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A first approach in solving initial-value problems in ODEs by elliptic fitting methods

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ABSTRACT

Exponentially-fitted and trigonometrically-fitted methods have a long successful history in the solution of initial-value problems, but other functions might be considered in adapted methods. Specifically, this paper aims at the derivation of a new numerical scheme for approximating initial value problems of ordinary differential equations using elliptic functions. The example considered is the undamped Duffing equation where the forcing term is of autonomous type affected by a perturbation parameter. The new scheme is constructed by considering a suitable approximation to the theoretical solution based on elliptic functions. The proposed elliptic fitting procedure has been tested on a variety of problems, showing its good performance.

1. Introduction

Consider the following IVP given by

$$x'' + ax + bx^3 = \epsilon\, g(x), \qquad x(t_0) = x_0, \qquad x'(t_0) = 0, \quad t \in [0, t_N], \tag{1}$$

which is a particular form of the undamped Duffing equation with forcing term $f(x) = \epsilon\, g(x)$. We assume that $a, b > 0$, which may be referred as a hardening stiffness system [1]. An equation of this type arises for example in the non-linear vibration of beams/plates subjected to axial/membrane loading.

It is well-known that an initial value problem of the form in (1) can be solved analytically just in a few cases (see [2–5]). Closed-form solutions to Eq. (1) for a general forcing function are not known. We are interested in finding an approximate discrete solution, say $x_n \simeq x(t_n)$, on the nodal points $t_n = nh;\ n = 0, 1, 2, \ldots, N$, where h is called the *step length*. Nonlinear oscillations are of great importance in physical science, mechanical structures and other engineering problems.

In the present paper, we present a new scheme for the numerical integration of the initial value problem (1). The proposed scheme has second order convergence. Numerical results are presented to validate the efficiency of the proposed method.

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2. Basic preliminaries

The differential problem

$$x'' + ax + bx^3 = 0, \qquad x(t_0) = x_0, \qquad x'(t_0) = 0, \tag{2}$$

where $a, b > 0$ are two parameters, is a particular form of the Duffing oscillator where the restoring force is characterized by a linear term plus a cubic nonlinear term. In this equation x stands for the displacement from the equilibrium position while the forcing strength is null.

In this case we have a single-well anharmonic potential $V(x)$ given by

$$V(x) = \frac{a}{2}x^2 + \frac{b}{4}x^4$$

and the true solution is given by the elliptic function $cn(t, k)$ where t denotes a real variable and k is the modulus, which depends on the initial conditions. Assuming a mass of unity, the constant total energy function is given by

$$H(x, x') = \frac{1}{2}(x')^2 + V(x) = E.$$

From this equation we can derive the period of oscillation T which is given by (see [6])

$$T = \int_{x_1}^{x_2} \sqrt{\frac{2}{E - V(x)}}\, dx$$

where x_1 and x_2 are such that $V(x_1) = V(x_2) = E$.

In view of the true solution of the problem in (2) it makes sense that in order to obtain an approximate solution of the problem in (1), we consider the Jacobi elliptic function cn. The main motivation in using such elliptic function comes from the fact that it is the exact solution of the nonlinear oscillator in (2), and thus we will derive an elliptic fitting method for solving the problem in (1).

3. The new scheme

In order to solve the problem in (1) we propose the following approximation to the theoretical solution $x(t)$ of (1) at $t = t_n + h$

$$x_{n+1} = \frac{1}{6}h^2(4f_n - f_{n-1}) + \frac{2}{D}\left(w^2 x_n \phi(h) - x_n'\phi'(h)\right) \tag{3}$$

$$\begin{aligned} x'_{n+1} &= \frac{1}{2}h(3f_n - f_{n-1}) \\ &= +\frac{2w^2}{D^2}\left(x_n\phi'(h)\left(2a + b\,s + b\,d\phi(h)^2\right) + x_n'\phi(h)\left(2a - b\,d + s\,b\,\phi(h)^2\right)\right) \end{aligned}$$

where we have considered the function $f(x) = \epsilon\, g(x)$, and as it is usual we set $f_n = f(x_n), f_{n-1} = f(x_{n-1})$. The function $\phi(t)$ has been taken as $\phi(t) = cn(wt, m)$ with

$$w = \sqrt{a + b\,x_0^2}, \qquad m = \frac{b\,x_0^2}{2w^2}$$

and the other parameters appearing in the method are given by

$$d = x_n^2 - x_0^2, \qquad s = x_n^2 + x_0^2, \qquad D = 2a + b\,s - d\,b\,\phi(h)^2.$$

Note that the above method consists in two formulas, one to follow the solution and another one to follow the derivative.

After some algebra, it can be shown that the numerical scheme in (3) is exact for solving the problem in (1) when $\epsilon = 0$. We can state this in a more rigorous form by means of the following statement.

Proposition. *The elliptic fitting method in (3) is exact, except round-off errors, for solving the initial-value problem*

$$x'' + ax + bx^3 = 0, \qquad x(t_0) = x_0, \qquad x'(t_0) = 0, \quad t \in [0, t_N].$$

In order to obtain the algebraic order of the above method we consider the following differential operator associated to the equation in (3)

$$\begin{aligned} L[z(t); h] = z(t+h) &- \frac{1}{6}h^2\left(4\left(z''(t) + az(t) + bz(t)^3\right) - \left(z''(t-h) + az(t-h) + bz(t-h)^3\right)\right) \\ &- \frac{2\left(w^2\phi(h)z(t) - \phi'(h)z'(t)\right)}{2a + b\left(z(t)^2 + z_0^2\right) - b\phi(h)^2\left(z(t)^2 - z_0^2\right)}. \end{aligned} \tag{4}$$

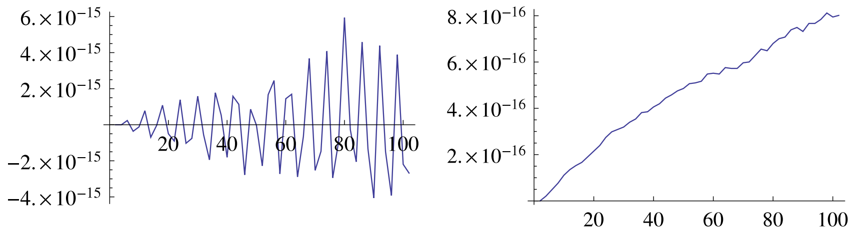

Fig. 1. Errors in the solution (left), and in the energy (right) for Example 1.

Considering the function $z(t) = x(t)$, the true solution of the problem in (1), and expanding by Taylor series $x(t_n + h)$ in the neighborhood of t_n, we finally obtain after simplifying that

$$L[x(t_n); h] = -\frac{h^4}{24}\left(3x_n\left(2ab\left(x_n^2 - x_0^2\right) + b^2\left(x_n^4 - x_0^4\right) + 2b\left(x'\right)_n^2\right) + f_n\left(a + 3bx_n^2\right) - 3f_n''\right) + \mathcal{O}(h^5). \tag{5}$$

Concerning the formula to follow the derivative we consider the corresponding differential operator and after expanding in Taylor around t_n we obtain similarly the local truncation error for this formula, which is given by

$$L[x'(t_n); h] = -\frac{h^3}{12}\left(6x_n\left(2ab\left(x_n^2 - x_0^2\right) + b^2\left(x_n^4 - x_0^4\right) + 2b\left(x'\right)_n^2\right) + 2f_n\left(a + 3bx_n^2\right) - 5f_n''\right) + \mathcal{O}(h^4). \tag{6}$$

Hence, the proposed scheme in (3) has convergence of second algebraic order.

4. Numerical examples

In this section, we present some numerical examples in order to see the performance of the proposed method. Only in Example 1 we know the exact solution. In order to get the errors we have considered the solutions provided by the Mathematica command `NDSolve`, where the options have been taken to obtain a solution with high precision (`WorkingPrecision->80`, `AccuracGoal->80`, which mean a high computational cost). As the problems are conservative, we have considered the error in the constant total energy as a measure of the efficiency of the methods.

4.1. Example 1

Consider the initial value problem given by

$$x'' + ax + bx^3 = 0, \qquad x(0) = x_0, \qquad x'(0) = 0, \tag{7}$$

which has the exact solution $x(t) = x_0 cn(wt, m)$, with $w = \sqrt{a + bx_0^2}$, $m = \frac{bx_0^2}{2w^2}$. For this problem the proposed method is exact, except for roundoff errors. We have taken $a = 1$, $b = 1$, $x_0 = 0.2$. For these values we have integrated the problem in the interval $[0, 100]$ with the proposed method taking $h = 2$. In Fig. 1 we show the errors in the solution and in the energy.

4.2. Example 2

As a second example we consider the problem

$$x'' + x + x^3 = \epsilon \cos(10^3 x) \qquad x(0) = 1, \qquad x'(0) = 0, \tag{8}$$

taking $\epsilon = 10^{-6}$. The problem has been solved in $[0, 20]$ with the proposed method and the classical fourth order Runge–Kutta method, considering a total number of steps $N = 60$. Fig. 2 shows the errors in the solution and in the energy. We see that the Runge–Kutta method performs worse even though it is of higher order.

4.3. Example 3

The last problem is a type of perturbed undamped Helmholtz–Duffing oscillator given by

$$x'' + 100x + 100x^3 = \epsilon\, x^2, \qquad x(0) = 1, \qquad x'(0) = 0, \tag{9}$$

where we take $\epsilon = 10^{-6}$ and the integration interval is $[0, 2]$. Fig. 3 shows the errors in the solution and in the energy. We observe anew that the Runge–Kutta method performs worse.

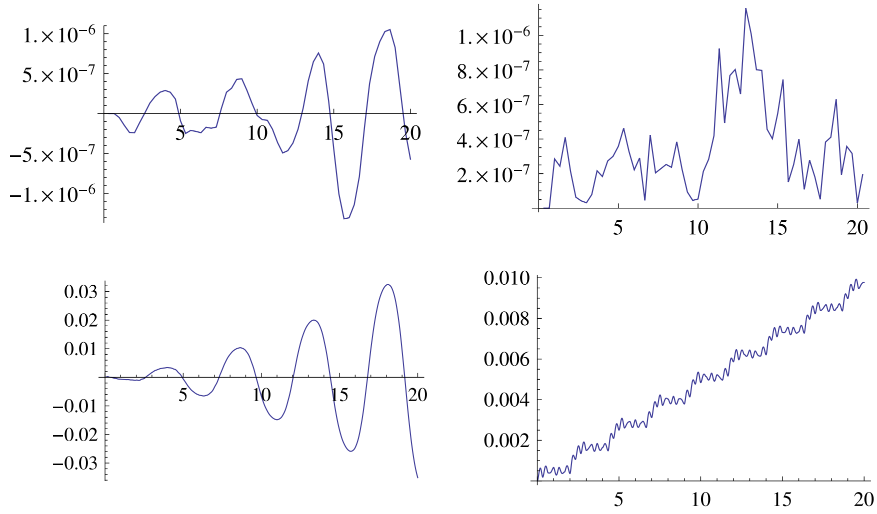

Fig. 2. Errors in the solution (left), and in the energy (right) for Example 2, using the proposed method (top) and the classical Runge–Kutta (bottom), taking $N = 60$ steps.

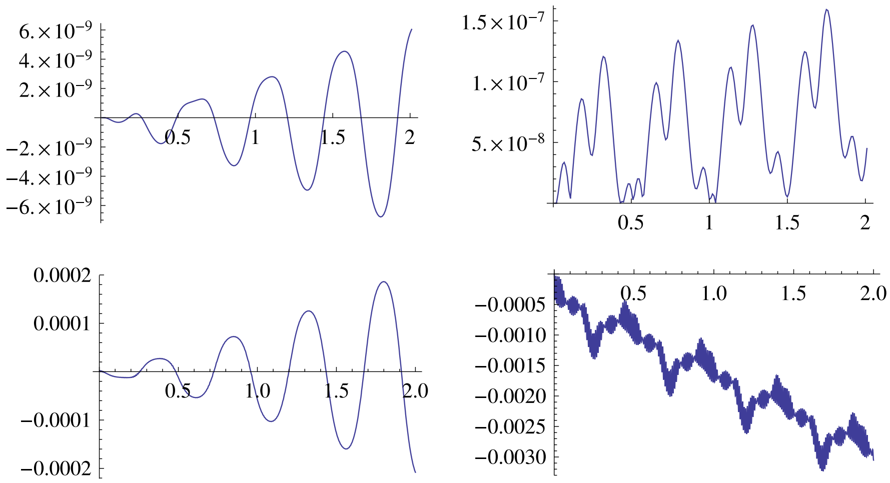

Fig. 3. Errors in the solution (left), and in the energy (right) for Example 3, using the proposed method (top) and the classical Runge–Kutta (bottom), taking $N = 200$ steps.

5. Conclusions

In this paper, we have presented a new explicit scheme to solve the autonomous undamped Duffing oscillator where the forcing term is perturbed by a small parameter. The proposed scheme is constructed by considering a suitable approximation to the theoretical solution of (1) when the forcing term is null, that is using the elliptic function $cn(t, k)$. The resulting method is of second order of convergence. As a byproduct we obtain not only the solution but also the derivative at each point on the discrete grid. The numerical results validate the good performance of the proposed scheme. In the examples presented we can see that the efficiency of the proposed method is very superior to the classical fourth order Runge–Kutta method, not only in accuracy but also in computational cost. In our future work it is our goal to extend this elliptic fitting approach for solving different kinds of problems.

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