mathbb{F}(u_2,-u_2),\\ (\mathcal{C}\times\mathcal{C})_{g_1g_2} = \mathbb{F}(v_3,v_3), & (\mathcal{C}\times\mathcal{C})_{g_1g_2h} = \mathbb{F}(v_3,-v_3),\\ (\mathcal{C}\times\mathcal{C})_{g_1^{-1}} = \mathbb{F}(v_1,v_1), & (\mathcal{C}\times\mathcal{C})_{g_1^{-1}h} = \mathbb{F}(v_1,-v_1),\\ (\mathcal{C}\times\mathcal{C})_{g_2^{-1}} = \mathbb{F}(v_2,v_2), & (\mathcal{C}\times\mathcal{C})_{g_2^{-1}h} = \mathbb{F}(v_2,-v_2),\\ (\mathcal{C}\times\mathcal{C})_{(g_1g_2)^{-1}} = \mathbb{F}(u_3,u_3), & (\mathcal{C}\times\mathcal{C})_{(g_1g_2)^{-1}h} = \mathbb{F}(u_3,-u_3). \end{array} \tag{5.2.4}$$

And the G-grading $\Gamma^2_{\mathcal{C}\times\mathcal{C}}(G, h, \overline{H})$ is determined by:

$$\begin{array}{ll} (\mathcal{C}\times\mathcal{C})_e = \mathbb{F}(1,1), & (\mathcal{C}\times\mathcal{C})_h = \mathbb{F}(1,-1),\\ (\mathcal{C}\times\mathcal{C})_{g_1} = \mathbb{F}(u,u), & (\mathcal{C}\times\mathcal{C})_{g_1h} = \mathbb{F}(u,-u),\\ (\mathcal{C}\times\mathcal{C})_{g_2} = \mathbb{F}(v,v), & (\mathcal{C}\times\mathcal{C})_{g_2h} = \mathbb{F}(v,-v),\\ (\mathcal{C}\times\mathcal{C})_{g_3} = \mathbb{F}(w,w), & (\mathcal{C}\times\mathcal{C})_{g_3h} = \mathbb{F}(w,-w),\\ (\mathcal{C}\times\mathcal{C})_{g_1g_2} = \mathbb{F}(vu,vu), & (\mathcal{C}\times\mathcal{C})_{g_1g_2h} = \mathbb{F}(vu,-v),\\ (\mathcal{C}\times\mathcal{C})_{g_1g_3} = \mathbb{F}(wu,wu), & (\mathcal{C}\times\mathcal{C})_{g_1g_3h} = \mathbb{F}(wu,-wv),\\ (\mathcal{C}\times\mathcal{C})_{g_2g_3} = \mathbb{F}(wv,wv), & (\mathcal{C}\times\mathcal{C})_{g_2g_3h} = \mathbb{F}(wv,-wv),\\ (\mathcal{C}\times\mathcal{C})_{g_1g_2g_3} = \mathbb{F}((wv)u,(wv)u), & (\mathcal{C}\times\mathcal{C})_{g_1g_2g_3h} = \mathbb{F}((wv)u,-(wv)u) \end{array} \tag{5.2.5}$$

where $\overline{H} = \langle \bar{g}_1, \bar{g}_2, \bar{g}_3 \rangle$.

The following result gives the classification of group-gradings, up to isomorphism, on $\mathcal{C} \times \mathcal{C}$ where it is graded-simple.

Theorem 5.2.8. *Any G-grading Γ by a group G on the cartesian product $\mathcal{C} \times \mathcal{C}$ of two Cayley algebras, such that $\mathcal{C} \times \mathcal{C}$ is graded-simple, is isomorphic to either $\Gamma^1_{\mathcal{C}\times\mathcal{C}}(G, h, \overline{\gamma})$ or to $\Gamma^2_{\mathcal{C}\times\mathcal{C}}(G, h, \overline{H})$ (for an element h in G of order 2, a triple $\overline{\gamma} = (\bar{g}_1, \bar{g}_2, \bar{g}_3)$ of elements of $\overline{G} = G/\langle h \rangle$ and a subgroup $\overline{H} \subset \overline{G}$ isomorphic to $(\mathbb{Z}/2)^3$). Moreover, no grading of the first type is isomorphic to one of the second type and*

- *$\Gamma^1_{\mathcal{C}\times\mathcal{C}}(G, h, \overline{\gamma})$ is isomorphic to $\Gamma^1_{\mathcal{C}\times\mathcal{C}}(G, h', \overline{\gamma}')$ if and only if $h = h'$ and $\overline{\gamma} \sim \overline{\gamma}'$ (see Definition of $\Gamma^1_{\mathcal{C}}(G, \gamma)$ in Chapter 4).*
- *$\Gamma^2_{\mathcal{C}\times\mathcal{C}}(G, h, \overline{H})$ is isomorphic to $\Gamma^2_{\mathcal{C}\times\mathcal{C}}(G, h', \overline{H}')$ if and only if $h = h'$ and $\overline{H} = \overline{H}'$.*

Proof. By Theorem 2.2.2, for $i = j = 1$, we have that $\Gamma^1_{\mathcal{C}\times\mathcal{C}}(G, h, \overline{\gamma})$ is isomorphic to $\Gamma^1_{\mathcal{C}\times\mathcal{C}}(G, h', \overline{\gamma}')$ if and only if $h = h'$ and the associated $\overline{G}$-gradings on $\mathcal{C}$, that is $\Gamma^1_{\mathcal{C}}(\overline{G}, \overline{\gamma})$ and $\Gamma^1_{\mathcal{C}}(\overline{G}, \overline{\gamma}')$, are isomorphic, which occurs if and only if $\overline{\gamma} \sim \overline{\gamma}'$ (Theorem 4.2.7). The proof for the grading of second type is analogous. □

Next result gives the classification of group-gradings, up to isomorphism, on $\mathcal{C} \times \mathcal{C}$ where it is not graded-simple.

Proposition 5.2.9. *Let G be a group and let Γ be a G-grading on the product of two Cayley algebras $\mathcal{C} \times \mathcal{C}$ such that it is not graded-simple, i.e. $\mathcal{C} \times 0$ and $0 \times \mathcal{C}$ are graded ideals. By Theorem 2.2.3 1. we have that Γ is isomorphic to a product G-grading $\Gamma^1 \times_G \Gamma^2$ for some G-gradings Γ^1 and Γ^2 on $\mathcal{C}$.*

Let Γ^1, Γ^2, Γ'^1 and Γ'^2 be G-gradings on $\mathcal{C}$. Then, the product G-gradings $\Gamma^1 \times_G \Gamma^2$ and $\Gamma'^1 \times_G \Gamma'^2$ are isomorphic if and only if $\Gamma^1 \simeq \Gamma'^1$ and $\Gamma^2 \simeq \Gamma'^2$ or $\Gamma^1 \simeq \Gamma'^2$ and $\Gamma^2 \simeq \Gamma'^1$.

□

Notice that the fine group-grading $\Gamma^1_{\mathcal{C}}(\mathbb{Z}^2, ((1,0),(0,1),(-1,-1)))$ is precisely $\Gamma^1_{\mathcal{C}}$ and the fine group-grading $\Gamma^2_{\mathcal{C}}$ is $\Gamma^2_{\mathcal{C}}(G, H)$ with $G = H = \left(\mathbb{Z}/2\right)^3$.

Finally we obtain the fine group-gradings on $\mathcal{C} \times \mathcal{C}$ up to equivalence.

Proposition 5.2.10. *Corollary 2.4.13 tells us that, up to equivalence, the fine group-gradings on $\mathcal{C} \times \mathcal{C}$ are:*

1. *The product group-grading $\Gamma^1_{\mathcal{C}} \times \Gamma^1_{\mathcal{C}}$ by its universal group $\mathbb{Z}^2 \times \mathbb{Z}^2 \simeq \mathbb{Z}^4$.*

2. *The product group-grading* $\Gamma^1_{\mathcal{C}} \times \Gamma^2_{\mathcal{C}}$ *by its universal group* $\mathbb{Z}^2 \times (\mathbb{Z}/2)^3$.

3. *The product group-grading* $\Gamma^2_{\mathcal{C}} \times \Gamma^2_{\mathcal{C}}$ *by its universal group* $(\mathbb{Z}/2)^3 \times (\mathbb{Z}/2)^3 \simeq (\mathbb{Z}/2)^6$.

4. *The grading* $\Gamma^1_{\mathcal{C}\times\mathcal{C}}(\mathbb{Z}/2 \times \mathbb{Z}^2, (\bar{1}, 0, 0), ((1, 0), (0, 1), (-1, -1)))$ *with universal group* $U = \mathbb{Z}/2 \times \mathbb{Z}^2$. *The group* $\overline{U} = U/\langle(\bar{1}, 0, 0)\rangle$ *is identified naturally with* $\mathbb{Z}^2$. *This grading is determined explicitly using Equation* (5.2.4).

5. *The grading* $\Gamma^2_{\mathcal{C}\times\mathcal{C}}\Big(\big(\mathbb{Z}/2\big)^4, (\bar{1}, \bar{0}, \bar{0}, \bar{0}), \big(\mathbb{Z}/2\big)^3\Big)$ *with universal group* $U = \big(\mathbb{Z}/2\big)^4$. *Here the group* $\overline{U} = U/\langle(\bar{1}, \bar{0}, \bar{0}, \bar{0})\rangle$ *is identified with* $\big(\mathbb{Z}/2\big)^3$. *This grading is determined explicitly using Equation* (5.2.5).

6. *The grading* $\Gamma^2_{\mathcal{C}\times\mathcal{C}}(\mathbb{Z}/4 \times (\mathbb{Z}/2)^2, (\widehat{2}, \bar{0}, \bar{0}), (\mathbb{Z}/2)^3)$. *Here we denote by* $\widehat{m}$ *the class of the integer* m *modulo* 4 *and restrict the usual notation* $\bar{m}$ *for the class of* m *modulo* 2. *The surjective group homomorphism* π *is the canonical homomorphism* $\mathbb{Z}/4 \times (\mathbb{Z}/2)^2 \to (\mathbb{Z}/2)^3$, $(\widehat{m}, \bar{n}, \bar{r}) \mapsto (\bar{m}, \bar{n}, \bar{r})$.

   Let us give a precise description of this grading. The nontrivial character χ *on* $\langle h = (\widehat{2}, \bar{0}, \bar{0})\rangle$ *extends to the character* χ *on* $U = \mathbb{Z}/4 \times (\mathbb{Z}/2)^2$ *by* $\chi(\widehat{m}, \bar{n}, \bar{r}) = \mathbf{i}^m$, *where* $\mathbf{i}$ *denotes a square root of* -1 *in* $\mathbb{F}$.

   The grading on the loop algebra $L_\pi(\mathcal{C})$ *is given by*

   $$L_\pi(\mathcal{C})_{(\widehat{m},\bar{n},\bar{r})} = (\mathcal{C})_{(\bar{m},\bar{n},\bar{r})} \otimes (\widehat{m}, \bar{n}, \bar{r})$$

   for the homogeneous components $\mathcal{C}_{(\bar{m},\bar{n},\bar{r})}$ *in Equation* (4.2.1), *and through the isomorphism in Theorem 2.1.9 our grading*

   $$\Gamma^2_{\mathcal{C}\times\mathcal{C}}(\mathbb{Z}/4 \times (\mathbb{Z}/2)^2, (\widehat{2}, \bar{0}, \bar{0}), (\mathbb{Z}/2)^3)$$

   on $\mathcal{C} \times \mathcal{C}$ *is given by*

   $$(\mathcal{C} \times \mathcal{C})_{(\widehat{m},\bar{n},\bar{r})} = \{(x, \mathbf{i}^m x) \mid x \in (\mathcal{C})_{(\bar{m},\bar{n},\bar{r})}\}.$$

   That is,

   $$\begin{aligned}
   &(\mathcal{C} \times \mathcal{C})_{(\widehat{0},\bar{0},\bar{0})} = \mathbb{F}(1, 1), && (\mathcal{C} \times \mathcal{C})_{(\widehat{2},\bar{0},\bar{0})} = \mathbb{F}(1, -1),\\
   &(\mathcal{C} \times \mathcal{C})_{(\widehat{0},\bar{1},\bar{0})} = \mathbb{F}(v, v), && (\mathcal{C} \times \mathcal{C})_{(\widehat{2},\bar{1},\bar{0})} = \mathbb{F}(v, -v),\\
   &(\mathcal{C} \times \mathcal{C})_{(\widehat{0},\bar{0},\bar{1})} = \mathbb{F}(w, w), && (\mathcal{C} \times \mathcal{C})_{(\widehat{2},\bar{0},\bar{1})} = \mathbb{F}(w, -w),\\
   &(\mathcal{C} \times \mathcal{C})_{(\widehat{0},\bar{1},\bar{1})} = \mathbb{F}(wv, wv), && (\mathcal{C} \times \mathcal{C})_{(\widehat{2},\bar{1},\bar{1})} = \mathbb{F}(wv, -wv),\\
   &(\mathcal{C} \times \mathcal{C})_{(\widehat{1},\bar{0},\bar{0})} = \mathbb{F}(u, \mathbf{i}u), && (\mathcal{C} \times \mathcal{C})_{(\widehat{3},\bar{0},\bar{0})} = \mathbb{F}(u, -\mathbf{i}u),\\
   &(\mathcal{C} \times \mathcal{C})_{(\widehat{1},\bar{1},\bar{0})} = \mathbb{F}(vu, \mathbf{i}vu), && (\mathcal{C} \times \mathcal{C})_{(\widehat{3},\bar{1},\bar{0})} = \mathbb{F}(vu, -\mathbf{i}vu),\\
   &(\mathcal{C} \times \mathcal{C})_{(\widehat{1},\bar{0},\bar{1})} = \mathbb{F}(wu, \mathbf{i}wu), && (\mathcal{C} \times \mathcal{C})_{(\widehat{3},\bar{0},\bar{1})} = \mathbb{F}(wu, -\mathbf{i}wu),\\
   &(\mathcal{C} \times \mathcal{C})_{(\widehat{1},\bar{1},\bar{1})} = \mathbb{F}((wv)u, \mathbf{i}(wv)u), && (\mathcal{C} \times \mathcal{C})_{(\widehat{3},\bar{1},\bar{1})} = \mathbb{F}((wv)u, -\mathbf{i}(wv)u).
   \end{aligned}$$

5.2.2 The tensor product of two Cayley algebras

In this section we will generate group-gradings on the tensor product of two Cayley algebras $\mathcal{C} \otimes \mathcal{C}$ from the group-gradings we already know on $\mathcal{C} \times \mathcal{C}$. This is enough in order to classify group-gradings on $\mathcal{C} \otimes \mathcal{C}$, since there is a correspondence between gradings on $\mathcal{C} \times \mathcal{C}$ and gradings on $\mathcal{C} \otimes \mathcal{C}$ (Theorem 4.3.7).

We will start by generating group-gradings on $\mathcal{C} \otimes \mathcal{C}$ from the group-gradings on $\mathcal{C} \times \mathcal{C}$ such that this product is graded-simple.

Let G be a group, let h be an element in G of order 2 and let $\overline{\gamma} = (\bar{g}_1, \bar{g}_2, \bar{g}_3)$ be a triple of elements in $\overline{G} := G/\langle h \rangle$. Consider the G-grading $\Gamma^1_{\mathcal{C}\times\mathcal{C}}(G, h, \overline{\gamma})$ (on $\mathcal{C} \times \mathcal{C}$ such that it is graded-simple) and take the restriction to the subalgebra $\mathcal{C}_0 \times \mathcal{C}_0$ (with the product given by the commutator $[\ ,\ ]$ which is graded-simple too):

$$\Gamma^1_{\mathcal{C}_0\times\mathcal{C}_0}(G, h, \overline{\gamma}) := \Gamma^1_{\mathcal{C}\times\mathcal{C}}(G, h, \overline{\gamma}) \mid_{\mathcal{C}_0\times\mathcal{C}_0} .$$

Then using the isomorphism

$$\begin{array}{ccc} \mathcal{C}_0 \times \mathcal{C}_0 & \longrightarrow & (\mathcal{C}_0 \otimes \mathbb{F}1) \oplus (\mathbb{F}1 \otimes \mathcal{C}_0) \\ (x, y) & \longmapsto & x \otimes 1 + 1 \otimes y \end{array} \tag{5.2.6}$$

we obtain a G-grading on $(\mathcal{C}_0 \otimes \mathbb{F}1) \oplus (\mathbb{F}1 \otimes \mathcal{C}_0)$ (with the product given by the commutator and it is graded-simple), which we will denote by

$$\Gamma^1_{(\mathcal{C}_0\otimes\mathbb{F}1)\oplus(\mathbb{F}1\otimes\mathcal{C}_0)}(G, h, \overline{\gamma})$$

where $\deg(x \otimes 1 + 1 \otimes y) = g$ for $g \in G$ if $(x, y) \in \mathcal{C}_0 \times \mathcal{C}_0$ is such that $\deg(x, y) = g$ in $\Gamma^1_{\mathcal{C}_0\times\mathcal{C}_0}(G, h, \overline{\gamma})$. Finally, by Remark 4.2.1, this last grading induces a G-grading on $\mathcal{C} \otimes \mathcal{C}$ (with the usual product) which we will denote by

$$\Gamma^1_{\mathcal{C}\otimes\mathcal{C}}(G, h, \overline{\gamma}).$$

Analogously, for an element h of order 2 in G and a subgroup $\overline{H} \subset \overline{G} = G/\langle h \rangle$ isomorphic to $(\mathbb{Z}/2)^3$, we can construct from the grading $\Gamma^2_{\mathcal{C}\times\mathcal{C}}(G, h, \overline{H})$ (on $\mathcal{C} \times \mathcal{C}$ such that it is graded-simple) a grading on $\mathcal{C} \otimes \mathcal{C}$ denoted by

$$\Gamma^2_{\mathcal{C}\otimes\mathcal{C}}(G, h, \overline{H}).$$

The following result gives the classification of group-gradings, up to isomorphism, on $\mathcal{C} \otimes \mathcal{C}$ such that the graded subspace $(\mathcal{C}_0 \otimes \mathbb{F}1) \oplus (\mathbb{F}1 \otimes \mathcal{C}_0)$ is graded-simple.

Corollary 5.2.11. *Let Γ be a grading by a group G on the tensor product $\mathcal{C} \otimes \mathcal{C}$ of two Cayley algebras. Suppose that for the induced G-grading on the algebra $(\mathcal{C}_0 \otimes \mathbb{F}1) \oplus (\mathbb{F}1 \otimes \mathcal{C}_0)$ with the product given by the commutator (see Lemma 5.2.1 b) and Definition 5.1.1) $(\mathcal{C}_0 \otimes \mathbb{F}1) \oplus (\mathbb{F}1 \otimes \mathcal{C}_0)$ is graded-simple. Then Γ is isomorphic to either $\Gamma^1_{\mathcal{C}\otimes\mathcal{C}}(G, h, \overline{\gamma})$ or to $\Gamma^2_{\mathcal{C}\otimes\mathcal{C}}(G, h, \overline{H})$ (for an element h in G of order 2, a triple $\overline{\gamma} = (\bar{g}_1, \bar{g}_2, \bar{g}_3)$ of elements in $\overline{G} = G/\langle h \rangle$ and a subgroup $\overline{H} \subset \overline{G}$ isomorphic to $(\mathbb{Z}/2)^3$). Moreover, no grading of the first type is isomorphic to one of the second type and*

- *$\Gamma^1_{\mathcal{C}\otimes\mathcal{C}}(G, h, \overline{\gamma})$ is isomorphic to $\Gamma^1_{\mathcal{C}\otimes\mathcal{C}}(G, h', \overline{\gamma}')$ if and only if $h = h'$ and $\overline{\gamma} \sim \overline{\gamma}'$ (see Definition of $\overline{\gamma} \sim \overline{\gamma}'$ in Chapter 4).*
- *$\Gamma^2_{\mathcal{C}\otimes\mathcal{C}}(G, h, \overline{H})$ is isomorphic to $\Gamma^2_{\mathcal{C}\otimes\mathcal{C}}(G, h', \overline{H}')$ if and only if $h = h'$ and $\overline{H} = \overline{H}'$.*

Proof. Let Γ be a G-grading on $\mathcal{C} \otimes \mathcal{C}$ such that for the induced G-grading Γ_0 on the algebra $(\mathcal{C}_0 \otimes \mathbb{F}1) \oplus (\mathbb{F}1 \otimes \mathcal{C}_0)$ with the product given by the commutator (see Lemma 5.2.1 b) and Definition 5.1.1) $(\mathcal{C}_0 \otimes \mathbb{F}1) \oplus (\mathbb{F}1 \otimes \mathcal{C}_0)$ is graded-simple.

Using the isomorphism from Equation (5.2.6) we obtain a G-grading $\Gamma_{\mathcal{C}_0\times\mathcal{C}_0}$ on $\mathcal{C}_0 \times \mathcal{C}_0$ isomorphic to Γ_0 where $\mathcal{C}_0 \times \mathcal{C}_0$ is a graded-simple algebra (again with the product given by the commutator). Finally, by Remark 4.2.1, $\Gamma_{\mathcal{C}_0\times\mathcal{C}_0}$ induces a grading $\Gamma_{\mathcal{C}\times\mathcal{C}}$ on $\mathcal{C} \times \mathcal{C}$ (with the usual product) such that $\mathcal{C} \times \mathcal{C}$ is graded-simple. The result follows from Theorem 5.2.8. □

Now we will generate group-gradings on $\mathcal{C} \otimes \mathcal{C}$ from group-gradings on $\mathcal{C} \times \mathcal{C}$ such that this cartesian product is not graded-simple, that is, such that $\mathcal{C} \times 0$ and $0 \times \mathcal{C}$ are graded ideals.

Let G be a group and let Γ be a G-grading on $\mathcal{C} \times \mathcal{C}$ such that $\mathcal{C} \times 0$ and $0 \times \mathcal{C}$ are graded ideals, then by Theorem 2.2.3

$$\Gamma \simeq \Gamma^1 \times_G \Gamma^2$$

for some G-gradings Γ^1 and Γ^2 on $\mathcal{C}$. Then we restrict the product G-grading to $\mathcal{C}_0 \times \mathcal{C}_0$:

$$\Gamma^1 \times_G \Gamma^2|_{\mathcal{C}_0\times\mathcal{C}_0} = \Gamma^1|_{\mathcal{C}_0} \times_G \Gamma^2|_{\mathcal{C}_0}$$

and using the isomorphism of Equation (5.2.6) we obtain a G-grading on $(\mathcal{C}_0 \otimes \mathbb{F}1) \oplus (\mathbb{F}1 \otimes \mathcal{C}_0)$. Finally since $(\mathcal{C}_0 \otimes \mathbb{F}1) \oplus (\mathbb{F}1 \otimes \mathcal{C}_0)$ generates $\mathcal{C} \otimes \mathcal{C}$ we get a G-grading on $\mathcal{C} \otimes \mathcal{C}$.

Next result gives the classification of group-gradings, up to isomorphism, on $\mathcal{C} \otimes \mathcal{C}$ such that the graded subspace $(\mathcal{C}_0 \otimes \mathbb{F}1) \oplus (\mathbb{F}1 \otimes \mathcal{C}_0)$ is not graded-simple.

Proposition 5.2.12. *Let G be a group and let Γ be a G-grading on the tensor product of two Cayley algebras $\mathcal{C}\otimes\mathcal{C}$. Suppose that for the induced G-grading Γ_0 on the algebra $(\mathcal{C}_0\otimes\mathbb{F}1)\oplus(\mathbb{F}1\otimes\mathcal{C}_0)$ with the product given by the commutator (see Lemma 5.2.1 b) and Definition 5.1.1) $(\mathcal{C}_0\otimes\mathbb{F}1)\oplus(\mathbb{F}1\otimes\mathcal{C}_0)$ is not graded-simple, i.e. $\mathcal{C}_0\otimes\mathbb{F}1$ and $\mathbb{F}1\otimes\mathcal{C}_0$ are G-graded ideals. By Theorem 2.2.3 1. we have that Γ_0 is isomorphic to a product G-grading $\Gamma_0^1\times_G\Gamma_0^2$ for some G-gradings Γ_0^1 on $\mathcal{C}_0\otimes\mathbb{F}1$ and Γ_0^2 on $\mathbb{F}1\otimes\mathcal{C}_0$.*

Let Γ and Γ' be G-gradings on $\mathcal{C}\otimes\mathcal{C}$ and let Γ_0 and Γ'_0 be the G-gradings induced on $(\mathcal{C}_0\otimes\mathbb{F}1)\oplus(\mathbb{F}1\otimes\mathcal{C}_0)$ by Γ and Γ', respectively. Let Γ_0^1 and $\Gamma_0'^1$ be G-gradings on $\mathcal{C}_0\otimes\mathbb{F}1$ and Γ_0^2 and $\Gamma_0'^2$ be G-gradings on $\mathbb{F}1\otimes\mathcal{C}_0$ such that $\Gamma_0\simeq\Gamma_0^1\times_G\Gamma_0^2$ and $\Gamma'_0\simeq\Gamma_0'^1\times_G\Gamma_0'^2$.

Then, Γ and Γ' are isomorphic if and only if $\Gamma_0^1\simeq\Gamma_0'^1$ and $\Gamma_0^2\simeq\Gamma_0'^2$ or $\Gamma_0^1\simeq\Gamma_0'^2$ and $\Gamma_0^2\simeq\Gamma_0'^1$.

□

Definition 5.2.13. *Let G and H be groups. Let Γ^1 be a G-grading on an algebra $\mathcal{A}$ and let Γ^2 be a H-grading on an algebra $\mathcal{B}$. Recall that $\mathcal{A}\otimes\mathcal{B}$ has a natural $G\times H$-grading given by $(\mathcal{A}\otimes\mathcal{B})_{(g,h)}=\mathcal{A}_g\otimes\mathcal{B}_h$. We will call this grading **tensor product of** Γ^1 **and** Γ^2 and denote it by $\Gamma^1\otimes\Gamma^2$.*

Finally we obtain the fine group-gradings on $\mathcal{C}\otimes\mathcal{C}$ up to equivalence.

Proposition 5.2.14. *By Corollary 2.4.13 we got the (six) different fine group-gradings, up to equivalence on $\mathcal{C}\times\mathcal{C}$ (see Proposition 5.2.10). Then we will have six different fine group-gradings, up to equivalence, on $\mathcal{C}\otimes\mathcal{C}$. Such group gradings are in correspondence with the ones in Proposition 5.2.10 and they are the following:*

1. *$\Gamma^1_{\mathcal{C}}\otimes\Gamma^1_{\mathcal{C}}$ by its universal group $\mathbb{Z}^2\times\mathbb{Z}^2\simeq\mathbb{Z}^4$.*
2. *$\Gamma^1_{\mathcal{C}}\otimes\Gamma^2_{\mathcal{C}}$ by its universal group $\mathbb{Z}^2\times(\mathbb{Z}/2)^3$.*
3. *$\Gamma^2_{\mathcal{C}}\otimes\Gamma^2_{\mathcal{C}}$ by its universal group $(\mathbb{Z}/2)^3\times(\mathbb{Z}/2)^3\simeq(\mathbb{Z}/2)^6$.*
4. *The grading $\Gamma^1_{\mathcal{C}\otimes\mathcal{C}}(\mathbb{Z}/2\times\mathbb{Z}^2,(\bar{1},0,0),((1,0),(0,1),(-1,-1)))$ on $\mathcal{C}\otimes\mathcal{C}$ by its universal group $\mathbb{Z}/2\times\mathbb{Z}^2$. This grading is generated by the following*

homogeneous components in $\mathcal{B} := (\mathcal{C}_0 \otimes \mathbb{F}1) \oplus (\mathbb{F}1 \otimes \mathcal{C}_0)$*:*

$$\begin{aligned}
\mathcal{B}_{(\bar{0},0,0)} &= \mathbb{F}((e_1 - e_2) \otimes 1 + 1 \otimes (e_1 - e_2)),\\
\mathcal{B}_{(\bar{0},1,0)} &= \mathbb{F}(u_1 \otimes 1 + 1 \otimes u_1),\\
\mathcal{B}_{(\bar{0},0,1)} &= \mathbb{F}(u_2 \otimes 1 + 1 \otimes u_2),\\
\mathcal{B}_{(\bar{0},1,1)} &= \mathbb{F}(v_3 \otimes 1 + 1 \otimes v_3),\\
\mathcal{B}_{(\bar{0},-1,0)} &= \mathbb{F}(v_1 \otimes 1 + 1 \otimes v_1),\\
\mathcal{B}_{(\bar{0},0,-1)} &= \mathbb{F}(v_2 \otimes 1 + 1 \otimes v_2),\\
\mathcal{B}_{(\bar{0},-1,-1)} &= \mathbb{F}(u_3 \otimes 1 + 1 \otimes u_3),\\
\mathcal{B}_{(\bar{1},0,0)} &= \mathbb{F}((e_1 - e_2) \otimes 1 - 1 \otimes (e_1 - e_2)),\\
\mathcal{B}_{(\bar{1},1,0)} &= \mathbb{F}(u_1 \otimes 1 - 1 \otimes u_1),\\
\mathcal{B}_{(\bar{1},0,1)} &= \mathbb{F}(u_2 \otimes 1 - 1 \otimes u_2),\\
\mathcal{B}_{(\bar{1},1,1)} &= \mathbb{F}(v_3 \otimes 1 - 1 \otimes v_3),\\
\mathcal{B}_{(\bar{1},-1,0)} &= \mathbb{F}(v_1 \otimes 1 - 1 \otimes v_1),\\
\mathcal{B}_{(\bar{1},0,-1)} &= \mathbb{F}(v_2 \otimes 1 - 1 \otimes v_2),\\
\mathcal{B}_{(\bar{1},-1,-1)} &= \mathbb{F}(u_3 \otimes 1 - 1 \otimes u_3).
\end{aligned}$$

5. *The grading* $\Gamma^2_{\mathcal{C}\otimes\mathcal{C}}\Big(\big(\mathbb{Z}/2\big)^4, (\bar{1},\bar{0},\bar{0},\bar{0}), \big(\mathbb{Z}/2\big)^3\Big)$ *on* $\mathcal{C}\otimes\mathcal{C}$ *by its universal group* $\big(\mathbb{Z}/2\big)^4$*. This grading is generated by the following homogeneous components in* $\mathcal{B} := (\mathcal{C}_0 \otimes \mathbb{F}1) \oplus (\mathbb{F}1 \otimes \mathcal{C}_0)$*:*

$$\begin{aligned}
\mathcal{B}_{(\bar{0},\bar{1},\bar{0},\bar{0})} &= \mathbb{F}(u \otimes 1 + 1 \otimes u),\\
\mathcal{B}_{(\bar{0},\bar{0},\bar{1},\bar{0})} &= \mathbb{F}(v \otimes 1 + 1 \otimes v),\\
\mathcal{B}_{(\bar{0},\bar{0},\bar{0},\bar{1})} &= \mathbb{F}(w \otimes 1 + 1 \otimes w),\\
\mathcal{B}_{(\bar{0},\bar{1},\bar{1},\bar{0})} &= \mathbb{F}(vu \otimes 1 + 1 \otimes vu),\\
\mathcal{B}_{(\bar{0},\bar{1},\bar{0},\bar{1})} &= \mathbb{F}(wu \otimes 1 + 1 \otimes wu),\\
\mathcal{B}_{(\bar{0},\bar{0},\bar{1},\bar{1})} &= \mathbb{F}(wv \otimes 1 + 1 \otimes wv),\\
\mathcal{B}_{(\bar{0},\bar{1},\bar{1},\bar{1})} &= \mathbb{F}((wv)u \otimes 1 + 1 \otimes (wv)u),\\
\mathcal{B}_{(\bar{1},\bar{1},\bar{0},\bar{0})} &= \mathbb{F}(u \otimes 1 - 1 \otimes u),\\
\mathcal{B}_{(\bar{1},\bar{0},\bar{1},\bar{0})} &= \mathbb{F}(v \otimes 1 - 1 \otimes v),\\
\mathcal{B}_{(\bar{1},\bar{0},\bar{0},\bar{1})} &= \mathbb{F}(w \otimes 1 - 1 \otimes w),\\
\mathcal{B}_{(\bar{1},\bar{1},\bar{1},\bar{0})} &= \mathbb{F}(vu \otimes 1 - 1 \otimes vu),\\
\mathcal{B}_{(\bar{1},\bar{1},\bar{0},\bar{1})} &= \mathbb{F}(wu \otimes 1 - 1 \otimes wu),\\
\mathcal{B}_{(\bar{1},\bar{0},\bar{1},\bar{1})} &= \mathbb{F}(wv \otimes 1 - 1 \otimes wv),\\
\mathcal{B}_{(\bar{1},\bar{1},\bar{1},\bar{1})} &= \mathbb{F}((wv)u \otimes 1 - 1 \otimes (wv)u).
\end{aligned}$$

6. *The grading* $\Gamma^2_{\mathcal{C}\otimes\mathcal{C}}(\mathbb{Z}/4 \times (\mathbb{Z}/2)^2, (\widehat{2},\bar{0},\bar{0}), (\mathbb{Z}/2)^3)$ *on* $\mathcal{C}\otimes\mathcal{C}$ *by its universal group* $\mathbb{Z}/4\times(\mathbb{Z}/2)^2$*. Here we denote by* $\widehat{m}$ *the class of the integer* m *modulo* 4 *and restrict the usual notation* $\bar{m}$ *for the class of* m *modulo* 2 *and* $\mathbf{i}$ *denotes a square root of* -1 *in* $\mathbb{F}$*. This grading is generated by*

the following homogeneous components in $\mathcal{B} := (\mathcal{C}_0 \otimes \mathbb{F}1) \oplus (\mathbb{F}1 \otimes \mathcal{C}_0)$*:*

$$
\begin{aligned}
\mathcal{B}_{(\widehat{0},\bar{1},\bar{0})} &= \mathbb{F}(v \otimes 1 + 1 \otimes v),\\
\mathcal{B}_{(\widehat{0},\bar{0},\bar{1})} &= \mathbb{F}(w \otimes 1 + 1 \otimes w),\\
\mathcal{B}_{(\widehat{0},\bar{1},\bar{1})} &= \mathbb{F}(wv \otimes 1 + 1 \otimes wv),\\
\mathcal{B}_{(\widehat{1},\bar{0},\bar{0})} &= \mathbb{F}(u \otimes 1 + \mathbf{i} \otimes u),\\
\mathcal{B}_{(\widehat{1},\bar{1},\bar{0})} &= \mathbb{F}(vu \otimes 1 + \mathbf{i} \otimes vu),\\
\mathcal{B}_{(\widehat{1},\bar{0},\bar{1})} &= \mathbb{F}(wu \otimes 1 + \mathbf{i} \otimes wu),\\
\mathcal{B}_{(\widehat{1},\bar{1},\bar{1})} &= \mathbb{F}((wv)u \otimes 1 + \mathbf{i} \otimes (wv)u),\\
\mathcal{B}_{(\widehat{2},\bar{1},\bar{0})} &= \mathbb{F}(v \otimes 1 - 1 \otimes v),\\
\mathcal{B}_{(\widehat{2},\bar{0},\bar{1})} &= \mathbb{F}(w \otimes 1 - 1 \otimes w),\\
\mathcal{B}_{(\widehat{2},\bar{1},\bar{1})} &= \mathbb{F}(wv \otimes 1 - 1 \otimes wv),\\
\mathcal{B}_{(\widehat{3},\bar{0},\bar{0})} &= \mathbb{F}(u \otimes 1 - \mathbf{i} \otimes u),\\
\mathcal{B}_{(\widehat{3},\bar{1},\bar{0})} &= \mathbb{F}(vu \otimes 1 - \mathbf{i} \otimes vu),\\
\mathcal{B}_{(\widehat{3},\bar{0},\bar{1})} &= \mathbb{F}(wu \otimes 1 - \mathbf{i} \otimes wu),\\
\mathcal{B}_{(\widehat{3},\bar{1},\bar{1})} &= \mathbb{F}((wv)u \otimes 1 - \mathbf{i} \otimes (wv)u).
\end{aligned}
$$

Appendix A

Gradings

In this chapter we give analogous definitions, to the ones given in Chapter 1, for the main concepts on gradings on algebras, with a view towards proving that the Definition 2.4.4 of "product grading" we gave in Chapter 2 is the natural one on a product of gradings.

A.1 Definitions

We start with an analogous definition of grading.

Definition A.1.1. *A* grading *on an algebra $\mathcal{A}$ over a field $\mathbb{F}$ is a set Γ of nonzero subspaces of $\mathcal{A}$ such that $\mathcal{A} = \bigoplus_{\mathcal{U}\in\Gamma} \mathcal{U}$ and for any $\mathcal{U}, \mathcal{V} \in \Gamma$, there is a $\mathcal{W} \in \Gamma$ such that $\mathcal{U}\mathcal{V} \subseteq \mathcal{W}$.*

There are several natural related notions in the situation of Definition A.1.1:

- The pair $(\mathcal{A}, \Gamma)$ is said to be a *graded algebra*.
- The elements of Γ are called the *homogeneous components*. The nonzero elements of the homogeneous components are called *homogeneous elements*.
- A subalgebra $\mathcal{S}$ of $\mathcal{A}$ is called a *graded subalgebra* if $\mathcal{S} = \bigoplus_{\mathcal{U}\in\Gamma} \mathcal{U} \cap \mathcal{S}$. In this case

  $$\Gamma|_{\mathcal{S}} := \{\mathcal{U} \cap \mathcal{S} \mid \mathcal{U} \in \Gamma \text{ and } 0 \neq \mathcal{U} \cap \mathcal{S}\}$$

  is the *induced grading* on $\mathcal{S}$. A *graded ideal* is an ideal which is, at the same time, a graded subalgebra.

- Given another grading Γ' on $\mathcal{A}$, Γ is said to be a *refinement* of Γ' (and Γ' a *coarsening* of Γ) if any subspace $\mathcal{U} \in \Gamma$ is contained in a subspace in Γ'. If at least one of these containments is proper, then the refinement is said to be proper. In this situation Γ is said to be *finer* than Γ', and Γ' *coarser* than Γ, and the map $\pi : \Gamma \to \Gamma'$, that sends any $\mathcal{U} \in \Gamma$ to the element $\mathcal{U}' \in \Gamma'$ that contains it, is a surjection. The refinement is proper if, and only if, π is not a bijection.

- The grading Γ is said to be *fine* if it admits no proper refinements.

  Any grading on a finite-dimensional algebra is a coarsening of a fine grading.

- Given two graded algebras $(\mathcal{A}, \Gamma)$ and $(\mathcal{A}', \Gamma')$, an *equivalence* $\varphi : (\mathcal{A}, \Gamma) \to (\mathcal{A}', \Gamma')$ is an isomorphism $\varphi : \mathcal{A} \to \mathcal{A}'$ such that $\varphi(\mathcal{U}) \in \Gamma'$ for each $\mathcal{U} \in \Gamma$.

Given a graded algebra $(\mathcal{A}, \Gamma)$, consider the abelian group $U(\Gamma)$ generated by the set Γ, subject to the relations $\mathcal{U}\mathcal{V}\mathcal{W}^{-1} = e$ (e denotes the neutral element) for each pair $\mathcal{U}, \mathcal{V}$ in Γ such that $0 \neq \mathcal{U}\mathcal{V} \subseteq \mathcal{W}$:

$$U(\Gamma) := \langle \Gamma \mid \mathcal{U}\mathcal{V}\mathcal{W}^{-1} = e \text{ if } 0 \neq \mathcal{U}\mathcal{V} \subseteq \mathcal{W} \rangle.$$

That is, $U(\Gamma)$ is the quotient of the free abelian group generated by Γ, modulo the normal subgroup generated by the elements $\mathcal{U}\mathcal{V}\mathcal{W}^{-1}$ above. Consider also the natural map:

$$\begin{aligned} \delta_\Gamma^U : \Gamma &\longrightarrow U(\Gamma) \\ \mathcal{U} &\mapsto [\mathcal{U}], \end{aligned}$$

where $[\mathcal{U}]$ denotes the class of $\mathcal{U}$ in $U(\Gamma)$.

Definition A.1.2. *The pair* $\big(U(\Gamma), \delta_\Gamma^U\big)$ *is called the* universal group *of the grading* Γ.

Example A.1.3. *The* trivial grading *on an algebra* $\mathcal{A}$ *is the grading* $\Gamma = \{\mathcal{A}\}$. *We have two possibilities for the universal group:*

- *if* $\mathcal{A}^2 \neq 0$, *then* $U(\Gamma)$ *is the trivial group.*

- *if* $\mathcal{A}^2 = 0$, *then* $U(\Gamma)$ *is the infinite cyclic group (isomorphic to* $\mathbb{Z}$*).*

Example A.1.4. *Consider the cartesian product* $\mathcal{A} = \mathbb{F} \times \mathbb{F} = \mathbb{F}e_1 \oplus \mathbb{F}e_2$, *for the orthogonal idempotents* $e_1 = (1, 0)$ *and* $e_2 = (0, 1)$. *Then* $\Gamma = \{\mathbb{F}e_1, \mathbb{F}e_2\}$ *is a grading. Denote* $u_i := \mathbb{F}e_i$, $i = 1, 2$. *Then*

$$U(\Gamma) = \langle u_1, u_2 \mid u_1^2 = u_1,\ u_2^2 = u_2 \rangle = \{e\}$$

is the trivial group, even though our grading Γ *is not trivial.*

A.2 Group-gradings

We are mainly interested in gradings by abelian groups:

Definition A.2.1. *Given an abelian group G, a G-grading on an algebra $\mathcal{A}$ is a triple (Γ, G, δ), where Γ is a grading on $\mathcal{A}$, and $\delta : \Gamma \to G$ is a one-to-one map, such that for any $\mathcal{U}, \mathcal{V}, \mathcal{W} \in \Gamma$ such that $0 \neq \mathcal{U}\mathcal{V} \subseteq \mathcal{W}$, $\delta(\mathcal{U})\delta(\mathcal{V}) = \delta(\mathcal{W})$.*

Given a G-grading (Γ, G, δ), define $\mathcal{A}_g = \mathcal{U}$ if $\delta(\mathcal{U}) = g$, and define $\mathcal{A}_g = 0$ if g is not in the range of δ. Then $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$, and we recover the usual expression for a G-grading. The map δ is called the *degree map*. For $\mathcal{A}_g \neq 0$, $\delta(\mathcal{A}_g)$ is simply g.

As for general gradings, there are several natural related notions in the situation of Definition A.2.1:

- The 4-tuple $(\mathcal{A}, \Gamma, G, \delta)$ is said to be a *G-graded algebra*. If the other components are clear from the context, we may refer simply to a G-graded algebra $\mathcal{A}$.

- The subset $\mathrm{Supp}_G(\Gamma) := \{g \in G : \mathcal{A}_g \neq 0\}$ is called the *support* of the G-grading. Thus $\Gamma = \{\mathcal{A}_g \mid g \in \mathrm{Supp}_G(\Gamma)\}$.

- Given a G-graded algebra $(\mathcal{A}, \Gamma, G, \delta)$, any graded subalgebra $\mathcal{S}$ of the graded algebra $(\mathcal{A}, \Gamma)$ gives rise to the G-graded algebra $(\mathcal{S}, \Gamma|_{\mathcal{S}}, G, \delta|_{\mathcal{S}})$, where $\Gamma|_{\mathcal{S}}$ consists of the nonzero subspaces of the form $\mathcal{U}\cap\mathcal{S}$ for $\mathcal{U} \in \Gamma$, with $\delta|_{\mathcal{S}}(\mathcal{U} \cap \mathcal{S}) = \delta(\mathcal{U})$. When referring to a G-graded subalgebra of $(\mathcal{A}, \Gamma, G, \delta)$ we will mean a graded subalgebra $\mathcal{S}$ of $(\mathcal{A}, \Gamma)$, endowed with the G-grading above. The same applies to G-graded ideals.

- Given two G-graded algebras $(\mathcal{A}, \Gamma, G, \delta)$ and $(\mathcal{A}', \Gamma', G, \delta')$, an *isomorphism* $\varphi : (\mathcal{A}, \Gamma, G, \delta) \to (\mathcal{A}', \Gamma', G, \delta')$ is an isomorphism $\varphi : \mathcal{A} \to \mathcal{A}'$ such that $\varphi(\mathcal{A}_g) = \mathcal{A}'_g$ for each $g \in G$.

Definition A.2.2. *A grading Γ on an algebra $\mathcal{A}$ is called a* group-grading *if there is an abelian group G and a G-grading of the form (Γ, G, δ) (i.e., the first component of the G-grading is Γ).*

In this situation, we say that Γ can be realized as a G-grading, or by the G-grading (Γ, G, δ), and we will talk about the group-graded algebra $(\mathcal{A}, \Gamma)$.

Remark A.2.3. *If* $\mathrm{char}\,\mathbb{F} = 2$*, then the algebra* $\mathbb{F}\times\mathbb{F}$ *admits a unique group-grading: the trivial one. Thus the trivial grading is a fine group-grading, but it* *is not a fine grading, because the grading considered in Example A.1.4 is finer. Note that* $\mathbf{Aut}_{\mathbb{F}}(\mathbb{F}\times\mathbb{F}) = \mathsf{C}_2$, *the constant group scheme corresponding to the cyclic group of order* 2: C_2, *which is not diagonalizable because* $\operatorname{char}\mathbb{F} = 2$.

The next result, whose proof is straightforward, characterizes group-gradings and explains the adjective *universal* in the definition of the universal group:

Theorem A.2.4. *Let* $(\mathcal{A}, \Gamma)$ *be a graded algebra, with universal group* $\big(U(\Gamma), \delta_\Gamma^U\big)$. *Then* Γ *is a group-grading if and only if* δ_Γ^U *is one-to-one. In this case* $(\mathcal{A}, \Gamma, U(\Gamma), \delta_\Gamma^U)$ *is a* $U(\Gamma)$*-graded algebra.*

Moreover, if Γ *can be realized by the* G*-grading* (Γ, G, δ), *then there is a unique group homomorphism* $\varphi : U(\Gamma) \to G$, *such that the diagram*

$$\begin{tikzcd} & U(\Gamma) \arrow[dd, "\varphi"] \\ \Gamma \arrow[ur, "\delta_\Gamma^U"] \arrow[dr, "\delta"'] & \\ & G \end{tikzcd}$$

is commutative. (In other words, there is a unique homomorphism $\big(U(\Gamma), \delta_\Gamma^U\big) \to (G, \delta)$.*)*

A.3 The group grading induced by a grading

Given any grading Γ, there is always a natural group-grading attached to it.

Definition A.3.1. *Let* Γ *be a grading on the algebra* $\mathcal{A}$, *and let* $\big(U(\Gamma), \delta_\Gamma^U\big)$ *be its universal group. The coarsening* Γ_{gr} *defined by*

$$\Gamma_{\mathrm{gr}} := \left\{ \sum_{\delta_\Gamma^U(\mathcal{U})=u} \mathcal{U} \;\middle|\; u \in \delta_\Gamma^U(\Gamma) \right\}$$

is called the group-grading induced by Γ. *The grading* Γ_{gr} *can be realized by the* $U(\Gamma)$*-grading* $\big(\Gamma_{\mathrm{gr}}, U(\Gamma), \delta_{\Gamma_{\mathrm{gr}}}^U\big)$, *where*

$$\delta_{\Gamma_{\mathrm{gr}}}^U\Big(\sum_{\delta_\Gamma^U(\mathcal{U})=u} \mathcal{U}\Big) = u$$

for any $u \in \delta_\Gamma^U(\Gamma)$.

Theorem A.2.4 implies the next result:

Theorem A.3.2. *Let* Γ *be a grading on an algebra* $\mathcal{A}$ *with universal group* $\big(U(\Gamma), \delta_\Gamma^U\big)$. *Then* Γ *is a group-grading if and only if* $\Gamma = \Gamma_{\mathrm{gr}}$.

A.4 Product gradings

Definition A.4.1. *Let* $(\mathcal{A}^i, \Gamma^i)$ *be a graded* $\mathbb{F}$*-algebra,* $i = 1, \dots, n$. *The grading on* $\mathcal{A}^1 \times \cdots \times \mathcal{A}^n$ *given by:*

$$\Gamma^1 \times \cdots \times \Gamma^n := \bigcup_{i=1}^{n} \left\{ 0 \times \cdots \times \mathcal{U} \times \cdots \times 0 \mid \mathcal{U} \in \Gamma^i \right\}.$$

is called the product grading *of the* Γ^i*'s.*

The universal group-grading of the product grading is

$$\left(U(\Gamma^1 \times \cdots \times \Gamma^n), \delta^U_{\Gamma^1 \times \cdots \times \Gamma^n}\right), \tag{A.4.1}$$

given by the following formulas:

$$U(\Gamma^1 \times \cdots \times \Gamma^n) = U(\Gamma^1) \times \cdots \times U(\Gamma^n),$$
$$\delta^U_{\Gamma^1 \times \cdots \times \Gamma^n}(0 \times \cdots \times \mathcal{U} \times \cdots \times 0) = \left(e, \cdots, \delta^U_{\Gamma^i}(\mathcal{U}), \cdots, e\right) \ \forall \mathcal{U} \in \Gamma^i, \ \forall i = 1, \dots, n.$$

Example A.4.2. *The grading in Example A.1.4, which is not a group-grading, is the product grading of the trivial gradings on the two copies of* $\mathbb{F}$.

As the previous example shows, even if $\Gamma^1, \dots, \Gamma^n$ are group-gradings, the product grading may fail to be so. Therefore we need a different definition of product grading for group-gradings.

Definition A.4.3. *Let* G^i *be an abelian group and let* $(\mathcal{A}^i, \Gamma^i, G^i, \delta^i)$ *be a* G^i*-group-graded algebra,* $i = 1, \dots, n$*, then the* product group-grading $(\Gamma^1, G^1, \delta^1) \times \cdots \times (\Gamma^n, G^n, \delta^n)$ *is the group-grading on* $\mathcal{A}^1 \times \cdots \times \mathcal{A}^n$ *by the abelian group* $G^1 \times \cdots \times G^n$ *with:*

$$\begin{aligned}
\left(\mathcal{A}^1 \times \cdots \times \mathcal{A}^n\right)_{(e,\dots,e)} &= \mathcal{A}^1_e \times \cdots \times \mathcal{A}^n_e, \\
\left(\mathcal{A}^1 \times \cdots \times \mathcal{A}^n\right)_{(e,\dots,g_i,\dots,e)} &= 0 \times \cdots \times \mathcal{A}^i_{g_i} \times \cdots \times 0, \ i = 1, \dots, n, \ e \neq g_i \in G^i \\
\left(\mathcal{A}^1 \times \cdots \times \mathcal{A}^n\right)_{(g_1,\dots,g_n)} &= 0, \ \text{if there are at least two indices } 1 \leq i < j \leq n \\
&\qquad \text{with } g_i \neq e \neq g_j.
\end{aligned}$$

Our next result shows the naturality of this definition (Definition 2.4.4).

Theorem A.4.4. *Let* Γ^i *be a group-grading on an algebra* $\mathcal{A}^i$*, and let* $\left(U(\Gamma^i), \delta^U_{\Gamma^i}\right)$ *be its universal group,* $i = 1, \dots, n$. *Then the product group-grading*

$$\left(\Gamma^1, U(\Gamma^1), \delta^U_{\Gamma^1}\right) \times \cdots \times \left(\Gamma^n, U(\Gamma^n), \delta^U_{\Gamma^n}\right)$$

coincides with the induced group-grading

$$\Big((\Gamma^1 \times \cdots \times \Gamma^n)_{\mathrm{gr}}, U(\Gamma^1 \times \cdots \times \Gamma^n), \delta^U_{(\Gamma^1 \times \cdots \times \Gamma^n)_{\mathrm{gr}}}\Big).$$

(That is, the group-grading induced from the product grading $\Gamma^1 \times \cdots \times \Gamma^n$*, with its universal grading group.)*

Proof. We already know (see Theorem A.3.2 and Equation (A.4.1)) that

$$U\big((\Gamma^1 \times \cdots \times \Gamma^n)_{\mathrm{gr}}\big) = U(\Gamma^1 \times \cdots \times \Gamma^n) = U(\Gamma^1) \times \cdots \times U(\Gamma^n).$$

Now everything follows from the definition of the group-grading induced by a grading. □

Conclusiones

Encontramos las graduaciones (por grupos) en el producto tensorial de un álgebra de Cayley $\mathcal{C}^1$ y un álgebra de Hurwitz $\mathcal{C}^2$. Para el caso donde $\dim \mathcal{C}^2 = 1$, i.e. $\mathcal{C}^2 = \mathbb{F}$, tenemos que $\mathcal{C}^2 \otimes \mathbb{F} \simeq \mathcal{C}^2$ y las graduaciones en el álgebra de Cayley ya son conocidas. Para los casos donde $\dim \mathcal{C}^2 = 2$ y 4 dimos una descripción explícita de graduaciones en $\mathcal{C}^1 \otimes \mathcal{C}^2$ usando las graduaciones en $\mathcal{C}^1$ y $\mathcal{C}^2$.

Si $\dim \mathcal{C}^2 = 2$ tenemos dos graduaciones finas, salvo equivalencia, en $\mathcal{C}^1 \otimes \mathcal{C}^2$ con grupos universales $(\mathbb{Z}/2)^4$ y $\mathbb{Z}^2 \times \mathbb{Z}/2$.

Si $\dim \mathcal{C}^2 = 4$ tenemos cuatro graduaciones finas, salvo equivalencia, en $\mathcal{C}^1 \otimes \mathcal{C}^2$ con grupos universales $(\mathbb{Z}/2)^5$, $(\mathbb{Z}/2)^3 \times \mathbb{Z}$, $\mathbb{Z}^2 \times (\mathbb{Z}/2)^2$ y $\mathbb{Z}^3$.

Para el caso donde $\mathcal{C}^2$ es un álgebra de Cayley primero clasificamos graduaciones en $\mathcal{C}^1 \times \mathcal{C}^2$ y, usando un isomorfismo entre los esquemas de automorfismos de $\mathcal{C}^1 \times \mathcal{C}^2$ y $\mathcal{C}^1 \otimes \mathcal{C}^2$, los transferimos a graduaciones en $\mathcal{C}^1 \otimes \mathcal{C}^2$. Tenemos seis graduaciones finas, salvo equivalencia, en $\mathcal{C}^1 \otimes \mathcal{C}^2$:

- tres tales que en la graduación inducida en $(\mathcal{C}^1)_0 \otimes \mathbb{F}1 \oplus \mathbb{F}1 \otimes (\mathcal{C}^2)_0$, dicha álgebra no es simple-graduada. Sus grupos universales son $\mathbb{Z}^4$, $\mathbb{Z}^2 \times (\mathbb{Z}/2)^3$ y $(\mathbb{Z}/2)^6$.

- tres tales que en la graduación inducida en $(\mathcal{C}^1)_0 \otimes \mathbb{F}1 \oplus \mathbb{F}1 \otimes (\mathcal{C}^2)_0$, dicha álgebra es simple-graduada y sus grupos universales son $\mathbb{Z}/2 \times \mathbb{Z}^2$, $(\mathbb{Z}/2)^4$ y $\mathbb{Z}/4 \times (\mathbb{Z}/2)^2$.

Para obtener graduaciones en $\mathcal{C}^1 \times \mathcal{C}^2$ generalizamos los resultados para encontrar una clasificación de graduaciones en un producto directo finito de álgebras simples de dimensión finita (álgebras semisimples). Obtuvimos algunos resultados acerca de dichas graduaciones usando la teoría de álgebras lazo. También dimos una clasificación de graduaciones finas en álgebras semisimples. Dimos algunos ejemplos de cómo aplicar estos resultados, uno de ellos es la clasificación de graduaciones en $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$. Además tenemos el ejemplo de la superálgebra de Jordan de Kac, que también ejemplifica el proceso seguido para obtener graduaciones en $\mathcal{C}^1 \otimes \mathcal{C}^2$.

La autora ha considerado como trabajo a futuro encontrar las graduaciones (por grupos) en las álgebras de Lie asociadas a las graduaciones obtenidas en $\mathcal{C}^1 \otimes \mathcal{C}^2$ donde $\mathcal{C}^1$ es un álgebra de Cayley y $\mathcal{C}^2$ es un álgebra de Hurwitz, esto usando la construcción TKK modificada. Esto resultará en graduaciones en las álgebras de Lie simples centrales de tipo F_4, E_6, E_7 y E_8.

En el presente estado del arte, las graduaciones en algunas de las álgebras estructurables simples centrales ya son conocidas. Las graduaciones en álgebras de Jordan (ver [EK13, Capítulo 5]) y en álgebras asociativas (ver [EK13, Capítulo 2]) son ya conocidas. Sólo algunas graduaciones en álgebras de matrices 2×2 construidas a partir del álgebra de Jordan de una forma cúbica admisible son conocidas. Graduaciones en álgebras con involución construidas a partir de una forma hermitiana no se conocen aún. El álgebra simple central de dimensión 35 construida a partir de un álgebra de octoniones ha sido estudiada por Diego Aranda-Orna pero los resultados no han sido publicados aún. Y nosotros hemos encontrado las graduaciones en el caso restante de álgebras estructurables: el producto tensorial de álgebras de Hurwitz, así que esto ayuda a completar la clasificación de graduaciones en álgebras estructurables y ayudará a construir las graduaciones en las álgebras de Lie asociadas.

Conclusions

We found the gradings (by groups) on the tensor product of a Cayley algebra $\mathcal{C}^1$ and a Hurwitz algebra $\mathcal{C}^2$. For the case where $\dim \mathcal{C}^2 = 1$, i.e. $\mathcal{C}^2 = \mathbb{F}$, we have $\mathcal{C}^2 \otimes \mathbb{F} \simeq \mathcal{C}^2$ and gradings on the Cayley algebra are already known. For the cases where $\dim \mathcal{C}^2 = 2, 4$ we gave an explicit description of gradings on $\mathcal{C}^1 \otimes \mathcal{C}^2$ from the gradings on $\mathcal{C}^1$ and $\mathcal{C}^2$.

If $\dim \mathcal{C}^2 = 2$ we have two fine gradings, up to equivalence, on $\mathcal{C}^1 \otimes \mathcal{C}^2$ with universal groups $(\mathbb{Z}/2)^4$ and $\mathbb{Z}^2 \times \mathbb{Z}/2$.

If $\dim \mathcal{C}^2 = 4$ we have four fine gradings, up to equivalence, on $\mathcal{C}^1 \otimes \mathcal{C}^2$ with universal groups $(\mathbb{Z}/2)^5$, $(\mathbb{Z}/2)^3 \times \mathbb{Z}$, $\mathbb{Z}^2 \times (\mathbb{Z}/2)^2$ and $\mathbb{Z}^3$.

For the case where $\mathcal{C}^2$ is a Cayley algebra we first classified gradings on $\mathcal{C}^1 \times \mathcal{C}^2$ and, using an isomorphism between the automorphism schemes of $\mathcal{C}^1 \times \mathcal{C}^2$ and $\mathcal{C}^1 \otimes \mathcal{C}^2$, we transferred them to gradings on $\mathcal{C}^1 \otimes \mathcal{C}^2$. We have six fine gradings, up to equivalence, on $\mathcal{C}^1 \otimes \mathcal{C}^2$:

- three such that for the induced grading on $(\mathcal{C}^1)_0 \otimes \mathbb{F}1 \oplus \mathbb{F}1 \otimes (\mathcal{C}^2)_0$, it is not graded-simple. Their universal groups are $\mathbb{Z}^4$, $\mathbb{Z}^2 \times (\mathbb{Z}/2)^3$ and $(\mathbb{Z}/2)^6$.
- three such that for the induced grading on $(\mathcal{C}^1)_0 \otimes \mathbb{F}1 \oplus \mathbb{F}1 \otimes (\mathcal{C}^2)_0$, it is graded-simple and their universal groups are $\mathbb{Z}/2 \times \mathbb{Z}^2$, $(\mathbb{Z}/2)^4$ and $\mathbb{Z}/4 \times (\mathbb{Z}/2)^2$.

In order to obtain gradings on $\mathcal{C}^1 \times \mathcal{C}^2$ we generalized the results to find a classification of gradings on a finite direct product of simple finite-dimensional algebras (semisimple algebras). We obtained some results about such gradings using the theory of loop algebras. We also gave a classification of fine gradings on semisimple algebras. We gave some examples of how to apply these results, one of them is the classification of gradings on $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$. We also have the example of the Kac's Jordan superalgebra, which also exemplify the process followed to obtain gradings on $\mathcal{C}^1 \otimes \mathcal{C}^2$.

The author has considered for future work to find the group-gradings on the Lie algebras associated to the obtained gradings on $\mathcal{C}^1 \otimes \mathcal{C}^2$ where $\mathcal{C}^1$ is a

Cayley algebra and $\mathcal{C}^2$ is a Hurwitz algebra, this by using the modified TKK construction. This will result in gradings on the central simple Lie algebras of type F_4, E_6, E_7 and E_8.

In the present state of the art, gradings on some of the central simple structurable algebras are known. The gradings on Jordan algebras (see [EK13, Chapter 5]) and on associative algebras (see [EK13, Chapter 2]) are already known. Only some gradings on 2×2 matrix algebras constructed from the Jordan algebra of an admissible cubic form are known. Gradings on algebras with involution constructed from an hermitian form are unknown yet. The 35-dimensional central simple algebra constructed from an octonion algebra has been studied by Diego Aranda-Orna but the results have not been published yet. And we have found the gradings on the remaining type of structurable algebras: the tensor product of Hurwiz algebras, so it helps to complete the classification of group-gradings on structurable algebras and will help to construct the group-gradings on the associated Lie algebras.

Bibliography

[Al78] B. N. Allison, *A class of nonassociative algebras with involution containing the class of Jordan algebras* Math. Ann. 237, 133-156, Springer-Verlag (1978).

[Al79] B. N. Allison, *Models of isotropic simple Lie algebras* Communications in Algebra, 7:17, 1835-1875 (1979).

[ABFP] B. Allison, S. Berman, J. Faulkner and A. Pianzola, *Realization of graded-simple algebras as loop algebras* (2013).

[Ara17] D. Aranda-Orna. *Gradings on simple exceptional Jordan systems and structurable algebras.* Doctoral Dissertation. Universidad de Zaragoza, 2017.

[AEK14] D. Aranda-Orna, Diego, A. Elduque and M. Kochetov. *A $\mathbb{Z}_3^4$-grading on a 56-dimensional simple structurable algebra and related fine gradings on the simple Lie algebras of type E.* Comment. Math. Univ. Carolin. **55** (2014), no. 3, 285–313.

[BE02] G. Benkart and A. Elduque, *A new construction of the Kac Jordan superalgebra*, Proc. Amer. Math. Soc. **130** (2002), no. 11, 3209–3217.

[BM92] S. Berman and R.V. Moody, *Lie algebras graded by finite root systems and the intersection matrix algebras of Slodowy.* Invent. Math. **108** (1992), 323–347.

[CDE18] Alejandra S. Córdova-Martínez, A. Darehgazani and A. Elduque, *On Kac's Jordan superalgebra*, arXiv:1701.06798.

[CDM10] A.J. Calderón Martín, C. Draper Fontanals, and C. Martín González, *Gradings on the Kac superalgebra*, J. Algebra **324** (2010), no. 12, 3249–3261.

[DE16] C. Draper and A. Elduque, *An overview of fine gradings on simple Lie algebras.* Note Mat. **36** (2016), suppl. 1, 15–34.

[DEM11] C. Draper Fontanals, A. Elduque and C. Martín González, *Fine gradings on exceptional simple Lie superalgebras*. Int J Math (2011). 22(12):1823-1855.

[Eld10] A. Elduque, *Fine gradings on simple classical Lie algebras*. J. Algebra **324** (2010), no. 12, 3532–3571.

[Eld15] A. Elduque, *Fine gradings and gradings by root systems on simple Lie algebras*. Rev. Mat. Iberoam. **31** (2015), no. 1, 245–266.

[EK13] A. Elduque and M. Kochetov, *Gradings on Simple Lie Algebras*. Mathematical Surveys and Monographs, **189**, American Mathematical Society, Providence, RI (2013).

[ELS07] A. Elduque, J. Laliena, and S. Sacristán, *The Kac Jordan superalgebra: automorphisms and maximal subalgebras* Proc. Amer. Math. Soc. **135** (2007), no. 12, 3805–3813.

[EM94] A. Elduque and H. C. Myung, *Mutations of Alternative Algebras*, Mathematics and Its Applications, Kluwer Academic Publishers, vol 278 (1994).

[Jac1] N. Jacobson, *Basic Algebra I*, W. H. Freeman and Company, New York (1985).

[Jac2] N. Jacobson, *Basic Algebra II*, W. H. Freeman and Company, New York (1989).

[Jac78] N. Jacobson, *Lie algebras*, Dover, New York (1979).

[Kac77] V.G. Kac, *Classification of simple $\mathbb{Z}$-graded Lie superalgebras and simple Jordan superalgebras*, Comm. Algebra **5** (1977), no. 13, 1375–1400.

[K72] I.L. Kantor, *Some generalizations of Jordan algebras*, Trudy Sem. Vektor. Tenzor. Anal. 16 (1972), 407-499 (Russian).

[MPP] P.J. Morandi, J.M. Pérez-Izquierdo and S. Pumplün, *On the tensor product of composition algebras*, Journal of Algebra **243**, 41-68 (2001).

[PZ89] J. Patera and H. Zassenhaus, *On Lie gradings. I*, Linear Algebra Appl. **112**, 87-159. MR MR976333 (90h:17048) (1989).

[RZ15] M.L. Racine and E.I. Zelmanov, *An octonionic construction of the Kac superalgebra K10*, Proc. Amer. Math. Soc. **143** (2015), no. 3, 1075–1083.

[Sm90] O. N. Smirnov, *Simple and semisimple structurable algebras*, Algebra and Logic 29, 331-336 (1990).

[Swe69] W. A. Benjamin, *Hopf algebras*, Mathematics Lecture Note Series, New York, MR MR0252485 (40 num.5705) (1969).

[Wat79] W. C. Waterhouse, *Introduction to affine group schemes*, Graduate Texts in Mathematics, vol 66, Springer-Verlag, New York, MR MR547117 (82e: 14003) (1979).