

Superintegrable deformations of superintegrable systems : Quadratic superintegrability and higher-order superintegrability

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Abstract

The superintegrability of four Hamiltonians $\widetilde{H}_r = \lambda H_r$, $r = a, b, c, d$, where H_r are known Hamiltonians and λ is a certain function defined on the configuration space and depending of a parameter κ , is studied. The new Hamiltonians, and the associated constants of motion J_{ri} , $i = 1, 2, 3$, are continuous functions of the parameter κ . The first part is concerned with separability and quadratic superintegrability (the integrals of motion are quadratic in the momenta) and the second part is devoted to the existence of higher-order superintegrability. The results obtained in the second part are related with the TTW and the PW systems.

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1 Introduction

A Hamiltonian system with n degrees of freedom is called Liouville (or Liouville-Arnold) integrable if it is endowed with n functionally independent integrals of motion in involution (including the Hamiltonian itself). Some integrable systems, as the harmonic oscillator or the Kepler problem, admit more constants of motion than degrees of freedom; they are called superintegrable. Therefore a Hamiltonian with two degrees of freedom is said to be superintegrable if it admits two fundamental integrals of motion, I_1 and I_2 , that Poisson commute and a third independent integral I_3 . The additional integral I_3 has vanishing Poisson bracket with H but not necessarily with I_1 and I_2 .

It is known that if a system is separable (Hamilton-Jacobi separable in the classical case or Schrödinger separable in the quantum case) then it is integrable with integrals of motion of at most second order in momenta. Thus, if a system admits multiseparability (separability in several different systems of coordinates) then it is endowed with “quadratic superintegrability” (superintegrability with linear or quadratic integrals of motion).

Fris et al. studied [1] the two-dimensional Euclidean systems, which admit separability in more than one coordinate systems and obtained four families of potentials V_r , $r = a, b, c, d$, possessing three functionally independent integrals of motion (they were mainly interested in the quantum two-dimensional Schrödinger equation but the results obtained are also valid at the classical level). Then other authors studied similar problems on higher-dimensional Euclidean spaces [2]–[4], on two-dimensional spaces with a pseudo-Euclidean metric (Drach potentials) [5]–[8], or on curved spaces [9]–[15] (see [16] for a recent review on superintegrability that includes a long list of references).

For some time the studies on superintegrability were mainly concerned with “quadratic superintegrability” but recent studies have proved the existence of certain systems endowed with higher order superintegrability, that is, with integrals of motion which are polynomials in the momenta of higher order than two. We mention the Calogero-Moser system whose superintegrability is related with a Lax equation [17]–[19] (this formalism is not considered in this paper) and three important systems that are separable but in only one system of coordinates, the generalized SW system (caged anisotropic oscillator) [20]–[22], the Tremblay-Turbine-Winternitz (TTW) system [23]–[37], and the Post-Winternitz (PW) system [38]–[39]; in these three cases two of the integrals are quadratic but the third one is of higher order.

One important point is that although the number of superintegrable systems can be considered as rather limited, they are not however isolated systems but, on the contrary, they frequently appear grouped in families; for example, everyone of the above mentioned potentials V_r , $r = a, b, c, d$, has structure of a three-dimensional vector space. Now in this paper we consider the idea of one-parameter deformations of a given Hamiltonian, that is, families of Hamiltonians depending of a real parameter κ that are superintegrable for all the values of κ (in the domain of the parameter) and that for $\kappa = 0$ they reduce to superintegrable Hamiltonians previously studied.

The main objective of this article is twofold. First, study the existence of superintegrable deformations of the four two-dimensional Euclidean systems (some of the results obtained are related with some nonlinear systems studied in [40]–[43]) and, second, study the existence of superintegrable deformations of the TTW and the PW systems. In these two cases we prove the superintegrability and we obtain the explicit expression of the third integral (that we recall is of higher-order) as the product of powers of two particular complex functions (this complex formalism is very similar to the approach presented in [30], [37], for the TTW system and in [39] for the PW system)

The plan of the article is as follows: Sec. 2 is devoted to recall the main characteristics of the four two-dimensional potentials which separability in two different coordinate systems in the Euclidean plane. Then Sec. 3 and 4 are concerned with quadratic superintegrability and Sec. 5 with higher order

constants of motion. In Sec. 3 we first introduce the idea of deformation (depending of one parameter κ) of a Hamiltonian and then we study four families of superintegrable families endowed with multiple separability. Sec. 5 has two parts. In the first part we study a κ -dependent Hamiltonian related with the harmonic oscillator, that can be considered as a deformation of the TFW system, and in the second part a κ -dependent Hamiltonian related with the Kepler problem, that can be considered as a deformation of the PW system. Finally in Sec. 6 we make some final comments.

2 Superintegrability with quadratic constants of motion in the Euclidean plane

Let us denote by V_r , $r = a, b, c, d$, the four two-dimensional potentials with separability in two different coordinate systems in the Euclidean plane.

The two first potentials, V_a y V_b , are related with the harmonic oscillator. They satisfy the equation $V_{xy} = 0$ and correspond, therefore, to a direct sum of one-degree of freedom systems.

(a) The following potential

$$V_a = \left(\frac{1}{2}\right)\alpha^2(x^2 + y^2) + \frac{k_2}{x^2} + \frac{k_3}{y^2} \quad (1)$$

is separable in (i) Cartesian coordinates and (ii) polar coordinates. The three constants of motion, I_{a1} , I_{a2} , y I_{a3} , are given by

$$\begin{aligned} I_{a1} &= p_x^2 + \alpha^2 x^2 + \frac{2k_2}{x^2}, \\ I_{a2} &= p_y^2 + \alpha^2 y^2 + \frac{2k_3}{y^2}, \\ I_{a3} &= (xp_y - yp_x)^2 + 2k_2\left(\frac{y}{x}\right)^2 + 2k_3\left(\frac{x}{y}\right)^2. \end{aligned}$$

(b) The following potential

$$V_b = \left(\frac{1}{2}\right)\alpha^2(4x^2 + y^2) + \frac{k_2}{x^2} + k_3x \quad (2)$$

is separable in (i) Cartesian coordinates and (ii) parabolic coordinates. The three constants of motion, I_{b1} , I_{b2} , y I_{b3} , are given by

$$\begin{aligned} I_{b1} &= p_x^2 + 4\alpha^2 x^2 + 2k_3x, \\ I_{b2} &= p_y^2 + \alpha^2 y^2 + \frac{2k_2}{y^2}, \\ I_{b3} &= (xp_y - yp_x)p_y - \alpha^2 xy^2 + k_2\left(\frac{2x}{y^2}\right) - \frac{k_3}{2y^2}. \end{aligned}$$

The two other potentials, V_c y V_d , are related with the Kepler problem.

(c) The following potential

$$V_c = \frac{k_1}{\sqrt{x^2 + y^2}} + \frac{k_2}{y^2} + \frac{k_3x}{y^2\sqrt{x^2 + y^2}} \quad (3)$$

is separable in (i) polar coordinates and (ii) parabolic coordinates. The first constant of motion is the Hamiltonian itself, that is $I_{c1} = H_c$, and the other two constants of motion, I_{c2} , and I_{c3} , are given by

$$\begin{aligned} I_{c2} &= (xp_y - yp_x)^2 + \frac{2k_2x^2}{y^2} + \frac{2k_3x\sqrt{x^2+y^2}}{y^2}, \\ I_{c3} &= (xp_y - yp_x)p_y + \frac{k_1x}{\sqrt{x^2+y^2}} + \frac{2k_2x}{y^2} + \frac{k_3(2x^2+y^2)}{y^2\sqrt{x^2+y^2}}. \end{aligned}$$

(d) The following potential

$$V_d = \frac{k_1}{\sqrt{x^2+y^2}} + k_2 \frac{[\sqrt{x^2+y^2}+x]^{1/2}}{\sqrt{x^2+y^2}} + k_3 \frac{[\sqrt{x^2+y^2}-x]^{1/2}}{\sqrt{x^2+y^2}} \quad (4)$$

is separable in (i) parabolic coordinates (a, b) and (ii) a second system of parabolic coordinates (α, β) obtained from (a, b) by a rotation. The first constant of motion is the Hamiltonian itself, that is $I_{d1} = H_d$, and the other two constants of motion, I_{d2} , and I_{d3} , are given by

$$\begin{aligned} I_{d2} &= (xp_y - yp_x)p_y + \frac{k_1x}{\sqrt{x^2+y^2}} - k_2 \frac{y[\sqrt{x^2+y^2}-x]^{1/2}}{\sqrt{x^2+y^2}} + k_3 \frac{y[\sqrt{x^2+y^2}+x]^{1/2}}{\sqrt{x^2+y^2}}, \\ I_{d3} &= (xp_y - yp_x)p_x - \frac{k_1y}{\sqrt{x^2+y^2}} - k_2 \frac{x[\sqrt{x^2+y^2}-x]^{1/2}}{\sqrt{x^2+y^2}} + k_3 \frac{x[\sqrt{x^2+y^2}+x]^{1/2}}{\sqrt{x^2+y^2}}. \end{aligned}$$

3 Four new superintegrable families endowed with multiple separability

Suppose we are given a Hamiltonian H ; then we can construct a new Hamiltonian \tilde{H} as $\tilde{H} = \lambda H$ where λ is a certain function defined on the configuration space. This new Hamiltonian represents a new and different dynamics; for example if H is defined on an Euclidean space then the new dynamics will be nonEuclidean. More particularly, we are interested in multipliers λ that preserve certain properties of H as integrability or separability. For example if H is defined in the Euclidean plane and is separable in Cartesian coordinates (x, y) and λ is of the form $\lambda = 1/\mu$, $\mu = f(x) + g(y)$, then \tilde{H} is separable in Cartesian coordinates as well, and if H is separable in polar coordinates (r, ϕ) and λ is of the form $\lambda = 1/\mu$, $\mu = f(r) + g(\phi)/r^2$, then \tilde{H} is also separable in polar coordinates. A more strong condition is that λ must preserve not just separability but multiple separability; this requirement will strongly restrict the form of the multiplier.

Another important property we wish to introduce is that the new Hamiltonian \tilde{H} must be a deformation of H . By deformation we mean that λ , and therefore \tilde{H} , will depend of a parameter κ in such a way that

- (i) The new Hamiltonian $\tilde{H}(\kappa)$ must be a continous function of κ (in a certain domain of the parameter).
- (ii) When $\kappa \rightarrow 0$ we have $\lambda \rightarrow 1$ and then the dynamics of the Euclidean Hamiltonian H is recovered.

In this section we study the separability and the superintegrability of four Hamiltonians $\tilde{H}_r(\kappa)$ obtained as deformations of the four Hamiltonians H_r , $r = a, b, c, d$.

3.1 Hamiltonian $\widetilde{H}_a(\kappa)$

Let us consider the Hamiltonian H_a

$$H_a = \frac{1}{2} (p_x^2 + p_y^2) + \frac{1}{2} \alpha^2 (x^2 + y^2) + \frac{k_2}{x^2} + \frac{k_3}{y^2} \quad (5)$$

and denote by λ_a the following multiplier

$$\lambda_a = \frac{1}{\mu_a}, \quad \mu_a(x, y) = 1 - \kappa r^2, \quad r^2 = x^2 + y^2.$$

Then the new κ -dependent Hamiltonian $\widetilde{H}_a(\kappa)$ defined as $\widetilde{H}_a(\kappa) = \lambda_a H_a$ is given by

$$\widetilde{H}_a(\kappa) = \frac{1}{2} \left(\frac{p_x^2 + p_y^2}{1 - \kappa r^2} \right) + \frac{1}{2} \alpha^2 \left(\frac{x^2 + y^2}{1 - \kappa r^2} \right) + \frac{k_2}{(1 - \kappa r^2) x^2} + \frac{k_3}{(1 - \kappa r^2) y^2}. \quad (6)$$

The parameter κ can take both positive and negative values. In the $\kappa < 0$ case the $\widetilde{H}_a(\kappa)$ dynamics is correctly defined for all the values of the variables; nevertheless when $\kappa > 0$, the Hamiltonian (and the associated dynamics) has a singularity at $1 - \kappa r^2 = 0$, so in this case the $\widetilde{H}_a(\kappa)$ dynamics is defined in the interior of the circle $r^2 = 1/\kappa$, $\kappa > 0$, that is the region in which kinetic term is positive definite.

(i) Cartesian separability

The Hamilton-Jacobi (H-J) equation takes the form

$$\frac{1}{2} \left(\frac{1}{1 - \kappa r^2} \right) \left[\left(\frac{\partial W}{\partial x} \right)^2 + \left(\frac{\partial W}{\partial y} \right)^2 \right] + \frac{1}{2} \alpha^2 \left(\frac{x^2 + y^2}{1 - \kappa r^2} \right) + \frac{k_2}{(1 - \kappa r^2) x^2} + \frac{k_3}{(1 - \kappa r^2) y^2} = E$$

so that if we assume that W can be written as $W = W_x(x) + W_y(y)$ then we can perform a separation of variables and arrive to the following two one-variable expressions

$$\begin{aligned} \left[\left(\frac{\partial W_x}{\partial x} \right)^2 + \alpha^2 x^2 + \frac{2k_2}{x^2} + 2\kappa E x^2 \right] &= K + E, \\ \left[\left(\frac{\partial W_y}{\partial y} \right)^2 + \alpha^2 y^2 + \frac{2k_3}{y^2} + 2\kappa E y^2 \right] &= -K + E, \end{aligned}$$

where K denotes the constant associated to the separability. Everyone of these two expressions determine a constant of motion. So the κ -dependent functions J_{a1} and J_{a2} given by

$$\begin{aligned} J_{a1} &= p_x^2 + \alpha^2 x^2 + \frac{2k_2}{x^2} + 2\kappa \widetilde{H}_a x^2, \\ J_{a2} &= p_y^2 + \alpha^2 y^2 + \frac{2k_3}{y^2} + 2\kappa \widetilde{H}_a y^2, \end{aligned}$$

are constants of motion satisfying the following properties

$$(i) \, dJ_{a1} \wedge dJ_{a2} \neq 0, \quad (ii) \, \{J_{a1}, J_{a2}\} = 0, \quad (iii) \, \widetilde{H}_a(\kappa) = \frac{1}{2} (J_{a1} + J_{a2}).$$

(ii) Polar separability

The Hamilton-Jacobi (H-J) equation takes the form

$$\frac{1}{2m} \left[\left(\frac{\partial W}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial W}{\partial \phi} \right)^2 \right] + \alpha^2 r^2 + \frac{k_2}{r^2 \cos^2 \phi} + \frac{k_3}{r^2 \sin^2 \phi} = (1 - \kappa r^2) E$$

so that if we assume that W is of the form $W = W_r(r) + W_\phi(\phi)$ we can perform a separation of variables and rewrite the equation as the sum of two one-variable summands

$$\left[r^2 \left(\frac{\partial W_r}{\partial r} \right)^2 + \alpha^2 r^4 - 2Er^2(1 - \kappa r^2) \right] + \left[\left(\frac{\partial W_\phi}{\partial \phi} \right)^2 + \frac{2k_2}{\cos^2 \phi} + \frac{2k_3}{\sin^2 \phi} \right] = 0$$

so that the following function

$$J_{a3} = p_\phi^2 + \frac{2k_2}{\cos^2 \phi} + \frac{2k_3}{\sin^2 \phi}$$

is a constant of motion.

We summarize the results in the following proposition.

Proposition 1 *The κ -dependent Hamiltonian $\widetilde{H}_a(\kappa)$ is H-J separable in Cartesian (x, y) and polar (r, ϕ) coordinates and it is endowed with the following three quadratic constants of motion*

$$\begin{aligned} J_{a1} &= \frac{1}{(1 - \kappa r^2)} \left((1 - \kappa y^2) p_x^2 + \kappa x^2 p_y^2 + \alpha^2 x^2 + \frac{2k_2(1 - \kappa y^2)}{x^2} + \frac{2k_3 x^2}{y^2} \right), \\ J_{a2} &= \frac{1}{(1 - \kappa r^2)} \left((1 - \kappa x^2) p_y^2 + \kappa y^2 p_x^2 + \alpha^2 y^2 + \frac{2k_2 y^2}{x^2} + \frac{2k_3(1 - \kappa x^2)}{y^2} \right), \\ J_{a3} &= (xp_y - yp_x)^2 + 2k_2 \left(\frac{y}{x} \right)^2 + 2k_3 \left(\frac{x}{y} \right)^2. \end{aligned}$$

We note that the two first functions satisfy the correct limit when $\kappa \rightarrow 0$, that is $\lim_{\kappa \rightarrow 0} J_{ai} = I_{ai}$, $i = 1, 2$. Concerning the third function J_{a3} it is κ -independent and it coincides with the original constant I_{a3} .

3.2 Hamiltonian $\widetilde{H}_b(\kappa)$

Let us consider the Hamiltonian H_b

$$H_b = \frac{1}{2} (p_x^2 + p_y^2) + \frac{1}{2} \alpha^2 (4x^2 + y^2) + \frac{k_2}{y^2} + k_3 x \quad (7)$$

and denote by λ_b the following multiplier

$$\lambda_b = \frac{1}{\mu_b}, \quad \mu_b(x, y) = 1 - \kappa x.$$

Then the new κ -dependent Hamiltonian $\widetilde{H}_b(\kappa)$ is thus given by

$$\widetilde{H}_b(\kappa) = \frac{1}{2} \left(\frac{p_x^2 + p_y^2}{1 - \kappa x} \right) + \frac{1}{2} \alpha^2 \left(\frac{4x^2 + y^2}{1 - \kappa x} \right) + \frac{k_2}{(1 - \kappa x) y^2} + \frac{k_3 x}{(1 - \kappa x)}. \quad (8)$$

(i) Cartesian separability

The H-J equation takes the form

$$\frac{1}{2} \left(\frac{1}{1 - \kappa x} \right) \left[\left(\frac{\partial W}{\partial x} \right)^2 + \left(\frac{\partial W}{\partial y} \right)^2 \right] + \frac{1}{2} \alpha^2 \left(\frac{4x^2 + y^2}{1 - \kappa x} \right) + \frac{k_2}{(1 - \kappa x) y^2} + \frac{k_3 x}{(1 - \kappa x)} = E$$

so that if we assume that the function W is of the form $W = W_x(x) + W_y(y)$ we can perform a separation of variables and arrive to

$$\left[\left(\frac{\partial W_x}{\partial x} \right)^2 + 4\alpha^2 x^2 + 2k_3 x + 2\kappa E x \right] + \left[\left(\frac{\partial W_y}{\partial y} \right)^2 + \alpha^2 y^2 + \frac{2k_2}{y^2} \right] = 2E.$$

This means that the following two functions

$$J_{b1} = p_x^2 + 4\alpha^2 x^2 + 2k_3 x + 2\kappa x \widetilde{H}_b, \quad J_{b2} = p_y^2 + \alpha^2 y^2 + \frac{2k_2}{y^2},$$

are constants of motion satisfying the following properties

$$(i) dJ_{b1} \wedge dJ_{b2} \neq 0, \quad (ii) \{J_{b1}, J_{b2}\} = 0, \quad (iii) \widetilde{H}_b(\kappa) = \frac{1}{2}(J_{b1} + J_{b2}).$$

(ii) Parabolic separability

If we introduce the following change

$$(x, y) \rightarrow (a, b), \quad x = a^2 - b^2, \quad y = 2ab,$$

then the κ -dependent Hamiltonian becomes

$$\widetilde{H}_b(\kappa) = \frac{1}{2} \frac{1}{1 - \kappa(a^2 - b^2)} \left(\frac{p_a^2 + p_b^2}{a^2 + b^2} \right) + \frac{1}{1 - \kappa(a^2 - b^2)} \left[\frac{1}{2} \alpha^2 \left(\frac{a^6 + b^6}{a^2 + b^2} \right) + \frac{k_2}{a^2 b^2} + k_3 \frac{a^4 - b^4}{a^2 b^2} \right]$$

and the H-J equation

$$\frac{1}{2} \frac{1}{(a^2 + b^2)} \left[\left(\frac{\partial W}{\partial a} \right)^2 + \left(\frac{\partial W}{\partial b} \right)^2 \right] + \left[\frac{1}{2} m \alpha^2 \left(\frac{a^6 + b^6}{a^2 + b^2} \right) + \frac{k_2}{a^2 b^2} + k_3 \frac{a^4 - b^4}{a^2 b^2} \right] = (1 - \kappa(a^2 - b^2))E$$

also admits separation of variables

$$\left[\left(\frac{\partial W_a}{\partial a} \right)^2 + \alpha^2 a^6 + \frac{2k_2}{a^2} + 2k_3 a^4 - 2(a^2 - \kappa a^4)E \right] + \left[\left(\frac{\partial W_b}{\partial b} \right)^2 + \alpha^2 b^6 + \frac{2k_2}{b^2} - 2k_3 b^4 - 2(b^2 + \kappa b^4)E \right] = 0$$

The result is that the following function

$$\begin{aligned} J_{b3} &= p_a^2 + \alpha^2 a^6 + \frac{2k_2}{a^2} + 2k_3 a^4 - 2(a^2 - \kappa a^4) \widetilde{H}_b(\kappa), \\ &= - \left[p_b^2 + \alpha^2 b^6 + \frac{2k_2}{b^2} - 2k_3 b^4 - 2(b^2 + \kappa b^4) \widetilde{H}_b(\kappa) \right] \end{aligned}$$

is also a constant of motion.

The following proposition summarizes these results.

Proposition 2 *The κ -dependent Hamiltonian $\widetilde{H}_b(\kappa)$ is H-J separable both in Cartesian coordinates (x, y) and in parabolic coordinates (a, b) , and it is endowed with the following three quadratic constants of motion*

$$\begin{aligned} J_{b1} &= \frac{1}{(1 - \kappa x)} \left(p_x^2 + \kappa x p_y^2 + \alpha^2 (4x + \kappa y^2) x + \frac{2k_2 \kappa x}{y^2} + 2k_3 x \right), \\ J_{b2} &= p_y^2 + \alpha^2 y^2 + \frac{2k_2}{y^2}, \\ J_{b3} &= (x p_y - y p_x) p_y - \kappa \frac{y^2 (p_x^2 + p_y^2)}{4(1 - \kappa x)} + \frac{1}{(1 - \kappa x)} \left(-\frac{\alpha^2}{4} y^2 (4x + \kappa y^2) + \frac{k_2}{2y^2} (4x - \kappa(4x^2 + y^2)) - k_3 \frac{y^2}{2} \right). \end{aligned}$$

3.3 Hamiltonian $\widetilde{H}_c(\kappa)$

Let us consider the Hamiltonian H_c

$$H_c = \frac{1}{2m} (p_x^2 + p_y^2) + \frac{k_1}{\sqrt{x^2 + y^2}} + \frac{k_2}{y^2} + k_3 \frac{x}{y^2 \sqrt{x^2 + y^2}} \quad (9)$$

and denote by λ_c the following multiplier

$$\lambda_c = \frac{1}{\mu_c}, \quad \mu_c(x, y) = 1 - \frac{\kappa}{r}.$$

Then the new κ -dependent Hamiltonian $\widetilde{H}_c(\kappa)$ is

$$\widetilde{H}_c(\kappa) = \frac{1}{2} \left(\frac{r}{r - \kappa} \right) (p_x^2 + p_y^2) + \frac{k_1}{r - \kappa} + \frac{k_2 r}{(r - \kappa) y^2} + \frac{k_3 x}{(1 - \kappa r) y^2}. \quad (10)$$

(i) Polar separability

The H-J equation takes the form

$$\frac{1}{2} \left(\frac{r}{r - \kappa} \right) \left[\left(\frac{\partial W}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial W}{\partial \phi} \right)^2 \right] + \frac{k_1}{r - \kappa} + \frac{k_2 r}{(r - \kappa) r \sin^2 \phi} + \frac{k_3 \cos \phi}{(r - \kappa) r \sin^2 \phi} = E$$

so that if we assume that W is of the form $W = W_r(r) + W_\phi(\phi)$ we can perform a separation of variables and rewrite the equation as the sum of two one-variable summands

$$\left[r^2 \left(\frac{\partial W_r}{\partial r} \right)^2 + 2k_1 r - 2r(r - \kappa)E \right] + \left[\left(\frac{\partial W_\phi}{\partial \phi} \right)^2 + \frac{2k_2}{\sin^2 \phi} + 2k_3 \frac{\cos \phi}{\sin^2 \phi} \right] = 0$$

so that the following function

$$J_{c2} = p_\phi^2 + \frac{2k_2}{\sin^2 \phi} + 2k_3 \frac{\cos \phi}{\sin^2 \phi}$$

is a constant of motion.

(ii) Parabolic separability

The Hamiltonian \widetilde{H}_c takes the following form in parabolic coordinates

$$\widetilde{H}_c(\kappa) = \frac{1}{2} \frac{1}{(a^2 + b^2) - \kappa} (p_a^2 + p_b^2) + \frac{1}{(a^2 + b^2) - \kappa} \left[2k_1 + k_2 \frac{a^2 + b^2}{a^2 b^2} + k_3 \frac{a^2 - b^2}{a^2 b^2} \right]$$

so that the H-J equation becomes

$$\left[\left(\frac{\partial W}{\partial a} \right)^2 + \left(\frac{\partial W}{\partial b} \right)^2 \right] + 2 \left[2k_1 + k_2 \frac{a^2 + b^2}{a^2 b^2} + k_3 \frac{a^2 - b^2}{a^2 b^2} \right] = 2((a^2 + b^2) - \kappa)E$$

and it leads to

$$\begin{aligned} \left[\left(\frac{\partial W_a}{\partial a} \right)^2 + 2k_1 + \frac{2k_2}{a^2} - \frac{2k_3}{a^2} - 2a^2 E \right] &= -\kappa E + K \\ \left[\left(\frac{\partial W_b}{\partial b} \right)^2 + 2k_1 + \frac{2k_2}{b^2} + \frac{2m k_3}{b^2} - 2b^2 E \right] &= -\kappa E - K \end{aligned}$$

so that the following two functions

$$J_{c3a} = p_a^2 + 2k_1 + \frac{2k_2}{a^2} - \frac{2k_3}{a^2} - 2a^2 \widetilde{H}_c(\kappa),$$

$$J_{c3b} = p_b^2 + 2k_1 + \frac{2k_2}{b^2} + \frac{2k_3}{b^2} - 2b^2 \widetilde{H}_c(\kappa),$$

are two constants of motion representing two different κ -deformations of I_{c3}

$$(i) dJ_{c3a} \wedge dJ_{c3b} \neq 0, \quad (ii) \{J_{c3a}, J_{c3b}\} = 0, \quad (iii) \lim_{\kappa \rightarrow 0} J_{c3a} = \lim_{\kappa \rightarrow 0} J_{c3b} = I_{c3}.$$

Nevertheless, as the only difference between them is a term proportional to $\widetilde{H}_c(\kappa)$ (with κ as coefficient), we consider as a more appropriate third constant the following function

$$\begin{aligned} J_{c3} &= p_a^2 + 2k_1 + \frac{2k_2}{a^2} - \frac{2k_3}{a^2} - 2a^2 \widetilde{H}_c(\kappa) + \kappa \widetilde{H}_c(\kappa) \\ &= - \left[p_b^2 + 2k_1 + \frac{2k_2}{b^2} + \frac{2k_3}{b^2} - 2b^2 \widetilde{H}_c(\kappa) + \kappa \widetilde{H}_c(\kappa) \right] \end{aligned}$$

that in a more detailed way is as follows

$$\begin{aligned} J_{c3} &= \frac{1}{a^2 + b^2 - \kappa} \left[(a^2 p_b^2 - b^2 p_a^2) + \frac{\kappa}{2} (p_a^2 - p_b^2) + 2k_1 (a^2 - b^2) \right. \\ &\quad \left. + \frac{k_2}{a^2 b^2} (2(a^4 - b^4) - \kappa(a^2 - b^2)) + \frac{k_3}{a^2 b^2} (2(a^4 + b^4) - \kappa(a^2 + b^2)) \right]. \end{aligned}$$

Proposition 3 *The κ -dependent Hamiltonian $\widetilde{H}_c(\kappa)$ is H - J separable in polar coordinates (r, ϕ) and parabolic coordinates (a, b) and it is endowed with the Hamiltonian as the first constant, that is $J_{c1} = \widetilde{H}_c(\kappa)$, and the following two additional quadratic constants of motion*

$$\begin{aligned} J_{c2} &= (xp_y - yp_x)^2 + \frac{2k_2 x^2}{y^2} + \frac{2k_3 x \sqrt{x^2 + y^2}}{y^2}, \\ J_{c3} &= \frac{1}{r - \kappa} \left[r (xp_y - yp_x) p_y + \frac{\kappa}{2} (xp_x^2 - xp_y^2 + 2yp_x p_y) + k_1 x \right. \\ &\quad \left. + \frac{k_2 x}{y^2} (2r - \kappa) + \frac{k_3}{y^2} (2x^2 + y^2 - \kappa r) \right]. \end{aligned}$$

The function J_{c2} it is κ -independent and it coincides with the original constant I_{c2} (the same situation we found in the case (a) with J_{a3}). Concerning J_{c3} , it is clear that it satisfies the limit $J_{c3} \rightarrow I_{c3}$ when $\kappa \rightarrow 0$.

3.4 Hamiltonian $\widetilde{H}_d(\kappa)$

Let us consider the Hamiltonian H_d

$$H_d = \frac{1}{2} (p_x^2 + p_y^2) + \frac{k_1}{r} + k_2 \frac{\sqrt{r+x}}{r} + k_3 \frac{\sqrt{r-x}}{r}, \quad r^2 = x^2 + y^2, \quad (11)$$

and denote by λ_d the following multiplier

$$\lambda_d = \frac{1}{\mu_d}, \quad \mu_d(x, y) = 1 - \frac{\kappa}{r}.$$

Thus, the κ -dependent Hamiltonian we will study is

$$\widetilde{H}_d(\kappa) = \frac{1}{2} \left(\frac{r}{r - \kappa} \right) (p_x^2 + p_y^2) + \frac{k_1}{r - \kappa} + \frac{k_2 \sqrt{r+x}}{r - \kappa} + \frac{k_3 \sqrt{r-x}}{r - \kappa}. \quad (12)$$

(i) Parabolic separability I

The Hamiltoniano $\widetilde{H}_d(\kappa)$ takes the following form when written in parabolic coordinates

$$\widetilde{H}_d(\kappa) = \frac{1}{2} \frac{1}{(a^2 + b^2) - \kappa} (p_a^2 + p_b^2) + \frac{1}{(a^2 + b^2) - \kappa} (k_1 + k_2 a + k_3 b)$$

so that the corresponding H-J Equation

$$\frac{1}{2} \frac{1}{(a^2 + b^2) - \kappa} \left[\left(\frac{\partial W}{\partial a} \right)^2 + \left(\frac{\partial W}{\partial b} \right)^2 \right] + \frac{1}{(a^2 + b^2) - \kappa} (k_1 + k_2 a + k_3 b) = E$$

admits separation of variables and it reduces to

$$\left[\left(\frac{\partial W_a}{\partial a} \right)^2 + k_1 + 2k_2 a - 2a^2 E \right] + \left[\left(\frac{\partial W_b}{\partial b} \right)^2 + k_1 + 2k_3 b - 2b^2 E \right] = -2\kappa E.$$

Thus, the following two functions

$$\begin{aligned} J_{d2a} &= p_a^2 + k_1 + 2k_2 a - 2a^2 \widetilde{H}_d(\kappa), \\ J_{d2b} &= p_b^2 + k_1 + 2k_3 b - 2b^2 \widetilde{H}_d(\kappa), \end{aligned}$$

that written with more detail are as follows

$$\begin{aligned} J_{d2a} &= \frac{1}{(a^2 + b^2) - \kappa} \left[a^2 p_b^2 - b^2 p_a^2 + \kappa p_a^2 + k_1 (a^2 - b^2 + \kappa) - 2k_2 a (b^2 - \kappa) + 2k_3 a^2 b \right] \\ J_{d2b} &= \frac{1}{(a^2 + b^2) - \kappa} \left[a^2 p_b^2 - b^2 p_a^2 - \kappa p_b^2 + k_1 (a^2 - b^2 - \kappa) - 2k_2 a b^2 + 2k_3 b (a^2 - \kappa) \right] \end{aligned}$$

are two independent constants of motion. Nevertheless as the only difference between them is just a term proportional to the Hamiltonian, that is $J_{d2b} - J_{d2a} = 2\kappa \widetilde{H}_d(\kappa)$, we consider more convenient to choose the following function

$$J_{d2} = J_{d2b} - \kappa \widetilde{H}_d(\kappa) = J_{d2a} + \kappa \widetilde{H}_d(\kappa)$$

that takes the form

$$J_{d2} = \frac{1}{a^2 + b^2 - \kappa} \left((a^2 p_b^2 - b^2 p_a^2) + \frac{\kappa}{2} (p_a^2 - p_b^2) \right) + \frac{1}{a^2 + b^2 - \kappa} \left(k_1 (a^2 - b^2) - k_2 a (2b^2 - \kappa) + k_3 b (2a^2 - \kappa) \right)$$

as the integral representing the κ -dependent version of I_{d2} .

(ii) Parabolic separability II

We can introduce a new system of parabolic coordinates by rotating the original system

$$(a, b) \rightarrow (\alpha, \beta) ; \quad a = \frac{1}{\sqrt{2}}(\alpha + \beta), \quad b = \frac{1}{\sqrt{2}}(\alpha - \beta),$$

in such a way that the Hamiltonian $\widetilde{H}_d(\kappa)$, when written in this new system, it takes the following form

$$\widetilde{H}_d(\kappa) = \frac{1}{2} \frac{1}{(\alpha^2 + \beta^2) - \kappa} (p_\alpha^2 + p_\beta^2) + \frac{1}{(\alpha^2 + \beta^2) - \kappa} \left[k_1 + k_2 \frac{1}{\sqrt{2}}(\alpha + \beta) + k_3 \frac{1}{\sqrt{2}}(\alpha - \beta) \right].$$

The H-J equation $\widetilde{H}_d(\partial W / \partial \alpha, \partial W / \partial \beta, \alpha, \beta) = E$ is separable as well, and it leads to the following two functions

$$J_{d3a} = p_\alpha^2 + k_1 + \sqrt{2}k_2\alpha + \sqrt{2}k_3\alpha - 2\alpha^2 \widetilde{H}_d(\kappa),$$

$$J_{d3b} = p_\beta^2 + k_1 + \sqrt{2}k_2\beta - \sqrt{2}k_3\beta - 2\beta^2\widetilde{H}_d(\kappa),$$

that recovering the coordinates (a, b) appear as follows

$$J_{d3a} = \frac{1}{(a^2 + b^2) - \kappa} \left[(ap_b - bp_a)(ap_a - bp_b) - \frac{\kappa}{2}(p_a + p_b)^2 - k_1(2ab + \kappa) + k_2(a+b)(b^2 - ab - \kappa) + k_3(a+b)(a^2 - ab - \kappa) \right]$$

$$J_{d3b} = \frac{1}{(a^2 + b^2) - \kappa} \left[(ap_b - bp_a)(ap_a - bp_b) + \frac{\kappa}{2}(p_a - p_b)^2 - k_1(2ab - \kappa) - k_2(a-b)(b^2 + ab - \kappa) + k_3(a-b)(a^2 + ab - \kappa) \right]$$

This situation is similar to the previous one, that is, they are independent but the only difference between them is κ times de Hamiltonian, that is $J_{d3b} - J_{d3a} = 2\kappa\widetilde{H}_d(\kappa)$; therefore we choose the following function J_{d3} as the integral of motion associated to the system (α, β)

$$J_{d3} = \frac{1}{a^2 + b^2 - \kappa} \left((ap_b - bp_a)(ap_a - bp_b) - \kappa p_a p_b \right) + \frac{1}{a^2 + b^2 - \kappa} \left(-2k_1 ab - k_2 b(a^2 - b^2 + \kappa) + k_3 a(a^2 - b^2 - \kappa) \right).$$

Proposition 4 *The κ -dependent Hamiltonian $\widetilde{H}_d(\kappa)$ is H-J separable in two different systems of parabolic coordinates, the original system (a, b) and the rotated system (α, β) , and it is endowed with three quadratic constants of motion; the Hamiltonian as the first constant, that is $J_{d1} = \widetilde{H}_d(\kappa)$, and the following two additional integrals*

$$\begin{aligned} J_{d2} &= \frac{1}{r - \kappa} \left[r(xp_y - yp_x)p_y + \frac{\kappa}{2}(xp_x^2 - xp_y^2 + 2yp_x p_y) + k_1 x \right. \\ &\quad \left. - k_2(y\sqrt{r-x} - \kappa\sqrt{r+x}) + k_3(y\sqrt{r+x} - \kappa\sqrt{r-x}) \right], \\ J_{d3} &= \frac{1}{r - \kappa} \left[r(xp_y - yp_x)p_x + \frac{\kappa}{2}(yp_x^2 - yp_y^2 - 2xp_x p_y) - k_1 y \right. \\ &\quad \left. - k_2(x + \kappa)\sqrt{r-x} + k_3(x - \kappa)\sqrt{r+x} \right]. \end{aligned}$$

3.5 Comments

The previous paragraphs can be considered as rather technical; so now we comment some of the characteristics of these new families of Hamiltonians and we analyze to what physics systems they seem to correspond.

- (i) All these “new Hamiltonians” are of the form of the original Hamiltonian multiplied by a factor. How was this factor found? The answer is that every λ_r is obtained by imposing separability in two different systems of coordinates and this property determines λ_r with the only ambiguity of the appropriate position and sign of the parameter κ .

It is clear that the factors λ_r introduce a change in the geometry of the configuration space. The new deformed Hamiltonians $\widetilde{H}_r(\kappa)$ describe dynamics in Riemannian manifolds (a particle moving in a curved space); the negative sign in front of κ has been chosen in order that the sign of κ coincides with the sign of the curvature.

- (ii) The expressions of the four Hamiltonians $\widetilde{H}_r(\kappa)$ can be considered as rather involved mainly because the nonlinearity affects to both the kinetic term and the potential. Nevertheless the fundamental point is that, in spite of their complex aspect, they are directly related with the two fundamental superintegrable systems, that is, harmonic oscillators and Kepler problems.

The two first Hamiltonians, $\widetilde{H}_a(\kappa)$ and $\widetilde{H}_b(\kappa)$, are related with the harmonic oscillator. More specifically, they are deformations of the nonlinear oscillators H_a and H_b . Therefore $\widetilde{H}_a(\kappa)$ and $\widetilde{H}_b(\kappa)$ describe nonlinear oscillators in non-Euclidean spaces.

The other two Hamiltonians, $\widetilde{H}_c(\kappa)$ and $\widetilde{H}_d(\kappa)$, are deformations of the nonlinear Hamiltonians H_c and H_d and therefore they are related with the Kepler problem. For that reason $\widetilde{H}_c(\kappa)$ and $\widetilde{H}_d(\kappa)$ describe two different versions of the Kepler problem in two different non-Euclidean spaces.

The following section studies these questions with more detail.

4 The harmonic oscillator and the Kepler problem on curved spaces

In differential geometric terms, the four Hamiltonians $\widetilde{H}_r(\kappa)$, $r = a, b, c, d$, describe dynamics on non-Euclidean spaces. The two first Hamiltonians, $\widetilde{H}_a(\kappa)$ or $\widetilde{H}_b(\kappa)$, are related with the harmonic oscillator and the other two, $\widetilde{H}_c(\kappa)$ or $\widetilde{H}_d(\kappa)$, with the Kepler problem. Now in this section we analyze these two important κ -dependent systems from a geometric perspective.

4.1 The harmonic oscillator on curved spaces

The following two equations

$$\frac{d^2x}{dt^2} + \frac{\kappa x}{1 - \kappa x^2} \left(\frac{dx}{dt} \right)^2 + \frac{\alpha^2 x}{1 - \kappa x^2} = 0, \quad (13)$$

$$\frac{d^2x}{dt^2} - \frac{\kappa x}{1 - \kappa x^2} \left(\frac{dx}{dt} \right)^2 + \frac{\alpha^2 x}{(1 - \kappa x^2)^3} = 0, \quad (14)$$

represent two one-dimensional nonlinear oscillators that can be considered as arising from the Lagrangians

$$L_1 = \frac{1}{2} \frac{v_x^2}{1 - \kappa x^2} - \frac{1}{2} \frac{\alpha^2 x^2}{1 - \kappa x^2} \quad \text{and} \quad L_2 = \frac{1}{2} (1 - \kappa x^2) v_x^2 - \frac{1}{2} \frac{\alpha^2 x^2}{1 - \kappa x^2},$$

with associated Hamiltonians

$$H_1 = \frac{1}{2} (1 - \kappa x^2) p_x^2 + \frac{1}{2} \frac{\alpha^2 x^2}{1 - \kappa x^2}, \quad (15)$$

and

$$H_2 = \frac{1}{2} \frac{p_x^2}{1 - \kappa x^2} + \frac{1}{2} \frac{\alpha^2 x^2}{1 - \kappa x^2}. \quad (16)$$

That is, the two potentials are the same (see Figure 1) but the kinetic terms are different (these two oscillators are studied in [44]–[49]).

The two-dimensional versions of these Lagrangians are

$$\begin{aligned} L_1(\kappa) &= \frac{1}{2} \left(\frac{v_r^2}{1 - \kappa r^2} + r^2 v_\phi^2 \right) - \frac{\alpha^2}{2} \left(\frac{r^2}{1 - \kappa r^2} \right) \\ &= \frac{1}{2} \left(\frac{1}{1 - \kappa r^2} \right) \left[v_x^2 + v_y^2 - \kappa (xv_y - yv_x)^2 \right] - \frac{\alpha^2}{2} \left(\frac{x^2 + y^2}{1 - \kappa r^2} \right), \end{aligned}$$

and

$$L_2(\kappa) = \frac{1}{2}(1 - \kappa r^2)(v_x^2 + v_y^2) - \frac{\alpha^2}{2} \left(\frac{x^2 + y^2}{1 - \kappa r^2} \right).$$

It is known that a symmetric bilinear form in the velocities (v_x, v_y) can be considered as associated to a two-dimensional metric ds^2 in \mathbb{R}^2 . In this particular case, the kinetic term in the Lagrangian $L_1(\kappa)$ considered as a bilinear form determines the following κ -dependent metric

$$ds_\kappa^2 = \left(\frac{1}{1 - \kappa r^2} \right) \left[(1 - \kappa y^2) dx^2 + (1 - \kappa x^2) dy^2 + 2\kappa xy dx dy \right].$$

The second Lagrangian $L_2(\kappa)$ represents the harmonic oscillator in a two-dimensional space with a metric ds_κ^2 conformally flat

$$ds_\kappa^2 = (1 - \kappa r^2)(dx^2 + dy^2).$$

The two-dimensional versions of these Hamiltonian are

$$H_1(\kappa) = \frac{1}{2} \left((1 - \kappa r^2)(p_x^2 + p_y^2) + \kappa J^2 \right) + \frac{1}{2} \alpha^2 \left(\frac{x^2 + y^2}{1 - \kappa r^2} \right), \quad (17)$$

and

$$H_2(\kappa) = \frac{1}{2} \left(\frac{p_x^2 + p_y^2}{1 - \kappa r^2} \right) + \frac{1}{2} \alpha^2 \left(\frac{x^2 + y^2}{1 - \kappa r^2} \right), \quad (18)$$

where J denotes the angular momentum. They satisfy the correct Euclidean limit

$$\lim_{\kappa \rightarrow 0} H_1(\kappa) = \lim_{\kappa \rightarrow 0} H_2(\kappa) = \frac{1}{2} (p_x^2 + p_y^2) + \frac{1}{2} \alpha^2 (x^2 + y^2),$$

and represent two different harmonic oscillators in two-dimensional curved spaces.

- (i) The Hamiltonian $H_1(\kappa)$, that has been studied in Refs. [48, 50, 51, 52] (although in some cases with a trigonometric-hyperbolic notation), represents the harmonic oscillator in a space of constant curvature κ (sphere S_κ^2 with $\kappa > 0$, and Hiperbolic plane H_κ^2 with $\kappa < 0$). We note that the kinetic term includes not only the factor $(1 - \kappa r^2)$ but also a contribution of the angular momentum J with the curvature κ as coefficient.
- (ii) The second Hamiltonian $H_2(\kappa)$, that is just $\widetilde{H}_a(\kappa)$ with $k_2 = k_3 = 0$, represents an harmonic oscillator in a curved space (two-dimensional space of nonconstant curvature) [40]–[43]. In two dimensions the tensor R_{abcd} only has one independent component which can be taken R_{1212}

$$R_{1212} = \frac{1}{2} (\partial_2 \partial_1 g_{21} - \partial_2^2 g_{11} + \partial_1 \partial_2 g_{12} - \partial_1^2 g_{22}) - g_{ef} (\Gamma_{11}^e \Gamma_{22}^f - \Gamma_{12}^e \Gamma_{21}^f).$$

The result is

$$R_{1212} = \frac{2\kappa}{1 - \kappa r^2}$$

and the scalar Gaussian curvature is given by

$$K = \frac{R_{1212}}{\det[g]} = \frac{2\kappa}{(1 - \kappa r^2)^3}.$$

4.2 The Kepler problem on curved spaces

The following two κ -dependent Hamiltonians

$$H_{K1}(\kappa) = \frac{1}{2} \left((1 - \kappa r^2)(p_x^2 + p_y^2) + \kappa J^2 \right) - g \left(\frac{\sqrt{1 - \kappa r^2}}{r} \right) \quad (19)$$

and

$$H_{K2}(\kappa) = \frac{1}{2} \left(\frac{r}{r - \kappa} \right) (p_x^2 + p_y^2) - \frac{g}{r - \kappa} \quad (20)$$

represent two different versions of the Kepler problem on curved spaces. They satisfy the correct Euclidean limit

$$\lim_{\kappa \rightarrow 0} H_{K1}(\kappa) = \lim_{\kappa \rightarrow 0} H_{K2}(\kappa) = \frac{1}{2} (p_x^2 + p_y^2) - \frac{g}{r}.$$

- (i) The Hamiltonian $H_{K1}(\kappa)$, that has been studied in Refs. [53, 54], represents the Kepler problem in the three spaces of constant curvature κ ; that is, sphere S_κ^2 ($\kappa > 0$), Euclidean plane E^2 ($\kappa = 0$), and Hiperbolic plane H_κ^2 ($\kappa < 0$). Also in this case, when $\kappa > 0$, the dynamics is restricted to the interior of the bounded region $x^2 + y^2 < 1/\kappa$.
- (ii) The Hamiltonian $H_{K2}(\kappa)$ is just the Hamiltonian of the Kepler problem in $\widetilde{H}_c(\kappa)$ or $\widetilde{H}_d(\kappa)$. In this case if κ is negative then the Hamiltonian is well defined for all the values $\kappa < 0$, but when $\kappa > 0$ then the dynamics is only well defined in the region $r > \kappa$ in such a way that when $\kappa \rightarrow 0$ then the dynamics is defined in the whole space. This system is endowed with the following two integrals of motion

$$\begin{aligned} J_2 &= \frac{1}{r - \kappa} \left[r (xp_y - yp_x) p_y - g x \right] + \frac{\kappa}{2(r - \kappa)} \left(xp_x^2 - xp_y^2 + 2yp_x p_y \right) \\ J_3 &= \frac{1}{r - \kappa} \left[r (xp_y - yp_x) p_x + g y \right] + \frac{\kappa}{2(r - \kappa)} \left(yp_x^2 - yp_y^2 - 2xp_x p_y \right) \end{aligned}$$

that represent the κ -dependent version of the (two-dimensional) Runge-Lenz vector. In fact, it can be verified that their Poisson bracket is given by

$$\{J_2, J_3\} = 2(xp_y - yp_x) H_{K2}(\kappa)$$

that is the same relation characterizing the Runge-Lenz vector in the Euclidean case.

The metric ds_κ^2 , that is also conformal, is given by

$$ds_\kappa^2 = \left(1 - \frac{\kappa}{r} \right) (dx^2 + dy^2).$$

We note the factor $(1 - \kappa/r)$ shows a certain similarity with the coefficient (related with the singularity) in the Schwarzschild metric. The curvature tensor R_{1212} takes the value

$$R_{1212} = \frac{\kappa}{2r^2(r - \kappa)}$$

and the Gaussian curvature (scalar) is given by

$$K = \frac{R_{1212}}{\det[g]} = \frac{\kappa}{2(r - \kappa)^3}.$$

5 Superintegrability with higher order constants of motion

We have proved that the two Hamiltonians $\widetilde{H}_a(\kappa)$ (related to the harmonic oscillator) and $\widetilde{H}_c(\kappa)$ (related to the Kepler problem) are superintegrable with quadratic constants of motion. Now, in this section, we will prove that they can be considered as particular cases of a more general situation; that is, they admit superintegrable generalizations that are separable but in only one system of coordinates (two quadratic constants) and are endowed with an additional constant of higher order.

At this point (as a previous comment to the next two subsections) we notice that a Hamiltonian of the form

$$H(\kappa) = \frac{1}{2} \left(\frac{1}{1 - \kappa r^2} \right) \left(p_r^2 + \frac{p_\phi^2}{r^2} \right) + \frac{1}{2} \alpha^2 \left(\frac{A(r)}{1 - \kappa r^2} \right) + \frac{1}{2} \frac{1}{1 - \kappa r^2} \frac{B(\phi)}{r^2},$$

where $A(r)$ and $B(\phi)$ are arbitrary functions, is H-J separable in polar coordinates (r, ϕ) and it is endowed with the following constant of motion

$$J = p_\phi^2 + B(\phi) = - \left[r^2 p_r^2 + \alpha^2 r^2 A(r) - 2r^2 (1 - \kappa r^2) H(\kappa) \right].$$

This property is true for all the values of the parameter κ .

5.1 A κ -dependent Hamiltonian related with the harmonic oscillator and the TTW system

The following potential

$$V_{ttw}(r, \phi) = \frac{1}{2} \omega_0^2 r^2 + \frac{1}{2r^2} \left(\frac{\alpha}{\cos^2(m\phi)} + \frac{\beta}{\sin^2(m\phi)} \right), \quad (21)$$

was studied by Tremblay, Turbiner, and Winternitz [23]–[24] and also by other authors [25]–[37]. In the general $m \neq 1$ case it is only separable in polar coordinates (r, ϕ) (m must be an integer or rational number) and therefore the third integral is not quadratic but a polynomial of higher order in the momenta (the degree of the polynomial depends of the value of m).

Let us consider the κ -dependent Hamiltonian

$$\widetilde{H}_{am}(\kappa) = \frac{1}{2} \left(\frac{1}{1 - \kappa r^2} \right) \left(p_r^2 + \frac{p_\phi^2}{r^2} \right) + U_{am}(r, \phi), \quad U_{am}(r, \phi) = \frac{1}{2} \alpha^2 \left(\frac{r^2}{1 - \kappa r^2} \right) + \frac{1}{2} \frac{F_m(\phi)}{(1 - \kappa r^2) r^2}, \quad (22)$$

where $F_m(\phi)$ denotes the following angular function

$$F_m(\phi) = \frac{k_a}{\cos^2(m\phi)} + \frac{k_b}{\sin^2(m\phi)}$$

and k_a and k_b are arbitrary constants. It represents a generalization of the Hamiltonian $\widetilde{H}_a(\kappa)$ in the sense that if $m = 1$ then we have $\widetilde{H}_{a1}(\kappa) = \widetilde{H}_a(\kappa)$. It is clear that this more general Hamiltonian is separable but only in polar (r, ϕ) coordinates. Therefore it is integrable with the Hamiltonian itself as the first integral and a second quadratic constant of motion J_2 associated to the Liouville integrability

$$\begin{aligned} J_1(\kappa) &= \left(\frac{1}{1 - \kappa r^2} \right) \left(p_r^2 + \frac{p_\phi^2}{r^2} \right) + \alpha^2 \left(\frac{r^2}{1 - \kappa r^2} \right) + \frac{F_m(\phi)}{(1 - \kappa r^2) r^2}, \\ J_2 &= p_\phi^2 + F_m(\phi). \end{aligned}$$

Let us denote by A_r and B_ϕ be the complex functions $A_r = A_{r1} + i A_{r2}$ and $B_\phi = B_{\phi1} + i B_{\phi2}$ with real and imaginary parts, A_{ra} and $B_{\phi a}$, $a = 1, 2$, be defined as

$$A_{r1} = \frac{2}{r} p_r \sqrt{J_2}, \quad A_{r2} = \frac{1}{(1 - \kappa r^2) r^2} \left(r^2 p_r^2 + \alpha^2 r^4 - (1 - 2\kappa r^2) J_2 \right) = J_1 - \frac{2}{r^2} J_2,$$

and

$$B_{\phi1} = J_2 \cos(2m\phi) + (k_b - k_a), \quad B_{\phi2} = p_\phi \sqrt{J_2} \sin(2m\phi).$$

First, let us comment that the moduli of these two complex functions, that are constant of motion of fourth order in the momenta, are given by

$$\begin{aligned} |A_r|^2 &= J_1^2 - 4\kappa J_1 J_2 - 4\alpha^2 J_2 \\ |B_\phi|^2 &= J_2^2 - 2(k_a + k_b) J_2 + (k_b - k_a)^2 \end{aligned}$$

The Poisson bracket of the function A_{r1} with $\widetilde{H}_{am}(\kappa)$ [time-derivative] is proportional to A_{r2} and the time-derivative of the A_{r2} is proportional to A_{r1} but with the opposite sign

$$\{A_{r1}, \widetilde{H}_{am}(\kappa)\} = -2\lambda_\kappa A_{r2}, \quad \{A_{r2}, \widetilde{H}_{am}(\kappa)\} = 2\lambda_\kappa A_{r1},$$

and this property is also true for the functions $B_{\phi1}$ and $B_{\phi2}$

$$\{B_{\phi1}, \widetilde{H}_{am}(\kappa)\} = -2m\lambda_\kappa B_{\phi2}, \quad \{B_{\phi2}, \widetilde{H}_{am}(\kappa)\} = 2m\lambda_\kappa B_{\phi1},$$

where the common factor λ_κ takes the value

$$\lambda_\kappa = \frac{1}{(1 - \kappa r^2) r^2} \sqrt{J_2}, \quad \lambda_0 = \frac{1}{r^2} \sqrt{J_2}.$$

Therefore, the time-evolution of the complex functions A_r and B_ϕ is given by

$$\frac{d}{dt} A_r = i 2 \lambda_\kappa A_r, \quad \frac{d}{dt} B_\phi = i 2 m \lambda_\kappa B_\phi.$$

Thus if we denote by K_m the complex function $A_r^m B_\phi^*$ then we have

$$\begin{aligned} \frac{d}{dt} K_m &= \frac{d}{dt} \left(A_r^m B_\phi^* \right) = m A_r^{(m-1)} \dot{A}_r B_\phi^* + A_r^m \dot{B}_\phi^* \\ &= A_r^{(m-1)} \left(m (i 2 \lambda_\kappa A_r) B_\phi^* + A_r (-i 2 m \lambda_\kappa B_\phi^*) \right) = 0. \end{aligned}$$

We summarize this result in the following proposition.

Proposition 5 *The κ -dependent Hamiltonian $\widetilde{H}_{am}(\kappa)$*

$$\widetilde{H}_{am}(\kappa) = \frac{1}{2} \left(\frac{1}{1 - \kappa r^2} \right) \left(p_r^2 + \frac{p_\phi^2}{r^2} \right) + \frac{1}{2} \alpha^2 \left(\frac{r^2}{1 - \kappa r^2} \right) + \frac{1}{2} \frac{F_m(\phi)}{(1 - \kappa r^2) r^2} \quad (23)$$

representing a generalization of the Hamiltonian $\widetilde{H}_a(\kappa)$, and also a κ -deformation of the Euclidean TTW Hamiltonian H_{ttw} , is superintegrable with two quadratic constants of motion and a third constant of motion of higher order.

(i) $\widetilde{H}_{am}(\kappa)$ is separable in polar coordinates (r, ϕ) and it possesses therefore two quadratic constants of motion associated to the Liouville integrability

$$\begin{aligned} J_1(\kappa) &= \left(\frac{1}{1 - \kappa r^2} \right) \left(p_r^2 + \frac{p_\phi^2}{r^2} \right) + \alpha^2 \left(\frac{r^2}{1 - \kappa r^2} \right) + \frac{F_m(\phi)}{(1 - \kappa r^2) r^2}, \\ J_2 &= p_\phi^2 + F_m(\phi). \end{aligned}$$

(ii) $\widetilde{H}_{am}(\kappa)$ admits a complex constant of motion K_m defined as

$$K_m = A_r^m B_\phi^*.$$

The function K_m can be written as $K_m = J_3 + iJ_4$ with J_3 and J_4 real constants of motion. One of them can be chosen as the third fundamental integral of motion.

5.2 A κ -dependent Hamiltonian related with the Kepler problem and the PW system

Let us first note that the potential V_c , that in polar coordinates becomes

$$V_c = -\frac{g}{r} + \frac{1}{r^2} \left(\frac{k_2}{\sin^2 \phi} + \frac{k_3 \cos \phi}{\sin^2 \phi} \right),$$

can also be written as follows

$$V_c = -\frac{g}{r} + \frac{1}{r^2} \left(\frac{\alpha}{\cos^2(\phi/2)} + \frac{\beta}{\sin^2(\phi/2)} \right), \quad k_2 = 2(\alpha + \beta), \quad k_3 = 2(\beta - \alpha).$$

Therefore, the angular-dependent functions in the potentials V_a and V_c appear as two particular cases, $m = 1$ and $m = 1/2$, of the general function $F_m(\phi)$. It seems therefore natural to conjecture that any integrable generalization (or deformation) of the potential V_a must determine a similar generalization (or deformation) of the potential V_c .

In fact there is another interesting system rather similar to the TTW system but that is related, not with the harmonic oscillator, but with the Kepler problem (hydrogen atom in the quantum case)

$$V_{pw}(r, \phi) = -\frac{g}{r} + \frac{1}{r^2} \left(\frac{\alpha}{\cos^2(m\phi)} + \frac{\beta}{\sin^2(m\phi)} \right). \quad (24)$$

The first study of the superintegrability of this new potential was presented by Post and Winternitz [38] by relating V_{pw} with V_{ttw} making use of the so-called coupling constant metamorphosis transformation (Stäckel transform) [55].

Let us now consider the κ -dependent Kepler-related potential

$$\widetilde{H}_{cm}(\kappa) = \frac{1}{2} \left(\frac{r}{r - \kappa} \right) \left(p_r^2 + \frac{p_\phi^2}{r^2} \right) + U_{cm}(r, \phi), \quad U_{cm}(r, \phi) = -\frac{g}{r - \kappa} + \frac{F_m(\phi)}{(r - \kappa)r}, \quad (25)$$

where $F_m(\phi)$ is the same angular function as in the oscillator $\widetilde{H}_{am}(\kappa)$ (and also in this case k_a and k_b are arbitrary constants). It represents a generalization of the Hamiltonian $\widetilde{H}_c(\kappa)$ in the sense that if $m = 1/2$ then we have $\widetilde{H}_{cm}(\kappa)|_{m=1/2} = \widetilde{H}_c(\kappa)$. It is clear that this more general Hamiltonian is separable but

only in polar (r, ϕ) coordinates. Therefore it is integrable with the Hamiltonian itself as the first integral and a second quadratic constant of motion J_2 associated to the Liouville integrability

$$\begin{aligned} J_1(\kappa) &= \left(\frac{r}{r-\kappa} \right) \left(p_r^2 + \frac{p_\phi^2}{r^2} \right) - \frac{2g}{r-\kappa} + \frac{2F_m(\phi)}{(r-\kappa)r}, \\ J_2 &= p_\phi^2 + 2F_m(\phi). \end{aligned}$$

We will prove the superintegrability of $\widetilde{H}_{cm}(\kappa)$ using as an approach the existence of a complex factorization for the additional constant of motion. The method, that is in fact a deformation of the formalism introduced in [39] for the superintegrability of the PW system, is similar to the one presented in the previous section for the Hamiltonian $\widetilde{H}_{am}(\kappa)$ (introducing the appropriate changes).

Now we denote by A_r and B_ϕ be the complex functions $A_r = A_{r1} + i A_{r2}$ and $B_\phi = B_{\phi1} + i B_{\phi2}$ with real and imaginary parts, A_{ra} and $B_{\phi a}$, $a = 1, 2$, be defined as

$$A_{r1} = p_r \sqrt{J_2}, \quad A_{r2} = \left(\frac{1}{r-\kappa} \right) \left(-g r + J_2 + \frac{\kappa}{2} (r p_r^2 - \frac{J_2}{r}) \right),$$

and

$$B_{\phi1} = J_2 \cos(2m\phi) + 2(k_b - k_a), \quad B_{\phi2} = p_\phi \sqrt{J_2} \sin(2m\phi).$$

First, we note that the moduli of these two complex functions are integrals of motion of fourth order in the momenta given by

$$\begin{aligned} |A_r|^2 &= J_1 J_2 + \left(\frac{\kappa}{2} \right)^2 J_1^2 - \kappa g J_2 + g^2 \\ |B_\phi|^2 &= J_2^2 - 4(k_a + k_b) J_2 + 4(k_a - k_b)^2 \end{aligned}$$

The time-derivative [Poisson bracket with $\widetilde{H}_{cm}(\kappa)$] of the function A_{r1} is proportional to A_{r2} and the time-derivative of the A_{r2} is proportional to A_{r1} but with the opposite sign

$$\{A_{r1}, \widetilde{H}_{cm}(\kappa)\} = -\lambda_\kappa A_{r2}, \quad \{A_{r2}, \widetilde{H}_{cm}(\kappa)\} = \lambda_\kappa A_{r1},$$

and this property is also true for the angular functions

$$\{B_{\phi1}, \widetilde{H}_{cm}(\kappa)\} = -2m \lambda_\kappa B_{\phi2}, \quad \{B_{\phi2}, \widetilde{H}_{cm}(\kappa)\} = 2m \lambda_\kappa B_{\phi1},$$

where the common factor λ_κ takes the value

$$\lambda_\kappa = \frac{1}{(r-\kappa)r} \sqrt{J_2}, \quad \lambda_0 = \frac{1}{r^2} \sqrt{J_2}.$$

Therefore, the time-evolution of the complex functions A_r and B_ϕ is given by

$$\frac{d}{dt} A_r = i \lambda_\kappa A_r, \quad \frac{d}{dt} B_\phi = i 2m \lambda_\kappa B_\phi.$$

Thus, if we denote by K_m the complex function $K_m = A_r^{(2m)} B_\phi^*$, then we have

$$\begin{aligned} \frac{d}{dt} K_m &= \frac{d}{dt} \left(A_r^{(2m)} B_\phi^* \right) = 2m A_r^{(2m-1)} \dot{A}_r B_\phi^* + A_r^{(2m)} \dot{B}_\phi^* \\ &= A_r^{(2m-1)} \left(2m (i \lambda_\kappa A_r) B_\phi^* + A_r (-i 2m \lambda_\kappa B_\phi^*) \right) = 0. \end{aligned}$$

We summarize this result in the following proposition.

Proposition 6 *The κ -dependent Hamiltonian $\widetilde{H}_{cm}(\kappa)$*

$$\begin{aligned} \widetilde{H}_{cm}(\kappa) &= \frac{1}{2} \left(\frac{r}{r-\kappa} \right) \left(p_r^2 + \frac{p_\phi^2}{r^2} \right) - \frac{g}{r-\kappa} + \frac{F_m(\phi)}{(r-\kappa)r} \\ F_m(\phi) &= \frac{k_a}{\cos^2(m\phi)} + \frac{k_b}{\sin^2(m\phi)} \end{aligned} \quad (26)$$

representing a generalization of the Hamiltonian $\widetilde{H}_c(\kappa)$, and also a κ -deformation of the Euclidean PW Hamiltonian H_{pw} , is superintegrable with two quadratic constants of motion and a third constant of motion of higher order.

(i) $\widetilde{H}_{cm}(\kappa)$ is separable in polar coordinates (r, ϕ) and it possesses therefore two quadratic constants of motion associated to the Liouville integrability

$$\begin{aligned} J_1(\kappa) &= \left(\frac{r}{r-\kappa} \right) \left(p_r^2 + \frac{p_\phi^2}{r^2} \right) - \frac{2g}{r-\kappa} + \frac{2F_m(\phi)}{(r-\kappa)r}, \\ J_2 &= p_\phi^2 + 2F_m(\phi). \end{aligned}$$

(ii) $\widetilde{H}_{cm}(\kappa)$ admits a complex constant of motion K_m defined as

$$K_m = A_r^{2m} B_\phi^*.$$

6 Final comments

We observed in the introduction that although the number of superintegrable systems can be considered as rather limited, they are not however isolated systems but, on the contrary, they frequently appear as grouped in families. Now we have proved the existence of four families of Hamiltonians $\widetilde{H}_r(\kappa)$, $r = a, b, c, d$, associated to previously known super-separable Hamiltonians H_r . The important point is that the multiplier λ (that is a function defined in the configuration space) is a continuous function of a parameter κ so that the superintegrability is preserved for all the values of κ and the integrals of motion (and therefore the associated symmetries) depend in a smooth way of the parameter. The fact that this continuous deformations also lead to generalizations of the TTW and the PW systems is certainly a very remarkable property.

We conclude with the following two comments. First, the κ -dependent constants of motion are consequence of the existence of κ -dependent symmetries; so it would be convenient to study the properties of these symmetries from a geometric approach (that is, symplectic formalism and Lie algebra of vector fields). Second, it is also convenient to study the quantum versions of these systems. It is clear that these Hamiltonians are systems with a position dependent mass (PDM) and therefore the quantization of these systems is not an easy matter. These two points are interesting questions deserving to be studied.

Figures

Acknowledgments

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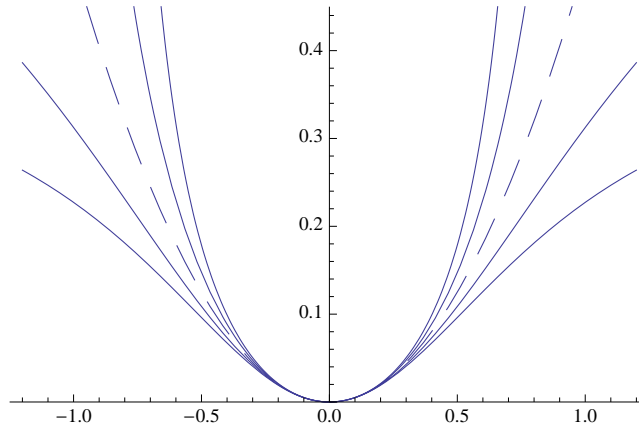


Figure 1: Plot of the Potential $V = (1/2)\alpha^2(x^2/(1 - \kappa x^2))$ as a function of x , for $\alpha = 1$ and $\kappa = 0$ (dash line), $\kappa > 0$ (inner upper curves), and $\kappa < 0$ (lower curves in the outside).

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