# An Introduction to Scheme Theory

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# Abstract

Desde su introducción en *Éléments de géométrie algébrique* por parte de Alexander Grothendieck la noción de esquema ha pasado a ser uno de los conceptos sobre los que descansa el desarrollo de la geometría algebraica moderna. El formalismo desarrollado en dicho trabajo, a pesar de su fama de inaccesible, fue clave en la resolución de importantes problemas del campo de la geometría algebraica como las Conjeturas de Weil y ha permitido relacionar este área con otras a primera vista distantes como puede ser la teoría de números, contribuyendo a la resolución del famoso Último Teorema de Fermat.

El siguiente trabajo, de carácter fundamentalmente expositivo, estudiará cómo nociones nociones relacionadas con los esquemas pueden trasladarse al campo de la geometría diferenciable para estudiarla decide un punto de vista algebráico y, tras esto, utilizará este hecho como motivación para generalizar las construcciones realizadas progresivamente hasta llegar a los esquemas.

En el primer capítulo se estudiará como dada una variedad diferenciable $(M, \mathcal{A})$, su anillo de funciones diferenciables $\mathcal{C}^\infty_M$ contiene toda su información topológica y diferencial.

- Se introducirá el concepto de haz de anillos sobre un espacio topológico y la consiguiente noción de espacio anillado y se darán definiciones equivalentes a las usuales de variedad diferenciable y aplicación diferenciable en dichos términos. Bajo esta perspectiva una variedad diferenciable pasará a ser un espacio localmente anillado que es localmente isomorfo a algún espacio euclideo $\mathbb{R}^n$ equipado con sus anillos de funciones diferenciables, y una aplicación diferenciable será vista como un morfismo de espacios localmente anillados entre variedades diferenciables.

- Veremos que $M$ es naturalmente biyectiva con el $\mathbb{R}$-espectro de $\mathcal{C}^\infty_M$, el conjunto de ideales maximales de $\mathcal{C}^\infty_M$ con cuerpo residuo $\mathbb{R}$. Además, la topología de $M$ equivale, a través de la biyección previamente establecida, con la topología de Zariski sobre el $\mathbb{R}$-espectro de $\mathcal{C}^\infty_M$.

- Finalmente se demostrará que la construcción de una variedad diferenciable en términos de espacios anillados puede realizarse tomando como punto de partida su $\mathbb{R}$-álgebra de funciones diferenciables y que, además, las aplicaciones diferenciables se corresponden con los homomorfismos entre dichas $\mathbb{R}$-álgebras.

El segundo capítulo tratará como se pueden generalizar las construcciones introducidas en el primero partiendo de otros tipos de anillos. Primero se considerarán las álgebras regulares sobre cuerpos algebráicamente cerrados, dando como resultado las variedades algebráicas clásicas vistas como espacios localmente anillados. Finalmente se verá brevemente que la teoría desarrollada para ese caso puede ampliarse, sin mucho trabajo, para englobar objetos derivados de anillos regulares arbitrarios, resultando en los esquemas.

# Contents

# Introduction

This bachelor's thesis for the mathematics degree from the University of Zaragoza, fundamentally of expository nature, is aimed to introduce scheme theory to a math student who is familiar with smooth manifolds and classic algebraic geometry at an undergraduate level. To do this the first part of the writing will focus on the relationship between smooth geometry and algebra, justifying tools that later will be used in other settings eventually leading to the definition and first properties of schemes. The second part will take the already introduced constructions and use them in the field of algebraic geometry, where they were developed. We will give an overview of the classic notion of algebraic variety using those ideas and then, from that point we will keep on generalizing previous notions going towards the construction of schemes.

Even though this work does not require strong categorical background to be read, concepts like category, functor, natural transformation, projective and inverse limits will be invoked every once in a while. Sheaf theory, which is closely related to the categorical framework, will be introduced here and can be considered the fundamental technical tool in this writing.

The learning process for this work has taken place spreadly during the academic year 2017-2018 and it was written mainly over the summer of 2018. The source material used here can be mainly found in the books [1], [2] and [3]. In there the fundamental ideas expressed here are developed profoundly across many more pages, so they are a good place to go after or in the middle of reading this work in the way of learning more about algebraic geometry and schemes.

For reaching the end of this bachelor's degree I owe to my parents everything that is to be owed. I also want to thank all the teachers that, in spite of my untidy and unorthodox ways, have at some point gone out of their way to help me, recognize my work or give me valuable advice.

I hope reading this will be worthwhile.

# Chapter 1

# Smooth manifolds: an algebraic approach

In this chapter we are going to show that it is possible to view smooth geometry as a part of algebra, proving that every manifold is characterized by its $\mathbb{R}$-algebra of smooth functions.

More precisely, we are going to show that the category $\mathbf{Man}^\infty$ of smooth manifolds is equivalent to the dual of some full subcategory of the category $\mathbb{R}$-**Alg** of $\mathbb{R}$-algebras. For doing so we will introduce various constructions that will be later generalized towards the definition of scheme. The here exposed ideas about the relation of smooth manifolds and their rings of functions can also be found in [3].

## 1.1 Classic definitions

The motivation behind the concept of **smooth manifold** is that of extending on the idea of a set in an Euclidean space $\mathbb{R}^n$ that is locally diffeomorphic to some fixed $\mathbb{R}^m$ with $m < n$. The simplest examples of this are smooth plane curves, smooth spacial curves and smooth surfaces in $\mathbb{R}^3$ (all of them with no auto intersections). We want to free ourselves from the need of an ambient space but keep having a set that is locally like an Euclidean space in a way that behaves well with smooth functions.

**Definition 1.1.1.** Given a topological space $M$, **a chart** $(\mathbf{U}, \mathbf{x})$ **on M** is a continuous map $x : U \to \mathbb{R}^n$ from an open set $U \subseteq M$ which is an homeomorphism onto its image. The integer $n$ is called **the dimension** of the chart.

Charts allow us to define local coordinates on $M$: for any $p \in U$ we say that the $n$-tuple $x(p)$ are the coordinates of $p$ by means of $x$.

**Definition 1.1.2.** Two charts $(U, x)$ and $(V, y)$ on $M$ are said to be **compatible** if they have the same dimension and $x \circ y^{-1}$ is a diffeomorphism from $y(U \cap V)$ onto $x(U \cap V)$.

Compatible charts have the same dimension (as we do not want $M$ to look like a line in some places and like a plane in others) and give rise to $\mathcal{C}^\infty$ coordinate change maps.

**Definition 1.1.3.** A family $\mathcal{A}$ of compatible charts $(U_\alpha, x_\alpha)$, where $\alpha$ ranges over some index set $J$, on $M$ is called an **atlas on M** if $M = \bigcup_{\alpha \in J} U_\alpha$. The **dimension** of $\mathcal{A}$ is the dimension of any of its charts. Two atlases on $M$ are called **compatible** if any chart from one of them is compatible with any chart from the another.

**Fact 1.1.1.** *The relation of compatibility between atlases is an equivalence relation. Each one of the equivalence classes contains exactly one maximal atlas, namely the union of all the atlases forming that equivalence class.*

**Definition 1.1.4.** A **smooth manifold** $(M, \mathcal{A})$, is a set $M$ equipped with a maximal atlas $\mathcal{A}_{max} = \bigcup_{\alpha \in J} (U_\alpha, x_\alpha)$ and the topology induced by it. The sets $U_\alpha$ with $\alpha \in J$ are called **coordinate domains** of $M$, and the **dimension** of the manifold is the dimension of $\mathcal{A}_{max}$

For the purpose of this work we will only study manifolds that satisfy the Hausdorff property and the second axiom of countability, i.e. the ones in where we can separate points by disjoint open sets and whose topology has a countable base.

Any open subset $U$ of an Euclidean space $\mathbb{R}^n$ can be equipped with a natural atlas, consisting of only one chart: the identity map on $U$. We will usually refer to the smooth manifold given by $U$ and this atlas as "$U$ with its usual structure".

Given an smooth manifold $(M, \mathcal{A})$, each open subset $U \subseteq M$ inherits a smooth manifold structure from $M$ which will be understood equipped to $U$ unless stated otherwise. This structure is the one given by the atlas $\mathcal{A}|_M = \{ \ (U \cap V, x|_{U \cap V}) \mid (V, x) \in \mathcal{A} \ \}$.

*Remark.* We will often abbreviate $(M, \mathcal{A})$ to $M$ when referring to a smooth manifold if the atlas $\mathcal{A}$ is understood or not relevant. It should be clear from the context when $M$ denotes the underlying topological space of the manifold and when it denotes the smooth manifold itself.

**Definition 1.1.5.** A **smooth map** $f$ between two smooth manifolds $(M, \mathcal{A})$ and $(N, \mathcal{B})$ is a map $f : M \to N$ such that for any $p \in M$, there are charts $(U, x) \in \mathcal{A}$ and $(V, y) \in \mathcal{B}$ such that $p \in U$, $f(p) \in V$ and satisfying that $y \circ f \circ x^{-1} : x(U) \to y(V)$ is a $\mathcal{C}^\infty$ function. A smooth map which is bijective and whose inverse is also a smooth map is called **diffeomorphism**.

Given a manifold $(M, \mathcal{A})$, any chart $(U, x) \in \mathcal{A}$ is a diffeomorphism into its image if we consider $x(U)$ with its usual structure. Thus $M$ is locally diffeomorphic to open subsets of an Euclidean space.

The most important kind of smooth maps for the purposes of this work will be the smooth maps from open subsets $U$ of a given smooth manifold $M$ to $\mathbb{R}$. We will call these maps **smooth functions over U** respectively for each $U$ and we will denote them by $\mathcal{C}^\infty(U)$.

To finish this section we will state without proof two results about smooth functions that will be useful for us later.

**Lemma 1.1.1.** *For any pair $F, G$ of disjoint closed sets from a smooth manifold $M$ there exists a smooth function $f \in \mathcal{C}^\infty(M)$ that satisfies $f(x) = 1$ in $F$, $f(x) = 0$ in $G$ and $f(M \setminus (F \cup G)) = (0, 1)$. In particular, for any non empty closed set $F$ from $\mathbb{R}^n$ there exists a function in $\mathcal{C}^\infty(M)$ that vanishes in $F$ and is positive and bounded by $1$ outside of $F$.*

**Lemma 1.1.2.** *Let $M$ be a smooth manifold, $U \subseteq M$ an open set and $f \in \mathcal{C}^\infty(M)$ a smooth function. Then for any closed set $V \subset U$ there exists a smooth function $g \in \mathcal{C}^\infty(M)$ such that $g|_V = f|_V$ and $g|_{M \setminus U} = 0$.*

**Lemma 1.1.3.** *For any smooth manifold $M$ there exists a function $f \in \mathcal{C}^\infty(M)$ such that $f^{-1}(\lambda)$ is compact for any $\lambda \in \mathbb{R}$.*

## 1.2 Smooth manifolds as ringed spaces

Given an open set $U$ of a smooth manifold $M$ let us consider the set $\mathcal{C}^\infty(U)$ **smooth functions over U**. It is clear that $\mathcal{C}^\infty(U)$ equipped with the point-wise sum and product is a commutative ring. The identity elements for the sum and product are the constant functions that satisfy $f(p) = 1$ and $g(p) = 0$ for any $p \in M$ respectively. Also, if $V \subseteq U$ are open sets from $M$ there exists a natural ring homomorphism from $\mathcal{C}^\infty(U)$ to $\mathcal{C}^\infty(V)$ which assigns to every function in $\mathcal{C}^\infty(U)$ its restriction to $V$. In this section we will elaborate on this relationship between the rings of smooth functions and the open sets from $M$. In regards to sheaf theory we will follow mainly pages 60-69 from [1] and pages 11-18 from [2].

**Definition 1.2.1.** Let $X$ be a topological space, $(\mathcal{O}(U))$ be a family of rings indexed by the open sets $U \subseteq X$ and $(res_{U,V})$ be a family of ring homomorphisms $res_{U,V} : \mathcal{O}(U) \to \mathcal{O}(V)$ indexed by the ordered pairs $U, V$ of open sets from $X$ such that $V \subseteq U$. Then we will say that the pair $\mathcal{O} = ((\mathcal{O}(U)), (res_{U,V}))$ is a **presheaf of rings over X** if it satisfies (1) $\mathcal{O}(\emptyset) = 0$, (2) $res_{U,U}$ is the identity map on $\mathcal{O}(U)$ and (3) $res_{U,W} = res_{V,W} \circ res_{U,V}$ for any open sets satisfying $W \subseteq V \subseteq U$.

The maps $res_{U,V}$ are called **restrictions** and the elements from $\mathcal{O}(U)$ are called the **sections** of $\mathcal{O}$ over $U$, while each $U$ will be referred to as the **domain** of the sections from $\mathcal{O}(U)$. Given $f \in \mathcal{O}(U)$ we will use the following notation: $f|_V = res_{U,V}(f)$.

Equivalently one can define a presheaf on $X$ to be a cotravariant functor from the category of open sets from $X$ (with a morphism $U \to V$ for each containment $V \subseteq U$) to the category of rings. Changing the target category to abelian groups we have the definition of a presheaf of abelian groups, and the same goes for vector spaces, algebras and so on.

*Remark.* We will follow the convention that rings have multiplicative unit and ring homomorphisms map multiplicative units to multiplicative units.

**Definition 1.2.2.** Given a presheaf of rings $\mathcal{O}$ over $X$ and an open set $U \subset X$ we will denote by $\mathcal{O}|_U$ to the presheaf of rings defined by $\mathcal{O}|_U(V) = \mathcal{O}(V)$ for each open set $V \subseteq U$ and which inherits its restriction morphisms from $\mathcal{O}$. This is called the **restriction of $\mathcal{O}$ to $U$**.

Also, presheaves can give rise to other presheaves by means of continuous maps:

**Definition 1.2.3.** Let $\mathcal{O}$ be a presheaf of rings over $X$ and $f : X \to Y$ be a continuous map. Then the rings $f_*\mathcal{O}(U) = \mathcal{O}(f^{-1}(U))$ for each open set $U \subseteq Y$, equipped with the restrictions from $\mathcal{O}$ $res_{f^{-1}(U),f^{-1}(V)}$ for each pair of open sets $V \subseteq U$ from $Y$ form a presheaf of rings over $Y$. This presheaf is called the **pushforward of the presheaf $\mathcal{O}$ along $f$** and will be denoted by $f_*\mathcal{O}$.

The considerations about smooth functions at the beginning of this section essentially define the **presheaf $\mathcal{C}^\infty_M$ of smooth functions over M**, given a smooth manifold $M$. We can also define another presheaf $\widehat{\mathcal{C}^\infty_M}$ of smooth funcions over $M$ contained in $\mathcal{C}^\infty_M$: the one given only by the global smooth functions over $M$ and their restrictions.

Analogously, we can define the presheaf of continuous functions (into $\mathbb{R}$) over a topological manifold and the presheaf of holomorphic functions (into $\mathbb{C}$) over a complex manifold. Presheaves of rings allow us to generalize the relationship between topological spaces and their associated rings of "nice" functions (in our examples "niceness" was smoothness, continuity and holomorphy respectively). In fact, as we will see, the generalization they provide is too much for our pruposes but they make for a good starting point.

From the definition of presheaves as contravariant functors arises naturally the definition of presheaf morphisms as the natural transformations between those functors. In a less categorical language:

**Definition 1.2.4.** Let $\mathcal{O}$ and $\mathcal{F}$ be presheafs of rings over the topological space $X$. A **presheaf morphism** $\varphi$ from $\mathcal{O}$ to $\mathcal{F}$ (we will use the notation $\varphi : \mathcal{O} \to \mathcal{F}$) is a family of ring homomorphisms $\varphi_U : \mathcal{O}(U) \to \mathcal{F}(U)$ indexed by the open sets $U \subseteq X$ (we will use the notation $\varphi = (\varphi_U)$) satisfying that given two open sets $V \subseteq U$ from $X$ the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{O}(U) & \xrightarrow{\varphi_U} & \mathcal{F}(U) \\ \big\downarrow{\scriptstyle res^{\mathcal{O}}_{U,V}} & & \big\downarrow{\scriptstyle res^{\mathcal{F}}_{U,V}} \\ \mathcal{O}(V) & \xrightarrow{\varphi_V} & \mathcal{F}(V) \end{array}$$

Each $\varphi_U$ is called a **component of $\varphi$**. The composition of two suitable presheaf morphisms is given by the composition of their components. This operation is associative.

We can also define the restriction and the pushforward of a presheaf morphism in a natural way:

**Definition 1.2.5.** Let $U \subseteq X$ be an open set and $\varphi : \mathcal{O} \to \mathcal{F}$ be a presheaf morphism between presheaves of rings over $X$. Then the restriction of $\varphi$ to $U$ is the presheaf morphism $\varphi|_U : \mathcal{O}|_U \to \mathcal{F}|_U$ given by the ring homomorphisms $(\varpi|_U)_V = \varphi_V$ for each open set $V \subseteq U$.

**Definition 1.2.6.** Let $\mathcal{O}$ and $\mathcal{F}$ be presheaves of rings over $X$, $\varphi : \mathcal{O} \to \mathcal{F}$ a presheaf morphism and $f : X \to Y$ a continuous map. Then the **pushforward of** $\varphi$ **along** $f$ is the presheaf morphism $f_*\varphi : f_*\mathcal{O} \to f_*\mathcal{F}$ given by the ring homomorphisms $(f_*\varphi)_U = \varphi_{f^{-1}(U)}$.

The pusforward along a given continuous map $f : X \to Y$ defines a functor between the categories of presheaves over the correspondent topological spaces, $f_* : \mathbf{PSh}(X) \to \mathbf{PSh}(Y)$. Given two suitable continuous maps $f$, $g$ we have that $(f \circ g)_* = f_* \circ g_*$. Also, it is trivially satisfied that if $f = Id_x$, then $f_*$ is the identity functor on $\mathbf{PSh}(X)$.

Now we can naturally give a mapping that associates to each smooth manifold $M$ the presheaf $\mathcal{C}^\infty_M$. The next thing to do is trying to characterize "image" of that mapping up to isomorphism. More specifically, we want to characterize the pairs $(X, \mathcal{O})$ of topological spaces $X$ and presheaves $\mathcal{O}$ over $X$ such that there exists an atlas $\mathcal{A}$ over $X$ satisfying that $\mathcal{O}$ is isomorphic to the presheaf of smooth functions over $(X, \mathcal{A})$.

To do this we should restrict ourselves to consider only presheaves of rings that are "like" presheaves of nice functions. There are two properties that such presheaves must satisfy. The first one is that, in the same way that a function which vanishes over an open covering of its domain trivially vanishes everywhere, a section whose restrictions over an open covering of its domain are zero should be zero itself:

(1) Given an open set $U$, and an open cover $\{U_i\}$ of $U$, if and element $f \in \mathcal{O}(U)$ satisfies $f|_{U_i} = 0$ for each $i$ then $f = 0$

The second one, less evident, comes from the fact that in practice we want "niceness" to be a local property (like continuity, smoothness, holomorphy), so the gluing of nice functions should give rise to nice functions. This, taken to the terrain of presheaves translates to:

(2) Given an open set $U$, an open covering $\{U_i\}$ of $U$ and elements $f_i \in \mathcal{O}(U_i)$ for each $i$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for each $i, j$ then there exists an element $f \in \mathcal{O}(U)$ that satisfies $f|_{U_i} = f_i$. (Note that (1) implies that such $f$ is unique.)

**Definition 1.2.7.** A presheaf of rings $\mathcal{O}$ over $X$ is called a **sheaf of rings** if it satisfies the above-mentioned properties (1) and (2). A presheaf morphism between sheaves is called a **sheaf morphism**.

**Example 1.2.1.** Given any smooth manifold $M$ the presheaf $\mathcal{C}^\infty_M$ is a sheaf. Sheaf property (1) holds trivially because of $\mathcal{C}^\infty_M$ being a presheaf of functions equipped with their usual restrictions: the only function on an open set $U \subseteq M$ which is zero locally around each point of $U$ is the zero function on $U$, while sheaf property (2) follows from the fact that any function on an open set $U \subseteq M$ that is locally smooth around each point of $U$ is a smooth function over $U$.

On the other hand, the presheaf $\widehat{\mathcal{C}}^\infty_M$ is never a sheaf. While sheaf property (1) holds for the same reason as in the case of $\mathcal{C}^\infty_M$, sheaf property (2) is never satisfied: let $U \subseteq M$ be a coordinate domain from $M$ and $p \in U$ whose associated chart is $(U, x)$. Then we can define a smooth function $f : U \setminus \{p\} \to \mathbb{R}$ given by $f(q) = \frac{1}{||x(p)-x(q)||}$. It is easy to see that $f$ cannot be extended smoothly (not even continuously) to $M$ as it has a non-removable discontinuity at $p$. This is equivalent to say that $f$ does not belong to $\widehat{\mathcal{C}}^\infty(U \setminus \{p\})$. But for each $q \in U \setminus \{p\}$ there is an open set $V_q \subset U$ such that $\overline{V}_q \subset U$ and $q \in V_q$ and using lemma 1.1.2 we can extend $f|_{V_q}$ to the whole $M$ smoothly. This shows that the local restrictions $f|_{V_q}$ belong to $\widehat{\mathcal{C}}^\infty(V_q)$ for each $q \in U \setminus \{p\}$ respectively, but there is no $g \in \widehat{\mathcal{C}}^\infty(U \setminus \{p\})$ such that $g|_{V_q} = f|_{V_q}$ for every $q \in U \setminus \{p\}$ as we would have that $g = f$.

**Definition 1.2.8.** A **ringed space** is a pair $(X, \mathcal{O})$ where $X$ is a topological space and $\mathcal{O}$ is a sheaf of rings over $X$.

Note that each open set $U \subseteq X$ can be given a natural ringed space structure by means of the restriction of the sheaf $\mathcal{O}$ to $U$: $(U, \mathcal{O}|_U)$. We will call them **open ringed subspaces of** $(X, \mathcal{O})$.

As with the case of smooth manifolds, we will sometimes denote a ringed manifold by its underlying topological space when its associated sheaf is understood or does not matter.

After this definition we can rephrase our previous mapping in the following form:

$$\text{Smooth manifolds } (M, \mathcal{A}) \mapsto \text{Ringed spaces } (M, \mathcal{C}^\infty_M)$$

Now, let $f : M \to N$ be a smooth map between them. Then $f$ induces ring homomorphisms $f^\#_U : \mathcal{C}^\infty(U) \to \mathcal{C}^\infty(f^{-1}(U))$ for each open set $U \subseteq N$ given by $f^\#_U(s) = s \circ f|_{f^{-1}(U)}$. It is trivially satisfied that for any open set $V \subseteq U$ $f^\#_U(s)|_{f^{-1}(V)} = f^\#_V(s|_V)$, so the family $f^\# = (f^\#_U)$ form a sheaf morphism from $\mathcal{C}^\infty_N$ to $f_*(\mathcal{C}^\infty_M)$. This consideration gives us a hint about what morphisms of ringed spaces should be:

**Definition 1.2.9.** Let $(X, \mathcal{O})$ and $(Y, \mathcal{F})$ be ringed spaces. A **morphism of ringed spaces** $(f, \varphi) : (X, \mathcal{O}) \to (Y, \mathcal{F})$ is a pair where $f : X \to Y$ is a continuous map and $\varphi : \mathcal{F} \to f_*\mathcal{O}$ is a sheaf morphism.

The composition of two suitable ringed space morphisms $(f, \varphi)$ and $(g, \psi)$ is defined by $(g, \psi) \circ (f, \varphi) = (g \circ f, g_*\varphi \circ \psi)$.

Trivially, if $f : M \to N$ is a smooth map, the pair $(f, f^\#) : (M, \mathcal{C}^\infty_M) \to (N, \mathcal{C}^\infty_M)$ is a ringed space morphism. This allows us to establish the following functor $\mathfrak{F}$ from the category $\mathbf{Man}^\infty$ of smooth manifolds to the one of ringed spaces:

$$\begin{aligned} &\text{Smooth manifolds } (M, \mathcal{A}) &&\mapsto \text{Ringed spaces } (M, \mathcal{C}^\infty_M) \\ &\text{Smooth maps } f : (M, \mathcal{A}) \to (N, \mathcal{B}) &&\mapsto \text{Ringed space morphisms } (f, f^\#) : (M, \mathcal{C}^\infty_M) \to (N, \mathcal{C}^\infty_N) \end{aligned}$$

Our next task is to give a full characterization of the essential image (i.e. the image up to isomorphism) of this functor.

**Definition 1.2.10.** Let $(X, \mathcal{O})$ be a ringed space. Then it will be called a **smooth ringed space** if there exists a smooth manifold $M$ such that $(X, \mathcal{O})$ is isomorphic to $(M, \mathcal{C}^\infty_M)$.

**Definition 1.2.11.** Let $(f, \varphi) : (X, \mathcal{O}) \to (Y, \mathcal{F})$ be a morphism of ringed spaces. Then, given two open sets $U \subseteq X$ and $V \subseteq Y$ such that $f(U) \subseteq V$ the restriction of $(f, \varphi)$ to $U$ and $V$ is the morphism of ringed spaces $(g, \psi) : (U, \mathcal{O}|_U) \to (V, \mathcal{F}|_V)$ given by the continuous map $g : U \to V$ defined by $g(p) = f(p)$ and the sheaf morphism $\psi$ whose components $\psi_W$ for every open set $W \subseteq V$ are defined by $\psi_W(s) = \varphi_W(s)|_{f^{-1}(W) \cap U}$.

**Lemma 1.2.1.** *Let $(f, \varphi) : (X, \mathcal{O}) \to (Y, \mathcal{F})$ be a morphism of ringed spaces. Then $(f, \varphi)$ is an isomorphism if and only if $f$ is an homeomorphism and $\varphi$ a sheaf isomorphism.*

*Proof.* If $(f, \varphi)$ is an isomorphism of inverse $(g, \psi)$ it is clear that $f$ is a homeomorphism of inverse $g$. Also, $g_*\varphi \circ \psi = Id_{\mathcal{O}}$ and $f_*\psi \circ \varphi = Id_{\mathcal{F}}$. Pushing the first equality forward along $f$ we obtain $f_*(g_*\varphi \circ \psi) = f_*Id_{\mathcal{O}}$ which is the same as $(f \circ g)_*\varphi \circ f_*\psi = Id_{f_*\mathcal{O}}$ and operating we obtain $\varphi \circ f_*\psi = Id_{f_*\mathcal{O}}$. So $\varphi$ is a sheaf isomorphism. Conversely, if $(f, \varphi)$ satisfy that $f$ is a homeomorphism and $\varphi$ is an isomorphism then $(f^{-1}, f^{-1}_*\varphi^{-1})$ is its both sided inverse. □

**Corollary 1.2.1.** *Let $(f,\varphi):(X,\mathcal{O})\to(Y,\mathcal{F})$ be an isomorphism of ringed spaces and $U\subseteq X$ an open set. Then the restriction $(g,\psi)$ of $(f,\varphi)$ to $U$ and $f(U)$ (note that $f(U)$ is open for $f$ being a homeomorphism) is also an isomorphism.*

**Lemma 1.2.2.** *Glueing lemma. Let $\{X_i\}$, $i\in J$, be an arbitrary family (possibly infinite) of ringed spaces and for each pair of indexes $i\neq j$ suppose given an open set $U_{i,j}\subseteq X_i$ with its induced ringed space structure. Suppose also given for each $i\neq j$ a ringed space isomorphism $\Phi_{i,j}:=(f_{i,j},\varphi_{i,j}):U_{i,j}\to U_{j,i}$ satisfying that (1) for each $i\neq j$ $\Phi_{i,j}^{-1}=\Phi_{j,i}$, and (2) for each $i,j,k$ such that $i\neq j$, $j\neq k$ and $k\neq i$ it holds that $f_{i,j}(U_{i,j}\cap U_{i,k})=U_{j,i}\cap U_{j,k}$. Then there exists a ringed space $X$ for which there are ringed space morphisms $\Psi_i:=(g_i,\psi_i):X_i\to X$ for each $i$ satisfying (1) for each $i$ $g_i(X_i)$ is open and $\Psi_i$ is an isomorphism of $X_i$ onto $g_i(X_i)$ with its induced ringed space structure, (2) the $g_i(X_i)$ cover $X$, (3) for each $i\neq j$ $g_i(U_{i,j})=g_i(X_i)\cap\psi_j(X_j)$, and (4) for each $i\neq j$ $\Psi_i=\Psi_j\circ\Phi_{i,j}$ over $U_{i,j}$. Such $X$ is unique up to isomorphism and we will call it* ***the glueing of the $X_i$ along the isomorphisms $\Phi_{i,j}$****.*

*Sketch of the proof.* For a full proof one can check [4].

Let $X$ be the quotient topological space of the disjoint union $\bigcup_{i\in J}X_i$ (here $X_i$ denotes the underlying topological space of the respective ringed space) modulo the equivalence relationship $\{p\in X_i,\ q\in X_j$ with $i\neq j$ and $f_{i,j}(p)=q \iff p=q\}$. This identification gives rise to continuous maps $g_i:X_i\to X$ by means of restricting the natural quotient map $\Pi:\bigcup_{i\in J}X_i\to X$ to each $X_i$.

Let $\mathcal{O}_i$ be the underlying sheaves of the ringed spaces $X_i$. Now consider for each open set $U\subseteq X$ the ring $\mathcal{O}(U)$ consisting of (1) the families $s_i, i\in J$ satisfying that for each $i$ $s_i\in\mathcal{O}_i(g_i^{-1}(U))$ and for each $i\neq j$ it holds $s_i|_{U_{i,j}\cap g_i^{-1}(U)}=s_j|_{U_{j,i}\cap g_j^{-1}(U)}$, equipped with the operations (2) $\{s_i\}+\{t_i\}:=\{s_i+t_i\}$ and $\{s_i\}\cdot\{t_i\}:=\{s_i\cdot t_i\}$ for every $\{s_i\},\{t_i\}\in\mathcal{O}(U)$. Also consider for each pair of open sets from $X$, $U$ and $V$ such that $V\subseteq U$ the ring homomorphisms $res_{U,V}:\mathcal{O}(U)\to\mathcal{O}(V)$ defined by $res_{U,V}(\{s_i\})=\{s_i|_{g_i^{-1}(V)}\}$. The rings $\mathcal{O}(U)$ together with those ring homomorphism form then a sheaf $\mathcal{O}$ of rings over $X$. Also, for each $j\in J$ there is a natural sheaf morphism $\psi_j:\mathcal{O}\to g_{j*}\mathcal{O}_j$ that maps each section from $\mathcal{O}$ to its $j$-th component. Then the ringed space $X\equiv(X,\mathcal{O})$ and the ringed space morphisms $\Psi_i:=(g_i,\psi_i)$ satisfy all the given conditions.

Suppose that there was another glueing of the $X_i$ along the $\Phi_{i,j}$, $Y\equiv(Y,\mathcal{F})$ and let $\Gamma_i=(h_i,\gamma_i)$ be the morphisms of ringed spaces that satisfy the given conditions for $Y$. Then, we can define a homeomorphism $F:X\to Y$ by means of imposing $\tau|_{\psi_i(X_i)}=\gamma_i\circ\psi_i^{-1}$. We can also define a sheaf isomorphism $\tau:\mathcal{F}\to F_*\mathcal{O}$ by means of defining, for each open set $U\subseteq Y$ $\tau_U(s)=\{\gamma_{i,U}(s)\}$. Thus, $X$ is isomorphic to $Y$ by means of $(F,\tau)$. □

**Corollary 1.2.2.** *Let $(X,\mathcal{O})$ be a ringed space, $\{U_i\}, i\in J$ be an open covering and for each $i$ let $X_i$ be a ringed space isomorphic to $U_i$ through a isomorphism $\Psi_i=(g_i,\psi_i)$. Then $X$ is the glueing of the $X_i$ along the morphisms $\Phi_{i,j}:g_i^{-1}(U_i\cap U_j)\to g_j^{-1}(U_i\cap U_j)$ for each $i\neq j$ given by $\Phi_{i,j}=\widehat{\Psi_j}^{-1}\circ\widehat{\Psi_i}$, where $\widehat{\Psi_k}$ denote the restriction of $\Psi_k$ to $g_i^{-1}(U_i\cap U_j)$ and to $U_i\cap U_j$ for each $k\in\{i,j\}$.*

**Lemma 1.2.3.** *Let $M$, $N$ and $W$ be smooth manifolds and $f:M\to N$, $g:N\to W$ smooth maps between them. Then it is satisfied that $(g\circ f,(g\circ f)^{\#})=(g,g^{\#})\circ(f,f^{\#})$. In particular if $f$ is a diffeomorphism then $(f,f^{\#})$ is an isomorphism.*

*Proof.* It suffices to show that $(g_*f^{\#})\circ g^{\#}$ is the same sheaf morphism as $(g\circ f)^{\#}$. Let $U\subseteq W$ be a open set and $s\in\mathcal{C}^\infty_W(U)$. Then $((g_*f^{\#})\circ g^{\#})_U(s)=(g_*f^{\#})_U\circ g^{\#}_U(s)=(g_*f^{\#})_U(s\circ g)=f^{\#}_{g^{-1}(U)}(s\circ g)=s\circ g\circ f=(f\circ g)^{\#}(s)$. The last part of the lemma comes from the fact that

$Id_M^{\#} = Id_{\mathcal{C}_M^\infty}$, where $Id_M$ is the identity map on $M$ and $Id_{\mathcal{C}_M^\infty}$ is the identity sheaf morphism on $\mathcal{C}_M^\infty$. Then, if $f$ is a diffeomorphism $(f, f^{\#})$ and $(f^{-1}, (f^{-1})^{\#})$ are inverse morphisms. □

Up until this point we have shown that the following assignation that sends each smooth manifold $(X, \mathcal{A})$ to the ringed space $(X, \mathcal{C}_X^\infty)$ and each smooth map $f$ to the morphism of ringed spaces $(f, f^{\#})$ is indeed a functor between the category of smooth manifolds and the one of ringed spaces.

**Proposition 1.2.1.** *Let $M$ and $N$ be smooth manifolds and $f : M \to N$ a continuous map. Then $f$ is smooth if and only if $s \circ f$ belongs to $\mathcal{C}_N^\infty(N)$ for any $s \in \mathcal{C}_M^\infty(M)$*

*Proof.* Let us assume that $f$ is not a smooth map and let $m$ and $n$ be the dimensions of $M$ and $N$ respectively. Then that there exists a point $p \in M$, and coordinate charts $(U, x), (V, y)$ from $M$ and $N$ respectively such that $p \in U$ and $f(p) \in V$ satisfying that $y \circ f \circ x^{-1}$ is not infinitely differentiable at $x(p)$. In particular, if we write $y = (y_1, ..., y_n)$ then for some $i \in \{1, ..., n\}$ the function $y_i \circ f \circ x^{-1}$ is not infinitely differentiable at $x(p)$. The idea now is to obtain a global smooth function on $N$ that is locally the same as $y_i$ around $f(p)$ so the proof would be concluded.

The open set $V$ is homeomorphic to some open set of $\mathbb{R}^n$, so we can take two open sets $W_1 \subset W_2 \subset V$ such that $f(p) \in W_1$. Let $s$ be a smooth function satisfying $s(\bar{W}_1) = \{1\}$, $s(N \setminus W_2) = \{0\}$ and $s(W_2 \setminus \bar{W}_1) = (0, 1)$. Such function exists because of lemma 1.1.1. Then we can define a smooth function $z$ by means of $z = s \cdot y_i$ in $W_2$ and $z = 0$ otherwise. It is satisfied that $z|_{W_1} = y_i|_{W_1}$, so the function $z \circ f \circ x^{-1}$ is not infinitely differentiable at $x(p)$. Thus the function $z \circ f$ is not a smooth function over $M$, while $z$ is a smooth function over $N$. □

**Theorem 1.2.1.** *Let $(X, \mathcal{O})$ be a ringed space. If $(X, \mathcal{O})$ is a smooth ringed space then $X$ satisfies the second axiom of countability and given any point $p \in X$ there exists an open set $U \subseteq X$ satisfying $p \in U$ and such that $(U, \mathcal{O}|_U)$ is isomorphic to $(\mathbb{R}^n, \mathcal{C}_{\mathbb{R}^n}^\infty)$ for some fixed $n$.*

*Proof.* It suffices to show that if $M$ is a smooth manifold then $(M, \mathcal{C}_M^\infty)$ satisfies the above mentioned-properties, as they are obviously conserved through isomorphisms.

The coordinate domains of $M$ that are diffeomorphic to balls of $\mathbb{R}^n$ form a base of the topology of $M$ and $M$ satisfies the second axiom of countability by definition, so we can take countable many coordinate domains of that kind to form an open covering $\{U_i\}$ of $M$. Also, open balls from $\mathbb{R}^n$ are diffeomorphic to $\mathbb{R}^n$ itself, so we can conclude that for each $i$ there are coordinate charts $(U_i, x_i)$ from $M$ such that $x_i(U_i) = \mathbb{R}^n$. Because of lemma 1.2.3 for each $i$, $(x_i, x_i^{\#})$ is an isomorphism of ringed spaces from $(U_i, \mathcal{C}_M^\infty|_{U_i})$ to $(R^n, \mathcal{C}_{\mathcal{R}^n}^\infty)$, as charts are local diffeomorphisms. □

## 1.3 Smooth manifolds as locally ringed spaces

In this section we are going to complete our characterization of smooth ringed spaces by proving the converse to last theorem. Let us consider a simple example to see what is stopping us from doing so:

Let $(X, \mathcal{O})$ be a ringed space and $X_1, X_2$ be two open sets such that $X = X_1 \cup X_2$ and $(X_1, \mathcal{O}|_{X_1}), (X_2, \mathcal{O}|_{X_2})$ are isomorphic to $(\mathbb{R}^n, \mathcal{C}_{\mathbb{R}^n}^\infty)$ through $(x_1, \varphi_1)$ and $(x_2, \varphi_2)$ respectively. We want to prove that $(X, \mathcal{O})$ is isomorphic to some $(M, \mathcal{C}_M^\infty)$. If $(X_1, x_1)$ and $(X_2, x_2)$ were compatible charts the work would be done: we would be able to equip $X$ with the atlas given by $x_1, x_2$ and then both $(X, \mathcal{C}_X^\infty)$ and $(X, \mathcal{O})$ would be glueings of $(X_1, \mathcal{O}|_{X_1}), (X_2, \mathcal{O}|_{X_2})$ along suitable morphisms so they would be isomorphic. So, as we have seen the problem relies in proving the compatibility of $x_1$ and $x_2$. Restricting accordingly we have the following chain of ringed space isomorphisms: $x_1(X_1 \cup X_2) \xrightarrow{(x_1,\varphi_1)^{-1}} X_1 \cup X_2 \xrightarrow{(x_2,\varphi_2)} x_2(X_1 \cup X_2)$. Through composition we end up with an isomorphism of the form $x_1(X_1 \cup X_2) \xrightarrow{(x_1^{-1} \circ x_2, \psi)} x_2(X_1 \cup X_2)$. We would like to say that $(x_1^{-1} \circ x_2)$ has to be necessarily smooth because of this. The reason why this line of reasoning works is that $(\mathbb{R}^n, \mathcal{C}^\infty_{\mathbb{R}^n})$ is in fact a *locally* ringed space. To define that property we need to introduce another important notion from sheaf theory:

**Definition 1.3.1.** Let $\mathcal{O}$ be a presheaf of rings on $X$. Then for any $x \in X$, the **stalk** $\mathcal{O}_x$ of $\mathcal{O}$ at $x$ is the direct limit of the rings $\mathcal{O}(U)$ for all open sets satisfying $x \in U$ through the restriction maps. That is $\mathcal{O}_x = \varinjlim_{x \in U} \mathcal{O}(U)$.

The elements of $\mathcal{O}_x$ can be represented by pairs $(f, U)$, where $f \in \mathcal{O}(U)$, $x \in U$ and two such pairs $(f, U)$ $(g, V)$ represent the same element from $\mathcal{O}_x$ if and only if there exists an open neighborhood $W \subseteq U \cap V$ of $x$ such that $f|_W = g|_W$. It is not hard to prove that $\mathcal{O}_x$ inherits a ring structure from the rings $\mathcal{O}(U)$, with $x \in U$ by means of defining $(f, U) + (g, V) = (f|_{U\cap V} + g|_{U\cap V}, U \cap V)$, $(f, U) \cdot (g, V) = (f|_{U\cap V} \cdot g|_{U\cap V}, U \cap V)$. This way arise natural ring homomorphisms $\rho_{x,U}$ from the rings $\mathcal{O}(U)$, $x \in U$ onto $\mathcal{O}_x$ that map each $f \in \mathcal{O}(U)$ to the element of $\mathcal{O}_x$ represented by the pair $(f, U)$, which will be denoted by $f_x$.

Stalks capture the local behaviour of a presheaf around a given point. In the particular case of the presheaf $\mathcal{C}^\infty_M$ over a smooth manifold $M$ we have they coincide with the **germs** of smooth functions at each $x$. In fact one can think about the concept of stalks as a generalization of germs.

**Definition 1.3.2.** Let $\varphi : \mathcal{O} \to \mathcal{F}$ be a morphism of presheaves of rings over $X$. Then, for each $x \in X$, the **map of stalks at** $x$ **induced by** $\varphi$ is the ring homomorphism $\varphi_x : \mathcal{O}_x \to \mathcal{F}_x$ given by $\varphi_x(s, U) = (\varphi_U(s), U)$. Analogously, if $(f, \varphi) : (X, \mathcal{O}) \to (Y, \mathcal{F})$ is a morphism of ringed spaces, for each $x \in X$ the **map of stalks at** $x$ **induced by** $(f, \varphi)$ is the ring homomorphism $\varphi_x : \mathcal{F}_{f(x)} \to \mathcal{O}_x$ given by $\varphi_x(s, U) = (\varphi_U(s), f^{-1}(U))$.

**Lemma 1.3.1.** *Let $\mathcal{O}$ be a sheaf of rings over $X$. It is satisfied that if a section $s$ of $\mathcal{O}$ over an open set $U$ satisfies $s_x = 0$ for every $x \in U$ then $s = 0$.*

*Proof.* By definition there exists an open neighbourhood $U_x \subseteq U$ for each $x \in U$ such that $s|_{U_x} = 0$. The sets $U_x$ form an open covering of $U$, so by sheaf property (1) $s = 0$. □

Because of this lemma we can identify each section $s \in \mathcal{O}(U)$ with a function $\tilde{s} : U \to \bigcup_{x \in X} \mathcal{O}_x$ defined by $\tilde{s}(x) = s_x$ for each $x \in U$. For each pair $s, t \in \mathcal{O}(U)$ we can define the sum and product of $\tilde{s}$ and $\tilde{t}$ as their point-wise sum and product. Then it is satisfied that $\widetilde{s + t} = \tilde{s} + \tilde{t}$ and $\widetilde{s \cdot t} = \tilde{s} \cdot \tilde{t}$. Also, for any open $V \subseteq U$, $\widetilde{s|_V} = \tilde{s}|_V$, where $\tilde{s}|_V$ denotes the restriction of $\tilde{s}$ to $V$ in the usual sense.

Note that the co-domain $\bigcup_{x \in X} \mathcal{O}_x$ has not been equipped with a ring structure, let alone a field one. If we want to do better we need to introduce the concept of local rings:

**Definition 1.3.3.** A **local ring** $R$ is a ring that contains a single maximal ideal (note that this implies $0_R \neq 1_R$), which we will denote by $\mathfrak{m}_R$. We will call to the field $R/\mathfrak{m}_R$ the **residue field of** $R$ and given an element $a \in R$ we will denote by $[a]$ to the equivalence class of $a$ in $R/\mathfrak{m}_R$. A **morphism of local rings** (or local homomorphism) $\alpha : R \to S$ of is a ring homomorphism between local rings satisfying $\alpha(\mathfrak{m}_S)^{-1} = \mathfrak{m}_R$.

The following will be two useful characterizations of local rings and morphisms of local rings:

**Lemma 1.3.2.** *Let $R$ be a ring. Then the $R$ is a local ring if and only if there exists a proper ideal $\mathfrak{m} \subset R$ such that for any $a \in R$ either $a$ is invertible or $a \in \mathfrak{m}$. In that case $\mathfrak{m} = \mathfrak{m}_R$.*

*Proof.* If $R$ is a local ring its clear that we can take $\mathfrak{m}_R$ as the $\mathfrak{m}$ in the statement. Conversely, if such $\mathfrak{m}$ exists it is clear that it is maximal, as for any $a \notin R$ the ideal generated by $\mathfrak{m}$ and $a$ is $R$ because of $a$ being invertible. Following the same argument $\mathfrak{m}$ must be the only maximal ideal of $R$ as any proper ideal of $R$ must be contained in $\mathfrak{m}$. □

**Lemma 1.3.3.** *Let $\alpha : R \to S$ be a ring homomorphism between local rings. Then $\alpha$ is a morphism of local rings if and only if $\alpha(\mathfrak{m}_R) \subseteq \mathfrak{m}_S$.*

*Proof.* If $\alpha$ is a local ring morphism then it is direct that $\alpha(\mathfrak{m}_R) \subseteq \mathfrak{m}_S$. Conversely, if $\alpha(\mathfrak{m}_R) \subseteq \mathfrak{m}_S$ we have that $\alpha^{-1}(\mathfrak{m}_S)$ is an ideal of $R$ that contains $\mathfrak{m}_R$, so $\alpha^{-1}(\mathfrak{m}_S) = \mathfrak{m}_R$. □

From the definition follows directly that the composition of morphisms of local rings, defined as their composition as maps, is also a morphism of local rings. It also follows that the isomorphisms of local rings are precisely the ring isomorphisms between local rings. Also, morphisms of local rings induce ring homomorphisms between the respective residue fields:

**Proposition 1.3.1.** *Let $\alpha : R \to S$ be a morphism of local rings and $\rho_R : R \to R/\mathfrak{m}_R$, $\rho_S : S \to S/\mathfrak{m}_S$ be the respective natural projections. Then there exists a unique ring homomorphism $\hat{\alpha} : R/\mathfrak{m}_R \to S/\mathfrak{m}_S$ such that the following diagram is commutative:*

$$\begin{array}{ccc} R & \xrightarrow{\quad\alpha\quad} & S \\ \downarrow{\scriptstyle \rho_R} & & \downarrow{\scriptstyle \rho_S} \\ R/\mathfrak{m}_R & \xrightarrow{\quad\hat{\alpha}\quad} & S/\mathfrak{m}_S \end{array}$$

*Proof.* If such map exists it is unique because of $\rho_R$ being surjective. The map $\hat{\alpha} : R/\mathfrak{m}_R \to S/\mathfrak{m}_S$ given by $\hat{\alpha}([a]) = [\alpha(a)]$ is well defined and is a ring homomorphism as for any $a \in R$, $\alpha(a + \mathfrak{m}_R) \subseteq \alpha(a) + \mathfrak{m}_S$. It is clear that $\hat{\alpha}$ satisfies the conditions we searched for. □

For the rest of the work we will keep denoting by $\hat{\alpha}$ to the field homomorphism induced by the morphism of local rings $\alpha$.

**Definition 1.3.4.** A **locally ringed space** $(X, \mathcal{O})$ is a ringed space such that the stalks of $\mathcal{O}$ are local rings. A **morphism of locally ringed spaces** $(f, \varphi)$, is a morphism of ringed spaces such that the maps of stalks induced by $(f, \varphi)$ are morphisms of local rings.

In consequence, if $(f, \varphi) : (X, \mathcal{O} \to (Y, \mathcal{F})$ is a morphism of locally ringed spaces then $\varphi$ induces naturally unique ring homomorphism between the residue fields of their stalks.

Locally ringed spaces allow us to "evaluate" each section at any point of its domain and obtain an element of some field as a result. That is, we can associate each section $s \in \mathcal{O}$ with a function $\hat{s} : U \to \bigcup_{x \in X} \mathcal{O}_x/\mathfrak{m}_{\mathcal{O}_x}$ defined by $\hat{s}(x) = [s_x]$. This association is analogous to the one between $s$ and the function $\tilde{s}$ in the sense that it respects restrictions and the ring structure on $\mathcal{O}(U)$. On the other hand, it is not and injective association in general, as we could have some section $s \neq 0$ such that $\hat{s} = 0$. We will give an special name to the locally ringed spaces where this does not happen.

**Definition 1.3.5.** We will call **geometric** to a locally ringed $(X, \mathcal{O})$ if any section $s$ of $\mathcal{O}$ such that $\hat{s} = 0$ satisfies $s = 0$.

**Proposition 1.3.2.** *Let $M$ be a smooth manifold. Then for any $x \in M$ the stalk $(\mathcal{C}^\infty_M)_x$ is a local ring whose maximal ideal is $\mathfrak{m}_x = \{s_x \mid s$ is a smooth function defined at $x$ and $s(x) = 0\}$ and whose residue field is isomorphic to $\mathbb{R}$.*

*Proof.* Note that $\mathfrak{m}_x$ is an ideal of $\mathcal{O}_x$. Also, for any $s_x \notin \mathfrak{m}_x$ we have that there exists an open neighbourhood $U$ of $x$ such that $s(y) \neq 0$ for every $y \in U$. Then $s|_U$ is invertible in $\mathcal{C}^\infty_M(U)$ with inverse $r$ and it is satisfied that $r_x = (s_x)^{-1}$. By lemma 1.3.2 $(\mathcal{C}^\infty_M)_x$ is a local ring whose maximal ideal is $\mathfrak{m}_x$. Also, the map $\alpha_x(\mathcal{C}^\infty_M)_x/\mathfrak{m}_x \to \mathbb{R}$ given by $\alpha_x([s_x]) = s(x)$ is well defined and it is a ring isomorphism. ☐

In consequence $(M, \mathcal{C}^\infty_M)$ is a locally ringed space. For each $x \in M$ we will denote by $\mathfrak{m}_x$ to the maximal ideal contained in $(\mathcal{C}^\infty_M)_x$. In the proof of last proposition we have shown that for each $x \in M$ there exist a ring isomorphism $\alpha_x$ from $(\mathcal{C}^\infty_M)_x/\mathfrak{m}_x$ to $R$ satisfying that for any smooth function $s$ defined at $x \quad \alpha_x(\hat{s}(x)) = s(x)$. In particular this means that $(M, \mathcal{C}^\infty_M)$ is geometric. We can prove that the isomorphisms $\alpha_x$ depend solely on the residue fields $(\mathcal{C}^\infty_M)_x/\mathfrak{m}_x$ so we can recover the value $s(x)$ from the stalk $(\mathcal{C}^\infty_M)_x$ algebraically.

**Lemma 1.3.4.** *The identity map is the only ring homomorphism from $\mathbb{R}$ into itself. If $R$ is a ring isomorphic to $\mathbb{R}$ then there exists only one ring isomorphism $\alpha : R \to \mathbb{R}$.*

*Proof.* Let $\phi : \mathbb{R} \to \mathbb{R}$ be a ring homomorphism. It is clear that for any $a \in \mathbb{Z}$, $\phi(a) = a$. Also, $ker\,\phi = \{0\}$. Then $\phi$ preserves sign: if $a > 0$ then, naming $b = \sqrt{a}$, $0 = a - b^2 = \phi(a) - \phi(b)^2$, so $\phi(a) > 0$. In consequence $\phi$ preserves order and $\phi$ is continuous: for any $n$, and $a, b \in \mathbb{R}$ such that $|a - b| < 1/n$ we have that $|\phi(a) - \phi(b)| < 1/n$. (Note that $|a - b| < 1/n$ means exactly $-1/n < (a - b) < 1/n$). As $\phi$ is continuous and it is the identity on $\mathbb{Q}$ it is the identity on $\mathbb{R}$.

For the second part, let $\phi$ and $\alpha$ be ring isomorphisms from $R$ to $\mathbb{R}$. Then $\phi \circ \alpha^{-1}$ and $\alpha \circ \phi^{-1}$ are both ring endomorphisms of $R$, so they must be the identity map and $\phi = \alpha$. ☐

**Proposition 1.3.3.** *Let $f : M \to N$ be a smooth map. Then the map $(f, f^\#) : (M, \mathcal{C}^\infty_M \to (N, \mathcal{C}^\infty_N)$ is a morphism of locally ringed spaces.*

*Proof.* Given any $x \in M$ and any open $U \subseteq N$ containing $f(x)$, we have that any smooth map $s \in \mathcal{C}^\infty_N(U)$ such that $s(f(x)) = 0$ satisfies $(f^\#_U(s))(x) = 0$, as $f^\#_U(s) = s \circ f|_U$. With this it is proven that any $s_{f(x)} \in \mathfrak{m}_{f(x)}$ satisfies $f^\#_x(s_{f(x)}) \in \mathfrak{m}_x$ so $f^\#_x(\mathfrak{m}_{f(x)}) \subseteq \mathfrak{m}_x$ and we have the result. ☐

Thus, the image of the functor $\mathfrak{F}$ previously defined lies in the category of locally ringed spaces: for each smooth manifold $M$, $(M, \mathcal{C}^\infty_M)$ is a locally ringed space and for each smooth map $f$, $(f, f^\#)$ is a morphism of locally ringed spaces. We can also prove that its image is full in said category, i.e. that any morphism of locally ringed spaces $(f, \varphi) : (M, \mathcal{C}^\infty_M) \to (N, \mathcal{C}^\infty_N)$ is of the form $(f, f^\#)$ for some smooth map $f$.

**Proposition 1.3.4.** *Let $(X, \mathcal{O})$ and $(Y, \mathcal{F})$ be geometric locally ringed spaces such that all the residue fields from the stalks of $\mathcal{O}$ and $\mathcal{F}$ are isomorphic to $\mathbb{R}$ ( $\mathcal{F}_y/\mathfrak{m}_y \overset{\alpha_y}{\simeq} \mathbb{R}$, $\mathcal{O}_x/\mathfrak{m}_x \overset{\beta_x}{\simeq} \mathbb{R}$ ). Then any local ring morphism $(f, \varphi)$ is completely determined by its associated continuous map $f$.*

*Proof.* It suffices to show that given a section $s \in \mathcal{F}(U)$ and a point $x \in f^{-1}(U)$ the value $\widehat{\varphi_U(s)}(x)$ is determined by $f$. Let us call $y = f(x)$, $t = \varphi_U(s)$ and $V = f^{-1}(U)$. The following diagram is commutative:

$$\begin{array}{ccccc}
\mathcal{F}(U) & & \xrightarrow[s \mapsto t]{\varphi_U} & & \mathcal{O}(V) \\
\downarrow{\scriptstyle s \mapsto \hat{s}(y)} & & & & \downarrow{\scriptstyle t \mapsto \hat{t}(x)} \\
\mathcal{F}_y/\mathfrak{m}_y & & \xrightarrow[\hat{s}(y) \mapsto \hat{t}(x)]{\widehat{\varphi_x}} & & \mathcal{O}_x/\mathfrak{m}_x \\
& \searrow^{\alpha_y} & & \nearrow^{(\beta_x)^{-1}} & \\
& & \mathbb{R} & &
\end{array}$$

So we have that $\widehat{\varphi_U(s)}(x) = (\beta_x)^{-1}(\alpha_y(\hat{s}(y)))$ does not depend on $\varphi$. $\square$

**Corollary 1.3.1.** *Let $M$ and $N$ be smooth manifolds and $(f, \varphi) : (M, \mathcal{C}^\infty_M) \to (N, \mathcal{C}^\infty_N)$ be a locally ringed space morphism. Then $f$ is a smooth map and $\varphi = f^\#$.*

*Proof.* Let $U \subseteq N$ be an arbitrary open set and $s \in \mathcal{C}^\infty_N(U)$ a smooth function. Then $\varphi_U(s)$ is a smooth function over $f^{-1}(U) \subseteq M$. Let $x \in f^{-1}(U)$ be a point and $y = f(x)$. Because of lemma 1.3.4 $s(y) = \alpha_y(\hat{s}(y))$ and $\varphi_U(s)(x) = \alpha_x(\widehat{\varphi_U(s)}(x))$. Also, in the proof of the last proposition we showed $\alpha_y(\hat{s}(y)) = \alpha_x(\widehat{\varphi_U(s)}(x))$, so $s(y) = \varphi_U(s)(x)$ and $\varphi_U(s) = s \circ f$. Then, because of lemma 1.2.1 $f$ is smooth and as shown $\varphi = f^\#$. $\square$

As we have shown $\mathfrak{F}$ is a full functor to the category of locally ringed spaces. $\mathfrak{F}$ is also faithful, which means that it sends different morphisms to different morphisms (if $f$ and $g$ are different smooth maps, trivially $(f, f^\#) \neq (g, g^\#)$). Then $\mathfrak{F}$ defines a equivalence of categories between $\mathbf{Man}^\infty$ and the full subcategory of the category $\mathbf{LocalRinS}^\infty$ of locally ringed spaces whose objects are the smooth ringed spaces. All that is left is to complete the characterization of these spaces. There exists a far deeper relationship between sheaves and their stalks (check [1] proposition 1.1, page 63), but the following is all we are going to need:

**Proposition 1.3.5.** *Let $\varphi : \mathcal{O} \to \mathcal{F}$ be an isomorphism of sheaves of rings over $X$. Then the maps induced by $\varphi$ on the stalks are ring isomorphisms.*

*Proof.* Let $x \in X$, and $(t, U) \in \mathcal{F}_x$. As $\varphi_U$ is surjective there exists $s \in \varphi_U^{-1}(t)$ and we have $\varphi_x(s_x) = t_x$, so $\varphi_x$ is surjective. Also, if for some $(s, U) \in \mathcal{O}_x$ it is satisfied $\varphi_x(s_x) = 0$ then it means that for some open neighbourhood $V \subseteq U$ of $x$ $(\varphi_U(s))|_V = 0$, or equivalently $\varphi_V(s|_V) = 0$. As $\varphi_V$ is injective we have that $s|_V = 0$ and $s_x = 0$, so $\varphi_x$ is also injective. $\square$

**Corollary 1.3.2.** *Any ringed space isomorphism between locally ringed spaces is an isomorphism of locally ringed spaces.*

**Theorem 1.3.1.** *Let $(X, \mathcal{O})$ be a ringed space such that $X$ satisfies the second axiom of countability and $(X, \mathcal{O})$ is locally isomorphic to $(\mathbb{R}^n, \mathcal{C}^\infty_{\mathbb{R}^n})$ for some fixed $n$. Then there exists some smooth atlas $\mathcal{A}$ over $X$ such that $(X, \mathcal{O})$ is isomorphic to $(X, \mathcal{C}^\infty_X)$.*

*Proof.* There exists a countable open cover $\{U_i\}$ of $X$ such that there exists a ringed space isomorphism $(x_i, \varphi_i) : (U_i, \mathcal{O}|_{U_i}) \to (\mathbb{R}^n, \mathcal{C}^\infty_{\mathbb{R}^n})$ for each $i$. It is clear that each pair $(U_i, x_i)$ is a chart on $X$. Let $i, j$ be such that $U_i \cap U_j \neq \emptyset$. Let us call $V = U_i \cap U_j$. Then the restriction $(z_\lambda, \psi_\lambda)$ of $(x_\lambda, \varphi_\lambda)$ to $V$ and $x(V)$ is also a ringed space isomorphism for each $\lambda \in \{i, j\}$. In particular this means that $(z_i, \psi_i) \circ (z_j, \psi_j)^{-1}$ is a ringed space isomorphism between locally ringed spaces and thus a locally ringed space isomorphism. In consequence $z_i \circ z_j^{-1}$ is smooth, $(U_j, x_j)$ and $(U_i, x_i)$ are compatible charts and the family $\mathcal{A} = \{(U_i, x_i)\}$ for $i$ ranging over $\mathbb{N}$ is an atlas. Also, it is satisfied that $(z_i, \psi_i) \circ (z_j, \psi_j)^{-1} = (z_i \circ z_j^{-1}, (z_i \circ z_j^{-1})^\#)$. So by proposition 1.2.2 $(X, \mathcal{O})$ its the glueing of $\{(\mathbb{R}^n, \mathcal{C}^\infty_{\mathbb{R}^n})\}, i \in \mathbb{N}$ along the morphisms $\mathcal{Z}_{i,j} := (z_i \circ z_j^{-1}, (z_i \circ z_j^{-1})^\#)$.

Then we have that the pair $(X, \mathcal{A})$ is a smooth manifold. Let us consider the locally ringed space $(X, \mathcal{C}^\infty_X)$. Then, for each $i$, its open ringed subspace $U_i$ is clearly isomorphic to $(\mathbb{R}^n, \mathcal{C}^\infty_{\mathbb{R}^n})$ by means of $(x_i, x_i^\#)$. By proposition 1.2.2, $(X, \mathcal{C}^\infty_X)$ is also the glueing of $\{(\mathbb{R}^n, \mathcal{C}^\infty_{\mathbb{R}^n})\}, i \in \mathbb{N}$ along the morphisms $\mathcal{Z}_{i,j}$, so by the uniqueness of the glueing $(X, \mathcal{O})$ is isomorphic to $(X, \mathcal{C}^\infty_X)$. $\square$

It is not hard to prove that the locally ringed space $(X, \mathcal{C}^\infty_X)$ and the isomorphism between $(X, \mathcal{O})$ and $(X, \mathcal{C}^\infty_X)$ are unique and can be given explicitly: if $(Id_X, \varphi) : (X, \mathcal{O}) \to (X, \mathcal{C}^\infty_X)$ is a locally ringed space morphism then $\varphi$ must send each section $s$ of $\mathcal{O}$ to a smooth function $s'$ from $\mathcal{C}^\infty_X$ satisfying $s'(p) = \alpha_p(\hat{s}(p))$. If this morphism is an isomorphism then $(X, \mathcal{C}^\infty_X)$ is completely determined. This way all smooth ringed spaces can be canonically realized as locally ringed spaces of the form $(X, \mathcal{C}_X^\infty)$.

## 1.4 The spectrum of an algebra and Zariski topology

For the next part of the chapter we are going to show that all the relevant information about a smooth ringed space $(X, \mathcal{O})$ is contained within its ring of global sections $\mathcal{O}$. As $\mathcal{O} \simeq \mathcal{C}_X^\infty$ for some smooth atlas on $X$ we can restrict ourselves to the study of smooth ringed spaces of the form $(M, \mathcal{C}_M^\infty)$. The here exposed ideas about geometric algebras and their realization as functions over their dual spaces can be found on the third chapter from [3].

The first goal of this section is to show that given a manifold $(M, \mathcal{A})$ we can recover the whole topological space $M$ from the ring $\mathcal{C}_M^\infty(M)$ algebraically.

**Definition 1.4.1.** Let $K$ be a field. Then a **$K$-algebra** is a pair $(A, \sigma)$ where $A$ is a ring and $\sigma : K \to A$ is a ring homomorphism. A **morphism of $K$-algebras** $\alpha : (A_1, \sigma_1) \to (A_2, \sigma_2)$ is a ring homomorphism from $A_1$ to $A_2$ satisfying $\sigma_2 = \alpha \circ \sigma_1$. The composition of two suitable $K$-algebra morphisms is given by their composition as ring homomorphisms.

Bluntly speaking, $K$-algebras are just rings that contain $K$ in a fixed way and their morphisms are just ring homomorphisms that respect the ways in which they contain $K$. Sometimes we will carry out the identification $\sigma(K) \simeq K$.

**Definition 1.4.2.** Let $(A, \sigma)$ be a $K$-algebra· A $K$-morphism of $(A, \sigma)$ is a ring homomorphism $\alpha : A \to K$ such that $\alpha \circ \sigma = Id_K$. The set of $K$-morphism of $(A, \sigma)$ is called its **dual space** $|A|$.

*Remark.* Because of lemma 1.3.4 every ring homomorphism between $\mathbb{R}$-algebras is a morphism of $\mathbb{R}$-algebras, and every ring homomorphism from a $\mathbb{R}$-algebra into $\mathbb{R}$ is a $\mathbb{R}$-morphism for that algebra (ring homomorphisms between rings isomorphic to $\mathbb{R}$ are unique).

If we identify $\mathbb{R}$ with the constant sections from $\mathcal{C}_M^\infty(M)$ in the obvious way we can consider the later ring to be a $\mathbb{R}$-algebra. Then there exists a canonical mapping from the points of $M$ into the dual space of $\mathcal{C}_M^\infty(M)$: the one which sends each point $p \in M$ to the "evaluation at $p$" homomorphism $|p|$, i.e the one who maps each smooth function over $M$ to its value at $p$. We can also map canonically this dual space to the set of maximal ideals of $A$ via the association $q \mapsto ker\, q$. This motivates the following definition:

**Definition 1.4.3.** Let $(A, \sigma)$ be a $K$-algebra. Then we will call the **$K$-spectrum** of $A$ to the following set of maximal ideals: $Spec_K\, A = \{\mathfrak{m} \lhd_m A \,|\, A/\mathfrak{m} \simeq K\}$.

**Proposition 1.4.1.** *Let $(A, \gamma)$ be a $K$-algebra. Then the map $G : |A| \to Spec_K A$ given by $q \mapsto ker\, q$ is a bijection.*

*Proof. G is injective:* Let $q$ be a $K$-morphism. For any arbitrary $s \in A$ and $\lambda \in K$ $\quad q(s) = \lambda \iff s - \gamma(\lambda) \in ker\, q$. Thus $q$ is determined by its $ker$ and different $K$-morphisms have different kernels.

*G is surjective:* Let $\mathfrak{m} \in Spec_K\, A$. Then, by definition there exists an isomorphism $\alpha : A/\mathfrak{m} \to K$. Let us denote by $\rho$ the natural projection $\rho : A \to A/\mathfrak{m}$. Then the composition $\beta := \alpha \circ \rho \circ \gamma$ is an automorphism on $K$ and the map $\beta^{-1} \circ \alpha \circ \rho$ is a $K$-morphism of $A$ whose kernel is $\mathfrak{m}$. □

Then, given any $K$-algebra $A$ we can write each element $\mathfrak{m}$ in $Spec_K\, A$ as $ker\, q$ for a unique $q \in |A|$. This makes the elements $s \in A$ induce functions $f_s : Spec_K\, A \to K$ by means of defining $f_s(ker\, q) = q(s)$. Note that we have obtained a ring of functions $\hat{A} := \{ f_s \,|s \in A \}$ equipped with the evident point-by-point sum and product, as $f_{s_1}(ker\, q) + f_{s_2}(ker\, q) = q(s_1) + q(s_2) = q(s_1 + s_2) = f_{s_1+s_2}(ker\, q)$ and analogously with the product.

This allows us to think of $Spec_K\, A$ as a set of points from where the functions of $\hat{A}$ are defined. Those functions also distinguish between different points: if $ker\, q_1 \neq ker\, q_2$ then there exists some $s \in A$ satisfying $q_1(s) \neq q_2(s)$ (as $q_1$ and $q_2$ are different maps) so $f_s(ker\, q_1) \neq f_s(ker\, q_1)$. Now arises the question on whether $A$ and $\hat{A}$ are canonically isomorphic through the map $H$ given by $s \mapsto f_s$. Even though surjectivity of the map follows directly from the definition, injectivity is not always archived. We have that: $f_s = 0 \iff q(s) = 0 \;\; \forall q \in |A| \iff s \in \bigcap_{q\in|A|} ker\, q \iff s \in \bigcap_{\mathfrak{m}\in Spec_K\, A} \mathfrak{m}$. Thus $H$ is injective if and only if $\bigcap_{\mathfrak{m}\in Spec_K\, A} \mathfrak{m} = \emptyset$. The $K$-algebras that satisfy this will be called **geometric**. Note that given any $K$-algebra $A$ we can always obtain a geometric one from it: $A/ \cap_{\mathfrak{m}\in Spec_K\, A} \mathfrak{m}$.

Returning to the case of $\mathbb{R}$-algebras of the form $\mathcal{C}^\infty_M(M)$, we can also prove the following:

**Theorem 1.4.1.** *Let $M$ be a smooth manifold. Then the map $F : M \to |\mathcal{C}^\infty_M(M)|$ given by $p \mapsto |p|$ is a bijection.*

*Proof. F is injective:* Let $p_1, p_2 \in M$ be two different points. Because of lemma 1.1.1 there exists a smooth function $s$ over $M$ such that $s(p_1) = 1$ and $s(p_2) = 0$, so $|p_1| \neq |p_2|$.

*F is surjective:* Let us assume that there exists an $\mathbb{R}$-morphism $q \in |\mathcal{C}^\infty_M(M)|$ such that $q \neq |p|$ for every $p \in M$. In particular, this means that $ker\, q \neq ker\, |p|$ for every $p$ because of last proposition. Thus, $\bigcap_{s\in ker\, q} s^{-1}(0) = \emptyset$. Let $r \in \mathcal{C}^\infty_M(M)$ be a smooth function over $M$ with compact level surfaces (such function exists because of lemma 1.1.3). Then $r - q(r) \in ker\, q$. We have that $(r - q(r))^{-1}(0) \cap (\bigcap_{s\in ker\, q} s^{-1}(O)) = \emptyset$. As $(r - q(r))^{-1}(0)$ is compact and $s^{-1}(0)$ is a closed set for any $s \in ker\, q$ there exist some $s_1, .., s_n \in ker\, q$ such that $(r - q(r))^{-1}(0) \cap (\bigcap_{i\in\{1,..,n\}} s_i^{-1}(O)) = \emptyset$. In consequence the smooth function $(r - q(r))^2 + \sum_{i=1}^n s_i^2$ is invertible (as it does not annihilate anywhere) and belongs to $ker\, q$, arriving to a contradiction. □

With the last two results we have proven that the map $\theta : M \to Spec_{\mathbb{R}}\, \mathcal{C}^\infty_M(M)$ given by $p \mapsto ker\, |p|$ is a bijection. It is worth noting that in the last proof, if $M$ happened to be compact then all of its maximal ideals would belong to its $\mathbb{R}$-spectrum. When $M$ is not compact that is never the case: consider the global smooth functions over $M$ with compact support. They form an ideal which is not included in any of the ideals of the form $\mathfrak{m}_p := \{f|f(p) = 0\}$, but applying Zorn's lemma we can conclude that ideal is contained in some maximal ideal of $\mathcal{C}^\infty_M(M)$. Thus in that case we can find a maximal ideal which does not belong the $\mathbb{R}$-spectrum.

Now we would like to define a topology over $Spec_{\mathbb{R}}\, \mathcal{C}^\infty_M(M)$ using only the algebraic properties of the ring $\mathcal{C}^\infty_M(M)$ that makes $\theta$ an homeomorphism. Lemma 1.1.2 hints us how: The topology over $M$ is precisely the one given by the sets of the form $s^{-1}(0)$ with $s \in \mathcal{C}^\infty_M(M)$ as a basis of closed sets (in fact the lemma says that all closed sets are of this form). On the other hand $p \in s^{-1}(0) \iff s \in ker\, |p|$. This implies that the topology we define over $Spec_{\mathbb{R}}\, \mathcal{C}^\infty_M(M)$ should have the sets of the form $V(s) := \{ \mathfrak{m} \in Spec_{\mathbb{R}}\, \mathcal{C}^\infty_M(M) \,|\, s \in \mathfrak{m} \}$ for every $s \in \mathcal{C}^\infty_M(M)$ as a basis of closed sets. The well known construction is the following [1] [2]:

**Definition 1.4.4.** Let $(A, \sigma)$ be a $K$-algebra. Then, the **Zariski topology** over $Spec_K\, A$ is the one given by the sets $V(S) := \{ \mathfrak{m} \in Spec_K\, A \,|\, S \subseteq \mathfrak{m} \}$ for each subset $S \subseteq A$ as its closed sets.

We can check that the Zariski topology is indeed a topology. Given an arbitrary subset $S \subseteq A$ it is straightforward that $V(S) = V(\langle S\rangle)$, where $\langle S\rangle$ is the ideal generated by $S$ in $A$.

It is also satisfied that $V(S_1) \cup V(S_2) = V(\langle S_1 \rangle \cdot \langle S_2 \rangle)$ and $\bigcap_{\alpha \in J} V(S_\alpha) = V(+_{\alpha \in J} \langle S_\alpha \rangle)$, so the $V(S)$ form a topology. Also, if $s \in S$, then $V(S) \subseteq V(s)$ ($V(s) = V(\{s\})$), so the sets $V(s)$ add up together to a basis of closed sets in the Zariski topology as desired.

From now on we will consider the $K$-spectrum of a $K$-algebra to be a topological space equipped with the Zariski topology.

**Proposition 1.4.2.** *Let $M$ be a smooth manifold. Then the map $\theta : M \to Spec_{\mathbb{R}}\, \mathcal{C}^\infty_M(M)$ given by $p \mapsto ker\,|p|$ is an homeomorphism.*

*Proof.* The map $\theta$ is bijective because of last two results. Given a closed set $N \subseteq M$, because of lemma 1.1.2 there exists a smooth function $s$ over $M$ such that $s^{-1}(0) = N$. Then $\theta(N) = V(s)$ and $\theta$ is a closed map. Conversely, let $V(S)$ be a closed set from $Spec_{\mathbb{R}}\, \mathcal{C}^\infty_M(M)$. Then $\theta^{-1}(V(S)) = \bigcap_{s \in S} s^{-1}(0)$, so $\theta$ is continuous. In consequence $\theta$ is an homeomorphism. □

Not only do the rings of global smooth functions carry all the topological information of their respective manifolds algebraically, but they also carry all the information about smooth maps between them:

**Proposition 1.4.3.** *Let $M$ and $N$ be smooth manifolds. For any smooth map $f : M \to N$ let us denote by $|f| : \mathcal{C}^\infty_N(N) \to \mathcal{C}^\infty_M(M)$ to the ring homomorphism given by $s \mapsto s \circ f$. Then the map between the set of smooth maps from $M$ to $N$ and the set of ring homomorphisms from $\mathcal{C}^\infty_N(N)$ to $\mathcal{C}^\infty_M(M)$ given by $f \mapsto |f|$ is a bijection.*

*Proof.* Let $\alpha : \mathcal{C}^\infty_N(N) \to \mathcal{C}^\infty_M(M)$. For each point $p \in M$ we have that $|p| \circ \alpha$ is a $\mathbb{R}$-morphism of $\mathcal{C}^\infty_N(N)$, so there exists a unique point $q \in M$ such that $|q| = |p| \circ \alpha$. This association defines a map $f : M \to N$. Then necessarily $\alpha$ is defined by $s \mapsto s \circ f$. By the definition of $f$ for any $s \in \mathcal{C}^\infty_N(N)$ and any $p \in M$, $s(f(p)) = \alpha(s)(p)$, so $\alpha(s) = s \circ f$. Then $f$ is smooth because of proposition 1.2.1 and $\alpha = |f|$. □

This way it is proven that the following contravariant functor $\mathfrak{G}$ from $\mathbf{LocalRinS}^\infty$ to the category $\mathbb{R}$-**Alg** of $\mathbb{R}$-algebras is a fully faithful one:

| | | |
|---|---|---|
| Smooth ringed spaces | $\to$ | $\mathbb{R}$-algebras |
| $(X, \mathcal{O})$ | $\mapsto$ | $\mathcal{O}(X)$ |
| Morphisms of locally ringed spaces | $\to$ | Morphisms of $\mathbb{R}$-algebras |
| $(f, \varphi) : (X, \mathcal{O}) \to (Y, \mathcal{F})$ | $\mapsto$ | $\varphi_Y : \mathcal{F}(Y) \to \mathcal{O}(X)$ |

As it is obviously fully faithful when restricted to objects of the form $(M, \mathcal{C}\infty_M)$ and the morphisms between them, and the category $\mathbf{LocalRinS}^\infty$ is just its repletion.

For information about the essential image of this functor -that is, the $\mathbb{R}$-algebras that are isomorphic to the ring of global smooth functions over some manifold- one can consult the characterization given in [3], chapters 3 and 7, in terms of $\mathcal{C}^\infty$ completeness or the one given in [5].

The construction of the whole sheaf $\mathcal{C}^\infty_M$ from its ring of global smooth functions follows easily from the fact that a function $s : U \to \mathbb{R}$ from some open set $U \subseteq M$ is smooth if and only if it is locally equal to some global smooth function around each point. This motivates the following definition:

**Definition 1.4.5.** Let $A$ be a geometric $K$-algebra. Then we will call the **soft structure sheaf of** $A$ to the sheaf $\mathcal{O}$ over $Spec_K$ whose rings of sections are given by

$$\mathcal{O}(U) := \{\, s : U \to K \text{ such that for every } \mathfrak{m} \in U \text{ there exists an open set } V \text{ satisfying } \mathfrak{m} \in V \subseteq U \text{ and an element } r \in A \text{ such that } s|_V = r|_V \,\}$$

and whose restrictions are given by the restrictions of its sections in the usual sense.

The reason why we call this sheaf soft is because, by construction, the natural projections from its global sections to its stalks are all surjective.

Up until now we have essentially proven the following:

**Theorem 1.4.2.** *Let $A = \mathcal{C}^{\infty}_{M}(M)$ for some smooth manifold $M$ and let $\mathcal{O}$ be the soft sheaf of $A$. Then $(\theta, \theta^{\#}) : (M, \mathcal{C}^{\infty}_{M}) \to (Spec_{\mathbb{R}}\, A, \mathcal{O})$ where $\theta$ is defined as in proposition 1.4.2 is a isomorphism of ringed spaces and $(Spec_{\mathbb{R}}\, A, \mathcal{O})$ is smooth.*

Thus we are able to construct locally ringed spaces for some particular kind of $\mathbb{R}$-algebras. To end this chapter, we are going to show the relationship between stalks of said ringed spaces and those algebras.

**Definition 1.4.6.** Let $A$ be a ring. Then a subset $S \subset A$ is called a **multiplicative set** if it satisfies that $1 \in S$ and for any pair $a, b \in S$ it is verified that $ab \in S$.

The **ring of fractions** $A_S$ of $A$ with denominators in $S$ is defined to be the set $\{\, a/b \,|\, a \in A, b \in S \,\}$ modulo the equivalence relation $\{a_1/b_1 = a_2/b_2 \iff \exists b \in S$ such that $b(b_2 a_1 + b_1 a_2) = 0$ in $A\,\}$, and equipped with the binary operations $a_1/b_1 + a_2/b_2 = b_2 a_1 + b_2 a_1$, $a_1/b_1 \cdot a_2/b_2 = a_1 a_2 / b_1 b_2$.

We can also add notation to include fractions in $A_S$ with denominators which are outside of $S$: for any $a \in A$ such that $a/1$ has an inverse in $A_S$ we will denote $1/a$ to that inverse. One can check that this notation is consistent with the given operations and equivalence relation on $A_S$.

We will call **localization map** from $A$ to $A_S$ to the ring homomorphism defined $\alpha : A \to S$ by means of $a \mapsto a/1$.

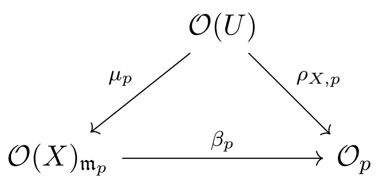
**Definition 1.4.7.** Let $A$ be a ring and $\mathfrak{p} \lhd A$ be a prime ideal. Then the localization $A_{\mathfrak{p}}$ of $A$ at $\mathfrak{p}$ is the ring of fractions with denominators in the multiplicative set $A \setminus \mathfrak{p}$.

**Fact 1.4.1.** *Let $A$ be a ring and $S$ be a multiplicative set of $A$. Then the ideals of $A$ non intersecting $S$ are in one to one correspondence with the ideals of $A_S$ by means of $I \mapsto \frac{I}{S} := \{\, a/s \in A_S \,|\, a \in I \,\}$.*

**Lemma 1.4.1.** *Let $A$ be a ring and $\mathfrak{p} \lhd A$ be a prime ideal. Then $A_{\mathfrak{p}}$ is a local ring.*

*Proof.* $\frac{\mathfrak{p}}{S}$ is an ideal from $A\mathfrak{p}$, and that every element $a \notin \frac{\mathfrak{p}}{S}$ is invertible as $a = b/c$ for some $b, c \notin \mathfrak{p}$, so $a^{-1} = c/b$. □

**Proposition 1.4.4.** *Let $(M, \mathcal{O})$ be a smooth ringed space. Then, for each point $p \in X$ the map $\beta_p : \mathcal{O}_p \to \mathcal{O}(X)_{\mathfrak{m}_p}$ from the stalk of $\mathcal{O}$ at $p$ to the localization of $\mathcal{O}(X)$ at the ideal $\mathfrak{m}_p$ given by $s/r \mapsto s_p/r_p$ is a ring isomorphism such that the following diagram commutes:*




*Where $\rho_{X,p}$ and $\mu_p$ denote the respective canonical maps between their sources and targets.*

*Proof.* First, let us see that $\beta_p$ is well defined. Given any $s/r \in \mathcal{O}(X)_{\mathfrak{m}_p}$ we have that $r \notin \mathfrak{m}_p$, so $r(p) \neq 0$ and $r_p$ does not belong to the maximal ideal in $\mathcal{O}_p$ because of proposition 1.3.2. Thus, $s_p/r_p$ is an element of $\mathcal{O}_p$ and $\beta_p$ is a ring homomorphism. $\beta_p$ is clearly surjective, as every element $a \in \mathcal{O}_p$ is of the form $s_p$ for some global section $s$, so $a = \beta_p(s/1)$. Now, let $s/r$ be such that $\beta_p(s/r) = 0$. That occurs if and only if $s_p = 0$ so there exists an open set $U \subseteq X$ with $p \in U$ satisfying $s|_U = 0$. Then, we can take a smooth function $t$ such that $t|_{X \setminus U} = 0$ and $t(p) = 1$ because of lemma 1.1.1 and we have that $t \cdot s = 0$. As $t \notin \mathfrak{m}_p$ it follows that $s/1 = 0$ in $\mathcal{O}(X)_{\mathfrak{m}_p}$ so $\beta_p$ is also injective.

Commutativity of the diagram is trivial as $\beta_p(\mu_p(s)) = \beta_p(s/1) = s_p$ for every global section $s$. □

# Chapter 2

# Towards generalization: Algebraic varieties and Schemes

Last chapter was about the following: we examined some geometric objects -smooth manifolds- and then realized that their rings of "good" functions encode algebraically all of their relevant characteristics, to the point that we are able to construct those objects from their respective associated rings. The procedure is as follows: we identified the points of any given manifold with some ideals of its respective ring of smooth functions, then we equipped that set of ideals with the Zariski topology and finally we constructed a sheaf that made it into a locally ringed space. In this chapter we are going to answer briefly to the question of what happens if we take different kind of rings as our starting point and follow the same path.

## 2.1 Algebraic varieties

The ideas that were exposed in the first chapter did not have their origins in the field of smooth geometry but in the one of algebraic geometry. There the 'basic' objects of study are the loci of zeros of polynomials and in consequence the relationship between algebra and geometry presents itself more evidently. This topic is profusely discussed in [1], chapter 1.

The rings that are of interest now are the finitely generated geometric algebras over algebraically closed fields. Let us explain the "finitely generated" part. We will say that $(A, \alpha)$ is an algebra generated by some subset $S \subseteq A$ if any element $a \in A$ can be written as a polynomial expression with variables in $S$ and coefficients in $K$, $a = \lambda_1 s_{1,1}^{e_{1,1}} \cdot .. \cdot s_{1,i_1}^{e_{1,i_1}} + .. + \lambda_n s_{n,1}^{e_{n,1}} \cdot .. \cdot s_{n,i_n}^{e_{n,i_n}}$ for some $n$ and $\lambda_1, .., \lambda_n \in K$, $s_1, .., s_n \in S$. Then we will say that $A$ is finitely generated (as a $K$ algebra) if is generated (as a $K$ algebra) by some of its finite subsets. The following is satisfied:

**Lemma 2.1.1.** *Let $K$ be a field and $A$ be a finitely generated $K$-algebra. Then $A \simeq K[x_1, .., x_n]/I$ for some $n$ and an ideal $I \lhd K[x_1, .., x_n]$..*

*Proof.* Let $a_1, .., a_n$ be generators of $A$. Then the map $\phi : K[x_1, .., x_n] \to A$ defined by $\phi(x_i) \mapsto a_i$ extends naturally to a $K$-algebra morphism $\Phi$. From there follows that $A \simeq K[x_1, .., x_n]/ker\, \Phi$. □

Conversely, is evident that any $K$-algebra of the form above described is finitely generated.

Rings of polynomials $K[x_1, .., x_n]$ over a field $K$ are geometric $K$-algebras: given any point $p \in K^n$, $p = (p_1, .., p_n)$ the 'evaluation at $p$' map, given by $x_i \mapsto p_i$ is clearly a $K$-morphism of $K[x_1, .., x_n]$, as it fixes the constant polynomials, i.e. the elements of $K$. It is satisfied that the only element whose evaluations are zero in every point is the zero polynomial, so that concludes the proof.

When $K$ is an algebraically closed field the following two known and important results hold:

**Theorem 2.1.1.** *Let $K$ be an algebraically closed field. Then the maximal ideals of $K[x_1, .., x_n]$ are all the ideals of the form $\langle x_1 - a_1, .., x_n - a_n \rangle$ with $a_i \in K$.*

In particular that means that if $I \lhd K[x_1, .., x_n]$ is an ideal then the set $Z(I) := \{ a \in K^n \mid f(a) = 0 \ \forall f \in I \}$ is not empty, as $I$ belongs to some maximal ideal.

Let $I \lhd A$ be an ideal of a ring $A$. Then we will denote $\sqrt{I} := \{a \in A \,|a^m \in I \, \text{for some } m \}$.

**Theorem 2.1.2.** ***Hilbert's Nullstellensatz*** *Let $K$ be an algebraically closed field and $I \lhd K[x_1, .., x_n]$ be an ideal. Then, if some polynomial $f \in K[x_1, .., x_n]$ satisfies that $f(a) = 0$ for every $a \in Z(I)$ it is satisfied that $f \in \sqrt{I}$.*

Note that the converse to this result is trivially true: if $f \in \sqrt{I}$ then $f^m \in I$ for some $m$ and the zeros of $f$ are contained in $Z(I)$.

The first theorem proves that we can identify $K^n$ with the set of maximal ideals of $A = K[x_1, .., x_n]$ called **maximal spectrum** $Spec_m\, A$, which in this case equals $Spec_K\, A$, by means of mapping $a = (a_1, .., a_n) \mapsto \mathfrak{m}_a := \langle x_1 - a_1, .., x_n - a_n \rangle$. The equality between both spectrums of $A$ comes from the fact that for each $a \in K^n$ the evaluation morphism $u_a : A \to K$ defined by $x_i \mapsto a_i$ has for $Ker\ \mathfrak{m}_a$, so $A/\mathfrak{m} \simeq K$ for every maximal ideal $\mathfrak{m}$. Note that now that $K \neq \mathbb{R}$ an ideal from the $K$-spectrum of $A$ does not unequivocally determine a ring isomorphism from $A$ to $K$ but it does determine a unique $K$-morphism. For example, lets take the maps $u, v : \mathbb{C}[x] \to \mathbb{C}$ defined by $u(f(x)) = f(1)$ and $v = \bar{u}$. It is clear that they both share $Ker\, u = Ker\, v = \langle x - 1 \rangle$ but only $u$ is a $\mathbb{C}$-morphism as it is the only one that fixes $\mathbb{C}$.

Now, let $A = K[x_1, .., x_n]/I$ for some ideal $I$. Then, if we call $X_i$ to the equivalence class of $x_i$ in $A$ we can denote $A = K[X_1, .., X_n]$. Given a polynomial $g \in K[x_1, .., x_n]$ we will denote by $G[X]$ to its equivalence class in $A$. As the maximal ideals in $A$ are in correspondence with maximal ideals from $K[x_1, .., x_n]$ containing $I$ it follows that the maximal ideals from $A$ are all the ones of the form $\mathfrak{m}_a = \langle X_1 - a_1, .., X_n - a_n \rangle$ with $f(a_1, .., a_n) = 0$ for each $f \in I$. That is, we can identify $Spec_m\, A$ with the locus of shared zeros from the elements of $I$ analogously to last paragraph. The $K$-morphism $u_a$ associated to each $\mathfrak{m}_a \in Spec\, A$ is the evaluation morphism that sends each $X_i$ to $a_i$ and fixes $K$, i.e. the one who maps $G[X] \mapsto g(a)$. Note that this is well defined if and only if $f(a) = 0$ for each $f \in I$. In this case we will denote $G(a) := g(a)$.

The second theorem tells us under what conditions $A = K[x_1, ..x_n]/I$ is geometric. Let $G(X) = G(X_1, .., X_n) \in A$ be a polynomial such that for all the $K$-morphisms $u_a$ of $A$ it is satisfied that $u_a(G(X)) = 0$. This is equivalent to $g(a) = 0$ for any $a \in K^n$ such that $f(a) = 0$ for every $f \in I$. In other words, $g(a) = 0$ for any $a \in Z(I)$, so by last theorem $g(x) \in \sqrt{I}$. So we conclude that $A$ is geometric if and only if , $I = \sqrt{I}$, which is equivalent to $\{0_A\} = \sqrt{\{0_A\}}$.

**Definition 2.1.1.** Let $A$ be a ring. Then we will call to $\sqrt{\{0_A\}}$ its **nilradical**. A ring will be called **regular** if its nilradical is trivial.

**Corollary 2.1.1.** *A finitely generated $K$-algebra $A$ over an algebraically closed field $K$ is geometric if and only if is regular. In particular, if $A$ is prime, $A$ is regular and geometric.*

Now that we have geometric algebras at hand we want to give some general construction, analogous to the ones in last chapter, that associates each of one them to a locally ringed space. To construct the underlying topological space the obvious thing to do is equipping $Spec_m\, A$ with the Zariski topology. From now on let $A = K[x_1, .., x_n]/I$ with $I = \sqrt{I}$ and the possibility that $I = \langle 0 \rangle$ and $A = K[x_1, .., x_n]/\langle 0 \rangle = K[x_1, .., x_n]$. Then because of the previous paragraphs $Spec_m\, A \simeq \{ a \in K^n \mid f(a) = 0 \ \forall f \in I \}$. Given any subset $S \subseteq A$ then its closed corresponding set $V(S)$ is defined by $V(S) := \{ \mathfrak{m}_a \in Spec_m\, A \mid g(a) = 0 \forall g \in S \}$.

Hilbert's Nullstellensatz can be easily extended to the algebras of our interest:

**Corollary 2.1.2.** *Let $A = K[x_1, .., x_n]/I$ and $J \lhd A$ some ideal. Then if some $G \in A$ satisfies $G \in \mathfrak{m}_a$ for every $\mathfrak{m}_a \in V(J)$ it is satisfied that $G \in \sqrt{J}$.*

*Proof.* Let us write $J$ in the form $J = \langle (h_i + J)_{i \in L} \rangle$ for some family of generators $(h_i + J)_{i \in L}$. Then, if $G \in \mathfrak{m}_a$ for every $\mathfrak{m}_a \in V(I)$ it means that $g(0)$ for each $a \in K^n$ such that $h_i(a) = 0$ for every $i \in L$ and $f(a) = 0$ for every $f \in I$. So by Hilbert's Nullstellensatz $g^l$ belongs to the ideal $\langle (h_i)_{i \in J} \rangle + I$ for some $l$, which implies that $G^l$ belongs to $J$. □

Let us consider briefly the case $K = \mathbb{C}$. Contrary to the case of smooth functions, now the Zariski topology over $Spec_m\, A$ does not coincide with its usual topology as a subset of $\mathbb{C}^n$. More so, the Zariski topology is strictly weaker that the usual one: given any basic closed set of the Zariski topology $V_f$, we have that $V_g \simeq \{ a \in Z(\langle f_1, .., f_m \rangle) \mid g(a) = 0 \}$ which is obviously closed respect to the usual topology in $Z(\langle f_1, .., f_m \rangle)$. More so, under certain conditions ($A$ being equal to some $K[x_1, .., x_n]/I$ with $I$ prime) it can be proved that the open sets from the Zariski topology are all dense respect the usual topology and in consequence respect the own Zariski topology. This is not the exception but the general rule: Zariski topologies are not usually a strong ones, and separation properties like being Hausdorff are generally out of their reach. In this regard, the malleability of smooth functions is the responsible for the "normalcy" of the Zariski topology over their $\mathbb{R}$-spectrum.

Hilbert's Nullstellensatz can be now rewritten in the following form: let $I, J$ be two ideals of $A$ such that $V(I) \subseteq V(J)$. Then $J \subseteq \sqrt{I}$. This has a nice consequence over the basic open sets $U_g := Spec_m\, A \setminus V_g$: it is satisfied that $U_h \subseteq U_g$ if and only if $h^l = fg$ for some $l$ and $f \in A$. It is worth noting that this property was not archieved in the case of smooth functions: Let us consider $\mathcal{C}^\infty_{(-1,1)}$. Then, if $g(x) = 0\chi_{[0]} + e^{-\frac{1}{x^2}}\chi_{(-1,0)\cup(0,1)}$ and $h(x) = x$, we have that $V_g = V_h \simeq 0$, but for any $n$, $\lim_{x\to 0^+} \frac{h(x)^n}{g(x)} = \infty$ (to see that just apply L'Hopital's rule until $(h^n)^{(m)}(0) \neq 0$, as $g$ satisfies $g^{(m)}(0) = 0$ for any $m$), so $h^n \neq fg$ for any $n$ and smooth function $f$.

Now we would like to give a method for equipping the spectrum $Spec_m\, A$ of a given geometric finitely generated $K$-algebra $A$ with a sheaf of rings $\mathcal{O}$ whose ring of global sections is precisely $A$. The resulting sheaf should satisfy such that $(Spec_m\, A, \mathcal{O})$ is a locally ringed with local rings precisely the localizations of $A$, $\mathcal{O}_\mathfrak{m} = A_\mathfrak{m}$. Things here are not as simple as with smooth functions. Here, the soft structure sheaf (see definition 1.4.5) of $A$ is not adequate for the task: if we let $A = \mathbb{C}[X]$ and consider the functions that coincide with polynomials locally (in the Zariski topology) then we end up with "copies" of $A$ for each open set $U \subset Spec_m\, A$. The reason for this the fact that if the values of two polynomials coincide in an open set they are the same polynomial. Again, the case of smooth functions should be more considered as an exception than as the rule in this regard. There the soft structure sheaf was enough because smooth functions are "locally invertible". That is to say that if a global smooth function $f$ does not annihilate at a point $p$ then there exists another global smooth function $g$ and an open set with $p \in U$ verifying $gf|_U = 1$.

Then what we need to do is to add inverses. For that we will make use of the localizations of $A$ over its elements:

**Definition 2.1.2.** Let $A$ be a ring and $f \in A$ an arbitrary element. Then we will call the **localization of** $A$ over $f$ to the ring $A_f$ of fractions of $A$ with denominators in the multiplicative set $S := \{f^n | n \in \mathbb{N}\} \cup \{1\}$.

Because of fact 1.4.1 it is satisfied that for any $f \in A$ there is a natural correspondence between $V_f$ and $Spec_m A_f$ given naturally by the localization map.$A_f$ is regular because of $A$ being so: let $a/f^k \in A_f$ be such that $(a/f^k)^n = 0$. Then for some $l$, $f^l a^n = 0$ which implies $(f^l a)^n = f^{(n-1)l}(f^l a^n) = 0$, so $f^l a = 0$ because the nilradical of $A$ is trivial. Thus, $a/1$ equals zero in $A_f$ and so does $a/f^k$. Also, $A_f$ is trivially a finitely generated algebra as well. This leads us to think that if we identify $U_f$ with $Spec\, A_f$ for each $f \in A$ then it would make sense that the "structure sheaf" of $A$ restricted to any $U_f$ were be the "structure sheaf" of $A_f$. Because of this, it is easy to see that for each $U_f$, its associated ring of sections should be $\mathcal{O}(U_f) = A_f$. Note that $A = A_1$. The restrictions between those sets are also easy to come up with: as we have seen, if $U_g \subset U_f$ then $g^m = lf$ for some $n$ and some $l \in A$. From this follows that the localization $(A_f)_g$ equals $A_g$, as $1/f = l/g^m$ in $(U_f)_g$ and for any $a \in A$, $f^n a = 0 \implies l^n f^n a = 0 \iff (g^m)^n a = 0$. Then we can define the restriction from $U_f$ to $U_g$ to be the localization map $A_f \to (A_f)_g$. With this we have a "sheaf-like" structure on $Spec_m\, A$, but only defined over a basis of open sets. The following can be proven:

**Definition 2.1.3.** Let $X$ be a topological space and $\mathcal{B}$ be a basis of open sets over $X$. Then a $\mathcal{B}$-presheaf $\mathcal{O}_{\mathcal{B}}$ over $X$ is given by a family of rings $(\mathcal{O}_{\mathcal{B}}(U))$ indexed by the open sets $U \in \mathcal{B}$ and a family restriction ring homomorphisms indexed by the pairs of contained open sets from $\mathcal{B}$ satisfying the properties (1), (2), (3) from the presheaf definition. A $\mathcal{B}$-sheaf is a $\mathcal{B}$-presheaf satisfying the following additional properties: let $U \in \mathcal{B}$ be an open set and $\{U_i\}$ be a open covering of $U$ with $U_i \in \mathcal{B}$ for every $i$. Then (1) the only section $s \in \mathcal{O}_{\mathcal{B}}(U)$ verifying $s|_{U_i} = 0$ for each $i$ is the zero section, and (2) if $(s_i)$ is a family of sections with $s_i \in \mathcal{O}_{\mathcal{B}}(U_i)$ for every $i$ satisfying that for each pair $i, j$ there exist an open set $V \in \mathcal{B}$, $V \subset U_i \cap U_j$ such that $s_i|_V = s_j|_V$ then there exists a section $s \in \mathcal{O}_{\mathcal{B}}(U)$ verifying $s|_{U_i} = s_i$ for each $i$.

**Proposition 2.1.1.** *Let $\mathcal{O}_{\mathcal{B}}$ be a $\mathcal{B}$-sheaf of rings over a topological space $X$ for some basis $\mathcal{B}$ of open sets. Then there exists a unique sheaf $\mathcal{O}$ over $X$ (up to isomorphism) such that $\mathcal{O}(U) = \mathcal{O}_{\mathcal{B}}(U)$ for every $U \in \mathcal{B}$.*

*Sketch of the proof.* For a full proof one can check [6]. Here we will just give a construction of the sheaf $\mathcal{O}$, that is as follows: for each point $p \in X$ we define the stalk $(\mathcal{O}_{\mathcal{B}})_p$ of $\mathcal{O}_{\mathcal{B}}$ at $p$ to be the inverse limit $\varinjlim_{U \in \mathcal{B},\, p \in U} \mathcal{O}_{\mathcal{B}}(U)$, analogously to the stalk definition for presheaves. Then, for any arbitrary open set $U \subset X$ we define the ring $\mathcal{O}$ to be:

$$\mathcal{O}(U) := \{\, f : U \to \cup_{p \in X} (\mathcal{O}_{\mathcal{B}})_p \text{ such that for every } p \in U \text{ there exist an open set } V \in \mathcal{B} \text{ satisfying } \\ p \in V \subseteq U \text{ and and a section } s \in \mathcal{O}_{\mathcal{B}}(V) \text{ such that } f(q) = s_q \text{ for every } q \in V\}$$

□

After that, we define the restrictions of $\mathcal{O}$ to be just the usual restrictions of its elements, which are functions.

Note the similarities with the definition of soft structure sheaves given in last chapter. The reason for them is that we are ultimately relying on constructions like the **étale space** associated to a given presheaf and the **sheafification** without introducing them.

**Proposition 2.1.2.** *Let $A$ be a finitely generated regular algebra over an algebraically closed field $K$. Let $\mathcal{B}$ be the basis of open sets from $Spec_m A$ consisting of the open sets of the form $U_f$, and let $\mathcal{O}_\mathcal{B}$ be the $\mathcal{B}$-presheaf over $Spec_m A$ given by $\mathcal{O}_\mathcal{B}(U_f) = A_f$ for each $f \in A$ equipped with the localization maps as restrictions. Then $\mathcal{O}_\mathcal{B}$ is a $\mathcal{B}$-sheaf. (Note that $U_0 = \emptyset$ and $A_0 = 0$)*

*Proof.* [2] page 19. □

**Definition 2.1.4.** We will call **the structure sheaf** of a finitely generated $K$-algebra over an algebraically closed field to the unique sheaf $\mathcal{O}_A$ over $Spec_m A$ that extends the $\mathcal{B}$-sheaf given in last proposition.

This construction is equivalent to the one of rational functions over algebraic subsets of $K^n$. Each section $h$ from the structure sheaf coincide locally around each point $\mathfrak{m}$ with a fraction of elements from $A$, $f/g$ (remember the construction given in proposition 2.1.1) such that $\mathfrak{m} \in U_g$. In particular, $g \notin \mathfrak{m}$.

**Proposition 2.1.3.** *Let $A$ be a finitely generated regular $K$-algebra over an algebraically closed field.Then the pair $(Spec_m A, \mathcal{O}_A)$ is a locally ringed space with stalks $(\mathcal{O}_A)_\mathfrak{m} = A_\mathfrak{m}$ and residue fields $K$.*

*Proof.* Let $\mathfrak{m} \in Spec_m A$. Every section from $\mathcal{O}_A$ can be locally represented by a fraction of elements from $A$ around $\mathfrak{m}$, so every $s \in (\mathcal{O}_A)_\mathfrak{m}$ is of the form $f_\mathfrak{m}/g_\mathfrak{m}$ where $f, g \in A$. We can proof that $1/f_\mathfrak{m}$ belongs to $(\mathcal{O}_A)_\mathfrak{m}$ if and only if $f \notin \mathfrak{m}$. If $f \notin \mathfrak{m}$ then $\mathfrak{m} \in U_f$ and $1/f \in A_f$, so $1/f_\mathfrak{m} \in \mathcal{O}_\mathfrak{m}$. Conversely, if $1/f_\mathfrak{m} \in (\mathcal{O}_A)_\mathfrak{m}$ for some $f \in A$, then $1/f = h/g^m$ in some basic open set $U_g$ containing $\mathfrak{m}$. Thus $g^l(g^m - fh) = 0$ for some $l$. Let $\mathfrak{n}$ be a maximal ideal containing $f$. Then $g^l fh \in \mathfrak{n}$ which means $g^{l+m} \in \mathfrak{n}$ and $g \in \mathfrak{n}$ because of $\mathfrak{n}$ being maximal. As $\mathfrak{m} \in U_g$, $\mathfrak{m}$ does not contain $g$ and in consequence $f \notin \mathfrak{m}$ either.

Sum and product of elements from the stalk do respect their natural representations as fractions: let $h_\mathfrak{m}/j_\mathfrak{m}, f_\mathfrak{m}/g_\mathfrak{m} \in (\mathcal{O}_A)_\mathfrak{m}$ for some $h, j, f, g \in A$. Clearly $(h/j)_\mathfrak{m} = h_\mathfrak{m}/j_\mathfrak{m}$ and $(f/g)_\mathfrak{m} = f_\mathfrak{m}/g_\mathfrak{m}$. We have that both $h/j$ and $f/g$ belong to $A_{jg}$, and there $h/j + f/g = (gh+jf)/jg$ and $h/j \cdot f/g = hf/jg$. Thus we can substitute $h, j, f, g$ for $h_\mathfrak{m}, j_\mathfrak{m}, f_\mathfrak{m}, g_\mathfrak{m}$ respectively in the previous equalities and they hold in $\mathcal{O}_\mathfrak{m}$.

Then, we have proven that the map $\phi : A_\mathfrak{m} \to (\mathcal{O}_A)_\mathfrak{m}$ given by $f/g \mapsto f_\mathfrak{m}/g_\mathfrak{m}$ is a surjective ring homomorphism. Let us see that it is indeed an isomorphism. Suppose that there is an element $f/g \in A_\mathfrak{m}$ such that $\phi(f/g) = 0$. This would imply that $f/g|_{U_h} = 0$ for some basic open set $U_h$ such that $\mathfrak{m} \in U_h \subseteq U_g$ (remember that necessarily $\mathfrak{m} \in U_g$). In consequence $h^l f = 0$ for some $l$, but this implies that $f/g = 0$ in $A_\mathfrak{m}$ as $h \notin \mathfrak{m}$. So we have finally obtained that $A_\mathfrak{m} \simeq \mathcal{O}_\mathfrak{m}$ for every $\mathfrak{m} \in Spec\, A$.

The last part follows trivially from the fact that the residue field of $A_\mathfrak{m}$ is precisely $A/\mathfrak{m}$. □

It is satisfied that $(Spec_m A, \mathcal{O}_A)$ is geometric as a locally ringed space. Given a section $s$ of $\mathcal{O}_A$ over some open set $U$, if the value of $s$ at the residue field of every point in $U$ is 0 then $s \in \mathfrak{m}$ for every $\mathfrak{m} \in U$. Let us consider a basic open covering of $U$, $\{U_{f_i}\}$. Then, for any $i$, $s|_{f_i} \in \mathfrak{m}$ for any $\mathfrak{m} \in U_{f_i}$. As $U_{f_i} \simeq Spec_m A_f$ and $\mathcal{O}_A(U_f) = A_f$ we have that $s|_{f_i} = 0$ because of $A_f$ being geometric. In consequence, $s = 0$.

Thus, we can consider all the sections of $\mathcal{O}_A$ to be functions from their respective domains to $K$ and do so in a way that the elements of each $A_f$ adopt their usual values in every point. That is to say that if we identify $Spec\, A$ with some subset of $K^n$ as described in this section, then the value of an element $g[x] \in A$ at a point $\mathfrak{m}_a$ is precisely $g(a)$.

Let us consider the following category. As objects we take the geometric locally ringed spaces $(X, \mathcal{O})$ where $\mathcal{O}$ is a sheaf of $K$-algebras and its residue fields are all $K$, and as morphisms we take the morphisms of locally ringed spaces $(f, \varphi)$ such that the components from $\varphi$ are morphisms of $K$-algebras. We will call to that category **the category of $K$-locally ringed spaces**, $K$-**LocalRingS**.

Up until now we have constructed an association between finitely generated $K$-algebras and objects form $K$-**LocalRingS**. Given an algebraically closed field will call **affine algebraic $K$-variety** to any object in $K$-**LocalRingS** of the form $(Spec_m\, A, \mathcal{O}_A)$ where $A$ is a geometric finitely generated $K$-algebra. We can extend this association to a fully faithful (contravariant) functor.

**Proposition 2.1.4.** *Let $(f, \varphi)$ be a morphism in $K$-**LocalRingS**. Then $(f, varphi)$ is completely determined by $f$.*

*Proof.* Exactly the same proof as in proposition 1.3.4. Just note that the condition of every map being a $K$-morphism implies that induced map on the residue field is the identity because $K$ has to be fixed. □

**Corollary 2.1.3.** *Let $(f, \varphi)$ be a morphism in $K$-**LocalRingS**. Then the components of $\varphi$ are given by composition with $f$. In other words, they are of the form $s \mapsto s \circ f$. We will denote $\varphi = f^{\#}$.*

*Proof.* The same as in corollary 1.3.1, knowing that all the endomorphisms over $K$ must be the identity. □

Let, $A$ and $B$ be two geometric finitely generated $K$-algebras for some algebraically closed field $K$. As in proposition 1.4.3, it is satisfied that the set of morphisms in $K$-**LocalRingS** between the algebraic varieties respectively associated to $A$ and $B$ is naturally bijective to the set of $K$-algebra morphisms from $B$ to $A$. The bijection is as follows: if $(f, \varphi) : (Spec_m\, A, \mathcal{O}_A) \to (Spec_m\, B, \mathcal{O}_B)$ is a morphism from $K$-**LocalRingS**, then $\varphi_B$ is a $K$-algebra morphism by definition. It has been proven that said association is injective, as $(f, \varphi)$ is determined by $f$ and $\varphi = f^{\#}$. Conversely:

**Proposition 2.1.5.** *Let $A$ and $B$ be two geometric finitely generated $K$-algebras and $\alpha : B \to A$ a $K$-algebra morphism. Then the map $|\alpha| : Spec_m\, A \to Spec_m\, B$ given by $|\alpha|(\mathfrak{m}) = \alpha^{-1}(\mathfrak{m})$ is continuous and $\alpha(f) = f \circ |\alpha|$ for any $f \in B$.*

*Proof.* For each point $\mathfrak{m} \in A$ it is defined a unique $K$-morphism for $A$ $u_{\mathfrak{m}}$ with $ker\, u_{\mathfrak{m}} = \mathfrak{m}$, namely the "evaluation at $\mathfrak{m}$" morphism. Then we can define the following $K$-morphism for $B$, $v : B \to K$ $v(f) = u_{\mathfrak{m}}(\alpha(f))$. Then $ker\, v \in Spec_m\, B$, $ker\, v = \alpha^{-1}(\mathfrak{m})$ and $|\alpha|$ is well defined.

To see that $|\alpha|$ it suffices to show that $|\alpha|^{-1}(U_f)$ is closed for each $f \in B$. Just note that for a given point $\mathfrak{m} \in Spec_m\, A$, $|\alpha|(\mathfrak{m}) \in U_f$ if and only if $\alpha(f) \notin \mathfrak{m}$. In consequence $|\alpha|^{-1}(U_f) = U_{\alpha(f)}$ and $|\alpha|$ is continuous.

For the second part note that given a point $\mathfrak{m} \in Spec_m\, A$ then the value of each element $f \in A$ at $\mathfrak{m}$ is given by the evaluation morphism $u_{\mathfrak{m}}$ with $ker\ \mathfrak{m}$. The map that sends each $g \in B$ to the evaluation of $\alpha(g)$ at $\mathfrak{m}$ is a $K$-morphism with $ker\ \alpha^{-1}(\mathfrak{m}) = |\alpha|(\mathfrak{m})$, so it is the evaluation morphism at $|\alpha|(\mathfrak{m})$. In other words, for any $g \in B$, the value of $\alpha(g)$ at $\mathfrak{m}$ equals the value of $g$ at $|\alpha|(\mathfrak{m})$, so $\alpha(g) = g \circ |\alpha|$. □

**Theorem 2.1.3.** *Let $A$ and $B$ be two geometric finitely generated $K$-algebras and $\alpha : B \to A$ a $K$-algebra morphism with $K$ an algebraically closed field. Then there exists a morphism in $K$-**LocalRingS** of the form $(|\alpha|, \varphi) : (Spec_m\, A, \mathcal{O}_A) \to (Spec_m\, B, \mathcal{O}_B)$ such that $\varphi_{Spec_m\, B} = \alpha$.*

*Sketch of the proof.* If such morphism exists then necessarily $\varphi = |\alpha|^\#$. Let us see that $|\alpha|^\#$ is well defined. Let $f$ be a section of $\mathcal{O}_B$ over some open set $U \subseteq Spec_m B$. Then we have to proof that $f \circ \alpha$ belongs to $\mathcal{O}_A(|\alpha|^{-1}(U))$. Let $\{U_{f_i}\}$ be a open cover of $U$. Then $f|_{U_{f_i}} = g_i/f_i^{n_i}$ for any $i$, as $\mathcal{O}_B(U_{f_i}) = B_{f_i}$. Trivially $(g_i/f_i^{n_i}) \circ |\alpha| = (g_i \circ |\alpha|)/(f_i^{n_i} \circ |\alpha|)$ and because of last proposition this equals $\alpha(g_i)/\alpha(f_i)^{n_i}$. Then, for any $i$ $|\alpha|^\#_{U_i}(f|_{U_i})$ is well defined as $\alpha(g_i)/\alpha(f_i)^{n_i}$ belongs to $\mathcal{O}_A(|\alpha|^{-1}(U_{f_i})) = \mathcal{O}_A(U_{\alpha(f_i)}) = A_{\alpha(f_i)}$. Thus, if we call $h = |\alpha|^\#_U(f)$, then the usual restriction of $h$ to each $U_{f_i}$ is $\alpha(g_i)/\alpha(f_i)^{n_i}$ respectively for each $i$. It is also satisfied that $\{U_{\alpha(f_i)}\}$ is an open covering of $|\alpha|^{-1}(U)$. As the restrictions of $h$ over an open covering of its domain are sections of $\mathcal{O}_A$ and $\mathcal{O}_A$ is a sheaf then $h$ itself is a section from $\mathcal{O}_A$. □

Thus, we have defined fully faithfull contravariant functors from the category of finitely generated $K$-algebras to $K$-**LocalRingS** for each algebraically closed field in the following way:

| | | |
|---|---|---|
| Finitely generated geometric $K$-algebras | $\to$ | Affine algebraic $K$-varieties |
| $A$ | $\mapsto$ | $(Spec_m A, \mathcal{O}_A)$ |
| $K$-algebra morphisms | $\to$ | Morphisms in $K$-**LocalRingS** |
| $\alpha : B \to A$ | $\mapsto$ | $(\lvert\alpha\rvert, \lvert\alpha\rvert^\#) : (Spec_m A, \mathcal{O}_A) \to (Spec_m B, \mathcal{O}_B)$ |

This way we have found again a strong connection between some geometrical objects, the affine algebraic $K$-varieties, and some kind of rings, the finitely generated geometric $K$-algebras.

It is often interesting to consider geometric objects that are not affine algebraic varieties but are locally isomorphic to them. This motivates the following definition:

**Definition 2.1.5.** Let $K$ be an algebraically closed field. A locally ringed space $(X, \mathcal{O})$ will be called **algebraic $K$-variety** if for any point $p \in X$ there is an open set $U \subseteq X$ such that $(U, \mathcal{O}|_U)$ is isomorphic to some affine algebraic $K$-variety.

One of the most important examples of algebraic $K$-varieties which are not affine are the projective spaces, which are geometrical objects also constructed over geometric finitelly generated $K$-algebras in a similar way to affine varieties, but considering homogeneous ideals instead of maximal ones and taking the fractions of homogeneous polynomials as the "good functions" instead of general polynomials and their fractions( [1], page 8).

## 2.2 Schemes

A logical step towards generalizing the constructions from last section is trying to apply them to general rings. That way we obtain schemes. The goal of this section will be to sketch the way in which previous notions have to be generalized to construct such objects. If one wants to check the constructions in full detail one may check [2], I.1, I.2, and [1], 2.2.

The first obstacle we find in the way is that, even though in the last section it worked well for us, in general preimages of maximal ideals through arbitrary ring homomorphisms are not maximal ideals themselves. A way to solve this is to consider prime ideals instead of maximal ones.

**Definition 2.2.1.** Let $A$ be a ring. Then **the spectrum of** $A$, $Spec\, A$ is the set of prime ideals from $A$.

Prime ideals have the following property that allow us to prove "Nullstellensatz-like" results:

**Lemma 2.2.1.** *Let $A$ be a ring and let $I \lhd A$ be an ideal. Then it is satisfied that* $\sqrt{I} = \bigcap_{\mathfrak{p} \in Spec\, A,\, I \subseteq \mathfrak{p}} \mathfrak{p}$. *In particular* $\sqrt{\langle 0 \rangle} = \bigcap_{\mathfrak{p} \in Spec\, A}$.

*Sketch of the proof.* Clearly, $\sqrt{I} \subseteq \mathfrak{p}$ for any prime ideal $\mathfrak{p}$ containing $I$. Conversely, if $a \notin \sqrt{I}$ using Zorn's lemma we can conclude that there exists a maximal element $\mathfrak{p}$ in the set of ideals containing $I$ and not containing $a$. One can then proof that $\mathfrak{p}$ has to be prime.

The last part of the lemma follows form the fact that every prime ideal contains the zero ideal. □

Now we would like to realize the elements of $A$ as functions over $Spec\,A$ analogously to last section. Given any $\mathfrak{p} \in Spec\,A$, the ring $A/\mathfrak{p}$ is not always a field but it is always an integral domain. So we can define for each $a \in A \quad a(\mathfrak{p}) = (a + \mathfrak{p})/1$, where $(a + \mathfrak{p})/1$ is the natural image of $(a + \mathfrak{p})$ in the field of fractions from $A/\mathfrak{p}$, $Frac(A/\mathfrak{p})$. After this we end up with functions over $Spec\,A$ whose image for each point always lie in some field but that field depends in general on the given point: let $\mathfrak{p}, \mathfrak{q}$ be two different points from $Spec\,A$. Then for any $a \in A$, $a(\mathfrak{p}) \in Frac(A/\mathfrak{p})$ and $a(\mathfrak{q}) \in Frac(A/\mathfrak{q})$ but in general $Frac(A/\mathfrak{p})$ does not equal $Frac(A/\mathfrak{q})$.

We should impose to our rings of interest some kind of property analogous to being geometric: if some $a \in A$ satisfies $a(\mathfrak{p}) = 0$ for every $\mathfrak{p} \in Spec\,A$ then $a$ should be the zero element from $A$. Note that $a(\mathfrak{p}) = 0 \iff a \in \mathfrak{p}$, so $a(\mathfrak{p}) = 0 \ \forall \mathfrak{p} \in Spec\,A \iff a \in \bigcap_{\mathfrak{p} \in Spec\,A} \iff a \in \sqrt{\langle 0 \rangle}$. So if we want to realize every non-zero element from $A$ as a non-zero function $A$ has to be a regular ring.

We can define the Zariski topology over $Spec\,A$ exactly in the same way as in previous sections, defining $V(S) := \{\, \mathfrak{p} \in Spec\,A \mid S \subseteq \mathfrak{p} \,\}$. We can also define the structure sheaf of $A$, $\mathcal{O}_A$ following the same steps as before, obtaining sections which are locally fractions of elements from $A$.

For any regular ring $A$, the pair $(Spec\,A, \mathcal{O}_A)$ is a geometric locally ringed space whose stalk at each point $\mathfrak{p} \in Spec\,A$ is $A_{\mathfrak{p}}$. This is also proven in exactly the same way as in the previous section. Locally ringed spaces of the form $(Spec\,A, \mathcal{O}_A)$ for a given regular ring $A$ are called **affine schemes**.

Analogously to before, ring homomorphisms $\alpha : A \to B$ between regular rings determine continuous functions $|\alpha| : Spec\,B \to Spec\,A$ by means of defining $|\alpha|(\mathfrak{p}) = \alpha^{-1}(\mathfrak{p})$ and the already given proofs still work.

It is also true that every ring homomorphism $\alpha : A \to B$ between regular rinds determines a unique morphism of locally ringed spaces $(|\alpha|, |\alpha|^{\#}) : (Spec\,B, \mathcal{O}_B) \to (Spec\,A, \mathcal{O}_A)$ between the respective affine schemes. This is a little trickier to proof ([1], page 73), as the residue fields from both locally ringed spaces differ from point to point. To close that gap we need to consider the naturally induced maps $\alpha_{\mathfrak{p}} : Frac(A/\alpha^{-1}(\mathfrak{p})) \to Frac(B/\mathfrak{p})$ for each $\mathfrak{p} \in B$ given by $\frac{a+\alpha^{-1}(\mathfrak{p})}{b+\alpha^{-1}(\mathfrak{p})} \mapsto \frac{\alpha(a)+\mathfrak{p}}{\alpha(b)+\mathfrak{p}}$. That way, if $f$ is a section of $\mathcal{O}_A$ over some open set $U \subseteq Spec\,A$ we can define $|\alpha|^{\#}_U(f)$ as the section from $\mathcal{O}_B(|\alpha|^{-1}(U))$ satisfying $|\alpha|^{\#}_U(f)(\mathfrak{p}) = \alpha_{\mathfrak{p}}((f \circ |\alpha|)(\mathfrak{p})) = \alpha_{\mathfrak{p}}(f(\alpha^{-1}(\mathfrak{p})))$.

That concludes a full contravariant functiorial association between the category of regular rings and the category of locally ringed spaces which is completely analogous to the one introduced in last section.

Finally, in general a **scheme** is defined to be a locally ringed space that is locally isomorphic to affine shemes around each point. That way we consider more general objects that still share nice properties with affine schemes. The example of projective algebraic varieties has its scheme-theoretic analogous in projective schemes, which are modeled after graded rings.

## Afterword

The starting point of this work were some objects modeled after smooth planes and curves in the space. Then we went through others modeled after the loci of zeros from ideals of polynomials and in the end we finished with some objects which do not have an easy intuitive notion associated to.

It may look like scheme theory is an over-generalization that derives from some other more sensible geometric theories and turns them into an abstract nonsense. The truth is that some of the more powerful tools which work in the more concrete settings work at their best in the framework of schemes, allowing surprising applications of the there found geometric notions in fields like number theory. The non-obvious connections with that field arise considering the affine scheme $Spec\,\mathbb{Z}$.

Impressively enough, this development eventually played a role in the Wile's proof of Fermat last theorem [7]. Scheme-theoretic notions are also used in modern elliptic curve theory ([8], [9]), which is an important field of investigation with applications in cryptography.

The goal of this work was to serve as an introduction towards this theory, which at first glance may seem unjustified but has plenty of richness further ahead. The gap between its rewarding applications and the initial need to introduce lots of technical tools can be felt as a tedious one but hopefully this writing helped bridging it.

Thank you for reading.

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