Drawings of the complete graph

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Resumen

Una de las conjeturas abiertas en la teoría de grafos, la conjetura de Harary-Hill, afirma que el número mínimo de cruces necesarios para trazar en el plano el grafo completo K_n (grafo de n vertices conectados todos entre si) es igual a

$$Z(n) := \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor$$

A lo largo de nuestro trabajo estudiamos las principales técnicas para calcular o estimar dicho valor mínimo del número de cruces, además de exponer algunos de los últimos avances acerca de dicha conjetura.

Dedicamos el primer capítulo a hacer una breve revisíon de las definiciones y conceptos más básicos de la teoría de grafos: qué es un grafo, cuales son sus principales características y cuales son los tipos más detacados de grafos.

Ya en el segundo cápitulo entramos de lleno al estudio del número de cruces. Hablamos de los grafos planos, aquellos que pueden ser trazados sin necesidad de cruces, destacando tres resultados clásicos en torno a ellos: la fórmula de Euler, el teorema de Fary-Wagner y el teorema de Kuratowski. A su vez, damos una demostración del famoso crossing lemma, que nos proporciona una cota inferior para el número de cruces de grafos con uno o más cruces en cualquiera de sus trazados (grafos no planos). Viendo además que aunque dicha cota es aceptable para el grafo completo, no es comparable a la conjeturada por Hill.

En el tercer capítulo hablamos en profundidad de los trazados topológicos. Definiendo una variente de ellos, los trazados simples, en nuestro camino en busca de una forma de minimizar el número de cruces. Su característica principal será que cada par de ejes se cruza a lo sumo una única vez. Definimos también dos tipos de isomorfismos de trazados. Se dice que dos trazados son débilmente isomorfos si existe una biyección tal que dos ejes se cruzan en un trazado si y solo si sus imágenes en el otro trazado también lo hacen. Existe un teorema que demuestra que para un grafo completo, las clases de isomorfía débil de los trazados simples están completamente determinadas por su sistema de rotación, un concepto computacionalmente más sencillo, que facilita en cierta medida la tarea de calcular el número de cruces, y es herramienta indispensable en otro tipo de problemas ligados al grafo completo.

Entender el grafo completo como una nube de n puntos dispuestos en el plano, e introducir aqui los conceptos de k-conjuntos o j-ejes, nos permite tratar de estimar el número de subconjuntos de cuatro elementos de la nube que forman un cuadrilátero, lo cual equivale a intentar determinar el número de cruces que aparecen cuando se quieren conectar los n puntos, todos entre si (las diagonales del cuadrilátero determinarán un cruce).

Aplicando estas ideas a trazados con líneas rectas, se demuestra en el cuarto capítulo que existe una diferencia asintóticamente significativa entre el número mínimo de cruces en trazados rectilíneos y el valor dado por la fórmula de Harary-Hill para trazados topológicos generales.

En el ultimo capítulo, estudiamos una gran familia de trazados de K_n, los trazados s-shellable. Usando una extensión del concepto de k-ejes para trazados topológicos del grafo completo se prueba que los trazados s-shellable para algún $s \geq n/2$ satisfacen la conjetura de Harary-Hill.

Finalmente, vemos que esta familia contiene a otros trazados importantes como los monotonos, los x-acotados, los 2-page book y los cilindricos (trazados propuestos por Hill para satisfacer su conjetura), y por tanto todos ellos satisfacen la conjetura de Harary-Hill.

Introduction

One of the open conjectures in the graph theory, the *Harary-Hill conjecture*, talks about the lowest number of crossings in any drawing of the complete graph in the plane. Another similar conjecture, the *Zarankizwicz's conjecture*, focuses on the minimal number of crossings in any drawing of the complete bipartite graph in the plane. Although both problems were formally exposed around 1950s, in recent years there has been a long list of articles presenting significant advances in this area.

Our aim will be to study the main techniques for calculating or estimating this minimal number of crossings and to expose the last results about the Harary-Hill conjecture, trying to unify several of the most outstanding approaches that have been given to these subjects.

The lowest number of edge-crossings of a topological drawing of a graph G is known as the *crossing number* $cr(G)$ of such graph. If the edges are represented as straight-line segments we will talk about *rectilinear crossing number*. In addition, our study will turn around two kind of graphs: The *complete graph* of order n, denoted by K_n, which has all its pairs of vertices connected; and the *complete bipartite graph*, denoted by $K(n_1,n_2)$, which has the vertices divided into two partitions (with n_1 and n_2 vertices each one) and all pairs of vertices from different partitions are connected. Having said that, we can give the formal statement for both conjectures.

Theorem (Harary-Hill conjecture)**.** *The crossing number of the complete graph K_n is equal to*

$$Z(n) := \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor$$

Theorem (Zarankiewicz's conjecture)**.** *The crossing number of the complete bipartite graph $K(n_1,n_2)$ is equal to*

$$Z(n_1,n_2) := \left\lfloor\frac{n_1}{2}\right\rfloor\left\lfloor\frac{n_1-1}{2}\right\rfloor\left\lfloor\frac{n_2}{2}\right\rfloor\left\lfloor\frac{n_2-1}{2}\right\rfloor$$

It has already proved that $Z(n)$ and $Z(n_1,n_2)$ are upper bounds of $cr(K(n))$ and $cr(K(n_1,n_2))$, respectively. The other sides of the conjectures are still open and they are currently called Harary-Hill conjecture and Zarankiewicz's conjecture.

We will devote the second chapter to study the classical results about the *crossing number* $cr(G)$ of a graph G.

On the one hand, we will study a particularly interesting case, *Planar graphs*, which have no crossings. The boundary between planar and non-planar graph are the *triangulated planar graphs* (*maximal*). There are three interesting results around (maximal) planar graphs : the *Euler's formula*, which relates the number of edges, vertices and regions of a planar graph in the plane by a simple linear equation; the *Fary-Wagner theorem*, which states, although it may sound obvious, that each planar graph can be draw without crossings in the plane using straight-line segments; and the *Kuratowski's theorem*, which gives a characterization of the planar graphs in relation to the non-planar graphs K_5 and $K(3,3)$.

On the other hand, determining the (rectilinear) crossing number for any graph is an *NP-Hard problem*. However, it is not difficult to find an upper bound, and there exist a well known result that gives a not bad lower bound (we will see that, for the complete graph, it is worse than the bound given by Hill).

Theorem (Crossing lemma)**.** *Given a graph G with n vertices and e edges such that $e > 4n$, the crossing number $cr(G)$ obeys the inequality*

$$cr(G) \geq \frac{e^3}{64n^2}$$

The third chapter focuses on topological drawings of graphs, giving a more rigorous definition.

Looking for a way to minimize the number of crossings in topological drawings of K_n, *good drawings* appear, which has the extra property that any two edges intersect at most once, either at a common endpoint or at a crossing. Good drawings also allows us to count the crossings by the overall sum of the number of points in which each pair of edges crossses instead of the number of crossing-edges pairs.

We will also introduce the concept of *isomorphism* of drawings, which preserve the vertex-edge-face incidences, *weak isomorphism*, which preserve the edge-crossings pairs, and *rotation system*, the set of the clockwise cyclic order of edges incident to a vertex for all vertices.

Beside, the main two theorems of this chapter state that for complete graphs, the rotation systems and the weak isomorphism classes are mutually determined. Thus, if it were possible to compute all the different rotation systems of a complete graph and then check which correspond to weak isomorphism classes, we could select those good drawings with the lowest number of crossings. Rotation systems also are very useful concepts in other open problems around the complete graph.

The problem of finding the rectilinear crossing number can be state in an alternative way by considering a set S of n points in the plane and trying to find the number of subsets of four points which form a *convex quadrilateral*. In the fourth chapter we will use that to give a result proving that the rectilinear crossing number and the crossing number differ in an assymptotically relevant term.

Theorem. *Let S be a set of n points in the plane in general position. Then the number of convex quadrilaterals determined by S is at least*

$$(3/8+\varepsilon)\binom{n}{4} + O(n^3) > 0.37501\binom{n}{4}$$

where $\varepsilon \approx 1.0872 \cdot 10^{-5}$.

Our main tools to prove this theorem will be the concepts of *k-sets* of S, subsets of cardinality k that can be separated from S by a line; and *j-edges* of S, an ordered edge determined by two points of S such that there are exactly j points on its right side.

A drawing of K_n is *s-shellable* if there exists a subset $S = \{v_1, v_2, ..., v_s\}$ of the vertices and a region R of the drawing such that for $1 \leq i \leq j \leq s$ the drawing obtained from the drawing by removing $v_1, v_2, ..., v_{i-1}, v_{j+1}, v_{j+2}, ..., v_s$ has a region that contains R with the vertices v_i and v_j on the boundary.

In the last chapter we will generalize the geometrical concept of k-edge to good drawings of K_n. This will allow us to prove that the Harary-Hill conjecture holds for s-shellable good drawings of K_n for some $s \geq n/2$.

In order to prove the conjecture, Hill described a way of drawing the complete graph in a cylinder (*cylindrical drawing*), dividing the vertices in both lids and using the three faces of the figure to draw the edges that connect the respective vertices. At the end of the chapter we will prove that every cylindrical, 2-page book, x-bounded or monotone drawing of K_n is s-shellable for some $s \geq n/2$. Thus, these drawings hold the Harary-Hill conjecture.

Contents

Chapter 1

Basic definitions and concepts

In this chapter we are going to do a quick review of the most basic ideas about the graph theory; the definition of a graph, its main characteristics and some outstanding types of graphs, which constitute the set of concepts necessary to go deeper into the subject at hand. All that is shown below comes from [1].

Definition 1.1. A *graph* is a pair $G = (V, E)$, where V is a nonempty set, whose elements are called *vertices*, and E is a possibly empty set of two-element subsets of distinct elements of V, whose elements are called *edges*.

We will consider $n = |V|$ and $m = |E|$ as the *order* and the *size* of G respectively. Also, we will use u_i with $i = 1, \ldots, n$ (also v_j) to talk about vertices and $e_k = u_i u_j$ with $k = 1, \ldots, m$ refering to edges.

The edge $e = uv$ is said to join the vertices u and v, then u and v are *adjacent vertices* while u and e are *incident*, as v and e. Furthermore, if e_1 and e_2 are distinct edges incident with a common vertex, then they are *adjacent edges*.

In addition, we will focus our study on *simple graphs*, which do not have *loops* (edges connecting a vertex with itself), *multiple edges* (different edges connecting the same vertices) or *direction* ($uv = e = vu$, which is not specified in the previous definition).

Broadly speaking, the *drawings of a graph* are the different ways to represent a graph according to the properties we are interested to study. We talk about a *topological drawing* of a graph when it is represented in the plane with points as vertices and edges as Jordan arcs connecting its vertices. The arcs are allowed to cross, but they do not pass through vertices except for their endpoints. If the edges are represented as straight-line segments, we have a *rectilinear drawing*.

The degree (or valency) $d(u)$ of a vertex u is the number of edges incident with it. A vertex is called odd or even depending on whether its degree is odd or even. Here we have a useful result.

Theorem 1.1. *In any graph, there is an even number of odd vertices.*

Proof. It follows from the fact that $\sum_{i=1}^{n} d(u_i) = 2m$. Since the sum of the degrees of all the vertices is even, there must be an even number of summands with an odd value. $\square$

Definition 1.2. A graph $G = (V, E)$ is *isomorphic* to a graph $G' = (V', E')$ if there exists a one-to-one mapping ϕ, called an isomorphism, from V onto V' such that ϕ preserves adjacency and nonadjacency; i.e., $uv \in E$ if and only if $\phi(u)\phi(v) \in E'$.

It is easy to see that "isomorphic to" is an equivalence relation on graphs.

Definition 1.3. A graph $G_1 = (V_1, E_1)$ is a *subgraph* of a graph $G = (V, E)$ if $V_1 \subseteq V$ and $E_1 \subseteq E$. We also say that G is a *supergraph* of G_1.

Whenever $V_1 = V$ we say that G_1 is a *spanning subgraph* of G. An important kind of subgraphs which we should consider are the *induced subgraphs*. The subgraph of G *induced by* the set of vertices V_1 (*vertex-induced*) is $G_1 = (V_1, E_1)$ with $E_1 = E \cap (V_1 \times V_1)$. In the same way the subgraph of G *induced by* the set of edges E_1 (*edge-induced*) is $G_1 = (V_1, E_1)$ with V_1 the set of vertices incident with at least one edge of E_1.

We denote by $G - u$ the subgraph obtained deleting the vertex u and its incident edges. $G - e$ is the subgraph built deleting e but keeping the vertices. The deletion of a set of vertices or set of edges is defined analogously. Also $G + e$ is the supergraph built adding a new edge $e = uv$ conecting u, v vertices of G.

Definition 1.4. A *u-v walk* of a graph G is a finite, sequence of vertices of G, beginning with u and ending with v, such that for every two consecutive vertices in the sequence there is a edge in E whose vertices are incident . A *u-v trial* is a u-v walk in which no edge is repeated, while if no vertex is repeated it is a *u-v path*.

Notice that every path is therefore a trial. A u-v walk is *closed* or *open* depending on whether $u = v$ or $u \neq v$. A non-trivial closed trial is a *circuit*. A circuit in which no vertices are repeated (except first and last) is called *cycle*. An *acyclic* graph has no cycles. The subgraph of G induced by the edges of a trial, path, circuit or cycle is also called a *trial*, *path*, *circuit* or *cycle* of G. the number of occurrences of edges in a walk is called its *length*. A cycle is *odd* or *even* depending on whether its length is odd or even.

Two vertices u and v of a graph G are *connected* if there exists a u-v path in G. The graph G itself is *connected* if every two of its vertices are connected. A graph which is not conected is *disconnected*.

Definition 1.5. A *tree* is an acyclic connected graph and a *forest* is an acyclic graph.

Thus each component of a forest is a tree.

Definition 1.6. A graph is *complete* if every two of its vertices are adjacent.

We denote by K_n the complete graph of order n. By definition the size of the this graph is $m = n(n-1)/2$.

A graph is *regular of degree r* if all its vertices have degree r. K_n is a regular graph of degree $n-1$. Moreover, every graph is a subgraph of a regular graph; indeed, every graph of order n is a spanning subgraph of K_n.

The complement $\overline{G}$ of a graph G is the graph with the same vertex set as that of G and such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

Definition 1.7. A graph G is *l-partite* , $l \geq 2$, if it is possible to partition V into l subsets V_1,V_2,...,V_l such that every element of E joins a vertex of V_i to a vertex of V_j with $i \neq j$.

For $l = 2$, such graph is called *bipartite*. A *complete n-partite* graph G is an n-partite graph with the added property that if $u \in V_i$ and $v \in V_j$ with $i \neq j$, then $e = uv \in E$. If $n_i = |V_i|$ with $i = 1, ..., l$, then this graph is denoted by $K(n_1, ..., n_l)$. Notice that a complete n-partite graph is complete if and only if $n_i = 1$ for all i, in which case it is K_l.

Chapter 2

Crossing number

When we think about how to draw a graph we immediately come across the problem of how to interpret the intersections between its edges, which is solved as soon as we take into account the following definition.

Definition 2.1. the *crossing number* $cr(G)$ of a graph G is the lowest number of edge-crossings of a topological drawing of G.

A variation of this concept, the *rectilinear crossing number*, requires a rectilinear drawing of G. Determining the crossing number and the rectilinear crossing number are both *NP-hard problems*. See [2] and references therein.

In addition, we can classify the graphs in two groups according to the crossing number. Let us consider a graph G with $n = cr(G)$, if $n = 0$ we will call G planar, otherwise if $n > 0$ we will say that G is non-planar. In this chapter we will study separately some interesting results about both kinds of graphs.We will use results from [1], [2] y [3].

2.1 Planar graphs

We will start by giving a more precise definition of this type of graphs.

Definition 2.2. A graph is *planar* if it can be embedded in the plane (or, equivalently, on the sphere), i.e., it can be drawn on the plane in such a way that its edges intersect only at their endpoints.

In other words, it can be drawn in such a way there are not edges-crossings. Such a drawing is called a *plane graph* or *planar embedding* of the graph.

The *regions* (or *faces*) of a plane graph are the different portions of the plane delimitated by the edges and vertex of the graph, which constitute the *boundary* of each of these regions. Every plane graph contains an unbounded region called the *exterior region*.

The following theorem allows us to relate the number of edges, vertices and regions of a connected plane graph using a simple linear equation. This equation does not have to be true for drawings of non-planar graphs.

Theorem 2.1 (Euler's formula)**.** *G is a connected plane graph with n vertices, m edges and r regions, then*

$$n - m + r = 2$$

Proof. By induction on m. For $m = 0$, it is obvious since in this case $n = 1$ and $r = 1$ (We cannot have more than 1 vertex without edges, the graph has to be connected). Assume the result is true for all connected plane graphs with fewer than m, where $m \geq 1$ and suppose G has m edges. There are two cases: If G is a tree, then $n = m+1$ and $r = 1$ so that the formula follows. If G is not a tree, let $e \in E$ be part of a cycle of the graph and consider $G-e$, a connected plane graph with n vertices, $m-1$ edges, and $r-1$ regions so that by the inductive hypothesis, $n-(m-1)+(r-1) = 2$, which implies that $n-m+r = 2$. □

An important consequence of this theorem is that any two planar embeddings of a planar graph have the same number of regions; thus we can talk about the number of regions of a planar graph.

A planar graph G is called *maximal planar* if for every pair of non-adjacent vertices u and v the graph $G+e$, where $e = uv$ is non-planar. Therefore, in any planar embedding of a maximal planar graph having order $n \geq 3$ the boundary of every region is a triangle. This is the reason why the maximal planar graphs are also named *triangulated planar graphs*. Also it is clear that any planar graph is a spanning graph of a maximal planar graph.

As a consecuence of the definition of triangulated planar graph and the Euler's Formula we have the following results.

Theorem 2.2. *If G is a maximal planar graph with n vertices and m edges, $n \geq 3$, then*

$$m = 3n-6$$

Proof. Denote by r the number of regions of G. In G the boundary of every region is a triangle, and each edge is on the boundary of two regions. Hence, if the number of edges on the boundary of a region is summed over all regions, the result is $3r$. On the other hand, such a sum counts each edge twice so that $3r = 2m$. Applying the Euler's formula, we obtain $m = 3n-6$. □

Corollary 2.3. *If G is a planar graph with n vertices and m edges, $n \geq 3$, then*

$$m \leq 3n-6$$

Proof. Add to G sufficiently many edges so that the resulting graph G' with n' vertices and m' edges is maximal planar. Clearly, $n = n'$ and $m \leq m'$. By theorem 2.2 , $m \leq m' = 3n'-6 = 3n-6$. □

Corollary 2.4. *Every planar graph contains a vertex of degree at most five.*

Proof. Let G be a planar graph with n vertices and m edges. Consider $V = \{u_1, u_2, ..., u_n\}$. If $p \leq 5$, then the result is obvious. Otherwise, $m \leq 3n-6$ implies that

$$\sum_{i=1}^{n} d(u_i) = 2m \leq 6n-12.$$

Not all n vertices of G have degree six or more, for then $2m \geq 6n$. Thus G contains a vertex of degree five or less. □

A good property of the planar graphs is that all of them admit to rectilinear drawings. We show it in the theorem below.

Theorem 2.5 (Fary-Wagner theorem)**.** *Each planar graph can be embedded in the plane so that every edge is a straight-line segment (Rectilinear drawing).*

Proof. It suffices to prove the theorem for maximal planar graphs. Assume every maximal planar graph can be embedded in the plane so that each of its edges is a straight-line segment, and let H be an arbitrary planar graph. We have observed that H is a spanning subgraph of a maximal planar graph G, which can be embedded in the plane so that each of its edges is a straight-line segment. By deleting the appropriate edges from G we can produce an embedding of H which each edge is a straight-line segment.

Let G_0 be a maximal planar graph and further assume it to be a plane graph. Without loss of generality, we suppose G_0 to have order $p \geq 4$. As we have already noted, the boundary of every region of G_0 (including the exterior region) is a triangle.

Our procedure will be to disassemble G_0 by removing one vertex at a time until only the exterior triangle remains. We then reassemble G_0 using only straight-line segments.

Select a vertex of G_0 not belonging to the exterior triangle and denote it by v_1. Delete v_1 and its incident edges from G_0 , and leave all other vertices and edges of G_0 exactly as originally placed. The boundary of all regions of $G_1 = G_0 - v_1$ are triangles except possibly one, the region in which v_1 would be inserted to reproduce G_0. Denote this region by R_1 and the cycle which forms its boundary by C_1. Among the vertices of C_1 not belonging to the exterior triangle of G_1, let v_2 be any one vertex having degree two (It is not the boundary of a triangular region). If there is no such vertex of degree two, select as v_2 any vertex of C_1 not on the exterior triangle. Let $G_2 = G_1 - v_2$, and denote by R_2 the region of G_2 into which v_2 would be inserted to obtain G_1. We continue this procedure until only the exterior triangle remains; thus we obtain vertices v_1, v_2,...,v_{p-3}, the plane graphs G_1, G_2,...,G_{p-3} and the regions R_1, R_2,...,R_{p-3}. From the manner in which the vertices v_i, $1 \leq i \leq p-3$, were selected, it follows that for each $i = 1,2,...,p-3$, the vertex v_i together with at least two edges are placed in region R_i to produce the graph G_{i-1}.

Before we proceed further with the proof, an observation is useful her. First, we define an additional term. A region of a plane graph is said to be *starlike* if there exists an open set $\mathscr{O}$ in the region such that for any point $x \in \mathscr{O}$, a straight-line segment can be drawn from x to each vertex on the boundary of the region such that the entire line segment (except the vertex on the boundary) lies completely in the region. then, it follows that if a vertex v is placed in the distinguished open set of a starlike region R and straight-line segments are drawn from v to two or more vertices on the boundary of R, the resulting regions on whose boundary v lies are all starlike.

Denote by G_{p-3} the graph K_3. Surely G_{p-3} may be embedded in the plane so that each of its edges is a straight-line segment. It is obvious that the interior region of G_{p-3} is starlike; indeed in this case, the open set of interest may be taken as the entire region. Place a vertex in the interior region R_{p-3} of G_{p-3} label it v_{p-3}, and join it to the two or three vertices of G_{p-3}, depending on the number of vertices of G_{p-3} to which v_{p-3} is adjacent in the final step of disassembling G_0. Denote the resulting graph G_{p-4}. As we have just observed, the two or three regions formed in constructing G_{p-4} are starlike. Label the appropriate region R_{p-4}, insert the vertex v_{p-4} in the suitable open set of R_{p-4}, and join v_{p-4} to the appropriate vertices of G_{p-4} by straight-line segments. We thus obtain an embedding of G_{p-5} in the plane in which every edge is a straight-line segment. We can continue this procedure until arriving at an embedding of G_0 in which every edge is a straight-line segment. □

To finish this section we will give a characterization of the planar graphs, but first we have to introduce a few concepts.

An *elementary subdivision* of a nonempty graph G is a graph obtained from it by removal of some edge $e = uv$ and the addition of a new vertex w and the edges $e_1 = uw$ and $e_2 = vw$. A *subdivision* of G is a graph obtained from it by a succession of elementary subdivisions.

A graph H is defined to be *homeomorphic from* G if either $H = G$ or H is a subdivison of G. A graph G_1 is *homeomorphic with* a graph G_2 if there exists a graph G_3 such that each of G_1 and G_2 is homeomorphic from G_3. Furthermore, if H is homeomorphic from G, then H is a *homeomorphic image* of G while if H is homeomorphic with G, then H is a *homeomorph* of G. "Homeomorphic with" is an equivalence relation on graphs. Thus we refer to two graph as being *homeomorphic* if either is homeomorphic with the other.

Theorem 2.6. *The graphs K_5 and $K(3,3)$ are non-planar.*

Proof. Since K_5 has $n = 5$ vertices and $m = 10$ edges,

$$10 = m > 3n - 6 = 3 \cdot 5 - 6 = 9$$

Corrollary 2.3 implies reciprocally that K_5 is non-planar. Now suppose, to the contrary, that $K(3,3)$ is planar, and consider any plane graph of it. Since the graph is bipartite, It has no triangles; thus each of its regions is bounded by at least four edges. Consider N the sum over all its regions r of the number of edges bounding them. Thus $N \geq 4r$. Since N counts each edge twice and $K(3,3)$ has $m = 9$ edges, $N = 18$ so that $r \leq 9/2$. However, $K(3,3)$ has $n = 6$ vertices and by the Euler's formula, $r = 5$ and this is a contradiction. Hence $K(3,3)$ is non-planar too. □

Theorem 2.7 (Kuratowski's theorem)**.** *A graph is planar if and only if it contains no subgraph homeomorphic with K_5 or $K(3,3)$.*

The proof of the theorem above is beyond the scope of our study.

2.2 Crossing lemma

As we said at the begining of the chapter, determining the (rectilinear) crossing number of a non-planar graph G is a NP-hard problem. But we can tackle this problem by looking for a lower and an upper bound of $cr(G)$. In this way we will have enclosed $cr(G)$ between two values (more or less separated).

In this section we will prove a previous lemma and a theorem which will give us a lower bound of $cr(G)$, finding a upper bound of $cr(G)$ will be much more easier.

Lemma 2.8. *Consider a graph G with $n \geq 3$ vertices and e edges, then*

$$cr(G) \geq e - 3n$$

Proof. Consider a topological drawing of G which has exactly $cr(G)$ crossings. Each of these crossings can be removed by removing an edge from G. Thus we can find a graph with at least $e - cr(G)$ edges and n vertices with no crossings, therefore G is a planar graph. By corollary 2.3 we have $e - cr(G) \leq 3n - 6 \leq 3n$ with $n \geq 3$, and the claim follows. □

Theorem 2.9 (Crossing lemma)**.** *Given a graph G with n vertices and e edges such that $e > 4n$, the crossing number $cr(G)$ obeys the inequality*

$$cr(G) \geq \frac{e^3}{64n^2}$$

Proof. We will use a probabilistic argument. Let $0 < p < 1$ be a probability parameter which we will choose later. Now we construct a random subgraph H of G by allowing each vertex of G to lie in H independently with probability p, and allowing an edge of G to lie in H if and only if its two vertices were chosen to lie in H. Let e_H, n_H and cr_H denote the number of edges, vertices and crossings of H, respectively. Since H is a subgraph of G, a topological drawing of G contains a topological drawing of H. By lemma 2.8 we have that

$$cr_H \geq e_H - 3n_H$$

Since each of the n vertices in G had a probability p of being in H, we can assume that $n_H \sim B(n, p)$. Similarly, each of the edges in G has a probability p^2 of remaining in H since both endpoints need to stay in H, so we can also assume that $e_H \sim B(e, p^2)$. Finally, every crossing in the drawing of G has a probability p^4 of remaining in H, since every crossing involves four distinct vertices (If we consider a drawing of G with $cr(G)$, then any two edges in the drawing with a common vertex are disjoint, otherwise we could interchange the intersecting parts of the two edges and reduce the crossing number by one). Thus we can assume that $cr_H \sim B(cr(G), p^4)$.

Now, taking expectations in the previous formula we obtain

$$E[cr_H] \geq E[e_H] - 3E[n_H]$$

Also we know that $E[X] = np$ when $X \sim B(n,p)$. Then we have

$$cr(G)p^4 \geq ep^2 - 3np$$

If we fix $0 < p = 4n/e < 1$ (since we assumed that $e > 4n$), just simplifying we obtain

$$cr(G) \geq \frac{e^3}{64n^2}$$

□

Example 2.3. As we saw in chapter 1 the complete graph K_n has $m = n(n-1)/2$. Also in chapter 3 we will prove that there are at most $\binom{n}{4}$ crossing in a good drawings of K_n. Combining this with the crossing lemma we get

$$\frac{n^4}{512} \approx \frac{(n(n-1)/2)^3}{64n^2} \leq cr(K_n) \leq \binom{n}{4} \approx \frac{n^4}{24}$$

We can obtain a similar result for the complete bipartite graph $K(n_1,n_2)$. We know that this graph has $n = n_1 + n_2$ vertices and $m = n_1 n_2$ edges. In chapter 3 we will also see that there are at most $n_1(n_1-1)n_2(n_2-1)/4$ crossing in a good drawing of $K(n_1,n_2)$. Combining again this with the crossing lemma we obtain

$$\frac{(n_1 n_2)^3}{64(n_1+n_2)^2} \leq cr(K(n_1,n_2)) \leq \frac{n_1(n_1-1)n_2(n_2-1)}{4}$$

In the special case of a complete bipartite graph with $n = n_1 = n_2$ we can simplify the inequality as follows

$$\frac{n^4}{256} \leq cr(K(n,n)) \leq \frac{n^4}{4}$$

Chapter 3

Topological drawings

In relation to the example at the end of the previous chapter, two of the most interesting open problems in the graph theory are the following.

Theorem 3.1 (Harary-Hill conjecture)**.** *The crossing number of the complete graph K_n is equal to*

$$Z(n) := \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor$$

Theorem 3.2 (Zarankiewicz's conjecture)**.** *The crossing number of the complete bipartite graph $K(n_1,n_2)$ is equal to*

$$Z(n_1,n_2) := \left\lfloor\frac{n_1}{2}\right\rfloor\left\lfloor\frac{n_1-1}{2}\right\rfloor\left\lfloor\frac{n_2}{2}\right\rfloor\left\lfloor\frac{n_2-1}{2}\right\rfloor$$

It has already proved that $cr(K(n))$ and $cr(K(n_1,n_2))$ are at most $Z(n)$ and $Z(n_1,n_2)$, respectively. The other sides of the conjectures are still open problems (They are currently called Harary-Hill conjecture and Zarankiewicz's conjecture).

This chapter constitute the first step to be able to explain in detail some of the latest avances around these conjectures, although in the following chapters we will focus our study on the complete graph. The content we will see below comes mainly from [4] and [5].

First of all we will give a more rigorous definition of topological drawing of a graph.

Definition 3.1. A *topological drawing D* of a graph *G* is a drawing of the graph in the plane (or, respectively, on the sphere), where the vertices are distinct points and the edges are Jordan arcs connecting two (end) points. Edges are allowed to cross, but they do not pass through other vertices, tangencies are forbbiden and no three edges pass through a single crossing.

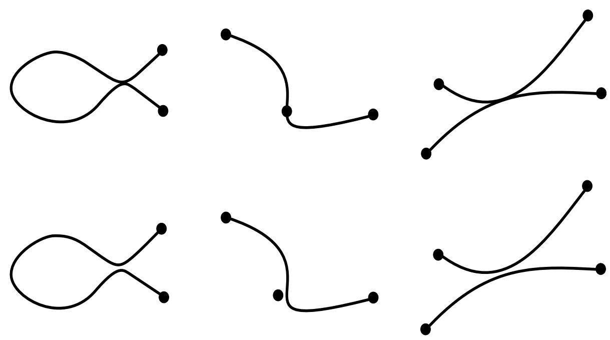


Figure 3.1: Degenerate cases forbidden in topological drawings and how to make them valid. Tangencies on both sides and a edge passing through other vertex in the middle.

As we will see below topological drawings still have some disadventages we should try to avoid. Just adding a few restrictions we obtain the following definition.

Definition 3.2. A *good drawing* is a topological drawing of a graph such that self-intersections of edges are forbidden, and any two edges intersect at most once, either at a common endpoint or at a proper crossing.

Our main motivation for considering good drawings comes from the problem of minimizing the number of crossings in drawings of K_n. Using these drawings crossings are counted by the overall sum of the number of crossing edges pairs, as opposed to the number of points in which each pair of edges crosses. Indeed, for any topological drawing of a graph, there exits a good drawing of the same graph with at most the same number of crossings.

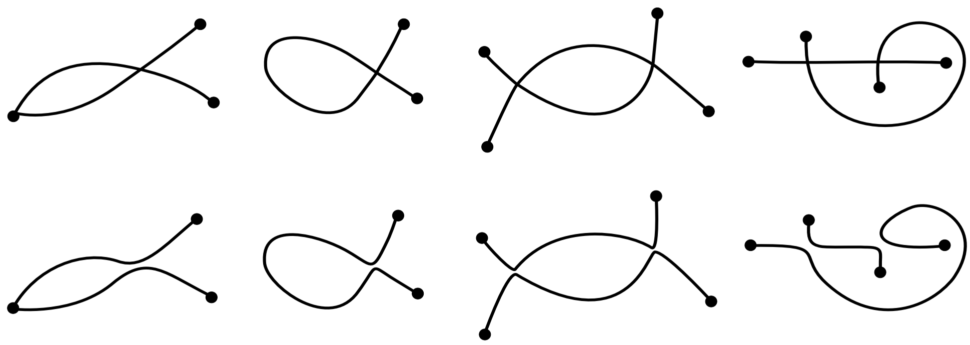


Figure 3.2: Forbidden crossings in good drawings and how to make them good. From left to right: incident edges, self-crossing edge, double-crossing.

Understading the differences between the following two types of isomorphism of drawings of a graph is essensial in this chapter.

Definition 3.3. Two drawings are *isomorphic* if there is a homeomorphism of the plane (or, equivalently, of the sphere) that transforms one drawing into the other.

This definition is equivalent to the fact that all vertex-edge-face incidences are the same.

Definition 3.4. Two good drawings are *weakly isomorphic* if there is an incidence-preserving bijection between the drawings such that two edges cross in one drawing if and only if their images in the other drawing cross as well.

If we have two good drawings of the same graph with different crossing number, for sure they are not weakly isomorphic.

Isomorphic drawings are also weakly isomorphic. Unlike the isomorphism, weak isomorphism can change the faces of the drawings. On the other hand, roughly speaking, weakly isomorphic drawings that are non-isomorphic differ in the order in which their edges intersect.

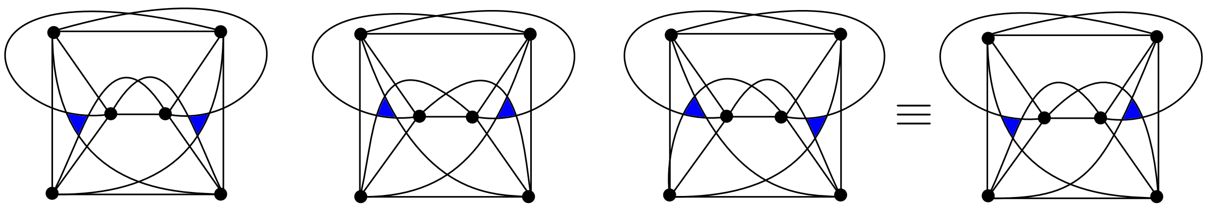


Figure 3.3: Four drawings of the same weak isomorphism class of K_6. The last two are also isomorphic (consider the labeling horizontally mirrored), the other pairs are not.

The definitions and theorems below are the tools that will allow us to deal with good drawings in a much easier way.

The *rotation* $\rho_D(v)$ (or $\rho(v)$ when D is clear from the context) of a vertex v in D is the clockwise cyclic order of edges incident to v, given as a sequence (which is interpreted circularly) of the second vertices of all edges at v (Note that if $G = K_n$ then $\rho_D(v)$ is a cyclic permutation of $V - \{v\}$).

Definition 3.5. The *rotation system* of D is the set of rotations of all vertices of D and it is denoted by $\mathscr{R}(D)$.

We consider that two rotation systems are equivalent if one can be obtained from the other by relabeling and optional inversion of all rotations. A rotation system is called *realizable* if it is the rotation system of a good drawing of a complete graph.

The following two results imply that for complete graphs, the rotation system uniquely determines the weak isomorphism class of a good drawing. Thus we can work with rotation systems instead of work with good drawings directly.

Theorem 3.3. *The rotation system of a good drawing of the complete graph determines the pairs of crossing edges.*

Theorem 3.4. *The set of crossings pairs of edges determines the equivalence class of the rotation system of a good drawing of the complete graph.*

Note that these two results are, in general, only true for complete graphs: Determining the crossing number of any graph knowing its rotation system is NP-complete. A result similar to the above ones is also known for isomorphism classes.

Theorem 3.5. *Two good drawings are isomorphic if and only if there exists a bijection between their vertices such that (i) they are weakly isomorphic, (ii) for each edge, the order of crossings along its image is the same, and (iii) for each crossing the radial order of the edge parts emanating to the four involved vertices is the same (or inverted for all crossings).*

It is not difficult to prove that K_4 has only two weak isomorphism classes. We denoted them by D_4^X and D_4^Y. We can see drawings of them with their rotation system in figure 3.4.

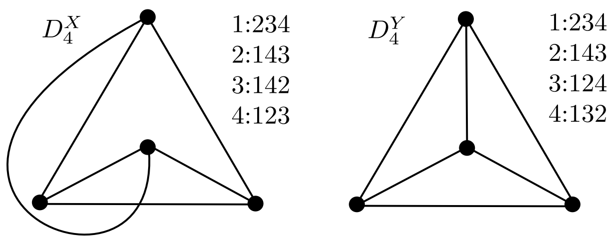


Figure 3.4: The two weak isomorphism classes of K_4.

From this property of K_4 we say that a rotation system is *consistent* if each rotation system restricted to any four of its vertices is either the one of D_4^X or D_4^Y. Still, there exist non-realizable consistent rotation systems.

Remark 3.6. For each set of 4 vertices of K_n there is at most one edge-crossing. Thus, $cr(K_n)$ can be as large as $\binom{n}{4}$. Something similar happens with $K(n_1,n_2)$. For each set of 4 vertices (2 from each partition) there is at most one edge-crossing. Hence, $cr(K(n_1,n_2))$ can be as large as $\binom{n_1}{2}\binom{n_2}{2}$.

In figure 3.5 we show drawings of the seven different rotation systems of K_5, in which all 4-tuples are realizable. However, only drawings D_5^1 to D_5^5 are different weak isomorphism classes of K_5. Drawings D_5^6 and D_5^7 are not good drawings, because they contain edges that cross twice. For more details see [6].

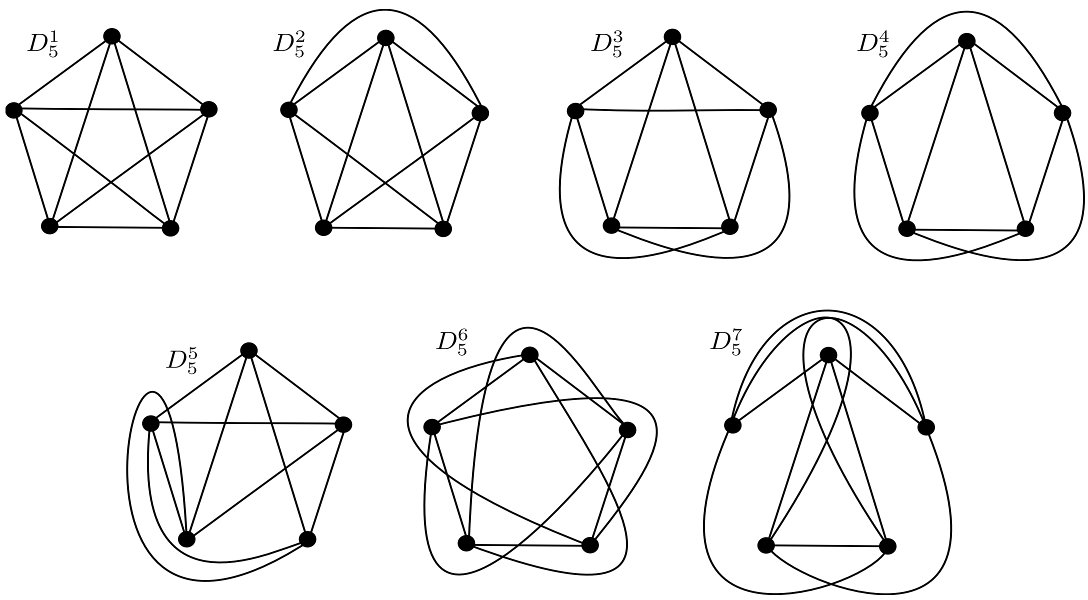


Figure 3.5: Drawings D_5^1 to D_5^7 of K_5.

Chapter 4

Rectilinear drawings

Let S be a set of n points in general position in the plane (no three points are collinear). The subset of four points in S which form the vertices of a *convex quadrilateral* are *convex*. We will be interested in studying the number of these subsets in S. As we saw in the previous chapter, it can be as large as $\binom{n}{4}$ (if S is in convex position), but what it is its the minimum?. Another way of stating the problem, more familiar to us, is to find the *rectilinear crossing number* of K_n.

We will prove at the end of the chapter that the number of convex quadrilaterals determined by the set S is at least

$$(3/8+\varepsilon)\binom{n}{4}+O(n^3) > 0.37501\binom{n}{4}$$

where $\varepsilon \approx 1.0872\cdot 10^{-5}$.

The small ε is significant for the following reason. If we skip the requirement that the edges have to be represented by straight line segments and instead allows to use Jordan arcs, then K_n can be drawn with $3/8\binom{n}{4}+O(n^3)$ crossings (Harary Hill conjecture). So, while there are studies which show that the crossing number and the rectilinear crossing number of K_n differ for specific values of n, this lower bound shows that the difference lies in the assymptotically relevant term.

We need to introduce few terms and results before give a proof of the theorem.

Definition 4.1. A *k-set* of S is a subset $T \subseteq S$ of cardinality k such that T can be separated from its complement $S\backslash T$ by a line. An i-set with $1 \leq i \leq k$ is called an $(\leq k)$-set.

Our main tool will be the following lower bound on the number of $(\leq k)$-set.

Theorem 4.1. *Let S be a set of n points in the plane in general position, and $k \leq \lfloor n/2 \rfloor$. Then the number of $(\leq k)$-sets of S is at least $3\binom{k+1}{2}$.*

We can find a proof in [8].

Despite the tempting result above, the following related object will be technically more convenient than using k-sets directly.

Definition 4.2. A *j-edge* of S is an ordered pair uv, with $u, v \in S$ and $u \neq v$, such that there are exactly j points of S on the right side of the line uv.

Let $E_j = E_j(S)$ denote the number of j-edges of S; it is well known and not hard to see that for $1 \leq k \leq n-1$, the number of k-sets is E_{k-1}. An i-edge with $i < j$ will be called a $(\leq j)$-*edge*; we denote the number of $(\leq j)$-edges by $E_{\leq j} = E_0 + \ldots + E_j$.

Let $\square$ denote the number of 4-tuples of points in S that are in convex position, and let $\triangle$ denote the number of those in concave position. The following lemma gives an expression of $\square$ (up to some error term) as a positive linear combination of the E_j's. Then we will be able to substitute any lower estimates for the numbers E_j to obtain a lower bound for $\square$.

Lemma 4.2. *For every set of n points in the plane in general position,*

$$\square = \sum_{j \leq \lfloor \frac{n}{2} \rfloor -1} E_j \left(\frac{n-2}{2} - j \right)^2 - \frac{3}{4}\binom{n}{3}$$

Proof. Clearly we have

$$\square + \triangle = \binom{n}{4}$$

We can get another linear equation between these quantities. Let us count, in two different ways, ordered 4-tuples (u,v,w,z) such that w is on the right of the line uv and z is on the left of this line. First, if (u,v,w,z) is in convex position, then we can order it in 4 ways to get such an ordered quadruple. Second, if (u,v,w,z) is in concave position, then it has 6 such orderings. Hence the number of such odered quadruples $4\square + 6\triangle$. On the other hand, any j-edges uv can be completed to such a quadruple in $j(n-j-2)$ ways. So we have

$$4\square + 6\triangle = \sum_{j=0}^{n-2} E_j (n-j-2) j$$

From these two equation we get that

$$\square = \frac{1}{2}\left(6\binom{n}{4} - \sum_{j=0}^{n-2} E_j (n-j-2) j \right)$$

Using that

$$\sum_{j=0}^{n-2} E_j = n(n-1)$$

we can write

$$6\binom{n}{4} = \sum_{j=0}^{n-2} E_j \frac{(n-2)(n-3)}{4}$$

to get

$$\square = \frac{1}{2}\left(\sum_{j=0}^{n-2} E_j \left(\frac{(n-2)(n-3)}{4} - j(n-j-2) \right) \right)$$

Adding and subtracting $\frac{(n-2)^2}{4}$ to the expression inside the second parenthesis, and simplifying we obtain

$$\square = \frac{1}{2}\left(\sum_{j=0}^{n-2} E_j \left(\left(\frac{n-2}{2} - j \right)^2 - \frac{(n-2)}{4} \right) \right)$$

Finally, from the fact that

$$\frac{1}{2}\sum_{j=0}^{n-2} E_j \left(\frac{n-2}{2} - j \right)^2 = \sum_{j \leq \lfloor \frac{n}{2} \rfloor -1} E_j \left(\frac{n-2}{2} - j \right)^2$$

and using again the equation about E_j the lemma follows by simple computation. □

We can obtain easily a similar expression of $\square$ in terms of $(\leq j)$-edges as follows.

Lemma 4.3. *For every set of n points in the plane in general position,*

$$\square = \sum_{j \leq \lfloor \frac{n}{2} \rfloor -1} E_{\leq j}(n-2j-3) - \frac{3}{4}\binom{n}{3} + c_n$$

where

$$c_n = \begin{cases} \frac{1}{4} E_{\leq \frac{n-3}{2}} & \text{if } n \text{ is odd} \\ E_{\leq \frac{n-2}{2}} & \text{if } n \text{ is even} \end{cases}$$

Proof. By the previous lemma we have that for every set of n points in the plane in general position,

$$\square = \sum_{j \le \lfloor \frac{n}{2} \rfloor - 1} E_j \left(\frac{n-2}{2} - j \right)^2 - \frac{3}{4}\binom{n}{3}$$

Using that $E_j = E_{\le j} - E_{\le j-1}$ we obtain

$$\begin{aligned} \square &= \sum_{j \le \lfloor \frac{n}{2} \rfloor - 1} (E_{\le j} - E_{\le j-1}) \left(\frac{n-2}{2} - j \right)^2 - \frac{3}{4}\binom{n}{3} \\ &= \sum_{j \le \lfloor \frac{n}{2} \rfloor - 1} E_{\le j} \left(\frac{n-2}{2} - j \right)^2 - \sum_{j \le \lfloor \frac{n}{2} \rfloor - 1} E_{\le j-1} \left(\frac{n-2}{2} - j \right)^2 - \frac{3}{4}\binom{n}{3} \end{aligned}$$

Making the change of variable $j-1=k$ in the second term we get

$$\begin{aligned} \square &= \sum_{j \le \lfloor \frac{n}{2} \rfloor - 1} E_{\le j} \left(\frac{n-2}{2} - j \right)^2 - \sum_{k \le \lfloor \frac{n}{2} \rfloor - 2} E_{\le k} \left(\frac{n-2}{2} - k - 1 \right)^2 - \frac{3}{4}\binom{n}{3} \\ &= \sum_{j \le \lfloor \frac{n}{2} \rfloor - 1} E_{\le j} \left(\left(\frac{n-2}{2} - j \right)^2 - \left(\frac{n-2}{2} - j - 1 \right)^2 \right) + E_{\le \lfloor \frac{n}{2} \rfloor - 1} \left(\frac{n-2}{2} - \left\lfloor \frac{n}{2} \right\rfloor \right)^2 - \frac{3}{4}\binom{n}{3} \end{aligned}$$

from which the lemma follows by simple computation. $\square$

We will just give the statement of the following theorem before we use it to prove the goal of this chapter.

Theorem 4.4. *Let S be a set of n points in the plane, and consider a (not necessarily contiguous) index set $K \subseteq \{1, 2, ..., j \le \lfloor n/2 \rfloor\}$. Then the total number of k-sets with $k \in K$ is at most*

$$2n\sqrt{2\textstyle\sum_{k \in K} k}$$

Theorem 4.5. *Let S be a set of n points in the plane in general position. Then the number of convex quadrilaterals determined by S is at least*

$$(3/8 + \varepsilon)\binom{n}{4} + O(n^3) > 0.37501\binom{n}{4}$$

where $\varepsilon \approx 1.0872 \cdot 10^{-5}$.

Proof. Let $m = \lfloor n/2 \rfloor$, and apply theorem 4.4 to the intervals of the form $\{j+2, j+3, ..., m\}$. Observing that E_i is precisely the number of $(i+1)$-sets, we obtain that for all $j \le m-1$,

$$E_{\le m-1} - E_{\le j} \le 2n\sqrt{\textstyle\sum_{i=j+2}^{m} i} = 2n\sqrt{m^2 + m - j^2 - 3j - 2}$$

and since $E_{\le m-1} \ge \binom{n}{2}$ and $m = \lfloor n/2 \rfloor \le n/2$,

$$E_{\le j} \ge \binom{n}{2} - 2n\sqrt{(n/2)^2 + n/2 - j^2 - 3j - 2}$$

Combining that with $E_{\le j} \ge 3\binom{j+2}{2}$ with $j < n/2$, we get

$$\begin{aligned} E_{\le j} &\ge 3\binom{j+2}{2} + \max\left(0, \binom{n}{2} - 3\binom{j+2}{2} - 2n\sqrt{(n/2)^2 + n/2 - j^2 - 3j - 2}\right) \\ &\ge 3\binom{j+2}{2} + n^2 \max\left(0, \frac{1 - 3(j/n)^2}{2} - \sqrt{1 - 4(j/n)^2}\right) + O(n). \end{aligned}$$

The "max" term is positive for $j/n \geq t_0 = \sqrt{(2\sqrt{13}-5)/9} \approx 0.4956$, so we do gain when j is very near $n/2$. Using last lemma, we get

$$\begin{aligned}
\square &= \sum_{j \leq m-1} E_{\leq j}(n-2j-3) + O(n^3) \\
&\geq \sum_{j \leq m-1} 3\binom{j+2}{2}(n-2j-3) \\
&\quad + n^3 \sum_{t_0 n \leq j \leq m} (1-2(j/n))\left(\frac{1-3(j/n)^2}{2} - \sqrt{1-4(j/n)^2}\right) + O(n^3) \\
&= \frac{3}{8}\binom{n}{4} + n^4 \int_{t_0}^{1/2} (1-2t)\left(\frac{1-3t^2}{2} - \sqrt{1-4t^2}\right) dt + O(n^3).
\end{aligned}$$

Thus,

$$\square \geq (3/8+\varepsilon)\binom{n}{4} + O(n^3)$$

where

$$\varepsilon = 24 \int_{t_0}^{1/2} (1-2t)\left(\frac{1-3t^2}{2} - \sqrt{1-4t^2}\right) dt \approx 1.0887 \cdot 10^{-5}.$$

$\square$

In this chapter we have collected the main results from [8].

Chapter 5

Conjectures

Paul Turán described in a letter to Richard Guy a problem that came to his mind when he was working in a brick factory near Budapest during World War II. Later it would be called the *brick factory problem* and along with the *houses-and-utilities problem* of unknown origin, they can be described mathematically as a problem of minimizing the number of edge-crossings on the complete bipartite graph.

In 1952, Kazimierz Zarankiewicz and Kazimierz Urbanik proposed solutions almost simultaneously. In addition, the first one stated that $Z(n_1,n_2) := \lfloor \frac{n_1}{2} \rfloor \lfloor \frac{n_1-1}{2} \rfloor \lfloor \frac{n_2}{2} \rfloor \lfloor \frac{n_2-1}{2} \rfloor$ and he observed that this crossing number can be obtained by dividing the n_1 vertices into two groups of equal (or almost equal) size and placing the two groups equally spaced on the x-axis on either side of the origin, doing the same with the n_2 vertices but placing them on the y-axis, and then joinining the appropriate pairs of vertices together using straight-line segments. Even so, it has only been possible to test the upper bound problem. More details in [11].

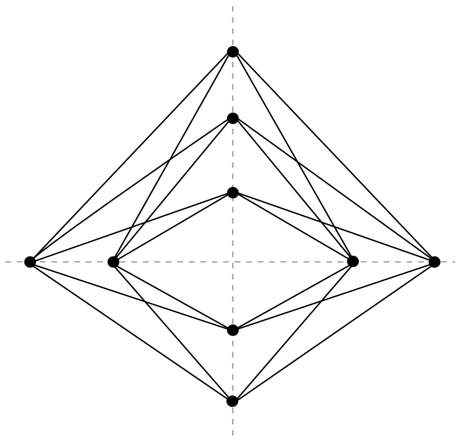


Figure 5.1: Zarankiewicz drawing of $K(4,5)$.

Around 1958, the British artist Anthony Hill got interested in producing drawings of the complete graph K_n with the least possible number of edges crossing. After several uncertain years of talking and working with friends and current graph theory experts Hill and Harary produced a joint paper, summarizing the progress on the (rectilinear) crossing number problem that had been made up to that point. He exposed there a technique that consisted of drawing K_n on a cylinder, a cycle with $n/2$ vertices on the rim of the top lid, and a cycle with the remaining $n/2$ vertices on the rim of the bottom lid. Then he draw the remaining edges joining vertices on the same lid using straight-line segments across the lid. Finally, for any two vertices on distinct lids, draw the edges joining them along the geodesic on the side of the cylinder. These kind of drawings of K_n are now called cylindrical drawings and they have exactly $Z(n) := \frac{1}{4} \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor \lfloor \frac{n-2}{2} \rfloor \lfloor \frac{n-3}{2} \rfloor$ crossings. We will see below a more concise definition.

At about the same time, Blažek and Koman got independently interested in drawings of K_n with as few crossings as possible. They developed a quite different constructions which also provide drawings of K_n with exactly $Z(n)$ crossings. They started by drawing a regular n-gon with the vertices, and then drawing all diagonals with positive slope as straight-line segments and all other edges outside the regular n-gon. As we will see below they are now known as 2-pages drawings. It should be noted that they also confirmed Harary-Hill conjecture as upper bound. More details in [11].

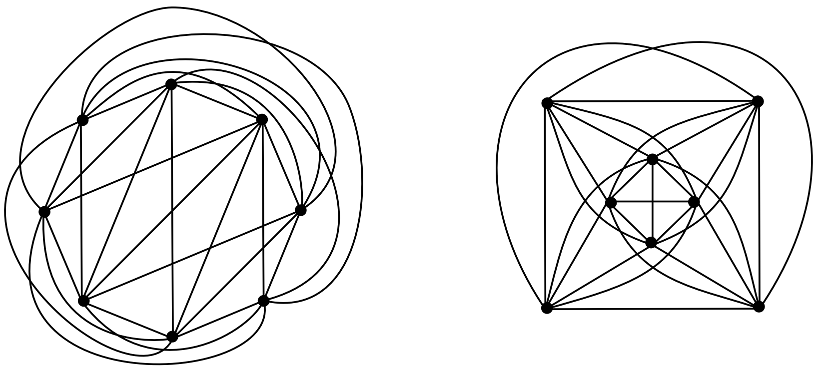


Figure 5.2: Left: A 2-pages drawing of K_8. Right: A cylindrical drawing of K_8.

In this chapter we will review the latest advances around the Harary-Hill conjecture presented in [9], we will also use results from [10]. We will introduce shellable drawings, a large class of drawings for which we will show that this conjecture holds. Shellability will also allow us to extend the lower bound to other kinds of drawings, including cylindrical, monotone, and x-bounded drawings. Let us start with some definitions.

If a drawing D of a graph is regarded as a subset of the plane, then region of D is a connected component of $\mathbb{R}^2 \setminus D$.

Definition 5.1. A drawing D of K_n is *s-shellable* if there exists a subset $S = \{v_1, v_2, ..., v_s\}$ of the vertices and a region R of D with the following property. For $1 \leq i \leq j \leq s$, if D_{ij} denotes the drawing obtained from D by removing $v_1, v_2, ..., v_{i-1}, v_{j+1}, v_{j+2}, ..., v_s$, then for all $1 \leq i \leq j \leq s$, the vertices v_i and v_j are on the boundary of the region of D_{ij} that contains R.

The set S is an *s-shelling* of D *witnessed* by R.

Definition 5.2. In a *2-page book drawing*, the vertices are placed on a line (the spine of the book), and each edge (except for its endvertices) lies entirely on a open halfplane spanned by the spine (one of the 2 pages of the book).

Definition 5.3. In a *cylindrical drawing*, there are two concentric circles that host all the vertices, and no edge is allowed to intersect these circles, other than at its endvertices.

Definition 5.4. A drawing is *monotone* if each vertical line intersecrs eac edge at most once.

Definition 5.5. A drawing is *x-bounded* if by labelling the vertices $v_1, v_2, ..., v_n$ in increasing order of their x-coordinates, for all $1 \leq i \leq j \leq n$ the edge v_iv_j is contained in the strip bounded by the vertical line that contains v_i and the vertical line that contains v_j.

Now we will generalize the geometrical concept of a k-edge seen in chapter 4 to good drawings of K_n, as follows. Let D be a good drawing of K_n, uv a directed edge of D, and w a vertex of D distinct from u and v. Then uvw denotes the oriented closed curve defined by the join of the edges uv, vw and wu. An oriented, simple, and closed curve in the plane is oriented counterclockwise (respectively, clockwise) if the bounded region it encloses is on the left (respectively, right) hand side of the curve. Further, w is on the left (respectively, right) side of uw if uvw is oriented counterclockwise (respectively, clockwise). We

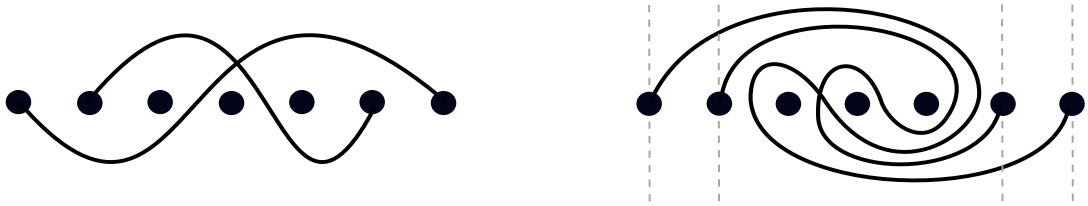


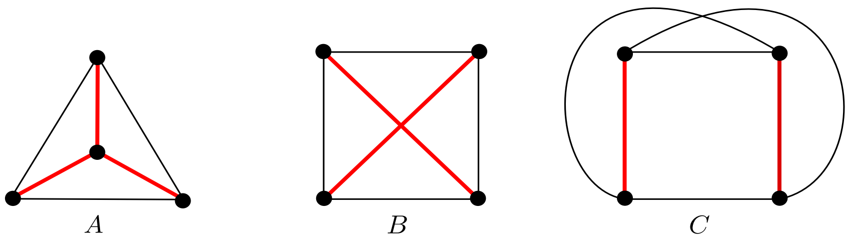
Figure 5.3: Left: monotone drawing. Right: x-bounded drawing.

say that the edge uv is a k-edge of D if it has exactly k points of D on one side (left or right), and thus $n-2-k$ points on the other side. Hence, a k-edge is also an $(n-2-k)$-edge. The direction of the edge pq is no longer relevant and every edge of D is a k-edge for some unique k such that $0 \leq k \leq \lfloor n/2 \rfloor - 1$. Here we have an interesting expression of $cr(D)$ in terms of the number of k-edges denoted by $E_k(D)$.

Theorem 5.1. *For any good drawing D of K_n in the plane the following identity holds,*

$$Cr(D) = 3\binom{n}{4} - \sum_{k=0}^{\lfloor n/2 \rfloor - 1} k(n-2-k)E_k(D)$$

Proof. In a good drawing of K_n, we say that an edge uv *separates* the vertices w_1 and w_2 if the orientations od the triangles uvw_1 and uvw_2 are opposite. In this case, we say that the set $\{uv, w_1, w_2\}$ is a *separation*. It is straightforward to check that, up to ambient isotopy equivalence, there are only three different good drawings A, B, C of K_4, which we can see below.



We denote N_A, N_B and N_C the number of vertex-induced subdrawings of D of type A, B and C, respectively. Then

$$N_A + N_B + N_C = \binom{n}{4}$$

and since the subdrawings of types B or C are in one-to-one correspondence with the crossing of D, it follows that

$$Cr(D) = N_B + N_C$$

We count the number of separations in D in two different ways. First, each subdrawing of type A has 3 separations, and each subdrawing of type B and C has 2 separations. This gives a total of $3N_A + 2N_B + 2N_C$ separations in D. Second, each k-edge belongs to exactly $k(n-2-k)$ separations. Summing over all k-edges for $0 \leq k \leq \lfloor n/2 \rfloor - 1$ gives a total of $\sum_{k=0}^{\lfloor n/2 \rfloor - 1} k(n-2-k)E_k(D)$ separations in D. Thus

$$3N_A + 2N_B + 2N_C = \sum_{k=0}^{\lfloor n/2 \rfloor - 1} k(n-2-k)E_k(D)$$

Combining the three previous equations we easily obtain the claimed result. $\square$

For each $0 \leq k \leq \lfloor n/2 \rfloor - 1$ we define the set of $(\leq k)$-edges of D as all j-edges in D for $j = 0, \ldots, k$. The number of $(\leq k)$-edges of D is denoted by

$$E_{\leq k}(D) := \sum_{j=0}^{k} E_j(D)$$

Similarly, we denote the number of $(\leq\leq k)$-edges of D by

$$E_{\leq\leq k}(D) := \sum_{j=0}^{k} E_{\leq j}(D) = \sum_{j=0}^{k}\sum_{i=0}^{j} E_i(D) = \sum_{i=0}^{k}(k+1-i)E_i(D)$$

To avoid special cases we define $E_{\leq\leq -1}(D) = E_{\leq\leq -2}(D) = 0$.

The following result restates theorem 5.1 in terms of the number of $(\leq\leq k)$-edges.

Proposition 5.2. *Let D be a good drawing of K_n. Then*

$$Cr(D) = 2\sum_{k=0}^{\lfloor n/2\rfloor -2} E_{\leq\leq k}(D) - \frac{1}{2}\binom{n}{2}\left\lfloor \frac{n-2}{2}\right\rfloor - \frac{1}{2}(1+(-1)^n)E_{\leq\leq\lfloor n/2\rfloor -2}(D)$$

Proof. First of all, note that for $2 \leq k \leq \lfloor n/2\rfloor - 1$ we have that $E_{\leq\leq k}(D) - E_{\leq\leq k-1}(D) = E_{\leq k}(D)$ and $E_{\leq k}(D) - E_{\leq k-1}(D) = E_k(D)$. Thus

$$E_k(D) = E_{\leq\leq k}(D) - 2E_{\leq\leq k-1}(D) + E_{\leq\leq k-2}(D).$$

We rewrite the last term in theorem 5.1.

$$\begin{aligned}
&\sum_{k=0}^{\lfloor n/2\rfloor -1} k(n-2-k)E_k(D) \\
= &\sum_{k=0}^{\lfloor n/2\rfloor -1} k(n-2-k)\left[E_{\leq\leq k}(D) - 2E_{\leq\leq k-1}(D) + E_{\leq\leq k-2}(D)\right] \\
= &\sum_{k=0}^{\lfloor n/2\rfloor -3} \left(k(n-2-k) - 2(k+1)(n-3-k) + (k+2)\right)E_{\leq\leq k}(D) \\
&+ \left(\left\lfloor\frac{n}{2}\right\rfloor - 1\right)\left(n-1-\left\lfloor\frac{n}{2}\right\rfloor\right)E_{\leq\leq\lfloor n/2\rfloor -1}(D) \\
&+ \left(-2\left(\left\lfloor\frac{n}{2}\right\rfloor - 1\right)\left(n-1-\left\lfloor\frac{n}{2}\right\rfloor\right) + \left(\left\lfloor\frac{n}{2}\right\rfloor - 2\right)\left(n-\left\lfloor\frac{n}{2}\right\rfloor\right)\right)E_{\leq\leq\lfloor n/2\rfloor -2}(D) \\
= &-2\sum_{k=0}^{\lfloor n/2\rfloor -3} E_{\leq\leq k}(D) + \left(\left\lfloor\frac{n}{2}\right\rfloor - 1\right)\left(n-1-\left\lfloor\frac{n}{2}\right\rfloor\right)E_{\leq\leq\lfloor n/2\rfloor -1}(D) \\
&+ \left(-2\left(\left\lfloor\frac{n}{2}\right\rfloor - 1\right)\left(n-1-\left\lfloor\frac{n}{2}\right\rfloor\right) + \left(\left\lfloor\frac{n}{2}\right\rfloor - 2\right)\left(n-\left\lfloor\frac{n}{2}\right\rfloor\right)\right)E_{\leq\leq\lfloor n/2\rfloor -2}(D)
\end{aligned}$$

Since $E_{\leq\leq\lfloor n/2\rfloor -1}(D) = E_{\leq\leq\lfloor n/2\rfloor -2}(D) + E_{\leq\lfloor n/2\rfloor -1}(D) = E_{\leq\leq\lfloor n/2\rfloor -2}(D) + \binom{n}{2}$, using theorem 5.1 we have that

$$\begin{aligned}
Cr(D) &= 3\binom{n}{4} - \sum_{k=0}^{\lfloor n/2\rfloor -1} k(n-2-k)E_k(D) = 3\binom{n}{4} + 2\sum_{k=0}^{\lfloor n/2\rfloor -3} E_{\leq\leq k}(D) \\
&+ \left(n+1-2\left\lfloor\frac{n}{2}\right\rfloor\right)E_{\leq\leq\lfloor n/2\rfloor -2}(D) - \left(\left\lfloor\frac{n}{2}\right\rfloor - 2\right)\left(n-\left\lfloor\frac{n}{2}\right\rfloor\right)\binom{n}{2} \\
&= 2\sum_{k=0}^{\lfloor n/2\rfloor -2} E_{\leq\leq k}(D) - \frac{1}{2}\binom{n}{2}\left\lfloor\frac{n-2}{2}\right\rfloor + \begin{cases} 2E_{\leq\leq\lfloor n/2\rfloor -2}(D) & \text{if n is odd} \\ E_{\leq\leq\lfloor n/2\rfloor -2}(D) & \text{if n is even}\end{cases}
\end{aligned}$$

which is equivalent to the equation of the statement.

□

The last step is to give a lower bound on the number of $(\leq\leq k)$-edges of s-shellable drawings of K_n for a certain interval of k determined by s.

Proposition 5.3. *Let D be an s-shellable good drawing of K_n, in which the region R that witnesses the s-shellability of D is its unbounded region. Then $E_{\leq\leq k}(D) \geq 3\binom{k+3}{3}$ for all $0 \leq k \leq min(s-2, \lfloor (n-3)/2 \rfloor)$.*

We can find a proof in [9].

At this point we can prove that s-shellable drawings of K_n, with restrictions on s, satisfy the Harary-Hill conjecture.

Theorem 5.4. *Let D be an s-shellable good drawing of K_n for some $s \geq n/2$. Then D has at least $Z(n)$ crossings.*

Proof. Using a suitable inversion, if it is needed, we transform D into a drawing D', with the same number of crossings as D, such that the region that witnesses the s-shellability of D' is the unbounded region. Since $\min(s-2, \lfloor (n-3)/2 \rfloor) = \lfloor (n-3)/2 \rfloor$, it follows from proposition 5.3 that $E_{\leq\leq k}(D') \geq 3\binom{k+3}{3}$ for all $0 \leq k \leq \lfloor (n-3)/2 \rfloor$.

Since D' is a good drawing, then by proposition 5.2 we have

$$Cr(D') = 2\sum_{k=0}^{\lfloor n/2 \rfloor -2} E_{\leq\leq k}(D') - \frac{1}{2}\binom{n}{2}\left\lfloor \frac{n-2}{2} \right\rfloor - \frac{1}{2}(1+(-1)^n)E_{\leq\leq \lfloor n/2 \rfloor -2}(D')$$

By combining both results we obtain

$$\begin{aligned} Cr(D') &\geq 2\sum_{k=0}^{\lfloor n/2 \rfloor -2} 3\binom{k+3}{3} - \frac{1}{2}\binom{n}{2}\left\lfloor \frac{n-2}{2} \right\rfloor - \frac{3}{2}(1+(-1)^n)\binom{\lfloor n/2 \rfloor +1}{3} \\ &= 6\binom{\lfloor n/2 \rfloor +2}{4} - \frac{1}{2}\binom{n}{2}\left\lfloor \frac{n-2}{2} \right\rfloor - \frac{3}{2}(1+(-1)^n)\binom{\lfloor n/2 \rfloor +1}{3} \\ &= \begin{cases} \frac{1}{64}(n-1)^2(n-3)^2 & \text{if n is odd} \\ \frac{1}{64}(n-2)^2(n-4)^2 & \text{if n is even} \end{cases} = Z(n) \end{aligned}$$

□

Using the following two results we will extend finally theorem 5.4 to the rest of the drawings defined at the begining of the chapter.

Lemma 5.5. *Let D be a drawing of K_n. Suppose that $C = v_1v_2...v_s$ is a cycle that satisfies the following: (i) the edge v_sv_1 has no crossings; and (ii) for $k = 1,...,s-1$ all crossings in the edge v_kv_{k+1} involve edges v_iv_j with $i < k$ and $j > k+1$. Then D is s-shellable.*

Proof. Let R be a region of D containing the edge v_sv_1 on its boundary. Let $1 \leq i < j \leq s$ and define D_{ij} as before. Let R' be the region of D_{ij} that contains R. Since the vertices $v_1, v_2, ..., v_{i-1}, v_{j+1}, v_{j+2}, ..., v_s$, and consequently any edge incident to one of these vertices, are removed to obtain D_{ij}, then v_1 and v_s are in the interior of R'. Moreover, it follows from the crossing properties of the edges of C that the edges v_1v_2, v_2v_3,...,$v_{i-1}v_i$, v_jv_{j+1}, $v_{j+1}v_{j+2}$,..., $v_{s-1}v_s$ do not intersected with any edge of D_{ij}. Therefore, the paths $v_1...v_{i-1}, v_i$ and $C = v_jv_{j+1}...v_s$ are completely contained in R' and hence v_i and v_j are on the boundary of R'. Thus, $\{v_1, v_2, ..., v_s\}$ is an s-shelling of D witnessed by R. □

Corollary 5.6. *If a drawing D of K_n has a crossing free cycle C of size s then D is s-shellable.*

Theorem 5.7. *Every cylindrical, 2-page book, x-bounded or monotone drawing of K_n has at least $Z(n)$ crossings.*

Proof. Let D be a cylindrical good drawing of K_n. Out of the two concentric cycles that contain all the vertices, let ρ be one that contains at least $n/2$ vertices. Let $v_1, v_2, \dots, v_s$ be the vertices on ρ, in counterclockwise order. Since no two edges cross each other more than once in a good drawing and there are not edges crossing ρ, it follows that $v_1 v_2 \dots v_s$ is a crossing free cycle of D. Since $s \geq n/2$, the result follows by theorem 5.4 and corollary 5.6 for cylindrical drawings.

A 2-page drawing is a particular (degenerate) kind of cylindrical drawing with all vertices on one of the concentric circles. Thus the previous paragraph immediately implies the result for 2-page book drawings.

It is a straightforward to check that and x-bounded drawing D of K_n satisfies the conditions of lemma 5.5. Thus the Harary-Hill conjecture holds for x-bounded drawings.

Since every monotone drawing is obviously x-bounded, this implies that Harary-Hill conjecture also holds for monotone drawings. $\square$

Bibliography

[1] M.BEHZAD AND G.CHARTRAND, *Introduction to the Theory of Graphs*, Allyn and Bacon, 1971.

[2] WIKIPEDIA, *Crossing number*, `https://en.wikipedia.org/w/index.php?title=Crossing_number_(graph_theory)&oldid=846520058`, June 19, 2018.

[3] WIKIPEDIA, *Crossing number inequality*, `https://en.wikipedia.org/w/index.php?title=Crossing_number_inequality&oldid=795538447`, August 17, 2017.

[4] B. ÁBREGO, O. AICHHOLZER, S. FERNÁNDEZ-MERCHANT, T. HACKL, J. PAMMER, A. PILZ, P. RAMOS, G. SALAZAR AND B. VOGTENHUBER, *All good drawings of small complete graphs*, In Proc. 31^{st} European Workshop on Computational Geometry EuroCG '15, pages 57-60, Ljubljana, Slovenia, 2015.

[5] BIRGIT VOGTENHUBER, *Topological Graphs: Good Drawings and Rotation Systems*, Discrete and Computational Geometry Workshop 2015-16, Institute of Software Technology, Graz University of Technology, Austria, January, 2016. `http://www.ist.tugraz.at/_attach/Publish/Dcg/GoodDrawingsRotationSystems1.pdf`.

[6] F.EBENFÜHRER, *Realizability of Rotation Systems*, Institute of Software Technology, Graz University of Technology, Graz, Austria, `http://diglib.tugraz.at/download.php?id=5988e7d46d6f2&location=browse`.

[7] J.PAMMER, *Rotation Systems and Good Drawings*, Institute of Software Technology, Graz University of Technology, Graz, Austria, `https://diglib.tugraz.at/download.php?id=576a7385e723c&location=browse`.

[8] L.LOVÁSZ, K.VESZTERGOMBI, U.WAGNER AND E.WELZL, *Convex Quadrilaterals and k-Sets*, Contemporary Mathematics Series, 342, AMS 2004, pages 139-148.

[9] B. ÁBREGO, O. AICHHOLZER, S. FERNÁNDEZ-MERCHANT, P. RAMOS AND G. SALAZAR, *Shellable drawings and the cylindrical crossing number of K_n*, Discrete and Computational Geometry, 52(4):743-753, 2014.

[10] B. ÁBREGO, O. AICHHOLZER, S. FERNÁNDEZ-MERCHANT, P. RAMOS AND G. SALAZAR, *The 2-Pages Crossing Number of K_n*, Discrete and Computational Geometry, 49(4):747-777, 2013.

[11] L. BEINEKE AND R. WILSON, *The early history of the brick factory problem*, The Mathematical Intelligencer, 32(2):41-48, 2010.